COMPOSITE FUNCTIONS

- **(a)** Let $z = f(x,y)$ and $x = \Phi(t), y = \Psi(\mathrm{t})$ so that z is function of x , y and x , y are function of third variable t.
- The three relations define z as a function of t. In such cases z is called a **composite function of t**.
- **e.g. (i)** $z = x^2 + y^2 , x = at^2 , y = 2at$
- **(ii)** $z = \mathrm{x}^2\mathrm{y} + \mathrm{xy}^2 \ , \ x = acost \ , \ y = bsint$

  define z as a composite function of t
- **Differentiation:** Let $z = f(x,y)$ posses continuous first order partial derivatives and $x = \Phi(t), y = \Psi(t)$ posses continuous first order derivatives then,

$$\frac{dz}{dt} = \frac{\partial z}{\partial x} . \frac{dx}{dt} + \frac{\partial z}{\partial y} . \frac{dy}{dt}$$

EXAMPLE-11

- If $u = x^2y^3, x = \log t, y = e^t$, find $\frac{du}{dt}$
- **Solution:** $u = x^2y^3, x = \log t, y = e^t$
- $$\frac{du}{dt} = \frac{\partial u}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dt}$$
- $$= (2xy^3)\frac{1}{t} + (3x^2y^2)e^t$$
- Substituting x and y,
- $$\frac{du}{dt} = 2(\log t)e^{3t} \cdot \frac{1}{t} + 3(\log t)^2e^{2t} \cdot e^t$$
- $$= \frac{2}{t}\log t\, e^{3t} + 3(\log t)^2e^{3t}$$

EXAMPLE-12

- If $u = xy + yz + zx$ where $x = \frac{1}{t}, y = e^t, z = e^{-t}$, find $\frac{du}{dt}$
- **Solution:** $u = xy + yz + zx, x = \frac{1}{t}, y = e^t, z = e^{-t}$
- $\frac{du}{dt} = \frac{\partial u}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial u}{\partial y} \cdot \frac{dy}{dt} + \frac{\partial u}{\partial z} \cdot \frac{dz}{dt}$
- $= (y + z)\left(-\frac{1}{t^2}\right) + (x + z)e^t + (y + x)(-e^{-t})$
- Substituting x, y and z,
- $\frac{du}{dt} = -\frac{1}{t^2}(e^t + e^{-t}) + \left(\frac{1}{t} + e^{-t}\right)e^t - \left(e^t + \frac{1}{t}\right)e^{-t}$
- $= \frac{1}{t^2}(e^t + e^{-t}) + \frac{1}{t}(e^t - e^{-t})$

EXAMPLE-13

- If $z = e^{xy}, x = t\cos t\,, y = t\sin t\,$, find $\frac{dz}{dt}$ at $t = \frac{\pi}{2}$
- **Solution:** $z = e^{xy}, x = t\cos t\,, y = t\sin t$

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial z}{\partial y}\cdot\frac{dy}{dt}$$

$$= e^{xy}y(\cos t - t\sin t) + e^{xy}x(\sin t + t\cos t)$$

- At $t = \frac{\pi}{2}, x = 0, y = \frac{\pi}{2}$
- Hence, $\left.\frac{dz}{dt}\right|_{t=\frac{\pi}{2}} = e^0\left[\frac{\pi}{2}\left(0 - \frac{\pi}{2}\right) + 0\right] = -\frac{\pi^2}{4}$`

COMPOSITE FUNCTIONS

- **(b)** Let $z = f(x, y)$ and $x = \Phi(u, v)$, $y = \Psi(\mathrm{u}, \mathrm{v})$ so that z is function of x , y and x , y are function of u , v.
- The three relations define z as a function of u , v. In such cases z is called a **composite function of $\boldsymbol{u}$, $\boldsymbol{v}$.**
- **e.g. (i)** $z = xy$, $x = e^{u} + e^{-v}$, $y = e^{-u} + e^{v}$
- **(ii)** $z = \mathrm{x}^2 - \mathrm{y}^2$, $x = 2u - 3v$, $y = 3u + 2v$
- define z as a composite function of u and v

- **Differentiation:** Let $z = f(x, y)$ possess continuous first order partial derivatives and $x = \Phi(u, v), y = \Psi(u, v)$ possess continuous first order partial derivatives then,
- $\dfrac{\partial z}{\partial u} = \dfrac{\partial z}{\partial x} . \dfrac{\partial x}{\partial u} + \dfrac{\partial z}{\partial y} . \dfrac{\partial y}{\partial u}$ and $\dfrac{\partial z}{\partial v} = \dfrac{\partial z}{\partial x} . \dfrac{\partial x}{\partial v} + \dfrac{\partial z}{\partial y} . \dfrac{\partial y}{\partial v}$

EXAMPLE-14

If $x^2 = au + bv,\ y^2 = au - bv\ and\ z = f(x,y)$, Prove that

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 2\left(u\frac{\partial z}{\partial u} + v\frac{\partial z}{\partial v}\right).$$

- **Solution:** $z = f(x,y),\ \ x^2 = au + bv,\ y^2 = au - bv$
- $\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u}$
- $= \frac{\partial z}{\partial x} \cdot \frac{a}{2x} + \frac{\partial z}{\partial y} \cdot \frac{a}{2y}$
- $u\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x} \cdot \frac{au}{2x} + \frac{\partial z}{\partial y} \cdot \frac{au}{2y}$ .... (1)
- $\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v}$
- $= \frac{\partial z}{\partial x} \cdot \frac{b}{2x} + \frac{\partial z}{\partial y}\left(-\frac{b}{2y}\right)$
- $v\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x} \cdot \frac{bv}{2x} - \frac{\partial z}{\partial y} \cdot \frac{bv}{2y}$ .... (2)

EXAMPLE-14

- Adding Eqs. (1) and (2),
- $u\dfrac{\partial z}{\partial u}+v\dfrac{\partial z}{\partial v}$
- $=\dfrac{\partial z}{\partial x}\cdot\dfrac{au}{2x}+\dfrac{\partial z}{\partial y}\cdot\dfrac{au}{2y}+\dfrac{\partial z}{\partial x}\cdot\dfrac{bv}{2x}-\dfrac{\partial z}{\partial y}\cdot\dfrac{bv}{2y}$
- $=\dfrac{\partial z}{\partial x}\left(\dfrac{au+bv}{2x}\right)+\dfrac{\partial z}{\partial y}\left(\dfrac{au-bv}{2y}\right)$
- $=\dfrac{\partial z}{\partial x}\left(\dfrac{x^2}{2x}\right)+\dfrac{\partial z}{\partial y}\left(\dfrac{y^2}{2y}\right)$
- $=\dfrac{1}{2}\left(x\dfrac{\partial z}{\partial x}+y\dfrac{\partial z}{\partial y}\right)$

EXAMPLE-15

- If $u = \log(x^2 + y^2)\,, v = \frac{y}{x}$, prove that $x\frac{\partial z}{\partial y} - y\frac{\partial z}{\partial x} = (1 + v^2)\frac{\partial z}{\partial v}$
- **Solution:** $z = f(u, v), u = \log(x^2 + y^2)\,, v = \frac{y}{x}$,
- $\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial z}{\partial v} \cdot \frac{\partial v}{\partial x}$
- $= \frac{\partial z}{\partial u} \cdot \frac{1}{x^2+y^2} \cdot 2x + \frac{\partial z}{\partial v}\left(-\frac{y}{x^2}\right)$
- $y\frac{\partial z}{\partial x} = \frac{2xy}{x^2+y^2} \cdot \frac{\partial z}{\partial u} - \frac{y^2}{x^2} \cdot \frac{\partial z}{\partial v}$ .... (1)
- $\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial y} + \frac{\partial z}{\partial v} \cdot \frac{\partial v}{\partial y}$
- $= \frac{\partial z}{\partial u} \cdot \frac{2y}{x^2+y^2} + \frac{\partial z}{\partial v} \cdot \frac{1}{x}$
- $x\frac{\partial z}{\partial y} = \frac{2xy}{x^2+y^2} \cdot \frac{\partial z}{\partial u} + \frac{\partial z}{\partial v}$ .... (2)
- Subtracting Eq. (1) from Eq. (2),
- Hence, $x\frac{\partial z}{\partial y} - y\frac{\partial z}{\partial x} = \frac{\partial z}{\partial v} + \frac{y^2}{x^2}\frac{\partial z}{\partial v} = (1 + v^2)\frac{\partial z}{\partial v}$

EXAMPLE-16

If $u = f(x^2 - y^2, y^2 - z^2, z^2 - x^2)$, prove that $\frac{1}{x}\frac{\partial u}{\partial x} + \frac{1}{y}\frac{\partial u}{\partial y} + \frac{1}{z}\frac{\partial u}{\partial z} = 0$.

- **Solution:** Let $l = x^2 - y^2, m = y^2 - z^2, n = z^2 - x^2$
- $\frac{\partial l}{\partial x} = 2x, \quad \frac{\partial m}{\partial x} = 0, \quad \frac{\partial n}{\partial x} = -2x$
- $\frac{\partial l}{\partial y} = -2y, \quad \frac{\partial m}{\partial y} = 2y, \quad \frac{\partial n}{\partial y} = 0$
- $\frac{\partial l}{\partial z} = 0, \quad \frac{\partial m}{\partial z} = -2z, \quad \frac{\partial n}{\partial z} = 2z$
- $u = f(x^2 - y^2, y^2 - z^2, z^2 - x^2) = f(l, m, n)$
- $\frac{\partial u}{\partial x} = \frac{\partial u}{\partial l} \cdot \frac{\partial l}{\partial x} + \frac{\partial u}{\partial m} \cdot \frac{\partial m}{\partial x} + \frac{\partial u}{\partial n} \cdot \frac{\partial n}{\partial x}$
- $= \frac{\partial u}{\partial l} \cdot 2x + \frac{\partial u}{\partial m} \cdot 0 + \frac{\partial u}{\partial n} \cdot (-2x)$
- $\frac{1}{x}\frac{\partial u}{\partial x} = 2\frac{\partial u}{\partial l} - 2\frac{\partial u}{\partial n}$ .... (1)

EXAMPLE-16

$$\frac{\partial u}{\partial y}=\frac{\partial u}{\partial l}\cdot\frac{\partial l}{\partial y}+\frac{\partial u}{\partial m}\cdot\frac{\partial m}{\partial y}+\frac{\partial u}{\partial n}\cdot\frac{\partial n}{\partial y}$$

$$=\frac{\partial u}{\partial l}(-2y)+\frac{\partial u}{\partial m}(2y)+\frac{\partial u}{\partial n}(0)$$

$$\frac{1}{y}\frac{\partial u}{\partial y}=-2\frac{\partial u}{\partial l}+2\frac{\partial u}{\partial m} \qquad \text{..........}\ (2)$$

$$\frac{\partial u}{\partial z}=\frac{\partial u}{\partial l}\cdot\frac{\partial l}{\partial z}+\frac{\partial u}{\partial m}\cdot\frac{\partial m}{\partial z}+\frac{\partial u}{\partial n}\cdot\frac{\partial n}{\partial z}$$

$$=\frac{\partial u}{\partial l}\cdot 0+\frac{\partial u}{\partial m}(-2z)+\frac{\partial u}{\partial n}(2z)$$

$$\frac{1}{z}\frac{\partial u}{\partial z}=-2\frac{\partial u}{\partial m}+2\frac{\partial u}{\partial n} \qquad \text{..........}\ (3)$$

Adding Eqs (1), (2) and (3),

$$\frac{1}{x}\frac{\partial u}{\partial x}+\frac{1}{y}\frac{\partial u}{\partial y}+\frac{1}{z}\frac{\partial u}{\partial z}=0$$

EXAMPLE-17

- If $x = e^u cosec\, v, y = e^u \cot v$ and z is a function of x and y, prove that
- $$\left(\frac{\partial z}{\partial x}\right)^2 - \left(\frac{\partial z}{\partial y}\right)^2 = e^{-2u}\left[\left(\frac{\partial z}{\partial u}\right)^2 - sin^2 v\left(\frac{\partial z}{\partial v}\right)^2\right]$$
- **Solution:** $z = f(x, y), x = e^u cosec\, v, y = e^u \cot v$
- $$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial u}$$
- $$= \frac{\partial z}{\partial x} e^u cosec\, v + \frac{\partial z}{\partial y} e^u \cot v$$
- $$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial v}$$
- $$= \frac{\partial z}{\partial x}(-e^u cosec\, v \cot v) + \frac{\partial z}{\partial y}(-e^u cosec\, v)$$

EXAMPLE-17

- R.H.S = $e^{-2u}\left[\left(\frac{\partial z}{\partial u}\right)^2 - \sin^2 v\left(\frac{\partial z}{\partial v}\right)^2\right]$
- $=$
  $e^{-2u}\left[\left(\frac{\partial z}{\partial x}\right)^2 e^{u}cosec^2\, v + \left(\frac{\partial z}{\partial y}\right)^2 e^{2u}\cot^2 v + 2\frac{\partial z}{\partial x}\cdot\frac{\partial z}{\partial y}e^{2u}cosec\, v\cot v\right.$
- $+(-\sin^2 v)\left(\frac{\partial z}{\partial x}\right)^2(e^{2u}cosec^2\, v\cot^2 v) +$
- $\left.(-\sin^2 v)\left(\frac{\partial z}{\partial y}\right)^2 e^{2u}\operatorname{cosec}^4 v + (-\sin^2 v)2\frac{\partial z}{\partial x}\cdot\frac{\partial z}{\partial y}e^{2u}cosec^3\, v\cot v\right]$

- $= \left(\frac{\partial z}{\partial x}\right)^2(cosec^2 - \cot^2 v) + \left(\frac{\partial z}{\partial y}\right)^2(\cot^2 v - cosec^2\, v)$
- $= \left(\frac{\partial z}{\partial x}\right)^2 - \left(\frac{\partial z}{\partial y}\right)^2$ = L.H.S.

EXAMPLE-18

- If $x+y=2e^{\theta}\cos\Phi$, $x-y=2\,i\,e^{\theta}\sin\Phi$, show that $\frac{\partial^2 u}{\partial\theta^2}+\frac{\partial^2 u}{\partial\Phi^2}=4xy\frac{\partial^2 u}{\partial x\partial y}$
- **Solution:** Adding the given results, $2x=2e^{\theta}(\cos\Phi+i\sin\Phi)$
- $\therefore x=e^{\theta}\cdot e^{i\Phi}=e^{\theta+i\Phi}$
- and subtracting results, $2y=2e^{\theta}(\cos\Phi-i\sin\Phi)$
- $\therefore y=e^{\theta-i\Phi}$
- Now, u is a function of x, y and x, y are functions of θ and Φ
- $\therefore \frac{\partial u}{\partial\theta}=\frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial\theta}+\frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial\theta}$
- $=\frac{\partial u}{\partial x}\cdot e^{\theta+i\Phi}+\frac{\partial u}{\partial y}\cdot e^{\theta-i\Phi}=\frac{\partial u}{\partial x}\cdot x+\frac{\partial u}{\partial y}\cdot y$ ………… (1)
- $\therefore \frac{\partial}{\partial\theta}\equiv x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}$ ………… (2)
- $\therefore \frac{\partial^2 u}{\partial\theta^2}=\frac{\partial}{\partial\theta}\left(\frac{\partial u}{\partial\theta}\right)=\frac{\partial}{\partial\theta}\left(x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}\right)$ ……………… [From (1)]
- $=\left(x\frac{\partial}{\partial x}+y\frac{\partial}{\partial y}\right)\left(x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}\right)$ ……………… [From (2)]
- $=x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial x^2}+x\frac{\partial u}{\partial y}+y\frac{\partial u}{\partial y}$ ………… (3)

EXAMPLE-18

- Also, $\frac{\partial u}{\partial \Phi} = \frac{\partial u}{\partial x} \cdot \frac{\partial x}{\partial \Phi} + \frac{\partial u}{\partial y} \cdot \frac{\partial y}{\partial \Phi}$
- $= \frac{\partial u}{\partial x} \cdot e^{\theta + i\Phi} \cdot i + \frac{\partial u}{\partial y} \cdot e^{\theta - i\Phi} \cdot (-i)$
- $= \frac{\partial u}{\partial x} \cdot ix - i \frac{\partial u}{\partial y} \cdot y$ ………… (4)
- $\therefore \frac{\partial}{\partial \Phi} \equiv i \left(x \frac{\partial}{\partial x} - y \frac{\partial}{\partial y} \right)$ ………… (5)
- $\therefore \frac{\partial^2 u}{\partial \Phi^2} = \frac{\partial}{\partial \Phi} \left(\frac{\partial u}{\partial \Phi} \right) = \frac{\partial}{\partial \Phi} \left[i \left(x \frac{\partial u}{\partial x} - y \frac{\partial u}{\partial y} \right) \right]$ ……………… [From (4)]
- $= i \left[i \left(x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} \right) \left(x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} \right) \right]$ ……………… [From (5)]
- $= - \left[x^2 \frac{\partial^2 u}{\partial x^2} - 2xy \frac{\partial^2 u}{\partial x \partial y} + y^2 \frac{\partial^2 u}{\partial y^2} + x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} \right]$
- $= -x^2 \frac{\partial^2 u}{\partial x^2} + 2xy \frac{\partial^2 u}{\partial x \partial y} - y^2 \frac{\partial^2 u}{\partial y^2} - x \frac{\partial u}{\partial x} - y \frac{\partial u}{\partial y}$ ………… (6)
- $\therefore$ Adding the two results, (5) and (6) we get,
- $\frac{\partial^2 u}{\partial \theta^2} + \frac{\partial^2 u}{\partial \Phi^2} = 4xy \frac{\partial^2 u}{\partial x \partial y}$