MATRIX THEORY: RANK OF MATRIX

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BASICS OF MATRICES

FY BTECH SEM-I

MODULE-2

SUB-MODULE 2.1/ PRE-REQUISITE/ SELF LEARNING

SOMAIYA VIDYAVIHAR UNIVERSITY
K J Somaiya College of Engineering

Somaiya TRUST

MATRIX: REVIEW OF BASIC CONCEPTS (Self-Learning)

- **Matrix** is a collection of information stored or arranged in an orderly fashion (Rectangular arrangement). It is denoted by Capital symbols and its elements are denoted by small letters with row and column indices. e.g. $A = \left[a_{ij}\right]$
- **Order of a Matrix:** (no. of rows) x (no. of columns), $A_{m\ X\ n}$
- **Basic Types of Matrices:**

1. **Row or Column Matrix:** Matrix containing only one row is called **Row Matrix** and Matrix containing only one column is called **column Matrix.**
2. **Rectangular Matrix:** Matrix with unequal number of rows and columns.

Basic Types of Matrices

3. **Null Matrix:** Matrix with all entries as zero entries. Denoted by $O_{m\ X\ n}$

4. **Square Matrix:** Matrix with same number of rows and columns (m = n)

$$\text{e.g. } A = \begin{bmatrix} 5 & 7 & -3 \\ -2 & 3 & 0 \\ 1 & 0 & -1 \end{bmatrix}$$

Diagonal and Principal diagonal entries:

Diagonal entries a_{ii} (e.g. $a_{11},\ a_{22}, a_{33}$) are called Principal diagonal entries.

5. **Diagonal Matrix:** Matrix whose all non-diagonal elements are zero.

$$\text{e.g. } A = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

6. **Scalar Matrix:** Diagonal matrix where all diagonal entries are equal to scalar k.

7. **Unit / Identity Matrix:** Scalar matrix whose all diagonal entries are equal to scalar 1.

8. **Upper Triangular Matrix:** Square Matrix where below diagonal entries are zero.

9. **Lower Triangular Matrix:** Square Matrix where above diagonal entries are zero.

Basic operations on Matrices

- **Equality of two matrices**: Two matrices are equal if all corresponding entries of them are equal.
- **No order relation**.
- **Addition/ subtraction**: Two matrices can be added (subtracted) by adding (subtracting) the corresponding elements of the two matrices. Both matrices must have same order (m x n).
- **Multiplication by scalar**: multiply each element by scalar k,
- e. g. $kA = \begin{bmatrix} ka_{11} & ka_{12} & ka_{13} \\ ka_{21} & ka_{22} & ka_{23} \\ ka_{31} & ka_{32} & ka_{33} \end{bmatrix}$

- **Product**: Two matrices can be multiplied together provided they are compatible with respect to their orders.
- The number of columns in the first matrix [A] must be equal to the number of rows in the second matrix [B]. The resulting matrix [C] will have the same number of rows as [A] and the same number of columns as [B].
- The entries are $c_{ij} = \sum_{k=1}^{n} a_{ik}\, b_{kj}$ and The order relation is
- $C_{m\,X\,p} = A_{m\,X\,n} . B_{n\,X\,p}$

Basic operations on Matrices

- **Trace of a Matrix**: Sum of all principal diagonal entries of a square matrix is called trace of matrix. $Trace(A) = \sum_i a_{ii}$
- **Transpose of a Matrix**: The transpose A^T or A' of a m x n matrix A is a n x m matrix obtained by interchanging rows with columns of A
- **Properties of Transpose**

i. $(A^T)^T = A$

ii. $(A + B)^T = A^T + B^T$

iii. $(AB)^T = B^T A^T$

- **Determinant of a Matrix**: It is denoted by $|A|$
- **Properties of determinant**

Det (AB) = Det (A).Det(B) ,

- $|A| = |A^T|$

- **Singular Matrix**: If $|A| = 0$
- **Non-singular Matrix**: If $|A| \neq 0$
- **Inverse of a Matrix**:
- It is denoted by A^{-1} and given by
- $A^{-1} = \frac{1}{|A|} adj\ A,$
- it exist only if $|A| \neq 0$
- **Properties of Inverse**

i. $(A^{-1})^{-1} = A$

ii. $(AB)^{-1} = B^{-1} A^{-1}$

Basic operations on Matrices

- **Conjugate of a Matrix**: Matrix obtained by replacing each element by its conjugate. (need not be square).
- It is denoted by $\bar{A}$
- **Example** $A = \begin{bmatrix} 2 & 2+3i & i \\ 7i & -3i & -i \end{bmatrix}$
- $Then\ \bar{A} = \begin{bmatrix} 2 & 2-3i & -i \\ -7i & 3i & i \end{bmatrix}$
- **Properties of Conjugate**

i. $\overline{(\bar{A})} = A$

ii. $\overline{(A+B)} = \bar{A} + \bar{B}$

iii. $\overline{(AB)} = \bar{A}.\bar{B}$ (check!!)

- **Tranjugate (Transposed conjugate) of a Matrix**: obtained by taking transpose of conjugate
- **Example** $A = \begin{bmatrix} 2 & 2+3i & i \\ 7i & -3i & -i \end{bmatrix}$
- $Then\ A^* = (\bar{A})^T = A^\theta = \begin{bmatrix} 2 & -7i \\ 2-3i & 3i \\ -i & i \end{bmatrix}$
- **Note:** need not be square matrix,
- Order of operation doesn't matter
- $(\bar{A})^T = \overline{(A^T)} = A^\theta$
- **Properties of Tranjugate**
- $(A^\theta)^\theta = A$
- $(A+B)^\theta = A^\theta + B^\theta$
- $(AB)^\theta = B^\theta A^\theta$ (check!!)

Comparative study of important Types: (self learning)

	Symmetric Matrix	Skew-symmetric Matrix
Definition	symmetric if i) A is square ii) $a_{ij} = a_{ji},\ \forall\ i, j$	Skew-symmetric if i) A is square ii) $a_{ij} = -a_{ji},\ \forall\ i, j$
Condition	A is symmetric iff $A^T = A$	A is skew-symmetric iff $A^T = -A$
Diagonal elements	$a_{ii} = a_{ii}$ No specific condition on elements	Condition $a_{ii} = -a_{ii}$, true for zero Diagonal elements must be zero.
Example	$A = \begin{bmatrix} 5+2i & 1+i & 2-i \\ 1+i & 3 & 7i \\ 2-i & 7i & 0 \end{bmatrix}$	$A = \begin{bmatrix} 0 & -1-i & -2-i \\ 1+i & 0 & 7i \\ 2+i & -7i & 0 \end{bmatrix}$
observation	Diagonal can be any no and sign of real and imaginary part of symmetric position no. are same	Diagonal is zero and sign of both real and imaginary part of symmetric position no. are different.

Comparative study of important Types:

	Hermitian Matrix	**Skew-Hermitian Matrix**
Definition	Hermitian if i) A is square ii) $a_{ij} = \overline{a_{ji}},\ \forall\, i,j$	Skew- Hermitian if i) A is square ii) $a_{ij} = \overline{-a_{ji}},\ \forall\, i,j$
Condition	A is Hermitian iff $A^{\theta} = A$	A is skew- Hermitian iff $A^{\theta} = -A$
Diagonal elements	$a_{ii} = \overline{a_{ii}}$ Diagonal elements are real.	$a_{ii} = \overline{-a_{ii}},$ Diagonal elements are imaginary/ zero.
Example	$A = \begin{bmatrix} 5 & 1-i & 2+i \\ 1+i & -3 & -7i \\ 2-i & 7i & 0 \end{bmatrix}$	$A = \begin{bmatrix} 5i & -1-i & -2 \\ 1-i & -3i & 7i \\ 2 & 7i & 0 \end{bmatrix}$
observation	Diagonal entries are real and sign of only imaginary part of symmetric position is different	Diagonal entries are imaginary and sign of only real part of symmetric position is different

Properties of Matrices

- Symmetric Matrices with real entries are Hermitian.
- Skew-Symmetric Matrices with real entries are skew-Hermitian.
- If A is Hermitian then $\bar{A} = A^T$ (equivalent condition)

Proof: $A^\theta = A \; Take\ transpose \; (\bar{A}^T)^T = A^T \;\Rightarrow\; \bar{A} = A^T$

- If A is skew-Hermitian then $\bar{A} = -A^T$ (equivalent condition)
- If A is Hermitian then *iA is* skew-Hermitian.

Proof: $(iA)^\theta = i^\theta A^\theta = -iA$

- If A is skew-Hermitian then *iA is* Hermitian.

Properties of Matrices

- If A is Hermitian then $\bar{A}$ is also Hermitian.

Proof: A is Hermitian. So $A^{\theta} = A$

Now, $(\bar{A})^{\theta} = \left(\bar{\bar{A}}\right)^{T} = A^{T} = \bar{A}$

- If A is skew-Hermitian then $\bar{A}$ is also skew-Hermitian.
- If A is any square matrix then

i. $A + A^{\theta}$ is Hermitian

ii. $A + A^{\theta}$ is skew-Hermitian.

Proof: $(A - A^{\theta})^{\theta} = A^{\theta} - \left(A^{\theta}\right)^{\theta} = A^{\theta} - A = -\left(A - A^{\theta}\right)$

$hence\ skew - Hermitian$

Properties of Matrices (self learning)

- If A and B are symmetric then AB is symmetric if and only if A and B are square matrices of same order and AB = BA
  $Since\ A^T = A\ and\ B^T = B, consider\ (AB)^T = B^T A^T = BA = AB$
- If A and B are skew-symmetric then AB is symmetric if and only if A and B are square matrices of same order and AB = BA
- If A is skew symmetric of order n then $|A| = 0$, if n is odd.

$$Since\ A^T = -A,\ take\ \det both\ side\ |A^T| = |-A|$$

$$\therefore |A| = (-1)^n |A| \therefore |A| = 0\ if\ n\ is\ odd$$

Properties of Matrices (self learning)

- If A is skew symmetric and X is a column matrix then $X^T AX$ is null matrix.

Pf: Let $X^T AX = B$, taking transpose $(X^T AX)^T = B^T$

$\therefore\ X^T A^T X = B^T$ But $A^T = -A$ *and*

$X^T{}_{1\,X\,n} A_{n\,X\,n} X_{n\,X\,1} = B_{1\,X\,1} \quad \therefore\ B^T = B$

$\therefore\ -X^T AX = B \quad \therefore X^T AX = -B \quad \therefore B = -B \quad \therefore B = 0$

- If A is any square matrix then

1. $A + A^T$ is symmetric
2. $A - A^T$ is skew-symmetric