K J Somaiya College of Engineering, Vidyavihar, Mumbai

(A Constituent College of SVU)

Engineering Mechanics Notes

Module 3 – Centroid of Wires, Laminas and Solids

Class: FY BTech — Division: C3

Professor: Aniket S. Patil — Date: 05/05/2023

Centre of Gravity is a point where the whole weight of the body is assumed to act; or, it is a point where the entire distribution of gravitational force (weight) is supposed to be concentrated. It is usually denoted by the letter G.

Centroid is the geometrical centre of an object, where the entire length of a wire, or area of a plane lamina, or volume of a solid is supposed to be concentrated.

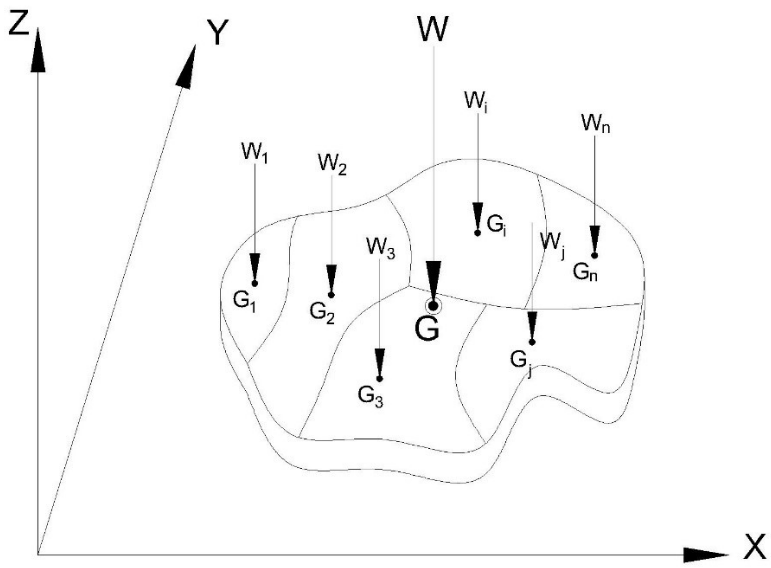
NOTE: For a uniformly distributed object (homogenous material), centre of gravity and centroid are the same point. But if an object is made of different composite materials, these points will be different.

Centre of Mass is a point where the entire mass is supposed to be concentrated.

NOTE: At the surface of the earth, centre of mass and centre of gravity will be the same point. But if the body is too large compared to the earth, then gravitational force will act differently at different parts of the body while the mass will remain as it is. Then, these points will be different.

Relation for Centre of Gravity:

Consider a body of weight W whose centre of gravity is located at G $(\bar{x}, \bar{y})$. If the body is split in "n" parts, each part will have its elemental weight W_i acting through its centre of gravity at G_i $(\bar{x}_i, \bar{y}_i)$.



The individual weights W_1, W_2, W_3,…, W_i,… W_n form a system of parallel forces.

$$W = W_1 + W_2 + \cdots + W_i + \cdots + W_n$$
$$W = \sum W_i$$

Using Varignon's theorem to find the location of resultant weight force W:

Taking moments about y-axis:

$$W \times \bar{x} = W_1 \times x_1 + W_2 \times x_2 + \cdots + W_i \times x_i + \cdots + W_n \times x_n$$
$$W \times \bar{x} = \sum W_i x_i$$

$$\bar{x} = \frac{\sum W_i x_i}{W} = \frac{\sum W_i x_i}{\sum W_i}$$

Similarly, taking moments about x-axis, we will get:

$$\bar{y} = \frac{\sum W_i y_i}{W} = \frac{\sum W_i y_i}{\sum W_i}$$

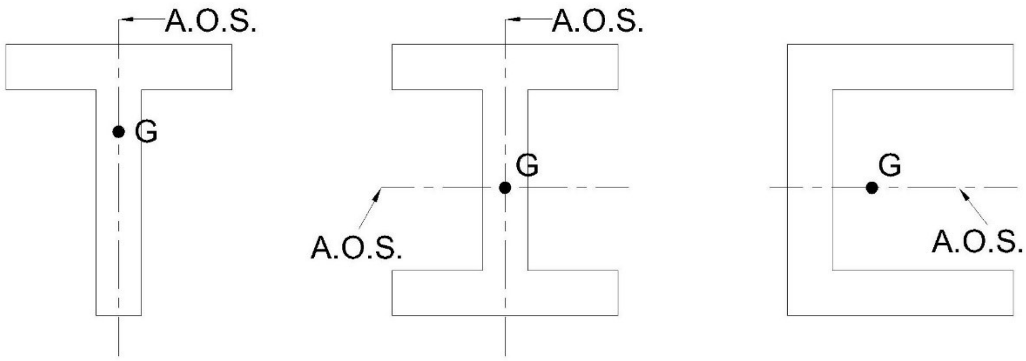
Relation for Centroid of Plane Laminas or Areas:

$$\text{Weight} = \text{mass} \times g$$
$$W = (\text{density} \times \text{volume}) \times g$$
$$W = (\text{density} \times \text{area} \times \text{thickness}) \times g$$
$$W = \rho \times A \times t \times g = (\rho \times t \times g) \times A$$

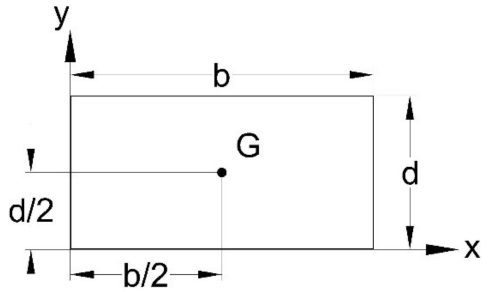
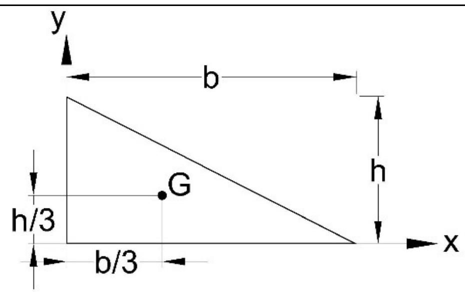
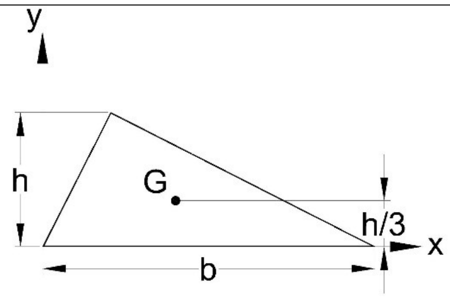
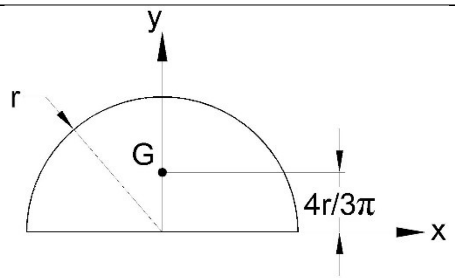
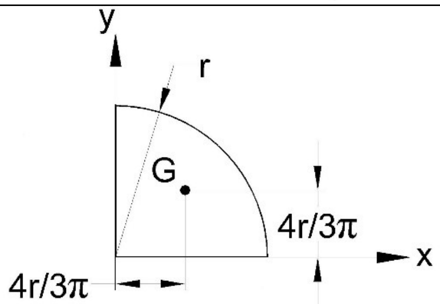
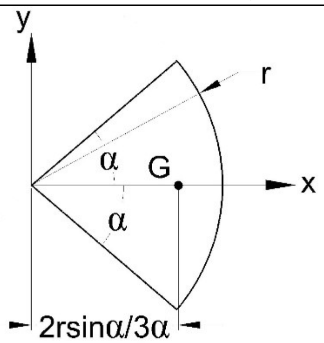
For uniform bodies (same density and thickness throughout), the centroid of the body is given by,

$$\bar{x} = \frac{\sum (\rho \times t \times g) A_i x_i}{\sum (\rho \times t \times g) A_i} = \frac{\sum A_i x_i}{\sum A_i}$$
$$\bar{y} = \frac{\sum (\rho \times t \times g) A_i y_i}{\sum (\rho \times t \times g) A_i} = \frac{\sum A_i y_i}{\sum A_i}$$

Axis of Symmetry: It is defined as the line which divides the figure into two equal parts such that each part is a mirror image of the other. If a body is symmetrical, then its centroid will lie on the axis of symmetry. If it has more than one axis of symmetry, the centroid will lie on the intersection of the axes of symmetry.



Centroids of Regular Plane Laminas:

Shape	Figure	Area	$\bar{x}$	$\bar{y}$
Rectangle		$b \times d$	$\frac{b}{2}$	$\frac{d}{2}$
Right Angled Triangle		$\frac{1}{2} \times b \times h$	$\frac{b}{3}$	$\frac{h}{3}$
Any Triangle		$\frac{1}{2} \times b \times h$	–	$\frac{h}{3}$
Semi-Circle		$\frac{\pi r^2}{2}$	0	$\frac{4r}{3\pi}$
Quarter-Circle		$\frac{\pi r^2}{4}$	$\frac{4r}{3\pi}$	$\frac{4r}{3\pi}$
Sector of a Circle		$r^2\alpha$	$\frac{2r\sin\alpha}{3\alpha}$	0

α in radians

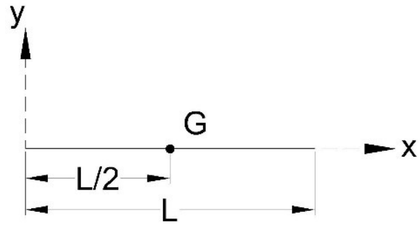
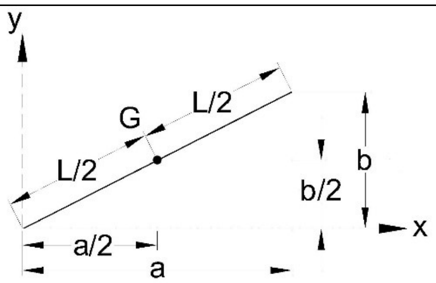
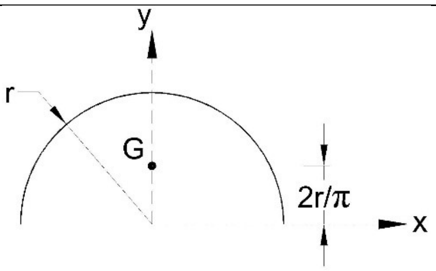
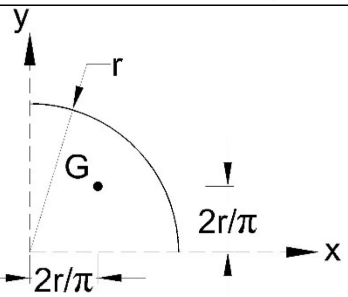
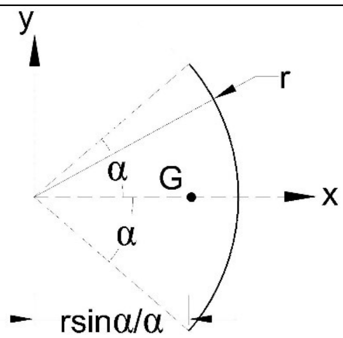
Relation for Centroid of Lines or Wires:

Lines are 1-D figures whose length is more prominent than its thickness, with the thickness being uniform throughout its length.

$$\bar{x} = \frac{\sum A_i x_i}{\sum A_i} = \frac{\sum L_i \times b \times x_i}{\sum L_i \times b} = \frac{\sum L_i x_i}{\sum L_i} \text{ \& Similarly, } \bar{y} = \frac{\sum L_i y_i}{\sum L_i}$$

The physical bodies which are equivalent to lines are bent up wires, pipe lines, etc.

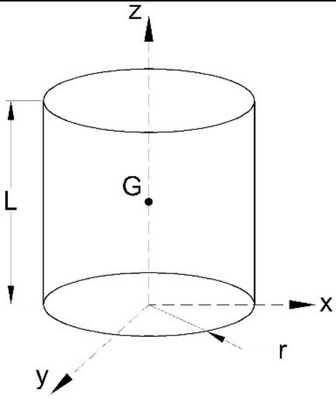
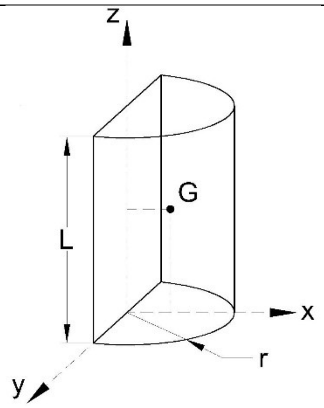
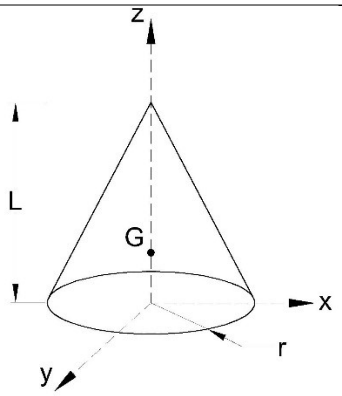
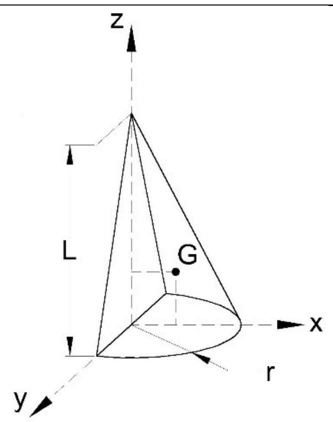
Centroids of Regular Lines:

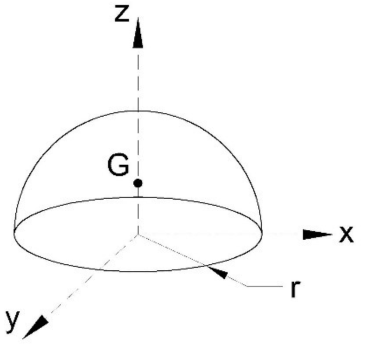
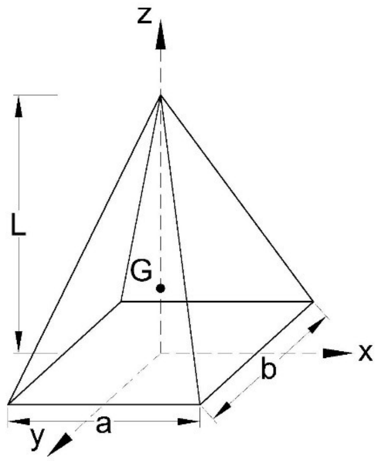
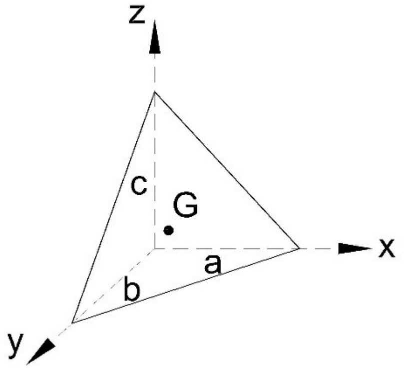
Shape	Figure	Length	$\bar{x}$	$\bar{y}$
Straight Horizontal Line		L	$\frac{L}{2}$	0
Straight Inclined Line		L	$\frac{a}{2}$	$\frac{b}{2}$
Semi-Circular Arc		πr	0	$\frac{2r}{\pi}$
Quarter-Circular Arc		$\frac{\pi r}{2}$	$\frac{2r}{\pi}$	$\frac{2r}{\pi}$
Circular Arc		$2r\alpha$	$\frac{r\sin\alpha}{\alpha}$ α in radians	0

Relation for Centroid of Solids:

$$\bar{x} = \frac{\sum W_i x_i}{\sum W_i} = \frac{\sum V_i \times \rho \times g \times x_i}{\sum V_i \times \rho \times g} = \frac{\sum V_i x_i}{\sum V_i} \text{ \& Similarly, } \bar{y} = \frac{\sum V_i y_i}{\sum V_i}, \bar{z} = \frac{\sum V_i z_i}{\sum V_i}$$

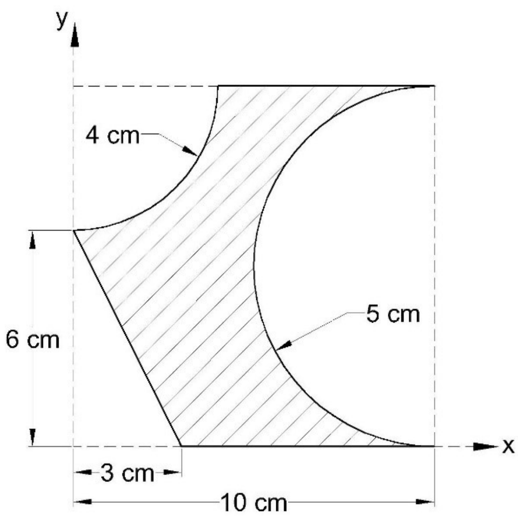
Centroids of Regular Solids:

Shape	Figure	Volume	$\bar{x}$	$\bar{y}$	$\bar{z}$
Cylinder		$\pi r^2 L$	0	0	$\frac{L}{2}$
Semi-Cylinder		$\frac{\pi r^2 L}{2}$	$\frac{4r}{3\pi}$	0	$\frac{L}{2}$
Cone		$\frac{1}{3}\pi r^2 L$	0	0	$\frac{L}{4}$
Half Cone		$\frac{\pi r^2 L}{6}$	$\frac{r}{\pi}$	0	$\frac{L}{4}$

Hemi-Sphere		$\frac{2}{3}\pi r^3$	0	0	$\frac{3r}{8}$
Square Pyramid		$\frac{1}{3}abL$	0	0	$\frac{L}{4}$
Regular Tetrahedron		$\frac{1}{6}abc$	$\frac{a}{4}$	$\frac{b}{4}$	$\frac{c}{4}$

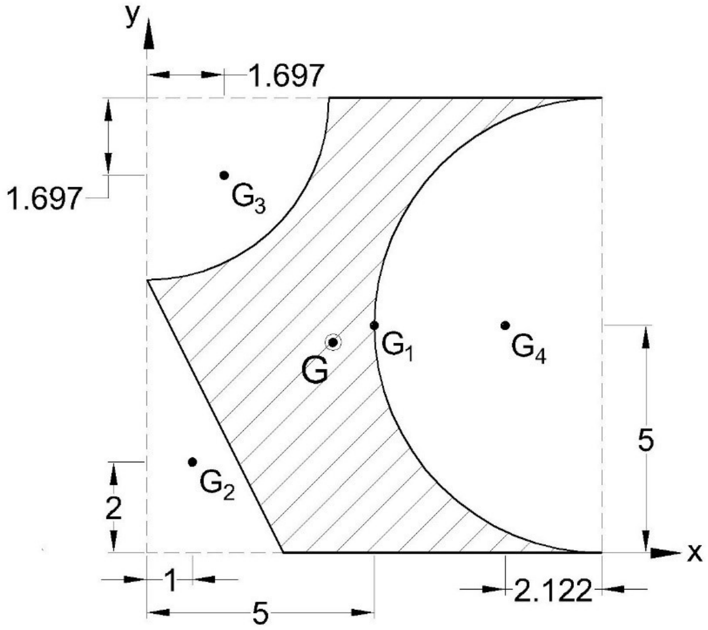
Numericals:

N1: Find the centroid of the shaded area as shown in the given figure.



Soln: The shaded region can be obtained by taking an entire square of 10 cm x 10 cm and subtracting a right triangle of 6 cm x 3 cm, a quarter-circle of radius 4 cm, and a semi-circle of radius 5 cm.

Section	**Area (A_i) cm^2**	**x_i cm**	**y_i cm**	**A_ix_i cm^3**	**A_iy_i cm^3**
Square	$10 \times 10 = 100$	$\frac{10}{2} = 5$	$\frac{10}{2} = 5$	500	500
Triangle	$-\frac{1}{2} \times 3 \times 6 = -9$	$\frac{3}{3} = 1$	$\frac{6}{3} = 2$	-9	-18
Quarter-Circle	$-\frac{\pi(4)^2}{4} = -12.57$	$\frac{4 \times 4}{3\pi} = 1.697$	$10 - \frac{4 \times 4}{3\pi} = 8.302$	-21.32	-104.33
Semi-Circle	$-\frac{\pi(5)^2}{2} = -39.27$	$10 - \frac{4 \times 5}{3\pi} = 7.878$	5	-309.37	-196.35
Total	$\boldsymbol{\sum A_i = 39.16}$			$\boldsymbol{\sum A_ix_i = 160.31}$	$\boldsymbol{\sum A_iy_i = 181.32}$



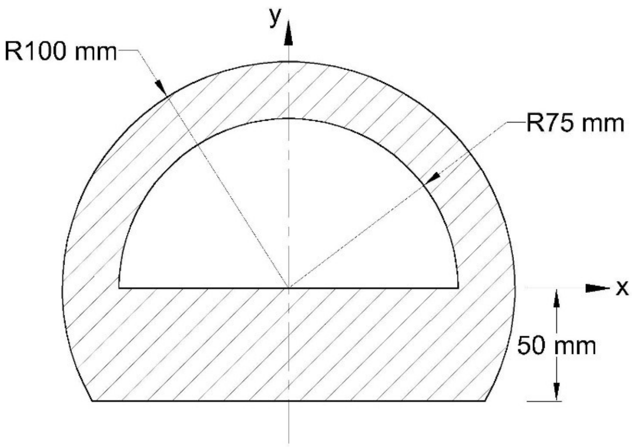
The co-ordinates of the centroid are given by,

$$\bar{x} = \frac{\sum A_ix_i}{\sum A_i} = \frac{160.31}{39.16} = 4.09 \text{ cm}$$

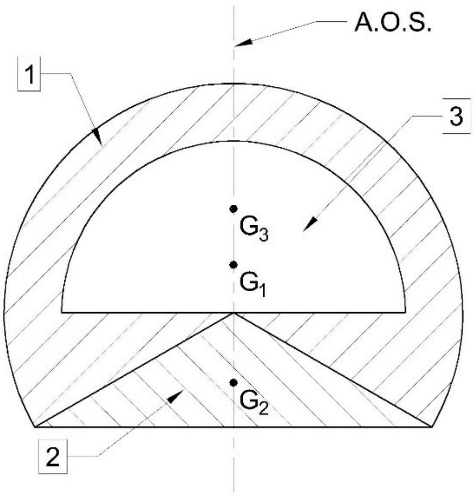
$$\bar{y} = \frac{\sum A_iy_i}{\sum A_i} = \frac{181.32}{39.16} = 4.63 \text{ cm}$$

Hence, the centroid of the shaded area is G (4.09, 4.63) cm.

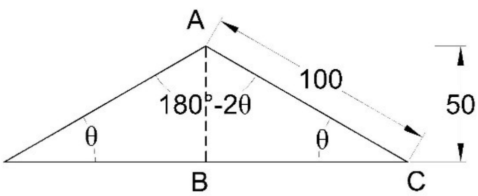
N2: A semi-circular section is removed from the plane area as shown. Find the centroid of the shaded area.



Soln: The shaded area can be obtained by adding a sector and a triangle and subtracting a semi-circle. All three areas are symmetrical about the y-axis; therefore, their centroids are going to lie on the y-axis, meaning x-coordinates are 0.



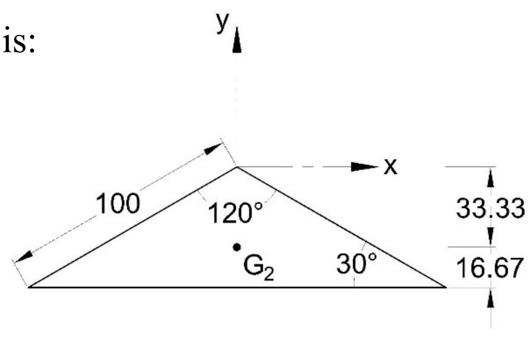
Let us first consider the triangular section. Within that consider the ΔABC:



$$\sin\theta = \frac{AB}{AC} = \frac{50}{100} \Rightarrow \theta = \sin^{-1} 0.5 = 30°$$

$$\text{And}, BC = \sqrt{100^2 - 50^2} = 86.6025 \text{ mm}$$

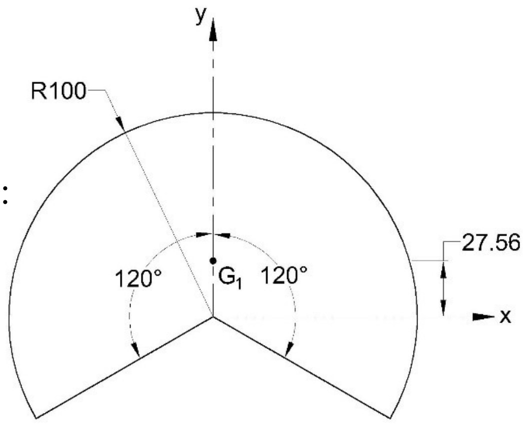
Hence, the area and y-coordinate for the triangle is:



$$A_2 = \frac{1}{2} \times (2 \times 86.6025) \times 50 = 4330.13 \text{ mm}^2$$

$$y_2 = 50 - \frac{1}{3} \times 50 = 50 - 16.67 = 33.33 \text{ mm}$$

The α for the sector will be 120° considering the angles found from the triangle, which in radians is 2.094395.

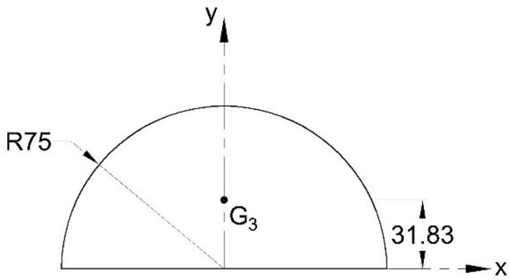


Hence, the area and y-coordinate for the sector is:

$$A_1 = 100^2 \times 2.094395 = 20943.95 \text{ mm}^2$$

$$y_1 = \frac{2}{3} \times \frac{100 \sin 2.094395}{2.094395} = 27.57 \text{ mm}$$

The area and y-coordinate for the semi-circle is:



$$A_1 = \frac{\pi}{2} \times 75^2 = 8835.73 \text{ mm}^2$$

$$y_1 = \frac{4}{3} \times \frac{75}{\pi} = 31.83 \text{ mm}$$

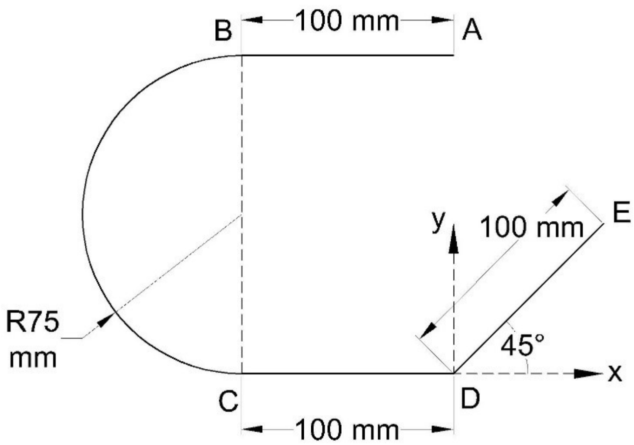
Section	**Area (A_i) mm^2**	**y_i mm**	**A_iy_i mm^3**
Sector	20943.95	27.57	577424.70
Triangle	4330.13	−33.33	−144323.23
Semi-Circle	−8835.73	31.83	−281241.29
Total	$\sum A_i = \mathbf{16438.35}$		$\sum A_i y_i = \mathbf{151860.18}$

The y-coordinate of the centroid,

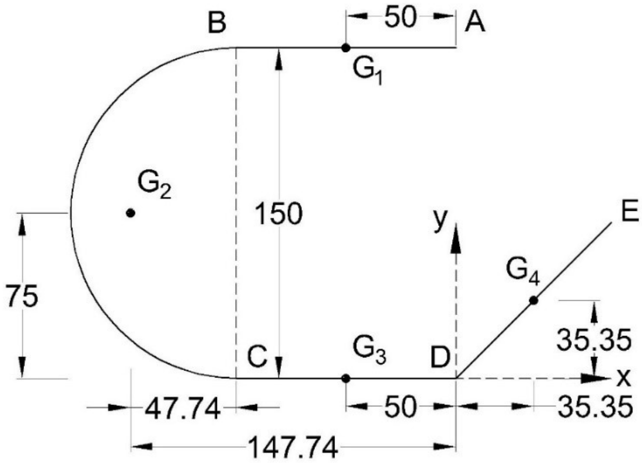
$$\bar{y} = \frac{\sum A_i y_i}{\sum A_i} = \frac{151860.18}{16438.35} = 9.24 \text{ mm}$$

Hence, the centroid of the shaded area is G (0, 9.24) mm.

N3: A uniform wire is bent into shape as shown. Calculate the C.G. of the wire.



Soln: The bent-up wire is made up of 2 straight horizontal portions AB and CD, a semi-circular portion BC and a straight inclined portion DE.



Section	Length (L_i) mm	x_i mm	y_i mm	L_ix_i mm^2	L_iy_i mm^2
AB	100	−50	150	−5000	15000
BC	$\pi \times 75 = 235.62$	−147.74	75	−34812	17671
CD	100	−50	0	−5000	0
DE	100	35.35	35.35	3535	3535
Total	$\sum L_i = \mathbf{535.62}$			$\sum L_i x_i = \mathbf{-41277}$	$\sum L_i y_i = \mathbf{36206}$

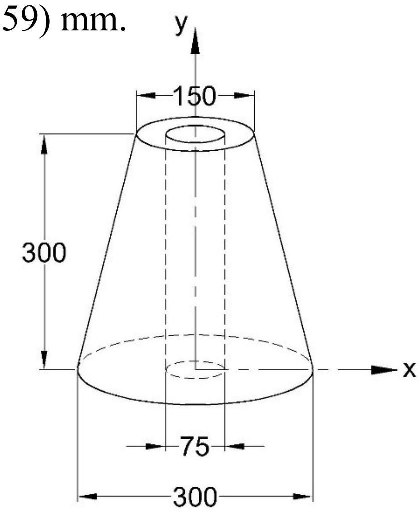
The co-ordinates of the centroid are given by,

$$\bar{x} = \frac{\sum L_i x_i}{\sum L_i} = \frac{-41277}{535.62} = -77.06 \text{ mm}$$

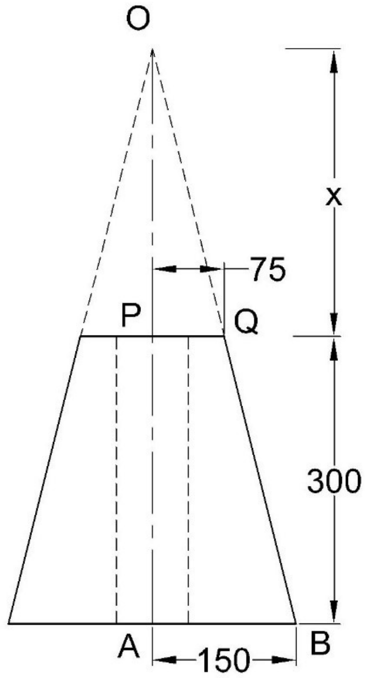
$$\bar{y} = \frac{\sum L_i y_i}{\sum L_i} = \frac{36206}{535.62} = 67.59 \text{ mm}$$

Hence, the centroid of the bent-up wire is G (-77.06, 67.59) mm.

<ins>N4</ins>: The frustum of a solid circular cone of smaller diameter 150 mm and larger diameter 300 mm with height 300 mm has an axial hole of 75 mm diameter. Determine the C.G. of the body.



Soln: The frustum of a cone can be considered as a bigger cone subtracted by a smaller cone. Considering the front projections of the cone as shown below:



In ΔOAB, AB is the radius of the larger cone.

$$\therefore AB = \frac{1}{2} \times 300 = 150 \text{ mm}$$

In ΔOPQ, PQ is the radius of the smaller cone.

$$\therefore PQ = \frac{1}{2} \times 150 = 75 \text{ mm}$$

Also, let OP, the height of smaller cone, be x mm.

$$\because \Delta OAB \sim \Delta OPQ, \therefore \frac{OA}{AB} = \frac{OP}{PQ}$$

$$\therefore \frac{x + 300}{150} = \frac{x}{75} \Rightarrow x = 300 \text{ mm}$$

Thus, the given body can be considered as a larger cone with height 600 mm and radius 150 mm subtracted by smaller cone with height 300 mm and radius 75 mm and subtracted by a cylinder with height 300 mm and radius 37.5 mm.

Also, the body is symmetric about y-axis, hence the C.G. will lie on it.

Section	**Volume (V_i) mm^3**	**y_i mm**	**$V_i y_i$ mm^4**
Larger Cone	$\frac{1}{3}\pi(150^2)(600) = 4500000\pi$	$\frac{600}{4} = 150$	675000000π
Smaller Cone	$-\frac{1}{3}\pi(75^2)(300) = -562500\pi$	$300 + \frac{300}{4} = 375$	-210937500π
Cylinder	$-\pi(37.5^2)(300) = -421875\pi$	$\frac{300}{2} = 150$	-63281250π
Total	$\boldsymbol{\sum V_i = 3515625\pi}$		$\boldsymbol{\sum V_i y_i = 400781250\pi}$

The y-coordinate of the centroid,

$$\bar{y} = \frac{\sum V_i y_i}{\sum V_i} = \frac{400781250\pi}{3515625\pi} = 114 \text{ mm}$$

Hence, the centroid of the shaded area is G (0, 114) mm.