UNIT III . PROBABILITY DISTRIBUTIONS

Bernoulli's Distribution :

-> Experiment is conducted once
-> It has only two possible outcomes . success , failure

P.m.f

x	0	1
$P(X=x)$	q	P

P-> probability of success
$q = 1-P =$ probability of failure $(P+q=1)$

$$P(X=x) = \begin{cases} p^x q^{1-x} & x=0,1 \\ 0 & \text{otherwise} \end{cases}$$

Moments

$$E(X^r) = 0^r . q + 1^r . P = P$$

$$E(X^2) = P, \quad E(X) = P$$

$$var = P - P^2 = P(1-P) = pq \; ; \quad \sigma = \sqrt{pq}$$

M.G.F :

$$e^{0.t} . q + e^{1.t} P = pe^t + q$$

Binomial Distribution :-

Assumptions .

① Experiment is conducted for a finite & fixed no. of times .

② n trials must be independent

③ P remains same in each trial .

④ each trial has only two outcomes success / failures

$$P(X=x) = \begin{cases} {}^nC_x \, p^x q^{n-x} & , x = 0, 1, \cdots n \\ 0 & , \text{otherwise} \end{cases}$$

$$\sum_{x=0}^{n} {}^nC_x \, p^x q^{n-x} = (p+q)^n = 1^n = 1$$

$$\text{mean} = \sum x \, P(X=x) = \sum_{x=}^{n} x \, {}^nC_x \, p^x q^{n-x}$$

$$= \sum x \frac{n!}{(n-x)! \, x(x-1)!} p^x q^{n-x}$$

$$= p \cdot \sum \frac{n(n-1)!}{((n-1)-(x-1))!(x-1)!} p^{(x-1)} q^{((n-1)-(x-1))}$$

$$= np \sum \frac{{}^{n-1}C_{x-1} \, p^{x-1} q^{(n-1)-(x-1)}}{\cancel{n-1}}$$

$$= np(p+q)^{n-1}$$

$$= np \cdot 1 = np$$

$$E(x^2) = \sum_{x=0}^{n} x^2 \, {}^nC_x \, p^x q^{n-x}$$

$$= \sum (x(x-1)+x) \, {}^nC_x \, p^x q^{n-x}$$

$$= \sum x(x-1) \, {}^nC_x \, p^x q^{n-x} + \sum x \, {}^nC_x \, p^x q^{n-x}$$

$$= p^2 \sum \frac{x(x-1) \, n(n-1)(n-2)! \, p^{x-2} q^{n-x}}{x(x-1)(x-2)! \, (n-2)-(x-2)!} + np$$

(2)

$$= p^2 n(n-1) \sum {}^{n-2}C_{x-2}\, p^{x-2}\, q^{((n-2)-(x-2))} + np$$

$$= p^2 n (p+q)^{n-2} + np$$

$$\text{Var} = p^2 n(n-1) + np - n^2p^2$$

$$= n^2p^2 - np^2 + np - n^2p^2$$

$$= np(-p+1)$$

$$\text{Var} = npq$$

$$\sigma = \sqrt{npq}$$

Q) can, $X \sim B(n, P)$ with mean $= 3$, var $= 4$

$np = 3$

$npq = 4$ $\Rightarrow q = \frac{4}{3} p,$

not possible

MOMENT GENERATING FUNCTION

$$M_x(t) = E(e^{tx}) = \sum e^{tx}\, {}^nC_x\, p^x q^{n-x}$$

$$= \sum_{x=0}^{n} {}^nC_x (pe^t)^x q^{n-x}$$

$$M_x(t) = (pe^t + q)^n$$

$$E(x) = (n(pe^t+q)^{n-1} \cdot pe^t)\Big|_{t=0} = np$$

1) The mean and variance of a binomial distribution are 4 and 4/3 respectively. Find $P(x \geq 1)$

$np = 4$

$\sqrt{npq} = 4/3$

$npq = 16/9$

$q = \frac{16}{4 \times 9}$ $\quad q = \frac{4}{9}$

$p = 1 - \frac{4}{9} = \frac{5}{9}$

$npq = 4/3$

$4q = \frac{4}{3}$

$q = 1/3$

$p = 1 - q$

$p = \frac{2}{3}$

$P(x = x) = 6C_x \left(\frac{2}{3}\right)^x \left(\frac{1}{3}\right)^{6-x}$

$P(x \geq 1) = 1 - P(x = 0)$

$= 1 - 6C_0 \left(\frac{2}{3}\right)^0 \left(\frac{1}{3}\right)^6$

$=$

2) A traffic control engineer reports that 75% of vehicles passing thru a checkpost are from within state. What is the prob fewer than 4 of the 9 are from out of the state

SOL

$n = 9$

$p = \frac{1}{4} \quad q = \frac{3}{4}$

$P(<4$ vehicles are out of state$) = P(X<4) = P(X=0) + P(X=1) + P(X=2) + P(X=3)$

$$= \sum_{x=0}^{3} {}^9C_x \left(\frac{1}{4}\right)^x \left(\frac{3}{4}\right)^{9-x}$$

$$= {}^9C_0 \left(\frac{1}{4}\right)^0 \left(\frac{3}{4}\right)^9 + {}^9C_1 \left(\frac{1}{4}\right)^1 \left(\frac{3}{4}\right)^8 + {}^9C_2 \left(\frac{1}{4}\right)^2 \left(\frac{3}{4}\right)^7 + {}^9C_3 \left(\frac{1}{4}\right)^3 \left(\frac{3}{4}\right)^6$$

$= 0.834$

POISSON DISTRIBUTION

-> when n is very large & p is very small

p.m.f of => taking $\boxed{np = \lambda}$ which is finite binomial distribution and as $\boxed{n \to \infty}$ $\boxed{p \to 0}$

$$\hookrightarrow P(X = x) = \frac{e^{-\lambda} \lambda^x}{x!} \qquad x = 0, 1, 2 \ldots$$

$X \sim P(\lambda)$

5%
rm
an

+P(x=3)

$\frac{1}{4})^3(\frac{3}{4})^6$

Mean :- $E(X) = \sum x \frac{e^{-\lambda}\lambda^x}{x!}$, $x = 0, 1, 2, 3 \ldots$

verify Pmf

$$\sum_{x=0}^{\infty} \frac{e^{-\lambda}\lambda^x}{x!} = e^{-\lambda} \sum \frac{\lambda^x}{x!} = e^{-\lambda} \cdot e^{\lambda} = 1$$

$$E(x) = e^{-\lambda} \sum_{x=0}^{\infty} \frac{x\,\lambda^{x-1}}{x\,(x-1)!} \cdot \lambda = \lambda e^{-\lambda} \sum_{x=0}^{\infty} \frac{\lambda^{x-1}}{(x-1)!}$$

$$= \lambda e^{-\lambda} e^{\lambda} = \lambda$$

$$E(x^2) = \sum_{x=0}^{\infty} \frac{x^2 e^{-\lambda}\lambda^x}{x!}$$ $x^2 = x(x-1) + x$

$$= e^{-\lambda}\left[\sum_{x=0}^{\infty} \frac{x(x-1)\,\lambda^x}{x(x-1)(x-2)!} + \sum_{x=0}^{\infty} \frac{x\,\lambda^x}{x(x-1)!}\right]$$

$$= e^{-\lambda} \sum_{x=0}^{\infty} \frac{\lambda^{x-2} \cdot \lambda^2}{(x-2)!} + e^{-\lambda} \sum_{x=0}^{\infty} \frac{\lambda^{x-1}}{(x-1)!}\lambda$$

$$= e^{-\lambda}\lambda^2(e^{\lambda}) + e^{-\lambda}\lambda e^{-\lambda}$$

$$= \lambda^2 + \lambda$$

$$\text{var} = E(x^2) - (E(x))^2$$
$$= \lambda^2 + \lambda - \lambda^2 = \lambda$$

M.G.F

$$E(e^{tx}) = \sum e^{tx} \frac{e^{-\lambda}\lambda^{x}}{x!}$$

$$= e^{-\lambda} \sum \frac{(e^{t}\lambda)^{x}}{x!} = e^{-\lambda} e^{e^{t}\lambda}$$

$$= e^{\lambda(e^{t}-1)}$$

1) if $P(1) = P(2)$ for a r v X which follows Poisson distribution find

i) $P(4)$

ii) $P(X \geq 1)$

iii) $P(1 < X < 4)$

SOL

$$P(x) = \frac{e^{-\lambda}\lambda^{x}}{x!}$$

$$P(1) = \frac{e^{-\lambda}\lambda^{1}}{1!}$$

$$P(2) = \frac{e^{-\lambda}\lambda^{2}}{2}$$

$$\frac{e^{-\lambda}\lambda}{1} = \frac{e^{-\lambda}\lambda^{2}}{2}$$

$$\lambda^2 - 2\lambda = 0$$

$$\lambda(\lambda - 2) = 0$$

$\lambda = 0$ $\quad \lambda = 2$

↳ only possible.

i) $P(4) = \frac{e^{-\lambda} \lambda^4}{4!}$

$= \frac{e^{-2} (2)^4}{24}$

$= \frac{0.13 \times 16}{24} = 0.08$

ii) $P(X \geq 1) = 1 - P(X < 1)$

$= 1 - P(X = 0)$

$= 1 - \frac{e^{-\lambda} \lambda^0}{0!} = 1 - \frac{1}{e^2}$

(iii) $P(X = 2) + P(X = 3)$

$= e^{-\lambda} \left(\frac{\lambda^x}{x!} \right)$

$= e^{-\lambda} \left(\frac{2^2}{2!} + \frac{2^3}{3!} \right)$

$= \frac{1}{e^2} \left(\frac{4}{2} + \frac{8}{6} \right) = 0.4511$

2) The Average number of phone calls received between 2pm and 4pm is 2.5. determine the Prob that during one particular minute, There will ~~be~~ be

i) 4 or fewer

ii) more than 6 calls

Sol

$E(x) = 2.5 = \lambda$

i) $P(X \leq 4)$

$$P(x) = \frac{e^{-\lambda}\lambda^{x}}{x!}$$

$$P(X \leq 4) = \sum_{x=0}^{4} e^{-2.5} \frac{(+2.5)^{x}}{x!}$$

$$= e^{-2.5}\left[\frac{(2.5)^0}{0!} + \frac{(2.5)^1}{1} + \frac{(2.5)^2}{2} + \frac{(2.5)^3}{6} + \frac{(2.5)^4}{24}\right]$$

$$= 0.08\left[1 + 2.5 + 3.125 + 2.6 + 1.62\right]$$

$$= 0.867$$

ii) $P(X > 6)$

$$= 1 - P(X \leq 6)$$

$$= 1 - \left(P(X \leq 4) + P(X=5) + P(X=6)\right)$$

$$= 0.08\left(\frac{(2.5)^5}{120} + \frac{(2.5)^6}{720}\right)$$

$$= 0.08(0.81 + 0.33)$$

$$= 1 - (0.867 + 0.091)$$

$$= 0.041$$

3) Manufacture produces bulbs that are packed into boxes of 100. If quality control studies indicate that 0.5% of bulbs produced are defective. What percent of boxes will contain

(i) non defective

(ii) 2 or more defective

SOL

$$\lambda = np = 100 \times \frac{0.5}{100} = 0.5$$

$$\lambda = 0.5$$

$$P(X=0) = \frac{e^{-0.5} \times (0.5)^0}{0!} = e^{-0.5} = 0.606$$

CONTINOUS UNIFORM DISTRIBUTION

Every point in the given interval has equal chance of occuring (discrete)

$$f(x) = \begin{cases} \frac{1}{b-a} & a \le x \le b \\ 0 & \text{otherwise} \end{cases}$$

$$\text{Pdf} = \frac{1}{b-a}\int_a^b dx = \frac{b-a}{b-a} = 1$$

$$\text{Mean : } E(x) = \int_a^b \frac{1}{b-a} x^r\, dx = \frac{1}{b-a}\left[\frac{x^{r+1}}{r+1}\right]_a^b$$

$$= \frac{1}{(b-a)(r+1)}\left[b^{(r+1)} - a^{(r+1)}\right]$$

$r=1$ $$E(x) = \frac{1}{2(b-a)}\left[b^2 - a^2\right] = \frac{b+a}{2}$$

$r=2$ $$E(x^2) = \frac{1}{3(b-a)}\left[b^3 - a^3\right] = \frac{1}{3(b-a)}(b-a)(a^2+b^2+ab)$$

$$\text{Variance} = E(x^2) - (E(x))^2$$

$$= \frac{1}{3(b-a)}\left[b^3 - a^3\right] - \left[\frac{1}{2(b-a)}\left[b^2 - a^2\right]\right]^2$$

$$= \frac{a^2+b^2+ab}{3} - \frac{(b+a)^2}{4}$$

$$= \frac{4a^2 + 4b^2 + 4ab - 3a^2 - 3b^2 - 6ab}{12}$$

$$= \frac{1}{12}(a^2 + b^2 - 2ab)$$

$$= \frac{(b+a)^2}{12}$$

$$var = \frac{(b-a)^2}{12}$$

M.G.F

$$= E(e^{tx}) = \frac{1}{b-a}\int_a^b e^{tx}\,dx$$

$$= \frac{1}{b-a}\left[\frac{e^{tx}}{t}\right]_a^b$$

$$= \frac{1}{t(b-a)}\left[e^{bt} - e^{at}\right]$$

1) If x is uniformly distributed with mean 2 and variance 12. Find prob of $P(x < 3)$

SOL

$E(x) = 2$
$Var = 12$

$$\frac{a+b}{2} = 2 \qquad \frac{(b-a)^2}{12} = 12$$

$$a+b = 4 \qquad (b-a)^2 = 144$$

$$a = 4-b \qquad b-a = 12$$

$$\boxed{a = -4} \qquad b = 12+4-b$$

$$2b = 16$$

$$\boxed{b = 8}$$

$P(X < 3)$

$$= \int_{-4}^{3} \frac{1}{12}\,dx = \frac{1}{12}[x]_{-4}^{3} = \frac{1}{12}[9] = \frac{7}{12}$$

2) A Random Variable x has uniform distribution over the interval -2 to 2. Find K for which Prob of $X > K$ is $1/2$

SOL

$$P(X > K) = \frac{1}{2}$$

$$\int_{K}^{b} \frac{1}{b-a}\,dx = \frac{1}{2}$$

and $\frac{1}{b-a}[x]_k^b = \frac{1}{2}$

$\frac{1}{b-a}[b-k] = \frac{1}{2}$

$\frac{1}{-2}[2-k] = \frac{1}{2}$

$2-k = -2$

$k = 0$

3) The amount of time in minutes that a person has to wait that is uniformly distributed from 0 to 15 mins

(i) what is the prob that a person waits fewer than 12.5 mins

(ii) on an average how long a person ~~weight~~ waits

SOL

$$\frac{1}{b-a}\int_a^{12.5} dx$$

$$= \frac{1}{15}[x]_0^{12.5} = \frac{12.5}{15}$$

ii) $E(x) = \frac{a+b}{2} = \frac{15}{2} = 7.5$

$$Var = \frac{(b-a)^2}{12} = \frac{(15)^2}{12} = 18.75$$

Std. dev = 4.33

4) A manufacturer produces sweets of length l where l has continous uniform distribution with range 15 mm to 30 mm

a) Find the prob that a randomly selected sweet has a length greater than 24 mm

b) The sweets are packed in a bags of 20 sweets. Find the prob that randomly selected bag will contain atleast 8 sweets with length greater than 24 mm?

SOL

a) $$\frac{1}{b-a}\int_{24}^{30} dx = \frac{1}{15}[x]_{24}^{30}$$

$$= \frac{6}{15}$$

b) $P(X \geq 8)$

$$n = 20 \quad p = \frac{6}{15} \quad q = \frac{9}{15}$$

$$= 1 - P(X<8) = 1 - \left(\sum_{x=0}^{7} {}^{20}C_x \left(\frac{6}{15}\right)^x \left(\frac{9}{15}\right)^{20-x}\right)$$

EXPONENTIAL DISTRIBUTION (CONTINOUS)

distribution of time until an event occurs

$$\text{p.d.f } f(x) = \begin{cases} \lambda e^{-\lambda x} & x > 0 \\ 0 & \text{otherwise} \end{cases}$$

verify p.d.f $\int_0^\infty \lambda e^{-\lambda x} dx = \lambda \int_0^\infty e^{-\lambda x} = \lambda \left[\frac{e^{-\lambda x}}{\lambda}\right]_0^\infty$

$= [0 - (-1)] = 1$

Mean :

$$E(X) = \int_0^\infty x \lambda e^{-\lambda x} dx = \lambda \int_0^\infty x e^{-\lambda x} dx$$

$$\int u\, dv = uv - \int v\, du$$

$$= \lambda \left[x \frac{e^{-\lambda x}}{-\lambda} - \int_0^\infty \frac{e^{-\lambda x}}{-\lambda} dx \right]$$

$$= \lambda \left[x \frac{e^{-\lambda x}}{-\lambda} - \frac{e^{-\lambda x}}{\lambda^2} \right]_0^\infty$$

$$= -\lambda \left[\frac{0-1}{\lambda^2} \right] = \frac{1}{\lambda}$$

$$E(x)^2 = \frac{2}{\lambda^2}$$

$Var = E(X^2) - [E(X)]^2 = \frac{2}{\lambda^2} - \frac{1}{\lambda^2} = \frac{1}{\lambda^2}$

M.G.F

$$E(e^{tx}) = \int_0^\infty \lambda e^{tx} e^{-\lambda x} \, dx$$

$$= \lambda \int_0^\infty e^{(-t+\lambda)x} \, dx$$

$$= \lambda \left[\frac{e^{-x(\lambda - t)}}{-(\lambda - t)} \right]_0^\infty$$

$$= -\lambda \left[0 - \frac{1}{\lambda - t} \right] = \frac{\lambda}{\lambda - t}$$

$$M_x(t) = \left(1 - \frac{t}{\lambda}\right)^r$$

$$= 1 + 1.\frac{t}{\lambda!} + \frac{2}{2!} \frac{t^2}{\lambda^2} + \dots$$

1) Telephone calls arrive at a switchboard following an exponential distribution with parameter $\lambda = 12$ per hour. If we are at the switchboard what is the prob that the waiting time for the call is (i) atleast 15 mins

(ii) not more than 10 mins

Sol

i. $P(X > \frac{1}{4}) = \lambda \int_{1/4}^{\infty} e^{-\lambda x} \, dx$

$= \cancel{\lambda} \left[\frac{-e^{-\lambda x}}{-\cancel{\lambda}} \right]_{1/4}^{\infty} = [0 - (-e^{-1/4 \times 12})$

$= 0 + e^{-3} = 0.04$

ii) $P(X < \frac{1}{6}) = \lambda \int_{0}^{1/6} e^{-\lambda x} \, dx$

$= \cancel{\lambda} \left[-\frac{e^{-\lambda x}}{+\cancel{\lambda}} \right]_{0}^{1/6}$

$= \left[-e^{-2} - (-e^{0}) \right] = -0.13 + 1 = 0.8647$

2) Suppose that lifetime of a smartphone has exp. dist with mean life of 4 years. Given that the phone has lasted for 3 years. what is the prob that it will last for 5 more years

Sol. mean $= 4 = \frac{1}{\lambda}$

$\lambda = \frac{1}{4}$

$$P(X>5+3 \mid X>3) = \frac{P(X>8 \cap X>3)}{P(X>3)}$$

$$= \frac{P(X>8)}{P(X>3)}$$

$$P(X>8) = \lambda \int_8^\infty e^{-\lambda x}\,dx$$

$$= \lambda \left[-\frac{e^{-\lambda x}}{\lambda}\right]_8^\infty$$

$$= -e^{-\infty} - [-e^{-8}] = + \quad e^{-2} = \cancel{0.0}$$
$$= 0.13$$

$$P(X>4) = \lambda \left[-\frac{e^{-\lambda x}}{\lambda}\right]_4^\infty$$

$$= -0 - [-e^{-4}] = e^{-1} = 0.36$$

Memory less property of exponential distribution

$$P(X > x+t \mid X > t)$$

$$= \frac{P(X>x+t)}{P(X>t)}$$

$$= \frac{e^{-\lambda(x+t)}}{e^{-\lambda t}} = e^{-\lambda x}$$

$$= P(X>x)$$

$= \lambda \int_0^x e^{-\lambda x} \, dx$

$= \lambda \left[\frac{e^{-\lambda x}}{-\lambda} \right]_0^x = -(e^{-\lambda x} - 1)$

$= 1 - e^{-\lambda x}$

$F(x) = P(\cancel{x} < x) = 1 - e^{-\lambda x}$

$F(x) = 1 - e^{-\lambda x}$

$P(X < x) = 1 - e^{-\lambda x}$

$P(X > x) = e^{-\lambda x}$

1) If X is a R.V that follows poisson distribution such that

$P(x = 2) = 9 P(x = 4) + 90 P(x = 6)$ find λ.

SOL

$$\frac{e^{-\lambda} \lambda^2}{2!} = \frac{9 \cdot e^{-\lambda} \lambda^4}{4!} + \frac{90 e^{-\lambda} \lambda^6}{6!}$$

$$\frac{1}{2!} = \frac{9\lambda^2}{4!} + \frac{90\lambda^4}{6!} \Rightarrow \lambda^4 + 3\lambda^2 - 4 = 0$$

$\lambda^2 = -4, 1$

$\lambda = 1$