MATRIX THEORY: RANK OF MATRIX
-
SYSTEM OF LINEAR EQUATIONS

FY BTECH SEM-I

MODULE:2

SUB-MODULE: 2.3

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A SYSTEM OF LINEAR EQUATIONS

- Consider a system of **m linear equations in n unknowns**, say $x_1, x_2, x_3, \dots . x_n$

$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1$

$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2$

$a_{31}x_1 + a_{32}x_2 + \cdots + a_{3n}x_n = b_3$

……………………………………………$a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m$

- This system can be written compactly in matrix notation as $\boldsymbol{AX = B}$

$\boldsymbol{A = [a_{ij}]_{m\,X\,n}}$: matrix of coefficients

$\boldsymbol{B = [b_1\ \ b_2\ \ b_3 \dots b_m]^T}$ is the column vector of order $(m \times 1)$

$\boldsymbol{X = [x_1\ x_2\ x_3 \dots x_n]^T}$ is the column vector of order $(n \times 1)$

- Any vector U satisfying AU = B is said to be a **solution** of AX = B.
- The matrix [A,B] i.e., the matrix formed by the coefficients and the constants is called the **augmented matrix.**
- A system $AX = B$ is

(i) **Homogeneous** if $B = O$ and

(ii) **Non – homogeneous** if $B \neq O$

Different cases for solution

Consistent
(possess at least one solution)

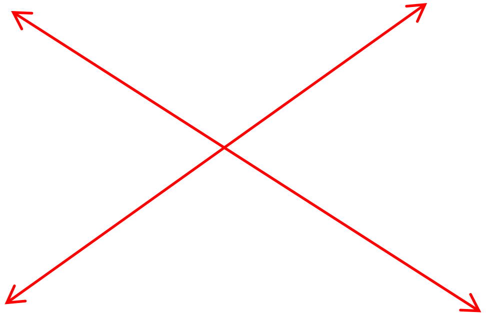
Inconsistent

• (i) Unique Solution:

Ex: $4x_1 + 3x_2 = 11$,

$$4x_1 - 3x_2 = 5$$

- Solving we get, $x_1 = 2$ and $x_2 = 1$

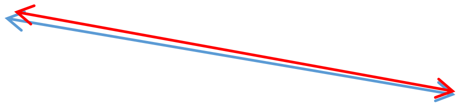


• (ii) Infinite Solutions:

Ex: $4x_1 + 3x_2 = 11$,

$$8x_1 + 6x_2 = 22$$

has more solutions, say $(2,1)^T$ or $(0,11/3)^T$.
$[k, \frac{11-4k}{3}]^T$ is a general solution for all k

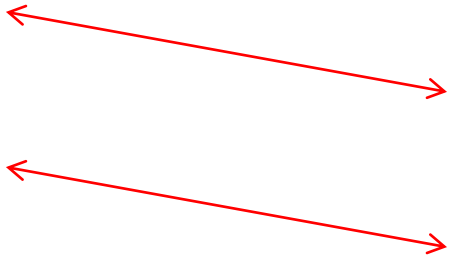


• (iii) No Solution:

Ex: $4x_1 + 3x_2 = 11$,

$$8x_1 + 6x_2 = 20$$

has no solution at all.



NON – HOMOGENEOUS SYSTEM OF LINEAR EQUATIONS

(i) n EQUATIONS IN n UNKNOWN:

Method 1: by finding A^{-1}

Consider $AX = B$, where A is a non – singular $n \times n$ matrix, X is $n \times 1$ vector and B is $n \times 1$ matrix then the system has **unique solution.**

- **Steps:**

1. Write $AX = B$

2. Check that $|A| \neq 0$

3. Find A^{-1} by any suitable method.

4. Solution is given by $\boldsymbol{X = A^{-1}B}$.

- **Note:** If A is singular matrix, then this inverse method fails. In that case the system may have infinitely many solutions or none at all.

NON – HOMOGENEOUS SYSTEM OF LINEAR EQUATIONS

- **(ii) m EQUATIONS IN n UNKNOWN:**
- **Method :2**
- **Case (i) :** $Rank\ A < Rank\ [A, B]$

In this case the equations are **inconsistent** i.e., they have no solution.

- **Case (ii):** $Rank\ A = Rank\ [A, B] = r$

In this case the equations are **consistent** i.e., they possess a solution.

- Further,

(a) If $r = n$ i.e if the rank of A is equal to the number of unknowns, the system has **unique solutions**.

(Also note that the system has unique solution if the coefficient matrix is non – singular).

(b) If $r < n$, if the rank of A is less than the number of unknown the system has **infinite solutions**.

- In this case $n - r$ unknowns called parameters can be assigned arbitrary values.
- The remaining unknowns then can be expressed in terms of these parameters.

NON – HOMOGENEOUS SYSTEM OF LINEAR EQUATIONS

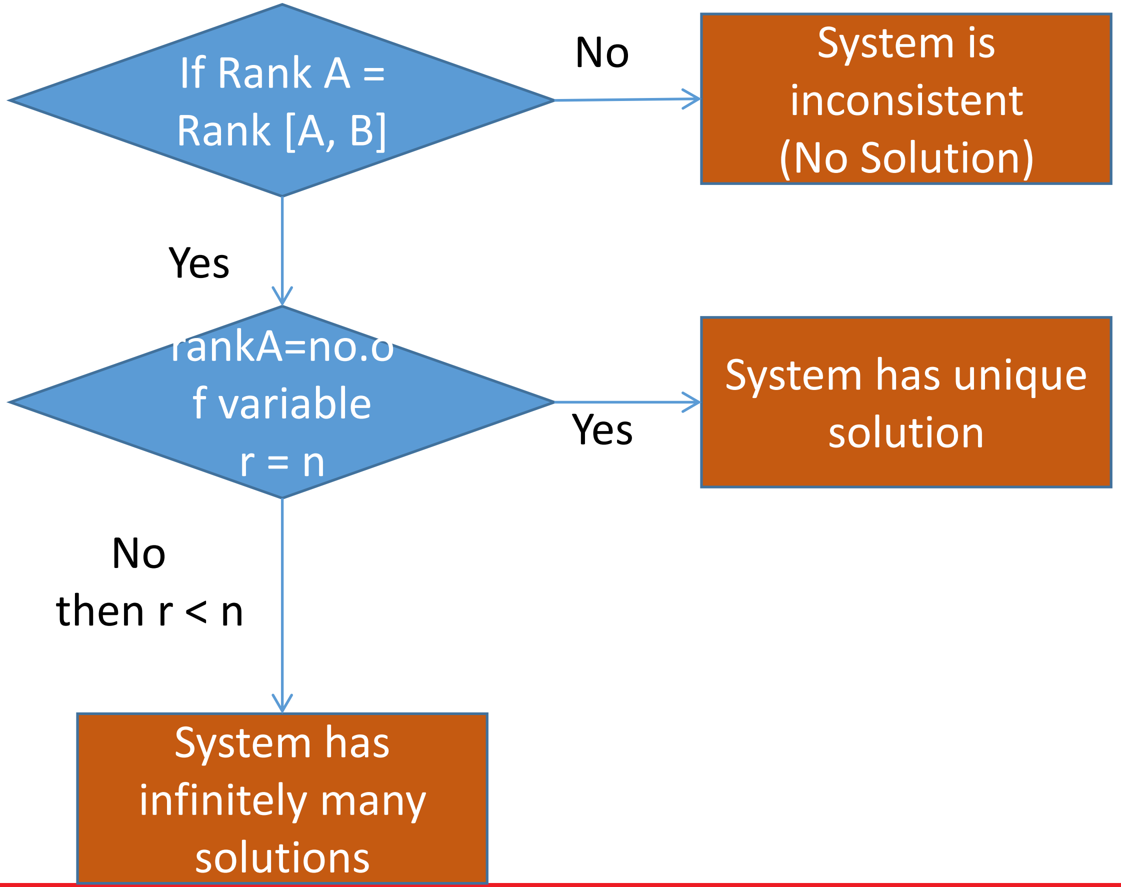
- **Working Rule:**

1. Write the given system in the matrix form $AX = B$.

2. Apply row transformations on A as well as on the column matrix B i.e. on the augmented matrix [A, B] till you get an row echelon form.

3. We know that the rank of a matrix in echelon form is equal to the number of non-zero rows.

Determine the rank of A and the rank of the augmented matrix [A, B] and find the solution by using following cases.



Example 1

- Test the consistency of the equations
- $x + y + z = 3, x + 2y + 3z = 4, x + 4y + 9z = 6.$
- **Solution:** The system of equations can be written as $AX = B$, i.e. $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 4 & 9 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \\ 6 \end{bmatrix}$
- ∴ the augmented matrix can be written as $[A: B] = \begin{bmatrix} 1 & 1 & 1 & 3 \\ 1 & 2 & 3 & 4 \\ 1 & 4 & 9 & 6 \end{bmatrix}$
- Applying $R_2 - R_1$ *and* $R_3 - R_1$ we get $[A: B] \sim \begin{bmatrix} 1 & 1 & 1 & 3 \\ 0 & 1 & 2 & 1 \\ 0 & 3 & 8 & 3 \end{bmatrix}$
- Applying $R_3 - 3R_2$, we get $[A: B] \sim \begin{bmatrix} 1 & 1 & 1 & 3 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 2 & 0 \end{bmatrix}$
- ∴ rank of $[A: B]$ = rank of $A = 3$.
- ∴ The system has unique solution.
- The reduced form of equations can be written as
- $x + y + z = 3, \quad y + 2z = 1, \quad 2z = 0$
- ∴ (iii) ⇒ $z = 0$ substituting this value in (ii) $y = 1$
- ∴ (i) ⇒ $x + 1 + 0 = 3 \Rightarrow x = 2$
- Hence the solution set can be written as $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$

Example 2

- Solve $\begin{matrix} x_1 + 2x_2 + x_3 = 2 \\ 2x_1 + 4x_2 + 3x_3 = 3 \\ 3x_1 + 6x_2 + 5x_3 = 4 \end{matrix}$
- The system of equations can be represented by the matrix equation as, $\begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 3 \\ 3 & 6 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$
- Where the augmented matrix is
- $[A:B] = \begin{bmatrix} 1 & 2 & 1 & 2 \\ 2 & 4 & 3 & 3 \\ 3 & 6 & 5 & 4 \end{bmatrix}$
- Applying elementary row transformations, the matrix $[A:B]$ can be reduced to Echelon form.
- Applying $R_2 - 2R_1$ and $R_3 - 3R_1$ we get

$$[A:B] \sim \begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 2 & -2 \end{bmatrix}$$

- Applying $R_3 - 2R_2$, we get

$$[A:B] \sim \begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

- Hence $\rho(A) = \rho[A:B]$, therefore the system is consistent.
- Further rank $r = 2 < 3$ (number of variables), therefore the system has infinite solutions.
- $\therefore (n - r) = 3 - 2 = 1$ (free variable)
- The reduced form of the linear equations can be written as, $x_1 + 2x_2 + x_3 = 2$, $x_3 = -1$
- Let $x_2 = k$, an arbitrary constant. $\therefore x_1 = 3 - 2k$
- Hence $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 - 2k \\ k \\ -1 \end{bmatrix}$ infinite solutions as k varies.

Example 3

- Are the following equations consistent? Justify.

$$2x + y + z = 4$$
$$x + y + z = 2$$
$$5x + 3y + 3z = 6$$

- **Solution:** The system of linear equations can be written in the matrix form $AX = B$ i.e

$$\begin{bmatrix} 2 & 1 & 1 \\ 1 & 1 & 1 \\ 5 & 3 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \\ 6 \end{bmatrix}$$

- ∴ the augmented matrix can be written as

$$[A:B] = \begin{bmatrix} 2 & 1 & 1 & 4 \\ 1 & 1 & 1 & 2 \\ 5 & 3 & 3 & 6 \end{bmatrix}$$

- Applying R_{12}, we get

$$[A:B] \sim \begin{bmatrix} 1 & 1 & 1 & 2 \\ 2 & 1 & 1 & 4 \\ 5 & 3 & 3 & 6 \end{bmatrix}$$

- Applying $R_2 - 2R_1$ and $R_3 - 5R_1$, we get

$$[A:B] \sim \begin{bmatrix} 1 & 1 & 1 & 2 \\ 0 & -1 & -1 & 0 \\ 0 & -2 & -2 & -4 \end{bmatrix}$$

- Applying $R_3 - 2R_1$, we get

$$[A:B] \sim \begin{bmatrix} 1 & 1 & 1 & 2 \\ 0 & -1 & -1 & 0 \\ 0 & 0 & 0 & -4 \end{bmatrix}$$

- $\Rightarrow rank\ of\ A = 2$ and rank of $[A:B] = 3$.
- i.e $\rho(A) \neq \rho[A:B]$.
- Hence the given system of linear equation is inconsistent and therefore has no solution.

Example 4

- Solve the system of equations.

$$x_1 - x_2 + 2x_3 + x_4 = 2$$
$$3x_1 + 2x_2 + x_4 = 1$$
$$4x_1 + x_2 + 2x_3 + 2x_4 = 3$$

- **Solution:** The system of equations can be represented by the matrix equation as,

$$\begin{bmatrix} 1 & -1 & 2 & 1 \\ 3 & 2 & 0 & 1 \\ 4 & 1 & 2 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}$$

- where the augmented matrix is
- $[A : B] = \begin{bmatrix} 1 & -1 & 2 & 1 & 2 \\ 3 & 2 & 0 & 1 & 1 \\ 4 & 1 & 2 & 2 & 3 \end{bmatrix}$

- Applying $R_3 - (R_1 + R_2)$,we get

$$[A : B] \sim \begin{bmatrix} 1 & -1 & 2 & 1 & 2 \\ 3 & 2 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

- Applying $R_2 - 3R_1$, we get

$$[A : B] \sim \begin{bmatrix} 1 & -1 & 2 & 1 & 2 \\ 0 & 5 & -6 & -2 & -5 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

- Hence $\rho(A) = \rho[A : B] = 2$, the system is consistent
- Rank $r = 2 < 4$ (Number of variables), therefore the system has infinite solutions.
- $\therefore (n - r) = 4 - 2 = 2$ (free variables)

Example 4 contd..

- The reduced form of the linear equations is
- $x_1 - x_2 + 2x_3 + x_4 = 2$,
- $5x_2 - 6x_3 - 2x_4 = -5$
- Let $x_3 = p$ and $x_4 = q$, an arbitrary constant for free variables
- Substituting back, we get
- $x_2 = \frac{1}{5}(-5 + 6p + 2q)$ and
- $x_1 = \frac{1}{5}(5 - 4p - 3q)$
- Hence the infinite values of solution are given by,
- $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} \frac{1}{5}(5 - 4p - 3q) \\ \frac{1}{5}(-5 + 6p + 2q) \\ p \\ q \end{bmatrix}$ as p and q varies.

Example 5

- Investigate for what values of a and b the following linear equations
- $x + 2y + 3z = 4, x + 3y + 4z = 5, x + 3y + az = b$, have **(i)** no solution, **(ii)** a unique solution,
- **(iii)** Infinite number of solutions.
- **Solution:** The system of linear equations can be written in the matrix form as $AX = B$

$$\begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 4 \\ 1 & 3 & a \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 4 \\ 5 \\ b \end{bmatrix}$$

where the augmented matrix is $[A:B] = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 4 & 5 \\ 1 & 3 & a & b \end{bmatrix}$

Applying $R_2 - R_1 \ and \ R_3 - R_1$, we get

$$[A:B] \sim \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 1 & 1 \\ 0 & 1 & a-3 & b-4 \end{bmatrix}$$

Applying $R_3 - R_2$, $[A:B] \sim \begin{bmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & a-4 & b-5 \end{bmatrix}$

- **(i) For no solution:**

  In this case $\rho\,(A) \neq \rho[A:B]$.

  Then we have $\rho(A) = 2 \ and \ \rho[A:B] = 3$.

  i.e. $a = 4$ and $b \neq 5$,

- **(ii) For unique solution:**

  we should have $\rho(A) = \rho[A:B] = n = 3$

  In this case the system is consistent. Further,

  Since $\rho(A) =$ number of unknowns, therefore the system possesses unique solution if $a \neq 4$ and for any value of b.

- **(iii) For infinite number of solutions:**

  we get $\rho[A:B] = \rho(A) = 2 < 3$, the number of unknowns,

  therefore system of equations is consistent and possesses an infinite number of solutions when $a = 4$ and $b = 5$,

Example 6

- For which values of λ following set of equations is consistent? Find and solve equations for those values
- $x + 2y + z = 3, x + y + z = \lambda, 3x + y + 3z = \lambda^2$
- **Solution:** consider $[A:B] = \begin{bmatrix} 1 & 2 & 1 & 3 \\ 1 & 1 & 1 & \lambda \\ 3 & 1 & 3 & \lambda^2 \end{bmatrix}$
- Applying $R_2 - R_1$ and $R_3 - 3R_1$, we get
  $[A:B] \sim \begin{bmatrix} 1 & 2 & 1 & 3 \\ 0 & -1 & 0 & \lambda - 3 \\ 0 & -5 & 0 & \lambda^2 - 9 \end{bmatrix}$
- Applying $R_3 - 5R_2$, we get
  $[A:B] \sim \begin{bmatrix} 1 & 2 & 1 & 3 \\ 0 & -1 & 0 & \lambda - 3 \\ 0 & 0 & 0 & \lambda^2 - 5\lambda + 6 \end{bmatrix}$
- For consistency of the equation, the rank of A and rank of $[A:B]$ must be the same.

From above reduced form of $[A:B]$ it is clear that the rank of $A = 2$. To have the rank of $[A:B] = 2$,

- Consider $\lambda^2 - 5\lambda + 6 = 0$ i.e., $(\lambda - 2)(\lambda - 3) = 0$
  $\Rightarrow \lambda = 2 \; and \; \lambda = 3$
- **(i) Solution for $\lambda = 2$**
- The reduced Echelon form of $[A:B]$ is
  $\begin{bmatrix} 1 & 2 & 1 & 3 \\ 0 & -1 & 0 & \lambda - 3 \\ 0 & 0 & 0 & \lambda^2 - 5\lambda + 6 \end{bmatrix}$
- Substituting $\lambda = 2$, we have $\begin{bmatrix} 1 & 2 & 1 & 3 \\ 0 & -1 & 0 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$
- The reduced form of linear equations is
- $x + 2y + z = 3 \; and \; y = 1$

Example 6 (contd…)

- Let $z = k$, an arbitrary constant, $\therefore x = 1 - k$
- Hence $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1-k \\ 1 \\ k \end{bmatrix}$ has infinite values as k varies
- **(ii) Solution for $\boldsymbol{\lambda = 3}$:**
- The reduced Echelon form of $[A: B]$
- By substituting $\lambda = 3$, we have $\begin{bmatrix} 1 & 2 & 1 & 3 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$
- The reduced form of linear equations is
- $x + 2y + z = 3 \; and \; y = 0$
- Let $z = c$, an arbitrary constant. $\therefore x = 3 - c$
- Hence $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3-c \\ 0 \\ c \end{bmatrix}$ has infinite solutions as c varies.

Example 7

- For what values of λ the equations $3x - 2y + \lambda z = 1, 2x + y + z = 2, x + 2y - \lambda z = -1$, will have no unique solution? Will the equations have any solutions for this value of λ.
- **Solution:** (taking the equations in reverse order)
- $$\begin{bmatrix} 1 & 2 & -\lambda \\ 2 & 1 & 1 \\ 3 & -2 & \lambda \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix}$$
- Applying $R_2 - 2R_1$ and $R_3 - 3R_1$, we get
  $$\begin{bmatrix} 1 & 2 & -\lambda \\ 0 & -3 & 1+2\lambda \\ 0 & -8 & 4\lambda \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1 \\ 4 \\ 4 \end{bmatrix} \text{..............(1)}$$
- The equations have unique solutions if the coefficient matrix is non – singular.
- $\therefore -12\lambda + 8 + 16\lambda \neq 0,\ 4\lambda \neq -8 \qquad \therefore \lambda \neq -2$
- $\therefore$ The equations have unique solutions if $\lambda \neq -2$
- and they have no unique solutions if $\lambda = -2$
- Further, if $\lambda = -2$, we have from (1)
  $$\begin{bmatrix} 1 & 2 & 2 \\ 0 & -3 & -3 \\ 0 & -8 & -8 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1 \\ 4 \\ 4 \end{bmatrix}$$
- Applying $R_3 - \frac{8}{3}R_2$, we get
  $$\begin{bmatrix} 1 & 2 & 2 \\ 0 & -3 & -3 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} -1 \\ 4 \\ -20/3 \end{bmatrix}$$
- $\therefore 0x + 0y + 0z = -20/3$ which is absurd
- Also the rank of A $= 2 < the\ rank\ of\ [A, B] = 3$
- $\therefore$ The equations are inconsistent, For $\lambda = -2$ there is no solution.