Recap:

Hamiltonian Cycle: n length cycle that visits each vertex exactly once.

Any 'local' tests for Hamiltonicity? Unlikely.

Given graph G, we dont have any efficient algo. to determine if G is Hamiltonian.

Any sufficient conditions for Hamiltonicity? which graphs are guaranteed to have a Hamiltonian cycle?

- complete graph on n vertices : K_n
- complete bipartite graph with $V = A \cup B$, $A \cap B = \phi$, $E = \{\{u,v\} : u \in A, v \in B\}$. $|A| = |B| = n$

Thm 23.2 : $n \geq 3$. If $\deg(v) \geq n/2$ for all $v \in V$,
[Dirac, 1952] then G has a Hamiltonian cycle.

By induction? Not clear.

Proof :

Obs. 1 : If $deg(v) \geq n/2$ for all $v \in V$, then G is connected.

Pf: Suppose G is not connected. Then it has at least one component whose size is at most $n/2$. $\Rightarrow$ degree of vertices in that component is $\leq n/2 - 1$. ∎

Consider the longest path in G:
$P = v_0 \ v_1 \ \cdots \ v_k$

The longest path / tour idea strikes again! However, you should think about whether the following approach works : consider the longest cycle, and if it doesn't contain all vertices, then there exists a longer cycle.

We will prove two things :

(a) there exists a cycle of length $k+1$ using P

(b) P contains all vertices i.e. $k = n-1$.

Obs 2: All neighbours of v_0 are in the set $\{v_1, v_2, \dots, v_k\}$, and all neighbours of v_k are in the set $\{v_0, v_1, \dots, v_{k-1}\}$.

Proof: Suppose not. Suppose w is a neighbour of v_0, and $w \notin \{v_1, v_2, \dots, v_k\}$. Then $(w, v_0, v_1, \dots, v_k)$ is a longer path. Similarly, if v_k has a neighbour in $V \setminus \{v_0, \dots, v_{k-1}\}$, then there exists a longer path. ∎

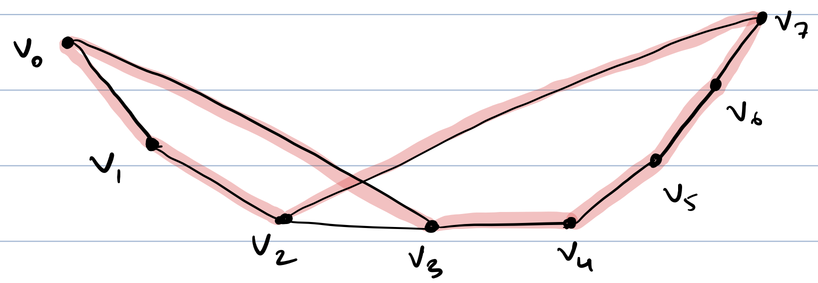
Claim: Suppose the longest path in G is
$|P| = k$ $P = (v_0, v_1, \dots, v_k)$, and vertices v_0, v_k have degree $\geq n/2$. Then there exists a cycle of length $k+1$.

Proof: If $\{v_0, v_k\} \in E$, then we have a $k+1$ cycle $(v_0\ v_1\ \dots\ v_k\ v_0)$.

Suppose $\{v_0, v_k\} \notin E$. Even then, we can find a $(k+1)$ cycle. The proof follows from Pigeonhole Principle, via the following obs.

__Obs 3__: $\exists\ j \in \{2, \ldots, k-1\}$ s.t. $\{v_0, v_j\} \in E$, and $\{v_{j-1}, v_k\} \in E$. As a result, we have a $(k+1)$ cycle

$$(v_0\ v_1\ \cdots\ v_{j-1}\ v_k\ v_{k-1}\ \cdots\ v_j\ v_0).$$



__Pf__: Suppose, on the contrary, $\nexists$ such a j.
Let $\{v_{i_1}, v_{i_2}, \ldots, v_{i_{n/2}}\}$ be the neighbours of v_0.
$1 = i_1 < i_2 < \cdots < i_{n/2} < k$.

Neighbours of v_k are in the set

$$T = \underbrace{\{v_1, \ldots, v_{k-1}\}}_{k-1 \text{ elements}} \setminus \underbrace{\{v_{i_2-1}, v_{i_3-1}, \ldots, v_{i_{n/2}-1}\}}_{n/2 - 1 \text{ elements}}.$$

Note that we didn't include v_{i_1-1} in this set since $i_1 = 1$.

$$|T| \leq k-1-(n/2-1) = k - n/2$$

But note that $k \leq n-1$, and as a result, $\deg(v_k) < n/2$. Contradiction. ∎

<u>Obs 4</u>: P contains all n vertices.

<u>Pf</u>: Suppose not. Then $\exists\, v' \in V \setminus \{v_0, \dots, v_k\}$.
Find $w \in \{v_0, \dots, v_k\}$ s.t. $\mathrm{dist}(v', w) \le \mathrm{dist}(v', u)$
for all $u \in \{v_0, \dots, v_k\}$.

Since graph is connected (Obs. 1),
$\mathrm{dist}(v', w) < n$.

We know (from Obs 3) that w is in a cycle of length $k+1$. Let us denote the cycle by $(w = w_0, w_1, \dots, w_k, w_0)$
This is just a renaming of the vertices $\{v_0, v_1, \dots, v_k\}$.

Let P_1 denote the shortest path from v' to w, and let P_2 denote the path $(w = w_0, w_1, \dots, w_k)$.

$|P_2| \ge k$, and $V(P_1) \cap V(P_2) = \{w_2\}$.
As a result, the concatenation of P_1 and P_2 is a path of length $\ge k+1$. ■

Hence, longest path is of length $n-1$, and there is a cycle of length n.

■

Trivia 1 : Dirac's theorem was generalized by Øystein Ore, who showed the following :

<u>Thm</u> : G : graph with $n \geq 3$ vertices s.t. for all non-adjacent vertices u, v s.t. $\{u, v\} \notin E$, $\deg(u) + \deg(v) \geq n$.
Then G has a Hamiltonian cycle.

Trivia 2 : Ore's theorem was further generalized by Bondy-Chvatal, who showed the following :

<u>Thm</u> : For any graph G, the closure of G is defined as follows :

$$\text{while} \begin{pmatrix} \exists \text{ pair of non-adjacent } u, v \text{ s.t.} \\ \deg(u) + \deg(v) \geq n \end{pmatrix}$$

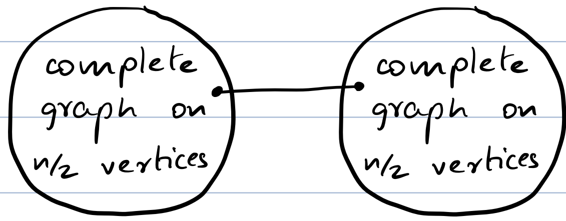
add $\{u, v\}$ to G.

(a) closure of G is well defined (i.e. the above algorithm always gives some final graph)

(b) G is Hamiltonian iff closure of G is Hamiltonian.

Last class, when we started our discussion on Hamiltonian cycles, we saw that a graph can be connected, have $\Theta(n^2)$ edges, and still not have a Hamiltonian cycle.

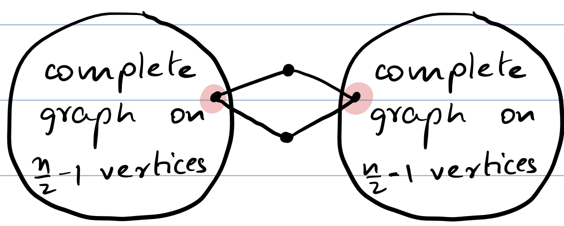
eg.



This graph has $\left(\frac{n}{2}\right)\left(\frac{n}{2}-1\right)+1$ edges, but has no Hamiltonian cycle.

There was a question in class: can we have a graph which has an Eulerian walk, and has $\Theta(n^2)$ edges? Consider the following graph if $n/2$ is even:

H :



This graph has $\Theta(n^2)$ edges, and an Eulerian walk. But it does not have a Hamiltonian cycle.

One main issue is that there are 2 vertices in this graph (marked in ●) whose removal makes the graph disconnected.

Graphs which have this 'vulnerability' can't have a Hamiltonian cycle.

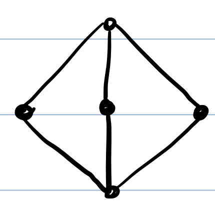
Def^n: A graph $G=(V,E)$ is 2-connected if, for every $v \in V$, the graph remains connected even after removal of v.

Check that the graph H on prev. page is not 2-connected.

Clearly, if G has a Hamiltonian cycle, then G is 2-connected.

Is the converse also true?

No.



This graph is 2-connected, but not Hamiltonian.

While 2-connectedness does not guarantee the existence of a Hamiltonian cycle, it guarantees that every pair of vertices lie on some cycle.

Thm: The following two statements are equivalent:

1. Graph G is 2-connected
2. For every $u, v \in V$, there exists a cycle containing u and v.

(2) => (1)

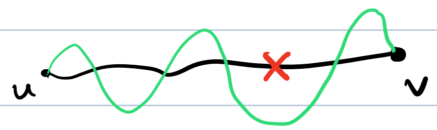
Suppose vertex w is removed. Pick any two vertices u, v. We will show that u and v are connected even after w is removed.

From (2), we get that u and v lie on a cycle. Therefore, there exist two VERTEX-DISJOINT paths from u to v (that is, $P_1 = u\, y_1\, y_2 \dots y_k\, v$ and $P_2 = u\, z_1\, z_2 \dots z_\ell\, v$ where $\{y_1, \dots y_k\, z_1 \dots z_\ell\}$ are all disjoint).

Hence, even if w is removed, there exists a path from u to v.

(1) $\Rightarrow$ (2).

Suppose graph is 2-connected. Consider any two vertices u, v. We know that there exists a path from u to v. If we remove any vertex w that is on this path, then we are guaranteed another



path from u to v. However, that doesn't immediately give us a cycle containing u & v.

Therefore, let us first look at some simpler cases to build intuition.

Suppose $\{u,v\} \in E$, and G is 2-connected.

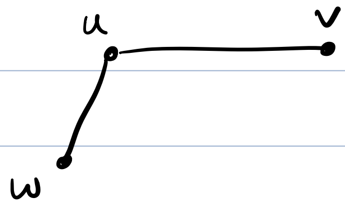
To prove: there exists a cycle containing $\{u,v\}$.

3 cases ((b) and (c) are overlapping and symmetric)

(a) $\deg(u) = \deg(v) = 1$. In this case, the graph has only 2 vertices.

(b) $\deg(u) > 1$ (c) $\deg(v) > 1$.

Case (b):



Since G is 2-connected, there exists a path from v to w even if u is removed. Let this path be $(v_0 = v, v_1, \ldots, v_r = w)$.

None of the v_is are equal to u.

$\therefore (v_0 = v, \ldots, v_r = w, u, v_0 = v)$ is a cycle containing $\{u, v\}$.

Case (c) is symmetric.

How to extend this proof if u, v are not adjacent vertices?

To be contd ...