Practice Problems

Type – 1

1. **(i)** If A and B are Hermitian matrices then prove that $(A+B)$ is also Hermitian matrix.

(ii) If A and B are skew – Hermitian matrices then prove that $(A+B)$ is skew – Hermitian matrix.

2. If A is any square matrix, then show that $A+A^{\theta}$ is Hermitian and $A-A^{\theta}$ is skew – Hermitian

3. If A is any matrix, then show that AA^{θ} and $A^{\theta}A$ are Hermitian matrices.

4. Show that the matrix $B^{\theta}AB$ is Hermitian or skew – Hermitian accordingly when A is Hermitian or skew – Hermitian matrix.

5. Prove that $\overline{A}$ is Hermitian or skew – Hermitian accordingly when A is Hermitian or skew – Hermitian

6. Show that every square matrix can be uniquely expressed as sum of Hermitian and skew Hermitian matrix.

7. Show that every square matrix can be uniquely expressed as P + iQ, where P and Q both are Hermitian matrices.

8. Show that every Hermitian matrix can be uniquely expressed as P + iQ, where P is real symmetric and Q is real skew symmetric matrix.

9. Show that every skew Hermitian matrix can be uniquely expressed as P + iQ, where P is real skew symmetric and Q is real symmetric matrix.

10. Express the following matrices as the sum of Hermitian and skew – Hermitian matrices

(i) $\begin{bmatrix} 2+i & -i & 3+i \\ 1+i & 3 & 6-2i \\ 3-2i & 6i & 4-3i \end{bmatrix}$ **(ii)** $\begin{bmatrix} 2 & 4+i & 4i \\ 3i & 6-i & 2 \\ 6 & 4-2i & 1-i \end{bmatrix}$ **(iii)** $\begin{bmatrix} 1+i & 2-3i & 2 \\ 3-4i & 4+5i & 1 \\ 5 & 3 & 3-i \end{bmatrix}$

11. Express following matrices as P + iQ, where P and Q both are Hermitian matrices.

(i) $A=\begin{bmatrix} 2 & 3-i & 1-i \\ 2-i & 3+i & 2+i \\ 1+i & 0 & -3i \end{bmatrix}$ **(ii)** $A=\begin{bmatrix} 1+2i & 2 & 3-i \\ 2+3i & 2i & 1-2i \\ 1+i & 0 & 3+2i \end{bmatrix}$

12. Express the following Hermitian matrices as $B+iC$ where B is real symmetric and C is real skew symmetric.

(i) $\begin{bmatrix} 4 & 3-2i & -1+i \\ 3+2i & 2 & 5+4i \\ -1-i & 5-4i & 7 \end{bmatrix}$ **(ii)** $A=\begin{bmatrix} 3 & 2-i & 1+2i \\ 2+i & 2 & 3-2i \\ 1-2i & 3+2i & 0 \end{bmatrix}$ **(iii)** $A=\begin{bmatrix} 1 & 2+i & -1+i \\ 2-i & 1 & 2i \\ -1-i & -2i & 0 \end{bmatrix}$

13. Express the following skew – Hermitian matrices as $P+iQ$ where P is real skew – symmetric and Q is real symmetric.

(i) $\begin{bmatrix} 2i & 3+i & 2-i \\ -3+i & 0 & 6i \\ -2-i & 6i & -2i \end{bmatrix}$ **(ii)** $A=\begin{bmatrix} 3i & -1+i & 3-2i \\ 1+i & -i & 1+2i \\ -3-2i & -1+2i & 0 \end{bmatrix}$

Type – II

1. Verify that the matrix A is orthogonal, where $A = \frac{1}{3}\begin{bmatrix} 1 & 2 & -2 \\ 2 & 1 & 2 \\ 2 & -2 & -1 \end{bmatrix}$ and hence find A^{-1}

2. Show that following matrices are orthogonal and hence find A^{-1}

(i) $\begin{bmatrix} cos\theta & 0 & sin\theta \\ 0 & 1 & 0 \\ -sin\theta & 0 & cos\theta \end{bmatrix}$ **(ii)** $\frac{1}{\sqrt{6}}\begin{bmatrix} 0 & \sqrt{3} & \sqrt{3} \\ 2 & 1 & -1 \\ \sqrt{2} & -\sqrt{2} & \sqrt{2} \end{bmatrix}$ **(ii)** $\begin{bmatrix} cos\emptyset & 0 & sin\emptyset \\ sin\theta.sin\emptyset & cos\theta & -sin\theta.cos\emptyset \\ -cos\theta.sin\emptyset & sin\theta & cos\theta.cos\emptyset \end{bmatrix}$

3. Determine the values of $\propto, \beta, \gamma$ when the matrix given by $A = \begin{bmatrix} \alpha & \beta & -\gamma \\ \alpha & -2\beta & 0 \\ \alpha & \beta & \gamma \end{bmatrix}$ is orthogonal

4. Determine the values of a, b, c when the matrix $\frac{1}{9}\begin{bmatrix} a & 1 & b \\ c & b & 7 \\ 1 & a & c \end{bmatrix}$ is orthogonal

5. If $A = \begin{bmatrix} 1/3 & 2/3 & a \\ 2/3 & 1/3 & b \\ 2/3 & -2/3 & c \end{bmatrix}$ is orthogonal then find a, b, c. Also find A^{-1}. State the rank of A^2

6. Is the following matrix orthogonal? If not, can it be converted into an orthogonal matrix? If yes how?

A = $\begin{bmatrix} 2 & 2 & 1 \\ -2 & 1 & 2 \\ 1 & -2 & 2 \end{bmatrix}$

7. If $(l_r, m_r, n_r), r = 1,2,3$ are the direction cosines of three mutually perpendicular lines referred to an orthogonal Cartesian coordinate system, then prove that the matrix $\begin{bmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{bmatrix}$ is an orthogonal matrix.

8. If A = $\begin{bmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{bmatrix}$ is orthogonal find the relations between $(l_r, m_r, n_r), r = 1,2,3$

9. Prove that the following matrices are unitary and hence find A^{-1}.

(i) $\begin{bmatrix} \frac{2+i}{3} & \frac{2i}{3} \\ \frac{2i}{3} & \frac{2-i}{3} \end{bmatrix}$ **(ii)** $\begin{bmatrix} \frac{1+i}{2} & \frac{i}{\sqrt{3}} & \frac{3+i}{2\sqrt{15}} \\ \frac{-1}{2} & \frac{1}{\sqrt{3}} & \frac{4+3i}{2\sqrt{15}} \\ \frac{1}{2} & \frac{-i}{\sqrt{3}} & \frac{5i}{2\sqrt{15}} \end{bmatrix}$

10. If N = $\begin{bmatrix} 0 & 1+2i \\ -1+2i & 0 \end{bmatrix}$, then show that $(I-N)(I+N)^{-1}$ is unitary.

11. Show that if A is Hermitian and P is unitary, then $P^{-1}AP$ is Hermitian.

ANSWERS

1. $\frac{1}{3}\begin{bmatrix} 1 & 2 & -2 \\ 2 & 1 & 2 \\ 2 & -2 & -1 \end{bmatrix}$

3. $\alpha = \pm\frac{1}{\sqrt{3}}, \beta = \pm\frac{1}{\sqrt{6}}$ $and\ \gamma = \pm\frac{1}{\sqrt{2}}$

4. $a = \pm 8,\ b = \pm 4,\ c = \pm 4$

5. $a = \pm\frac{2}{3}$, $b = \pm\frac{2}{3}$, $c = \pm\frac{1}{3}$ $A^{-1} = \frac{1}{3}\begin{bmatrix} 1 & 2 & \pm 2 \\ 2 & 1 & \pm 2 \\ 2 & -2 & \pm 1 \end{bmatrix}$, rank of $A^2 = 3$

6. $\begin{bmatrix} 2/3 & 2/3 & 1/3 \\ -2/3 & 1/3 & 2/3 \\ 1/3 & -2/3 & 2/3 \end{bmatrix}$

8. $l_1^2 + m_1^2 + n_1^2 = 1, l_1 l_2 + m_1 m_2 + n_1 n_2 = 0$ etc

Type – III

1. Find the ranks of the following matrices

(i) $\begin{bmatrix} 1 & 2 & -2 & 3 \\ -1 & -3 & 2 & -2 \\ 0 & -1 & 0 & 1 \\ -1 & -4 & 2 & -1 \end{bmatrix}$ **(ii)** $\begin{bmatrix} 3 & 4 & 5 & 6 & 7 \\ 4 & 5 & 6 & 7 & 8 \\ 5 & 6 & 7 & 8 & 9 \\ 10 & 11 & 12 & 13 & 14 \\ 15 & 16 & 17 & 18 & 19 \end{bmatrix}$ **(iii)** $\begin{bmatrix} 2 & -3 & 5 \\ 4 & -6 & 10 \\ -8 & 12 & -20 \\ 6 & -9 & 15 \end{bmatrix}$

(iv) $\begin{bmatrix} 1 & 2 & 3 & 1 \\ 2 & 4 & 3 & 2 \\ 3 & 2 & 1 & 3 \\ 6 & 8 & 7 & 6 \end{bmatrix}$ **(v)** $\begin{bmatrix} -1 & 2 & 3 & -2 \\ 2 & -5 & 1 & 2 \\ 3 & -8 & 5 & 2 \\ -5 & -12 & -1 & 6 \end{bmatrix}$ **(vi)** $\begin{bmatrix} 2 & 3 & 1 & 4 \\ 5 & 2 & 3 & 0 \\ 9 & 8 & 0 & 8 \end{bmatrix}$

(vii) $\begin{bmatrix} 25 & 31 & 17 & 43 \\ 75 & 94 & 53 & 132 \\ 75 & 94 & 54 & 134 \\ 25 & 32 & 20 & 48 \end{bmatrix}$

2. Reduce the following matrices to their normal form and hence obtain their ranks.

(i) $\begin{bmatrix} 1 & 2 & -1 & 3 \\ 3 & 4 & 0 & -1 \\ -1 & 0 & -2 & 7 \end{bmatrix}$ **(ii)** $\begin{bmatrix} 3 & -3 & 0 & -1 & -7 \\ 0 & 2 & 2 & 1 & -5 \\ 1 & -2 & -3 & -2 & 1 \\ 0 & 1 & 2 & 1 & -6 \end{bmatrix}$ **(iii)** $\begin{bmatrix} 8 & 1 & 3 & 6 \\ 0 & 3 & 2 & 2 \\ -8 & 1 & -3 & 4 \end{bmatrix}$

(iv) $\begin{bmatrix} -2 & -1 & -3 & -1 \\ 1 & 2 & 3 & -1 \\ 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \end{bmatrix}$ **(v)** $\begin{bmatrix} 0 & 1 & 2 & 3 & 4 \\ 1 & 1 & 2 & 3 & 3 \\ 2 & 2 & 0 & 2 & 2 \\ 3 & 3 & 2 & 1 & 1 \\ 4 & 3 & 2 & 1 & 0 \end{bmatrix}$ **(vi)** $\begin{bmatrix} 6 & 1 & 3 & 8 \\ 4 & 2 & 6 & -1 \\ 10 & 3 & 9 & 7 \\ 16 & 4 & 12 & 15 \end{bmatrix}$

(vii) $\begin{bmatrix} 2 & -4 & 3 & 1 & 0 \\ 1 & -2 & 1 & -4 & 2 \\ 0 & 1 & -1 & 3 & 1 \\ 4 & -7 & 4 & -4 & 5 \end{bmatrix}$ **(viii)** $\begin{bmatrix} 3 & 2 & 5 & 7 & 12 \\ 1 & 1 & 2 & 3 & 5 \\ 3 & 3 & 6 & 9 & 15 \end{bmatrix}$ **(ix)** $\begin{bmatrix} 1 & 2 & -2 & 3 & 1 \\ 1 & 3 & -2 & 3 & 0 \\ 2 & 4 & -3 & 6 & 4 \\ 1 & 1 & -1 & 4 & 6 \end{bmatrix}$

(x) $\begin{bmatrix} 1 & 3 & 5 & 7 \\ 4 & 6 & 8 & 10 \\ 15 & 27 & 39 & 51 \\ 6 & 12 & 18 & 24 \end{bmatrix}$ **(xi)** $\begin{bmatrix} 1 & 1 & 0 \\ 2 & 2 & 0 \\ 0 & 1 & 1 \end{bmatrix}$ **(xii)** $\begin{bmatrix} 2 & 1 & -3 & -6 \\ 3 & -3 & 1 & 2 \\ 1 & 1 & 1 & 2 \end{bmatrix}$

(xiii) $\begin{bmatrix} 2 & 15 & 14 & 15 \\ 6 & 24 & 18 & 30 \\ 1 & 4 & 2 & 5 \end{bmatrix}$ **(xiv)** $\begin{bmatrix} 1 & 0 & 2 & -2 \\ 2 & -1 & 0 & -1 \\ 1 & 0 & 2 & -1 \\ 4 & -1 & 3 & -1 \end{bmatrix}$

3. Find the rank of A by reducing it to the normal form, where $A = \begin{bmatrix} 2 & 1 & 3 & 4 \\ 3 & -1 & 2 & 2 \\ 4 & 1 & 0 & -1 \\ 9 & 1 & 5 & 6 \end{bmatrix}$

Hence find the rank of A^2

4. Reduce the following matrices to Echelon Forms and hence find the ranks.

(i) $\begin{bmatrix} 0 & 1 & -3 & -1 \\ 1 & 0 & 1 & 1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & -2 & 0 \end{bmatrix}$ **(ii)** $\begin{bmatrix} 1 & 2 & -1 & 3 \\ 3 & 4 & 0 & -1 \\ -1 & 0 & -2 & 7 \end{bmatrix}$

5. Find the values of P for which the matrix $A = \begin{bmatrix} P & 2 & 2 \\ 2 & P & 2 \\ 2 & 2 & P \end{bmatrix}$ will have **(i)** rank 1, **(ii)** rank 2, **(iii)** rank 3,

6. The rank of the matrix $\begin{bmatrix} \lambda & -1 & 0 \\ 0 & \lambda & -1 \\ -1 & 0 & \lambda \end{bmatrix}$ is 2. Find the value of λ, where λ is real.

7. Find the rank of A = $\begin{bmatrix} x-1 & x+1 & x \\ -1 & x & 0 \\ 0 & 1 & 1 \end{bmatrix}$ where x is real.

9. Find non – singular matrices P and Q such that PAQ is in normal form, hence obtain rank of A where A is

(i) $\begin{bmatrix} 1 & 2 & -1 & 2 \\ 2 & 5 & -2 & 3 \\ 1 & 2 & 1 & 2 \end{bmatrix}$ **(ii)** $\begin{bmatrix} 1 & -1 & 2 & -1 \\ 4 & 2 & -1 & 2 \\ 2 & 2 & -2 & 0 \end{bmatrix}$ **(iii)** $\begin{bmatrix} 3 & -1 & 2 & 1 \\ 1 & 4 & 6 & 1 \\ 7 & -11 & -6 & 1 \\ 7 & 2 & 12 & 3 \end{bmatrix}$

(iv) $\begin{bmatrix} 2 & 1 & 4 \\ 3 & 2 & 2 \\ 7 & 4 & 10 \\ 8 & 5 & 8 \end{bmatrix}$ **(v)** $\begin{bmatrix} 4 & 3 & 1 & 6 \\ 2 & 4 & 2 & 2 \\ 12 & 14 & 5 & 16 \end{bmatrix}$ **(vi)** $\begin{bmatrix} 2 & 1 & -3 & -6 \\ 3 & -3 & 1 & 2 \\ 1 & 1 & 1 & 2 \end{bmatrix}$

(vii) $\begin{bmatrix} 1 & 3 & 5 & 7 \\ 4 & 6 & 8 & 10 \\ 15 & 27 & 39 & 51 \\ 6 & 12 & 18 & 24 \end{bmatrix}$ **(viii)** $\begin{bmatrix} 2 & 1 & 4 & 3 \\ 2 & 3 & 6 & 4 \\ 6 & 5 & 15 & 10 \end{bmatrix}$ **(ix)** $\begin{bmatrix} 0 & 1 & -3 & -1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & -2 & 0 \end{bmatrix}$

(x) $\begin{bmatrix} 2 & 1 & 4 & 3 \\ 1 & 0 & 2 & 2 \\ 4 & 1 & 9 & 7 \end{bmatrix}$ **(xi)** $\begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 2 & 4 & 3 \\ 5 & 6 & 10 & 2 \end{bmatrix}$ **(xii)** $\begin{bmatrix} 1 & 3 & 6 & -1 \\ 1 & 4 & 5 & 1 \\ 1 & 5 & 4 & 3 \end{bmatrix}$

10. Find non – singular matrices P and Q such that PAQ is in normal form. Hence find **(i)** rank of A, **(ii)** A^{-1} , where A is $\begin{bmatrix} 1 & 2 & 1 & 1 \\ 0 & 1 & 0 & -1 \\ 2 & 1 & 3 & 3 \\ 1 & 1 & 1 & 1 \end{bmatrix}$

11. If A = $[a_{ij}]$ is a square matrix of order 3 where $a_{ij} = i + j$, find the rank of A

12. Find the rank of A = $[a_{ij}]_{3\times 3}$ where $a_{ij} = \frac{i}{j}$

ANSWERS

1. **(i)** 2 **(ii)** 2 **(iii)** 1 **(iv)** 3 **(v)** 2 **(vi)** 3 **(vii)** 3

2. **(i)** 2 **(ii)** 4 **(iii)** 3 **(iv)** 2 **(v)** 4 **(vi)** 2 **(vii)** 3
(viii) 2 **(ix)** 4 **(x)** 2 **(xi)** **2** **(xii)** **3** **(xiii)** **3** **(xiv)** 2

3. $\rho(A) = 4 \quad \rho(A^2) = 4$ 4. **(i)** 2 **(ii)** 2

5. rank of $A = 1$ for $P = 2$, rank of $A = 2$ for $P = -4$,
rank of $A = 3$ for any value of P other than 2 and -4.

6. $\lambda = 1$ **7.** 3 8**.** $r = 3\ if\ x \neq 1$

9. **(i)** 3 **(ii)** 3 **(iii)** 3 **(iv)** 2 **(v)** 3 **(vi)** 3 **(vii)** 2
(viii) **3** **(ix)** **2** **(x)** **3** **(xi)** **3** **(xii)** 2

10. $\begin{bmatrix} -2 & 0 & -1 & 5 \\ 1 & 0 & 0 & -1 \\ 0 & 1 & 1 & -2 \\ 1 & -1 & 0 & -1 \end{bmatrix}$ **11.** 2 **12.** 1