MATRIX THEORY: RANK OF MATRIX

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FY BTECH SEM-I

MODULE-2

ELEMENTARY TRANFORMATIONS

(i) Interchanging any two rows or any two columns:

R_{ij} denotes the interchange of i^th and j^th rows and

C_{ij} denotes the interchange of i^th and j^th columns.

(ii) Multiplication of each element of i^th row by non zero k, i. e. kR_i

Multiplication of each element of i^th column by non zero k, kC_i

(iii) Adding row $(R_i + kR_j)$ / Adding columns $\left(C_i + kC_j\right)$.

These are only valid transformations.

Two matrices A and B are said to be **Equivalent Matrices** if the matrix B is obtained by performing elementary transformations on the matrix A.

Denoted by, $A \sim B$ (A is equivalent to B).

RANK OF A MATRIX

- **Minor of order r/ sub-matrix of order r** – If we select any r rows and r columns in Given m X n matrix then a matrix formed by these r rows and r columns is called a square sub-matrix of order r.
- **Determinant of this square sub-matrix of order r is called Minor of order r**
- **Definition of rank of 'A':** A number 'r' is said to be the rank of matrix A, if
- **(i)** There exists at least one sub – matrix of A of order r whose determinant is non – zero
- **(ii)** Every sub – matrix of A whose determinant with order (r + 1), if it exists, should be zero.
- **In short,** the rank of matrix is the order of any highest order non – vanishing minor.
- The rank 'r' of a matrix A is denoted by $\rho(A)$.

RANK OF A MATRIX

- **Properties**

(i) The rank of a null matrix is always zero.

(ii) If A is a non zero square matrix of order n, then $1 \leq \rho(A) \leq n$.

(iii) If A is a matrix of order $m \times n$, then $1 \leq \rho(A) \leq$min (m, n)

(iv) Rank of a non – singular matrix is always equal to its order. i.e. If $|A| \neq 0 \; then \;\; \rho(A) = n$

(v) Rank of a matrix is always unique.

(vi) $\rho(A) = \rho(A')$

(vii) $\rho(AB) \leq \rho(A) \; and \;\; \rho(AB) \leq \rho(B)$

(viii) Rank is invariant under elementary transformations. i.e. If $A \sim B \; then \;\; \rho(A) = \rho(B)$

(ix) Rank of A = Rank of (kA), where k is any scalar

(x) If $A_{n \times n}$ is non – singular i.e., $|A| \neq 0$ then rank of $A \; = \; n$ and rank of $A^2 = n$

Since $|A^2| = |A.A| = |A|.|A| \neq 0$

Examples
Determine the ranks of the following matrices

1) $Let\ A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 0 & 2 & 2 \end{bmatrix}$

- We have

$|A| = 1(6-8) - 2(4-0) + 3(4-0)$

$= -2 - 8 + 12$

$= 2 \neq 0$

Thus A is non – singular matrix,

i.e., $|A|$ is the highest order non – vanishing minor of order 3.

Hence rank of A is 3.

2) $Let\ A = \begin{bmatrix} 1 & -2 & 3 \\ -2 & 4 & -1 \\ -1 & 2 & 7 \end{bmatrix}$

$|A| = 1(28+2) - (-2)(-14-1) + 3(-4+4)$

$= 0$

Here the minor of order 3 is zero.

Can we find at least one minor of order 2 which is non zero?

$\begin{vmatrix} 1 & -2 \\ -2 & 4 \end{vmatrix} = 0,$

$but\ \begin{vmatrix} -2 & 3 \\ 4 & -1 \end{vmatrix} = -10 \neq 0$

i.e., at least one minor of order 2 is non – zero.
Hence rank of A is 2.

Examples
Determine the ranks of the following matrices

- 3) $Let\ A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ -3 & -6 & -9 \end{bmatrix}$
- Since we have $|A| = 0$ i.e., the minor of order 3 is zero.
- All minors of order 2 are also zero. Minor of order one is not zero.
- Hence rank of A is 1.
- **Observation:** Here, observe that all rows are identical, so when all the rows of a given matrix are identical
- Rank of such matrices are 1.

- 4) $Let\ A = \begin{bmatrix} 2 & 4 & 3 & 2 \\ 1 & -1 & 0 & 3 \\ 3 & 5 & 1 & 6 \end{bmatrix}_{3\times 4}$
- Here, A is the matrix of order 3×4.
- Therefore $1 \leq \rho(A) \leq \min(3,4),\ i.e\ 3.$
- Now, consider the minor. $\begin{vmatrix} 2 & 4 & 3 \\ 1 & -1 & 0 \\ 3 & 5 & 1 \end{vmatrix}$
- $= 2(-1-0) - 4(1-0) + 3(5+3)$
- $= -2 - 4 + 24 = 18 \neq 0$
- Hence rank of A is 3.

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EXAMPLES
Find the ranks of the following matrices

- **(i)** $\begin{bmatrix} 6 & 1 & 3 & 8 \\ 4 & 2 & 6 & -1 \\ 10 & 3 & 9 & 7 \\ 16 & 4 & 12 & 15 \end{bmatrix}$
- $R_4 - (R_1 + R_3)$, $\sim \begin{bmatrix} 6 & 1 & 3 & 8 \\ 4 & 2 & 6 & -1 \\ 10 & 3 & 9 & 7 \\ 0 & 0 & 0 & 0 \end{bmatrix}$
- $R_3 - (R_1 + R_2)$, $\sim \begin{bmatrix} 6 & 1 & 3 & 8 \\ 4 & 2 & 6 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$
- $\therefore$ Minor of order 4 is zero. All minors of order 3 are zero
- Consider the minor of order two $\begin{vmatrix} 6 & 1 \\ 4 & 2 \end{vmatrix} = 12 - 4 = 8 \neq 0$ Hence, the rank of matrix is 2.

- **(ii)** $\begin{bmatrix} 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 6 \\ 4 & 5 & 6 & 7 \\ 9 & 10 & 11 & 12 \end{bmatrix}$
- $\left.\begin{matrix} R_4 - R_1 \\ R_3 - R_1 \\ R_2 - R_1 \end{matrix}\right\} \sim \begin{bmatrix} 2 & 3 & 4 & 5 \\ 1 & 1 & 1 & 1 \\ 2 & 2 & 2 & 2 \\ 7 & 7 & 7 & 7 \end{bmatrix}$
- $\left.\begin{matrix} R_4 - 7R_2 \\ R_3 - 2R_2 \end{matrix}\right\} \sim \begin{bmatrix} 2 & 3 & 4 & 5 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$
- $\therefore$ Minor of order 4 is zero. All minors of order 3 are zero
- Consider the minor of order two $\begin{vmatrix} 2 & 3 \\ 1 & 1 \end{vmatrix} = 2 - 3 = -1 \neq 0$ Hence, the rank of matrix is 2.

Finding rank by row echelon method

- We know If $A \sim B$, then A and B have same rank.
- consider $A = \begin{bmatrix} 1 & -2 & 3 \\ -2 & 4 & -1 \\ -1 & 2 & 7 \end{bmatrix}$ whose rank is 2
- Now, we will obtain an equivalent matrix B of A by performing elementary transformations.
- Applying $R_2 + 2R_1$ and $R_3 + R_1$, we get $A \sim \begin{bmatrix} 1 & -2 & 3 \\ 0 & 0 & 5 \\ 0 & 0 & 10 \end{bmatrix}$
- Again, applying $R_3 - 2R_2$, we get $A \sim \begin{bmatrix} 1 & -2 & 3 \\ 0 & 0 & 5 \\ 0 & 0 & 0 \end{bmatrix}$
- $Let\ B = \begin{bmatrix} 1 & -2 & 3 \\ 0 & 0 & 5 \\ 0 & 0 & 0 \end{bmatrix}$ we have $|B| = 0$
- Consider the minor $\begin{vmatrix} -2 & 3 \\ 0 & 5 \end{vmatrix} = -10 \neq 0$
- Therefore, the rank of B is 2.
- Hence, $A \sim B$, and the rank of A = the rank of B.

ECHELON FORM OF A MATRIX

- **Definition:** If a matrix A is reduced to a matrix B by using elementary row transformations alone, then B is said to be row equivalent to A.
- **Defn:** The **Echelon form** or **Canonical form** of a matrix A is a row equivalent matrix of rank 'r' in which
- **(a)** One or more elements of each of the first r rows are non – zero while all other rows have only zero elements, (i.e all zero rows, if any, are placed at the bottom of the matrix so that the first r rows form an upper triangular matrix).
- **(b)** The number of zero before the first non – zero element in a row is less than the number of such zeros in the next row.
- **In short,** by performing only row transformations, a given matrix that is reduced to an **upper triangular form** is called its **Echelon form**.
- **Note:** Rank of a given matrix is equal to the number of non – zero rows in the Echelon form.

EXAMPLES

- Reduce the matrix $\begin{bmatrix} 3 & 4 & 1 & 1 \\ 2 & 4 & 3 & 6 \\ -1 & -2 & 6 & 4 \\ 1 & -1 & 2 & -3 \end{bmatrix}$ to Echelon Forms and hence find the ranks.
- **Solution:** $\begin{bmatrix} 3 & 4 & 1 & 1 \\ 2 & 4 & 3 & 6 \\ -1 & -2 & 6 & 4 \\ 1 & -1 & 2 & -3 \end{bmatrix}$

$$R_{14} \quad \sim \begin{bmatrix} 1 & -1 & 2 & -3 \\ 2 & 4 & 3 & 6 \\ -1 & -2 & 6 & 4 \\ 3 & 4 & 1 & 1 \end{bmatrix}$$

- $$\left.\begin{matrix} R_2 - 2R_1 \\ R_3 + R_1 \\ R_4 - 3R_1 \end{matrix}\right\} \sim \begin{bmatrix} 1 & -1 & 2 & -3 \\ 0 & 6 & -1 & 12 \\ 0 & -3 & 8 & 1 \\ 0 & 7 & -5 & 10 \end{bmatrix}$$

$$R_2 - R_4 \quad \sim \begin{bmatrix} 1 & -1 & 2 & -3 \\ 0 & -1 & 4 & 2 \\ 0 & -3 & 8 & 1 \\ 0 & 7 & -5 & 10 \end{bmatrix}$$

- $$\left.\begin{matrix} R_3 - 3R_2 \\ R_4 + 7R_2 \end{matrix}\right\} \sim \begin{bmatrix} 1 & -1 & 2 & -3 \\ 0 & -1 & 4 & 2 \\ 0 & 0 & -4 & -5 \\ 0 & 0 & 23 & 24 \end{bmatrix}$$

$$R_4 + \frac{23}{4}R_3 \quad \sim \begin{bmatrix} 1 & -1 & 2 & -3 \\ 0 & -1 & 4 & 2 \\ 0 & 0 & -4 & -5 \\ 0 & 0 & 0 & -9/4 \end{bmatrix}$$

- This is Echelon form of the given matrix, in which the number of non – zero rows is 4.
- Hence the rank of the matrix is 4.