Problems based on Algebraic Functions

Monday, March 8, 2021 11:30 AM

① Find the n^{th} derivative of $\dfrac{x}{(x-1)(x-2)(x-3)}$

Solⁿ :- Let $y = \dfrac{x}{(x-1)(x-2)(x-3)} = \dfrac{a}{x-1} + \dfrac{b}{x-2} + \dfrac{c}{x-3}$

$\therefore x = a(x-2)(x-3) + b(x-1)(x-3) + c(x-1)(x-2)$

put $x=1$, $1 = a(-1)(-2) \Rightarrow a = 1/2$

put $x=2$, $2 = b(1)(-1) \Rightarrow b = -2$

put $x=3$, $3 = c(2)(1) \Rightarrow c = 3/2$

$\therefore \quad y = \frac{1}{2}\cdot\left(\frac{1}{x-1}\right) - 2\cdot\left(\frac{1}{x-2}\right) + \frac{3}{2}\left(\frac{1}{x-3}\right)$

By result : $y = \dfrac{1}{ax+b}$ then $y_n = \dfrac{(-1)^n \cdot n! \, a^n}{(ax+b)^{n+1}}$

we get $y_n = \frac{1}{2}\cdot\left[\frac{(-1)^n n! (1)^n}{(x-1)^{n+1}}\right] - 2\left[\frac{(-1)^n n! (1)^n}{(x-2)^{n+1}}\right]$

$+ \frac{3}{2}\left[\frac{(-1)^n n! (1)^n}{(x-3)^{n+1}}\right]$

$= \frac{(-1)^n n!}{2}\left\{\left(\frac{1}{(x-1)^{n+1}}\right) - 4\left(\frac{1}{(x-2)^{n+1}}\right) + 3\left(\frac{1}{(x-3)^{n+1}}\right)\right\}$

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② Find the n^{th} derivative of $\dfrac{x^2}{(x+2)(2x+3)}$

Solⁿ x^2

$(x+2)(2x+3)$

Soln :- $y = \dfrac{x^2}{(x+2)(2x+3)}$

we have to express the given expression in terms of partial fractions.

Since the degree of numerator is equal to the degree of denominator, we first divide the numerator by the denominator and then obtain the partial fraction

$$y = \frac{x^2}{2x^2+7x+6} = \frac{1}{2}\left[1 - \frac{7x+6}{2x^2+7x+6}\right]$$

Now $\dfrac{7x+6}{2x^2+7x+6} = \dfrac{A}{x+2} + \dfrac{B}{2x+3}$

By solving $A = 8, \quad B = -9$

$$y = \frac{1}{2}\left[1 - \frac{8}{x+2} + \frac{9}{2x+3}\right] = \frac{1}{2} - 4\left(\frac{1}{x+2}\right) + \frac{9}{2}\left(\frac{1}{2x+3}\right)$$

By Result : $y = \dfrac{1}{ax+b}$ then $y = \dfrac{(-1)^n n!\, a^n}{(ax+b)^{n+1}}$

$$y_n = 0 - 4\left[\frac{(-1)^n n!\,(1)^n}{(x+2)^{n+1}}\right] + \frac{9}{2}\left[\frac{(-1)^n n!\, 2^n}{(2x+3)^{n+1}}\right]$$

$$= \frac{(-1)^n n!}{2}\left[\frac{9(2)^n}{(2x+3)^{n+1}} - \frac{8}{(x+2)^{n+1}}\right]$$

$$\overline{\quad 2 \quad} \left[\frac{}{(2x+3)^{n+1}} \quad \frac{}{(x+2)^{n+1}} \right]$$

③ $y = \dfrac{1}{1+x+x^2+x^3}$ find y_n

sol^n:- $y = \dfrac{1}{(1+x)+x^2(1+x)} = \dfrac{1}{(1+x)(1+x^2)}$

$$y = \frac{1}{(x+1)(x+i)(x-i)} = \frac{a}{x+1} + \frac{b}{x+i} + \frac{c}{x-i}$$

$$1 = a(x+i)(x-i) + b(x+1)(x-i) + c(x+1)(x+i)$$

put $x=-1$, $1 = a(-1+i)(-1-i) \Rightarrow a = \frac{1}{2}$

put $x = i$, $1 = c(i+1)(2i) \Rightarrow c = \dfrac{1}{2i(1+i)} = \dfrac{-1}{4}(i+1)$

put $x = -i$, $1 = b(-i+1)(-2i) \Rightarrow b = \dfrac{1}{-2i(1-i)} = \dfrac{i-1}{4}$

$$y = \frac{1}{2}\left(\frac{1}{x+1}\right) + \frac{i-1}{4}\left(\frac{1}{x+i}\right) - \frac{(i+1)}{4}\left(\frac{1}{x-i}\right)$$

By result: $y = \dfrac{1}{ax+b}$, $y_n = \dfrac{(-1)^n n!\, a^n}{(ax+b)^{n+1}}$

$$y_n = \frac{1}{2}\left[\frac{(-1)^n n!}{(x+1)^{n+1}}\right] + \frac{(i-1)}{4}\left[\frac{(-1)^n n!}{(x+i)^{n+1}}\right] - \frac{(i+1)}{4}\left[\frac{(-1)^n n!}{(x-i)^{n+1}}\right]$$

④ $y = \dfrac{8x}{x^3-2x^2-4x+8}$ find y_n

Soln :- $y = \dfrac{8x}{x^2(x-2)-4(x-2)} = \dfrac{8x}{(x-2)(x^2-4)} = \dfrac{8x}{(x+2)(x-2)^2}$

$$y = \frac{8x}{(x+2)(x-2)^2} = \frac{A}{x+2} + \frac{B}{(x-2)^2} + \frac{C}{x-2}$$

$$8x = A(x-2)^2 + B(x+2) + C(x+2)(x-2)$$

put $x=2$, $8(2) = 4B$ $\quad \therefore B = 4$

put $x=-2$, $8(-2) = A(-4)^2 \Rightarrow A = -1$

put $x=0$, $0 = A(4) + B(2) + C(-4)$

$0 = -4 + 8 - 4C$

$4C = 4 \Rightarrow C = 1$

$$y = \frac{-1}{x+2} + \frac{4}{(x-2)^2} + \frac{1}{x-2}$$

By result : $y = \dfrac{1}{ax+b}$ then $y_n = \dfrac{(-1)^n n!\, a^n}{(ax+b)^{n+1}}$

$y = \dfrac{1}{(ax+b)^m}$ then $y_n = \dfrac{(-1)^n (m+n-1)!}{(m-1)!} \cdot \dfrac{a^n}{(ax+b)^{m+n}}$

$$y_n = -1\left[\frac{(-1)^n n!}{(x+2)^{n+1}}\right] + 4\left[\frac{(-1)^n (2+n-1)!}{(2-1)!} \frac{1}{(x-2)^{n+2}}\right] + \frac{(-1)^n n!}{(x-2)^{n+1}}$$

$$= 4\left[\frac{(-1)^n (n+1)!}{\quad}\right] + \frac{(-1)^n n!}{\quad} - \frac{(-1)^n n!}{\quad}$$

$$= 4\left[\frac{(-1)^n (n+1)!}{(x-2)^{n+2}}\right] + \frac{(-1)^n n!}{(x-2)^{n+1}} - \frac{(-1)^n n!}{(x+2)^{n+1}}$$

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(5) $y = \dfrac{x}{(x+1)^4}$ find y_n

$\underline{\text{soln}}$:- $y = \dfrac{x}{(x+1)^4} = \dfrac{(x+1)-1}{(x+1)^4} = \dfrac{1}{(x+1)^3} - \dfrac{1}{(x+1)^4}$

By the result:

$y = \dfrac{1}{(ax+b)^m}$ then $y_n = (-1)^n \dfrac{(m+n-1)!}{(m-1)!} \dfrac{a^n}{(ax+b)^{m+n}}$

put $m = 3, 4$ in this formula $a=1$, $b=1$

$$y_n = (-1)^n \frac{(3+n-1)!}{(3-1)!} \frac{(1)^n}{(x+1)^{n+3}} - (-1)^n \frac{(4+n-1)!}{(4-1)!} \frac{(1)^n}{(x+1)^{n+4}}$$

$$= (-1)^n \frac{(n+2)!}{2} \cdot \frac{1}{(x+1)^{n+3}} - (-1)^n \frac{(n+3)!}{6} \cdot \frac{1}{(x+1)^{n+4}}$$

$$= \frac{(-1)^n (n+2)!}{6(x+1)^{n+4}} (3x-n)$$

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(6) prove that the value of n^{th} differential coefficient of $\dfrac{x^3}{x^2-1}$ for $x=0$ is 0 if n is even and is $-n!$ if n is odd and greater than 1.

Sol^n :- $y = \frac{x^3}{x^2-1} = \frac{x^3 - x + x}{x^2-1} = x + \frac{x}{x^2-1}$

$$\frac{x}{x^2-1} = \frac{a}{x-1} + \frac{b}{x+1} \Rightarrow a = \frac{1}{2},\ b = \frac{1}{2}$$

$$y = x + \frac{1}{2}\left[\frac{1}{x-1} + \frac{1}{x+1}\right]$$

By result : $y = \frac{1}{ax+b}$, $y_n = \frac{(-1)^n n!\, a^n}{(ax+b)^{n+1}}$

$$y_n = 0 + \frac{1}{2}\left[\frac{(-1)^n n!}{(x-1)^{n+1}} + \frac{(-1)^n n!}{(x+1)^{n+1}}\right]$$

putting $x=0$,

$$y_n(0) = \frac{1}{2}\left[\frac{(-1)^n n!}{(-1)^{n+1}} + \frac{(-1)^n n!}{(1)^{n+1}}\right] \quad \text{—— (1)}$$

If n is even, $(n+1)$ is odd

$$y_n(0) = \frac{1}{2}\left[\frac{1 \cdot n!}{(-1)} + \frac{1 \cdot n!}{1}\right] = \frac{n!}{2}\left[-1+1\right]$$

$\therefore y_n(0) = 0$ when n is even

If n is odd, $(n+1)$ is even using (1)

$$y_n(0) = \frac{1}{2}\left[\frac{(-1) n!}{(1)} + \frac{(-1) n!}{(1)}\right]$$

$$= \frac{n_0^1}{2}\left[-1-1\right] = -n_0^1$$

$y_n(0) = -n_0^1$ when n is odd $n > 1$