Matrix Theory: Rank of Matrix

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System of Linear Equations

FY BTech SEM-I

Module-2

Sub-Module 2.3

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2	**Matrix Theory: Rank of Matrix**		8	CO 2
	2.1	Types of matrices: Hermitian, Skew-Hermitian, Unitary and Orthogonal matrix		
	2.2	Rank of a matrix using row echelon forms, reduction to normal form, and PAQ form		
	2.3	System of homogeneous and non-homogeneous equations, their consistency and solutions		
	2.4	Linearly dependent and independent vectors		
	2.5	Solution of system of linear algebraic equations by **(a)** Gauss Seidal method **(b)** Jacobi iteration method		
		#Self-learning topics: Symmetric, Skew-symmetric matrices and properties, Properties of adjoint and inverse of a matrix		

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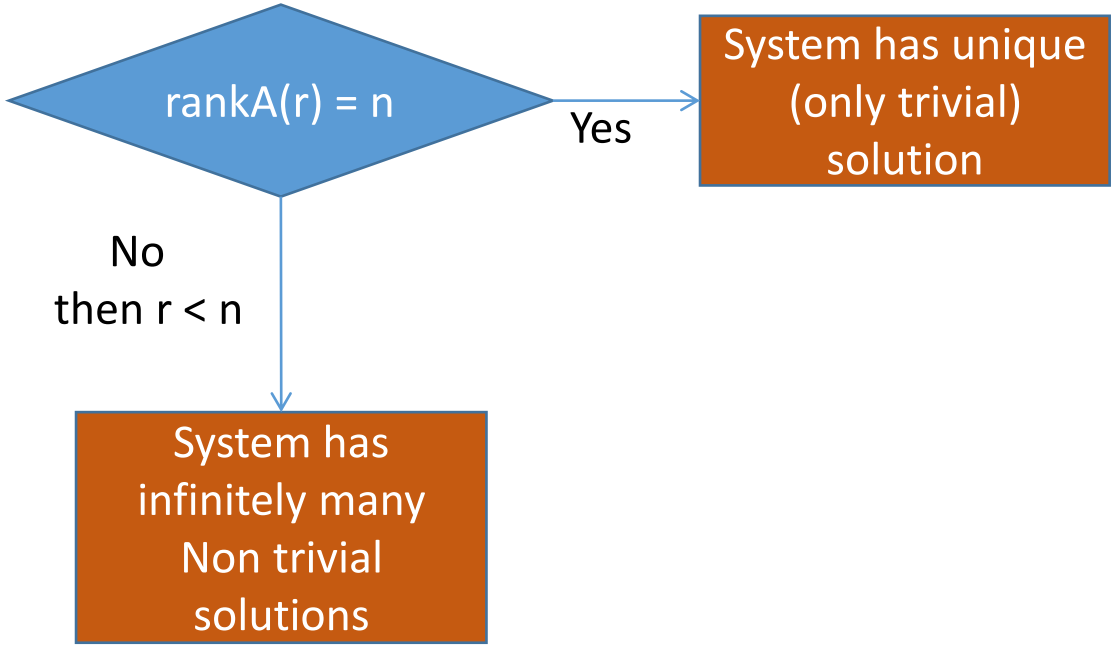
HOMOGENEOUS SYSTEM OF LINEAR EQUATIONS

- **Solutions of $AX = 0$**
- The null column matrix is obviously a solution of $AX = 0$. The solution $X = 0$ is called the **trivial solution** or the **zero – solution**.
- **Note:** 1. These equations do not have constant term or intercept. Hence geometrically all these equations are lines passing through origin.
- 2. Since, they have at least one point of intersection. We will not have case of **No solution** here.
- If we could find a non-zero solution $X \neq 0$, then it is called **non – trivial solution.**
- If we have a Homogeneous system of m equations in n unknowns. Then, the matrix A will be $m \times n$.
- Let r be the rank of the matrix A.
- **Case I :** If $r = n$ i.e., rank (A) is equal to the number of unknowns then this is case of unique solution.
- $x_1 = x_2 \ldots = x_n = 0$ i.e., only possible solution is **zero – solution** or **trivial solution**.
- **Case II:** If $r < n$ i.e., rank (A) is less than the number of unknowns then the system has Infinitely many
- **non – trivial solutions**. The no of independent solutions i.e parameters is equal to $n - r$

RULE TO SOLVE HOMOGENEOUS SYSTEM OF LINEAR EQUATIONS

- **1.** Write the given system in the matrix form $AX = 0.$
- **2.** Apply **row transformations only** and reduce the coefficient matrix A to **row echelon form**.
- **3.** We know that the rank of a matrix in echelon form is equal to the number of non-zero rows.
- Determine the rank of A = r and find the solution by using following cases.



Example 1

- Find the solution of the system given by
$$x_1 - x_2 + x_3 = 0$$
$$x_1 + 2x_2 + x_3 = 0$$
$$2x_1 + x_2 + 3x_3 = 0$$
- **Solution:** The system can be written as $\begin{bmatrix} 1 & -1 & 1 \\ 1 & 2 & 1 \\ 2 & 1 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$
- Applying elementary row transformations on A we will obtain the Echelon form.
- Applying $R_2 - R_1$ *and* $R_3 - 2R_1$, we have $\begin{bmatrix} 1 & -1 & 1 \\ 0 & 3 & 0 \\ 0 & 3 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$
- Applying $R_3 - R_2$, we have $\begin{bmatrix} 1 & -1 & 1 \\ 0 & 3 & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$
- i.e. the coefficient matrix is non – singular. Hence $\rho(A) = 3$, r = n Hence unique solution
- Therefore there exists a trivial solution $x_1 = x_2 = x_3 = 0$

Example 2

- Solve the following system of linear equations
  $x_1 + 2x_2 + 4x_3 + x_4 = 0$
  $2x_1 + x_2 + 5x_3 + 8x_4 = 0$
  $x_1 + 4x_2 + 6x_3 - 3x_4 = 0$
- **Solution:** The system can be written as

$$\begin{bmatrix} 1 & 2 & 4 & 1 \\ 2 & 1 & 5 & 8 \\ 1 & 4 & 6 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

- Applying $R_2 - 2R_1 \; and \; R_3 - R_1$, we have

$$\begin{bmatrix} 1 & 2 & 4 & 1 \\ 0 & -3 & -3 & 6 \\ 0 & 2 & 2 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

- Applying $R_3 + \frac{2}{3} R_2$, we have

$$\begin{bmatrix} 1 & 2 & 4 & 1 \\ 0 & -3 & -3 & 6 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

- Hence $\rho(A) = 2 < 4$ (Number of unknowns).
- So the system has infinitely non – trivial solution.
- $(n - r) = (4 - 2) = 2$ free variables
- Then reduced from of system of equations is
- $x_1 + 2x_2 + 4x_3 + x_4 = 0$ ……………..(i)
  $-3x_2 - 3x_3 + 6x_4 = 0$ …………….(ii)
- (ii) $\Rightarrow x_2 + x_3 - 2x_4 = 0 \quad \Rightarrow x_2 = -x_3 + 2x_4$
- Substituting in (i), we get $x_1 + 2(-x_3 + 2x_4) + 4x_3 + x_4 = 0 \quad \Rightarrow x_1 = -2x_3 - 5x_4$

Example 2 (contd…)

- Let $x_3 = k_1$ and $x_4 = k_2$, where k_1 and k_2 are some parameters.
- $\therefore$ We have $x_1 = -2k_1 - 5k_2 \; and \; x_2 = -k_1 + 2k_2$
- Hence $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -2k_1 - 5k_2 \\ -k_1 + 2k_2 \\ k_1 \\ k_2 \end{bmatrix}$ has infinite solutions as $k_1 \; and \; k_2$ vary.

Example 3

- Solve
$$3x_1 + 4x_2 - x_3 - 9x_4 = 0$$
$$2x_1 + 3x_2 + 2x_3 - 3x_4 = 0$$
$$2x_1 + x_2 - 14x_3 - 12x_4 = 0$$
$$x_1 + 3x_2 + 13x_3 + 3x_4 = 0$$

- **Solution:** The system can be written as $AX = 0$

i.e., $\begin{bmatrix} 3 & 4 & -1 & -9 \\ 2 & 3 & 2 & -3 \\ 2 & 1 & -14 & -12 \\ 1 & 3 & 13 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

- Applying R_{14}, we get

$$\begin{bmatrix} 1 & 3 & 13 & 3 \\ 2 & 3 & 2 & -3 \\ 2 & 1 & -14 & -12 \\ 3 & 4 & -1 & -9 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

- Applying $R_2 - 2R_1, R_3 - 2R_1, R_4 - 3R_1$, we get

$$\begin{bmatrix} 1 & 3 & 13 & 3 \\ 0 & -3 & -24 & -9 \\ 0 & -5 & -40 & -18 \\ 0 & -5 & -40 & -18 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

- Applying $R_4 - R_3$, we get

$$\begin{bmatrix} 1 & 3 & 13 & 3 \\ 0 & -3 & -24 & -9 \\ 0 & -5 & -40 & -18 \\ 0 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

- Applying $\frac{R_2}{-3}$, we get

$$\begin{bmatrix} 1 & 3 & 13 & 3 \\ 0 & 1 & 8 & 3 \\ 0 & -5 & -40 & -18 \\ 0 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Example 3 (contd…)

- Applying $R_3 + 5R_2$, we get
$$\begin{bmatrix} 1 & 3 & 13 & 3 \\ 0 & 1 & 8 & 3 \\ 0 & 0 & 0 & -3 \\ 0 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$
- Hence $\rho(A) = 3 < 4$ (number of variables).
- $\therefore$ the system has infinite non trivial solutions.
- The reduced form of system of equations can be written as
- $x_1 + 3x_2 + 13x_3 + 3x_4 = 0$ ………….(i)
$x_2 + 8x_3 + 3x_4 = 0$ ………………..(ii)
$-3x_4 = 0$ …………(iii)
- Now, since $n = 4, r = 3,$
- $(n - r) = (4 - 3) = 1$ free variable
- (iii) $\Rightarrow x_4 = 0,\quad Let\ \ x_3 = k$ (arbitrary)
- $\therefore$ (ii) $\Rightarrow x_2 + 8k + 0 = 0 \ \Rightarrow\ x_2 = -8k$
- And (i) $\Rightarrow x_1 - 24k + 13k + 0 = 0 \Rightarrow x_1 = 11k$
- Hence $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 11k \\ -8k \\ k \\ 0 \end{bmatrix}$ will have infinite many solutions as k varies.

Example 4

- Determine the values of λ for which the following system of equations possess a non – trivial solution and obtain these solutions for each value of λ.

$$3x_1 + x_2 - \lambda x_3 = 0$$
$$4x_1 - 2x_2 - 3x_3 = 0$$
$$2\lambda x_1 + 4x_2 + \lambda x_3 = 0$$

- **Solution:** The system can be written as $\begin{bmatrix} 3 & 1 & -\lambda \\ 4 & -2 & -3 \\ 2\lambda & 4 & \lambda \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$
- System will possess non trivial solution if rank of coefficient matrix is less than number of variables
- i.e., $r < 3$ if $|A| = 0$
- $\therefore \begin{vmatrix} 3 & 1 & -\lambda \\ 4 & -2 & -3 \\ 2\lambda & 4 & \lambda \end{vmatrix} = 0$
- $\therefore 3(-2\lambda + 12) - (4\lambda + 6\lambda) - \lambda(16 + 4\lambda) = 0$
- $\therefore \lambda^2 + 8\lambda - 9 = 0 \qquad \therefore (\lambda + 9)(\lambda - 1) = 0$
- $\therefore \lambda = -9 \;\; and \; \lambda = 1$ for which the system possesses a non – trivial solution.

Example 4 (contd…)

- **Case (i) For $\boldsymbol{\lambda = -9}$**
- the system can be written as
$$\begin{bmatrix} 3 & 1 & 9 \\ 4 & -2 & -3 \\ -18 & 4 & -9 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$
- Applying $R_2 - \frac{4}{3}R_1, R_3 + 6R_1$, we have
$$\begin{bmatrix} 3 & 1 & 9 \\ 0 & -10/3 & -15 \\ 0 & 10 & 45 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$
- Applying $R_3 + 3R_2$, we have
$$\begin{bmatrix} 3 & 1 & 9 \\ 0 & -10/3 & -15 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$
- $\therefore$ The reduced form of system of equations is
- $3x_1 + x_2 + 9x_3 = 0$ and $-\left(\frac{10}{3}\right)x_2 - 15x_3 = 0$
- $\therefore x_2 = -(9/2)x_3$
- $\Rightarrow 3x_1 = (9/2)x_3 - 9x_3 = -(9/2)x_3$
- $\therefore x_1 = -(3/2)x_3$
- Let $x_3 = a$ (arbitrary)
- Hence $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -(3/2)a \\ -(9/2)a \\ a \end{bmatrix}$ has infinite solutions as 'a' varies

Example 4 (contd…)

- **Case (i) For $\boldsymbol{\lambda = 1}$**
- System can be written as $\begin{bmatrix} 3 & 1 & -1 \\ 4 & -2 & -3 \\ 2 & 4 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$
- Applying $R_1 - R_2$, we have $\begin{bmatrix} 1 & -3 & -2 \\ 4 & -2 & -3 \\ 2 & 4 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$
- Applying $R_2 - 4R_1, R_3 - 2R_1$, we have $\begin{bmatrix} 1 & -3 & -2 \\ 0 & 10 & 5 \\ 0 & 10 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$
- Applying $R_3 - R_2$, we have $\begin{bmatrix} 1 & -3 & -2 \\ 0 & 10 & 5 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$
- $\therefore$ The reduced form of system of equations is
- $x_1 - 3x_2 - 2x_3 = 0\ , \quad 10x_2 + 5x_3 = 0,$
- $\therefore x_2 = -(1/2)x_3$
- $\therefore x_1 - 3x_2 - 2x_3 = 0$
- $\Rightarrow x_1 = -(3/2)x_3 + 2x_3 = (1/2)x_3$
- $\therefore x_1 = (1/2)x_3$
- Let $x_3 = b$ (arbitrary)
- Hence $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} (1/2)b \\ -(1/2)b \\ b \end{bmatrix}$ has infinite solutions as ‘b’ varies.

Example 5

- Discuss for all values of k, the following system of equations possesses trivial and non – trivial solutions
- $$\begin{gathered} 2x + 3ky + (3k+4)z = 0 \\ x + (k+4)y + (4k+2)z = 0 \\ x + 2(k+1)y + (3k+4)z = 0 \end{gathered}$$
- **Solution:** The given system of equations can be written as $AX = O$ i.e., $\begin{bmatrix} 2 & 3k & 3k+4 \\ 1 & k+4 & 4k+2 \\ 1 & 2(k+1) & 3k+4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$
- Applying R_{12}, we have $\begin{bmatrix} 1 & k+4 & 4k+2 \\ 2 & 3k & 3k+4 \\ 1 & 2(k+1) & 3k+4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$
- Applying $R_2 - 2R_1, R_3 - R_1$, we have
- $\begin{bmatrix} 1 & k+4 & 4k+2 \\ 0 & k-8 & -5k \\ 0 & k-2 & -k+2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$ ....(1)
- If the given system of equations is to posses non – trivial Solutions then the coefficient matrix A must be of rank less than 3. i.e., $|A|$ must be zero.
- i.e $(k-8)(-k+2) + 5k(k-2) = 0$
- i.e $-k^2 + 10k - 16 + 5k^2 - 10k = 0$
- i.e $4k^2 - 16 = 0$ i.e $k^2 = 4$ i.e $k = \pm 2$
- Now three cases arise:
- **Case (i):** If $k \neq \pm 2$,
- then given system of equations possesses only trivial solution
- i.e. $x = y = z = 0$ is the only solution.

Example 5 (contd…)

- **Case (ii):** If $k = 2,$
- then (1) $\Rightarrow \begin{bmatrix} 1 & 6 & 10 \\ 0 & -6 & -10 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$
- The coefficient matrix is of rank 2, the given system of equations now possesses non – trivial solutions.
- The reduced form of the equations is
- $x + 6y + 10z = 0\ ,\ \ 6y + 10z = 0$
- Let $z = c$ (arbitrary)
- Then we have $y = \frac{-5}{3} c \ and\ x = 0$
- Hence the general solution of the given system can be written as $x = 0, y = \frac{-5}{3} c, z = c$
- **Or** $x = 0, y = -5c', z = 3c'\ ,\ \ c' = \frac{c}{3}$ is parameter

- **Case (iii):** If $k = -2$
- then (1) $\Rightarrow \begin{bmatrix} 1 & 2 & -6 \\ 0 & -10 & 10 \\ 0 & -4 & 4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$
- Applying $\frac{R_2}{10}\ and\ \frac{R_3}{4}$ $\begin{bmatrix} 1 & 2 & -6 \\ 0 & -1 & 1 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$
- Applying $R_3 - R_2,$ $\begin{bmatrix} 1 & 2 & -6 \\ 0 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$
- The coefficient matrix is of rank 2, the given system of equations now possesses non – trivial solutions.
- The reduced form of the equations is
- $x + 2y - 6z = 0\ ,\ \ -y + z = 0$
- $\Rightarrow y = z\ \ and\ x = 4z,$ Let $z = b,$
- then $x = 4b, y = b, z = b$ is the general soln