MATRIX THEORY: RANK OF MATRIX
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TYPES OF MATRICES

FY BTECH SEM-I
MODULE-2
SUB-MODULE 2.1

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Results on sum of Matrices

- **Show that every square matrix can be uniquely expressed as sum of Hermitian and skew Hermitian matrices.**

- **Proof:** Let A be any square matrix.

Consider $A = \frac{1}{2}(A + A^{\theta}) + \frac{1}{2}(A - A^{\theta})$

= P + Q, Where,

$P = \frac{1}{2}(A + A^{\theta})$

$Q = \frac{1}{2}(A - A^{\theta})$

Part I To Prove : P is Hermitian

$P^{\theta} = \left[\frac{1}{2}(A + A^{\theta})\right]^{\theta}$

$= \frac{1}{2}\left(A^{\theta} + (A^{\theta})^{\theta}\right)$

$= \frac{1}{2}(A^{\theta} + A) = P$

Part II To Prove : Q is skew-Hermitian

$Q^{\theta} = \left[\frac{1}{2}(A - A^{\theta})\right]^{\theta}$

$= \frac{1}{2}(A^{\theta} - A)$

$= -\frac{1}{2}(A - A^{\theta}) = -Q$

Part III: To prove uniqueness

Suppose there is another representation of A = R + S, where R is Hermitian ($R^{\theta} = R$) and S is skew Hermitian ($S^{\theta} = -S$)

Now, $A^{\theta} = (R + S)^{\theta}$

$= R^{\theta} + S^{\theta}$

$= R - S$

For uniqueness, we need to prove P=R & Q=S

$P = \frac{1}{2}(A + A^{\theta}) = \frac{1}{2}(R + S + R - S)$

$= R$

$Q = \frac{1}{2}(A - A^{\theta}) =$

$\frac{1}{2}(R + S - (R - S)) = S$

- Hence A = R + S =P+Q . So given representation is unique.

Results on sum of Matrices

- **Show that every square matrix can be uniquely expressed as sum of symmetric and skew symmetric matrices.**

- **Proof:** Let A be any square matrix.

$$A = \frac{1}{2}(A + A') + \frac{1}{2}(A - A')$$

= P + Q, Where,

$$P = \frac{1}{2}(A + A')$$

$$Q = \frac{1}{2}(A - A')$$

Part I To Prove : P is symmetric

$$P' = \left[\frac{1}{2}(A + A')\right]'$$

$$= \frac{1}{2}(A' + (A')')$$

$$= \frac{1}{2}(A' + A) = P$$

Part II To Prove : Q is skew-Hermitian

$$Q' = \left[\frac{1}{2}(A - A')\right]'$$

$$= \frac{1}{2}(A' - A)$$

$$= -\frac{1}{2}(A - A') = -Q$$

Part III: To prove uniqueness

Suppose there is another representation of A = R + S, where R is symmetric ($R' = R$) and S is skew symmetric ($S' = -S$)

Now, $A' = (R + S)'$

$$= R' + S'$$

$$= R - S$$

For uniqueness, we need to prove P=R & Q=S

$$P = \frac{1}{2}(A + A')$$

$$= \frac{1}{2}(R + S + R - S) = R$$

$$Q = \frac{1}{2}(A - A') = \frac{1}{2}(R + S - (R - S)) = S$$

- Hence A = R + S =P+Q . So given representation is unique.

Results on sum of Matrices

- **Show that every square matrix can be uniquely expressed as P + iQ, where P and Q both are Hermitian matrices.**
- **Proof:** Let A be any square matrix. Consider $A = \frac{1}{2}\left(A + A^{\theta}\right) + i\left[\frac{1}{2i}\left(A - A^{\theta}\right)\right]$ = P + iQ

Where, $P = \frac{1}{2}\left(A + A^{\theta}\right) \quad and \quad Q = \left[\frac{1}{2i}\left(A - A^{\theta}\right)\right]$

Part I To Prove : P is Hermitian

$$P^{\theta} = \left[\frac{1}{2}\left(A + A^{\theta}\right)\right]^{\theta} = \frac{1}{2}\left(A^{\theta} + \left(A^{\theta}\right)^{\theta}\right) = \frac{1}{2}\left(A^{\theta} + A\right) = P$$

Part II To Prove : Q is Hermitian

$$Q^{\theta} = \left[\frac{1}{2i}\left(A - A^{\theta}\right)\right]^{\theta} = \left(\frac{1}{2i}\right)^{\theta}\left(A^{\theta} - A\right) = -\frac{1}{2i}\left(A^{\theta} - A\right) = \frac{1}{2i}\left(A - A^{\theta}\right) = Q$$

Part III: To prove uniqueness

Consider another representation, say $A = R + iS$, where R and S are Hermitian. $(R^{\theta} = R\ ,\ S^{\theta} = S\ )$

Then $A^{\theta} = (R + iS)^{\theta} = R^{\theta} + i^{\theta}S^{\theta} = R - iS$

Now consider, $P = \frac{1}{2}\left(A + A^{\theta}\right) = \frac{1}{2}(R + iS + R - iS) = R$

and $Q = \frac{1}{2i}\left(A - A^{\theta}\right) = \frac{1}{2i}(R + iS - (R - iS)) = S$

Thus we establish R is same as P and S is same as Q. Hence given representation is unique.

Results on sum of Matrices

- **Show that every Hermitian matrix can be uniquely expressed as P + iQ, where P is real symmetric and Q is real skew symmetric matrix.**

Proof: Let A be any Hermitian matrix. Consider $A = \frac{1}{2}(A+\bar{A}) + i\left[\frac{1}{2i}(A-\bar{A})\right]$ = P + iQ

Where, $P = \frac{1}{2}(A+\bar{A}) \quad and \quad Q = \left[\frac{1}{2i}(A-\bar{A})\right]$

Part I To Prove : P is real symmetric, we show $\bar{P} = P \;\; and \;\; P^T = P$

$\bar{P} = \overline{\frac{1}{2}(A+\bar{A})} = \frac{1}{2}\overline{(A+\bar{A})} = \frac{1}{2}(\bar{A}+A) = P$ Hence, P is real

$P^T = \left[\frac{1}{2}(A+\bar{A})\right]^T = \frac{1}{2}(A^T+(\bar{A})^T) = \frac{1}{2}\left(A^T+A^\theta\right)$ (Since A is Hermitian,

$= \frac{1}{2}(\bar{A}+A) = P$ $(A^\theta = A \; and \; A^T = \bar{A}$) Hence P is symmetric .

Part II To Prove : Q is real Skew-symmetric we show $\bar{Q} = Q \;\; and \;\; Q^T = -Q$

$\bar{Q} = \overline{\frac{1}{2i}(A-\bar{A})} = -\frac{1}{2i}\overline{(A-\bar{A})} = -\frac{1}{2i}(\bar{A}-A) = \frac{1}{2i}(A-\bar{A}) = Q$ Hence, Q is real.

$Q^T = \left[\frac{1}{2i}(A-\bar{A})\right]^T = \frac{1}{2i}(A^T-(\bar{A})^T) = \frac{1}{2i}\left(A^T-A^\theta\right)$ (Since A is Hermitian,

$= \frac{1}{2i}(\bar{A}-A) = -Q$ $A^\theta = A \; and \; A^T = \bar{A}$)

Results on sum of Matrices

Part III: To prove uniqueness

Consider another representation, say $A = R + iS$, where R is real symmetric and S is real skew symmetric.

Then $\bar{A} = \overline{R + iS} = \bar{R} + \bar{i}\bar{S} = R - iS$ (since $\bar{R} = R$, $\bar{S} = S$)

Now consider, $P = \frac{1}{2}(A + \bar{A}) = \frac{1}{2}(R + iS + R - iS) = R$

and $Q = \frac{1}{2i}(A - \bar{A}) = \frac{1}{2i}(R + iS - (R - iS)) = S$

Thus, we establish R is same as P and S is same as Q. Hence given representation is unique.

- **Show that every skew Hermitian matrix can be uniquely expressed as P + iQ, where P is real skew symmetric and Q is real symmetric matrix.**
- (Try yourself)

Table of results on unique representation

Matrix	Expressed As	Unique Representation
Square	Symmetric + skew symmetric	$A = \frac{1}{2}(A + A^T) + \frac{1}{2}(A - A^T)$
Square	Hermitian + skew Hermitian	$A = \frac{1}{2}(A + A^\theta) + \frac{1}{2}(A - A^\theta)$
Square	P + iQ, P and Q both Hermitian	$A = \frac{1}{2}(A + A^\theta) + i\left[\frac{1}{2i}(A - A^\theta)\right]$
Hermitian	P + iQ, P real symmetric Q real skew symmetric	$A = \frac{1}{2}(A + \bar{A}) + i\left[\frac{1}{2i}(A - \bar{A})\right]$
Skew Hermitian	P + iQ, P real skew symmetric Q real symmetric	$A = \frac{1}{2}(A + \bar{A}) + i\left[\frac{1}{2i}(A - \bar{A})\right]$

Example-1

- **Express following Matrix as sum of Hermitian and skew Hermitian**

Matrices. $\boldsymbol{A = \begin{bmatrix} 2 & 3-i & 2+i \\ i & 0 & 1-i \\ 1+2i & -1 & 3i \end{bmatrix}}$

- Solution: As we know the unique representation, $\boxed{A = \frac{1}{2}\left(A + A^{\theta}\right) + \frac{1}{2}\left(A - A^{\theta}\right)}$ = P + Q,

Where, $P = \frac{1}{2}\left(A + A^{\theta}\right)$ $\;and\;$ $Q = \frac{1}{2}\left(A - A^{\theta}\right)$

Now, $A^{\theta} = \begin{bmatrix} 2 & -i & 1-2i \\ 3+i & 0 & -1 \\ 2-i & 1+i & -3i \end{bmatrix}$

$\therefore P = \frac{1}{2}\left\{\begin{bmatrix} 2 & 3-i & 2+i \\ i & 0 & 1-i \\ 1+2i & -1 & 3i \end{bmatrix} + \begin{bmatrix} 2 & -i & 1-2i \\ 3+i & 0 & -1 \\ 2-i & 1+i & -3i \end{bmatrix}\right\}$ = $\frac{1}{2}\begin{bmatrix} 4 & 3-2i & 3-i \\ 3+2i & 0 & -i \\ 3+i & i & 0 \end{bmatrix}$

P is Hermitian as $a_{ij} = \overline{a_{ji}},\ \forall\, i, j$

- $Q = \frac{1}{2}\left\{\begin{bmatrix} 2 & 3-i & 2+i \\ i & 0 & 1-i \\ 1+2i & -1 & 3i \end{bmatrix} - \begin{bmatrix} 2 & -i & 1-2i \\ 3+i & 0 & -1 \\ 2-i & 1+i & -3i \end{bmatrix}\right\}$

$= \frac{1}{2}\begin{bmatrix} 0 & 3 & 1+3i \\ -3 & 0 & 2-i \\ -1+3i & -2-i & 6i \end{bmatrix}$

- $\boxed{\text{Q is skew Hermitian } a_{ij} = -\overline{a_{ji}},\ \forall\, i,j}$

Hence we get the unique expression,

$$A = \begin{bmatrix} 2 & 3-i & 2+i \\ i & 0 & 1-i \\ 1+2i & -1 & 3i \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 4 & 3-2i & 3-i \\ 3+2i & 0 & -i \\ 3+i & i & 0 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 0 & 3 & 1+3i \\ -3 & 0 & 2-i \\ -1+3i & -2-i & 6i \end{bmatrix}$$

Example 2

- Express following Matrix as P + iQ, where P and Q are both Hermitian matrix.

$$A = \begin{bmatrix} 2 & 3-i & 2+i \\ i & 0 & 1-i \\ 1+2i & 1 & 3i \end{bmatrix}$$

- **Solution:** As we know the unique representation, $A = \frac{1}{2}(A + A^\theta) + i\left[\frac{1}{2i}(A - A^\theta)\right]$
- say, A = P + iQ, Where, $P = \frac{1}{2}(A + A^\theta)$ and $Q = \frac{1}{2i}(A - A^\theta)$
- $A^\theta = \begin{bmatrix} 2 & -i & 1-2i \\ 3+i & 0 & 1 \\ 2-i & 1+i & -3i \end{bmatrix}$
- $\therefore P = \frac{1}{2}\left\{\begin{bmatrix} 2 & 3-i & 2+i \\ i & 0 & 1-i \\ 1+2i & 1 & 3i \end{bmatrix} + \begin{bmatrix} 2 & -i & 1-2i \\ 3+i & 0 & 1 \\ 2-i & 1+i & -3i \end{bmatrix}\right\} = \frac{1}{2}\begin{bmatrix} 4 & 3-2i & 3-i \\ 3+2i & 0 & 2-i \\ 3+i & 2+i & 0 \end{bmatrix}$
- $\therefore Q = \frac{1}{2i}\left\{\begin{bmatrix} 2 & 3-i & 2+i \\ i & 0 & 1-i \\ 1+2i & 1 & 3i \end{bmatrix} - \begin{bmatrix} 2 & -i & 1-2i \\ 3+i & 0 & 1 \\ 2-i & 1+i & -3i \end{bmatrix}\right\} = \frac{1}{2}\begin{bmatrix} 0 & 3 & 1+3i \\ -3 & 0 & -i \\ -1+3i & -i & 6i \end{bmatrix}$
- For all elements P & Q , $\boldsymbol{a_{ij}} = \overline{\boldsymbol{a_{ji}}}$. Hence P and Q are Hermitian.

Example 3

- Express following skew Hermitian Matrix as P + iQ, where P is real skew symmetric and Q is real symmetric matrix. $A = \begin{bmatrix} 3i & -1+i & 3-2i \\ 1+i & -i & 1+2i \\ -3-2i & -1+2i & 0 \end{bmatrix}$
- **Solution:** As we know the unique representation, $\boxed{A = \frac{1}{2}(A+\bar{A}) + i\left[\frac{1}{2i}(A-\bar{A})\right]}$
- say, A = P + iQ, Where, $P = \frac{1}{2}(A+\bar{A})$ and $Q = \frac{1}{2i}(A-\bar{A})$
- Now, Consider $\bar{A} = \begin{bmatrix} -3i & -1-i & 3+2i \\ 1-i & i & 1-2i \\ -3+2i & -1-2i & 0 \end{bmatrix}$
- $\therefore P = \frac{1}{2}\left\{\begin{bmatrix} 3i & -1+i & 3-2i \\ 1+i & -i & 1+2i \\ -3-2i & -1+2i & 0 \end{bmatrix} + \begin{bmatrix} -3i & -1-i & 3+2i \\ 1-i & i & 1-2i \\ -3+2i & -1-2i & 0 \end{bmatrix}\right\} = \frac{1}{2}\begin{bmatrix} 0 & -2 & 6 \\ 2 & 0 & 2 \\ -6 & -2 & 0 \end{bmatrix}$
- All elements of P are real and $\boldsymbol{a_{ij}} = \boldsymbol{a_{ji}}$. Hence P is real skew symmetric.

$$Q = \frac{1}{2i}\left\{\begin{bmatrix} 3i & -1+i & 3-2i \\ 1+i & -i & 1+2i \\ -3-2i & -1+2i & 0 \end{bmatrix} - \begin{bmatrix} -3i & -1-i & 3+2i \\ 1-i & i & 1-2i \\ -3+2i & -1-2i & 0 \end{bmatrix}\right\}$$

$$Q = \frac{1}{2i}\begin{bmatrix} 6i & 2i & -4i \\ 2i & -2i & 4i \\ -4i & 4i & 0 \end{bmatrix} = \begin{bmatrix} 3 & 1 & -2 \\ 1 & -1 & 2 \\ -2 & 2 & 0 \end{bmatrix}$$

All elements of Q are real and $\boldsymbol{a_{ij}} = \boldsymbol{a_{ji}}$. So, Q is real symmetric.

Hence we get the unique expression, A = P + iQ,

$$A = \begin{bmatrix} 3i & -1+i & 3-2i \\ 1+i & -i & 1+2i \\ -3-2i & -1+2i & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -1 & 3 \\ 1 & 0 & 1 \\ -3 & -1 & 0 \end{bmatrix} + i\begin{bmatrix} 3 & 1 & -2 \\ 1 & -1 & 2 \\ -2 & 2 & 0 \end{bmatrix}$$

Example 4

- Express following Matrix as P + iQ, where P is real symmetric and Q as real skew symmetric matrix.

$$A = \begin{bmatrix} 2 & 1+i & -i \\ 1-i & 0 & -3-i \\ i & -3+i & -1 \end{bmatrix}$$

Solution: As we know the unique representation, $A = \frac{1}{2}(A+\bar{A}) + i\left[\frac{1}{2i}(A-\bar{A})\right]$

say, A = P + iQ, Where, $P = \frac{1}{2}(A+\bar{A})$ and $Q = \frac{1}{2i}(A-\bar{A})$

Now, Consider $\bar{A} = \begin{bmatrix} 2 & 1-i & i \\ 1+i & 0 & -3+i \\ i & -3-i & -1 \end{bmatrix}$

$$\therefore P = \frac{1}{2}\left\{\begin{bmatrix} 2 & 1+i & -i \\ 1-i & 0 & -3-i \\ i & -3+i & -1 \end{bmatrix} + \begin{bmatrix} 2 & 1-i & i \\ 1+i & 0 & -3+i \\ i & -3-i & -1 \end{bmatrix}\right\} = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 0 & -3 \\ 0 & -3 & -1 \end{bmatrix}$$

All elements of P are real and $\boldsymbol{a_{ij}} = \boldsymbol{a_{ji}}$. Hence P is symmetric.

$$Q = \frac{1}{2i}\left\{\begin{bmatrix} 2 & 1+i & -i \\ 1-i & 0 & -3-i \\ i & -3+i & -1 \end{bmatrix} - \begin{bmatrix} 2 & 1-i & i \\ 1+i & 0 & -3+i \\ i & -3-i & -1 \end{bmatrix}\right\}$$

$$Q = \frac{1}{2i}\begin{bmatrix} 0 & 2i & -2i \\ -2i & 0 & -2i \\ 2i & 2i & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & -1 \\ -1 & 0 & -1 \\ 1 & 1 & 0 \end{bmatrix}$$

All elements of Q are real and $\boldsymbol{a_{ij}} = -\boldsymbol{a_{ji}}$. So, Q is real skew symmetric.

Hence we get the unique expression, A = P + iQ,

$$A = \begin{bmatrix} 2 & 1+i & -i \\ 1-i & 0 & -3-i \\ i & -3+i & -1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 1 & 0 \\ 1 & 0 & -3 \\ 0 & -3 & -1 \end{bmatrix} + i\begin{bmatrix} 0 & 1 & -1 \\ -1 & 0 & -1 \\ 1 & 1 & 0 \end{bmatrix}$$