SOMAIYA VIDYAVIHAR UNIVERSITY
K J Somaiya College of Engineering

Somaiya TRUST

PARTIAL DIFFERENTIATION

FYBTECH SEM-I

MODULE-4

Partial Derivatives of the first order

- Let $z = f(x, y)$ be a function of two independent variables x and y.
- If we keep y constant and allow only x to vary then derivative, if it exists, so obtained is called the **partial derivative of z with respect to x** and it is denoted by $\frac{\partial z}{\partial x}$ or $\frac{\partial f}{\partial x}$ or f_x.
- Thus, $\frac{\partial z}{\partial x} = \lim_{\delta x \to 0} \frac{f(x+\delta x, y) - f(x,y)}{\delta x}$
- Similarly, the derivative of z with respect to y keeping x constant, if it exists is called the **partial derivative of z with respect to y** and it is denoted by $\frac{\partial z}{\partial y}$ or $\frac{\partial f}{\partial y}$ or f_y.
- Thus, $\frac{\partial z}{\partial y} = \lim_{\delta y \to 0} \frac{f(x, y+\delta y) - f(x,y)}{\delta y}$

Partial Derivatives of Higher Order

- The partial derivatives of higher order, if they exist, can be obtained from partial derivatives of the first order by using the above definitions again.
- Thus, $\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right)$ is the second order partial derivative of z w.r.t. x and is denoted by $\frac{\partial^2 z}{\partial x^2}$ or $\frac{\partial^2 f}{\partial x^2}$ or f_{xx}.
- Similarly, we have $\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial^2 z}{\partial y^2} = f_{yy}$,

$$\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial^2 z}{\partial y \partial x}$$

And $$\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial^2 z}{\partial x \partial y}$$

Note

(1) If $u = f(x, y)$ possesses **continuous** second-order partial derivatives $\frac{\partial^2 u}{\partial x \partial y}$ and $\frac{\partial^2 u}{\partial y \partial x}$ then $\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}$.

This is called commutative property

(2) Standard rules for differentiation of sum, difference, product and quotient are also applicable for partial differentiation

Differentiation of a function of a function

- Let $z = f(u)$and $u = \Phi(x, y)$ so that z is function of u and u itself is a function of two independent variables x and y.
- The two relations define z as a function of x and y. In such cases z may be called a **function of a function** of x and y.
- **e.g. (i)** $z = \frac{1}{u}$ and $u = \sqrt{x^2 + y^2}$

  (ii) $z = \tan u$ and $u = x^2 + y^2$

define z as a function of a function of x and y.

Differentiation of a function of a function

If $z = f(u)$ is differentiable function of u and $u = \Phi(x, y)$possesses first order partial derivatives then,

- $\frac{\partial z}{\partial x} = \frac{dz}{du} . \frac{\partial u}{dx}$ i.e. $\frac{\partial z}{\partial x} = f'(u) \frac{\partial u}{\partial x}$

Similarly $\frac{\partial z}{\partial y} = \frac{dz}{du} . \frac{\partial u}{\partial y}$ i.e. $\frac{\partial z}{\partial y} = f'(u) \frac{\partial u}{\partial y}$

- e.g. If $z = (ax + by)^n$ then

$\frac{\partial z}{\partial x} = n(ax + by)^{n-1} . a$ and

$$\frac{\partial z}{\partial y} = n(ax + by)^{n-1} . b$$

EXAMPLE-1

- If $u = \cos\left(\sqrt{x} + \sqrt{y}\right)$, prove that
- $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + \frac{1}{2}\left(\sqrt{x} + \sqrt{y}\right) sin\left(\sqrt{x} + \sqrt{y}\right) = 0.$

$$\frac{\partial u}{\partial x} = -\sin(\sqrt{x} + \sqrt{y})\frac{1}{2\sqrt{x}}$$

$$\frac{\partial u}{\partial y} = -\sin\left(\sqrt{x} + \sqrt{y}\right)\frac{1}{2\sqrt{y}}$$

EXAMPLE-1

- If $u = \cos\left(\sqrt{x} + \sqrt{y}\right)$, prove that
- $x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} + \frac{1}{2}\left(\sqrt{x} + \sqrt{y}\right) sin\left(\sqrt{x} + \sqrt{y}\right) = 0.$
- **Solution:** We have

$$\frac{\partial u}{\partial x} = -\sin\left(\sqrt{x} + \sqrt{y}\right)\frac{1}{2\sqrt{x}}$$

$$\frac{\partial u}{\partial y} = -\sin\left(\sqrt{x} + \sqrt{y}\right)\frac{1}{2\sqrt{y}}$$

$$\therefore \quad x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = -\sin\left(\sqrt{x} + \sqrt{y}\right)\cdot\frac{1}{2}\left(\sqrt{x} + \sqrt{y}\right)$$

$$\therefore x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + \frac{1}{2}(\sqrt{x} + \sqrt{y})\sin(\sqrt{x} + \sqrt{y}) = 0$$

EXAMPLE-2

- If $z(x+y)=x^2+y^2$, prove that $\left(\frac{\partial z}{\partial x}-\frac{\partial z}{\partial y}\right)^2=4\left(1-\frac{\partial z}{\partial x}-\frac{\partial z}{\partial y}\right)$

- **Solution:** Since $z=\frac{x^2+y^2}{x+y}$

$$\frac{\partial z}{\partial x}=\frac{(x+y)2x-(x^2+y^2)}{(x+y)^2}$$

$$=\frac{x^2+2xy-y^2}{(x+y)^2}$$

$$\frac{\partial z}{\partial y}=\frac{(x+y)2y-(x^2+y^2)}{(x+y)^2}$$

$$=\frac{-x^2+2xy+y^2}{(x+y)^2}$$

EXAMPLE-2

❖ If $z(x+y) = x^2 + y^2$, prove that $\left(\frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right)^2 = 4\left(1 - \frac{\partial z}{\partial x} - \frac{\partial z}{\partial y}\right)$

❖ **Solution:** Since $z = \frac{x^2+y^2}{x+y}$

$$\frac{\partial z}{\partial x} = \frac{x^2+2xy-y^2}{(x+y)^2};\ \frac{\partial z}{\partial y} = \frac{-x^2+2xy+y^2}{(x+y)^2}$$

$$\therefore \text{LHS} = \left[\frac{x^2+2xy-y^2+x^2-2xy-y^2}{(x+y)^2}\right]^2$$

$$= \left[2 \cdot \frac{(x^2-y^2)}{(x+y)^2}\right]^2$$

$$= \left[2 \cdot \frac{(x-y)}{(x+y)}\right]^2 = 4\frac{(x-y)^2}{(x+y)^2}$$

Putting the values of $\frac{\partial z}{\partial x}, \frac{\partial z}{\partial y}$

$$\text{RHS} = 4\left[1 - \frac{x^2+2xy-y^2}{(x+y)^2} - \frac{-x^2+2xy+y^2}{(x+y)^2}\right]$$

$$= 4\left[\frac{x^2-2xy+y^2}{(x+y)^2}\right] = 4\frac{(x-y)^2}{(x+y)^2}$$

$$\therefore \text{LHS} = \text{RHS}$$

EXAMPLE-3

- If $u = \log(x^3 + y^3 + z^3 - 3xyz)$, prove that $\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)^2 u = -\frac{9}{(x+y+z)^2}$.

Solution: LHS $= \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right) u$

$= \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)\left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}\right)$ ....(i)

Now, $\frac{\partial u}{\partial x} = \frac{1}{x^3+y^3+z^3-3xyz}(3x^2 - 3yz)$, $\frac{\partial u}{\partial y} = \frac{3y^2-3zx}{x^3+y^3+z^3-3xyz}$, $\frac{\partial u}{\partial z} = \frac{3z^2-3xy}{x^3+y^3+z^3-3xyz}$

$$\therefore \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 3\left(\frac{x^2 + y^2 + z^2 - xy - yz - zx}{x^3 + y^3 + z^3 - 3xyz}\right) = \frac{3}{(x + y + z)}$$

$$\{\because (x^2 + y^2 + z^2 - xy - yz - zx)(x + y + z) = x^3 + y^3 + z^3 - 3xyz\}$$

Hence from (1), LHS $= \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right) \cdot \frac{3}{(x+y+z)}$

$= 3\left[\frac{-1}{(x+y+z)^2} + \frac{-1}{(x+y+z)^2} + \frac{-1}{(x+y+z)^2}\right]$

$= -\frac{9}{(x+y+z)^2} =$ RHS

EXAMPLE-4

- If $z = x^y + y^x$, verify that $\frac{\partial^2 z}{\partial x\,\partial y} = \frac{\partial^2 z}{\partial y\,\partial x}$.

Solution: Differentiating z partially w.r.t. y we get,

$$\frac{\partial z}{\partial y} = x^y \log x + xy^{x-1}$$

Differentiating this partially w.r.t. x we get,

$$\frac{\partial^2 z}{\partial x\,\partial y} = yx^{y-1} \cdot \log x + x^y \cdot \frac{1}{x} + 1 \cdot y^{x-1} + xy^{x-1} \log y$$

$$= yx^{y-1} \cdot \log x + x^{y-1} + y^{x-1} + xy^{x-1} \log y$$

Now, differentiating z partially w.r.t. x, we get,

$$\frac{\partial z}{\partial x} = yx^{y-1} + y^x \log y$$

Differentiating this again partially w.r.t. y, we get,

$$\frac{\partial^2 z}{\partial y\,\partial x} = x^{y-1} + y \cdot x^{y-1} \log x + \frac{y^x}{y} + xy^{x-1} \log y$$

$$= yx^{y-1} \log x + x^{y-1} + y^{x-1} + xy^{x-1} \log y$$

EXAMPLE-5

❖ If $u = e^{x^2+y^2+z^2}$, prove that $\dfrac{\partial^3 u}{\partial x\, \partial y\, \partial z} = 8\, xyzu$.

Solution: $$\frac{\partial u}{\partial z} = e^{x^2+y^2+z^2} \cdot 2z$$

$$\begin{aligned}
\frac{\partial^2 u}{\partial y\, \partial z} &= \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial z}\right) \\
&= 2z \cdot e^{x^2+y^2+z^2} \cdot 2y \\
&= 4yz \cdot e^{x^2+y^2+z^2}
\end{aligned}$$

$$\begin{aligned}
\frac{\partial^3 u}{\partial x\, \partial y\, \partial z} &= \frac{\partial}{\partial y}\left(\frac{\partial^2 u}{\partial y\, \partial z}\right) \\
&= 4yz \cdot e^{x^2+y^2+z^2} \cdot 2x \\
&= 8xyz \cdot e^{x^2+y^2+z^2} \\
&= 8xyzu
\end{aligned}$$

EXAMPLE-6

- If $\theta = t^n e^{-r^2/4t}$, find n which will make $\frac{\partial\theta}{\partial t} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\theta}{\partial r}\right)$.
- **Solution:** $\frac{\partial\theta}{\partial t} = nt^{n-1}\cdot e^{-r^2/4t} + t^n e^{-r^2/4t}\cdot\left(\frac{r^2}{4t^2}\right)$
- $= \frac{n}{t}\theta + \frac{r^2}{4t^2}\theta = \left(\frac{n}{t} + \frac{r^2}{4t^2}\right)\theta$ ....(1)
- Also, $\frac{\partial\theta}{\partial r} = t^n e^{-r^2/4t}\cdot\left(-\frac{2r}{4t}\right) = -\frac{r\theta}{2t}$
- $\therefore r^2\frac{\partial\theta}{\partial r} = -\frac{r^3\theta}{2t}$
- $\therefore \frac{\partial}{\partial r}\left(r^2\frac{\partial\theta}{\partial r}\right) = \frac{\partial}{\partial r}\left(-\frac{r^3\theta}{2t}\right) = -\frac{1}{2t}\frac{\partial}{\partial r}(r^3\theta)$
- $= -\frac{1}{2t}\left[r^3\frac{\partial\theta}{\partial r} + 3r^2\theta\right]$
- $= -\frac{1}{2t}\left[r^3\frac{r\theta}{2t} + 3r^2\theta\right]$
- $= -\frac{1}{2t}\left[\frac{r^4\theta}{2t} + 3r^2\theta\right]$
- $\therefore \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\theta}{\partial r}\right) = -\frac{1}{2t}\left[-\frac{r^2\theta}{2t} + 3\theta\right]$ ....(2)
- $\therefore$ Equating (1) and (2), we get, $\frac{n}{t} = -\frac{3}{2t}$ $\quad\therefore n = -\frac{3}{2}$

EXAMPLE-7

- If $u = 3(ax + by + cz)^2 - (x^2 + y^2 + z^2)$ *and* $a^2 + b^2 + c^2 = 1$, Prove that $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0$.
- **Solution:** $u = 3(ax + by + cz)^2 - (x^2 + y^2 + z^2)$

Differentiating u partially w.r.t. x, $\frac{\partial u}{\partial x} = 6(ax + by + cz)a - 2x$

Differentiating $\frac{\partial u}{\partial x}$ partially w.r.t. x, $\frac{\partial^2 u}{\partial x^2} = 6a \cdot a - 2 = 6a^2 - 2$

Differentiating u partially w.r.t. y, $\frac{\partial u}{\partial y} = 6(ax + by + cz)b - 2y$

Differentiating $\frac{\partial u}{\partial y}$ partially w.r.t. y, $\frac{\partial^2 u}{\partial y^2} = 6b \cdot b - 2 = 6b^2 - 2$

Differentiating u partially w.r.t. z, $\frac{\partial u}{\partial z} = 6(ax + by + cz)c - 2z$

Differentiating $\frac{\partial u}{\partial z}$ partially w.r.t. z, $\frac{\partial^2 u}{\partial z^2} = 6c \cdot c - 2 = 6c^2 - 2$

Hence, $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 6(a^2 + b^2 + c^2) - 6$

$= 6(1) - 6$ $[\because a^2 + b^2 + c^2 = 1]$

$= 0$

EXAMPLE-8

- If $u = f(r), r^2 = x^2 + y^2 + z^2$, prove that $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = f''(r) + \frac{2}{r} f'(r)$.
- **Solution:** $u = f(r)$

Differentiating u partially w.r.t. x,

$$\frac{\partial u}{\partial x} = \frac{\partial}{\partial x} f(r) = \frac{d}{dr} f(r) \cdot \frac{\partial r}{\partial x}$$

$$= f'(r) \cdot \frac{\partial r}{\partial x} \quad \ldots\ldots\ldots\ldots\ (1)$$

But $r^2 = x^2 + y^2 + z^2$

Differentiating r^2 partially w.r.t. x, $2r\frac{\partial r}{\partial x} = 2x \quad \Rightarrow \quad \frac{\partial r}{\partial x} = \frac{x}{r}$

Substituting in Eq. (1), $\frac{\partial u}{\partial x} = f'(r) \cdot \frac{x}{r}$

Differentiating $\frac{\partial u}{\partial x}$ partially w.r.t. x,

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial}{\partial x}\left[f'(r) \cdot \frac{x}{r}\right]$$

$$= f''(r)\frac{\partial r}{\partial x} \cdot \frac{x}{r} + \frac{f'(r)}{r} + x f'(r)\left(-\frac{1}{r^2}\right) \cdot \frac{\partial r}{\partial x}$$

$$= f''(r)\frac{x}{r}\frac{x}{r} + \frac{f'(r)}{r} - \frac{x}{r^2} f'(r) \cdot \frac{x}{r}$$

$$= f''(r)\frac{x^2}{r^2} + \frac{f'(r)}{r} - \frac{x^2}{r^3} f'(r) \quad \ldots\ldots\ldots\ldots\ (2)$$

EXAMPLE-8

- Similarly, $\dfrac{\partial^2 u}{\partial y^2} = f''(r)\dfrac{y^2}{r^2} + \dfrac{f'(r)}{r} - \dfrac{y^2}{r^3} f'(r)$ .... (3)
- and $\dfrac{\partial^2 u}{\partial z^2} = f''(r)\dfrac{z^2}{r^2} + \dfrac{f'(r)}{r} - \dfrac{z^2}{r^3} f'(r)$ .... (4)

- Adding Eqs (2), (3) and (4),
- $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} + \dfrac{\partial^2 u}{\partial z^2}$
- $= \dfrac{f''(r)}{r^2}(x^2 + y^2 + z^2) + \dfrac{3f'(r)}{r} - \dfrac{(x^2+y^2+z^2)}{r^3} f'(r)$
- $= \dfrac{f''(r)}{r^2} \cdot r^2 + \dfrac{3f'(r)}{r} - \dfrac{r^2}{r^3} f'(r)$
- $= f''(r) + \dfrac{2f'(r)}{r}$

EXAMPLE-9

- If $z = u(x,y)\ e^{ax+by}$ where $u(x,y)$ is such that $\frac{\partial^2 u}{\partial x \partial y} = 0$, find the constants a, b
- such that $\frac{\partial^2 z}{\partial x \partial y} - \frac{\partial z}{\partial x} - \frac{\partial z}{\partial y} + z = 0.$
- **Solution:** We have, from $z = u(x,y)e^{ax+by}$ .... (1)
- $\frac{\partial z}{\partial x} = \frac{\partial u}{\partial x} \cdot e^{ax+by} + u \cdot e^{ax+by} \cdot a = e^{ax+by}\left(\frac{\partial u}{\partial x} + au\right)$ .... (2)
- $\frac{\partial z}{\partial y} = \frac{\partial u}{\partial y} \cdot e^{ax+by} + u \cdot e^{ax+by} \cdot b = e^{ax+by}\left(\frac{\partial u}{\partial y} + bu\right)$ .... (3)
- Differentiating (3) partially w.r.t. x,
- $\frac{\partial^2 z}{\partial x \partial y} = e^{ax+by} \cdot a \cdot \left(\frac{\partial u}{\partial y} + bu\right) + e^{ax+by}\left(\frac{\partial^2 u}{\partial x \partial y} + b \cdot \frac{\partial u}{\partial x}\right)$ .... (4)
- But since by data $\frac{\partial^2 u}{\partial x \partial y} = 0$, we get,
- $\frac{\partial^2 z}{\partial x \partial y} = e^{ax+by}\left(a \cdot \frac{\partial u}{\partial y} + b \cdot \frac{\partial u}{\partial x} + abu\right)$ .... (5)

EXAMPLE-9

Further by data $\frac{\partial^2 z}{\partial x \partial y} - \frac{\partial z}{\partial x} - \frac{\partial z}{\partial y} + z = 0$ .... (6)

- Putting the values from (5), (2), (3) and (1) in (6), we get,
- $$e^{ax+by}\left[a \cdot \frac{\partial u}{\partial y} + b \cdot \frac{\partial u}{\partial x} + abu - \frac{\partial u}{\partial x} - au - \frac{\partial u}{\partial y} - bu + u\right] = 0$$
- $\therefore e^{ax+by}\left[(a-1)\frac{\partial u}{\partial y} + (b-1)\frac{\partial u}{\partial x} + au(b-1) - u(b-1)\right] = 0$
- $\therefore e^{ax+by}\left[(a-1)\frac{\partial u}{\partial y} + (b-1)\frac{\partial u}{\partial x} + (b-1) \cdot u(b-1)\right] = 0$
- Since $u \neq 0, \frac{\partial u}{\partial x} \neq 0$ and $\frac{\partial u}{\partial y} \neq 0$
- We should have $a - 1 = 0, b - 1 = 0$ i.e., $a = 1, b = 1$

EXAMPLE-10

❖ If $a^2x^2 + b^2y^2 = c^2z^2$, evaluate $\frac{1}{a^2}\frac{\partial^2 z}{\partial x^2} + \frac{1}{b^2}\frac{\partial^2 z}{\partial y^2}$

Solution: $a^2x^2 + b^2y^2 = c^2z^2$

Differentiating partially w.r.t. x,

$$2a^2x = 2c^2z \cdot \frac{\partial z}{\partial x}$$

$$\frac{\partial z}{\partial x} = \frac{a^2x}{c^2z}$$

Differentiating $\frac{\partial z}{\partial x}$ partially w.r.t. x,

$$\frac{\partial^2 z}{\partial x^2} = \frac{a^2}{c^2}\left(\frac{1}{z} - \frac{x}{z^2} \cdot \frac{\partial z}{\partial x}\right) = \frac{a^2}{c^2z}\left(1 - \frac{x}{z} \cdot \frac{a^2x}{c^2z}\right)$$

$$\frac{1}{a^2}\frac{\partial^2 z}{\partial x^2} = \frac{1}{c^2z}\left(1 - \frac{a^2x^2}{c^2z^2}\right)$$

Similarly, $\frac{1}{b^2}\frac{\partial^2 z}{\partial y^2} = \frac{1}{c^2z}\left(1 - \frac{b^2y^2}{c^2z^2}\right)$

Hence, $\frac{1}{a^2}\frac{\partial^2 z}{\partial x^2} + \frac{1}{b^2}\frac{\partial^2 z}{\partial y^2} = \frac{1}{c^2z}\left(2 - \frac{a^2x^2+b^2y^2}{c^2z^2}\right)$

$$= \frac{1}{c^2z}\left(2 - \frac{c^2z^2}{c^2z^2}\right)$$

$$= \frac{1}{c^2z}(2-1) = \frac{1}{c^2z}$$

COMPOSITE FUNCTIONS

- **(a)** Let $z = f(x, y)$ and $x = \Phi(t), y = \Psi(\mathrm{t})$ so that z is function of x , y and x , y are function of third variable t.
- The three relations define z as a function of t. In such cases z is called a **composite function of $\boldsymbol{t}$**.
- **e.g. (i)** $z = x^2 + y^2 , x = at^2 , y = 2at$
- **(ii)** $z = \mathrm{x}^2\mathrm{y} + \mathrm{xy}^2$, $x = acost$, $y = bsint$

  define z as a composite function of t
- **Differentiation:** Let $z = f(x, y)$ posses continuous first order partial derivatives and $x = \Phi(t), y = \Psi(t)$ posses continuous first order derivatives then,

$$\frac{dz}{dt} = \frac{\partial z}{\partial x} . \frac{dx}{dt} + \frac{\partial z}{\partial y} . \frac{dy}{dt}$$

EXAMPLE-11

- If $u = x^2y^3, x = \log t\,, y = e^t$, find $\frac{du}{dt}$
- **Solution:** $u = x^2y^3, x = \log t\,, y = e^t$
- $$\frac{du}{dt} = \frac{\partial u}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dt}$$
- $$= (2xy^3)\frac{1}{t} + (3x^2y^2)e^t$$
- Substituting x and y,
- $$\frac{du}{dt} = 2(\log t)e^{3t} \cdot \frac{1}{t} + 3(\log t)^2e^{2t} \cdot e^t$$
- $$= \frac{2}{t}\log t\, e^{3t} + 3(\log t)^2e^{3t}$$

EXAMPLE-12

- If $u = xy + yz + zx$ where $x = \frac{1}{t}, y = e^t, z = e^{-t}$, find $\frac{du}{dt}$
- **Solution:** $u = xy + yz + zx, x = \frac{1}{t}, y = e^t, z = e^{-t}$
- $\frac{du}{dt} = \frac{\partial u}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dt} + \frac{\partial u}{\partial z} \cdot \frac{dz}{dt}$
- $= (y+z)\left(-\frac{1}{t^2}\right) + (x+z)e^t + (y+x)(-e^{-t})$
- Substituting x, y and z,
- $\frac{du}{dt} = -\frac{1}{t^2}(e^t + e^{-t}) + \left(\frac{1}{t} + e^{-t}\right)e^t - \left(e^t + \frac{1}{t}\right)e^{-t}$
- $= \frac{1}{t^2}(e^t + e^{-t}) + \frac{1}{t}(e^t - e^{-t})$

EXAMPLE-13

- If $z = e^{xy}, x = t\cos t\,, y = t\sin t\,$, find $\frac{dz}{dt}$ at $t = \frac{\pi}{2}$
- **Solution:** $z = e^{xy}, x = t\cos t\,, y = t\sin t$

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial z}{\partial y}\cdot\frac{dy}{dt}$$

$$= e^{xy}y(\cos t - t\sin t) + e^{xy}x(\sin t + t\cos t)$$

- At $t = \frac{\pi}{2}, x = 0, y = \frac{\pi}{2}$
- Hence, $\left.\frac{dz}{dt}\right|_{t=\frac{\pi}{2}} = e^0\left[\frac{\pi}{2}\left(0 - \frac{\pi}{2}\right) + 0\right] = -\frac{\pi^2}{4}$