

Department of Science and Humanities
F.Y.B.Tech Physics Semester I, 2023

Module 4:- Electromagnetism

Topic:- The Fundamental Theorem of Calculus

The basic format of the fundamental theorem:

The integral of a derivative over some region is given by the value of the function at the end points (boundaries). In vector calculus there are three species of derivative (gradient, divergence, and curl), and each has its own “fundamental theorem,” with essentially the same format.

➤ **The Fundamental Theorem for Gradients**

Suppose we have a scalar function of three variables $T(x, y, z)$. Starting at point **a**, we move a small distance $d\mathbf{l}_1$ (Fig. 1.26). According to Eq. 1.37, the function T will change by an amount $dT = (\nabla T) \cdot d\mathbf{l}_1$.

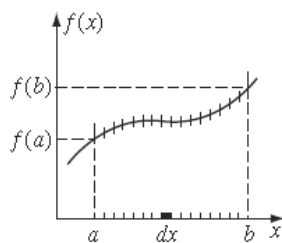


FIGURE 1.25

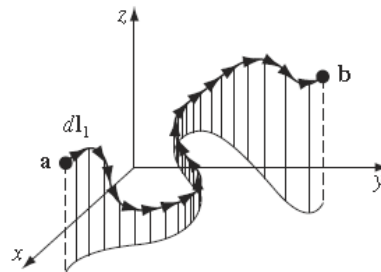


FIGURE 1.26

Now we move a little further, by an additional small displacement $d\mathbf{l}_2$; the incremental change in T will be $(\nabla T) \cdot d\mathbf{l}_2$. In this manner, proceeding by infinitesimal steps, we make the journey to point **b**. At each step we compute the gradient of T (at that point) and dot it into the displacement $d\mathbf{l}$. . . this gives us the change in T . Evidently the *total* change in T in going from **a** to **b** (along the path selected) is

$$\int_a^b (\nabla T) \cdot d\mathbf{l} = T(\mathbf{b}) - T(\mathbf{a}).$$

This is the **fundamental theorem for gradients**; like the “ordinary” fundamental theorem, it says that “*the integral (here a line integral) of a derivative (here the gradient) is given by the value of the function at the boundaries (a and b)*”.

Corollary 1: $\int_a^b (\nabla T) \cdot d\mathbf{l}$ is independent of the path taken from **a** to **b**.

Corollary 2: $\oint (\nabla T) \cdot d\mathbf{l} = 0$, since the beginning and end points are identical, and hence $T(\mathbf{b}) - T(\mathbf{a}) = 0$.

Example 1.9. Let $T = xy^2$, and take point **a** to be the origin $(0, 0, 0)$ and **b** the point $(2, 1, 0)$. Check the fundamental theorem for gradients.

Solution

Although the integral is independent of path, we must *pick* a specific path in order to evaluate it. Let's go out along the x axis (step i) and then up (step ii) (Fig. 1.27). As always, $d\mathbf{l} = dx \hat{\mathbf{x}} + dy \hat{\mathbf{y}} + dz \hat{\mathbf{z}}$; $\nabla T = y^2 \hat{\mathbf{x}} + 2xy \hat{\mathbf{y}}$.

(i) $y = 0$; $d\mathbf{l} = dx \hat{\mathbf{x}}$, $\nabla T \cdot d\mathbf{l} = y^2 dx = 0$, so

$$\int_{\text{i}} \nabla T \cdot d\mathbf{l} = 0.$$

(ii) $x = 2$; $d\mathbf{l} = dy \hat{\mathbf{y}}$, $\nabla T \cdot d\mathbf{l} = 2xy dy = 4y dy$, so

$$\int_{\text{ii}} \nabla T \cdot d\mathbf{l} = \int_0^1 4y dy = 2y^2 \Big|_0^1 = 2.$$

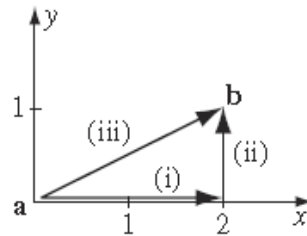


FIGURE 1.27

The total line integral is 2. Is this consistent with the fundamental theorem? Yes: $T(\mathbf{b}) - T(\mathbf{a}) = 2 - 0 = 2$.

Now, just to convince you that the answer is independent of path, let me calculate the same integral along path iii (the straight line from **a** to **b**):

(iii) $y = \frac{1}{2}x$, $dy = \frac{1}{2} dx$, $\nabla T \cdot d\mathbf{l} = y^2 dx + 2xy dy = \frac{3}{4}x^2 dx$, so

$$\int_{\text{iii}} \nabla T \cdot d\mathbf{l} = \int_0^2 \frac{3}{4}x^2 dx = \frac{1}{4}x^3 \Big|_0^2 = 2.$$

Problem 1.32 Check the fundamental theorem for gradients, using $T = x^2 + 4xy + 2yz^3$, the points $\mathbf{a} = (0, 0, 0)$, $\mathbf{b} = (1, 1, 1)$, and the three paths in Fig. 1.28:

(a) $(0, 0, 0) \rightarrow (1, 0, 0) \rightarrow (1, 1, 0) \rightarrow (1, 1, 1)$;

(b) $(0, 0, 0) \rightarrow (0, 0, 1) \rightarrow (0, 1, 1) \rightarrow (1, 1, 1)$;

(c) the parabolic path $z = x^2$; $y = x$.

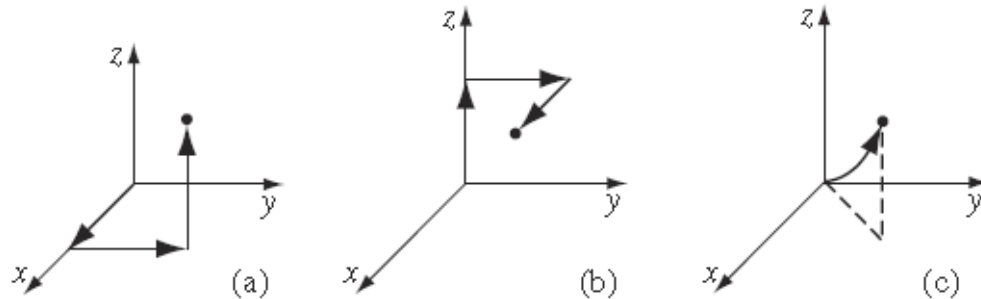


FIGURE 1.28

➤ The Fundamental Theorem for Divergences

The fundamental theorem for divergences states that:

$$\int_V (\nabla \cdot \mathbf{v}) d\tau = \oint_S \mathbf{v} \cdot d\mathbf{a}.$$

“The integral of a derivative (in this case the divergence) over a region (in this case a volume, V) is equal to the value of the function at the boundary (in this case the surface S that bounds the volume)”.

This theorem has at least three special names:

Gauss’s theorem, **Green’s theorem**, or simply the **divergence theorem**.

Notice that the boundary term is itself an integral (specifically, a surface integral). This is reasonable: the “boundary” of a *line* is just two end points, but the boundary of a *volume* is a (closed) surface.

Example 1.10. Check the divergence theorem using the function

$$\mathbf{v} = y^2 \hat{\mathbf{x}} + (2xy + z^2) \hat{\mathbf{y}} + (2yz) \hat{\mathbf{z}}$$

and a unit cube at the origin (Fig. 1.29).

Solution

In this case

$$\nabla \cdot \mathbf{v} = 2(x + y),$$

and

$$\int_V 2(x + y) d\tau = 2 \int_0^1 \int_0^1 \int_0^1 (x + y) dx dy dz,$$

$$\int_0^1 (x + y) dx = \frac{1}{2} + y, \quad \int_0^1 (\frac{1}{2} + y) dy = 1, \quad \int_0^1 1 dz = 1.$$

Thus,

$$\int_V \nabla \cdot \mathbf{v} d\tau = 2.$$

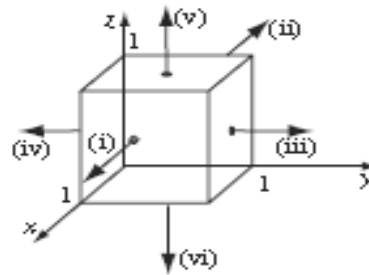


FIGURE 1.29

So much for the left side of the divergence theorem. To evaluate the surface integral we must consider separately the six faces of the cube:

$$(i) \quad \int \mathbf{v} \cdot d\mathbf{a} = \int_0^1 \int_0^1 y^2 dy dz = \frac{1}{3}.$$

$$(ii) \quad \int \mathbf{v} \cdot d\mathbf{a} = - \int_0^1 \int_0^1 y^2 dy dz = -\frac{1}{3}.$$

$$(iii) \quad \int \mathbf{v} \cdot d\mathbf{a} = \int_0^1 \int_0^1 (2x + z^2) dx dz = \frac{4}{3}.$$

$$(iv) \quad \int \mathbf{v} \cdot d\mathbf{a} = - \int_0^1 \int_0^1 z^2 dx dz = -\frac{1}{3}.$$

$$(v) \quad \int \mathbf{v} \cdot d\mathbf{a} = \int_0^1 \int_0^1 2y dx dy = 1.$$

$$(vi) \quad \int \mathbf{v} \cdot d\mathbf{a} = - \int_0^1 \int_0^1 0 dx dy = 0.$$

So the total flux is:

$$\oint_S \mathbf{v} \cdot d\mathbf{a} = \frac{1}{3} - \frac{1}{3} + \frac{4}{3} - \frac{1}{3} + 1 + 0 = 2,$$

as expected.

➤ The Fundamental Theorem for Curls

The fundamental theorem for curls, which goes by the special name of **Stokes' theorem**, states that

$$\int_S (\nabla \times \mathbf{v}) \cdot d\mathbf{a} = \oint_P \mathbf{v} \cdot d\mathbf{l}.$$

“The integral of a derivative (here, the curl) over a region (here, a patch of surface, S) is equal to the value of the function at the boundary (here, the perimeter of the patch, P)”.

As in the case of the divergence theorem, the boundary term is itself an integral—specifically, a closed line integral.

Geometrical Interpretation: Recall that the curl measures the “twist” of the vectors \mathbf{v} ; a region of high curl is a whirlpool—if you put a tiny paddle wheel there, it will rotate. Now, the integral of the curl over some surface (or, more precisely, the *flux* of the curl *through* that surface) represents the “total amount of swirl,” and we can determine that just as well by going around the edge and finding how much the flow is following the boundary (Fig. 1.31).

is sometimes called the **circulation** of \mathbf{v} .

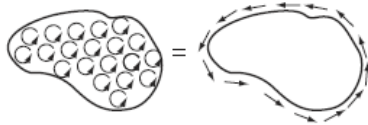


FIGURE 1.31

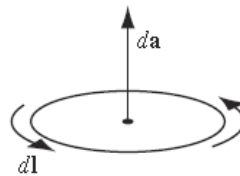


FIGURE 1.32

For a *closed* surface (as in the divergence theorem), $d\mathbf{a}$ points in the direction of the *outward* normal; but for an *open* surface, which way is “out”? Consistency in Stokes' theorem (as in all such matters) is given by the right-hand rule: if your fingers point in the direction of the line integral, then your thumb fixes the direction of $d\mathbf{a}$ (Fig. 1.32).

Corollary 1: $\int (\nabla \times \mathbf{v}) \cdot d\mathbf{a}$ depends only on the boundary line, not on the particular surface used.

Corollary 2: $\oint (\nabla \times \mathbf{v}) \cdot d\mathbf{a} = 0$ for any closed surface, since the boundary line, like the mouth of a balloon, shrinks down to a point, and hence the right side of Eq. 1.57 vanishes.

Example 1.11. Suppose $\mathbf{v} = (2xz + 3y^2)\hat{\mathbf{y}} + (4yz^2)\hat{\mathbf{z}}$. Check Stokes' theorem for the square surface shown in Fig. 1.33.

Solution

Here

$$\nabla \times \mathbf{v} = (4z^2 - 2x)\hat{\mathbf{x}} + 2z\hat{\mathbf{z}} \quad \text{and} \quad d\mathbf{a} = dy \, dz \hat{\mathbf{x}}.$$

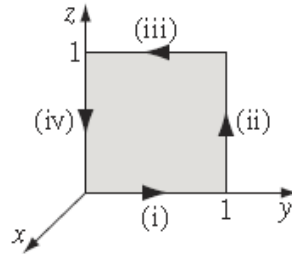


FIGURE 1.33

(In saying that $d\mathbf{a}$ points in the x direction, we are committing ourselves to a counterclockwise line integral. We could as well write $d\mathbf{a} = -dy \, dz \hat{\mathbf{x}}$, but then we would be obliged to go clockwise.) Since $x = 0$ for this surface,

$$\int (\nabla \times \mathbf{v}) \cdot d\mathbf{a} = \int_0^1 \int_0^1 4z^2 \, dy \, dz = \frac{4}{3}.$$

Now, what about the line integral? We must break this up into four segments:

$$(i) \quad x = 0, \quad z = 0, \quad \mathbf{v} \cdot d\mathbf{l} = 3y^2 \, dy, \quad \int \mathbf{v} \cdot d\mathbf{l} = \int_0^1 3y^2 \, dy = 1,$$

$$(ii) \quad x = 0, \quad y = 1, \quad \mathbf{v} \cdot d\mathbf{l} = 4z^2 \, dz, \quad \int \mathbf{v} \cdot d\mathbf{l} = \int_0^1 4z^2 \, dz = \frac{4}{3},$$

$$(iii) \quad x = 0, \quad z = 1, \quad \mathbf{v} \cdot d\mathbf{l} = 3y^2 \, dy, \quad \int \mathbf{v} \cdot d\mathbf{l} = \int_1^0 3y^2 \, dy = -1,$$

$$(iv) \quad x = 0, \quad y = 0, \quad \mathbf{v} \cdot d\mathbf{l} = 0, \quad \int \mathbf{v} \cdot d\mathbf{l} = \int_1^0 0 \, dz = 0.$$

So

$$\oint \mathbf{v} \cdot d\mathbf{l} = 1 + \frac{4}{3} - 1 + 0 = \frac{4}{3}.$$

It checks.

A point of strategy: notice how I handled step (iii). There is a temptation to write $d\mathbf{l} = -dy \hat{\mathbf{y}}$ here, since the path goes to the left. You can get away with this, if you absolutely insist, by running the integral from $0 \rightarrow 1$. But it is much safer to say $d\mathbf{l} = dx \hat{\mathbf{x}} + dy \hat{\mathbf{y}} + dz \hat{\mathbf{z}}$ *always* (never any minus signs) and let the limits of the integral take care of the direction.