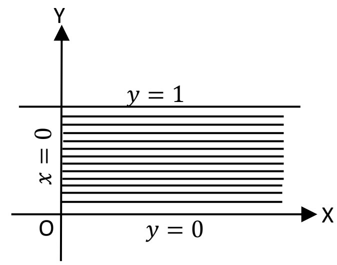
SOME MORE SOLVED EXAMPLES ON DOUBLE INTEGRATION

TYPE-1: EVALUATION OVER GIVEN LIMITS

1. $\int_0^\infty dx \int_0^1 e^{-x^a y} dy$

Solution: Since the limits are constant and integration with respect to y first and then with respect to x leads to a complicated integral, we reverse the order of integration



$\therefore I = \int_0^1 dy \int_0^\infty e^{-x^a y}\, dx$

Now, we put $x^a y = t$ $\quad \therefore x = \left(\frac{t}{y}\right)^{1/a}$

$\therefore dx = \frac{1}{a}\cdot\frac{t^{(1/a)-1}}{y^{1/a}} dt$

When $x = 0, t = 0$; when $x = \infty, t = \infty$

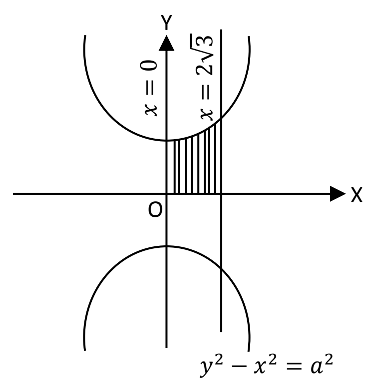
$= \int_0^1 dy \int_0^\infty e^{-t}\cdot\frac{1}{a\cdot y^{1/a}}\cdot t^{(1/a)-1}\, dt$

$= \frac{1}{a}\int_0^1 y^{-1/a}\, dy \cdot \int_0^\infty e^{-t}\cdot t^{(1/a)-1}\, dt$

$= \frac{1}{a}\left[\frac{y^{-(1/a)+1}}{-(1/a)+1}\right]_0^1 \cdot \overline{|\tfrac{1}{a}} = \frac{\overline{|1/a}}{a-1}$ [By defintion of $\overline{|n}$]

2. $\int_0^{a\sqrt{3}} \int_0^{\sqrt{x^2+a^2}} \frac{x}{y^2+x^2+a^2} dy\, dx$

Solution: $I = \int_0^{a\sqrt{3}} \int_0^{\sqrt{x^2+a^2}} \frac{x}{y^2+(x^2+a^2)} dy\, dx$



$= \int_0^{a\sqrt{3}} \left[\frac{1}{\sqrt{x^2+a^2}}\tan^{-1}\left(\frac{y}{\sqrt{x^2+a^2}}\right)\right]_0^{\sqrt{x^2+a^2}} x\ dx$

$= \int_0^{a\sqrt{3}} \frac{1}{\sqrt{x^2+a^2}}\cdot\left[\frac{\pi}{4}-0\right]\cdot x\ dx$

$= \frac{\pi}{4}\int_0^{a\sqrt{3}} \frac{x}{\sqrt{x^2+a^2}}\ dx = \frac{\pi}{4}\left[\sqrt{x^2+a^2}\right]_0^{a\sqrt{3}}$

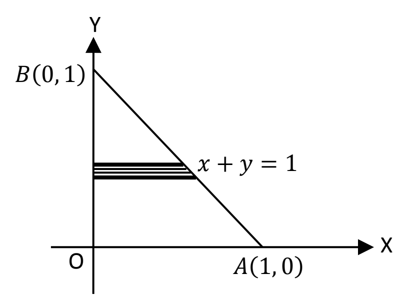
$= \frac{\pi}{4}[2a-a] = \frac{\pi}{4}a$

TYPE-2: EVALUATION OVER GIVEN REGION

1. $\iint_R xy\sqrt{1-x-y}\, dx\, dy$ over the area of the triangle formed by $x = 0,\ y = 0,\ x + y = 1$

Solution: The region of integration is shown in the figure. Now, consider a strip parallel to the x –axis. On this strip x varies from $x = 0$ to $x = 1-y$. Then y varies from $y = 0$ to $y = 1$

$\therefore I = \int_0^1 \int_0^{1-y} y\cdot x\sqrt{(1-y)-x}\cdot dxdy$



Now, put $x = (1-y)t$ $\quad \therefore dx = (1-y)dt$

When $x = 0, t = 0$; when $x = 1-y, t = 1$

$= \int_0^1 \int_0^1 y(1-y)\cdot t\sqrt{(1-y)-(1-y)t}\cdot(1-y)\, dtdy$

$= \int_0^1 \int_0^1 y(1-y)^2\cdot\sqrt{1-y}\cdot t\sqrt{1-t}\cdot dtdy$

$= \int_0^1 \int_0^1 y(1-y)^{5/2}\cdot\left[t(1-t)^{1/2}\right]\cdot dtdy$

$= \left(\int_0^1 y(1-y)^{5/2}\, dy\right)\left(\int_0^1 t(1-t)^{1/2}\, dt\right)$

Put $t = \sin^2\theta$ and $y = \sin^2\theta$

$\therefore dt = 2\sin\theta\cos\theta\, d\theta, dy = 2\sin\theta\cos\theta\, d\theta$

$$\therefore I = \left(\int_0^{\pi/2} \sin^2\theta\cos^5\theta \cdot 2\sin\theta\cos\theta\, d\theta\right) \times \left(\int_0^{\pi/2} \sin^2\theta\cos\theta \cdot 2\sin\theta\cos\theta\, d\theta\right)$$

$$= \left(2\int_0^{\pi/2} \sin^3\theta\cos^6\theta\, d\theta\right) \times \left(2\int_0^{\pi/2} \sin^3\theta\cos^2\theta\, d\theta\right)$$

$$= 4 \cdot \left(\frac{1}{2} \cdot \frac{\left|\overline{2}\right. \left|\overline{\tfrac{7}{2}}\right.}{\left|\overline{\tfrac{11}{2}}\right.}\right) \cdot \left(\frac{1}{2} \cdot \frac{\left|\overline{2}\right. \left|\overline{\tfrac{3}{2}}\right.}{\left|\overline{\tfrac{7}{2}}\right.}\right)$$

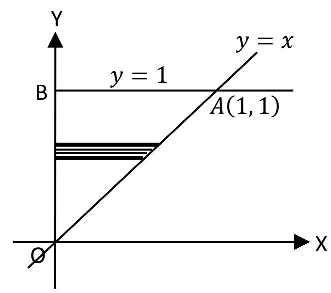
$$= \frac{\left|\overline{\tfrac{3}{2}}\right.}{\left|\overline{\tfrac{11}{2}}\right.} = \frac{\left|\overline{\tfrac{3}{2}}\right.}{\left(\frac{9}{2}\right)\left(\frac{7}{2}\right)\left(\frac{5}{2}\right)\left(\frac{3}{2}\right)\left|\overline{\tfrac{3}{2}}\right.} = \frac{16}{945}$$

2. $\iint_R \frac{2xy^5}{\sqrt{1+x^2y^2-y^4}}\, dx\, dy$ where R is the region of the triangle whose vertices are $(0,0), (1,1), (0,1)$

Solution: Let $O(0,0), A(1,1), B(0,1)$ be the vertices of the trinalge OAB. Now, the equation of the line AB is $y = 1$ and the equation of the line OA is $\frac{x-0}{0-1} = \frac{y-0}{0-1}$ i.e. $x = y$

Now, consider a strip parallel to the x −axis.

On this strip x varies from $x = 0$ to $x = y$. The strip moves from $y = 0$ to $y = 1$



$$\therefore I = \int_0^1 \int_0^y \frac{2y^5 \cdot x}{\sqrt{(1-y^4)+x^2y^2}}\, dxdy$$

$$= \int_0^1 \int_0^y \frac{1}{y} \cdot \frac{2y^5 \cdot x}{\sqrt{\frac{1-y^4}{y^2}+x^2}}\, dxdy$$

$$= \int_0^1 2y^4 \left[\sqrt{\frac{1-y^4}{y^2}+x^2}\right]_0^y dy$$

$$= \int_0^1 2y^4 \left[\sqrt{\frac{1-y^4}{y^2}+y^2} - \sqrt{\frac{1-y^4}{y^2}}\right] dy$$

$$= \int_0^1 2y^4 \left[\frac{1}{y} - \frac{\sqrt{1-y^4}}{y}\right] dy$$

$$= 2\int_0^1 \left[y^3 - \sqrt{1-y^4} \cdot y^3\right] dy = 2\left[\frac{y^4}{4} + \frac{1}{4} \cdot \frac{(1-y^4)^{3/2}}{3/2}\right]_0^1$$

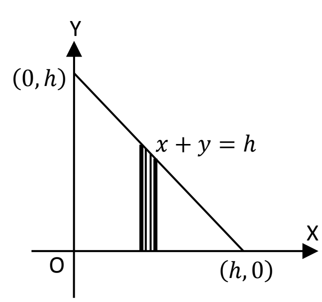
$$= 2\left[\frac{1}{4} - \frac{1}{4} \cdot \frac{2}{3}\right] = \frac{1}{2} - \frac{1}{3} = \frac{1}{6}$$

3. $\iint x^{m-1}y^{n-1} dx\, dy$ over the region bounded by $x + y = h,\ x = 0,\ y = 0$

Solution: The region is bounded by the x −axis, y −axis and the line $x + y = h$

On this strip y varies from 0 to $h - x$ and then strip moves from $x = 0$ to $x = h$

$$\therefore I = \int_0^h \int_0^{h-x} x^{m-1}y^{n-1}\, dydx$$



Let $I_1 = \int_0^{h-x} y^{n-1}\, dy = \left[\frac{y^n}{n}\right]_0^{h-x} = \frac{1}{n}(h-x)^n$

Now, $I = \int_0^h x^{m-1} \cdot \frac{1}{n}(h-x)^n\, dx$

Put $x = ht$

$$= \int_0^1 h^{m-1} \cdot t^{m-1} \cdot \frac{1}{n} h^n (1-t)^n \cdot h\, dt$$

$$= \frac{h^{m+n}}{n} \int_0^1 t^{m-1}(1-t)^n\, dt$$

$$= \frac{h^{m+n}}{n} \cdot \frac{\left|\overline{m}\right. \left|\overline{n+1}\right.}{\left|\overline{m+n+1}\right.} = \frac{h^{m+n}\left|\overline{m}\right. \left|\overline{n}\right.}{(m+n)\left|\overline{m+n}\right.}$$

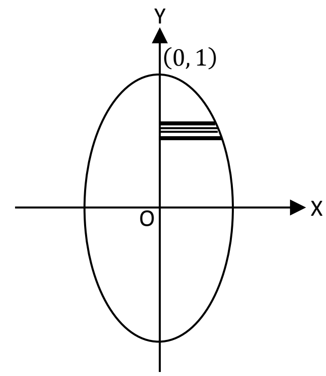
4. $\iint_R \frac{1}{\sqrt{1-x^2-y^2}} dx\, dy$ where R is the region of the first quadrant of the ellipse $2x^2 + y^2 = 1$

Solution: The ellipse $2x^2 + y^2 = 1$ i.e. $\frac{x^2}{1/2} + \frac{y^2}{1} = 1$ has semi-major axis

$a = \frac{1}{\sqrt{2}}$ and semi-minor axis $b = 1$

If we consider a strip parallel to the x —axis, on this strip x varies from $x = 0$ to $x = \sqrt{1-y^2}/\sqrt{2}$

This strip moves from $y = 0$ to $y = 1$



$$\therefore I = \int_0^1 \int_0^{\sqrt{(1-y^2)/2}} \frac{dxdy}{\sqrt{(1-y^2)-x^2}}$$

$$= \int_0^1 \sin^{-1}\left[\frac{x}{\sqrt{1-y^2}}\right]_0^{\sqrt{(1-y^2)/2}} dy$$

$$= \int_0^1 \sin^{-1}\left(\frac{1}{\sqrt{2}}\right) dy = \int_0^1 \frac{\pi}{4} dy \qquad = \frac{\pi}{4}[y]_0^1 = \frac{\pi}{4}$$

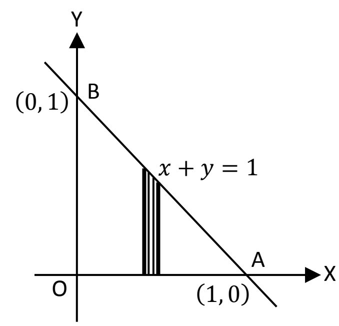
5. $\iint_R (y - \sqrt{x})\, dA$ where R is the region cut-off the line $x + y = 1$ of the xy —plane in the first quadrant

Solution: The region of integration is the triangle OAB.

Consider a strip parallel to the y —axis.

On this strip y varies from 0 to $(1 - x)$.

Then the strip moves from $x = 0$ to $x = 1$



$$\therefore I = \int_0^1 \int_0^{1-x} (y - \sqrt{x})\, dydx$$

$$= \int_0^1 \left[\frac{y^2}{2} - \sqrt{x}\cdot y\right]_0^{1-x} dx$$

$$= \int_0^1 \left[\frac{(1-x)^2}{2} - \sqrt{x}(1-x)\right] dx$$

$$= \int_0^1 \left[\frac{1}{2}(1 - 2x + x^2) - \sqrt{x} + x^{3/2}\right] dx$$

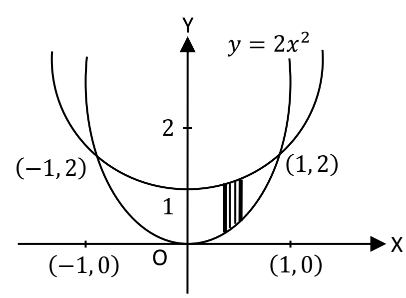
$$= \left[\frac{1}{2}\left(x - x^2 + \frac{x^3}{3}\right) - \frac{x^{3/2}}{3/2} + \frac{x^{5/2}}{5/2}\right]_0^1 \qquad = \frac{1}{2}\left(1 - 1 + \frac{1}{3}\right) - \frac{2}{3} + \frac{2}{5} = \frac{1}{6} - \frac{2}{3} + \frac{2}{5} = -\frac{1}{10}$$

6. Evaluate $\iint (x + 2y)\, dA$ over the region bounded by the parabolas $y = 2x^2$ and $y = 1 + x^2$

Solution: The parabola $y = 1 + x^2$ i.e. $y - 1 = x^2$ has the vertex at $(0, 1)$ and it opens upwards.

The two parabolas intersect when $2x^2 = x^2 + 1$ i.e. $x = \pm 1$

Now, $I = \iint (x + 2y)\, dA$



$$= \int_{-1}^1 \int_{2x^2}^{x^2+1} (x + 2y)\, dydx$$

$$= \int_{-1}^1 [xy + y^2]_{2x^2}^{x^2+1} dx$$

$$= \int_{-1}^1 \{[x(x^2+1) + (x^2+1)^2] - (2x^3 + 4x^4)\}\, dx$$

$$= \int_{-1}^1 (-3x^4 - x^3 + 2x^2 + x + 1)\, dx$$

$$= \left[-\frac{3x^5}{5} - \frac{x^4}{4} + \frac{2x^3}{3} + \frac{x^2}{2} + x\right]_{-1}^1$$

$$= \left[\left(-\frac{3}{5} - \frac{1}{4} + \frac{2}{3} + \frac{1}{2} + 1\right) - \left(\frac{3}{5} - \frac{1}{4} - \frac{2}{3} + \frac{1}{2} - 1\right)\right] = -\frac{6}{5} + \frac{4}{3} + 2$$

$$= \frac{-18+20+30}{15} = \frac{32}{15}$$

TYPE 3: CHANGE OF ORDER OF INTEGRATIONS

1. $\int_0^1 \int_0^{\sqrt{1-y^2}} \frac{cos^{-1}x}{\sqrt{1-x^2}\sqrt{1-x^2-y^2}} dx\, dy$

Solution: The limit for y are 0 and 1 and for x are 0 and $x = \sqrt{1-y^2}$ i.e. $x^2 + y^2 = 1$.

Hence, the region of integration is first quadrant of the cirlce $x^2 + y^2 = 1$

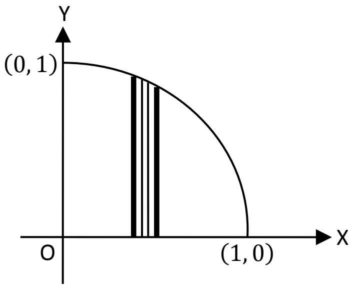
Now, if we change the order of integration y varies from 0 to $\sqrt{1-x^2}$

and x varies from 0 to 1. Hence,

$$\therefore I = \int_0^1 \int_0^{\sqrt{1-x^2}} \frac{\cos^{-1}x}{\sqrt{1-x^2}\sqrt{1-x^2-y^2}} dy\, dx$$

$$= \int_0^1 \frac{\cos^{-1}x}{\sqrt{1-x^2}} \left[\sin^{-1}\frac{y}{\sqrt{1-x^2}}\right]_0^{\sqrt{1-x^2}} dx$$

$$= \frac{\pi}{2}\int_0^1 \frac{\cos^{-1}x}{\sqrt{1-x^2}}$$



Put $\cos^{-1}x = t,\ \therefore\ \frac{dx}{\sqrt{1-x^2}} = -dt$

$$= -\frac{\pi}{2}\int_{\pi/2}^{0} t\, dt = \frac{\pi}{2}\int_0^{\pi/2} t\, dt$$

$$= \frac{\pi}{2}\left[\frac{t^2}{2}\right]_0^{\pi/2} = \frac{\pi^3}{16}$$

2. $\int_0^a \int_0^x \frac{e^y}{\sqrt{(a-x)(x-y)}} dy\, dx$

Solution: Since integration with respect to y is complicated we change the order of integration.

The limits for y are $y = 0$ and $y = x$ for x are $x = 0$ to $x = a$.

The region of integration is the triangel OAB

Now, consider a strip parallel to the x —axis.

On this strip x varies from $x = y$ to $x = a$ and for the strip y varies from $y = 0$ to $y = a$

$$\therefore I = \int_0^a \int_y^a \frac{e^y}{\sqrt{(a-x)(x-y)}} dxdy$$

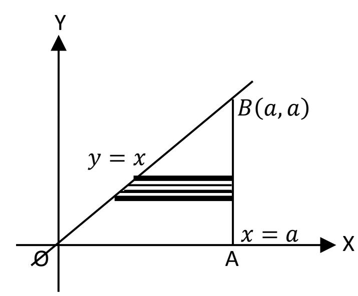
$$= \int_0^a \int_y^a \frac{e^y}{\sqrt{-ay-[x^2-(a+y)x]}} dxdy$$

$$= \int_0^a \int_y^a \frac{e^y}{\sqrt{\left(\frac{a-y}{2}\right)^2 - \left(x - \frac{a+y}{2}\right)^2}} dxdy$$

$$= \int_0^a e^y \left[\sin^{-1}\left\{\frac{x-(a+y)/2}{(a-y)/2}\right\}\right]_y^a dy$$

$$= \int_0^a e^y \left[\frac{\pi}{2} - \left(-\frac{\pi}{2}\right)\right] dy = \pi \int_0^a e^y\, dy$$

$$= \pi[e^y]_0^a = \pi(e^a - 1)$$



Aliter:

The integral can also be evalated by putting $x - y = t^2$ i.e. $x = y + t^2$ $\therefore dx = 2tdt$

When $x = y, t = 0$; when $x = a, t = \sqrt{a-y}$

$$\therefore I = \int_0^a \int_0^{\sqrt{a-y}} e^y \cdot \frac{1}{\sqrt{[(a-y)-t^2]\cdot t^2}} \cdot 2t\, dtdy$$

$$= \int_0^a e^y\, dy \int_0^{\sqrt{a-y}} \frac{2\, dt}{\sqrt{(a-y)-t^2}}$$

$$= 2\int_0^a e^y \cdot \left[\sin^{-1}\left(\frac{t}{\sqrt{a-y}}\right)\right]_0^{\sqrt{a-y}} dy$$

$$= 2\int_0^a e^y \cdot \left[\frac{\pi}{2} - 0\right] dy = \pi\int_0^a e^y\, dy \quad = \pi[e^a - 1]$$

3. $\int_0^a \int_0^x \frac{1}{(y+a)\sqrt{(a-x)(x-y)}} dx\, dy$

Solution: The given region of integration is the same as above example

$\therefore I \quad = \int_0^a \int_y^a \frac{dx\, dy}{(y+a)\sqrt{(a-x)(x-y)}}$

Putting $x - y = t^2 \qquad \therefore dx = 2t\, dt$

When $x = y, t = 0$, when $x = a, t = \sqrt{a-y}$

$\therefore I \quad = \int_0^a \int_y^{\sqrt{a-y}} \frac{dy}{(y+a)} \cdot \frac{2t\, dt}{\sqrt{(a-y-t^2)\cdot t}}$

$= \int_0^a \int_y^{\sqrt{a-y}} \frac{dy}{(y+a)} \cdot \frac{2dt}{\sqrt{(a-y)-t^2}}$

$= 2\int_0^a \frac{dy}{(y+a)} \left[\sin^{-1}\frac{t}{\sqrt{a-y}}\right]_0^{\sqrt{a-y}}$

$= 2\int_0^a \frac{dy}{y+a} \cdot \frac{\pi}{2} = \pi \int_0^a \frac{dy}{y+a}$

$= \pi[\log(y+a)]_0^a$

$= \pi[\log 2a - \log a] = \pi \log 2$

4. $\int_0^\pi \int_0^x \frac{\sin y}{\sqrt{(\pi-x)(x-y)}} dy\, dx$

Solution: The given region of integration is the same as above example

$\therefore I \quad = \int_0^\pi \int_y^\pi \frac{\sin y\, dxdy}{\sqrt{(\pi-x)(x-y)}}$

Put $x - y = t^2 \qquad \therefore dx = 2t\, dt$

When $x = y, t = 0$; when $x = \pi, t = \sqrt{\pi - y}$

$= \int_0^\pi \int_0^{\sqrt{\pi-y}} \sin y \cdot \frac{2t\, dt}{\sqrt{(\pi-y)-t^2\cdot t}}$

$= \int_0^\pi \int_0^{\sqrt{\pi-y}} \sin y \cdot \frac{dt}{\sqrt{(\pi-y)-t^2}} dy$

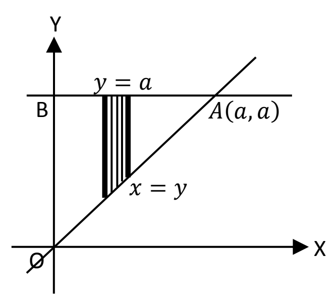
$= 2\int_0^\pi \sin y \left[\sin^{-1}\left(\frac{t}{\sqrt{\pi-y}}\right)\right]_0^{\sqrt{\pi-y}} dy$

$= 2\int_0^\pi \frac{\pi}{2} \sin y\, dy = \pi[-\cos y]_0^\pi = 2\pi$

5. $\int_0^a \int_0^y \frac{x}{\sqrt{(a^2-x^2)(a-y)(y-x)}} dx\, dy$

Solution: The region of integration is bounded by $x = 0$, the y –axis, $x = y$, the line passing through origin, $y = 0$, the x –axis and $y = a$, a line parallel to the x –axis. To change the prder of integration consider a strip parallel to the y –axis

On the strip y varies from $y = x$ to $y = a$ and then x varies from $x = 0$ to $x = a$



$\therefore I \quad = \int_0^a \int_x^a \frac{x\, dydx}{\sqrt{(a^2-x^2)}\sqrt{(a-y)(y-x)}}$

Put $y - x = t^2 \qquad \therefore dy = 2t\, dt$

When $y = x, t = 0$; when $y = a, t = \sqrt{a-x}$

$$= \int_0^a \int_0^{\sqrt{a-x}} \frac{x\,dx}{\sqrt{(a^2-x^2)}} \frac{2t\,dt}{\sqrt{(a-x)-t^2}\cdot t}$$

$$= 2\int_0^a \frac{x\,dx}{\sqrt{a^2-x^2}} \left[\sin^{-1}\left(\frac{t}{\sqrt{a-x}}\right)\right]_0^{\sqrt{a-x}}$$

$$= 2\int_0^a \frac{x}{\sqrt{a^2-x^2}} \frac{\pi}{2} dx = \pi\left[-\sqrt{a^2-x^2}\right]_0^a$$

$$= \pi[-0+a] = \pi a$$

6. $\int_0^a \int_0^x \frac{\sin y}{\sqrt{(a-x)(x-y)(4-5\cos y)^2}} dy\,dx$

Solution: The region of integration is the same as above example

$$\therefore I = \int_0^a \int_y^a \frac{\sin y}{(4-5\cos y)} \cdot \frac{dx}{\sqrt{[(a-x)(x-y)]}} dy$$

Put $x-y=t^2$ $\therefore dx = 2t\,dt$

When $x=y$, $t=0$; When $x=a, t=\sqrt{a-y}$

$$= 2\int_0^a \frac{\sin y}{4-5\cos y} \cdot \left[\sin^{-1}\left(\frac{t}{\sqrt{a-y}}\right)\right]_0^{\sqrt{a-y}} dy$$

$$= \pi\int_0^a \frac{\sin y}{4-5\cos y} dy = \pi\left[-\frac{1}{5}\log(4-5\cos y)\right]_0^a$$

$$= \frac{\pi}{5}[-\log(4-5\cos a)+0] = -\frac{\pi}{5}\log(4-5\cos a)$$

7. $\int_0^2 \int_{\sqrt{2x}}^2 \frac{y^2}{\sqrt{y^4-4x^2}} dy\,dx$

Solution: Here, the region of integration is bounded by $y=\sqrt{2x}$ i.e. $y^2=2x$ a parabola;

$y=2$, a line parallel the x −axis; $x=0$, the y −axis and $x=2$, the line parallel to the y −axis.

If we consider a strip parallel to the x −axis,

on this strip x varies $x=0$ to $x=y^2/2$ and then y varies from $y=0$ to $y=2$

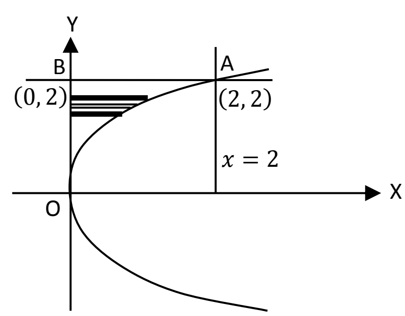
$$\therefore I = \int_0^2 \int_0^{y^2/2} \frac{y^2 dydx}{\sqrt{y^4-4x^2}}$$

$$= \frac{1}{2}\int_0^2 \int_0^{y^2/2} \frac{y^2}{\sqrt{(y^2/2)^2-x^2}} dxdy$$

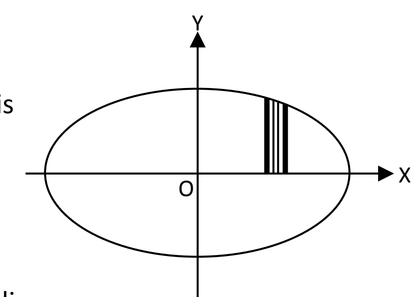
$$= \frac{1}{2}\int_0^2 y^2 \left[\sin^{-1}\left(\frac{x}{y^2/2}\right)\right]_0^{y^2/2} dy$$

$$= \frac{1}{2}\int_0^2 y^2[\sin^{-1}1 - \sin^{-1}0]\,dy$$

$$= \frac{1}{2}\int_0^2 \frac{\pi}{2}\cdot y^2\,dy = \frac{\pi}{4}\left[\frac{y^3}{3}\right]_0^2 = \frac{2\pi}{3}$$



8. $\int_0^{1/2} \int_0^{\sqrt{1-4y^2}} \frac{1+x^2}{\sqrt{1-x^2}\sqrt{1-x^2-y^2}} dx\,dy$



Solution: Here, the region of integration is bounded by $x=0$ i.e. the y −axis

$x=\sqrt{1-4y^2}$ $\therefore x^2 = 1-4y^2$ $\therefore x^2+\frac{y^2}{(1/4)}=1$

It is an ellipse with semi-major axis 1 and semi-minor axis $1/2$.

$y=0$, the x −axis and $y=1/2$, a line parallel the x −axis.

Thus, the region of integration is the first quadrant of the above ellipse.

To change the order of integration, consider a strip parallel to the y −axis.

On this strip y varies from $y = 0$ to $y = \frac{\sqrt{1-x^2}}{2}$. Then x varies from 0 to 1

$$\therefore I = \int_0^1 \int_0^{\sqrt{1-x^2}/2} \frac{1+x^2}{\sqrt{1-x^2}} \cdot \frac{dy}{\sqrt{(1-x^2)-y^2}}\, dx$$

$$= \int_0^1 \frac{1+x^2}{\sqrt{1-x^2}} \cdot \left[\sin^{-1}\left(\frac{y}{\sqrt{1-x^2}}\right)\right]_0^{\sqrt{1-x^2}/2} dx$$

$$= \int_0^1 \frac{1+x^2}{\sqrt{1-x^2}} \cdot \left[\sin^{-1}\frac{1}{2} - \sin^{-1} 0\right] dx$$

$$= \frac{\pi}{6}\int_0^1 \frac{1+x^2}{\sqrt{1-x^2}}\, dx$$

To find the integral, put $x = \sin\theta$, $dx = \cos\theta\, d\theta$

$$= \frac{\pi}{6}\int_0^{\pi/2} \frac{1+\sin^2\theta}{\cos\theta} \cdot \cos\theta\, d\theta$$

$$= \frac{\pi}{6}\int_0^{\pi/2} (1+\sin^2\theta)\, d\theta$$

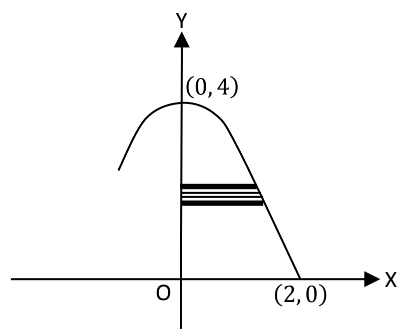
$$= \frac{\pi}{6}\left[\{\theta\}_0^{\pi/2} + \frac{1}{2}\frac{\Gamma\left(\frac{3}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{4}{2}\right)}\right] = \frac{\pi}{6}\left[\frac{\pi}{2} + \frac{1}{2}\frac{\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma(2)}\right] = \frac{\pi}{6}\left[\frac{\pi}{2} + \frac{\pi}{4}\right] = \frac{\pi^2}{8}$$

9. Change the order of integration and evaluate the integral $\int_0^2 \int_0^{4-x^2} \frac{xe^{2y}}{4-y}\, dy\, dx$

Solution: The curve $y = 4 - x^2$ i.e. $y - 4 = -x^2$ is a parabola with vertex at $(0, 4)$, opening downwards

To reserve the order of integration, consider a strip parallel to the x −axis.

On this strip x varies from 0 to $\sqrt{4-y}$ and the strip moves parallel to itself from $y = 0$ and $y = 4$



$$\therefore I = \int_{y=0}^{4} \int_{x=0}^{\sqrt{4-y}} \frac{xe^{2y}}{4-y}\, dx\, dy$$

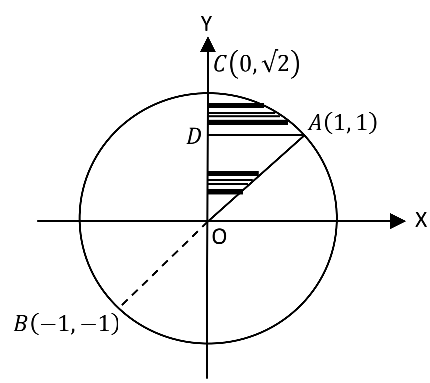
$$= \int_0^4 \left[\frac{x^2}{2}\right]_0^{\sqrt{4-y}} \cdot \frac{e^{2y}}{4-y}\, dy$$

$$= \frac{1}{2}\int_0^4 (4-y) \cdot \frac{e^{2y}}{(4-y)}\, dy$$

$$= \frac{1}{2}\int_0^4 e^{2y}\, dy = \frac{1}{2}\left[\frac{e^{2y}}{2}\right]_0^4$$

$$= \frac{e^8-1}{4}$$

10. $\int_0^1 dx \int_x^{\sqrt{2-x^2}} \frac{x}{\sqrt{x^2+y^2}}\, dy$



Solution: The limit for y are x and $\sqrt{2-x^2}$, and those for x are 0 and 1.

We, therefore, draw the curves $y = x$ which is a stright line and $y = \sqrt{2-x^2}$ which is the upper half of the circle $x^2 + y^2 = 2$.

The region of integration is $OACD$.

Solving the equations $y = x$ and $x^2 + y^2 = 2$, we get the points of intesection $A(1, 1)$ and $B(-1, -1)$

If we consider a strip parallel to the x −axis the region has to be divided into two parts OAD and ADC

In the region ODA, x varies from 0 to y and y varies from 0 to 1

In the region ADC, x varies from 0 to $\sqrt{2-y^2}$ and y varies from 1 to $\sqrt{2}$

$$\therefore I = \int_0^1 dy \int_0^y \frac{x}{\sqrt{x^2+y^2}}\, dx + \int_1^{\sqrt{2}} dy \int_0^{\sqrt{2-y^2}} \frac{x}{\sqrt{x^2+y^2}}\, dx$$

Now, $I_1 = \int_0^1 dy\left[\sqrt{x^2+y^2}\right]_0^y = \int_0^1 (\sqrt{2}\cdot y - y)\,dy = (\sqrt{2}-1)\left[\frac{y^2}{2}\right]_0^1 = \frac{1}{2}(\sqrt{2}-1)$

and $I_2 = \int_1^{\sqrt{2}} dy\left[\sqrt{x^2+y^2}\right]_0^{\sqrt{2-y^2}} = \int_1^{\sqrt{2}}(\sqrt{2}-y)\,dy = \left[\sqrt{2}y - \frac{y^2}{2}\right]_1^{\sqrt{2}} = \frac{3}{2}-\sqrt{2}$

$\therefore I = I_1 + I_2 = 1 - \frac{1}{\sqrt{2}}$

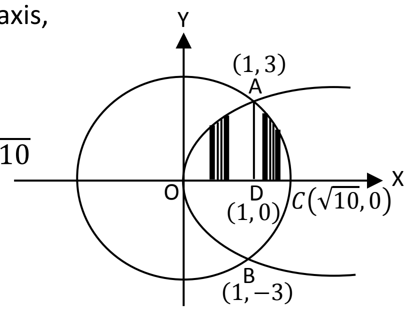
11. $\int_0^3 \int_{y^2/9}^{\sqrt{10-y^2}} dxdy$

Solution: The limits for x are $y^2/9$ and $\sqrt{10-y^2}$ and those for y are 0 and 3.

We, therefore, draw the curves $x = y^2/9$ i.e. the parabola $y^2 = 9x$ and $x = \sqrt{10-y^2}$ i.e. the upper half of the circle $x^2+y^2 = 10$ The region of integration is thus $OACD$.

Solving $y^2 = 9x$ and $x^2+y^2 = 10$, we get the points of intesection $A(1,3)$ and $B(1,-3)$

Now, to change the order, if we consider a strip parallel to the y –axis, the region has to be divided into two parts ODA and ADC.



In the region ODA, y varies from 0 to $3\sqrt{x}$ and x varies from 0 to 1

In the region ADC, y varies from 0 to $\sqrt{10-x^2}$ and x varies 1 to $\sqrt{10}$

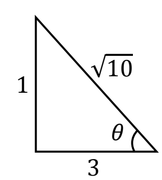
$\therefore I \quad = \int_0^1 \int_0^{3\sqrt{x}} dydx + \int_1^{\sqrt{10}} \int_0^{\sqrt{10-x^2}} dydx$

Now, $I_1 = \int_0^1 [y]_0^{3\sqrt{x}}\,dx = \int_0^1 3\sqrt{x}\,dx = 3\left[\frac{2}{3}x^{3/2}\right]_0^1 = 2$

$\therefore I_2 \quad = \int_1^{\sqrt{10}} [y]_0^{\sqrt{10-x^2}}\,dx = \int_1^{\sqrt{10}} \sqrt{10-x^2}\cdot dx$

$= \left[\frac{x}{2}\sqrt{10-x^2} + \frac{10}{2}\sin^{-1}\frac{x}{\sqrt{10}}\right]_1^{\sqrt{10}}$

$= 5\cdot\frac{\pi}{2} - \frac{3}{2} - 5\sin^{-1}\frac{1}{\sqrt{10}}$



$\therefore I = I_1 + I_2 = 2 + 5\left(\frac{\pi}{2} - \sin^{-1}\frac{1}{\sqrt{10}}\right) - \frac{3}{2}$

$= \frac{1}{2} + 5\sin^{-1}\left(\frac{3}{\sqrt{10}}\right)$

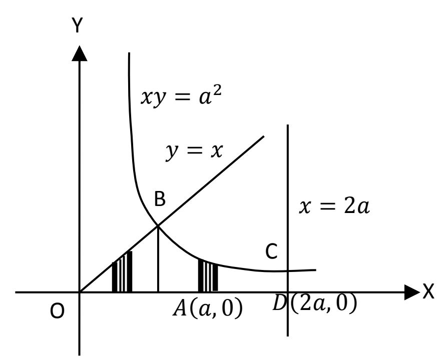
12. $\int\int_R x^2 dx\,dy$ where R is the region in the first quadrant bounded by $xy = a^2$, $x = 2a$, $y = 0$ and $y = x$

Solution: The region of integration is bounded by $xy = a^2$, a rectangular hyperbola;

$x = 2a$, a line is parallel to the y –axis; $y = 0$, the x –axis and $y = x$, a line passing through the origin.

The region is $OBCDO$

If we change the order of integration, the region is split into two parts. OAB and $ABCD$.



Now, consider a strip in the region OAB, parallel to the y –axis.

On this strip y –varies from $y = 0$ to $y = x$.

Then x varies from $x = 0$ to $x = a$

Also consider a strip in the region $ABCD$, parallel to the y –axis.

On this strip, y varies from $y = 0$ to $y = a^2/x$.

Then x varies from $x = a$ to $x = 2a$

$$\therefore I = \int_0^a \int_0^x x^2\, dydx + \int_0^{2a} \int_0^{a^2/x} x^2\, dydx$$

$$= \int_0^a x^2 [y]_0^x\, dx + \int_a^{2a} x^2 [y]_0^{a^2/x}\, dx$$

$$= \int_0^a x^3\, dx + \int_a^{2a} a^2 x\, dx$$

$$= \left[\frac{x^4}{4}\right]_0^a + \left[\frac{a^2x^2}{2}\right]_a^{2a}$$

$$= \frac{a^4}{4} + 2a^4 - \frac{a^4}{2} = \frac{7a^4}{4}$$