Solved Problems of Module 4

Sem I Physics

Divergence:-

Q.1) Calculating Divergence at a point.
if $\vec{F}(x,y,z) = e^x\hat{i} + yz\hat{j} + yz^2\hat{k}$, then find the divergen of $\vec{F}$ at $(0,2,-1)$

Sol^n:- The divergence of $\vec{F}$ is.

$$\vec{\nabla}\cdot\vec{F} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\cdot\left(e^x\hat{i} + yz\hat{j} - yz^2\hat{k}\right)$$

$$= \frac{\partial(e^x)}{\partial x} + \frac{\partial}{\partial y}(yz) - \frac{\partial}{\partial z}(yz^2)$$

$$\vec{\nabla}\cdot\vec{F} = e^x + z - 2yz$$

At pt $(0,2,-1)$,

$$\vec{\nabla}\cdot\vec{F} = e^0 - 1 + 4 \quad (+\text{ve div}).$$

$$\vec{\nabla}\cdot\vec{F} = 4$$

If $\vec{F}$ represents the velocity of a fluid, then more fluid is flowing out than flowing in at pt $(0,2,-1)$.

Q.2) Is it possible for $\vec{F}(x,y) = x^2y\,\hat{x} + (y - xy^2)\,\hat{y}$ to be magnetic field?

Sol^n: If $\vec{F}$ were magnetic, then its divergence would be zero. The

$$\vec{\nabla}\cdot\vec{F} = \left(\hat{x}\frac{\partial}{\partial x} + \hat{y}\frac{\partial}{\partial y} + \hat{z}\frac{\partial}{\partial z}\right)\cdot\left((x^2y)\hat{x} + (y - xy^2)\hat{y}\right)$$

$$\vec{\nabla}\cdot\vec{F} = \frac{\partial}{\partial x}(x^2y) + \frac{\partial}{\partial y}(y - xy^2)$$

$$= 2xy + 1 - 2xy$$

$$= 1 \neq 0$$

$\therefore$ $\vec{F}$ Cannot model a magnetic field.

Q.3) Find the divergence of an Electric field $\vec{E}(x,y,z) = e^{-xy}\hat{i} + e^{xz}\hat{j} + e^{yz}\hat{k}$ at pt $(3,2,0)$.

Q.4) Find the divergence of an Electric field $\vec{E} = (y^2+z^2)\hat{i} + y^2 \sin z\,\hat{j} + (y+2z)\hat{k}$ at pt $(1,2,0)$.

Q.5) Find the divergence of an Electric field $\vec{E} = xyz\,\hat{i} + x^2y^2z^2\hat{j} + y^2z^3\hat{k}$ at pt $(1,2,2)$.

Q.6) Find the divergence of an Electric field $\vec{E} = e^{x}\sin y\,\hat{i} - e^{x}\cos y\,\hat{j}$ at pt $(0,0,3)$.

Curl:-

Q.1) Find the curl of magnetic field

$$\vec{B} = x^2 z\,\hat{x} + (e^y + xz)\,\hat{y} + xyz\,\hat{z}$$

Soln:- $\text{Curl } \vec{B} = \vec{\nabla} \times \vec{B}$

$$= \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 z & (e^y + xz) & xyz \end{vmatrix}$$

$$\vec{\nabla} \times \vec{B} = (xz - x)\,\hat{x} + (x^2 - yz)\,\hat{y} + z\,\hat{z}$$

Q.2) Use the curl to determine whether $\vec{B}(x, y, z) = yz\,\hat{x} + xz\,\hat{y} + xy\,\hat{z}$ is conservative.

Soln:- Curl of $\vec{B}$ is

$$\vec{\nabla} \times \vec{B} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz & xz & xy \end{vmatrix}$$

$$= \left(\frac{\partial}{\partial y}(xy) - \frac{\partial}{\partial z}(xz)\right)\hat{x} + \left(\frac{\partial}{\partial y}(yz) - \frac{\partial}{\partial z}(xy)\right)\hat{y}$$
$$+ \left(\frac{\partial}{\partial y} xz - \frac{\partial}{\partial z} yz\right)\hat{z}$$

$$\vec{\nabla} \times \vec{B} = (x - x)\hat{x} + (y - y)\hat{y} + (z - z)\hat{z}$$

$$\vec{\nabla} \times \vec{B} = 0$$

i.e. $\vec{B}$ is conservative.

Q.3) Find the curl of $\vec{F}$ at given pt (1,2,3)

$$\vec{F} = xyz\,\hat{x} + y\,\hat{y} + x\,\hat{z}$$

Sol^n:-

$$\vec{\nabla} \times \vec{F} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xyz & y & x \end{vmatrix}$$

$$= \hat{x}(0 - 0) + \hat{y}(xy - 0) + \hat{z}(0 - xz)$$

$$\vec{\nabla} \times \vec{F} = xy\,\hat{y} - xz\,\hat{z}$$

$$\vec{\nabla} \times \vec{F}_{(1,2,3)} = 2\hat{y} - 3\hat{z}$$

Q.4) Find the curl of $\vec{F}$ at given pts.

a) $\vec{F}(x,y,z) = xy\,\hat{x} + yz\,\hat{y} + xz\,\hat{z}$ at (1,2,4)

b) $\vec{F}(x,y,z) = (x - y)\hat{x} + (y - z)\hat{y} + (z - x)\hat{z}$ pt (1,2,1)

c) $\vec{F}(x,y,z) = 3xyz^2\,\hat{x} + y^2 \sin z\,\hat{y} + x\,\hat{z}$ at (1,1,0)

Gradient:

Q.1) The potential in the region of space near the point $P(-2,4,6)$ is.

$$V = 80x^2 + 60y^2 \ V$$

a) Find out the electric field vector in the region

b) Find out the Electric field vector at pt P.

c) What is the value of potential at pt P.

Solⁿ: a) We have. $E = -\nabla V$ $\quad [E(r) = -\nabla V(r)]$

$$E_x = -\frac{\partial V}{\partial x} = -160x$$

$$E_y = -\frac{\partial V}{\partial y} = -120y$$

$$E_z = -\frac{\partial V}{\partial z} = 0.$$

So Electric field vector $= (-160x)\hat{i} + (-120y)\hat{j}$

$$\therefore \vec{E} = -160x\,\hat{i} - 120y\,\hat{j}$$

b) $\vec{E}$ at pt P $= 320\hat{i} - 480\hat{j}$

c) $V = 80x^2 + 60y^2 = 320 + 960 = 1280$ Volts

Q.2) The temperature at any point in space is given by $T = xy + yz + zx$. Determine the gradient at pt $(1,1,1)$

Sol^n: $T = xy + yz + zx$

$$\nabla T = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)T$$

$$= \left(i\frac{\partial T}{\partial x} + j\frac{\partial T}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)$$

$$= i(y+z) + j(x+z) + k(y+x)$$

$$\nabla T \text{ at } pt_{(1,1,1)} = 2\hat{i} + 2\hat{j} + 2\hat{k}$$

$\Rightarrow$ Find its directional derivative in the direction $\vec{A} = 3\vec{i} - 4\vec{k}$

$$dT = \nabla\phi \cdot \hat{d}$$

$$\because \hat{A} = \frac{3\vec{i} - 4\vec{k}}{\sqrt{9+16}}$$

$$= \nabla T \cdot \hat{A}$$

Dir. Derivative $= \nabla T \cdot \hat{A}$

$$\hat{A} = \frac{3\hat{i} - 4\hat{k}}{5}.$$

$$= (2\hat{i} + 2\hat{j} + 2\hat{k}) \cdot \frac{(3\hat{i} - 4\hat{k})}{5}$$

$$= \frac{1}{5}(6 - 8)$$

Directional Derivative $= -\frac{2}{5}$

Find the unknown constants.

Q.1) Find the value of a for which the vector B is solenoidal, where

$$\vec{B} = (x+2y)\hat{i} + (2ay - z)\hat{j} + (4x+2z)\hat{k}$$

Sol^n:- $\vec{\nabla}\cdot\vec{B} = 0$ for Solenoidal vector function

$$\therefore \vec{\nabla}\cdot\vec{B} = \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right)\cdot (x+2y)i + (2ay-z)j + (4x+2z)\hat{k}$$

$$= 1 + 2a + 2$$

$$\nabla\cdot B \Rightarrow 3 + 2a = 0$$

$$2a = -3$$

$$\therefore \boxed{a = -\frac{3}{2}} \Rightarrow \boxed{a = -1.5}$$

Q.2) Find the value of a for which the vector E is irrotational, where

$$\vec{E} = (3x^2y + az)\hat{i} + x^3\hat{j} + (3x + 3z^2)\hat{k} ?$$

Sol^n: $\vec{\nabla}\times\vec{E} = 0$ for Irrotational vector function

$$\vec{\nabla}\times\vec{E} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ (3x^2y + az) & x^3 & 3x + 3z^2 \end{vmatrix} = 0$$

$$\Rightarrow \hat{i}(0-0) + \hat{j}(a+3) + \hat{z}(3x^2 - 3x^2) = 0$$

$$\Rightarrow (a+3)\hat{y} = 0.$$

$$\therefore \quad a+3 = 0 \Rightarrow a = -3.$$

$$\therefore \boxed{a = -3}$$

Q.3) Find the value of b for vector $\vec{F}$ which is irrotational.

$$\vec{F} = (2xyz)\hat{i} + (x^2z + bxy)\hat{j} + x^2y\,\hat{k}.$$

Problems

Q.1) Find $\vec{\nabla} \cdot \vec{F}$, given that $\vec{F} = \nabla f$, where $f(x,y,z) = xy^3z^2$, at pt $(1,2,-1)$

Soln:-

$$\vec{F} = \nabla f = \left(\hat{i}\frac{\partial f}{\partial x} + \hat{j}\frac{\partial f}{\partial y} + \hat{k}\frac{\partial f}{\partial z}\right)$$

$$\vec{F} = y^3z^2\hat{i} + 3xy^2z^2\hat{j} + 2xy^3z\hat{k}$$

$$\vec{\nabla} \cdot \vec{F} = \left(\hat{i}\frac{\partial}{\partial x} + \hat{j}\frac{\partial}{\partial y} + \hat{k}\frac{\partial}{\partial z}\right) \cdot \vec{F}$$

$$= 0 + 6xyz^2 + 2xy^3$$

$$\vec{\nabla} \cdot \vec{F} = 6xyz^2 + 2xy^3$$

$$\vec{\nabla} \cdot \vec{F} \text{ at pt } (1,2,-1) = 6 \times 1 \times 2 \times (-1)^2 + 2 \times 1 \times 2^3$$

$$= 12 + 16$$

$$\vec{\nabla} \cdot \vec{F} = \underline{\underline{28}}$$

Problems for Practice :-

Q.1) Find the value of a if vector F is irrotational, where

$$\vec{F} = (x + 2y + 4z)\hat{i} + (2ax - z)\hat{j} + (4x - y + 2z)\hat{k}$$

Q.2) Find the value of b if vector B is solenoidal vector function at pt (2, 1, 2)

$$\vec{B} = bxy\hat{i} - xy^2\hat{j} + z^2\hat{k}$$

Q.3) Find $\vec{\nabla}\cdot\vec{F}$, given that $\vec{F} = \nabla f$, where

a) $f = x^2 + 2xy + 3z^2$ at pt (1, 2, 3)

b) $f = 3x^2y - y^3z^2$ at pt (1, -2, -1)