

BDA Project

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1. Project Description and Motivation

Studying the relationship between income and education has been the focus of many studies. The studies have concluded that a strong connection exists between higher education and income (Card, 1999). In general, individuals with stronger education are more likely to be employed and earn a big salary compared to less educated people (Card, 1999). For that reason, education is described as an investment in human capital (Wolla & Sullivan, 2017).

This study examines this phenomenon from the perspective of people that have acquired their education from colleges in the United States. As the connection between education and income has been shown in the existing literature, this study strives to further examine the associations between college related features and income level years after graduation. This project is not limited to only considering educational aspects but expands it to family backgrounds.

In this study, a Bayesian approach is taken to observe the bond between the educational and family related features and earnings in a multivariate linear regression setting. It is in our interest to find out how accurately the selected features can predict future income for the students. Furthermore, finding well-predictive features among the vast number of variables is pursued, and then evaluating predictive performance using these features with selected statistical models.

As this study is conducted in a university environment by university students and presented mainly to other students and faculty, the findings of this study could be especially meaningful for the members of the group and peers on the course.

2. Data description

LAITETTAVA TARKEMPI LÄHDE DATALLE (MISTÄ LÖYTYY)!

The used dataset is the most recent institutional-level college scorecard data from the US Department of Education¹. The institutional-level dataset contains aggregate data for each educational institution and includes data on institutional characteristics, enrollment, student aid, costs and student outcomes. The dataset has 6681 observations on 2989 variables.

This dataset was chosen because it seemed to provide information that could answer interesting education-related questions in the project, and also because the dataset has a large number of observations and a large number of variables. The large amount of variables permits for a lot of flexibility when it comes to modelling with the data and also having enough data is important in order to be able to make valid inferences.

After investigating the dataset, the most intriguing analysis problem was to study how college related factors affect the median earnings of alumni 10 years after entry, which is our dependent variable:

Name	Data type	Description
MD_EARN_WNE_P10	double	Median earnings of students 10 years after entry

In the next section, we outline how features were selected

Feature selection

The initial data set has almost 3000 features, which is a large number compared to the total number of observations of 6681. Moreover, there are many missing values for certain variables in the dataset and not all variables are interesting when trying to predict future earnings. For these reasons, there was a clear need to prune down features.

¹Available at: <https://collegescorecard.ed.gov/data>

We conducted feature selection in three phases: in the first phase, a subset of features was selected based on features commonly used in previous studies that have examined the relationship between educational factors and earnings. Moreover, we used common sense when trying to come up with additional interesting features that could predict future earnings. In the second phase, we assessed the relationship of the features with our dependent variable, and accounted for any multicollinearity arising from mutually correlated features. Furthermore, we visualized categorical variables and engineered new features to be added to the model. In the last phase, we utilized stepwise regression in streamlining our model to include only the most significant features in terms of the Akaike Information Criterion (AIC).

Phase 1 - Feature selection based on literature and common sense

The initial set of features chosen based on past literature and common sense is outlined in the table below:

	Name	Data type	Description
1	SATVRMID	integer	Midpoint of SAT scores at the institution (critical reading)
2	SATMTMID	integer	Midpoint of SAT scores at the institution (math)
3	SATWRMID	integer	Midpoint of SAT scores at the institution (writing)
4	MD_FAMINC	double	Median family income
5	AGE_ENTRY	double	Average age of entry
6	FEMALE	double	Share of female students
7	FIRST_GEN	double	Share of first-generation students
8	PCT_WHITE	double	Percent of the population from students' zip codes that is White
9	DEBT_MDN_SUPP	integer	Median debt, suppressed for n=30
10	C150_4	double	Completion rate for first-time, full-tim students
11	COSTT4_A	integer	Average cost of attendance (academic year institutions)
12	POVERTY_RATE	double	Poverty rate
13	UNEMP_RATE	double	Unemployment rate
14	MARRIED	double	Share of married students
15	VETERAN	double	Share of veteran students
16	LOCALE	categorical	Locale of institution
17	CCBASIC	categorical	Carnegie Classification – basic
18	CONTROL	categorical	Control of institution

Descriptive statistics for the numerical variables:

Table 1: Descriptive Statistics of the variables. Variable is the name of the variable, Min is the minimum, 1st Qu. is the 1st quartile, Median is the median, Mean is the arithmetic mean, 3rd Qu. is the 3rd quartile, Max is the maximum and St.dev is the sample standard deviation.

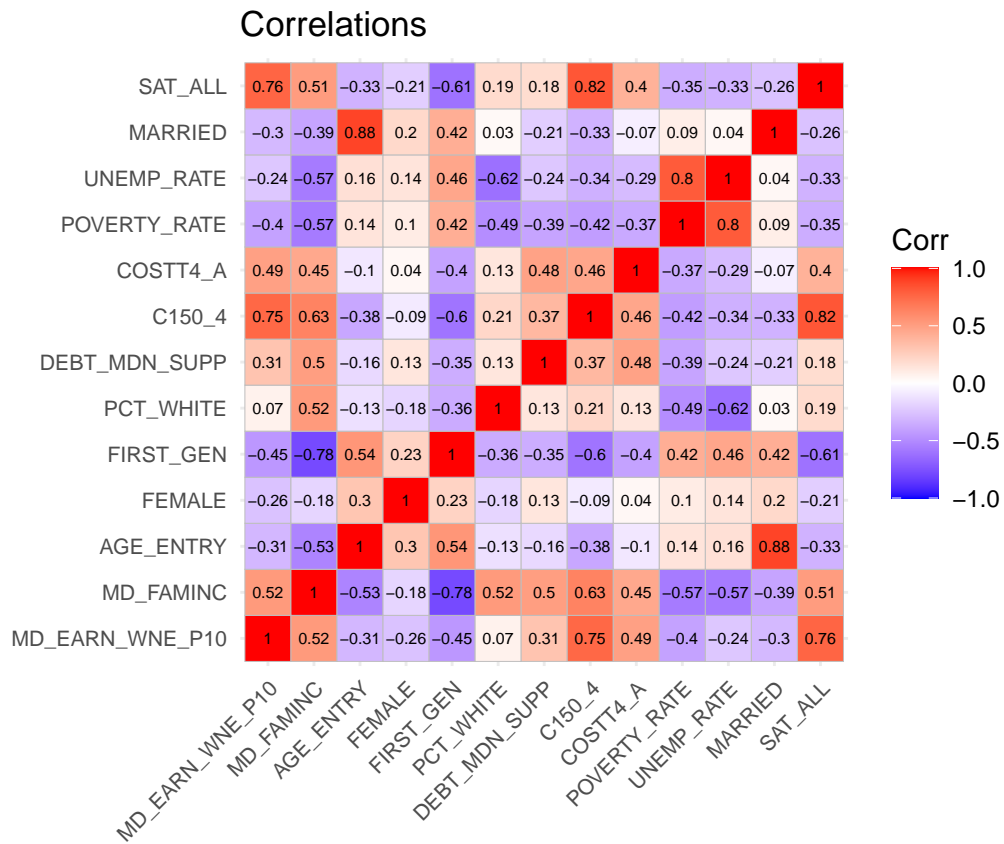
	Min	1st Qu.	Median	Mean	3rd Qu.	Max	St.dev
MD_EARN_WNE_P10	24209.00	43093.00	49240.50	51246.87	56516.75	103246.00	11587.58
MD_FAMINC	10702.00	35386.12	44858.50	48915.83	61402.38	121852.50	18732.71
AGE_ENTRY	19.55	20.92	22.04	22.53	23.51	33.82	2.18
FEMALE	0.12	0.54	0.59	0.59	0.64	0.98	0.10
FIRST_GEN	0.10	0.27	0.34	0.33	0.39	0.62	0.09
PCT_WHITE	24.24	71.45	80.24	77.62	87.62	95.96	13.09
DEBT_MDN_SUPP	3688.00	13971.00	16000.00	16190.37	19000.00	26800.00	3595.32
C150_4	0.09	0.46	0.56	0.57	0.68	0.98	0.16
COSTT4_A	11704.00	23250.25	31904.00	35719.42	45932.00	79750.00	15267.87
POVERTY_RATE	3.58	6.44	7.74	8.68	9.87	45.73	3.81
UNEMP_RATE	2.12	3.01	3.34	3.48	3.78	7.92	0.74
MARRIED	0.01	0.03	0.06	0.08	0.10	0.38	0.07
SAT_ALL	395.50	513.25	545.00	553.57	584.62	760.00	59.93

Phase 2 - Feature selection with correlation and linear associativity

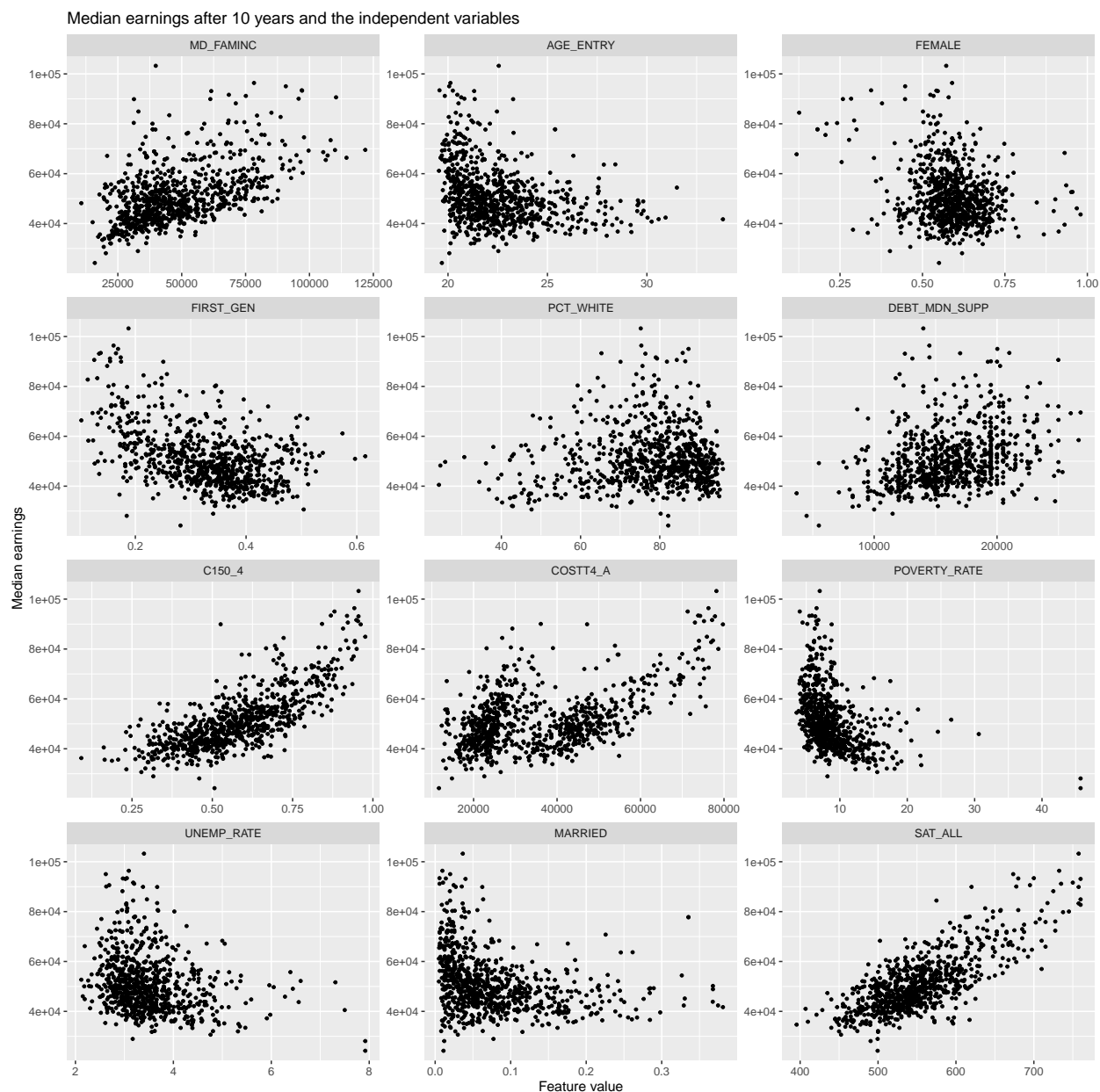
In the second phase of feature selection, a subset of features was selected from the 18 variables resulting from the first phase. We examined the correlations between the dependent variable and the features as well as between-features correlations. Moreover, we examined whether there were any linear relationships between the features and the dependent variable. Lastly, we visually examined categorical variables and engineered new features that we believed to have predictive ability over future earnings.

- **VOISI MUUTTA NIIN, ETTÄ PURETAAN TAULUKOT JA KUVAAJAT JA LOPUKSI YHTEENVETO MITÄ VALITTIIN**

Correlation analysis



Bivariate plotting



SAT scores were combined as one variable, because they were correlated and viewed as one entity. However, writing SAT scores had too few observations, due to discontinued tracking, so the variable SAT_ALL was formed by summing the math and critical thinking SAT scores. SAT scores were included because the correlation was high with the dependent variable.

For the rest of the numerical variables, if the correlation between a feature and the dependent variable was low and there was no observable dependency between the two in the scatter plot, the feature was excluded. To avoid multicollinearity, we removed independent variables that were highly correlated with other independent variables, especially if they were not clearly correlated with the dependent variable and we couldn't form a believable hypothesis for the mechanism through which that variable affected income after college.

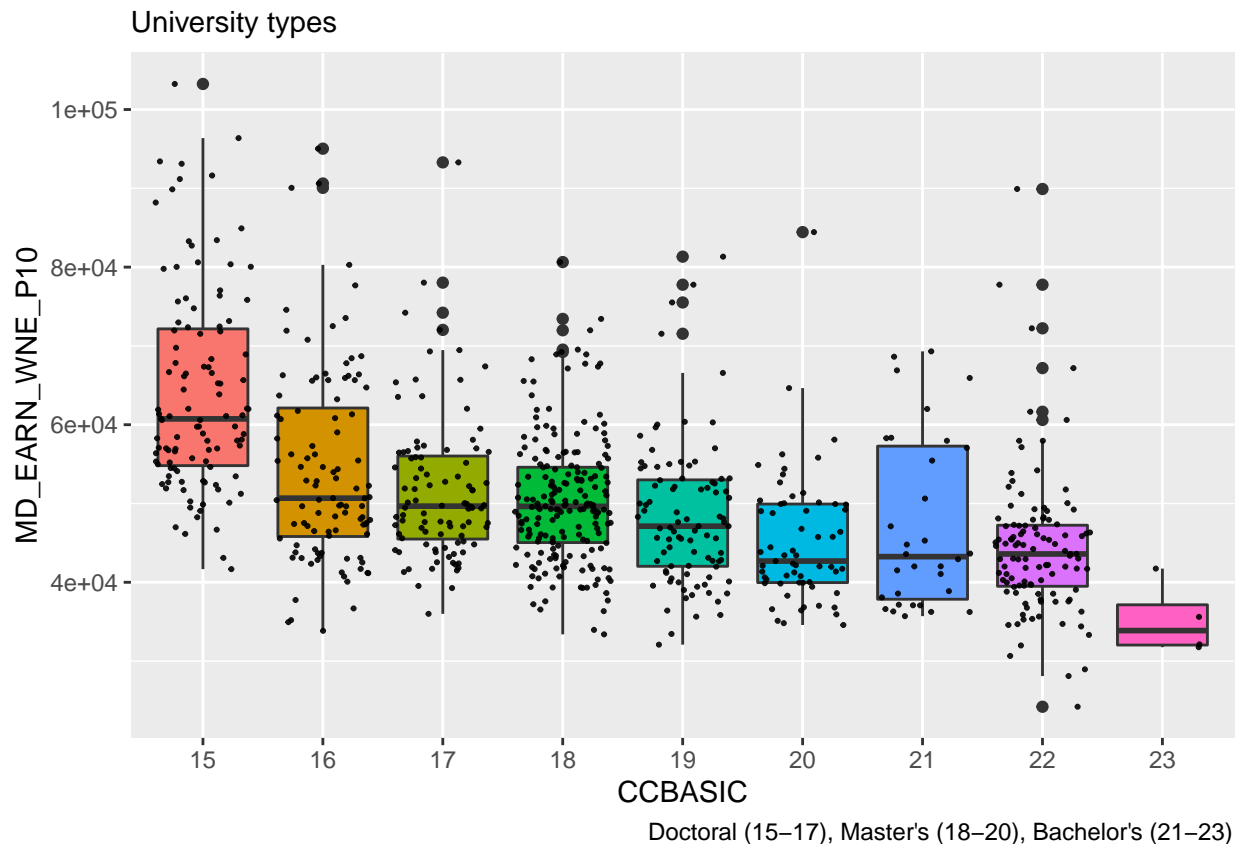
The selected numerical variables were: SAT_ALL (median sum of math and critical thinking SAT scores),

MD_FAMINIC (median family income of the student), AGE_ENTRY (median age of starting at the college), COSTT4_A (median cost of college), and POVERTY_RATE (poverty rate in the area the college is located).

Categorical variables

The categorical variable CCBASIC which represented the Carnegie Classification was divided into two binary MASTER and DOCTORAL. These variables represent if the college is classified as Master's college and university or Doctoral's college and university. If a college does not belong to either of those, it is a Bachelor's college and university. Other special focus colleges and universities were discarded from the dataset to keep the model more simple and avoid unnecessary outliers due to unconventional nature of some very specialized colleges. Our model does not attempt to provide accurate predictions for specialized colleges.

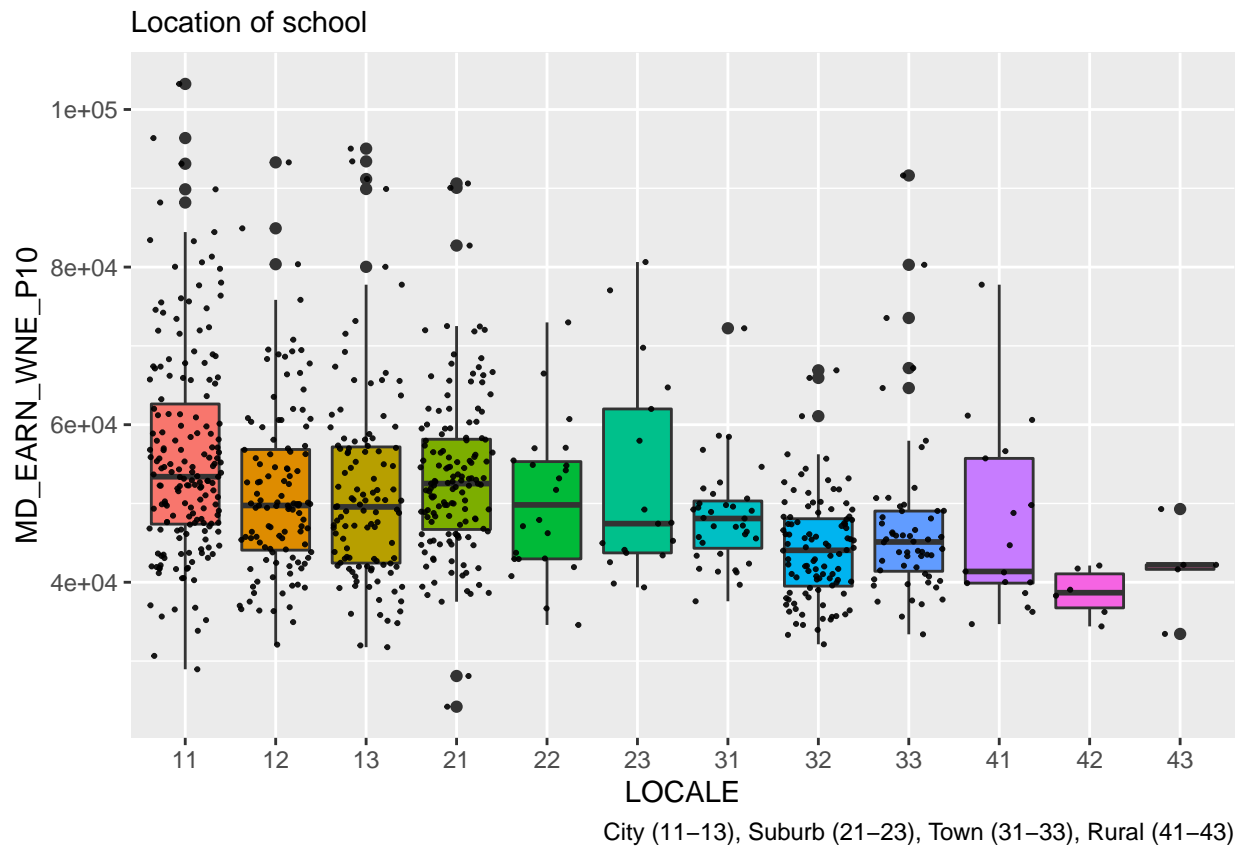
```
ggplot(melted.categorical.ccbasic, aes(x=value, y=MD_EARN_WNE_P10, fill=value)) +
  geom_boxplot() +
  geom_jitter(color="black", size=0.4, alpha=0.9) +
  theme(
    legend.position="none",
    plot.title = element_text(size=11)
  ) +
  ggtitle("University types") +
  xlab("CCBASIC") +
  labs(caption = "Doctoral (15-17), Master's (18-20), Bachelor's (21-23)")
```



The categorical variables were visualized with box plots to see if income after college differed among the categories. Based on an analysis on the variable LOCALE (whether college located in metropolis, city,

suburb, town, or rural area) a new binary variable URBAN was generated, where value 1 represents any area that is not categorized as rural.

```
ggplot(melted.categorical.locale, aes(x=value, y=MD_EARN_WNE_P10, fill=value)) +
  geom_boxplot() +
  geom_jitter(color="black", size=0.4, alpha=0.9) +
  theme(
    legend.position="none",
    plot.title = element_text(size=11)
  ) +
  ggtitle("Location of school") +
  xlab("LOCALE") +
  labs(caption = "City (11-13), Suburb (21-23), Town (31-33), Rural (41-43)")
```

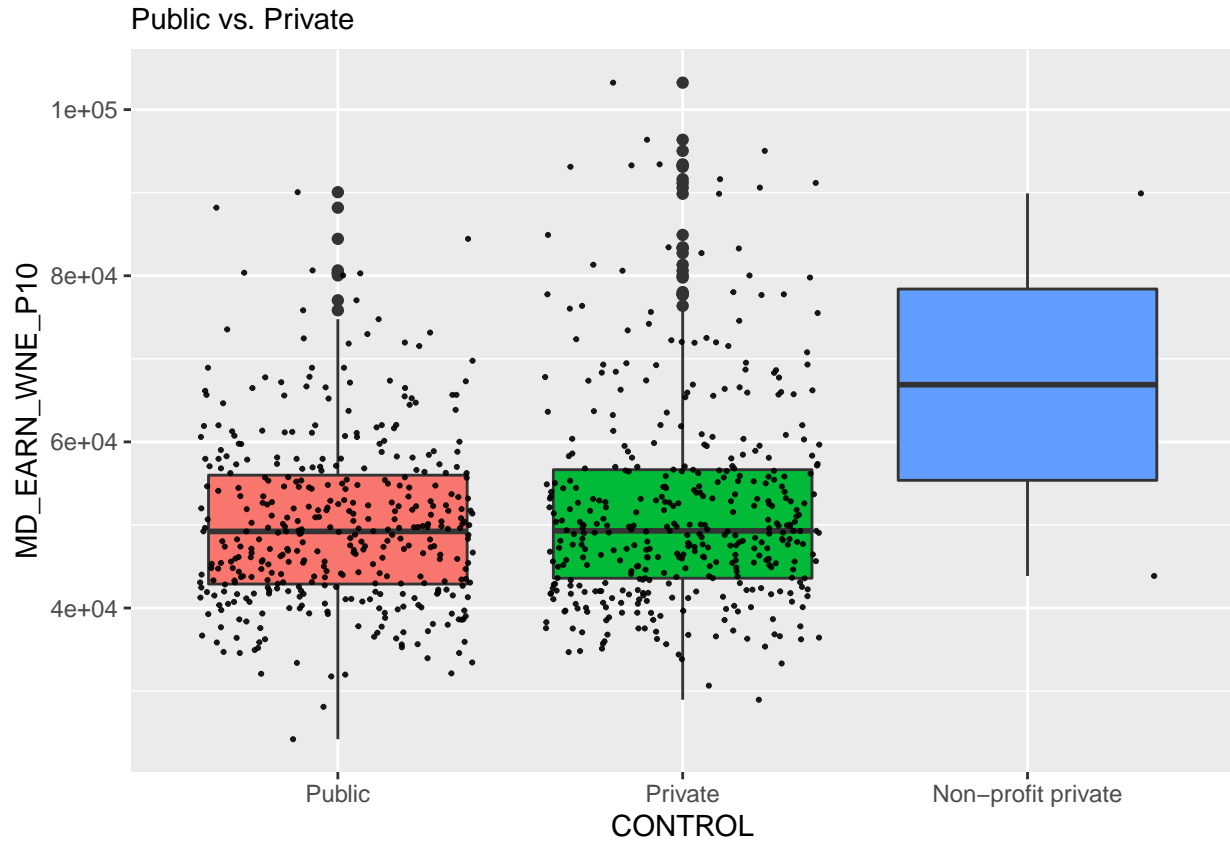


The categorical variable CONTROL had three classes: public, for-profit private and nonprofit private. CONTROL was modified into a binary variable PRIVATE representing if the school is private or public school. There were only few non-profit private observations, so those were merged together with the private observations.

```
ggplot(melted.categorical.control, aes(x=value, y=MD_EARN_WNE_P10, fill=value)) +
  geom_boxplot() +
  geom_jitter(color="black", size=0.4, alpha=0.9) +
  theme(
    legend.position="none",
    plot.title = element_text(size=11)
  ) +
```



```
ggtitle("Public vs. Private") +
xlab("CONTROL") +
scale_x_discrete(labels = c("Public","Private","Non-profit private"))
```



The selected categorical variables were: URBAN, DOCTORAL, MASTER, PRIVATE.

Phase 3 - Feature selection with stepwise regression

In the third and final phase, all remaining variables were used with stepwise regression to test which subset of features perform the best, whether the received coefficients are reasonable, and if there are signs of overfitting.

The stepwise regression suggested using all of the variables, except CITY and DOCTORAL. The output of the stepwise regression can be found in the Appendix. Because stepwise regression suggested leaving out DOCTORAL, for the sake of consistency we decided to also leave out MASTER as it was a the middle level in our institutional classifications and wouldn't have made much sense on its own.

The final features are listed in the table below:

	Name	Data type	Description
1	SAT_ALL	float	Midpoint of SAT scores at the institution (critical reading , math)
2	MD_FAMINC	float	Median family income
3	AGE_ENTRY	float	Average age of entry
4	COSTT4_A	float	Average cost of attendance (academic year institutions)
5	POVERTY_RATE	float	Poverty rate
6	PRIVATE	binary	Carnegie Classification – –basic

3. Description of the models

MATEMATIIKKA TARKISTETTAVA

Description of the separate model

In the separate model, posteriors for the parameters are constructed. In the context of the project, the separate model considers all regions independent from each other, meaning that each region have individual parameters (μ , σ).

Mathematical description:

$$y_{ij}|\mu_j, \sigma_j \sim \mathcal{N}(\mu_j, \sigma_j^2)$$

where $\mu_j = \alpha_j + \beta_j \mathbf{X}$

The parameters of the parameter vector are given in the section 4 with their priors.

Description of the pooled model

As in the separate model, the pooled model constructs posteriors for the parameters. However, the regions are considered as one entity meaning that all regions share the same distribution and parameter values.

Mathematical description:

$$y_i|\mu, \sigma \sim \mathcal{N}(\mu, \sigma^2)$$

where $\mu = \alpha + \beta \mathbf{X}$

The parameters of the parameter vector are given in the section 4 with their priors.

Description of the hierarchical model

Contrary to the other two models, in the hierarchical model posteriors are constructed for the prior parameters. With the hierarchical model, the regions are considered as individual but similar. In the context of the project, the regions share same sigma and the parameters forming mu are similarly distributed (sharing the hyperparameters).

Mathematical description:

$$y_{ij}|\mu_j \sim \mathcal{N}(\mu_j, \sigma^2)$$

where $\mu_j|\mu_P, \sigma_P^2 \sim \mathcal{N}(\mu_P, \sigma_P^2)$

The hyperparameters and parameters are given in the section 4 with their priors.

Description of the linear model

MIKÄ TÄÄ ON?

4. Informative or weakly informative priors, and justification of their choices.

We use weakly informative priors, as we do not possess enough information about the dependency between the independent variables and the dependent variable. When selecting the priors, proper distribution type was considered for each variable. The prior standard deviations were selected based on the magnitude of absolute values of the variables and what kind of impact they could have for income, with loose enough estimates to avoid limiting the values too much. Alternative approach could have been to standardize all variables, which would have reduced the need to think about magnitudes of absolute values.

$SAT_ALL \sim \text{Normal}(43, 500)$

Justification: We agreed that it was somewhat reasonable to expect scores in the SAT exam to be associated with higher income. Due to our approach of using weakly informative priors, we set a high standard deviation, but set a positive mean due to expected positive association.

The mean is calculated with the following formula: Median individual income in the United States / Average SAT Score (math and writing).

The median individual income in the US was approximately 31 000 in 2020 (Data Commons, 2020). The average SAT score in the US in 2020 were 523 in Math and 582 in Evidence-Based Reading and Writing (Number2, 2020).

$$\mu = 31\,000 / (523 + 582/2) = 43.2$$

$MD_FAMINIC \sim \text{Normal}(0, 100)$

Justification: Weakly informative prior is again selected as we don't possess enough information on the dependency. The absolute values are in general high, (for example compared to AGE_OF_ENTRY) and thus the standard deviation is set lower for this variable. However, there could be cases where $MD_FAMINIC$ is low, due to for example unemployment, so the standard deviation is still set considerably high.

Ehdotus

This is a variable we will try to transform to logarithmic scale as impact of say, 10 000\$ more income can be far greater for a low-income family than for a high-income family.

$AGE_ENTRY \sim \text{Normal}(0, 2500)$

Justification: We don't have a strong expectation of direction or magnitude of effect on income. As the age of entry could be somewhere in the range of 15 - 50 without considering outliers, a few dozen years difference could have dramatic changes in income either way, the standard deviation is set high.

$COSTT4_A \sim \text{Normal}(0, 500)$

Justification: Weakly informative prior is selected as we don't possess enough information on the dependency. The average cost per academic year is likely to take lower values than median family income, but higher values than average age of entry and thus the standard deviation is set between the standard deviations of those.

$POVERTY_RATE \sim \text{Normal}(0, 2500)$

Tää ei oikein täsmää. Jos tää sais ison painon pienillä arvoilla niin se tarkoittais että kun liikutaan esim 2% köyhyydestä 3% köyhyyteen niin on tosi isot vaikutukset tuloihin. Todellisuudessa alueet on varmaan aika samanlaisia hyvinvoivia alueita. Nollaköyhyys on intercept-arvo josta sitten lähetään askeleittain katomaan että miten tulot muuttuu kun alueen köyhyys lisääntyy. RATKAISUE-HDOTUS: Lasketaan sd:tä. Ei tarvii olla yhtä dramaattinen kuin binäärisillä muuttujilla.

Justification: The possible values are between 0 and 100. There could be situations where poverty rate in an area is really low, for instance 0.5%. For that reason, the standard deviation in the prior is set high to enable possibly high weight for a small value.

MASTER \sim Normal(0, 2500)

PRIVATE \sim Normal(0, 2500)

Justification: For MASTER and PRIVATE weakly informative prior, is selected as we don't possess enough information on the dependency with the dependent variable. Because the values are always either 0 and 1, the standard deviation is set high.

5. Stan code

TÄHÄN JOTAIN TRAIN TEST SPLITISTÄ?

We used cmdstanr for our models. Source code for all three models can be found in in the Appendix.

Separate model

Summary of model fit for main parameters:

Pooled model

Summary of model fit for main parameters:

Hierarchical model

Summary of model fit for main parameters:

6. How to the Stan model was run, that is, what options were used.

This is also more clear as combination of textual explanation and the actual code line.

We used the following options:

1. Seed: 1234
2. Chains: 4
3. Iterations per chain: 2000

7. Convergence diagnostics (rhat, ESS, divergences) and what was done if the convergence was not good with the first try.

Rhat

We used the `rhat()` function for all models to get the rhat convergence diagnostic. The function compares the between chain and within chain estimates for model parameters and other univariate quantities of interest. If the between chain and within chain estimates agree, it is said that the chains have mixed well. This happens when R-hat is less than 1.05 or less than 1.01 depending on the standard. We decided not be strict with our multivariate model and chose the less strict threshold of 1.05.

Separate model

Pooled model

```
params <- pooled.model$variables()$parameters %>% names()
rhat.df <- tibble()
for (param in params) {
  rhats <- extract_variable_matrix(pooled.fit$draws(), variable = param) %>% apply(2, rhat)
  row <- tibble("Parameter" = param,
               "Chain 1" = rhats[1],
               "Chain 2" = rhats[2],
               "Chain 3" = rhats[3],
               "Chain 4" = rhats[4])
  rhat.df <- rbind(rhat.df, row)
}
rhat.df
```

Hierarchical model

```
rhat.df <- tibble()

params <- c("alpha[1]", "beta_SAT_ALL[1]", "beta_MD_FAMINIC[1]",
           "beta_AGE_ENTRY[1]", "beta_COSTT4_A[1]", "beta_POVERTY_RATE[1]",
           "beta_MASTER[1]", "beta_PRIVATE[1]")

for (param in params) {
  rhats <- extract_variable_matrix(hierarchical.fit$draws(), variable = param) %>% apply(2, rhat)
  row <- tibble("Parameter" = param,
               "Chain 1" = rhats[1],
               "Chain 2" = rhats[2],
               "Chain 3" = rhats[3],
               "Chain 4" = rhats[4])
  rhat.df <- rbind(rhat.df, row)
}

rhat.df
```

Effective sample size (ESS)

Effective sample size (ESS) evaluates the uncertainty in estimates that is caused by autocorrelation of the chains. The value of effective sample size represents the number of independent samples that poses the same predictive power as all the autocorrelated samples. In other words, if the number of effective sample size is high, then the number of independent samples is high.

As can be seen on the tables below, for the separate model the values for the first region are range from 54 to 96. In this region, there are the sample size N is xxx. This means that...

For the pooled model the ESS values for parameters range from 508 to 954. This means that most of the samples for all parameters are independent, although there is also autocorrelation in most cases. According to (Stan) if the effective sample size is larger than the number of samples, this can be due to antithetic Markov Chains which have negative autocorrelations on odd lags.

For the hierarchical model the ESS values for parameters are between 266 and 1080. This means that for some parameters there samples are more autocorrelated samples than independent ones. The analysis for the highest values is the same as for pooled.

Source: https://mc-stan.org/docs/2_19/reference-manual/effective-sample-size-section.html

Separate Model

```
params <- separate.model$variables()$parameters %>% names()
ess.df <- tibble()

params <- c("alpha[1]", "beta_SAT_ALL[1]", "beta_MD_FAMINIC[1]",
           "beta_AGE_ENTRY[1]", "beta_COSTT4_A[1]", "beta_POVERTY_RATE[1]",
           "beta_MASTER[1]", "beta_PRIVATE[1]")

for (param in params) {
  ess <- extract_variable_matrix(separate.fit$draws(), variable = param) %>% apply(2, ess_basic)
  row <- tibble("Parameter" = param,
               "ESS" = ess)
  ess.df <- rbind(ess.df, row)
}
ess.df <- ess.df %>% group_by(Parameter) %>% summarise(ESS = sum(ESS)/n(),)

ess.df
```

Pooled Model

```
#pooled.fit$cmdstan_summary()

params <- pooled.model$variables()$parameters %>% names()
ess.df <- tibble()

for (param in params) {
  ess <- extract_variable_matrix(pooled.fit$draws(), variable = param) %>% apply(2, ess_basic)
  row <- tibble("Parameter" = param,
               "ESS" = ess)
  ess.df <- rbind(ess.df, row)
```

```

}
ess.df <- ess.df %>% group_by(Parameter) %>% summarise(ESS = sum(ESS)/n(),)

ess.df

```

Hierarchical Model

```

params <- hierarchical.model$variables()$parameters %>% names()
ess.df <- tibble()

params <- c("alpha[1]", "beta_SAT_ALL[1]", "beta_MD_FAMINIC[1]",
           "beta_AGE_ENTRY[1]", "beta_COSTT4_A[1]", "beta_POVERTY_RATE[1]",
           "beta_MASTER[1]", "beta_PRIVATE[1]")

for (param in params) {
  ess <- extract_variable_matrix(hierarchical.fit$draws(), variable = param) %>% apply(2, ess_basic)
  row <- tibble("Parameter" = param,
               "ESS" = ess)
  ess.df <- rbind(ess.df, row)
}
ess.df <- ess.df %>% group_by(Parameter) %>% summarise(ESS = sum(ESS)/n(),)

ess.df

```

HMC specific convergence diagnostics for all models

Based on the HMC specific convergence diagnostics, the separate model reached the initial maximum treedepth of 10 in 100% of the transitions. The diagnostics state that this leads to premature termination of trajectories and slow exploration. The proposed action of increasing the limit was tested, but it resulted to too long running time for the already time consuming fitting of the separate model. The rest of the diagnostics consisting of divergences, E-BMFI, effective sample size and split R-hat satisfactory.

For the pooled model, treedepth, divergences, E-BFMI, effective sample size and split R-hat were satisfactory.

For the hierarchical model, only 4 out of 4000 hit the maximum treedepth and 23 out of 4000 transitions did not converge. Due to relatively low numbers (0.1% and 0.57%) no actions were taken for the hierarchical model. The rest of the diagnostics were satisfactory.

```

separate.fit$cmdstan_diagnose()

pooled.fit$cmdstan_diagnose()

hierarchical.fit$cmdstan_diagnose()

```

8. Posterior predictive checks and model comparison

Posterior predictive checks

When doing posterior predictive checking we look for systematic discrepancies between the real observations and the data we get from simulating replicated data under the fitted model (Gelman and Hill, 2006). Posterior

predictive checking is a form of internal validation that is a helpful phase of model building and checking in assessing whether our fitted model makes sense.

```
separate.extract <- separate.fit$draws(variable = "y_rep")

y <- data.joined.stan$MD_EARN_WNE_P10

y_rep <- matrix(data = separate.extract[,1,], nrow = 1000)

separate.ppctitle <- ggtitle("Separate model",
                             "Comparing densities of y and y_rep")

ppc_dens_overlay(y, y_rep) + separate.ppctitle

# Pooled

pooled.extract <- pooled.fit$draws(variable = "y_rep")

y_rep <- matrix(data = pooled.extract[,1,], nrow = 1000)

pooled.ppctitle <- ggtitle("Pooled model",
                           "Comparing densities of y and y_rep")

ppc_dens_overlay(y, y_rep) + pooled.ppctitle

# Hierarchical

hierarchical.extract <- hierarchical.fit$draws(variable = "y_rep")

y_rep <- matrix(data = hierarchical.extract[,1,], nrow = 1000)

hierarchical.ppctitle <- ggtitle("Hierarchical model",
                                "Comparing densities of y and y_rep")

ppc_dens_overlay(y, y_rep) + hierarchical.ppctitle
```

LOO

The ELPD is the theoretical expected log pointwise predictive density for a new dataset (Eq 1 in VGG2017), which can be estimated, e.g., using cross-validation. `elpd_loo` is the Bayesian LOO estimate of the expected log pointwise predictive density (Eq 4 in VGG2017) and is a sum of N individual pointwise log predictive densities. Probability densities can be smaller or larger than 1, and thus log predictive densities can be negative or positive. For simplicity the ELPD acronym is used also for expected log pointwise predictive probabilities for discrete models. Probabilities are always equal or less than 1, and thus log predictive probabilities are 0 or negative.

LOO (Model comparison)

Yläpuolella kopsattu, kirjoita uusiks. Lähde alla

Lähde: <https://mc-stan.org/loo/reference/loo-glossary.html>

With $elpd_{loo-cv}$ we should select the model with the highest value. From the tables below we can see that in this case the suggested model would be separate model, the second best hierarchical model and the third best pooled model.

```
separate.loo <- separate.fit$loo()

pooled.loo <- pooled.fit$loo()

hierarchical.loo <- hierarchical.fit$loo()

loo_compare(separate.loo, pooled.loo, hierarchical.loo)
```

K hat

```
pareto_k_table(separate.fit$loo())

pareto_k_table(pooled.fit$loo())

pareto_k_table(hierarchical.fit$loo())

plot(separate.fit$loo(), main = "Separate model PSIS diagnostics")

plot(pooled.fit$loo(), main = "Pooled model PSIS diagnostics")

plot(hierarchical.fit$loo(), main = "Hierarchical model PSIS diagnostics")
```

Based on \hat{K} the predictions are ... too optimistic/reliable? - Why pooled model looks like that - overfitting or something?

9. Predictive performance assessment

Root Mean Squared Error (RMSE)

We use RMSE for assessing the predictive ability of our model. The assessment was conducted as follows. First we split the complete data set of 793 observations into a training set of 763 observations and a test set of 30 observations. Second we fit the models to the training set. Then we use the median of the posterior distribution for each parameter in our model to predict the dependent variable based on the values of the independent variables in the test set. Lastly, after obtaining the predicted values, we calculate the RMSE with respect to the actual observed values of the dependent variable in the test set.

Pooled model

```
pooled.fit.coeff.df <- pooled.fit$summary() %>% slice(2:8)

alpha.hat <- pooled.fit.coeff.df %>% slice(1) %>% select(median)
beta.hat <- pooled.fit.coeff.df %>% slice(2:8) %>% select(median) %>% as.matrix()
```

```

test.X <- data.joined.stan.test %>%
  select(SAT_ALL, MD_FAMINC, AGE_ENTRY, COSTT4_A, POVERTY_RATE, PRIVATE) %>%
  as.matrix()

N <- nrow(data.joined.stan.test)
y.hat <- numeric(N)
y <- data.joined.stan.test$MD_EARN_WNE_P10
se <- numeric(N)
for (i in 1:N) {

  y.hat[i] <- alpha.hat+beta.hat*test.X[i,]
  se[i] <- (y.hat[[i]]-y[i])^2

}

y.hat <- y.hat %>%
  as.matrix()

```

Plot of squared errors:

```
se %>% plot(type="b", pch=20)
```

RMSE:

```
rmse <- sqrt(mean(se))
rmse
```

10. Sensitivity analysis with respect to prior choices (i.e. checking whether the result changes a lot if prior is changed). This should be reported for all models.

To test whether our results are sensitive to prior choices, and to see how changing the priors affects our results, we ran all the models with both wider and narrower priors. The wide priors were obtained by multiplying each standard deviation by three, and the narrow priors were obtained by dividing the original priors by two.

Separate model:

```

separate.model <- cmdstan_model(stan_file = "./Stan/separate.stan")

scale_param <- 3

separate.model.data <- list(N = nrow(data.joined.stan),
                           K = length(unique(data.joined.stan$REGION)),
                           x = data.joined.stan$REGION,

                           SAT_ALL = data.joined.stan$SAT_ALL,
                           MD_FAMINC = data.joined.stan$MD_FAMINC,
                           AGE_ENTRY = data.joined.stan$AGE_ENTRY,
                           COSTT4_A = data.joined.stan$COSTT4_A,

```

```

    POVERTY_RATE = data.joined.stan$POVERTY_RATE,
    PRIVATE = data.joined.stan$PRIVATE,
    y = data.joined.stan$MD_EARN_WNE_P10,

    pm_alpha = 0,
    ps_alpha = 10000 * scale_param,
    pm_SAT_ALL = 50,
    ps_SAT_ALL = 500 * scale_param,
    pm_MD_FAMINC = 0,
    ps_MD_FAMINC = 100 * scale_param,
    pm_AGE_ENTRY = 0,
    ps_AGE_ENTRY = 2500 * scale_param,
    pm_COSTT4_A = 0,
    ps_COSTT4_A = 500 * scale_param,
    pm_POVERTY_RATE = 0,
    ps_POVERTY_RATE = 2500 * scale_param,
    pm_PRIVATE = 0,
    ps_PRIVATE = 2500 * scale_param,
    pm_sigma = 10000 * scale_param,
    ps_sigma = 10000 * scale_param

)

separate.fit <- separate.model$sample(data = separate.model.data, seed = 1234, refresh = 1e3)

separate.fit

separate.model <- cmdstan_model(stan_file = "./Stan/separate.stan")

scale_param <- 0.5

separate.model.data <- list(N = nrow(data.joined.stan),
    K = length(unique(data.joined.stan$REGION)),
    x = data.joined.stan$REGION,

    SAT_ALL = data.joined.stan$SAT_ALL,
    MD_FAMINC = data.joined.stan$MD_FAMINC,
    AGE_ENTRY = data.joined.stan$AGE_ENTRY,
    COSTT4_A = data.joined.stan$COSTT4_A,
    POVERTY_RATE = data.joined.stan$POVERTY_RATE,
    PRIVATE = data.joined.stan$PRIVATE,
    y = data.joined.stan$MD_EARN_WNE_P10,

    pm_alpha = 0,
    ps_alpha = 10000 * scale_param,
    pm_SAT_ALL = 50,
    ps_SAT_ALL = 500 * scale_param,
    pm_MD_FAMINC = 0,
    ps_MD_FAMINC = 100 * scale_param,
    pm_AGE_ENTRY = 0,
    ps_AGE_ENTRY = 2500 * scale_param,
    pm_COSTT4_A = 0,

```

```

        ps_COSTT4_A = 500 * scale_param,
        pm_POVERTY_RATE = 0,
        ps_POVERTY_RATE = 2500 * scale_param,
        pm_PRIVATE = 0,
        ps_PRIVATE = 2500 * scale_param,
        pm_sigma = 10000 * scale_param,
        ps_sigma = 10000 * scale_param
    )

separate.fit <- separate.model$sample(data = separate.model.data, seed = 1234, refresh = 1e3)

separate.fit

```

Pooled model:

```

pooled.model <- cmdstan_model(stan_file = "./Stan/pooled.stan")

scale_param <- 3

pooled.model.data <- list(N = nrow(data.joined.stan),

    SAT_ALL = data.joined.stan$SAT_ALL,
    MD_FAMINC = data.joined.stan$MD_FAMINC,
    AGE_ENTRY = data.joined.stan$AGE_ENTRY,
    COSTT4_A = data.joined.stan$COSTT4_A,
    POVERTY_RATE = data.joined.stan$POVERTY_RATE,
    PRIVATE = data.joined.stan$PRIVATE,
    y = data.joined.stan$MD_EARN_WNE_P10,

    pm_alpha = 0,
    ps_alpha = 10000 * scale_param,
    pm_SAT_ALL = 50,
    ps_SAT_ALL = 500 * scale_param,
    pm_MD_FAMINC = 0,
    ps_MD_FAMINC = 100 * scale_param,
    pm_AGE_ENTRY = 0,
    ps_AGE_ENTRY = 2500 * scale_param,
    pm_COSTT4_A = 0,
    ps_COSTT4_A = 500 * scale_param,
    pm_POVERTY_RATE = 0,
    ps_POVERTY_RATE = 2500 * scale_param,
    pm_PRIVATE = 0,
    ps_PRIVATE = 2500 * scale_param,
    pm_sigma = 10000 * scale_param,
    ps_sigma = 10000 * scale_param
)

pooled.fit <- pooled.model$sample(data = pooled.model.data, seed = 1234, refresh = 1e3)

```

```
pooled.fit
```

```
pooled.model <- cmdstan_model(stan_file = "./Stan/pooled.stan")

scale_param <- 0.5

pooled.model.data <- list(N = nrow(data.joined.stan),

  SAT_ALL = data.joined.stan$SAT_ALL,
  MD_FAMINIC = data.joined.stan$MD_FAMINC,
  AGE_ENTRY = data.joined.stan$AGE_ENTRY,
  COSTT4_A = data.joined.stan$COSTT4_A,
  POVERTY_RATE = data.joined.stan$POVERTY_RATE,
  PRIVATE = data.joined.stan$PRIVATE,
  y = data.joined.stan$MD_EARN_WNE_P10,

  pm_alpha = 0,
  ps_alpha = 10000 * scale_param,
  pm_SAT_ALL = 50,
  ps_SAT_ALL = 500 * scale_param,
  pm_MD_FAMINC = 0,
  ps_MD_FAMINC = 100 * scale_param,
  pm_AGE_ENTRY = 0,
  ps_AGE_ENTRY = 2500 * scale_param,
  pm_COSTT4_A = 0,
  ps_COSTT4_A = 500 * scale_param,
  pm_POVERTY_RATE = 0,
  ps_POVERTY_RATE = 2500 * scale_param,
  pm_PRIVATE = 0,
  ps_PRIVATE = 2500 * scale_param,
  pm_sigma = 10000 * scale_param,
  ps_sigma = 10000 * scale_param

)

pooled.fit <- pooled.model$sample(data = pooled.model.data, seed = 1234, refresh = 1e3)

pooled.fit
```

Hierarchical model:

```
hierarchical.model <- cmdstan_model(stan_file = "./Stan/hierarchical.stan")

scale_param <- 3

hierarchical.model.data <- list(N = nrow(data.joined.stan),
  K = length(unique(data.joined.stan$REGION)),
  x = data.joined.stan$REGION,

  SAT_ALL = data.joined.stan$SAT_ALL,
  MD_FAMINIC = data.joined.stan$MD_FAMINC,
  AGE_ENTRY = data.joined.stan$AGE_ENTRY,
```

```

COSTT4_A = data.joined.stan$COSTT4_A,
POVERTY_RATE = data.joined.stan$POVERTY_RATE,
PRIVATE = data.joined.stan$PRIVATE,
y = data.joined.stan$MD_EARN_WNE_P10,

pm_alpha = 0,
ps_alpha = 10000 * scale_param,
pm_s_alpha = 100,
ps_s_alpha = 100 * scale_param,

pm_SAT_ALL = 43,
ps_SAT_ALL = 500 * scale_param,
pm_s_SAT_ALL = 100,
ps_s_SAT_ALL = 100 * scale_param,

pm_MD_FAMINC = 0,
ps_MD_FAMINC = 100 * scale_param,
pm_s_MD_FAMINC = 10,
ps_s_MD_FAMINC = 100 * scale_param,

pm_AGE_ENTRY = 0,
ps_AGE_ENTRY = 2500 * scale_param,
pm_s_AGE_ENTRY = 500,
ps_s_AGE_ENTRY = 500 * scale_param,

pm_COSTT4_A = 0,
ps_COSTT4_A = 500 * scale_param,
pm_s_COSTT4_A = 50,
ps_s_COSTT4_A = 500 * scale_param,

pm_POVERTY_RATE = 0,
ps_POVERTY_RATE = 2500 * scale_param,
pm_s_POVERTY_RATE = 500,
ps_s_POVERTY_RATE = 500 * scale_param,

pm_PRIVATE = 0,
ps_PRIVATE = 2500 * scale_param,
pm_s_PRIVATE = 500,
ps_s_PRIVATE = 500 * scale_param,

pm_sigma = 10000,
ps_sigma = 10000 * scale_param,
pm_s_sigma = 500,
ps_s_sigma = 500) * scale_param

hierarchical.fit <- hierarchical.model$sample(data = hierarchical.model.data,
                                             seed = 1234, refresh = 1e3)

hierarchical.fit

hierarchical.model <- cmdstan_model(stan_file = "./Stan/hierarchical.stan")

scale_param <- 0.5

```

```

hierarchical.model.data <- list(N = nrow(data.joined.stan),
                               K = length(unique(data.joined.stan$REGION)),
                               x = data.joined.stan$REGION,

                               SAT_ALL = data.joined.stan$SAT_ALL,
                               MD_FAMINIC = data.joined.stan$MD_FAMINC,
                               AGE_ENTRY = data.joined.stan$AGE_ENTRY,
                               COSTT4_A = data.joined.stan$COSTT4_A,
                               POVERTY_RATE = data.joined.stan$POVERTY_RATE,
                               PRIVATE = data.joined.stan$PRIVATE,
                               y = data.joined.stan$MD_EARN_WNE_P10,

                               pm_alpha = 0,
                               ps_alpha = 10000 * scale_param,
                               pm_s_alpha = 100,
                               ps_s_alpha = 100 * scale_param,

                               pm_SAT_ALL = 43,
                               ps_SAT_ALL = 500 * scale_param,
                               pm_s_SAT_ALL = 100,
                               ps_s_SAT_ALL = 100 * scale_param,

                               pm_MD_FAMINC = 0,
                               ps_MD_FAMINC = 100 * scale_param,
                               pm_s_MD_FAMINC = 10,
                               ps_s_MD_FAMINC = 100 * scale_param,

                               pm_AGE_ENTRY = 0,
                               ps_AGE_ENTRY = 2500 * scale_param,
                               pm_s_AGE_ENTRY = 500,
                               ps_s_AGE_ENTRY = 500 * scale_param,

                               pm_COSTT4_A = 0,
                               ps_COSTT4_A = 500 * scale_param,
                               pm_s_COSTT4_A = 50,
                               ps_s_COSTT4_A = 500 * scale_param,

                               pm_POVERTY_RATE = 0,
                               ps_POVERTY_RATE = 2500 * scale_param,
                               pm_s_POVERTY_RATE = 500,
                               ps_s_POVERTY_RATE = 500 * scale_param,

                               pm_PRIVATE = 0,
                               ps_PRIVATE = 2500 * scale_param,
                               pm_s_PRIVATE = 500,
                               ps_s_PRIVATE = 500 * scale_param,

                               pm_sigma = 10000,
                               ps_sigma = 10000 * scale_param,
                               pm_s_sigma = 500,
                               ps_s_sigma = 500) * scale_param

hierarchical.fit <- hierarchical.model$sample(data = hierarchical.model.data,

```

```
seed = 1234, refresh = 1e3)
```

```
hierarchical.fit
```

11. Discussion of issues and potential improvements.

During the project, the large amount of variables also posed challenges, as it was arguably rather slow and burdensome to find the most relevant variables to use in our analysis. It was somewhat surprising how fast the observation count started to shrink in data cleaning process, so in hindsight more attention could have been paid to cleanliness, as this dataset had for example a lot of missing values.

It was a shame we couldn't use a larger testing dataset because all data used for testing would be away from training the model. We tried to reduce the need to have a very long dataset by working very hard on removing independent variables. That work also ended up expanding our observation count from roughly 200 to over 700 observations, as we found variables with less null values and removed variables with too much missing data. For the feature selection, with more time we could have experimented with alternative methods, like the LASSO or Ridge regression.

As we were building a predictive model, we had to take certain ethical questions into account, especially to avoid building a discriminatory model. For example, if our model was used to by an employer to assess how well alumni from a certain school would perform in their career, it would be discriminatory to have a model that predicts lower performance for a university with say, high female or black student population as the real reason for differences in our data might be something else entirely than skin color or gender. In the end, this wasn't a big issue as for example the correlations between both gender vs income and share of white population vs income were very low and they would have been dropped from our model no matter what.

- VOISI EHKÄ MAINITA JOTAIN PREDICTIVE ASSESSMENTISTA, ESIM. KAIKILLE MALLEILLE JONKUNLAINEN ASSESSMENT TAI MONIPUOLISEMPI

12. Conclusion what was learned from the data analysis.

13. Self-reflection of what the group learned while making the project.

In hindsight taking on a task of creating three multivariate models was very ambitious, because multivariable models had not been covered very much during the course. That ambition paid off, however, and we think we have now a much stronger grasp of how the different models - pooled, separate and hierarchical - work and also know how to build multivariate bayesian linear models in general. We also learned a lot about feature selection, because the 3000 variables in the dataset forced us to create sensible procedures for selection. The combination of using logic and feature engineering tools like stepwise regression is a very powerful skill we developed while making this project.

I think this project made us think really hard about what do we want to show the reader to make our work as understandable as possible and we learned to visualize data and results in interesting ways and apply the visualization techniques covered on the course. Making the visualizations also aided our own thinking and certain visualizations gave valuable insight on our results and their quality.

Working with Stan is something none of us had done before this course. Stan is clearly a very powerful tool for statistical and data science use and we feel this project really ingrained basic Stan workflows and made working with Stan feel more of an efficient routine than an obstacle.

14. References

- Card, D. (1999). THE CAUSAL EFFECT OF EDUCATION ON EARNINGS. Wolla, S. A., & Sullivan, J. (2017). Education, Income, and Wealth. <https://fred.stlouisfed.org/graph/?g=7yKu>.
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- Gelman, Andrew, and Jennifer Hill. Data analysis using regression and multilevel/hierarchical models. Cambridge university press, 2006.
- Number2. (2020). Average SAT Score [Blog Post]. Retrieved from: <https://www.number2.com/average-sat-score/>

15. Appendix

Stepwise regression

```
# PRELIMINARY ANALYSIS ----
# preliminary model with all numerical vars (not yet categorical)
# MODELING ----

data.joined.model <- data.joined.dropna %>%
  mutate(URBAN = case_when(LOCALE %in% c(seq(11,13), seq(21,23)) ~ 1,
    TRUE ~ 0),
    PRIVATE = case_when(CONTROL %in% c(2,3) ~ 1,
    TRUE ~ 0),
    DOCTORAL = case_when(CCBASIC %in% seq(15,17) ~ 1,
    TRUE ~ 0),
    MASTER = case_when(CCBASIC %in% seq(18,20) ~ 1,
    TRUE ~ 0)
  )

numerical.vars.model <- c("MD_EARN_WNE_P10", "SAT_ALL", "MD_FAMINC", "AGE_ENTRY",
  "COSTT4_A", "POVERTY_RATE")
categorical.vars.model <- c("URBAN", "PRIVATE", "DOCTORAL", "MASTER")

# data with REGION identifier for STAN
data.joined.stan <- data.joined.model %>%
  select(REGION, numerical.vars.model, categorical.vars.model)

# data for linear regression model in R
data.joined.model <- data.joined.stan %>%
  select(-REGION)

# baseline model
model <- lm(MD_EARN_WNE_P10 ~ ., data = data.joined.model)
summary(model)

##
```

```
## Call:
## lm(formula = MD_EARN_WNE_P10 ~ ., data = data.joined.model)
##
## Residuals:
##      Min       1Q   Median       3Q      Max
## -21261.2  -4348.1   -879.1   3665.5  31049.9
##
## Coefficients:
##              Estimate Std. Error t value Pr(>|t|)
## (Intercept) -1.284e+04  4.825e+03  -2.661 0.007946 **
## SAT_ALL      9.444e+01  5.901e+00  16.005 < 2e-16 ***
## MD_FAMINC    4.303e-02  2.019e-02   2.131 0.033362 *
## AGE_ENTRY   -1.192e+01  1.409e+02  -0.085 0.932603
## COSTT4_A     3.823e-01  3.409e-02  11.212 < 2e-16 ***
## POVERTY_RATE -2.011e+02  7.838e+01  -2.566 0.010469 *
## URBAN        2.172e+03  5.666e+02   3.834 0.000136 ***
## PRIVATE     -8.111e+03  9.801e+02  -8.276 5.44e-16 ***
## DOCTORAL     4.038e+02  8.149e+02   0.495 0.620389
## MASTER       1.058e+03  6.924e+02   1.529 0.126790
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 6616 on 786 degrees of freedom
## Multiple R-squared:  0.6777, Adjusted R-squared:  0.674
## F-statistic: 183.6 on 9 and 786 DF,  p-value: < 2.2e-16
```

```
# step wise regression implied "best" model in terms of AIC
step(model, direction = "backward")
```

```
## Start:  AIC=14015.21
## MD_EARN_WNE_P10 ~ SAT_ALL + MD_FAMINC + AGE_ENTRY + COSTT4_A +
##      POVERTY_RATE + URBAN + PRIVATE + DOCTORAL + MASTER
##
##              Df Sum of Sq      RSS   AIC
## - AGE_ENTRY    1 3.1328e+05 3.4407e+10 14013
## - DOCTORAL     1 1.0747e+07 3.4418e+10 14014
## <none>                                3.4407e+10 14015
## - MASTER       1 1.0227e+08 3.4509e+10 14016
## - MD_FAMINC    1 1.9887e+08 3.4606e+10 14018
## - POVERTY_RATE 1 2.8825e+08 3.4695e+10 14020
## - URBAN        1 6.4344e+08 3.5050e+10 14028
## - PRIVATE      1 2.9983e+09 3.7405e+10 14080
## - COSTT4_A     1 5.5026e+09 3.9910e+10 14131
## - SAT_ALL      1 1.1213e+10 4.5620e+10 14238
##
## Step:  AIC=14013.22
## MD_EARN_WNE_P10 ~ SAT_ALL + MD_FAMINC + COSTT4_A + POVERTY_RATE +
##      URBAN + PRIVATE + DOCTORAL + MASTER
##
##              Df Sum of Sq      RSS   AIC
## - DOCTORAL     1 1.0445e+07 3.4418e+10 14012
## <none>                                3.4407e+10 14013
## - MASTER       1 1.0407e+08 3.4511e+10 14014
## - MD_FAMINC    1 2.7912e+08 3.4686e+10 14018
```

```
## - POVERTY_RATE 1 2.9723e+08 3.4704e+10 14018
## - URBAN 1 6.4550e+08 3.5053e+10 14026
## - PRIVATE 1 3.3040e+09 3.7711e+10 14084
## - COSTT4_A 1 5.6436e+09 4.0051e+10 14132
## - SAT_ALL 1 1.1213e+10 4.5620e+10 14236
##
## Step: AIC=14011.46
## MD_EARN_WNE_P10 ~ SAT_ALL + MD_FAMINC + COSTT4_A + POVERTY_RATE +
## URBAN + PRIVATE + MASTER
##
##           Df Sum of Sq      RSS   AIC
## <none>                3.4418e+10 14012
## - MASTER      1 1.1892e+08 3.4537e+10 14012
## - MD_FAMINC   1 2.7162e+08 3.4689e+10 14016
## - POVERTY_RATE 1 3.0210e+08 3.4720e+10 14016
## - URBAN       1 7.2639e+08 3.5144e+10 14026
## - PRIVATE     1 3.6055e+09 3.8023e+10 14089
## - COSTT4_A    1 5.7805e+09 4.0198e+10 14133
## - SAT_ALL     1 1.2613e+10 4.7030e+10 14258
##
## Call:
## lm(formula = MD_EARN_WNE_P10 ~ SAT_ALL + MD_FAMINC + COSTT4_A +
## POVERTY_RATE + URBAN + PRIVATE + MASTER, data = data.joined.model)
##
## Coefficients:
## (Intercept)      SAT_ALL      MD_FAMINC      COSTT4_A  POVERTY_RATE
## -1.341e+04    9.533e+01    4.313e-02    3.846e-01   -2.012e+02
##          URBAN      PRIVATE          MASTER
##  2.240e+03   -8.246e+03    8.218e+02
```

```
stepwise.model <- lm(formula = MD_EARN_WNE_P10 ~ SAT_ALL + MD_FAMINC + COSTT4_A +
  POVERTY_RATE + URBAN + PRIVATE + MASTER, data = data.joined.model)
summary(stepwise.model)
```

```
##
## Call:
## lm(formula = MD_EARN_WNE_P10 ~ SAT_ALL + MD_FAMINC + COSTT4_A +
## POVERTY_RATE + URBAN + PRIVATE + MASTER, data = data.joined.model)
##
## Residuals:
##      Min       1Q   Median       3Q      Max
## -21303.3  -4335.7   -872.9   3705.7  30767.8
##
## Coefficients:
##              Estimate Std. Error t value Pr(>|t|)
## (Intercept) -1.341e+04  2.960e+03  -4.531 6.78e-06 ***
## SAT_ALL      9.533e+01  5.610e+00  16.993 < 2e-16 ***
## MD_FAMINC    4.313e-02  1.729e-02   2.494 0.01284 *
## COSTT4_A     3.846e-01  3.343e-02  11.504 < 2e-16 ***
## POVERTY_RATE -2.012e+02  7.652e+01  -2.630 0.00871 **
## URBAN        2.240e+03  5.492e+02   4.078 5.00e-05 ***
## PRIVATE     -8.246e+03  9.076e+02  -9.086 < 2e-16 ***
```

```
## MASTER      8.218e+02  4.980e+02  1.650  0.09934 .
## ---
## Signif. codes:  0 '***' 0.001 '**' 0.01 '*' 0.05 '.' 0.1 ' ' 1
##
## Residual standard error: 6609 on 788 degrees of freedom
## Multiple R-squared:  0.6776, Adjusted R-squared:  0.6747
## F-statistic: 236.6 on 7 and 788 DF,  p-value: < 2.2e-16
```