Recap :

$G = (V, E)$ undirected graph, connected

Eulerian walk : visits each edge exactly once
start point = end point

Last class : If G has an Eulerian walk, then every vertex has even degree.

Today, we will prove the converse.

Thm 23.1 : If $G = (V, E)$ is connected and every vertex has even degree, then G has an Eulerian walk.

Attempt 1 using induction : Take any vertex v, consider the reduced graph with v removed.

2 potential issues :

- $G \setminus \{v\}$ may not be connected.
- all vertices may not have even deg.

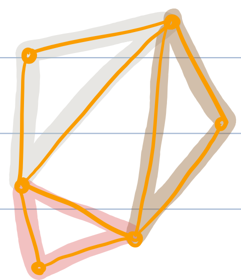
This approach can lead to a formal proof.
You are encouraged to explore this direction.

We will see two different proofs of Thm 23.1.

Proof 1: using cycle partitions

Claim 23.1: If $G = (V, E)$ and every vertex in V has even degree, then $E = E_1 \cup E_2 \cup \dots \cup E_k$ (for some k) where $E_i \cap E_j = \phi$, and each E_i forms a cycle in G.

eg.



Note: we have not assumed G is connected. Connectedness is not needed for this claim.

Claim 23.2: If $G = (V, E)$ where G is connected, and $E = E_1 \cup E_2 \cup \dots \cup E_k$, each E_i forms a cycle and $E_i \cap E_j = \phi$, then G has an Eulerian walk.

<u>Proof</u> of <u>Claim 23.1</u> : Strong induction on number of edges.

Predicate $P(m) :=$ For all graphs $G = (V, E)$ s.t. $|E| = m$ and each vertex has even degree, $\exists$ partitioning $E = E_1 \cup E_2 \cup \dots \cup E_k$ s.t. each E_i forms a cycle.

$P(0) :=$ True.

Suppose $P(j) =$ True for all $0 \le j < m$.

To prove: $P(m)$:
Consider any graph $G = (V, E)$ s.t. $|E| = m$, and each vertex of G has even degree.

Observation 1 : There exists a cycle in G.
<u>Proof</u>: We will give an algorithm that finds a cycle in G. Pick an arbitrary vertex $v_0 \in V$ with deg. > 0. There exists such a vertex since $m > 0$.
Pick an arbitrary neighbour of v_0, say v_1. Set $i = 1$.

while (true) :

if v_i has a neighbour in $\{v_0, v_1, \ldots, v_{i-2}\}$, say v_k, then we have a cycle. exit loop.
(v_k v_{k+1} ... v_{i-2} v_{i-1} v_i v_k).

else

v_i has a neighbour in $V \setminus \{v_0, \ldots, v_{i-2}, v_{i-1}\}$.

v_i has even no. of neighbours.
v_{i-1} is a neighbour of v_i
and none of $\{v_0, v_1, \ldots, v_{i-2}\}$ are neighbours of v_i. Therefore it has at least one nbr. in $V \setminus \{v_0, \ldots, v_{i-1}\}$.

pick an arbitrary neighbour of v_i, say v_{i+1}. $i = i+1$.

Observe that this must terminate in at most n iterations, and when it terminates, we have a cycle.

End of proof of Observation 1 ∎

Let $(u_0, u_1, \ldots, u_\ell, u_0)$ be a cycle in G.

Consider $G' = (V, E')$ where

$$E' = E \setminus \{\{u_0, u_1\}, \{u_1, u_2\}, \ldots, \{u_{\ell-1}, u_\ell\}, \{u_\ell, u_0\}\}$$

$$|E'| = m - (\ell+1) < m.$$

Also, observe that each vertex of G' has even degree. Hence, using $P(m-\ell-1)$, we get that E' can be partitioned into $E_1 \cup E_2 \cup \ldots \cup E_k$ s.t. each E_i is a cycle in G' (and therefore a cycle in G).

$$E_{k+1} = \{\{u_0, u_1\}, \{u_1, u_2\}, \ldots, \{u_{\ell-1}, u_\ell\}, \{u_\ell, u_0\}\}$$

$E = E_1 \cup E_2 \cup \ldots \cup E_k \cup E_{k+1}$, and each E_i forms a cycle.

Hence, using strong induction, Claim 23.1 holds.

$\blacksquare$

Proof of Claim 23.2 :

Given : $G = (V,E)$, where $E = E_1 \cup \dots \cup E_k$ and G is connected.

To compute : Eulerian walk in G.

We will show this using the following algorithm.

Notations : $V(E_i)$ = vertices present in E_i.
$V_1 = V(E_1)$, $F_1 = E_1$, $G_1 = (V_1, F_1)$
$S = \{E_2, \dots, E_k\}$

G_1 has Eulerian walk. We will iteratively construct $G_2, \dots, G_k = G$, and we will construct Eulerian walks for $G_2, G_3, \dots, G_k$

For $i = 2$ to k

1. Find cycle $E_j \in S$ s.t. $V_{i-1} \cap V(E_j) \neq \phi$.

   Such a j always exists since G is connected. If $V_{i-1} \cap V(E_j) = \phi$ for all E_j in S, then consider $A = V_{i-1}$, $B = \bigcup_{E_j \in S} V(E_j)$. A & B are disconnected components of G.

2. $V_i = V_{i-1} \cup V(E_j)$, $F_i = F_{i-1} \cup E_j$
   $G_i = (V_i, F_i)$ Note that $F_{i-1} \cap E_j$
   $S = S \setminus \{E_j\}$ is empty.
   Also note, for all $E_r \in S$, $F_i \cap E_r = \phi$.

3. Suppose $W_{i-1} = (v_0, v_1, \ldots, v_t)$ is an Eulerian walk in G_{i-1},

Let s be the smallest index s.t. $v_s \in V_{i-1}$ and $v_s \in V(E_j)$.

Let $E_j = (v_s \; u_1 \cdots u_r \; v_s)$

cycle E_j starting and ending at v_s.

Consider walk $W_i =$

$(v_0 \; v_1 \cdots v_s \; u_1 \ldots u_r \; v_s \; v_{s+1} \ldots v_t)$.

W_i is an Eulerian walk on G_i.

W_i visits all edges in F_i exactly once. Here, we use the fact that $F_{i-1} \cap E_j = \phi$.

When the algorithm terminates, $G_k = G$, and therefore we have an Eulerian walk in G.

∎

Proof 2: A more direct proof

Consider the longest walk W in G that does not repeat any edge. Let us call such a walk that doesn't repeat edges a TOUR.

If $\text{len}(W) = |E|$, then we have an Eulerian walk. Suppose $G = (V, E)$ is connected, every vertex has even degree, and longest tour has length t.

$$W = v_0 \; v_1 \; \cdots \; v_t$$

Claim 23.3: $v_0 = v_t$

Proof: If $v_0 \neq v_t$, then we can get a longer tour. This is because v_0 has even degree.

$$W = v_0 \, w_1 \cdots w_2 v_0 \, w_3 \cdots w_4 v_0 \, w_5 \cdots w_{2i} \, v_0 \, w_{2i+1} \cdots v_t$$

Note that $w_1, w_2, \ldots w_{2i+1}$ are all distinct because edges are not repeated in a tour.

$\{v_0, w_j\} \in E$ for each $j \leq 2i+1$.

Since v_0 has even deg., if $v_0 \neq v_t$, then $\exists\ w \in V \setminus \{w_1 \dots w_{2i+1}\}$ s.t. $\{v_0, w\} \in E$.

$\Rightarrow W' = w\ v_0\ w_1 \cdots\cdots w_2 v_0\ w_3 \cdots\cdots w_4 v_0\ w_5 \cdots\cdots w_{2i}\ v_0\ w_{2i+1} \cdots v_t$

is a longer tour. ∎

<u>Claim 23.4</u>: Suppose V' is the set of all vertices in the longest tour W. Then $V' = V$.

<u>Proof</u>: Suppose not. Then $\exists\ u \in V \setminus V'$.

G is connected, so $dist(u, w) \leq n-1$ for all $w \in V'$.

Let $w \in V'$ be such that $dist(u, w) \leq dist(u, v)$ for all $v \in V'$.

Since $dist(u, w) \leq dist(u, v)$ for all $v \in V'$, none of the vertices on the (u, w) path are in V'.

Let $(u = u_0, u_1, \dots, u_r = w)$ be the shortest path from u to w.

In Claim 23.3, we showed that the longest tour must be a cycle. Let $W = (w_0 = w, w_1, w_2, \dots, w_t = w)$ be the tour.

Now consider the following walk:
$W = (u_0, u_1, \dots, u_r = w, w_1, \dots, w_t)$

W is a tour since no edge is repeated, and W is longer than T.

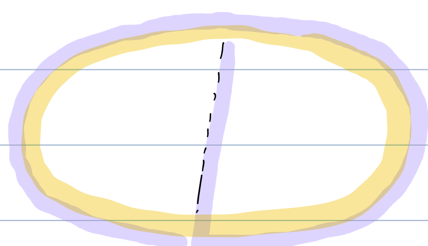
Hence we have a contradiction.

$\blacksquare$

Claim 23.5 : len(longest tour W) $= |E|$.

Proof: Let $W = (w_0, w_1, \dots, w_{t-1}, w_0)$ be the longest tour in G. In Claims 23.3 and 23.4, we proved that this longest tour starts and ends at the same vertex, and it contains all the vertices of V.

This figure can be a bit misleading since it shows the longest tour as a cycle. The longest tour can



: longest tour W

: edge not in W

: longer tour

have repeated vertices, the formal proof does not assume that W is a cycle.

Suppose edge $e = \{u, v\} \in E$, but for all i, $\{w_{i-1}, w_i\} \neq e$. Since W contains all the vertices, let i be the smallest index s.t. $u = w_i$, and let j be the smallest index s.t. $v = w_j$.

Consider $W' = (w_i, w_j, w_{j+1}, \ldots, w_{t-1}, w_0, w_1, \ldots w_{j-1}, w_j)$

$\text{len}(W') = \text{len}(W) + 1$ and W' is a tour.
Hence contradiction. ∎

Key idea: look at the longest tour, remaining proof just 'flows' from this starting point.

<u>Conclusion</u>: G has an Eulerian walk

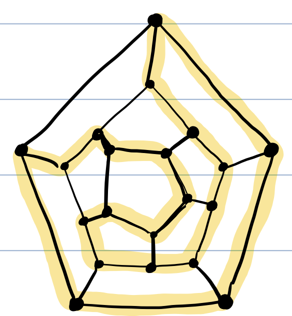
$$\Updownarrow$$

G is connected, and every vertex has even degree.

Why is this result interesting? Eulerianism can be tested using very 'local' tests - just look at the degree of each vertex.

Our next problem looks similar to Eulerian walks, but is very different in behaviour.

Hamiltonian Cycle : n length cycle that visits each vertex exactly once.



Any 'local' tests for Hamiltonicity ? Unlikely.

Given graph G, we dont have any efficient algo. to determine if G is Hamiltonian.

Any sufficient conditions for Hamiltonicity ? Which graphs are guaranteed to have a Hamiltonian cycle ?

- complete graph on n vertices : K_n

- complete bipartite graph with $V = A \cup B$, $A \cap B = \phi$
  $E = \{\{u,v\} : u \in A, v \in B\}$. $|A| = |B| = n$

Thm 23.2 : $n \geq 3$. If $\deg(v) \geq n/2$ for all $v \in V$, then G has a Hamiltonian cycle.

[Dirac, 1952]

By induction ? If we remove any vertex, remaining graph has $n-1$ vertices, but some vertices can have deg. $n/2 - 1$. Therefore we cant use induction hypothesis.

To be continued ...