SOMAIYA VIDYAVIHAR UNIVERSITY

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HOMOGENEOUS FUNCTIONS

FYBTECH SEM-I

MODULE-5

Module 5

- CO5. Apply Euler's theorem to prove results related to Homogeneous functions.

5	**Homogeneous Functions**		**4**	**CO5**
	5.1	Euler's theorem on homogeneous functions with two and three independent variables (statement only) and problems		
	5.2	Deductions(Corollaries) from Euler's theorem (statements only) and problems		

Homogeneous Functions

❖ **Definition:**

For two variables: $u = f(x, y)$ is called homogeneous function of degree n if $u = x^n f(\frac{y}{x})$ where n is any real number.

For three variable : $u = f(x, y, z)$ is homogeneous function of degree n if it can be expressed as $u = x^n f\left(\frac{y}{x}, \frac{z}{x}\right)$.

❖ **Working Rule:**

1. Replacing $x = xt, y = yt\ and\ z = zt$
2. Find $f(xt, yt, zt)$
3. If $f(xt, yt, zt) = t^n f(x, y, z)$ then $u = f(x, y, z)$ is called homogeneous function of degree n.

Check whether homogenous?

EULER'S THEOREM

- If u is a homogeneous function of two variables x and y of degree n then

$$x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=nu$$

- **Proof:** $u=f(x,y)$ is a homogeneous function of degree n then

$$\therefore u=x^n\emptyset\left(\frac{y}{x}\right) \quad \text{.....................(i)}$$

- Differentiate partially w.r.t. x we get

$$\frac{\partial u}{\partial x}=nx^{n-1}\emptyset\left(\frac{y}{x}\right)+x^n\emptyset'\left(\frac{y}{x}\right)\left(-\frac{y}{x^2}\right)$$

$$\therefore x\frac{\partial u}{\partial x}=nx^n\emptyset\left(\frac{y}{x}\right)-yx^{n-1}\emptyset'\left(\frac{y}{x}\right) \quad \text{................ (ii)}$$

Differentiate (i) partially w.r.t. y we get $\frac{\partial u}{\partial y}=x^n\emptyset'\left(\frac{y}{x}\right).\frac{1}{x}=x^{n-1}\emptyset'\left(\frac{y}{x}\right)$

$$\therefore y\frac{\partial u}{\partial y}=yx^{n-1}\emptyset'\left(\frac{y}{x}\right) \quad \text{................ (iii)}$$

Adding (ii) & (iii) $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=nx^n\emptyset\left(\frac{y}{x}\right)=nu$

Euler's Theorem

- **For Function of TWO variables:**

If u = f(x, y) is homogeneous function of degree n, then

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu$$

- **For Function of Three variables:**

If u = f(x, y, z) we get, $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = nu$

This theorem can be extended to n variables.

Corollary 1

- If u = f(x, y) is a homogeneous function of two variables x & y of degree n, then

$$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=n(n-1)u$$

- For u = f(x, y, z) we get,

$$x^2\frac{\partial^2 u}{\partial x^2}+y^2\frac{\partial^2 u}{\partial y^2}+z^2\frac{\partial^2 u}{\partial z^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+2yz\frac{\partial^2 u}{\partial y\partial z}+2zx\frac{\partial^2 u}{\partial z\partial x}$$
$$=n(n-1)u$$

Corollary 1(proof)

- If z is a homogeneous function of two variables x and y of degree n then

$$x^2\frac{\partial^2 z}{\partial x^2}+2xy\frac{\partial^2 z}{\partial x\partial y}+y^2\frac{\partial^2 z}{\partial y^2}=n(n-1)z$$

- **Proof:** Since z is a homogeneous function of degree n in x and y

by Euler's Theorem, $x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}=nz$ .... (i)

Differentiating (i) partially w.r.t. x, we get $\left(x\frac{\partial^2 z}{\partial x^2}+\frac{\partial z}{\partial x}.1\right)+y\frac{\partial^2 z}{\partial x\partial y}=n\frac{\partial z}{\partial x}$

$\therefore\ x\frac{\partial^2 z}{\partial x^2}+y\frac{\partial^2 z}{\partial x\partial y}=(n-1)\frac{\partial z}{\partial x}$ .... (ii)

Differentiating (i) partially w.r.t. y, we get $x\frac{\partial^2 z}{\partial y\partial x}+\left(y\frac{\partial^2 z}{\partial y^2}+\frac{\partial z}{\partial y}.1\right)=n\frac{\partial z}{\partial y}$

$\therefore\ x\frac{\partial^2 z}{\partial y\partial x}+y\frac{\partial^2 z}{\partial y^2}=(n-1)\frac{\partial z}{\partial y}$ .... (iii)

multiplying (ii) by x and (iii) by y and adding, we get,

$$x^2\frac{\partial^2 z}{\partial x^2}+2xy\frac{\partial^2 z}{\partial x\partial y}+y^2\frac{\partial^2 z}{\partial y^2}=(n-1)\left[x\frac{\partial z}{\partial x}+y\frac{\partial z}{\partial y}\right]$$

$$=(n-1)nz \qquad \text{[by (i)]}$$

Further, if u is a homogeneous function of three variables $x, y\ z$ of degree n then we can Prove that

$$x^2\frac{\partial^2 u}{\partial x^2}+y^2\frac{\partial^2 u}{\partial y^2}+z^2\frac{\partial^2 u}{\partial z^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+2yz\frac{\partial^2 u}{\partial y\partial z}+2zx\frac{\partial^2 u}{\partial z\partial x}=n(n-1)u.$$

Corollary 2

- For function of two variables: If f(u) is homogeneous function of degree n, then $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}=n\frac{f(u)}{f\prime(u)}$
- For function of three variable: If f(u) is homogeneous function of degree n, then we get $x\frac{\partial u}{\partial x}+y\frac{\partial u}{\partial y}+z\frac{\partial u}{\partial z}=n\frac{f(u)}{f\prime(u)}$
- Note:
- Here u is not homogenous function but f(u) is homogenous function. For eg.

\- $u=\sin^{-1}x^2y$ is not homogenous but f(u)=sinu $=x^2y$ is homogenous of degree 3.

- $u=\log\frac{x^2+y^2}{x+y}$ is not homogenous but $f(u)=e^u=\frac{x^2+y^2}{x+y}$ is homogenous with degree 1.

Corollary 2(proof)

- If z is homogeneous function of degree n in x and y, and $z = f(u)$ then $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = n\frac{f(u)}{f'(u)}$
- **Proof:** By Euler's Theorem,

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = nz = nf(u) \quad \text{..............(i)}$$

Since $z = f(u)$

$$\therefore \quad \frac{\partial z}{\partial x} = f'(u)\frac{\partial u}{\partial x} \quad \text{and} \quad \frac{\partial z}{\partial y} = f'(u)\frac{\partial u}{\partial y}$$

Putting these values in (i), we get,

$$xf'(u)\frac{\partial u}{\partial x} + yf'(u)\frac{\partial u}{\partial y} = nf(u)$$

$$\therefore x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = n\frac{f(u)}{f'(u)}$$

Corollary 3

- For function of two variables: If f(u) is homogeneous function of degree n , then

$$x^2\frac{\partial^2 u}{\partial x^2}+2xy\frac{\partial^2 u}{\partial x\partial y}+y^2\frac{\partial^2 u}{\partial y^2}=g(u)[g'(u)-1]$$

$$where\ g(u)=n\frac{f(u)}{f'(u)}$$

Corollary 3(proof)

- If z is homogeneous function of degree n in x and y, and $z = f(u)$ then
  $$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = \mathrm{g(u)[g'(u) - 1]} \quad \text{where} \quad \mathrm{g(u)} = \mathrm{n}\frac{\mathrm{f(u)}}{\mathrm{f'(u)}}$$
- **Proof:** By Corollary (2) we have $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = n\frac{f(u)}{f'(u)} = g(u)$ .... (i)

Differentiating (i) partially w.r.t. x, we get $\left(x\frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial x}.1\right) + y\frac{\partial^2 u}{\partial x \partial y} = g'(u)\frac{\partial u}{\partial x}$

$$\therefore \quad x\frac{\partial^2 u}{\partial x^2} + y\frac{\partial^2 u}{\partial x \partial y} = (g'(u) - 1)\frac{\partial u}{\partial x} \qquad \text{................ (ii)}$$

Differentiating (i) partially w.r.t. y, we get $x\frac{\partial^2 u}{\partial y \partial x} + y\frac{\partial^2 u}{\partial y^2} + \frac{\partial u}{\partial y} = g'(u)\frac{\partial u}{\partial y}$

$$\therefore \quad x\frac{\partial^2 u}{\partial y \partial x} + y\frac{\partial^2 u}{\partial y^2} = (g'(u) - 1)\frac{\partial u}{\partial y} \qquad \text{................ (iii)}$$

multiplying (ii) by x and (iii) by y and adding, we get,

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = (g'(u) - 1)\left[x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y}\right]$$
$$= g(u)[g'(u) - 1] \qquad \text{[by (i)]}$$