MATRIX THEORY: RANK OF MATRIX
-
SYSTEM OF LINEAR EQUATIONS

FY BTECH SEM-I

MODULE-2

SUB-MODULE 2.4

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2	**Matrix Theory: Rank of Matrix**		**8**	**CO 2**
	2.1	Types of matrices: Hermitian, Skew-Hermitian, Unitary and Orthogonal matrix		
	2.2	Rank of a matrix using row echelon forms, reduction to normal form, and PAQ form		
	2.3	System of homogeneous and non-homogeneous equations, their consistency and solutions		
	2.4	Linearly dependent and independent vectors		
	2.5	Solution of system of linear algebraic equations by **(a)** Gauss Seidal method **(b)** Jacobi iteration method		
		#Self-learning topics: Symmetric, Skew-symmetric matrices and properties, Properties of adjoint and inverse of a matrix		

VECTORS

- **Definition:** An ordered set of n elements x_i is called n – dimensional vector or a vector of order n denoted by X.
- $X = [x_1 \ \ x_2 \ \ x_3 \ldots\ldots x_n]$ The elements $x_1, x_2, x_3, \ldots.., x_n$ are called components of X.
- X is denoted by row matrix or column matrix.
- It is more convenient to denote it as column matrix
- $X = [x_1 \ \ x_2 \ \ x_3 \ldots \ldots x_n]^T = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \ldots \\ x_n \end{bmatrix}$
- The vector, all of whose components are zero, is called a zero or null vector and is denoted by 0.

VECTORS

- LINEARLY DEPENDENT VECTORS
- **Definition:** The set of vectors $X_1,\ X_2, X_3, \dots .X_m$ is said to be **Linearly Dependent** if there exist m scalars $k_1, k_2, k_3 \dots .k_m$, not all zero, such that $k_1X_1 + k_2X_2 + k_3X_3 + \cdots . + k_mX_m = 0$
- If $k_1 \neq 0$, then
- $-k_1X_1 = k_2X_2 + k_3X_3 + \cdots . + k_mX_m$
- $\therefore X_1 = \mu_2X_2 + \mu_3X_3 + \cdots .. + \mu_mX_m$
- where $\mu_i = -\frac{k_i}{k_1};\quad i = 2,3,4, \dots .., m$
- X_1 is expressed as linear combination of $X_2, \dots .X_m$
- **Note:** if the set of vectors $X_1, X_2, \dots \dots X_m$ is linearly dependent then any one vectors can be expressed as the linear combination of other vectors.

LINEARLY INDEPENDENT VECTORS

- **Definition:** The set of vectors $X_1, X_2, X_3, \dots .X_m$ is said to be **Linearly Independent** if they are not dependent.
- i.e., if $k_1X_1 + k_2X_2 + k_3X_3 + \cdots .. + k_mX_m = O$
- $\Rightarrow k_i = 0$ for all $i = 1,2, \dots .., m$
- then $X_1, X_2, X_3 \dots \dots X_m$ are said to be Linearly Independent.

VECTORS

- For any given set of vectors $X_1, X_2, \dots \dots X_m,$
- $k_1X_1 + k_2X_2 + k_3X_3 + \cdots .. + k_mX_m = O$ will form a homogeneous system of equation, say AX = 0. Here unknowns are $k_1, k_2, \dots . k_m$ and coefficient matrix is made up of vectors $X_1, X_2, \dots \dots X_m$ as columns.
- Apply **row transformations only** on A and reduce the coefficient matrix A to **row echelon form**. Then find the rank A (r)
- **Case(i):** when rank A (r) is equal to number of variables then system has only trivial solution and vectors are independent.
- **Case(ii):** when rank A (r) is less than number of variables then system has infinite non-trivial solutions and they can be obtained by assigning $n - r$ variables as parameter and vectors are dependent.

Example 1

- Are the vector $X_1 = [1\ \ 3\ \ 4\ \ 2]$, $X_2 = [3\ \ -5\ \ 2\ \ 6], X_3 = [2\ \ -1\ \ 3\ \ 4]$ linearly dependent? If so, express X_1 as a linear combination of the others.
- **Solution:** Consider the matrix equation $k_1X_1 + k_2X_2 + k_3X_3 = 0$ .... (i)
- $k_1[1\ 3\ 4\ 2] + k_2[3\ \ -5\ 2\ 6] + k_3[2\ \ -1\ 3\ 4] = [0\ 0\ 0\ 0]$
- $\therefore\ k_1 + 3k_2 + 2k_3 = 0, 3k_1 - 5k_2 - k_3 = 0,$
- $4k_1 + 2k_2 + 3k_3 = 0, 2k_1 + 6k_2 + 4k_3 = 0$
- which can be written in matrix form as

$$\begin{bmatrix} 1 & 3 & 2 \\ 3 & -5 & -1 \\ 4 & 2 & 3 \\ 2 & 6 & 4 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

- Applying $R_2 - 3R_1, R_3 - 4R_1, R_4 - 2R_1$, we get

$$\begin{bmatrix} 1 & 3 & 2 \\ 0 & -14 & -7 \\ 0 & -10 & -5 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

- Applying $(-1/7)R_2, (-1/5)R_3$, we get

$$\begin{bmatrix} 1 & 3 & 2 \\ 0 & 2 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

- Applying $R_3 - R_2$, we get

$$\begin{bmatrix} 1 & 3 & 2 \\ 0 & 2 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Example 1 (contd..)

- $k_1 + 3k_2 + 2k_3 = 0, \quad 2k_2 + k_3 = 0$
- if we put $k_3 = -2t$, we get $k_2 = t, k_1 = t$
- Now, from (i), we get, $tX_1 + tX_2 - 2tX_3 = 0$
  $\therefore X_1 + X_2 - 2X_3 = 0$
- Since k_1, k_2, k_3 are not all zero, the vectors are linearly dependent and $X_1 = -X_2 + 2X_3$

Example 2

- Examine whether the vectors

$X_1 = [3 \quad 1 \quad 1], X_2 = [2 \quad 0 \quad -1], X_3 = [4 \quad 2 \quad 1]$ are linearly dependent or independent.

- **Solution:** Consider the matrix equation $k_1X_1 + k_2X_2 + k_3X_3 = 0$................(i)
- $\therefore 3k_1 + 2k_2 + 4k_3 = 0 , \qquad k_1 + 0k_2 + 2k_3 = 0, \quad k_1 - k_2 + k_3 = 0$
- This is a homogeneous system of equations
$$\begin{bmatrix} 3 & 2 & 4 \\ 1 & 0 & 2 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$
- By $R_{13,}$ $we\ get$
$$\begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & 2 \\ 3 & 2 & 4 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$
- By $R_2 - R_1, \ R_3 - 3R_1$, we get
$$\begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & 1 \\ 0 & 5 & 1 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$
- By $R_3 - 5R_2$, we get $\begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & -4 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$
- $\Rightarrow k_1 - k_2 + k_3 = 0 , \quad k_2 + k_3 = 0, \quad -4k_3 = 0$
- $\Rightarrow k_3 = 0$ and hence $k_2 = 0$ and $k_1 = 0$
- Thus non zero values of k_1, k_2, k_3 do not exist which can satisfy equation (i).
- Hence by definition the given system of vectors is linearly independent.

Example 3

- Show that the vectors X_1, X_2, X_3 are linearly independent and vector X_4 depends upon them, where, $X_1 = [1\ 2\ 4\ ]$, $X_2 = [2 - 1\ 3]$, $X_3 = [0\ 1\ \ 2]$, $X_4 = [-3\ 7\ 2]$
- **Solution:** consider $k_1X_1 + k_2X_2 + k_3X_3 = 0$
- $\therefore k_1[1\ \ 2\ \ 4] + k_2[2\ \ -1\ \ 3] + k_3[0\ \ 1\ \ 2] = [0\ \ 0\ \ 0]$
- $\therefore k_1 + 2k_3 + 0k_3 = 0, \quad 2k_1 - k_2 + k_3 = 0, \quad 4k_1 + 3k_2 + 2k_3 = 0,$
- $\therefore \begin{bmatrix} 1 & 2 & 0 \\ 2 & -1 & 1 \\ 4 & 3 & 2 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$
- Applying $R_2 - 2R_1, R_3 - 4R_1$, we get

$$\begin{bmatrix} 1 & 2 & 0 \\ 0 & -5 & 1 \\ 0 & -5 & 2 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

- Applying $R_3 - R_2$ we get
$$\begin{bmatrix} 1 & 2 & 0 \\ 0 & -5 & 1 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$
- $\therefore k_1 + 2k_2 = 0, \quad -5k_2 + k_3 = 0, \quad k_3 = 0$
- $\therefore k_3 = 0, \ \ k_2 = 0, \ \ k_1 = 0.$
- Since, the rank = 3 = the number of unknowns, $\therefore$ only trivial solution is possible.
- $\therefore X_1, X_2, X_3$ are linearly independent.

Example 3 (contd…)

- Now, consider the matrix equation
- $k_1X_1 + k_2X_2 + k_3X_3 + k_4X_4 = 0$ ....(i)
- $\therefore k_1[1\ 2\ 4] + k_2[2\ -1\ 3] + k_3[0\ 1\ 2] + k_4[-3\ \ 7\ \ 2] = 0$
- $\therefore k_1 + 2k_2 + 0k_3 - 3k_4 = 0, \quad 2k_1 - k_2 + k_3 + 7k_4 = 0, \quad 4k_1 + 3k_2 + 2k_3 + 2k_4 = 0$
- $\therefore \begin{bmatrix} 1 & 2 & 0 & -3 \\ 2 & -1 & 1 & 7 \\ 4 & 3 & 2 & 2 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \\ k_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$
- Applying $R_2 - 2R_1, R_3 - 4R_1$, we get

$$\begin{bmatrix} 1 & 2 & 0 & -3 \\ 0 & -5 & 1 & 13 \\ 0 & -5 & 2 & 14 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \\ k_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

- Applying $R_3 - R_2$ we get

$$\begin{bmatrix} 1 & 2 & 0 & -3 \\ 0 & -5 & 1 & 13 \\ 0 & 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} k_1 \\ k_2 \\ k_3 \\ k_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

- $\therefore k_1 + 2k_2 - 3k_4 = 0, \quad -5k_2 + k_3 + 13k_4 = 0$, $k_3 + k_4 = 0$ Let $k_4 = t$ $\therefore k_3 = -t$
- $\therefore -5k_2 - t + 13t = 0 \qquad \therefore k_2 = \frac{12}{5}t$
- $\therefore k_1 + \frac{24}{5}t - 3t = 0 \qquad \therefore k_1 = -\frac{9}{5}t$
- Putting the values of k_1, k_2, k_3, k_4 in (i) we get, $-\frac{9}{5}tX_1 + \frac{12}{5}tX_2 - tX_3 + tX_4 = 0$
- $\therefore 9X_1 - 12X_2 + 5X_3 - 5X_4 = 0$
- Hence, X_1, X_2, X_3, X_4 are linearly dependent and $X_4 = \frac{9}{5}X_1 - \frac{12}{5}X_2 + X_3$