Problems on Series expansion

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Methods of Expansions of functions in power series

1) using Maclaurin's series
2) using standard expansions
3) Method of Inversion
4) Method of differentiation or Integration of known series
5) Method of Substitution
6) Method using Leibnitz theorem

Ex-1 By Maclaurin's series expand $\log(1+e^x)$ in powers of x upto x^4

Sol^n:-

$$f(x) = \log(1+e^x) \qquad f(0) = \log(2)$$

$$f'(x) = \frac{1}{1+e^x} \cdot e^x \qquad f'(0) = \frac{1}{2}$$

$$f''(x) = \frac{(1+e^x).e^x - e^x.e^x}{(1+e^x)^2}$$

$$= \frac{e^x}{(1+e^x)^2} \qquad f''(0) = \frac{1}{4}$$

$$f'''(x) = \frac{(1+e^x)^2.e^x - 2e^x(1+e^x).e^x}{(1+e^x)^4} = \frac{e^x - e^{2x}}{(1+e^x)^3}$$

$$f^{(iv)}(x) = \frac{(1+e^x)^3(e^x - 2e^{2x}) - (e^x - e^{2x}) \cdot 3(1+e^x)^2 . e^x}{(1+e^x)^6}$$

$$= \frac{(1+e^x)(e^x - 2e^{2x}) - (e^x - e^{2x}) \cdot 3e^x}{(1+e^x)^4}$$

$f(0) = \log 2$, $f'(0) = \frac{1}{2}$, $f''(0) = \frac{1}{4}$, $f'''(0) = 0$, $f^{(iv)}(0) = -\frac{1}{8}$

Using Maclaurin's series

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \frac{x^4}{4!} f^{(iv)}(0) + \cdots$$

$$\log(1+e^x) = \log 2 + \frac{1}{2}x + \frac{1}{8}x^2 - \frac{1}{192}x^4 + \cdots$$

Ex:- Prove that, $\log \sec x = \frac{1}{2}x^2 + \frac{1}{12}x^4 + \frac{1}{45}x^6 + \cdots$

Soln:- Let $y = \log \sec x$ $\qquad y(0) = \log \sec 0 = \log 1 = 0$

$y_1 = \frac{1}{\sec x} \cdot \sec x \tan x = \tan x$ $\qquad y_1(0) = \tan 0 = 0$

$y_2 = \sec^2 x = 1 + \tan^2 x = 1 + y_1^2$ $\qquad y_2(0) = 1 + y_1(0)^2 = 1$

$y_3 = 2y_1 y_2$ $\qquad y_3(0) = 2y_1(0)y_2(0) = 0$

$y_4 = 2[y_2 y_2 + y_1 y_3]$ $\qquad \therefore y_4(0) = 2[y_2(0)^2 + y_1(0)y_3(0)]$

$= 2y_2^2 + 2y_1 y_3$ $\qquad y_4(0) = 2$

$y_5 = 4y_2 \cdot y_3 + 2y_2 y_3 + 2y_1 y_4$ $\qquad y_5(0) = 0$

$= 6y_2 y_3 + 2y_1 y_4$

$y_6 = 6y_3 y_3 + 6y_2 y_4 + 2y_2 y_4 + 2y_1 y_5$ $\qquad y_6(0) = 6(0) + 8(1)(2)$

$$= 6y_3^2 + 8y_2y_4 + 2y_1y_5$$

$$+ 2(0)$$

$$y_6(0) = 16$$

Hence by Maclaurin's series

$$y = y(0) + x y_1(0) + \frac{x^2}{2!} y_2(0) + \frac{x^3}{3!} y_3(0) + \frac{x^4}{4!} y_4(0)$$
$$+ \frac{x^5}{5!} y_5(0) + \frac{x^6}{6!} y_6(0) + \cdots$$

$$\log \sec x = \frac{1}{2}x^2 + \frac{1}{12}x^4 + \frac{x^6}{45} + \cdots$$

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Ex-3 :- Prove that, $\log(1+\sin x) = x - \frac{x^2}{2} + \frac{x^3}{6} + \cdots$

solution :- $f(x) = \log(1+\sin x)$ $\quad f(0) = \log(1) = 0$

$$f'(x) = \frac{\cos x}{1+\sin x} \qquad f'(0) = 1$$

$$f''(x) = \frac{(1+\sin x)(-\sin x) - \cos x(\cos x)}{(1+\sin x)^2} = \frac{-\sin x - 1}{(1+\sin x)^2} = \frac{-1}{1+\sin x}$$

$$f'''(x) = \frac{1}{(1+\sin x)^2} \cdot \cos x$$

$$\therefore f(0) = 0,\ f'(0) = 1,\ f''(0) = -1,\ f'''(0) = 1$$

Using Maclaurin's series

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \cdots$$

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \cdots$$

$$\log(1+\sin x) = x - \frac{x^2}{2} + \frac{x^3}{6} + \cdots$$

Aliter :- $\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$

$$\log(1+\sin x) = \sin x - \frac{(\sin x)^2}{2} + \frac{(\sin x)^3}{3} - \frac{(\sin x)^4}{4} + \cdots$$

$$= \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} \cdots\right) - \frac{1}{2}\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} \cdots\right)^2$$

$$+ \frac{1}{3}\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots\right)^3 - \cdots$$

$$= x - \frac{1}{2}x^2 + \left(-\frac{x^3}{6} + \frac{x^3}{3}\right) + \cdots$$

$$\log(1+\sin x) = x - \frac{1}{2}x^2 + \frac{x^3}{6} + \cdots$$

Ex-4 Prove that $\sec^2 x = 1 + x^2 + \frac{2x^4}{3} + \cdots$

Soln :- $f(x) = \sec^2 x =$ $\qquad f(0) = \sec^2 0 = 1$

$f'(x) = 2\sec^2 x \tan x =$ $\qquad f'(0) = 0$

$f''(x) = 2\left[\sec^4 x + 2\sec^2 x \tan^2 x\right]$ $\qquad f''(0) = 2$

$= 2\sec^4 x + 4\sec^2 x \tan^2 x$

$f'''(x) = 8\sec^4 x \tan x + 8\sec^2 x \tan^3 x + 8\sec^4 x \tan x$

$= 16\sec^4 x \tan x + 8\sec^2 x \tan^3 x$ $\qquad f'''(0) = 0$

$$f^{iv}(x) = 64\sec^4 x\tan^2 x + 16\sec^6 x + 16\sec^2 x\tan^4 x + 24\sec^4 x\tan^2 x$$

$$f^{(iv)}(0) = 16$$

$$f(0)=1,\quad f'(0)=0,\quad f''(0)=2,\quad f'''(0)=0,\quad f^{(iv)}(0)=16$$

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \frac{x^4}{4!}f^{(iv)}(0) + \cdots\cdots$$

$$\sec^2 x = 1 + x^2 + \frac{2x^4}{3} + \cdots\cdots$$

Method of standard expansion

<u>Ex-5</u> :- Expand in powers of x, $e^{x\sin x}$

<u>Soln</u> :- let $x\sin x = y$

$$e^{x\sin x} = e^y = 1 + y + \frac{y^2}{2!} + \frac{y^3}{3!} + \frac{y^4}{4!} + \cdots\cdots$$

$$e^{x\sin x} = 1 + (x\sin x) + \frac{1}{2!}(x\sin x)^2 + \frac{1}{3!}(x\sin x)^3 + \cdots\cdots$$

$$= 1 + x\left[x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots\right] + \frac{1}{2!}x^2\left[x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots\right]^2$$

$$+ \frac{1}{3!}x^3\left[x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots\right]^3 + \cdots\cdots$$