UNIT - II

STATISTICAL AVERAGES

Mean and variances

Mean of a R.V X is denoted by $E(x)$ (expected value of X) defined as

$$E(X) = \sum_{i=1}^{n} x_i \ast P(X = x_i)$$

$$\sigma^2 = E(x^2) - E(X)^2$$

PROPERTIES

1) $E(c) = \sum_{i=1}^{n} P(X = x_i) = c \sum P(X = x_i) = c \cdot 1 = c$

2) $E(x+y) = E(X) + E(Y)$

3) $E(ax+b) = a\cancel{E(x+b)}$ $\quad a \cdot E(x) + b$

4) $E(XY) = E(X) \cdot E(Y)$ are independent

1) Find mean, $E(x^2)$, $E(4x^2)$, $E(3x+2x^2)$

for

X	0	1	2	3
$P(X=x)$ ~~0.2~~	0.2	0.1	0.4	0.3

SOL

$$E(g(x)) = \sum_{i=1}^{n} g(x) \, P(X = x_i)$$

$$= \quad E(x) = 0 \times 0.2 + (1 \times 0.1) + (2 \times 0.4) + 3(0.3)$$

$$E(x^2) = \cancel{(0 \times 0.2)^2 + (1 \times 0.1)^2 + (2 \times 0.4)^2 + 3(0.3)^2}$$

$$0^2 \times 0.2 + 1^2 \times 0.1 + 2^2 \times 0.4 + 3^2 \times 0.3^2$$

Variance : denoted by $\sigma^2 = E((X - E(X))^2)$

$$= E\left(X^2 - 2XE(X) + E(X)^2\right)$$

$$= E(X^2) - 2E(X)E(X) + (E(X))^2$$

$$= E(X^2) - (E(X))^2$$

Mean and variance of a continuous RV X

$$E(X) = \int_{-\infty}^{\infty} x\, f(x)\, dx$$

$$var(X) = \int_{-\infty}^{\infty} x^2 f(x)\, dx - \left[\int_{-\infty}^{\infty} x\, f(x)\, dx\right]^2$$

$E(x^2) - E(x)^2)$

PROPERTIES OF VARIANCE

1) $Var(c) = V(c) = 0$

2) $Var(ax) = a^2\, var(x)$

3) $Var(ax+b) = a^2\, var(x)$

4) $Var(x+y) = Var(x) + var(y)$
if x and y are independent.

1) let
$$F(x) = \begin{cases} 0 & \text{if } x \le 1 \\ K(x-1)^4 & \text{if } 1 < x \le 3 \\ 1 & \text{if } x > 3 \end{cases}$$

Find i) $f(x)$ ii) K iii) mean

Sol

i)
$$f(x) = \begin{cases} 0 & \text{if } x \le 1 \\ 4K (x-1)^3 & 1 \le x \le 3 \\ 0 & \text{if } x > 3 \end{cases}$$

ii)
$$\int_{-\infty}^{\infty} f(x)\, dx = 1 \Rightarrow \int_{1}^{3} 4K(x-1)^3\, dx = 1$$

$$= 4K\left[\frac{(x-1)^4}{4}\right]_1^3 = 1$$

$$4K\left[\frac{16-0}{4}\right] = 1$$

$K = \frac{1}{16}$

iii) $E(X) = 4\int_1^3 x \cdot \frac{1}{16}(x-1)^3 \, dx$

$= \frac{1}{4}\int_1^3 x(x-1)^3 \, dx$

$x - 1 = t$

$x = t + 1$

$dx = dt$

$= \frac{1}{4}\int_0^2 (t+1)(t)^3 \, dt$

$= \frac{1}{4}\int_0^2 (t^4 + t^3) \, dt = \frac{1}{4}\left[\frac{t^5}{5} + \frac{t^4}{4}\right]_0^2$

$= \frac{1}{4}\left[\frac{32}{5} + \frac{16}{4}\right] = \frac{1}{4}\left[\frac{32}{5} + 4\right] = \frac{13}{5}$

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2) $f(x) = \begin{cases} x & 0 \leq x \leq 1 \\ 2 - x & 1 < x \leq 2 \\ 0 & \text{elsewhere} \end{cases}$

find $E(x)$ & $Var(x)$

SOL

$E(x) = \int_0^1 x f(x) \, dx + \int_1^2 x f(x) \, dx$

$2x - x^2$

$= \int_0^1 x(x) \, dx + \int_1^2 x(2-x) \, dx$

$= \left[\frac{x^3}{3}\right]_0^1 + \left[x^2 - \frac{x^3}{3}\right]_1^2 = 1$

$= \frac{1}{3} + 4 - \frac{4}{3} - 1 + \frac{1}{3} = 1$

$$E(x^2) = \int_0^1 x^2 \cdot x\,dx + \int_1^2 x^2(2-x)\,dx = \frac{7}{6}$$

$$var = \frac{7}{6} - 1^2 = \frac{7}{6} - 1 = \frac{1}{6}$$

3) $f(x) = \frac{1}{3}x^2,\ -1 \le x \le 2,\ g(x) = 4x+5$
find $E(g(x))$.

<u>SOL</u>

$$E(g(x)) = \int_{-1}^{2} (4x+5)\frac{1}{3}x^2\,dx$$

$$= \int_{-1}^{2} \frac{4x^3}{3} + \frac{5x^2}{3}\,dx$$

$$= \left[\frac{x^4}{3} + \frac{5x^3}{9}\right]_{-1}^{2}$$

$$= \left[\frac{16}{3} + \frac{40}{9} - \frac{1}{3} + \frac{5}{9}\right]$$

$$= \left[\frac{15}{3} + \frac{45}{9}\right] = 5+5 = 10$$

4) $f(x) = Ke^{-2x},\ 0 \le x < \infty$ find K and $E(x)$

<u>SOL</u>

$$\int_0^\infty Ke^{-2x} = 1$$

$$K\left[\frac{e^{-2x}}{-2}\right]_0^\infty = 1$$

$$K\left[\frac{1}{2}\right] = 1$$

$$K = 2$$

$$E(x) = \int_0^\infty 2x\, e^{-2x}\, dx$$

$u = x \qquad dv = e^{-2x}\, dx$

$u' = 1 \qquad v = \frac{e^{-2x}}{-2} \qquad v' = \frac{e^{-2x}}{4}$

$\cancel{E(x)} \qquad \int u\,dv = \cancel{\int uv} - v \int du$

$= \int$

$= u \cdot v - u'v' + u'' v'' \ldots$

$= x\left(\frac{e^{-2x}}{-2}\right) - 1\left(\frac{e^{-2x}}{4}\right)$

$$= \left[\frac{-x e^{-2x}}{2} - \frac{e^{-2x}}{4}\right]_0^\infty$$

$\left[0 + \frac{1}{4}\right] = \frac{1}{4}$

Moments → non central

quantitative measures which describe the distribution

Raw (or) non-central (or) moments about origin

$\mu_r' = \sum_{n=1}^{\infty} x^r p(x)$ for discrete

$(x^r = (x-0)^r)$

$\int_{-\infty}^{\infty} x^r f(x)\, dx$ for continuous

$\underline{r=1}$

$\mu_1' = \text{Mean} = \underline{E(X) = \mu}$

$\mu_2' = E(x^2)$

in general, $\mu_r' = E(x^r)$

$\mu_3' = E(x^3)$, $\mu_4' = E(x^4)$

- central moments (or) moments about mean

$\mu_r = \sum_{n=1}^{\infty} (x - E(x))^r \cdot p(x)$ or $\sum_{n=1}^{\infty} (x-\mu)^r p(x)$

$$\boxed{\mu_r = E((x-\mu)^r)}$$

$\underline{r=1}$

$\mu_1 = E(x-\mu)$

$= E(x) - E(\mu)$

$= \mu - \mu = 0$

$\underline{r=2}$

$\mu_2 = E((x-\mu)^2)$

$= E(x^2) - (E(x))^2$

$\sigma^2 = \mu_2 = \mu_2' - (\mu_1')^2$

$r = 3$

$\mu_3 = E((x-\mu)^3)$

$= E(x^3) - E(\mu))^3$

$= \mu_3' - \mu_1^3$

$= E(x^3 - 3x^2\mu + 3x\mu^2 - \mu^3)$

$= E(x^3) - 3E(x^2)\mu + 3E(x)\mu^2 - \mu^3$

$= \mu_3' - 3\mu_2'\mu_1' + 3\mu_1'\mu_1'^2 - \mu_1'^3$

$= \mu_3' - 3\mu_2'\mu_1' + 2\mu_1'^3$

$a^4 - 4a^3b + 6a^2b^2 - 4ab^3 + b^4$

$\mu_4 = \mu_4' - 4\mu_3'\mu_1' + 6\mu_2'\mu_1'^2 - 3\mu_1'^4$

1) Given that first four moments about origin are 1, 4, 10 & 46 respectively find central moments (μ_r for $r = 1$ to 4)

SOL

$\mu_1' = 1$

$\mu_2' = 4$

$\mu_3' = 10$

$\mu_4' = 46$

$\mu_1 = 0$

$\mu_2 = \mu_2' - (\mu_1')^2 = 4 - (1)^2 = 3$

$\mu_3 = \mu_3' - 3\mu_2'\mu_1' + (2\mu_1')^3$

$= 10 - 3(4)(1) + (2(1))^3$

$= 10 - 12 + 8 =$

$$\mu_4 = \mu_4' - 4\mu_3'\mu_1' + 6\mu_2'(\mu_1')^2 - 3\mu_1'^4$$

$$= 27$$

2) For following P.m.f find 3^{rd} non central moments

X	1	2	3
$P(x)$	$\frac{1}{2}$	$\frac{1}{3}$	$\frac{1}{6}$

$$\mu_3' = E(x^3) = 1^3 \cdot \frac{1}{2} + 2^3 \cdot \frac{1}{3} + 3^3 \cdot \frac{1}{6}$$

$$= \frac{1}{2} + \frac{8}{3} + \frac{27}{6} = \frac{23}{3}$$

$$E(x) = \frac{5}{3}$$

$$\mu_3 = \Sigma \left(x - \frac{5}{3}\right)^3 P(x)$$

$$= \left(1 - \frac{5}{3}\right)^3 \frac{1}{2} + \left(2 - \frac{5}{3}\right)^3 \cdot \frac{1}{3} + \left(3 - \frac{5}{3}\right) \cdot \frac{1}{6} = \frac{7}{27}$$

3) $f(x) = \begin{cases} \dfrac{4x(9-x^2)}{81} & 0 \le x \le 3 \\ 0 & \text{otherwised} \end{cases}$

find first 4 non - central, 4 central momens

SOL

$$\mu_r' = \int_0^3 x^r \frac{4x(9-x^2)}{81} \, dx$$

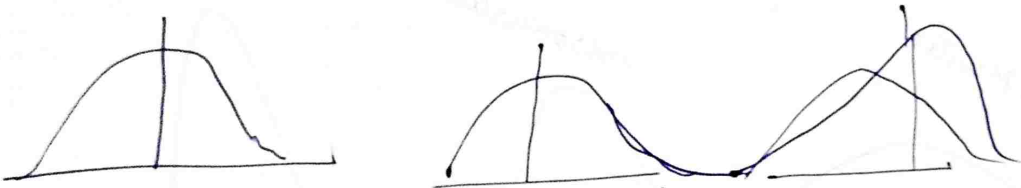
SOL

$$\int x^r f(x)\,dx \begin{cases} E(x) = \frac{8}{5} = \mu = \mu_1' \\ E(x^2) = 3 = \mu_2' \\ E(x^3) = \frac{216}{35} = \mu_3' \\ E(x^4) = \frac{27}{2} = \mu_4' \end{cases}$$

relation b/w central nd non central
$$\begin{cases} \mu_1 = 0 \\ \mu_2 = \frac{11}{25} \\ \mu_3 = \frac{-32}{875} \\ \mu_4 = \frac{3693}{8750} \end{cases}$$

Skewness (3rd central moment)

- Measure of asymetry of distribution



→ amt of deviation

$B_1 = \left(\frac{\mu_3^2}{\mu_2^3}\right)$ (absolute measure) $\gamma_1 > 0$ +ve skew data

$\gamma_2 < 0$ for -ve skew

$\gamma_2 = 0$ for symmetric data

$\sqrt{B_1} = \gamma_1 = \sqrt{\frac{(\mu_3)^2}{(\mu_2)^3}} = \frac{\mu_3}{(\mu_2)^{3/2}} = \frac{\mu_3}{\sigma^3}$

direction of deviation

sign of μ_3 decides the direction of skewness

β_1 - Measure of skewness

γ_1 = coefficient of skewness (to decide direction)

If $-0.5 \le \gamma_1 \le 0.5$ approximately symmetric

If $\gamma_1 = 0 \Rightarrow$ distribution is symmetric

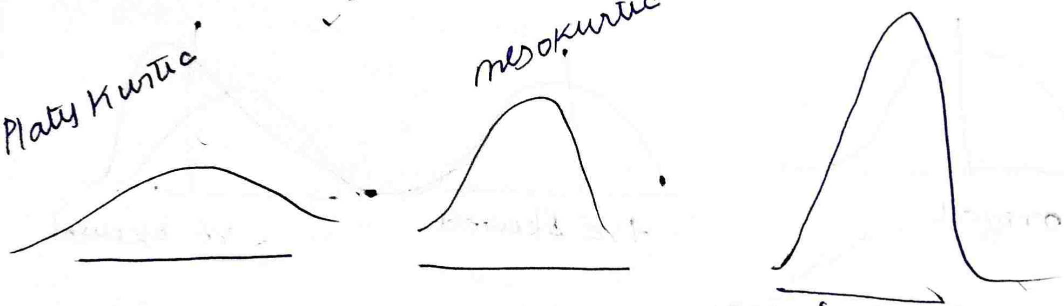
$\gamma_1 < -1$ or $\gamma_1 > 1$ then highly skewed

$-1 < \gamma_1 < -0.5$ & $0.5 < \gamma_1 < 1 \Rightarrow$ moderately skewed

Kurtosis (4th central moment)

→ Measure of tailedness (or) peakedness denoted by $\beta_2 = \frac{\mu_4}{\mu_2^2}$

Platy Kurtic | mesokurtic



tails are thin $\beta_2 < 3$ | $\beta_2 = 3$ | $\beta_2 > 3$ lepto Kurtic

coefficient of kurtosis (subtract extra kurtosis)

$\gamma_2 = \beta_2 - 3$

β

$\rightarrow \beta_2 = 3, \gamma_2 = 0 \rightarrow$ Mesokurtic

$\rightarrow \beta_2 > 3, \gamma_2 > 0 \rightarrow$ leptokurtic

$\rightarrow \beta_2 < 3, \gamma_2 < 0 \rightarrow$ Platykurtic

1) The First 4 raw moments are, 2, 136, 320 and 40,000 Find out skewness and curtosis.

SOL $\mu_1' = 2$ $\mu_2' = 136$, $\mu_3' = 320$, $\mu_4' = 40,000$

$\mu_1 = 0$

$\mu_2 = \mu_2' - (\mu_1')^2 = 136 - 4 = 132$

$\mu_3 = \mu_3' - 3\mu_2'\mu_1' + (2\mu_1')^3$

$= 320 - 3(136)(2) \cancel{+2}(\cancel{2}(2))^3$

$= 320 - 816 + \overset{16}{\cancel{64}} = \cancel{-432}$ -480

$\mu_4 = \mu_4' - 4\mu_3'\mu_1' + 6\mu_2'\mu_1'^2 - 3\mu_1'^4$

$= 40,000 - 4(320)(2) + 6(136)(2)^2 - 3(2)^4$

$= 40,000 - 2560 + 3264 - 48$

$= 40656$

wrong

Skewness

$$\beta_1 = \frac{\mu_3^2}{\mu_2^3} = \frac{(320)^2}{(136)^3} = \frac{230400}{2299968}$$

$= 0.1$ approx sym

Curtosis

$$\beta_2 = \frac{\mu_4}{\mu_2^2} = \frac{40656}{17424} = 2.3$$

platykurtic

MOMENT GENERATING FUNCTION

MGF of a R.V X denoted by $M_X(t) = E(e^{tx})$

→ used to generate moments

→ for discrete $M_x(t) = \sum e^{tx} p(x)$

for continuous $M_x(t) = \int e^{tx} f(x)\,dx$ } Function of t

$$E(e^{tx}) = E\left(1 + \frac{tx}{1!} + \frac{t^2x^2}{2!} + \frac{t^3x^3}{3!} + \dots\right)$$

$$E(e^{tx}) = M_x(t) = E(1) + \frac{t\,E(x)}{1!} + \frac{t^2\,E(x^2)}{2!} + \frac{t^3\,E(x^3)}{3!}$$

M_r' or $E(x^r)$ is coefficient of $\frac{t^r}{r!}$

$M_1' = E(x)$ is coefficient of $\frac{t}{1!}$

$M_2' = E(x^2)$ is coefficient of $\frac{t^2}{2!}$

(or)

$$\left.\frac{d\,M_x(t)}{dt}\right|_{t=0} = \Big[0 + E(x) + t(Ex^2) \dots\Big]_{t=0} = E(x)$$

$$\left.\frac{d^2 M_x(t)}{dt^2}\right|_{t=0} = \Big[0 + 0 + E(x^2) + t\,E(x^3) \dots\Big]_{t=0} = E(x^2)$$

$$E(x^r) = \left[\frac{d^r M_x(t)}{dt^r}\right]_{t=0}$$

$$(1-x)^{-1} = 1 + x + x^2 + x^3 \dots$$

$$(1+x)^{-1} = 1 - x + x^2 - x^3 + \dots$$

$$(1-x)^{-2} = 1 + 2x + 3x^2 + 4x^3 + \dots$$

$$(1+x)^{-2} = 1 - 2x + 3x^2 - 4x^3 \dots$$

i) $f(x) = \begin{cases} e^{-x} & 0 < x < \infty \\ 0 & \text{elsewhere} \end{cases}$

$$E(e^{tx}) = M_x(t) = \int_0^\infty e^{tx} e^{-x}\, dx$$

$t < 1$ coz otherwise it'll become ∞

$$= \int_0^\infty e^{-x(1-t)}\, dx$$

$$= \left[\frac{-e^{-x(1-t)}}{(1-t)} \right]_0^\infty = \left[0 - \left(-\frac{1}{1-t} \right) \right] = \frac{1}{1-t}$$

$$\frac{1}{1-t} = (1-t)^{-1} = 1 + 1 \cdot \frac{t}{1!} + 2! \frac{t^2}{2!} + 3! \frac{t^3}{3!} \ldots$$

$E(x) = 1$ $E(x^2) = 2$

$E(x^3) = 6$ $E(x^4) = 4! = 24$

(or) $\frac{d(1-t)^{-1}}{dt}\Big|_{t=0} = \left[(-1)(-1)(1-t)^{-2} \right]_{t=0} = 1$

$$\frac{d^2(1-t)^{-1}}{dt^2}\Big|_{t=0} = \left[(-1)(-2)(1-t)^{-3} \right]_{t=0} = 2$$

$$\frac{d^3(1-t)^{-1}}{dt^3}\Big|_{t=0} = \left[(2)(-)(-3)(1-t)^{4}) \right]_{t=0} = 6$$

2)

x	1	2
$P(x)$	$\frac{1}{3}$	$\frac{2}{3}$

find M.G.F hence find $E(x)$ & var

SOL

$$E(e^{tx}) = M_x(t) = \Sigma e^{tx} p(x)$$

$$= e^{t(1)} \frac{1}{3} + e^{t(2)} \frac{2}{3}$$

$$= \frac{1}{3} e^{t} + \frac{2}{3} e^{2t}$$

$$E(x) \frac{d M_x(t)}{dt}\Big|_{t=0} = \left[\frac{e^t}{3} + \frac{2}{3} 2 . e^{2t}\right]_{t=0} = 1 . \frac{1}{3} + \frac{4}{3} = \frac{5}{3}$$

$E(x^2)$

$$\frac{d^2 M_x(t)}{dt}\Big|_{t=0} = \left[\frac{e^t}{3} + 8 \frac{e^{2t}}{3}\right]_{t=0} = \frac{1}{3} + \frac{8}{3} = 3$$

$$Var = E(x^2) - (E(x))^2 = 3 - \left(\frac{5}{3}\right)^2 = 3 - \frac{25}{9} = \frac{2}{9}$$

3) If a R.V has M.G.F $\frac{2}{2-t}$ Find the variance of x

$$= \frac{1}{1-\frac{t}{2}} = \left(1 - \frac{t}{2}\right)^{-1}$$

SOL

$$\left(1 - \frac{t}{2}\right)^{-1} = 1 + \frac{t}{2} + \frac{t^2}{4} + \frac{t^3}{8} \dots$$

$$E(x) = \frac{d M_x(t)}{dt}\Big|_{t=0} = \left[0 + \frac{1}{2} + \frac{t}{2} + \frac{3t^2}{8}\right]_{t=0} = \frac{1}{2}$$

$$E(x^2) = \frac{d^2 M_x(t)}{dt^2}\Big|_{t=0} = \left[0 + 0 + \frac{1}{2} + \frac{6}{8} t\right] = \frac{1}{2}$$

$E(x^2) - (E(x))^2$

$= \frac{1}{2} - \frac{1}{4} = \frac{1}{2}$

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4) $f(x) = \begin{cases} 1/3 & , -1 < x < 2 \\ 0 & \text{elsewhere} \end{cases}$

Find M.G.F & $E(x)$, $Var(x)$

SOL

$M_X(t) = \int e^{tx} f(x)\, dx$

$= \int_{-1}^{2} e^{tx} \, 1/3 = \frac{1}{3}\left[\frac{e^{tx}}{t}\right]_{-1}^{2} = \frac{1}{3}\left[\frac{e^2}{}\right.$

$= \left[\frac{1}{3}\frac{e^{tx}}{t}\right]_{-1}^{2} = \frac{1}{3}\left[\frac{e^{2t}}{t} - \frac{e^{-t}}{t}\right]$

$\cancel{\frac{d M_X(t)}{dt}\Big|_{t=0}} =$

$uv' + u'v$

$= \frac{1}{3}\left[e^{2t} t^{-1} - e^{-t} t^{-1}\right]$

$\frac{d M_X(t)}{dt} = \frac{1}{3}\left[e^{2t} \cdot (-1) t^{-2} + 2 e^{2t} t^{-1} - (e^{-t} t^{-2} + \frac{e^{-t}}{-1} t^{-1})\right)$

$= \frac{1}{3}\left[-e^{2t} t^{-2} + 2e^{2t} t^{-1} + e^{-t} t^{-2} + e^{-t} t^{-1}\right]$

$E(X) = \frac{1}{3}[0] = 0$

$\frac{d^2 M(x) t}{dt} = \frac{1}{3}\left[t^{-1}(2e^{2t} + e^{-t}) + t^{-2}(-e^{2t} + e^{-t})\right)$

$= \frac{1}{3}\left[-t^{-2}(2e^{2t} + e^{-t}) + t^{-1}(4e^{2t} - e^{t})\right)$

$+ (-2t^{-3}(-e^{2t} + e^{-t}) + t^{-2}(-2e^{2t} - e^{-t}$

$= 0$