K J Somaiya College of Engineering, Vidyavihar, Mumbai

(A Constituent College of SVU)

Engineering Mechanics Notes

Module 1 – System of Forces

Module Section 1.2 – Forces in Space

Class: FY BTech   Division: C3
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References: Engineering Mechanics, by M. D. Dayal & Engineering Mechanics – Statics and Dynamics, by N. H. Dubey.

Vectors:

1. Basic Vector Operations:

$$\bar{P} = x_1\hat{i} + y_1\hat{j} + z_1\hat{k}$$
$$\bar{Q} = x_2\hat{i} + y_2\hat{j} + z_2\hat{k}$$

a) Dot Product

$$\bar{P}\cdot\bar{Q} = x_1x_2 + y_1y_2 + z_1z_2 \quad \text{or}$$
$$\bar{P}\cdot\bar{Q} = |\bar{P}||\bar{Q}|\cos\theta$$

b) Cross Product

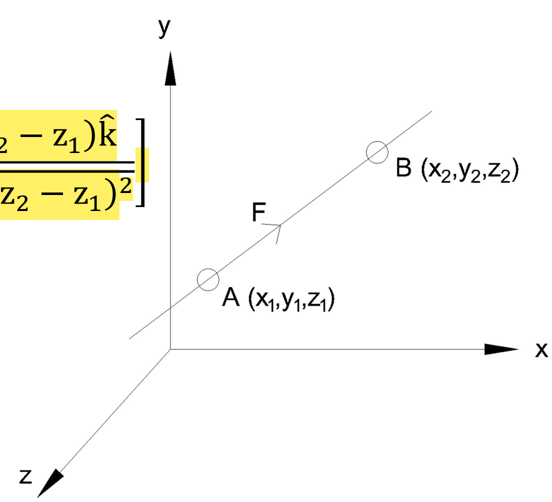
$$\bar{P}\times\bar{Q} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{vmatrix} \quad \text{or}$$
$$\bar{P}\times\bar{Q} = |\bar{P}||\bar{Q}|\sin\theta\,\hat{n}$$

(where $\hat{n}$ is the unit vector normal to the plane of $\bar{P}$ & $\bar{Q}$)

2. Force Vector:

$$\bar{F} = (F)(\hat{e}_{AB})$$
$$\bar{F} = (F)\left[\frac{(x_2-x_1)\hat{i} + (y_2-y_1)\hat{j} + (z_2-z_1)\hat{k}}{\sqrt{(x_2-x_1)^2 + (y_2-y_1)^2 + (z_2-z_1)^2}}\right]$$
$$\bar{F} = F_x\hat{i} + F_y\hat{j} + F_z\hat{k}$$

(where $\hat{e}_{AB}$ is the unit vector in the direction of AB)



3. Magnitude of Force & Direction Angles:

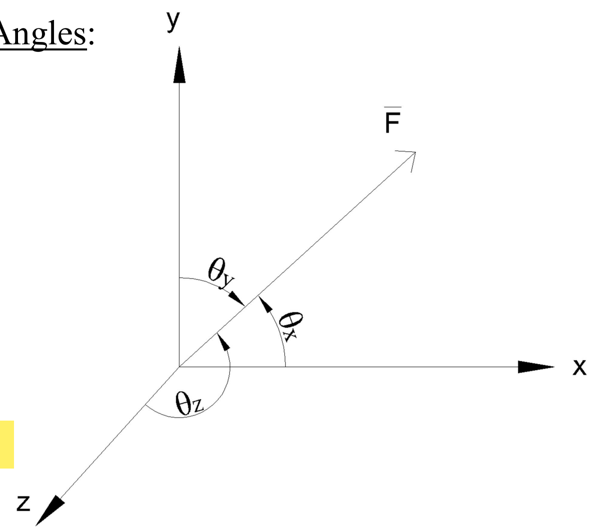
$$\bar{F} = F_x\hat{\imath} + F_y\hat{\jmath} + F_z\hat{k}$$

$$|\bar{F}| \text{ or } F = \sqrt{{F_x}^2 + {F_y}^2 + {F_z}^2}$$

$$F_x = F\cos\theta_x$$
$$F_y = F\cos\theta_y$$
$$F_z = F\cos\theta_z$$

By direction cosine rule,

$$\cos{\theta_x}^2 + \cos{\theta_y}^2 + \cos{\theta_z}^2 = 1$$



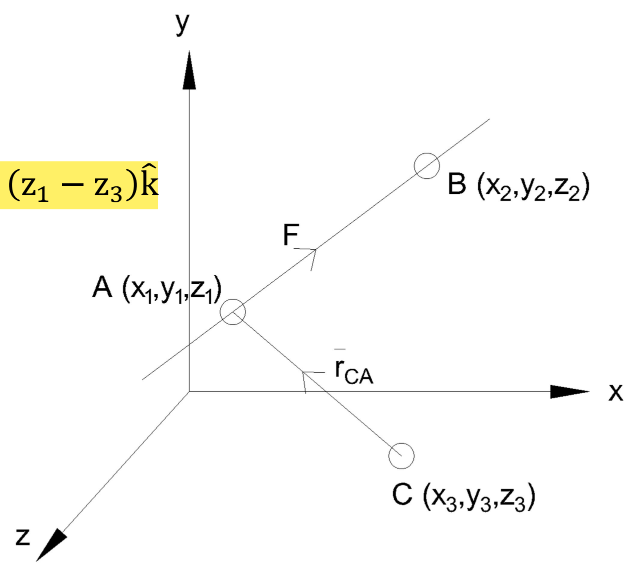
4. Moment Vector:

$$\bar{F} = F_x\hat{\imath} + F_y\hat{\jmath} + F_z\hat{k}$$

$$\bar{r}_{CA} = (x_1 - x_3)\hat{\imath} + (y_1 - y_3)\hat{\jmath} + (z_1 - z_3)\hat{k}$$
$$\bar{r}_{CA} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$$

$$\bar{M}_C = \bar{r}_{CA} \times \bar{F} = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ x & y & z \\ F_x & F_y & F_z \end{vmatrix}$$

$$\bar{M}_C = M_x\hat{\imath} + M_y\hat{\jmath} + M_z\hat{k}$$



5. Vector Component of a Force along a given line:

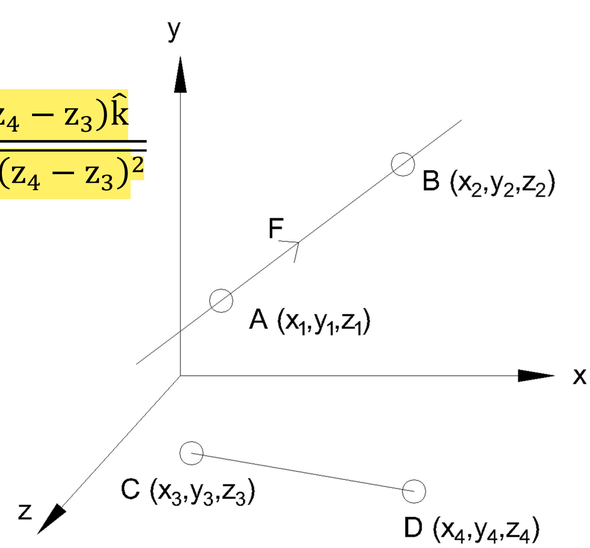
$$\bar{F} = (F)(\hat{e}_{AB})$$

$$\hat{e}_{CD} = \frac{(x_4 - x_3)\hat{\imath} + (y_4 - y_3)\hat{\jmath} + (z_4 - z_3)\hat{k}}{\sqrt{(x_4 - x_3)^2 + (y_4 - y_3)^2 + (z_4 - z_3)^2}}$$

$$\hat{e}_{CD} = x\hat{\imath} + y\hat{\jmath} + z\hat{k}$$

$$F_{CD} = \bar{F} \cdot \hat{e}_{CD}$$
$$F_{CD} = F_x x + F_y y + F_z z$$

$$\bar{F}_{CD} = (F_{CD})(\hat{e}_{CD})$$



6. <u>Moment of a Force about a given line</u>:

$\bar{M}_C = \bar{r}_{CA} \times \bar{F}$

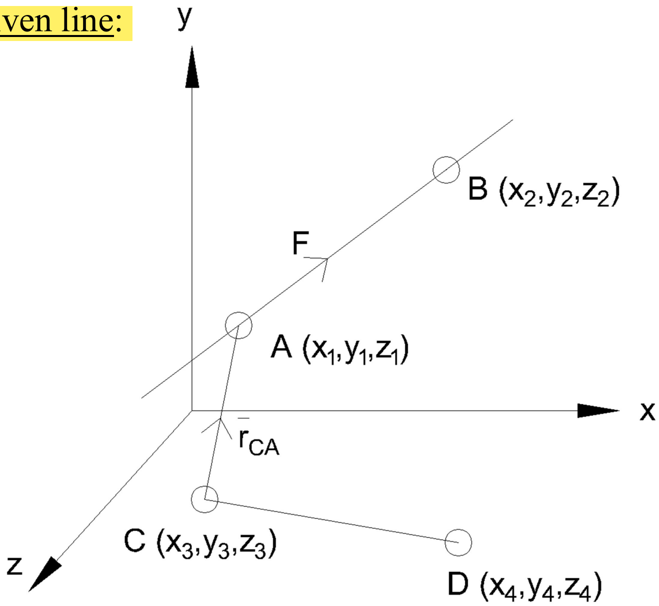
$\bar{M}_C = M_x\hat{i} + M_y\hat{j} + M_z\hat{k}$

$\hat{e}_{CD} = x\hat{i} + y\hat{j} + z\hat{k}$

$M_{CD} = \bar{M}_C \cdot \hat{e}_{CD}$

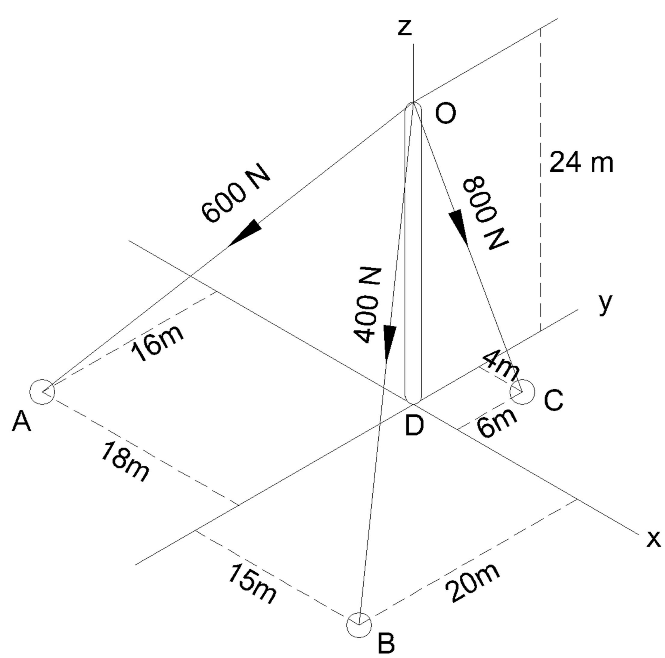
$M_{CD} = M_x x + M_y y + M_z z$

$\bar{M}_{CD} = (M_{CD})(\hat{e}_{CD})$



Numericals:

<u>N1</u>: A tower is being held in place by three cables. If the force of each cable acting on the tower is shown in figure, determine the resultant.



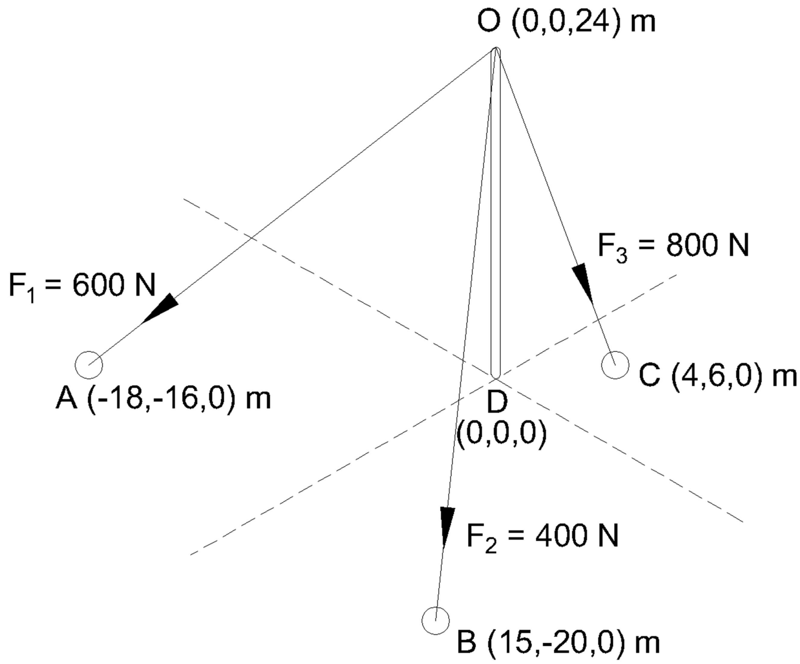
Soln: This is a concurrent space force system of 3 forces acting at O.

Let $\bar{F}_1$, $\bar{F}_2$, and $\bar{F}_3$ be the forces in the cables OA, OB and OC respectively.

$\therefore F_1 = 600$ N, $\quad F_2 = 400$ N, $\quad F_3 = 800$ N

And the co-ordinates of the points based on their distances from origin D are:

D (0,0,0) m
O (0,0,24) m
A (-18,-16,0) m
B (15,-20,0) m
C (4,6,0) m



$$\therefore \bar{F}_1 = (F_1)(\hat{e}_{OA}) = 600\left[\frac{(-18-0)\hat{\imath} + (-16-0)\hat{\jmath} + (0-24)\hat{k}}{\sqrt{(-18)^2 + (-16)^2 + (-24)^2}}\right]$$

$$\bar{F}_1 = (-317.6\hat{\imath} - 282.4\hat{\jmath} - 423.5\hat{k})\ \text{N}$$

$$\therefore \bar{F}_2 = (F_2)(\hat{e}_{OB}) = 400\left[\frac{(15-0)\hat{\imath} + (-20-0)\hat{\jmath} + (0-24)\hat{k}}{\sqrt{(15)^2 + (-20)^2 + (-24)^2}}\right]$$

$$\bar{F}_2 = (+173.1\hat{\imath} - 230.8\hat{\jmath} - 277\hat{k})\ \text{N}$$

$$\therefore \bar{F}_3 = (F_3)(\hat{e}_{OC}) = 800\left[\frac{(4-0)\hat{\imath} + (6-0)\hat{\jmath} + (0-24)\hat{k}}{\sqrt{(4)^2 + (6)^2 + (-24)^2}}\right]$$

$$\bar{F}_3 = (+127.7\hat{\imath} + 191.5\hat{\jmath} - 766.2\hat{k})\ \text{N}$$

Resultant force in vector form is simply given by vector addition of the forces.

$$\begin{aligned}\therefore \bar{R} = \bar{F}_1 + \bar{F}_2 + \bar{F}_3 &= (-317.6\hat{\imath} - 282.4\hat{\jmath} - 423.5\hat{k}) \\ &\quad +(+173.1\hat{\imath} - 230.8\hat{\jmath} - 277\hat{k}) \\ &\quad +(+127.7\hat{\imath} + 191.5\hat{\jmath} - 766.2\hat{k}) \\ \mathbf{\bar{R}} &= \mathbf{(-16.8\hat{\imath} - 321.7\hat{\jmath} - 1466.7\hat{k})\ N}\end{aligned}$$

N2: The lines of actions of three forces concurrent at origin O pass respectively through point A (-1,2,4), B (3,0,-3), C (2,-2,4). Force F_1 = 40 N passes through A, F_2 = 10 N passes through B, F_3 = 30 N passes through C. Find the magnitude and direction of their resultant.

Soln: In this concurrent space force system, putting the forces in vector form we get,

$$\bar{F}_1 = (F_1)(\hat{e}_{OA}) = 40\left[\frac{-1\hat{\imath} + 2\hat{\jmath} + 4\hat{k}}{\sqrt{1^2 + 2^2 + 4^2}}\right] = (-8.729\hat{\imath} + 17.457\hat{\jmath} + 34.915\hat{k})\ \text{N}$$

$$\bar{F}_2 = (F_2)(\hat{e}_{OB}) = 10\left[\frac{3\hat{\imath} + 0\hat{\jmath} - 3\hat{k}}{\sqrt{3^2 + 0^2 + 3^2}}\right] = (+7.071\hat{\imath} + 0\hat{\jmath} - 7.071\hat{k})\ \text{N}$$

$$\bar{F}_3 = (F_3)(\hat{e}_{OC}) = 30\left[\frac{2\hat{\imath} - 2\hat{\jmath} + 4\hat{k}}{\sqrt{2^2 + 2^2 + 4^2}}\right] = (+12.247\hat{\imath} - 12.247\hat{\jmath} + 24.495\hat{k})\ \text{N}$$

Resultant of these forces is,

$$\therefore \bar{R} = (-8.729\hat{\imath} + 17.457\hat{\jmath} + 34.915\hat{k}) + (+7.071\hat{\imath} + 0\hat{\jmath} - 7.071\hat{k}) + (+12.247\hat{\imath} - 12.247\hat{\jmath} + 24.495\hat{k})$$

$$\bar{R} = (+10.589\hat{\imath} + 5.27\hat{\jmath} + 52.339\hat{k})\ \text{N}$$

Magnitude of the resultant force,

$$\mathbf{R} = \sqrt{R_x^{\,2} + R_y^{\,2} + R_z^{\,2}} = \sqrt{10.589^2 + 5.27^2 + 52.339^2} = \mathbf{53.66\ N}$$

Direction of the resultant force is given by the angles θ_x, θ_y, and θ_z.

$R_x = R\cos\theta_x$

$\Rightarrow 10.589 = 53.66\cos\theta_x$

$\Rightarrow\ \boldsymbol{\theta_x = 78.62°}$

$R_y = R\cos\theta_y$

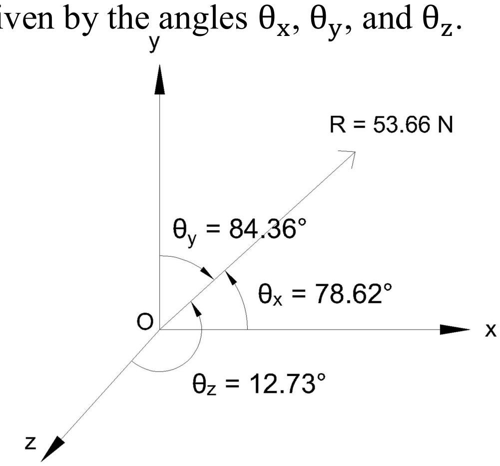
$\Rightarrow 5.27 = 53.66\cos\theta_y$

$\Rightarrow\ \boldsymbol{\theta_y = 84.36°}$

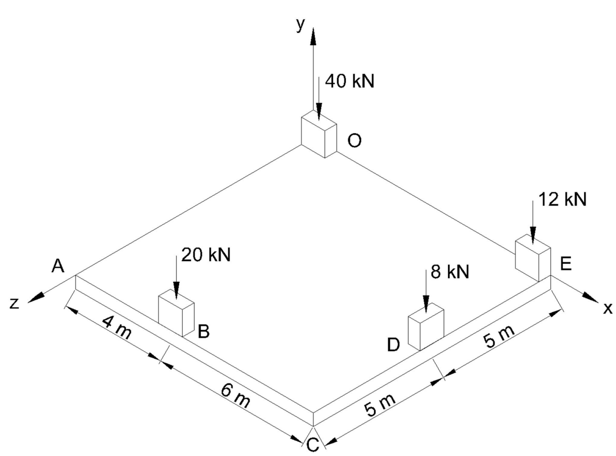
$R_z = R\cos\theta_z$

$\Rightarrow 52.339 = 53.66\cos\theta_z$

$\Rightarrow\ \boldsymbol{\theta_z = 12.73°}$



N3: A square foundation mat supports the four columns as shown. Determine the magnitude and point of application of the resultant of the four loads.



Soln: This is a parallel space force system with 4 forces. The co-ordinates of the points through which the forces act are as follows,

O (0,0,0), B (4,0,10), D (10,0,5), E (10,0,0)

Let the forces 20 kN, 8 kN, 12 kN & 40 kN be F_1, F_2, F_3 & F_4 respectively.

All the forces are parallel to y axis and in the downward direction; hence all of them will have $-\hat{j}$ in their vector forms.

$\bar{F}_1 = -20\hat{j}\ \text{kN}; \qquad \bar{F}_2 = -80\hat{j}\ \text{kN}; \qquad \bar{F}_3 = -12\hat{j}\ \text{kN}; \qquad \bar{F}_4 = -40\hat{j}\ \text{kN}$

Resultant, $\bar{R} = \bar{F}_1 + \bar{F}_2 + \bar{F}_3 + \bar{F}_4 = -20\hat{j} - 80\hat{j} - 12\hat{j} - 40\hat{j} = -80\hat{j}\ \text{kN}$

For point of application, we first need to find out the moment of all forces about a point (let's take it from origin). So, the position vectors for each force will be,

$\bar{r}_{OB} = (4-0)\hat{i} + (0-0)\hat{j} + (10-0)\hat{k} = \left(4\hat{i} + 10\hat{k}\right)\ \text{m}$

$\bar{r}_{OD} = \left(10\hat{i} + 5\hat{k}\right)\ \text{m}; \qquad \bar{r}_{OE} = \left(10\hat{i} + 0\hat{k}\right) = 10\hat{i}\ \text{m}; \qquad \bar{r}_{OO} = 0\ \text{m}$

Let resultant act at a point P (x,0,z) m. $\therefore \bar{r}_{OB} = \left(x\hat{i} + z\hat{k}\right)\ \text{m}$

Now, the moment vectors of the forces about the origin,

$$\bar{M}_O^{F_1} = \bar{r}_{OB} \times \bar{F}_1 = \left(4\hat{i} + 10\hat{k}\right) \times (-20\hat{j}) = -80(\hat{i} \times \hat{j}) - 200\left(\hat{k} \times \hat{j}\right)$$
$$\bar{M}_O^{F_1} = \left(200\hat{i} - 80\hat{k}\right)\ \text{kNm} \qquad \{\because \hat{i} \times \hat{j} = \hat{k}, \qquad \hat{k} \times \hat{j} = -\hat{i}\}$$

$$\bar{M}_O^{F_2} = \bar{r}_{OD} \times \bar{F}_2 = \left(10\hat{i} + 5\hat{k}\right) \times (-8\hat{j}) = \left(40\hat{i} - 80\hat{k}\right)\ \text{kNm}$$

$$\bar{M}_O^{F_3} = \bar{r}_{OE} \times \bar{F}_3 = (10\hat{i}) \times (-12\hat{j}) = \left(-120\hat{k}\right)\ \text{kNm}$$

$$\bar{M}_O^{F_4} = 0\ \{\because \bar{F}_4 \text{ passes through the origin}\}$$

And the moment of resultant about the origin in terms of x and z,

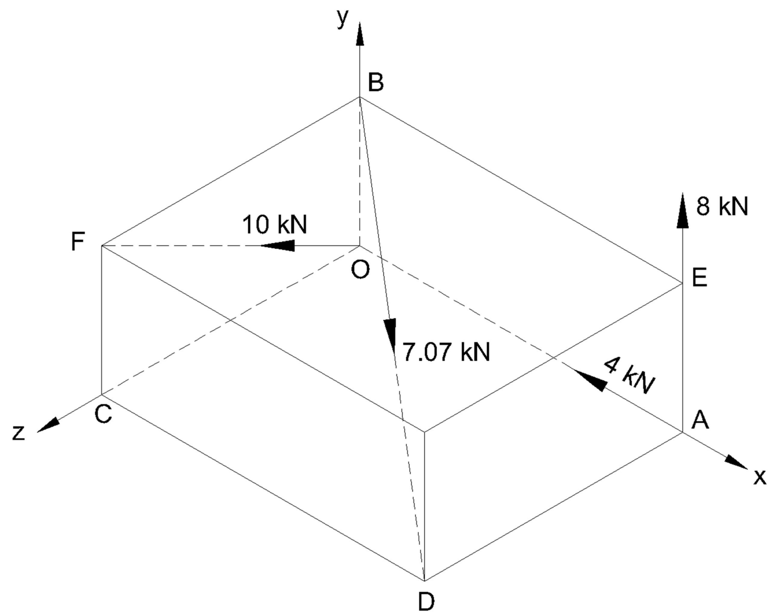
$$\bar{M}_O^R = \bar{r}_{OP} \times \bar{R} = \left(x\hat{i} + z\hat{k}\right) \times (-40\hat{j}) = \left[(80z)\hat{i} - (80x)\hat{k}\right]\ \text{kNm}$$

From Varignon's theorem,

$$\sum \bar{M}_O^F = \bar{M}_O^R \Rightarrow \bar{M}_O^{F_1} + \bar{M}_O^{F_2} + \bar{M}_O^{F_3} + \bar{M}_O^{F_4} = \bar{M}_O^R$$
$$\Rightarrow \left(200\hat{i} - 80\hat{k}\right) + \left(40\hat{i} - 80\hat{k}\right) + \left(-120\hat{k}\right) + 0 = (80z)\hat{i} - (80x)\hat{k}$$
$$\Rightarrow 240\hat{i} - 280\hat{k} = (80z)\hat{i} - (80x)\hat{k}$$
$$\Rightarrow 80z = 240\ \& \quad -80x = -280$$
$$\Rightarrow z = 3\ \text{m} \quad \& \quad x = 3.5\ \text{m}$$

Hence, the magnitude of the resultant is $\mathbf{R = 80\ kN}$ and passes through point $\mathbf{P\ (3.5, 0, 3)\ m}$.

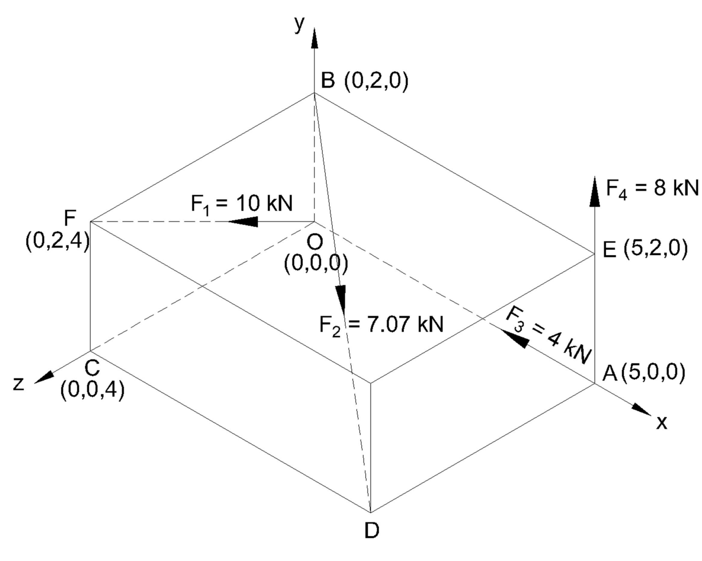
N4: A rectangle parallelepiped carries four forces as shown in the figure. Reduce the force system to a resultant force applied at the origin and moment around the origin. OA = 5 m, OB = 2 m, OC = 4 m.



Soln: The given system is a general space force system of 4 forces.

Let forces 10 kN, 7.07 kN, 4 kN & 8 kN be labelled as F_1, F_2, F_3 & F_4 respectively.

The co-ordinates of the various points through which the forces pass are:



Now, putting the forces in vector form,

$$\bar{F}_1 = (F_1)(\hat{e}_{OF}) = 10\left[\frac{0\hat{\imath} + 2\hat{\jmath} + 4\hat{k}}{\sqrt{0^2 + 2^2 + 4^2}}\right] = \left(0\hat{\imath} + 4.472\hat{\jmath} + 8.944\hat{k}\right)\text{ kN}$$

$$\bar{F}_2 = (F_2)(\hat{e}_{BD}) = 7.07\left[\frac{5\hat{\imath} - 2\hat{\jmath} + 4\hat{k}}{\sqrt{5^2 + 2^2 + 4^2}}\right] = \left(5.27\hat{\imath} - 2.108\hat{\jmath} + 4.216\hat{k}\right)\text{ kN}$$

$\bar{F}_3 = (F_3)(\hat{e}_{AO}) = 4\,(-\hat{\imath}) = (-4\hat{\imath})$ kN $\{\because$ it is along x − axes towards origin$\}$
$\bar{F}_4 = (F_4)(\hat{e}_{AE}) = 8\,(\hat{\jmath}) = (8\hat{\jmath})$ kN $\{\because$ it is along y − axes upwards$\}$

The resultant force, $\bar{R} = \bar{F}_1 + \bar{F}_2 + \bar{F}_3 + \bar{F}_4$
$\bar{R} = \left(0\hat{\imath} + 4.472\hat{\jmath} + 8.944\hat{k}\right) + \left(5.27\hat{\imath} - 2.108\hat{\jmath} + 4.216\hat{k}\right) + (-4\hat{\imath}) + (8\hat{\jmath})$
$\bar{\mathbf{R}} = \left(\mathbf{1.27\hat{\imath} + 10.364\hat{\jmath} + 13.16\hat{k}}\right)\ \mathbf{kN}$

Taking moments of all force about the origin,

$\bar{M}_O^{F_1} = 0\ \{\because \bar{F}_1$ passes through the origin$\}$

$$\bar{M}_O^{F_2} = \bar{r}_{OB} \times \bar{F}_2 = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 0 & 2 & 0 \\ 5.27 & -2.108 & 4.216 \end{vmatrix} = \left(8.432\hat{\imath} + 0\hat{\jmath} - 10.54\hat{k}\right) \text{kNm}$$

$\bar{M}_O^{F_3} = \bar{r}_{OA} \times \bar{F}_3 = 0\ \{\because \bar{r}_{OA} \,\&\, \bar{F}_3$ are along the same directions$\}$

$$\bar{M}_O^{F_4} = \bar{r}_{OA} \times \bar{F}_4 = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 5 & 0 & 0 \\ 0 & 8 & 0 \end{vmatrix} = \left(40\hat{k}\right) \text{kNm}$$

The resultant moment about the origin, $\bar{M}_O = \bar{M}_O^{F_1} + \bar{M}_O^{F_2} + \bar{M}_O^{F_3} + \bar{M}_O^{F_4}$
$\bar{M}_O = 0 + \left(8.432\hat{\imath} + 0\hat{\jmath} - 10.54\hat{k}\right) + 0 + \left(40\hat{k}\right)$
$\bar{\mathbf{M}}_{\mathbf{O}} = \left(\mathbf{8.432\hat{\imath} + 29.46\hat{k}}\right)\ \mathbf{kNm}$

Hence, the resultant force and moment at origin is,

$\bar{\mathbf{R}} = \left(\mathbf{1.27\hat{\imath} + 10.364\hat{\jmath} + 13.16\hat{k}}\right)\ \mathbf{kN}$

$\bar{\mathbf{M}}_{\mathbf{O}} = \left(\mathbf{8.432\hat{\imath} + 29.46\hat{k}}\right)\ \mathbf{kNm}$