

Maclaurin's Series

Wednesday, March 17, 2021 11:30 AM

we know that using Taylor's series expansion, we can write

$$f(x+h) = f(h) + x f'(h) + \frac{x^2}{2!} f''(h) + \frac{x^3}{3!} f'''(h) + \dots + \frac{x^n}{n!} f^{(n)}(h) + \dots \infty$$

If we put $h=0$ in the above expansion, we will get

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \dots + \frac{x^n}{n!} f^{(n)}(0) + \dots \infty$$

This series is known as Maclaurin's Series

Using Maclaurin's Series, we can find expansion of many standard functions

① Expansion of $\sin x$

$$\text{let } f(x) = \sin x$$

$$f'(x) = \cos x$$

$$f''(x) = -\sin x$$

$$f'''(x) = -\cos x$$

$$f^{(iv)}(x) = \sin x$$

$$f^{(v)}(x) = \cos x$$

$$f(0) = \sin 0 = 0$$

$$f'(0) = \cos 0 = 1$$

$$f''(0) = 0$$

$$f'''(0) = -1$$

$$f^{(iv)}(0) = 0$$

$$f^{(v)}(0) = 1$$

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \frac{x^4}{4!} f^{(iv)}(0) + \frac{x^5}{5!} f^{(v)}(0) + \dots$$

$$\sin x = 0 + x(1) + \frac{x^2}{2!}(0) + \frac{x^3}{3!}(-1) + \frac{x^4}{4!}(0) + \frac{x^5}{5!}(1) + \dots$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots$$

$$\text{similarly } \cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots$$

Similarly $\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots$

H.W $\tanh x = x + \frac{x^3}{3} + \frac{2x^5}{15} + \dots$

Expansion of e^x

$$f(x) = e^x, \quad f'(x) = e^x, \quad f''(x) = e^x, \quad f'''(x) = e^x$$

$$f(0) = 1, \quad f'(0) = 1, \quad f''(0) = 1, \quad f'''(0) = 1$$

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \dots$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$

Similarly replace x by $-x$

$$e^{-x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \frac{x^4}{4!} + \dots$$

also, $a^x = e^{x(\log a)}$

$$a^x = 1 + x(\log a) + \frac{x^2}{2!} (\log a)^2 + \frac{x^3}{3!} (\log a)^3 + \dots$$

* Expansion of $\sinh x$

$$\sinh 0 = \frac{e^0 - e^{-0}}{2} = 0$$

let $f(x) = \sinh x, \quad f(0) = 0$

$$f'(x) = \cosh x, \quad f'(0) = 1$$

$$\cosh 0 = \frac{e^0 + e^{-0}}{2} = 1$$

$$f''(x) = \sinh x, \quad f''(0) = 0$$

$$f'''(x) = \cosh x$$

$$f'''(0) = 1$$

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \dots$$

$$= 0 + x + \frac{x^2}{2!} (0) + \frac{x^3}{3!} + \dots$$

$$\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \dots$$

$$\text{Similarly } \cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \dots$$

$$\text{H.W } \tanh x = x - \frac{x^3}{3} + \frac{2x^5}{15} + \dots$$

Expansion of $\log(1+x)$

$$\text{Let } f(x) = \log(1+x)$$

$$f(0) = \log(1) = 0$$

$$f'(x) = \frac{1}{1+x}$$

$$f'(0) = 1$$

$$f''(x) = \frac{-1}{(1+x)^2}$$

$$f''(0) = -1$$

$$f'''(x) = \frac{2}{(1+x)^3}$$

$$f'''(0) = 2$$

$$f^{(iv)}(x) = \frac{-6}{(1+x)^4}$$

$$f^{(iv)}(0) = -6$$

$$f^{(v)}(x) = \frac{24}{(1+x)^5}$$

$$f^{(v)}(0) = 24$$

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \frac{x^4}{4!} f^{(iv)}(0) + \dots$$

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \frac{x^4}{4!} f^{(iv)}(0) + \dots$$

$$= 0 + x(1) + \frac{x^2}{2!}(-1) + \frac{x^3}{3!}(2) + \frac{x^4}{4!}(-6) + \frac{x^5}{5!}(24) + \dots$$

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \frac{x^5}{5} - \dots$$

Replacing x by $-x$

$$\log(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \frac{x^5}{5} - \dots$$

* Expansion of $\tanh^{-1}x$

$$\tanh^{-1}x = \frac{1}{2} \log\left(\frac{1+x}{1-x}\right)$$

$$= \frac{1}{2} \left[\log(1+x) - \log(1-x) \right]$$

$$= \frac{1}{2} \left[x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} - \dots + x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \dots \right]$$

$$\tanh^{-1}x = x + \frac{x^3}{3} + \frac{x^5}{5} + \dots$$

$$\sin^{-1}x = x + \frac{1}{2} \frac{x^3}{3} + \frac{1}{2} \cdot \frac{3}{4} \cdot \frac{x^5}{5} + \dots$$

$$\cos^{-1}x = \frac{\pi}{2} - \left[x + \frac{1}{2} \frac{x^3}{3} + \frac{1}{2} \cdot \frac{3}{4} \frac{x^5}{5} + \dots \right]$$

$$\tan^{-1}x = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \dots$$

$$\sinh^{-1}x = x - \frac{1}{2} \frac{x^3}{3} + \frac{1}{2} \frac{3}{4} \frac{x^5}{5} - \dots$$

Expansion of $(1+x)^m$

$$f(x) = (1+x)^m$$

$$f(0) = 1$$

$$f'(x) = m(1+x)^{m-1}$$

$$f'(0) = m$$

$$f''(x) = m(m-1)(1+x)^{m-2}$$

$$f''(0) = m(m-1)$$

$$f'''(x) = m(m-1)(m-2)(1+x)^{m-3}$$

$$f'''(0) = m(m-1)(m-2)$$

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \dots$$

$$(1+x)^m = 1 + mx + \frac{m(m-1)}{2!} x^2 + \frac{m(m-1)(m-2)}{3!} x^3 + \dots$$

If m is positive integer, we get finite number of term in the rhs of above series.

If $m = -1$, we get from the above expansion

$$(1+x)^{-1} = \frac{1}{1+x} = 1 - x + \frac{x^2}{2!} - \frac{x^3}{3!} + \dots$$

If we change x to $-x$ in above expansion

$$(1-x)^{-1} = \frac{1}{(1-x)} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$