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## Inverse Laplace Transform

\* If  $L[f(t)] = \phi(s)$

$$= \int_0^\infty e^{-st} \cdot f(t) dt$$

then  $f(t)$  is called inverse laplace transform of  $\phi(s)$

$$f(t) = L^{-1}[\phi(s)]$$

- Formulae for Laplace Inverse ↴

$$\rightarrow L[1] = \frac{1}{s} \quad \therefore L^{-1}\left[\frac{1}{s}\right] = 1 //$$

$$\rightarrow L[e^{at}] = \frac{1}{s-a} \quad \therefore L^{-1}\left[\frac{1}{s-a}\right] = e^{at} //$$

Similarly,  $L^{-1}\left[\frac{1}{s+a}\right] = e^{-at}$  //

$$\rightarrow L[\cos(at)] = \frac{s}{s^2+a^2} \quad \therefore L^{-1}\left[\frac{s}{s^2+a^2}\right] = \cos(at) //$$

Similarly,

$$L[\sin(at)] = \frac{a}{s^2+a^2} \quad \therefore L^{-1}\left[\frac{a}{s^2+a^2}\right] = \frac{\sin(at)}{a} //$$

$$\rightarrow L[\cosh(at)] = \frac{s}{s^2-a^2} \quad \therefore L^{-1}\left[\frac{s}{s^2-a^2}\right] = \cosh(at) //$$

Similarly,

$$L[\sinh(at)] = \frac{a}{s^2-a^2} \quad \therefore L^{-1}\left[\frac{a}{s^2-a^2}\right] = \frac{\sinh(at)}{a} //$$

$$\rightarrow L[t^n] = \frac{[n+1]}{s^{n+1}} \quad \therefore L^{-1} \left[ \frac{1}{s^{n+1}} \right] = t^n \text{ (from 2.1 part 2)}$$

OR  $L^{-1} \left[ \frac{1}{s^n} \right] = t^{n-1}$  (from 2.1 part 2)

$(n+2) \Phi = [(t^n) + t^{n-1}] + \text{mult.}$

- Linearity Property  $\Rightarrow$

$$(L^{-1}[\Phi])^T = (\Phi)^T \quad \text{by 2.1}$$

$$L^{-1}[a \cdot \phi_1(s) + b \cdot \phi_2(s)]^T = a L^{-1}[\phi_1(s)] + b L^{-1}[\phi_2(s)]$$

$$[(\Phi)^T]^T =$$

Q. Find  $L^{-1} \left[ \frac{2s-5}{4s^2+2s} - \frac{(4s-18)}{9-s^2} + \frac{1}{s+1} + \frac{5}{s^4+2s} \right]^T$

Ans.  $= \frac{L^{-1}}{4} \left[ 2 \left( \frac{s}{s^2 + (s/2)^2} \right) - \frac{5}{4} \left( \frac{1}{s^2 + (s/2)^2} \right) + 4 \left( \frac{s}{s^2 - 9} \right) - 18 \left( \frac{1}{s^2 - 9} \right) + \frac{1}{s+1} + \frac{5}{s^4} \right]$

$$= \frac{1}{2} L^{-1} \left[ \left( \frac{s}{s^2 + (s/2)^2} \right) \right] - \frac{5}{4} L^{-1} \left[ \frac{1}{s^2 + (s/2)^2} \right] + 4 L^{-1} \left[ \frac{s}{s^2 - 9} \right] - 18 L^{-1} \left[ \frac{1}{s^2 - 9} \right] + L^{-1} \left[ \frac{1}{s+1} \right] + 5 L^{-1} \left[ \frac{1}{s^4} \right]$$

$$= \frac{1}{2} \times \cos(st/2) - \frac{5}{8} \times \frac{1}{s^2} \sin(st/2) + 4 \cosh(3t) - \frac{18}{3} \sinh(3t) + e^{-t} + \frac{5t^3}{14}$$

$$= \frac{\cos(st/2)}{2} - \frac{\sin(st/2)}{2} + 4 \cosh(3t) - 6 \sinh(3t) + e^{-t} + \frac{5t^3}{6}$$

$$\frac{1}{2} s^2 - \frac{5}{8} s^4 - \frac{18}{3} s^6 + \frac{5}{6} s^8$$

• First Shifting Property  $\rightarrow$

We know that if,

$$\mathcal{L}[f(t)] = \phi(s)$$

$$\text{then } \mathcal{L}[e^{at} f(t)] = \phi(s+a)$$

$$\therefore \text{If } f(t) = \mathcal{L}^{-1}[\phi(s)]$$

$$\text{then } \mathcal{L}^{-1}[\phi(s+a)] = e^{at} f(t)$$

$$= e^{at} \cdot \mathcal{L}^{-1}[\phi(s)]$$

$$Q. \quad \mathcal{L}^{-1}\left[\frac{s+3}{(s+3)^2+9}\right]$$

$$\text{Ans.} \quad = e^{-3t} \cdot \mathcal{L}^{-1}\left[\frac{s}{s^2+9}\right]$$

$$= e^{-3t} \cdot \cos(3t)$$

Q. Find inverse Laplace of following functions:

(i)  $s^2$

$$(s-1)^3$$

(ii)  $\frac{s}{s^4+s^2+1}$

(iii)  $\frac{s^2+16s-24}{s^4+20s^2+54}$

$$Q.1 \quad L\left[ \frac{s^2}{(s-1)^3} \right]$$

Ans.  $= L^{-1} \left[ \frac{(s-1)^2 + (2s-1)}{1+2s_2(s-1)^3 + 2s_2} \right] = \frac{2}{(1+2s_2)(1+2+s_2)} = \dots$

 $= L^{-1} \left[ \frac{(s-1)^2 + 2(s-1) + 1}{(s-1)^3} \right] = \frac{L^{-1}\left[\frac{1}{s}\right] + 2L^{-1}\left[\frac{1}{(s-1)^2}\right] + L^{-1}\left[\frac{1}{(s-1)^3}\right]}{s\left(\frac{2}{s}\right) + s\left(\frac{1}{s}\right)} =$ 
 $= L^{-1}\left[\frac{1}{s-1}\right] + 2L^{-1}\left[\frac{1}{(s-1)^2}\right] + L^{-1}\left[\frac{1}{(s-1)^3}\right] =$ 
 $= e^t - L^{-1}\left[\frac{1}{s}\right] + 2e^t L^{-1}\left[\frac{1}{(s-1)^2}\right] = \frac{1}{s} = \frac{1}{s\left(\frac{2}{s}\right) + s\left(\frac{1}{s}\right)} =$ 
 $= e^t + 2 \cdot e^t \cdot L^{-1}\left[\frac{1}{s^2}\right] + e^t \cdot L^{-1}\left[\frac{1}{s^3}\right] =$ 
 $= e^t + 2 \cdot e^t \cdot \frac{1}{s^2} + e^t \cdot \frac{1}{s^3} =$ 
 $= \left(\frac{e^t}{s^2}\right) \cancel{\left(\frac{1}{s^2} \cdot \sqrt{2}\right)} \cdot \cancel{\left(\frac{1}{s^2} \cdot \sqrt{3}\right)} + \left(\frac{e^t}{s^3}\right) \cancel{\left(\frac{1}{s^3} \cdot \sqrt{2}\right)} \cdot \cancel{\left(\frac{1}{s^3} \cdot \sqrt{3}\right)} =$ 
 $= e^t \left[ 1 + 2t + \frac{t^2}{2} \right] =$

Q.2

$$\frac{[s^{1/2} - s^{1/2}]}{L[s]} \cdot \left(\frac{1}{s}\right)^{1/2} =$$

$$\left(\frac{1}{s}\right)^{1/2} \cdot \left(\frac{1}{s}\right)^{1/2} \cdot \frac{1}{s} =$$

Q.2

$$\frac{s}{s^4+s^2+1}$$

$$\text{Ans.} = \frac{s}{(s^2+s+1)(s^2-s+1)} = \frac{1}{2} \left[ \frac{(1-2s)+s(s-2)}{s^2-s+1(1-2)} \right] =$$

$$= \frac{1}{2} \left[ \frac{1}{(s-\frac{1}{2})^2 + (\frac{\sqrt{3}}{2})^2} \right] - \frac{1}{2} \left[ \frac{1}{(s+\frac{1}{2})^2 + (\frac{\sqrt{3}}{2})^2} \right]$$

$\therefore$  Taking Laplace Inverse,  $L^{-1}[s(s-2)]$

$$= \frac{1}{2} L^{-1} \left[ \frac{1}{(s-\frac{1}{2})^2 + (\frac{\sqrt{3}}{2})^2} \right] - \frac{1}{2} L^{-1} \left[ \frac{1}{(s+\frac{1}{2})^2 + (\frac{\sqrt{3}}{2})^2} \right]$$

$$= \frac{1}{2} \cdot e^{\frac{t}{2}} \cdot L^{-1} \left[ \frac{1}{s^2 + (\frac{\sqrt{3}}{2})^2} \right] - \frac{1}{2} e^{\frac{t}{2}} L^{-1} \left[ \frac{1}{s^2 + (\frac{\sqrt{3}}{2})^2} \right]$$

$$= \frac{1}{2} \cdot e^{\frac{t}{2}} \cdot \frac{z}{\sqrt{3}} \cdot \sin(\sqrt{3}t/2) - \frac{1}{2} \cdot e^{\frac{t}{2}} \cdot \frac{z}{\sqrt{3}} \cdot \sin(\sqrt{3}t/2)$$

$$= \sin\left(\frac{\sqrt{3}t}{2}\right) \cdot \left[ \frac{e^{\frac{t}{2}} - e^{-\frac{t}{2}}}{\sqrt{3}} \right]$$

$$= \frac{2}{\sqrt{3}} \sin\left(\frac{\sqrt{3}t}{2}\right) \cdot \left[ \frac{e^{\frac{t}{2}} - e^{-\frac{t}{2}}}{2} \right]$$

$$= \frac{2}{\sqrt{3}} \cdot \sin\left(\frac{\sqrt{3}t}{2}\right) \cdot \sinh\left(\frac{t}{2}\right)$$

$$\text{Q.3} \quad \frac{s^2 + 16s - 24}{s^4 + 20s^2 + 64}$$

Ans.

→ utrueg pmtial basis  
 $a = -\frac{4}{3}, b = \frac{10}{3}$   
 $c = \frac{4}{3}, d = -\frac{7}{3}$

hints

$$\text{if tent. } (2)\phi = \left[ \begin{array}{c} As+B \\ s^2+4 \end{array} \right] + \left[ \begin{array}{c} Cs+D \\ s^2+16 \end{array} \right]$$

$$\left. \begin{array}{l} s < t \text{ rot. } (s-t) \cdot \frac{d}{ds} = (t-s)e \\ s > t \text{ rot. } 0 \end{array} \right\} =$$

$$(2)\phi \cdot \frac{2s}{s^2+9} = [(t-s)e] \cdot \frac{d}{ds} \text{ tent.}$$

$$[(t-s)e] \cdot \frac{2s}{s^2+9} =$$

$$[(2)\phi]^{1-1} = (t-s)e \cdot 2s \cdot \frac{d}{ds}$$

$$(t-s)e = [(2)\phi]^{1-1} \text{ tent.}$$

$$\left. \begin{array}{l} s < t \text{ rot. } (s-t) \cdot \frac{d}{ds} = \\ s > t \text{ rot. } 0 \end{array} \right\} =$$

$$\left[ \begin{array}{c} 2s \\ s^2+28+s^2 \end{array} \right]^{1-1} \text{ bnf test.} \quad \text{Q.}$$

$$\left[ \begin{array}{c} 1 \\ s^2+28+s^2 \end{array} \right] = (2)\phi + ts \cdot \frac{d}{ds}$$

$$\left[ \begin{array}{c} 1 \\ s^2+28+s^2 \end{array} \right]^{1-1} = [(2)\phi]^{1-1} \text{ tent.}$$

$$\left[ \begin{array}{c} 1 \\ s^2+28+s^2 \end{array} \right] \cdot \frac{2s}{s^2+9} =$$

$$(t-s)e = (t-s)e \cdot 2s \cdot \frac{d}{ds} =$$

[utrueg pmtial basis]       $(t-s)e = [(2)\phi \cdot \frac{2s}{s^2+9}]^{1-1} \text{ tent.}$

$$\left. \begin{array}{l} s < t \text{ rot. } (s-t)s \cdot \frac{d}{ds} \cdot \frac{(s-t)s}{s^2+9} = \\ s > t \text{ rot. } 0 \end{array} \right\} =$$

$$\left. \begin{array}{l} s < t \text{ rot. } (s-t)s \cdot \frac{d}{ds} \cdot \frac{(s-t)s}{s^2+9} = \\ s > t \text{ rot. } 0 \end{array} \right\} =$$

$$\int e^{-st} f(t) dt = \phi(s)$$

### • Second Shifting Property $\rightarrow$

we know that if,

$$L[f(t)] = \phi(s)$$

$$\text{and } g(t) = \begin{cases} f(t-a) & \text{for } t > a \\ 0 & \text{for } t < a \end{cases}$$

$$\begin{aligned} \text{then } L[g(t)] &= e^{-as} \cdot \phi(s) \\ &= e^{-as} \cdot L[f(t)] \end{aligned}$$

$$\therefore \text{If } f(t) = L^{-1}[\phi(s)]$$

$$\text{then } L^{-1}[e^{-as} \phi(s)] = g(t)$$

$$= \begin{cases} f(t-a) & \text{for } t > a \\ 0 & \text{for } t < a \end{cases}$$

Q. Find  $L^{-1}\left[\frac{e^{-2s}}{s^2+8s+25}\right]$

Ans. Let  $\phi(s) = \frac{1}{s^2+8s+25} = \frac{1}{(s+4)^2+3^2}$

$$\therefore L^{-1}[\phi(s)] = L^{-1}\left[\frac{1}{(s+4)^2+3^2}\right]$$

$$= e^{-4t} \cdot L^{-1}\left[\frac{1}{s^2+3^2}\right]$$

$$= e^{-4t} \cdot \frac{\sin(3t)}{3} = f(t)$$

$$\begin{aligned} \therefore L^{-1}\left[e^{-2s} \cdot \phi(s)\right] &= g(t) \quad [2^{\text{nd}} \text{ shifting property}] \\ &= \begin{cases} f(t-2) & \text{for } t > 2 \\ 0 & \text{for } t < 2 \end{cases} \end{aligned}$$

$$= \begin{cases} \frac{e^{-4(t-2)} \cdot \sin(3(t-2))}{3} & \text{for } t > 2 \\ 0 & \text{for } t < 2 \end{cases}$$

## Laplace Inverse of derivatives

We know that if,

$$L[f(t)] = \phi(s)$$

then  $L[t^n f(t)] = (-1)^n \frac{d^n}{ds^n} [\phi(s)]$

$$\therefore \text{If } f(t) = L^{-1}[\phi(s)]$$

$$\text{then } L^{-1}[\phi^{(n)}(s)] = (-1)^n t^n \cdot f(t)$$

$$= (-1)^n t^n \phi(s)$$

Q. Find Laplace Inverse of following functions:

$$(i) \frac{s+3}{(s^2+6s+10)^2}$$

$$(iii) \log\left(\frac{(s^2+1)}{s(s+1)}\right)$$

$$(vii) 2\tanh^{-1}(s)$$

$$(iv) \tan^{-1}(2/s^2)$$

Q.1

$$\frac{s+3}{(s^2+6s+10)^2}$$

Ansatzes nach normal angeg.

Hilfsworten

Ans. Let  $\phi(s) = \frac{1}{s^2+6s+10}$

$$(s^2+6s+10) = (s+3)^2 + 1$$

$$s^2+6s+10 = s^2 + 2 \cdot 3 \cdot s + 3^2 + 1 = (s+3)^2 + 1 \text{ nach}$$

$$\therefore \phi'(s) = \frac{-(2s+6)}{(s^2+6s+10)^2} = \frac{-2(s+3)}{(s^2+6s+10)^2}$$

$$\therefore L^{-1}[\phi'(s)] = -2 \cdot L^{-1}\left[\frac{1}{(s+3)^2 + 1}\right] \text{ nach } L^{-1}[1/(s^2+1)] = \cos(t)$$

$$= -t \cdot L^{-1}[\phi(s)] \quad (\text{by second Laplace inverse of derived})$$

$$\therefore L^{-1}\left[\frac{s+3}{(s^2+6s+10)^2}\right] = -\frac{1}{2} \times -t \times L^{-1}[\phi(s)]$$

2. Hilfswort zu denkt: sofern nicht sonst

$$= \frac{t}{2} L^{-1}\left[\frac{1}{(s+3)^2 + 1^2}\right] \quad \begin{array}{l} \text{(i)} \\ \text{(s+3)^2 + 1^2} \end{array}$$

$$= \frac{t}{2} \cdot e^{-3t} \cdot L^{-1}\left[\frac{1}{s^2 + 1^2}\right] \quad \begin{array}{l} ((1+s_2)) \text{ col. (ii)} \\ (1+2s_2) \end{array}$$

$$= \frac{t}{2} \cdot e^{-3t} \cdot \sin(t) \quad \begin{array}{l} (2)^{\text{nd}} \text{ dient } (iii) \\ (s_2/s) \text{ Finst. (vi)} \end{array}$$

$$\text{Q.2} \quad \log\left(\frac{s^2+1}{s(s+1)}\right)$$

$$(2)^{-1} \cdot d(10) - 2 \quad 8.0$$

Ans. Let  $\phi(s) = \log\left(\frac{s^2+1}{s(s+1)}\right)$

$$(2)^{-1} \cdot d(10) - 2 \rightarrow (2)\phi + \dots$$

$$= \log(s^2+1) - \log(s) - \log(s+1)$$

$$\therefore \phi'(s) = \frac{2s}{s^2+1} - \frac{1}{s} - \frac{1}{s+1}$$

$$\therefore L^{-1}[\phi'(s)] = 2L^{-1}\left[\frac{s}{s^2+1}\right] - L^{-1}\left[\frac{1}{s}\right] - L^{-1}\left[\frac{1}{s+1}\right]$$

↓  
by Laplace  
Inverse of  
Derivatives

$$-t \cdot f(t) = 2 \cdot \cos(t) - 1 - e^{-t}$$

$$\therefore f(t) = \frac{1}{t} \left[ 2 \cos(t) - 1 - e^{-t} \right] (2)\phi$$

$$\frac{t-3-t_0}{t} = F(s)\phi$$

Q.3  $2 \tanh^{-1}(s)$

$$\frac{(1+s)}{(1-s)} = \frac{e^s + e^{-s}}{e^s - e^{-s}}$$

Ans. Let  $\phi(s) = 2 \tanh^{-1}(s)$

$$= 2 \times \frac{1}{2} \log \left( \frac{1+s}{1-s} \right)$$

$$= \log(1+s) - \log(1-s)$$

$$\therefore \phi'(s) = \frac{1}{1+s} - \frac{1}{1-s}$$

$$\therefore L^{-1}[\phi'(s)] = L^{-1}\left[\frac{1}{s+1} - \frac{1}{s-1}\right]$$

↓  
by Laplace  
Inverse of  
Derivatives

$$-t \cdot L^{-1}[\phi(s)] = e^{-t} - e^{t-200s}$$

$$\therefore L^{-1}[\phi(s)] = \frac{e^{-t} - e^{t-200s}}{t}$$

Q.4  $\tan^{-1}\left(\frac{2}{s^2}\right)$

+ important information

Ans. Let  $\phi(s) = \tan^{-1}\left(\frac{2}{s^2}\right)$

~~$(2), \phi = ((0), 2) + tE$~~

~~$(2), \phi = (0, 2) + bno$~~

~~$\therefore \phi'(s) = \frac{1}{(s^2)^2} \cdot \frac{-4}{s^3}$~~

~~$= -4 \cdot s^{-4} = -\frac{4}{s^4}$~~

~~$\therefore L^{-1}[\phi'(s)] = -4 L^{-1}\left[\frac{s^4}{(s^2+2)^2 - 4s^2}\right]$~~

~~$\therefore L^{-1}[\phi'(s)] = -4 L^{-1}\left[\frac{s}{(s^2-2s+2)(s^2+2s+2)}\right]$~~

~~$= -4 L^{-1}\left[\frac{4s}{(s^2-2s+2)(s^2+2s+2)}\right]$~~

~~$= -4 \left[ L^{-1}\left[\frac{1}{s^2-2s+2}\right] - L^{-1}\left[\frac{1}{s^2+2s+2}\right] \right]$~~

by Laplace  
Inverse of  
Derivatives

 $= L^{-1}\left[\frac{1}{s^2+2s+2}\right] - L^{-1}\left[\frac{1}{s^2-2s+2}\right]$

~~$\therefore L^{-1}\left[\frac{1}{s^2+2s+2}\right] = L^{-1}\left[\frac{(s+1)^2 + 1^2}{(s+1)^2 + 1^2}\right]$~~

$= e^{-t} \cdot L^{-1}\left[\frac{1}{s^2+1^2}\right] - e^t L^{-1}\left[\frac{1}{s^2+1^2}\right] \quad (\text{Using First shifting property})$

$= e^{-t} \cdot \sin(t) - e^t (\sin(t))$

$-t \cdot L^{-1}[\phi(s)] = \sin(t) (e^{-t} - e^t)$

$\therefore L^{-1}[\phi(s)] = \sin(t) \left( \frac{e^t - e^{-t}}{t} \right)$

• Convolution Theorem  $\rightarrow$

$$\text{If } L[f_1(t)] = \phi_1(s)$$

$$\text{and } L[f_2(t)] = \phi_2(s)$$

$$\text{then } L[\phi_1(s) \cdot \phi_2(s)] = \int_0^t f_1(u) \cdot f_2(t-u) du$$

$$\text{where, } f_1(t) = L^{-1}[\phi_1(s)] \text{ and } f_2(t) = L^{-1}[\phi_2(s)].$$

Q. Find inverse Laplace using convolution theorem.

$$(i) \frac{s^2 + 2s + 3}{(s+2s+3)(s+2s+5)}$$

$$\frac{(s^2+1)(s^2+2s+2)}{(s+2s+2)(s+2s+5)}$$

$$(ii) \frac{s^2 + 1}{(s+2s+2)^2}$$

$$\frac{(s^2+1)(s^2+2s+2)}{(s+2s+2)^2}$$

★ Take Lengthy Laplace function as  $\phi_1(s)$  and simpler one as  $\phi_2(s)$

$$\frac{(s^2+1)(s^2+2s+2)}{(s+2s+2)^2}$$

$$(s^2+1)(s^2+2s+2) =$$

$$(s^2+1)(s^2+2s+2) = ((s+1)^2 + 1)((s+1)^2 + 3)$$

$$(s^2+1)(s^2+2s+2) = ((s+1)^2 + 1)((s+1)^2 + 3)$$

$$\Phi(s) = \frac{s^2 + 2s + 3}{(s^2 + 2s + 2)(s^2 + 2s + 5)} \quad 2+^2 = (2)\phi \quad \text{C. D.}$$

$$\text{Ans.} = \frac{(s+1)^2 + 2}{((s+1)^2 + 1)((s+1)^2 + 4)}$$

$$\therefore L^{-1}[\Phi(s)] = e^{-t} \cdot L^{-1}\left[\frac{s^2 + 2}{(s^2 + 1)(s^2 + 4)}\right] \quad (\text{by first shifting property})$$

$$= e^{-t} \left[ L^{-1}\left[\frac{s^2}{(s^2 + 1)(s^2 + 4)}\right] + \frac{2}{3} L^{-1}\left[\frac{1}{s^2 + 1} - \frac{1}{s^2 + 4}\right] \right]$$

$$= e^{-t} \left[ L^{-1}\left[\frac{s^2 + 2}{(s^2 + 1)(s^2 + 4)}\right] + \frac{2}{3} \left( \sin(t) - \frac{1}{2} \sin(2t) \right) \right]$$

Finding  $L^{-1}\left(\frac{s^2}{(s^2 + 1)(s^2 + 4)}\right)$  using convolution,

$$\text{Let } \phi_1(s) = \frac{s}{s^2 + 4} \quad \therefore L^{-1}[\phi_1(s)] = \cos(2t)$$

$$\phi_2(s) = \frac{s}{s^2 + 1} \quad \therefore L^{-1}[\phi_2(s)] = \cos(t)$$

$$\therefore L^{-1}[\phi_1(s) \phi_2(s)] = \int_0^t \cos(2u) \cdot \cos(t-u) du$$

$$= \frac{1}{2} \int_0^t [\cos(3u-t) + \cos(u+t)] du$$

$$= \frac{1}{2} \left[ \frac{1}{2} [\sin(u+t) + \frac{\sin(3u-t)}{3}] \right]_0^t$$

$$\text{Putting limits} \rightarrow + \Rightarrow \frac{1}{2} \left[ \frac{4}{3} \sin(2t) - \frac{2}{3} \sin(t) \right]$$

$$\therefore L^{-1}[\Phi(s)] = e^{-t} \left[ \left( \frac{2}{3} \sin(2t) - \frac{1}{3} \sin(t) \right) + \frac{2}{3} \sin(t) - \frac{1}{3} \sin(2t) \right]$$

$$= e^{-t} \left[ \frac{1}{3} \sin(2t) + \frac{1}{3} \sin(t) \right]$$

$$L^{-1}[\Phi(s)] = \frac{e^{-t}}{3} [\sin(2t) + \sin(t)]$$

$$\text{Q.2} \quad \phi(s) = \frac{s^2 + s}{(s^2 + 1)(s^2 + 2s + 2)} = \frac{s+2s+s^2}{(s^2+1)(s^2+2s+2)} = \frac{s+2(s+1)}{(s^2+1)(s^2+2s+2)}$$

Ans.

$$= \frac{s(s+1)}{(s^2+1)(s^2+2s+2)} = \frac{s+2(s+1)}{(s^2+1)(s^2+2s+2)}$$

$$\text{Let } \phi_1(s) = \frac{s+1}{(s+1)^2 + 1}, \quad \phi_2(s) = \frac{s+2}{s^2+1}$$

$$\therefore L^{-1}[\phi_1(s)] = L^{-1}\left[\frac{s+1}{(s+1)^2 + 1}\right]$$

$$\text{By first shifting property } \Rightarrow f_1(t) = e^{-t} \cdot L^{-1}\left[\frac{s}{s^2+1}\right] = e^{-t} \cdot \cos(\omega_0 t)$$

$$L^{-1}[\phi_2(s)] = L^{-1}\left[\frac{s}{s^2+4}\right] = \cos(\omega_0 t)$$

By convolution theorem,  $f(t) = f_1(t) * f_2(t)$

$$L^{-1}[\phi_1(s)\phi_2(s)] = \int_0^t f_1(u) f_2(t-u) du$$

$$= \int_0^t e^{-u} \cos(u) \cdot \cos(t-u) du$$

$$= \frac{1}{2} \int_0^t e^{-u} (\cos(t) + \cos(2u-t)) du$$

$$(1) \text{ m2} = \frac{1}{2} \left[ \int_0^t e^{-u} \cos(t) du + \int_0^t e^{-u} \cos(2u-t) du \right]$$

$$(2) \text{ m12} = \frac{1}{2} \left[ -\cos(t)e^{-u} + \frac{e^{-u}}{1+4} \cdot (-\cos(2u-t) + 2\sin(2u-t)) \right]_0^t$$

$$[(1) \text{ m2} + (2) \text{ m12}] \Big|_0^t = [(2) \text{ m12}] \Big|_0^t$$

(next page)

$$\begin{aligned}
 \text{Putting limit} &= \frac{1}{2} \left[ -\cos(t) (e^{-t} - 1) + e^{-t} (-\cos t + 2\sin t) - \frac{1}{5} (-\cos t - 2\sin t) \right] \\
 &= \frac{1}{2} \left[ \cos(t) - e^{-t} \cdot \cos(t) - e^{-t} \cdot \cos(t) + \frac{2e^{-t} \sin t}{5} + \frac{\cos(t)}{5} + \frac{2\sin(t)}{5} \right] \\
 &= \frac{1}{2} \left[ \cos(t) \left( \frac{6}{5} - \frac{6e^{-t}}{5} \right) + \sin(t) \left( \frac{2}{5} + \frac{2e^{-t}}{5} \right) \right] \\
 &= \frac{3 \cos(t)}{5} \left[ 1 - e^{-t} \right] + \frac{1}{5} \sin(t) \left[ 1 + e^{-t} \right] \\
 &\quad \text{Hence } 2 = (2)_{s^2} \phi \quad \text{and } (3)_{s^2} \phi = 1
 \end{aligned}$$

$$(us)_{s^2} \omega_{200} = [(2)_{s^2} \phi]^{1/2} = [(2)_{s^2} \phi]^{1/2}$$

merely initial value problem

$$\rho b(u-v) + (v)^+ = [(2)_{s^2} \phi \cdot (2)_{s^2} \phi]^{1/2}$$

$$\rho b \left( \frac{(u-v)}{25-25} \omega_{200} + (us) \omega_{200} \right)^{1/2} =$$

$$\left[ \rho b \left( \frac{(u-v)}{25-25} \omega_{200} \right)^2 + \rho b (us) \omega_{200} \right]^{1/2} =$$

$$\left[ \left[ (u-v) \rho b \omega_{200} \right]^{1/2} \cdot \left[ \rho b \cdot (us) \omega_{200} \right]^{1/2} \right] =$$

$$\left[ \left[ (us) \omega_{200} + (us) \omega_{200} \right]^{1/2} + (us) \omega_{200} \right] = \leftarrow \text{final profit}$$

$$(us) \omega_{200} \cdot 1 + (us) \omega_{200} \cdot \frac{1}{2} =$$

Q. Find  $L^{-1} \left[ \frac{(s+3)^2}{(s^2+6s+5)^2} \right]$  using convolution theorem.

$$\text{Ans: } \therefore \phi(s) = \frac{(s+3)^2}{(s^2+4s+4)^2}$$

By first shifting property,

$$L^{-1}[\phi(s)] = e^{-3t} \left[ \frac{s^2}{(s^2-4)^2} \right]$$

$$\text{Let } \phi_1(s) = \phi_2(s) = \frac{s}{s^2-4}$$

$$\therefore L^{-1}[\phi_1(s)] = L^{-1}[\phi_2(s)] = \cosh(2u)$$

By convolution theorem,

$$\begin{aligned} L^{-1}[\phi_1(s) \cdot \phi_2(s)] &= \int_0^t f_1(u) f_2(t-u) du \\ &= \int_0^t \cosh(2u) \cdot \cosh\left(\frac{2(t-u)}{2}\right) du \\ &= \frac{1}{2} \left[ \int_0^t \cosh(2t) du + \int_0^t \cosh(uu-2t) du \right] \\ &= \frac{1}{2} \left[ \cosh(2t) \cdot [u]_0^t + \left[ \frac{\sinh(uu-2t)}{4} \right]_0^t \right] \end{aligned}$$

$$\begin{aligned} \text{Putting limits} \rightarrow &= \frac{1}{2} \left[ t \cdot \cosh(2t) + \left( \frac{\sinh(2t)}{4} + \frac{\sinh(2t)}{4} \right) \right] \\ &= \frac{t}{2} \cdot \cosh(2t) + \frac{1}{4} \cdot \sinh(2t) \end{aligned}$$

$$\text{Q. Find } \phi(s) = \frac{1}{s} \log \left( \frac{s+3}{s+2} \right)$$

$\Leftrightarrow$  natural logarithm

$$(s+3)^{-1} - (s+2)^{-1} = \text{FE}$$

$$\left[ \frac{1}{s+3} \right]_{0}^{\infty} - \left[ \frac{1}{s+2} \right]_{0}^{\infty} = [(sF)]_0^{\infty} \text{ inst}$$

$$\left. \begin{array}{l} s+3 \rightarrow 0 \\ s+2 \rightarrow 0 \end{array} \right\} = (sF)_0^{\infty} \text{ inst}$$

$$\left[ \frac{1}{s+3} \right]_{0}^{\infty} - \left[ \frac{1}{s+2} \right]_{0}^{\infty} = [(sF)]_0^{\infty} \text{ inst}$$

$$\left[ \frac{1}{s+3} \right]_{0}^{\infty} + \left[ \frac{1}{s+2} \right]_{0}^{\infty} = \frac{1}{(s+3-1)2}$$

$$\left( \frac{1}{s+3} + \frac{1}{s+2} \right) \frac{1}{2} = \frac{1}{(s+3-1)2}$$

$$\left( \frac{1}{s+3} + \frac{1}{s+2} \right) \frac{1}{2} = \frac{1}{(s+3-1)2}$$

$$\left( \frac{1}{s+3-1} + \frac{1}{s+2-1} \right) \frac{1}{2} = \frac{1}{(s+3-1)(s+2-1)2}$$

(reversing the replacement need no  $s^{20}$ ,  $s^{19}$ ,  $s^{18}$  etc.)

$$\frac{1}{s+2} - \frac{1}{s+3} \times \frac{1}{2} = [(sF)]_0^{\infty}$$

$$\frac{1}{s+2} \left( \frac{1}{s+3} - \frac{1}{s+2} \right) \times \frac{1}{2} =$$

$$\left( \frac{1}{s+2} \right) \frac{1}{s+3} \times \frac{1}{2} =$$

$$\left( \frac{1}{s+2} \right) \frac{1}{s+3} \times \frac{1}{2} = [(sF)]_0^{\infty}$$

- Periodic function  $\rightarrow$

If  $f(t) = f(t+a)$ ,

$$\text{then } L[f(t)] = \frac{1}{1-e^{-as}} \int_0^a e^{-st} \cdot f(t) dt$$

Q.  $f(t) = \begin{cases} k & \text{if } 0 < t < a \\ -k & \text{if } a < t < 2a \end{cases}$

$$\begin{aligned} \text{Ans. } L[f(t)] &= \frac{1}{1-e^{-2as}} \int_0^{2a} e^{-st} \cdot f(t) dt \\ &= \frac{1}{1-e^{-2as}} \left[ \int_0^a k \cdot e^{-st} dt + \int_a^{2a} -k \cdot e^{-st} dt \right] \\ &= \frac{k}{s(1-e^{-2as})} (-e^{-as} + 1 + e^{-2as} - e^{-as}) \\ &= \frac{k}{s(1-e^{-2as})} (e^{-as} - 1)^2 \\ &\therefore \frac{k(-e^{-as} + 1)^2}{s(1-e^{-as})(1+e^{-as})} = \frac{+k}{s} \times \frac{(e^{-as} - 1)(1-e^{-as})}{(1+e^{-as})} \end{aligned}$$

(Multiply by  $e^{as/2}$  on both numerator and denominator)

$$\begin{aligned} \therefore L[f(t)] &= \frac{k}{s} \times \frac{e^{as/2} - e^{-as/2}}{e^{as/2} + e^{-as/2}} \\ &= \frac{k}{s} \times \frac{(e^{as/2} - e^{-as/2})/2}{(e^{as/2} + e^{-as/2})/2} \\ &= \frac{k}{s} \times \frac{\sinh(\frac{as}{2})}{\cosh(\frac{as}{2})} \end{aligned}$$

$$L[f(t)] = \frac{k}{s} \times \tanh\left(\frac{as}{2}\right)$$

Q1. Find Laplace transform of full wave rectification of  $\sin(wt)$ .

$$f(t) = |\sin(wt)|, t \geq 0$$

Ans. Period of function =  $\frac{\pi}{w}$

$$\therefore L[f(t)] = \frac{1}{1 - e^{-\pi s/w}} \int_0^{\pi s/w} e^{-st} \cdot \sin(wt) dt$$

$$= \frac{1}{1 - e^{-\pi s/w}} \cdot \left[ \frac{e^{-st}}{s^2 + w^2} (-s \cdot \sin(wt) - w \cdot \cos(wt)) \right]_0^{\pi s/w}$$

$$= \frac{1}{1 - e^{-\pi s/w}} \left[ \frac{e^{-\pi s/w}}{s^2 + w^2} ((-s \cdot 0) - w(-1)) - \frac{1}{s^2 + w^2} (0 - w) \right]$$

=

$$\therefore L[f(t)] = \frac{1}{1 - e^{-\pi s/w}} \left( w \left( \frac{e^{-\pi s/w}}{(s^2 + w^2)} + \frac{1}{(s^2 + w^2)} \right) \right)$$

$$= \frac{w \cdot (1 + e^{-\pi s/w})}{(s^2 + w^2) \cdot (1 - e^{-\pi s/w})}$$

$\therefore f(t)H = \frac{1 + e^{-\pi s/w}}{1 - e^{-\pi s/w}}$

$$(f)H = \frac{1 + e^{-\pi s/w}}{1 - e^{-\pi s/w}}$$

$$\therefore f = (f)H$$

- Heaviside Unit Step Function  $\rightarrow$  forward shifted part.

The function is defined by,

$$H(t-a) = \begin{cases} 1 & \text{for } t \geq a \\ 0 & \text{for } t < a \end{cases}$$

If  $a=0$ , then

$$H(t) = \begin{cases} 1 & \text{for } t \geq 0 \\ 0 & \text{for } t < 0 \end{cases}$$

$$\rightarrow L[H(t-a)] = \frac{e^{-as}}{s}$$

$$\therefore L^{-1}\left[\frac{e^{-as}}{s}\right] = H(t-a)$$

At  $a=0$ ,

$$L^{-1}\left[\frac{1}{s}\right] = H(t)$$

$$\therefore H(t) = 1$$

\* • Important Result  $\Rightarrow H(t) = (1 - e^{-st}) \cdot f(t)$  (from 1.1 Unit 1)

If  $L[f(t)] = \phi(s)$ , then  $L[f(t-a) \cdot H(t-a)] = e^{-as} \cdot \phi(s)$

then  $L[f(t-a) \cdot H(t-a)] = e^{-as} \cdot \phi(s)$

ALSO ↴

$$\text{If } f(t) = \begin{cases} f_1(t) & \text{for } 0 \leq t < a \\ f_2(t) & \text{for } a \leq t < b \\ f_3(t) & \text{for } t > b \end{cases}$$

then,

$$\begin{aligned} f(t) &= f_1(t) [H(t) - H(t-a)] \\ &\quad + f_2(t) [H(t-a) - H(t-b)] \\ &\quad + f_3(t) [H(t-b)] \end{aligned}$$

Q. Find  $L \left[ \sin(t) (H(t - \pi/2) - H(t - 3\pi/2)) \right]$

Ans.  $= \int_{\pi/2}^{3\pi/2} e^{-st} \cdot \sin(t) dt$

$$= \left[ \frac{e^{-st}}{s^2 + 1} (-s \cdot \sin(t) - \cos(t)) \right]_{\pi/2}^{3\pi/2}$$

$$= \left( \frac{e^{-3\pi s/2}}{s^2 + 1} \right) (s) - \left( \frac{e^{-\pi s/2}}{s^2 + 1} \right) (-s)$$

$$= \frac{s}{s^2 + 1} \times \left( e^{-3\pi s/2} + e^{-\pi s/2} \right)$$

Q. Express the following function, in terms of unit step function and obtain its Laplace Transform.

$$\text{Ans. } f(t) = \begin{cases} t^2 & , 0 < t < 1 \\ 4t & , t > 1 \end{cases}$$

$$\therefore f(t) = t^2 \cdot [H(t) - H(t-1)] + 4t [H(t-1)] = (t^2 + 4t) H(t-1)$$

$$\therefore L[f(t)] = L[t^2 \cdot H(t)] + L[(4t - t^2) \cdot H(t-1)]$$

We know,

$$L[f(t) \cdot H(t-a)] = e^{-as} \cdot L[f(t+a)]$$

$$\therefore L[f(t)] = L[t^2] + e^{-s} \cdot L[4(t+1) - (t+1)^2]$$

$$(1) \boxed{3 = \frac{1}{s^2} + e^{-s} \cdot L[2t+3+t^2]}$$

$$\therefore L[f(t)] = \frac{2}{s^3} + e^{-s} \left( \frac{2}{s^2} + \frac{3}{s} - \frac{2}{s^3} \right)$$

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Date 19/9/22

## Dirac Delta Function (Unit Impulse Function)

Consider  $F(t) = \begin{cases} \frac{1}{\epsilon} & \text{for } -\epsilon < t < \epsilon \\ 0 & \text{otherwise} \end{cases}$

$$\therefore \lim_{\epsilon \rightarrow 0} F(t) = \begin{cases} \infty & \text{at } t=0 \\ 0 & \text{otherwise} \end{cases}$$

$$(1-s)H(s-t) + sH(s-t) = \delta(t)$$

$$\left[ \int_0^{\infty} \delta(t-a) dt = 1 \right] \quad \text{, called SVD}$$

$$\rightarrow L[\delta(t-a)] = e^{-as}$$

If  $a=0$ ,

$$\text{then } L[\delta(t)] = 1 \Rightarrow L[1] = \delta(t)$$

$$\rightarrow L[f(t) \cdot \delta(t-a)] = e^{-as} f(a)$$

Q. Find  $L[t \cdot H(t-a) + t^2 \cdot \delta(t-a)]$

Ans.  $= L\left[\underset{f(t)}{\tilde{t}} \cdot H(t-a)\right] + L\left[\underset{f(t)}{t^2} \cdot \delta(t-a)\right]$   $\therefore [t^n] = [t^n] \quad \text{①}$

$$= e^{-as} \cdot [t^1] [(t+a)] + 2e^{-as} [a^2] \cdot s_2 = [(t)^{n+1}] \quad \text{②}$$

$$\therefore e^{-as} \left( \frac{1}{s} + \frac{a}{s^2} \right) + e^{-as} [a^2] \cdot s_2 = [(t)^{n+1}] \quad \text{③}$$

Q. Evaluate:  $\int t \cdot e^{2t} \cdot \sin(3t) \cdot \delta(t-2) dt$

$\Omega = (0)_{\mu}$  b/w  $\mu = (0)_{\mu}$  norm

Ans.  $\therefore \int_0^\infty \tilde{e}^{(2t)} \cdot t \cdot \sin(3t) \cdot \delta(t-2) dt$

$$= \left[ \frac{t^2}{2} + \frac{t^3}{6} \right] \cdot f(t) + [t^2] \cdot s + [t^3] \quad \text{.2NA}$$

$$\Omega = L \left[ \tilde{t} \cdot f(t) \right]_{s=-2}^{s=1+2} = (0)_V - \bar{e} \cdot 2 \cdot s + [(0)^1 V - (0)V \cdot 2 - \bar{e} \cdot s_2] =$$

$$\therefore L \left[ \underbrace{t \cdot \sin(3t)}_{f(t)} \cdot \delta(t-2) \right]_{s=-2} = (0)_V - \bar{e} \cdot 2 \cdot s + [(0)^1 V - (0)V \cdot 2 - \bar{e} \cdot s_2] \quad \text{.2NA}$$

We know.

$$= \bar{e} \cdot L[f(t) \cdot \delta(t-a)] = \bar{e}^{-as} \cdot f(a) \quad \text{.2NA}$$

$$\therefore L[f_1(t) \cdot \delta(t-2)] = \bar{e}^{2s} \cdot f_1(2)$$

$$= \frac{s=-2}{a=+2} \bar{e}^{2s} \cdot f_1(2) = \bar{e}^{2s} \cdot f_1(2)$$

$$= \bar{e}^{2s} \cdot 2 \cdot \sin(s)$$

$$= \frac{s=0}{a=+2} \bar{e}^{2s} \cdot 2 \cdot \sin(s) = \bar{e}^4 \cdot 2 \cdot \sin(2)$$

$$= 2 \cdot e^4 \cdot \sin(2)$$

$$= \frac{2 \cdot e^4 \cdot \sin(2)}{s(s+2)} \quad \text{.2NA}$$

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Ex 9.2

• Laplace for solving Differential Equations (part 1)

$$\text{① } L[f'(t)] = s \cdot L[f(t)] - f(0) \quad [L(f(t))] = \frac{1}{s} f'(0) + \int_0^s f(t) dt$$

$$\text{② } L[f''(t)] = s^2 \cdot L[f(t)] - s \cdot f(0) - f'(0)$$

$$\text{③ } L[f'''(t)] = s^3 \cdot L[f(t)] - s^2 \cdot f(0) - s \cdot f'(0) - f''(0)$$

Q. Solve  $\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + y = 3 \cdot t \cdot e^t$

given  $y(0) = 4$  and  $y'(0) = 2$

$$\text{Ans. } L[y''] + 2L[y'] + L[y] = 3L[t \cdot e^t]$$

$$= [s^2 \bar{y} - s \cdot y(0) - y'(0)] + 2[s \cdot \bar{y} - y(0)] + (\bar{y}) = \frac{3}{(s+1)^2}$$

But  $y(0) = 4$  and  $y'(0) = 2$

$$= [\bar{y}(s^2 + 2s + 1) - 4s - 2] + 2[\bar{y}(s + 4)] + \bar{y} = \frac{3}{(s+1)^2}$$

$$= \bar{y}(s^2 + 2s + 1) - 4s - 10 = \frac{3}{(s+1)^2}$$

$$\therefore \bar{y} = \frac{3}{(s+1)^4} + \frac{4s+10}{(s+1)^2}$$

Breaking further  $\rightarrow \bar{y} = \frac{3}{(s+1)^4} + \frac{4}{s+1} + \frac{6}{(s+1)^2}$

$$\begin{aligned} \therefore y = f(t) &= L^{-1} \left[ \frac{3}{(s+1)^4} + \frac{6}{(s+1)^2} + \frac{4}{s+1} \right] \\ &= 3e^{-t} L^{-1} \left[ \frac{1}{s^4} \right] + 6e^{-t} L^{-1} \left[ \frac{1}{s^2} \right] + 4e^{-t} L^{-1} \left[ \frac{1}{s} \right] \\ &= 3e^{-t} \cdot t^3 + 36e^{-t} \cdot t^2 + 4e^{-t} (1) \end{aligned}$$

~~$\left[ (t^2)(t^2) \right] + 4 \cdot 2(t^2) \cdot t^2 = t^2(12 + 4t^2) \right] + [4] \right]$~~

Q. Solve  $y''' + y = t$   $\left[ (0)e^{-0.2}s + (0)e^{-0.2} \cdot s^2 \right] + \left[ (0)e^{-0.2} - (0)e^{-0.2} \cdot s^2 \right] \dots$

Ans. Given:  $y(\pi) = 0$  and  $y'(0) = 1$

Ans. Let  $y(0) = \alpha$ .  $\therefore y(s) = \frac{1}{s^2+1} + \frac{\alpha s}{s^2+1} + \frac{1}{s+1}$

$\therefore$  Taking Laplace on both sides,

$$L[y'''] + L[y] = tL[t] + \dots$$

Using formulae,  $L[t^3] = \frac{1}{s^4}$ ,  $L[t] = \frac{1}{s^2}$

$$\left[ s^2 \bar{y} - s y(0) - y'(0) \right] + \bar{y} = \frac{1}{s^2} + \frac{1}{s+1}$$

$$y(0) = \alpha, y'(0) = 1$$

$$\therefore s^2 \bar{y} + \alpha s - 1 + \bar{y} = \frac{1}{s^2} + \frac{1}{s+1}$$

$$(s^2+1)\bar{y} + \alpha s - 1 = \frac{1}{s^2} + \frac{1}{s+1}$$

$$\bar{y}(s^2+1) = \frac{1}{s^2} + \alpha s + 1$$

$$(s^2+1)\bar{y} + \alpha s - 1 = \frac{1}{s^2} + \frac{1}{s+1}$$

$$\bar{y} = \frac{1}{s^2(s^2+1)} + \frac{\alpha s}{s^2+1} = \frac{1}{s^2} - \frac{1}{s^2+1} + \frac{\alpha s}{s^2+1} + \frac{1}{s^2+1}$$

$$L[f(t)] = \frac{1}{s^2} - \frac{1}{s^2+1} + \frac{\alpha s}{s^2+1} + \frac{1}{s^2+1}$$

$$\therefore f(t) = L^{-1} \left( \frac{1}{s^2} \right) + \alpha L^{-1} \left( \frac{s}{s^2+1} \right)$$

$$f(t) = t + \alpha \cos(t) \quad [\text{but } y(\pi) = 0]$$

$$\therefore f(\pi) = \pi + \alpha \cdot (-1) \quad \therefore \alpha = \pi$$

$$f(t) = t + \pi \cos(t)$$

$$y'' + 2y' + 5y = 8\sin(t) + 4\cos(t)$$

Q.  $y'' + 2y' + 5y = 8\sin(t) + 4\cos(t)$   
Given  $y(0) = 1$  and  $y(\pi/4) = \sqrt{2}$

Ans. Let  $y'(0) = \alpha$ .

Taking Laplace on both sides,

$$L[y''] + 2L[y'] + 5L[y] = 8L[\sin(t)] + 4L[\cos(t)]$$

$$\therefore [s^2\bar{y} - s \cdot y(0) - y'(0)] + 2[s\bar{y} - y(0)] + 5\bar{y} = \frac{8}{s^2+1} + \frac{4s}{s^2+1}$$

$$(s^2+2s+5)\bar{y} - s - \alpha - 2 = 4s+8$$

$$\therefore \bar{y} = \frac{4s+8}{s^2+2s+5} + \frac{s+\alpha+2}{s^2+2s+5}$$

$$\bar{y} = 2\left(\frac{1}{s^2+1} - \frac{1}{s^2+2s+5}\right) + \frac{(s+1)+(\alpha+1)}{(s+1)^2+4}$$

$$\therefore yf(t) = 2L^{-1}\left(\frac{1}{s^2+1}\right) - 2L^{-1}\left(\frac{1}{s^2+2s+5}\right) + L^{-1}\left(\frac{s+1}{(s+1)^2+4}\right) + (s+1)L^{-1}\left(\frac{1}{(s+1)^2+4}\right)$$

$$= 2\sin(t) - 2e^{-t}\sin(2t) + e^{-t}\cos(2t) + (\alpha+1)e^{-t}\sin(2t)$$

$$f(t) = 2\sin(t) + e^{-t}\cos(2t) + (\alpha-1)e^{-t}\sin(2t)$$

By  $y(\pi/4) = \sqrt{2}$

$$\therefore \sqrt{2} = \sqrt{2} + 0 + (\alpha-1)e^{-\pi/4} \quad \therefore \alpha = 1$$

$$\therefore f(t) = 2\sin(t) + e^{-t}\cos(2t)$$

$$= 2\sin(t) + e^{-t}(2\cos^2 t + \sin^2 t) = 2\sin(t) + e^{-t}(3\cos^2 t - 1) = 2\sin(t) + e^{-t}(2\cos^2 t - 1) = 2\sin(t) + e^{-t}\cos(2t)$$

$$= 2\sin(t) + e^{-t}\cos(2t) = 2\sin(t) + e^{-t}(2\cos^2 t - 1) = 2\sin(t) + e^{-t}(2\cos^2 t - 1) = 2\sin(t) + e^{-t}\cos(2t)$$

$$Q. \frac{dy}{dx} + 2y + \int_0^t y dt = \sin(t)$$

$\left\{ \begin{array}{l} \text{Ans. } y(t) \\ y(t) = -\frac{3}{2}e^{-t} \cdot t \end{array} \right.$

Given  $y(0) = 1$

Ans.  $\therefore y' + 2y + \left[ \frac{y^2}{2} \right]_0^t = \sin(t)$

~~$y' + 2y = \sin(t) - \frac{t^2}{2}$~~

$$L[y'] + 2L[y] + L \left[ \int_0^t y dt \right] = \frac{\text{Net assigned error}}{s^2+1}$$

$$\therefore [s \cdot \bar{y} - y(0)] + 2\bar{y} + \frac{\bar{y}}{s} = \frac{\bar{E}}{s^2+1} + \lambda = \bar{E}$$

$$\therefore (s+2+\frac{1}{s})\bar{y} = \frac{\bar{E}-1}{s^2+1} + \frac{1}{s_2} = \frac{\bar{E}}{s^2+1}$$

$$\left(\frac{s^2+2s+1}{s}\right)\bar{y} = \frac{1}{s^2+1} + \frac{1}{s_2} = \frac{1}{s_2}$$

$$\therefore \bar{y} = \frac{s}{(s^2+1)(s+1)^2} + \frac{s+1}{(s+1)^2} + \frac{1}{s_2} = (\text{?}) + \dots$$

$$\frac{s+1}{(s+1)^2} + \frac{1}{s_2} = (\text{?}) + \dots$$

PP

CS P. O.S

Q. Solve  $y(t) = kt + \int_0^t y(u) \cdot \sin(t-u) du$

Ans. By convolution,

$$\begin{aligned} L\left[\int_0^t f_1(u) f_2(t-u) du\right] &= \phi_1(s) \cdot \phi_2(s) \\ &= L[f_1(t)] \cdot L[f_2(t)] \end{aligned}$$

: Taking Laplace on both sides,

$$\bar{y} = \frac{k}{s^2} + \frac{\bar{y}}{s^2+1} = \frac{1}{2} s + \frac{1}{2} s + \frac{1}{2} (s^2+1)$$

$$\bar{y} \left(1 - \frac{1}{s^2+1}\right) = \frac{k}{s^2} \quad \therefore \bar{y} = \frac{k(s^2+1)}{s^4} \left(1 + \frac{1}{s^2}\right)$$

$$\bar{y} = \frac{k}{s^2} + \frac{k}{s^4} \quad \therefore \bar{y} = \frac{1}{2} s + \frac{1}{2} s + \frac{1}{2} (s^2+1)$$

$$\therefore f(t) = \frac{k t^{2-1}}{1} + \frac{k t^{4-1}}{s(s+1)(s+2)} = \frac{k t}{2} + \frac{k t^3}{s(s+1)(s+2)}$$

$$f(t) = kt + \frac{kt^3}{6}$$

29/09/22 विषय

Q. Solve  $\frac{d^2y}{dt^2} + 4y = g(t)$ , given  $y(0) = 0$  and  $y'(0) = 1$

$$\text{and } g(t) = \begin{cases} 1 & \text{for } 0 < t \leq 1 \\ 0 & \text{for } t > 1 \end{cases}$$

Ans: निम्नलिखित बहुपद से यहां पर्याप्त है।

$$y(t) = \frac{1}{2} \sin(2t) + \frac{(1-\cos(2t))}{2} \cdot [1 - \cos(2(t-1))] + H(t-1)$$

इस नियम का उपयोग एवं इसका प्रतिवर्ती एवं अवकलन करने की विधि

Ans.

(JS+D, 0) द्वारा दिए गए नियमों का अनुपालन करते हुए (1)

$$(JS+x)^2 = (x)^2 + \text{बाकी}$$

ज्ञात किया गया JS नियम का अनुपालन करते हुए (2)

(JS+D, 0) द्वारा दिए गए नियमों का अनुपालन करते हुए (3)

ज्ञात किया गया JS नियम का अनुपालन करते हुए (4)

ज्ञात किया गया अनुपालन का अनुपालन करते हुए (5)

ज्ञात किया गया JS नियम का अनुपालन करते हुए (6)

$$\left[ \begin{array}{l} \text{ज्ञात किया गया} \\ \text{नियम } (x)^2 = x^2 \\ \text{ज्ञात किया गया } (x)^2 = x^2 \\ \text{ज्ञात किया गया } (x)^2 = x^2 \end{array} \right] \rightarrow \left[ \begin{array}{l} [(x\sin(x))^2 + (x\cos(x))^2] x^2 + x^2 = (x)^2 \\ [(x\sin(x))^2 + (x\cos(x))^2] x^2 + x^2 = (x)^2 \\ [(x\sin(x))^2 + (x\cos(x))^2] x^2 + x^2 = (x)^2 \end{array} \right]$$

$$\left[ \begin{array}{l} x^2 (x)^2 \\ x^2 (x)^2 \end{array} \right] \stackrel{1 = \alpha}{=} \text{अनुपालन}$$

$$\left[ \begin{array}{l} x^2 (x)^2 \\ x^2 (x)^2 \end{array} \right] \stackrel{1 = \alpha}{=} \text{अनुपालन}$$

$$\left[ \begin{array}{l} x^2 (x)^2 \\ x^2 (x)^2 \end{array} \right] \stackrel{1 = \alpha}{=} \text{अनुपालन}$$