MATRIX THEORY: RANK OF MATRIX
-
ORTHOGONAL & UNITARY MATRICES

FY BTECH SEM-I

MODULE-2

SUB-MODULE 2.1

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ORTHOGONAL MATRIX

- **Definition:** A real square matrix A is called orthogonal

if $AA^T = A^TA = I$

- **Properties:**
- If A is orthogonal matrix then $|A| = \pm 1$

Proof: Since A is orthogonal, $AA^T = I$. taking determinants,

$|AA^T| = |I|$

$\therefore |A||A^T| = |I|$ (Since $|AB| = |A||B|$)

$\therefore |A||A| = 1$ $|A^T| = |A|$ & $|I| = 1$

$\therefore |A|^2 = 1$ $\therefore |A| = \pm 1$

- If A is orthogonal then $A^{-1} = A^T$
- If A is orthogonal then A^{-1}, A^T are also orthogonal.

UNITARY MATRIX

- **Definition:** A square matrix A is called unitary if $AA^{\theta} = A^{\theta}A = I$
- **Properties:**
- If A is Unitary then $A^{-1} = A^{\theta}$
- If A is unitary matrix of order n then A^{T} unitary.

Proof: Since A is unitary, $A^{\theta}A = I$

taking transpose on both sides, $\left(A^{\theta}A\right)^{T} = I^{T}$ $\quad A^{T}\left(A^{\theta}\right)^{T} = I$

$\therefore (A^{T})(A^{T})^{\theta} = I$ Hence, A^{T} is unitary

- If A and B are unitary matrices of order n then$A^{-1}, A^{\theta}, AB \text{ and } BA$ are also unitary.

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UNITARY MATRIX

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- If A is unitary matrix then Its determinant is of unit modulus
- **Proof:** Since A is unitary, $AA^{\theta} = I$ taking determinant

$$\left|AA^{\theta}\right| = |I|$$

$\therefore |A|\left|A^{\theta}\right| = |I|$ (Since $|AB| = |A||B|$)

$\therefore |A||(\bar{A})^{T}| = |I|$

$\therefore |A||\bar{A}| = 1$ (Since $|A^{T}| = |A|,$ $|I| = 1$)

$\therefore |A|\overline{|A|} = 1$ ($|\bar{A}| = \overline{|A|}$: check)

Now, we know that for complex number z, $z\bar{z} = (mod\ z)^{2}$

Hence, $(mod\ |A|)^{2} = 1$ $\therefore mod\ |A| = \pm 1,$

But modulus is never negative. $\therefore mod\ |A| = 1$

i.e. determinant of unitary matrix is of unit modulus.

Example 1

- Prove that following matrix is orthogonal and hence find A^{-1},

$$A = \begin{bmatrix} \cos\alpha & 0 & \sin\alpha \\ 0 & 1 & 0 \\ -\sin\alpha & 0 & \cos\alpha \end{bmatrix}$$

Soln: $A^T = \begin{bmatrix} \cos\alpha & 0 & -\sin\alpha \\ 0 & 1 & 0 \\ \sin\alpha & 0 & \cos\alpha \end{bmatrix}$

$$\therefore \quad AA^T = \begin{bmatrix} \cos\alpha & 0 & \sin\alpha \\ 0 & 1 & 0 \\ -\sin\alpha & 0 & \cos\alpha \end{bmatrix} \begin{bmatrix} \cos\alpha & 0 & -\sin\alpha \\ 0 & 1 & 0 \\ \sin\alpha & 0 & \cos\alpha \end{bmatrix}$$

$$= \begin{bmatrix} \cos^2\alpha + \sin^2\alpha & 0 & -\cos\alpha\sin\alpha + \cos\alpha\sin\alpha \\ 0 & 1 & 0 \\ -\cos\alpha\sin\alpha + \cos\alpha\sin\alpha & 0 & \sin^2\alpha + \cos^2\alpha \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$

Also, $A^T A$

$$= \begin{bmatrix} \cos\alpha & 0 & -\sin\alpha \\ 0 & 1 & 0 \\ \sin\alpha & 0 & \cos\alpha \end{bmatrix} \begin{bmatrix} \cos\alpha & 0 & \sin\alpha \\ 0 & 1 & 0 \\ -\sin\alpha & 0 & \cos\alpha \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= I$$

Since $A^T A = AA^T$, A is orthogonal.

For orthogonal matrix

$$A^{-1} = A^T = \begin{bmatrix} \cos\alpha & 0 & -\sin\alpha \\ 0 & 1 & 0 \\ \sin\alpha & 0 & \cos\alpha \end{bmatrix}$$

Example 2

- Is the following matrix orthogonal? If not, can it be converted into orthogonal matrix?

$$A = \begin{bmatrix} \sqrt{2} & 1 & \sqrt{3} \\ \sqrt{2} & -2 & 0 \\ \sqrt{2} & 1 & -\sqrt{3} \end{bmatrix}$$

Soln: $A^T = \begin{bmatrix} \sqrt{2} & \sqrt{2} & \sqrt{2} \\ 1 & -2 & 1 \\ \sqrt{3} & 0 & -\sqrt{3} \end{bmatrix}$

$$\therefore \quad AA^T = \begin{bmatrix} \sqrt{2} & 1 & \sqrt{3} \\ \sqrt{2} & -2 & 0 \\ \sqrt{2} & 1 & -\sqrt{3} \end{bmatrix} \begin{bmatrix} \sqrt{2} & \sqrt{2} & \sqrt{2} \\ 1 & -2 & 1 \\ \sqrt{3} & 0 & -\sqrt{3} \end{bmatrix}$$

$$= \begin{bmatrix} 2+1+3 & 2-2 & 2+1-3 \\ 2-2 & 2+4 & 2-2 \\ 2+1-3 & 2-2 & 2+1+3 \end{bmatrix}$$

$$AA^T = \begin{bmatrix} 6 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 6 \end{bmatrix} = 6I \neq I$$

Thus Given matrix A is not orthogonal.

But it can be converted into an orthogonal matrix as follow

$$\therefore AA^T = 6I \qquad \therefore \frac{1}{6} AA^T = I$$

$$\therefore \left(\frac{1}{\sqrt{6}} A\right) . \left(\frac{1}{\sqrt{6}} A^T\right) = I$$

Hence $\frac{1}{\sqrt{6}} A = \frac{1}{\sqrt{6}} \begin{bmatrix} \sqrt{2} & 1 & \sqrt{3} \\ \sqrt{2} & -2 & 0 \\ \sqrt{2} & 1 & -\sqrt{3} \end{bmatrix}$ is the orthogonal matrix.

Example 3

- If $A = \frac{1}{3}\begin{bmatrix} 1 & 2 & a \\ 2 & 1 & b \\ 2 & -2 & c \end{bmatrix}$ is orthogonal, then find a, b, c

Soln: Consider $A^T = \frac{1}{3}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ a & b & c \end{bmatrix}$

Since A is orthogonal we have, $AA^T = I$

$$\therefore AA^T = \frac{1}{9}\begin{bmatrix} 1 & 2 & a \\ 2 & 1 & b \\ 2 & -2 & c \end{bmatrix}\begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & -2 \\ a & b & c \end{bmatrix}$$

$$= \frac{1}{9}\begin{bmatrix} 5+a^2 & 4+ab & -2+ac \\ 4+ab & 5+b^2 & 2+bc \\ -2+ac & 2+bc & 8+c^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Comparing we get total 6 distinct equations,

Example 3(Contd..)

- $\frac{5+a^2}{9} = 1 \quad \therefore a^2 = 9 - 5 = 4 \quad \therefore a = \pm 2,$
- $\frac{5+b^2}{9} = 1 \quad \therefore b^2 = 9 - 5 = 4 \quad \therefore b = \pm 2$
- $\frac{8+c^2}{9} = 1 \quad \therefore c^2 = 9 - 8 = 1 \quad \therefore c = \pm 1$
- $4 + ab = 0 \quad \therefore ab = -4 \quad \rightarrow when\ a = +2, \quad b = -2 \quad \&$
  $when\ a = -2, \quad b = +2$
- Also, $-2 + ac = 0 \quad \therefore ac = 2 \quad \rightarrow when\ a = +2, \quad c = +1$
  $and\ when\ a = -2, \quad c = -1$
- Hence (2, -2, 1) and (-2, 2, -1) are the required pairs.

- **(Note Observation)** For orthogonal matrix, column vectors are orthonormal to each other

Example 4

Show matrix A is unitary and hence find A^{-1} where $A = \begin{bmatrix} \frac{2+i}{3} & \frac{2i}{3} \\ \frac{2i}{3} & \frac{2-i}{3} \end{bmatrix}$

- **Soln:** T. S. T. A is unitary, we have to show $AA^{\theta} = I$
- Consider $A = \frac{1}{3}\begin{bmatrix} 2+i & 2i \\ 2i & 2-i \end{bmatrix}$ $\therefore A^{\theta} = \frac{1}{3}\begin{bmatrix} 2-i & -2i \\ -2i & 2+i \end{bmatrix}$
- $AA^{\theta} = \frac{1}{9}\begin{bmatrix} 2+i & 2i \\ 2i & 2-i \end{bmatrix}\begin{bmatrix} 2-i & -2i \\ -2i & 2+i \end{bmatrix}$
- $= \frac{1}{9}\begin{bmatrix} (2+i)(2-i)-(2i)(2i) & -2i(2+i)+2i(2+i) \\ 2i(2+i)-2i(2+i) & -(2i)(2i)+(2-i)(2+i) \end{bmatrix} = \frac{1}{9}\begin{bmatrix} 5+4 & 0 \\ 0 & 5+4 \end{bmatrix}$
- $= I$
- Hence the given matrix is unitary. For unitary matrix $A^{-1} = A^{\theta} = \frac{1}{3}\begin{bmatrix} 2-i & -2i \\ -2i & 2+i \end{bmatrix}$

Example 5

Show matrix A is unitary and hence find A^{-1} where, $A = \frac{1}{2}\begin{bmatrix} \sqrt{2} & -i\sqrt{2} & 0 \\ i\sqrt{2} & -\sqrt{2} & 0 \\ 0 & 0 & 2 \end{bmatrix}$

Soln: $A^{\theta} = \frac{1}{2}\begin{bmatrix} \sqrt{2} & -i\sqrt{2} & 0 \\ i\sqrt{2} & -\sqrt{2} & 0 \\ 0 & 0 & 2 \end{bmatrix} = A$ and

$$AA^{\theta} = \frac{1}{4}\begin{bmatrix} \sqrt{2} & -i\sqrt{2} & 0 \\ i\sqrt{2} & -\sqrt{2} & 0 \\ 0 & 0 & 2 \end{bmatrix}^2 = \frac{1}{4}\begin{bmatrix} 4 & -2i+2i & 0 \\ 2i-2i & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix} = I$$

- Hence the given matrix is unitary.
- For unitary matrix $A^{-1} = A^{\theta} = \frac{1}{2}\begin{bmatrix} \sqrt{2} & -i\sqrt{2} & 0 \\ i\sqrt{2} & -\sqrt{2} & 0 \\ 0 & 0 & 2 \end{bmatrix}$

Example 6

Show that $A = \begin{bmatrix} \alpha + i\gamma & -\beta + i\delta \\ \beta + i\delta & \alpha - i\gamma \end{bmatrix}$ is unitary if $\alpha^2 + \beta^2 + \gamma^2 + \delta^2 = 1$

- **Soln:** Since, $A^\theta = \begin{bmatrix} \alpha - i\gamma & \beta - i\delta \\ -\beta - i\delta & \alpha + i\gamma \end{bmatrix}$ Let us check,

- $AA^\theta = \begin{bmatrix} \alpha + i\gamma & -\beta + i\delta \\ \beta + i\delta & \alpha - i\gamma \end{bmatrix} \begin{bmatrix} \alpha - i\gamma & \beta - i\delta \\ -\beta - i\delta & \alpha + i\gamma \end{bmatrix}$

$$= \begin{bmatrix} \alpha^2 + \beta^2 + \gamma^2 + \delta^2 & (\alpha + i\gamma)(\beta - i\delta) - (\beta - i\delta)(\alpha + i\gamma) \\ (\beta + i\delta)(\alpha - i\gamma) - (\beta + i\delta)(\alpha - i\gamma) & \alpha^2 + \beta^2 + \gamma^2 + \delta^2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

- By comparing, we get the required condition,

  that A is unitary if $\alpha^2 + \beta^2 + \gamma^2 + \delta^2 = 1$