Module :3
Matrices
Properties:
Eigen Values & Eigen Vectors

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❖ Singular Matrices have Zero Eigenvalues.

Statement: *Suppose A is a square matrix. Then A is singular if and only if $\lambda=0$ is an eigenvalue of A.*

Proof: A is singular $\Leftrightarrow$ there exists $x\neq 0$,

$\Leftrightarrow$ there exists $x\neq 0$, $Ax=0x$

$\Leftrightarrow$ $\lambda=0$ is an eigenvalue of A

❖ Eigenvalues of the Transpose of a Matrix

Statement: *Suppose* A *is a square matrix and* λ *is an eigenvalue of* A. *Then* λ *is an eigenvalue of the matrix* A^T. *i.e. A and* A^T have *eigenvalues.*

Proof: Suppose A has order n.

Now, $\det(A - \lambda I)^T = \det(A^T - \lambda I^T) = \det(A^T - \lambda I)$

$\therefore \det(A - \lambda I)^T = 0 \Leftrightarrow \det(A^T - \lambda I) = 0$

So A and A^T have the same characteristic polynomial and their eigenvalues are identical and have equal algebraic multiplicities.

❖ Eigenvalues of a Scalar Multiple of a Matrix

Statement: *Suppose A is a square matrix and λ is an eigenvalue of A. Then αλ is an eigenvalue of αA.*

Proof: Let $x \neq 0$ be one eigenvector of A for λ. Then

$$(\alpha A)x = \alpha(Ax)$$
$$= \alpha(\lambda x)$$
$$= (\alpha\lambda)x$$

So $x \neq 0$ is an eigenvector of αA for the eigenvalue αλ.

Eigenvalues Of Matrix Powers

Statement: *Suppose A is a square matrix of order n and λ is an eigenvalue of A. Then for any integer $s \geq 0$, λ^s is an eigenvalue of A^s.*

Proof: Let $x \neq 0$ be one eigenvector of A for λ.

We will prove the theorem by mathematical induction on s.

Step 1: For $s=0$,

$$A^s x = A^0 x = I_n x = x$$

$$= 1x = \lambda^0 x = \lambda^s x$$

∴The theorem is true for $s=0$.

Step 2: Assume the theorem is true for s, then we will prove its true for $s+1$.

$$A^{s+1}x = A^s A x = A^s(\lambda x) = \lambda(A^s x) = \lambda(\lambda^s x) = (\lambda\lambda^s)x = \lambda^{s+1}x$$

So, $x \neq 0$ is an eigenvector of A^{s+1} corresponding to eigenvalue λ^{s+1}

By Mathematical induction, Theorem is true for all $s \geq 0$.

❖ Eigenvalues of the Polynomial of a Matrix

Statement: *Suppose A is a square matrix and λ is an eigenvalue of A. Let q(x) be a polynomial in the variable x. Then q(λ) is an eigenvalue of the matrix q(A).*

Proof : Let $x \neq 0$ be one eigenvector of A for λ, and write $q(x)=a_0+a_1x+a_2x^2+\cdots+a_mx^m$

Then $q(A)x=(a_0+a_1A+a_2A^2+\cdots+a_mA^m)x$

$=a_0(A^0x)+a_1(A^1x)+a_2(A^2x)+\cdots+a_m(A^mx)$

$=a_0(\lambda^0x)+a_1(\lambda^1x)+a_2(\lambda^2x)+\cdots+a_m(\lambda^mx)$

$=(a_0+a_1\lambda+a_2\lambda^2+\cdots+a_m\lambda^m)x$

$=q(\lambda)x$

So $x \neq 0$ is an eigenvector of q(A) for the eigenvalue q(λ).

❖ Eigenvalues of the Inverse of a Matrix

Statement: *Suppose A is a square nonsingular matrix and λ is an eigenvalue of A. Then λ^{-1} is an eigenvalue of the matrix A^{-1}.*

Proof: since A is assumed nonsingular, A^{-1} exists and $1/\lambda$ does not involve division by zero.

Let $x \neq 0$ be one eigenvector of A for λ. Suppose A has order n. Then

$$A^{-1}x = A^{-1}(1x) = A^{-1}\left(\frac{1}{\lambda}\lambda x\right)$$

$$= \frac{1}{\lambda}A^{-1}(\lambda x)$$

$$= \frac{1}{\lambda}A^{-1}(Ax) = \frac{1}{\lambda}(A^{-1}Ax)$$

$$= \frac{1}{\lambda}(I_n x) = \frac{1}{\lambda}x$$

So $x \neq 0$ is an eigenvector of A^{-1} for the eigenvalue $\frac{1}{\lambda}$.

❖ Eigenvalues of the Adjoint of a Matrix

Statement: *Suppose* A *is a square nonsingular matrix and* λ *is an eigenvalue of* A. *Then* $\frac{|A|}{\lambda}$ *is an eigenvalue of the matrix* Adj.(A).

Proof: We know that A adj.A=$|A|$I

Premultiplying by A^{-1}

A^{-1} A adj.A= $A^{-1}\,|A|$

∴ adj.A= $A^{-1}\,|A|$

Post multiplying by x, x is the eigenvector corresponding to λ .

(adj.A)x = $|A|$ A^{-1}x= $|A|$ $\frac{1}{\lambda}$ x =$\frac{|A|}{\lambda}$ x

∴ $\frac{|A|}{\lambda}$ *is an eigenvalue of the matrix* Adj.(A).

❖ The eigenvalues of a unitary matrix are of unit modulus.

Statement: *Suppose A is a Unitary matrix and λ is an eigenvalue of A. Then* $|\lambda| = 1$.

Proof: But A is Unitary. ∴ $AA^{\theta} = A^{\theta}A = I$

Let $x \neq 0$ be one eigenvector of A corresponding to eigenvalue λ.

∴ $Ax = \lambda x$……(1)

Taking complex conjugate transpose on both sides

$(Ax)^{\theta} = (\lambda x)^{\theta}$

$x^{\theta} A^{\theta} = \bar{\lambda} x^{\theta}$ ……. (2)

Multiplying (1) & (2)

$(x^{\theta} A^{\theta})(A x) = (\bar{\lambda} x^{\theta})(\lambda x)$

∴ $x^{\theta} (A^{\theta} A) x = \bar{\lambda} \lambda (x^{\theta} x)$

∴ $x^{\theta} x = \bar{\lambda} \lambda (x^{\theta} x)$

Since X is an eigen vector, $x^{\theta} x \neq 0$

∴ $\lambda\bar{\lambda} = 1$ ∴ $|\lambda| = 1$

❖ Hermitian Matrices have Real Eigenvalues

Statement: *Suppose that A is a Hermitian matrix and λ is an eigenvalue of A. Then λ∈R.*

Proof : Let $x \neq 0$ be one eigenvector of A corresponding to eigenvalue λ.

we know $Ax = \lambda x$

Premultiplying by x^{θ} we get

$x^{\theta} Ax = x^{\theta} \lambda x = \lambda x^{\theta} x$ 1

Taking complex congugate transpose on both sides

$(x^{\theta} Ax)^{\theta} = (\lambda x^{\theta} x)^{\theta}$

$x^{\theta} A^{\theta} (x^{\theta})^{\theta} = \bar{\lambda} x^{\theta} (x^{\theta})^{\theta}$ 2

But A is Hermitian. ∴ $A = A^{\theta}$

By 2, $x^{\theta} A x = \bar{\lambda} x^{\theta} x$ 3

From (1) & (3), we get

$\lambda x^{\theta} x = \bar{\lambda} x^{\theta} x$

Since X is an eigen vector, $x^{\theta} x \neq 0$

$\therefore \lambda = \bar{\lambda}$

Corollaries

- Determinent of Harmitian mtrix is real
- Real Symmetric Matrices have Real Eigenvalues
- Eigenvalues of a Skew-Hermitian Matrix are either purely imaginary or zero.
- The eigen values of a real skew-symmetric matrix are purely imaginary or zero.

- The eigenvalues of an Orthogonal Matrix are of unit modulus.

Statement: *Suppose* A *is a Orthogonal matrix and* λ *is an eigenvalue of* A. *Then* $|\lambda| = 1.$

Hint: If the element of a unitary matrix A are all real, then A becomes the orthogonal Matrix.

Ex. Find the eigenvalues of Adj. A and inv. A Where

$$A = \begin{bmatrix} 1 & 2 & 3 & -2 \\ 0 & 2 & 4 & 6 \\ 0 & 0 & 4 & -5 \\ 0 & 0 & 0 & 6 \end{bmatrix}$$

- Eigenvalues of A are λ= 1,2,4,6

By properties,

Eigenvalues of Adj.A = $\frac{|A|}{\lambda}$ & $|A|$=Product of eigenvalues

∴ Eigenvalues of Adj.A = $\frac{48}{1}, \frac{48}{2}, \frac{48}{4}, \frac{48}{6}$

∴ Eigenvalues of Adj.A =48, 24,12, 8

Eigenvalues of Inv.A = $\frac{1}{\lambda} = 1, \frac{1}{2}, \frac{1}{4}, \frac{1}{6}$ … , … , … , …

Ex. 1 Find the eigenvalues and eigen vectors of $6A^{-1}+A^3+2I$. Where A = $\begin{bmatrix} -1 & 2 \\ 0 & -1 \end{bmatrix}$

Eigenvalues of A are λ= -1, -1 and corresponding eigenvector is X= [1,0]' (Can be found by students)

By properties,

Eigenvalues of $\mathbf{6}A^{-1}+A^3+2I = \mathbf{6}\frac{1}{\lambda} + \lambda^3 + 2(1)$

∴ Eigenvalues of **6** $A^{-1}+A^3+2I$ are $(-6)+(-1)^3+2 = 5, 5$ with and corresponding eigenvector is X= [1,0]'.

Ex. Find eigen values of A and prove they are of unit modulus.

$$A = \begin{bmatrix} \frac{1+i}{2} & -\frac{1-i}{2} \\ \frac{1+i}{2} & \frac{1-i}{2} \end{bmatrix}$$

The characteristic equation of A is

$$\begin{vmatrix} \frac{1+i}{2} - \lambda & -\frac{1-i}{2} \\ \frac{1+i}{2} & \frac{1-i}{2} - \lambda \end{vmatrix} = 0$$

$$\therefore \left(\frac{1+i}{2} - \lambda\right)\left(\frac{1-i}{2} - \lambda\right) + \left(\frac{1+i}{2}\right)\left(\frac{1-i}{2}\right) = 0$$

$$\therefore \left\{\left(\frac{1}{2} - \lambda\right)^2 - \left(\frac{i}{2}\right)^2\right\} + \left\{\left(\frac{1}{2}\right)^2 - \left(\frac{i}{2}\right)^2\right\} = 0$$

$\therefore \lambda^2 - \lambda + 1 = 0$

Eigen values are: $\lambda = \frac{1+i\sqrt{3}}{2}, \frac{1-i\sqrt{3}}{2}$ & their modulus are 1.

Ex. If A = $\begin{bmatrix} \sin\theta & \mathrm{cosec}\,\theta & 1 \\ \sec\theta & \cos\theta & 1 \\ \tan\theta & \cot\theta & 1 \end{bmatrix}$ **then prove that there does not exists a real value of θ for which characteristic roots of A are -1,1 & 4.**

Soln. Sum of A= sum of diagonal elements

= $\sin\theta + \cos\theta + 1$………..(1)

By properties, trace of A = sum of eigen values =4…..(2)

By (1) & (2),

$\sin\theta + \cos\theta + 1 = 4$ ………(3)

For no real value of θ equation (3) holds true.

Hence the proof.