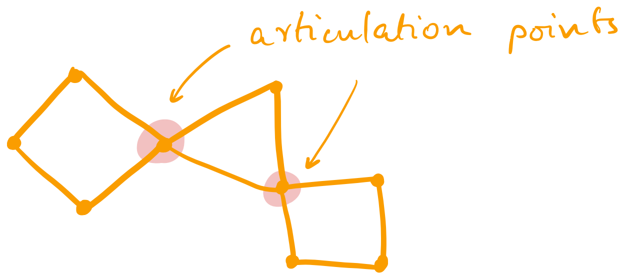
Def: [2-connected graph]: $G = (V, E)$ undirected graph is 2-connected (also called 2-vertex-connected) if for all $v \in V$, $G \setminus \{v\}$ is a connected graph.

vertices whose removal disconnects the graph are called articulation points.



Last time: if G has a Hamiltonian cycle, then it is 2-connected.

In fact, the condition "G has a Hamiltonian cycle," can be relaxed as follows.

Observation: If G satisfies the following property: for every $u, v \in V$, there exists a cycle containing u and v then G is 2-connected.

In fact, the converse also holds true, and therefore, this is a clean characterization of 2-connected graphs.

Thm 25.1: For any graph G, the following two conditions are equivalent:

1. Graph is 2-connected (i.e. remains connected even if any vertex is removed)
2. For any $u, v \in V$, there exists a cycle in G containing u & v.

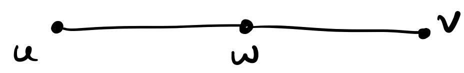
This theorem also explains why we call such graphs 2-connected. If we only had property (1), then maybe a better name would be "1-disconnected" (since removal of 1 vertex makes the graph disconnected). However, property (2) says that for any two vertices u, v, there are two vertex-disjoint paths from u to v (paths which dont have any common vertices other than u and v). Hence the name 2-vertex-connected, or simply 2-connected graphs.

We have seen that (2) $\Rightarrow$ (1).

(1) $\Rightarrow$ (2)

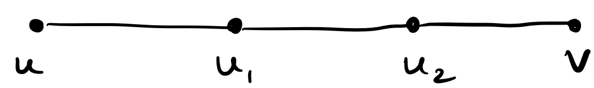
Last class, we proved that if u, v are connected by an edge, and G is 2-connected, then there exists a cycle containing u & v.

What if u and v are not connected by an edge? What if u & v have a common neighbour?



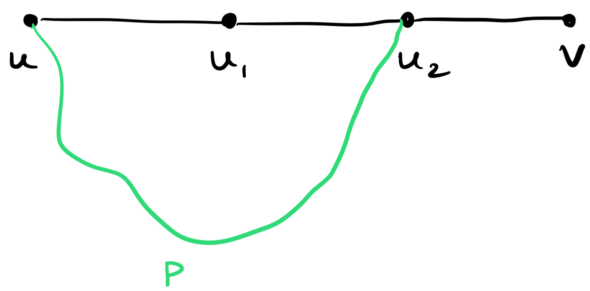
Suppose w fails. Since G is 2-connected, there exists a path $P = (u_0 = u, u_1, \ldots, u_k = v)$ from u to v that doesnt contain w. As a result, $(u_0 = u, u_1, \ldots, u_k = v, w, u)$ is a cycle containing u and v.

What if u and v dont have a common neighbour? What if the shortest from u to v has length 3?

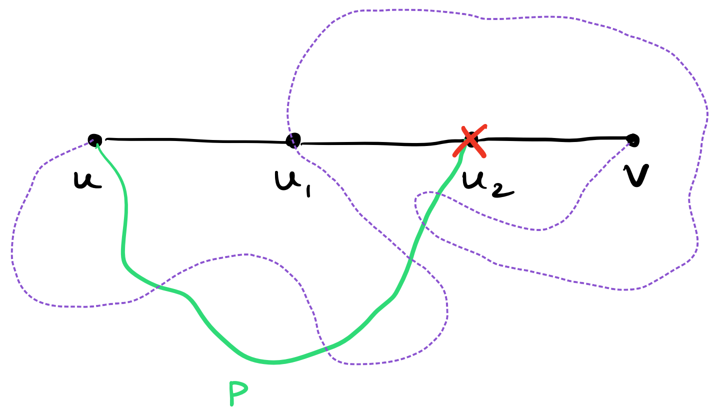


If we consider u_1 failing, then we get a path from u to v avoiding u_1, but that alternate path can be $(u, \ldots, u_2, v)$. This will not give us a cycle containing u and v. Similar issue if we consider u_2 failing - we might have $(u, u_1, \ldots, v)$ as the alternate path.

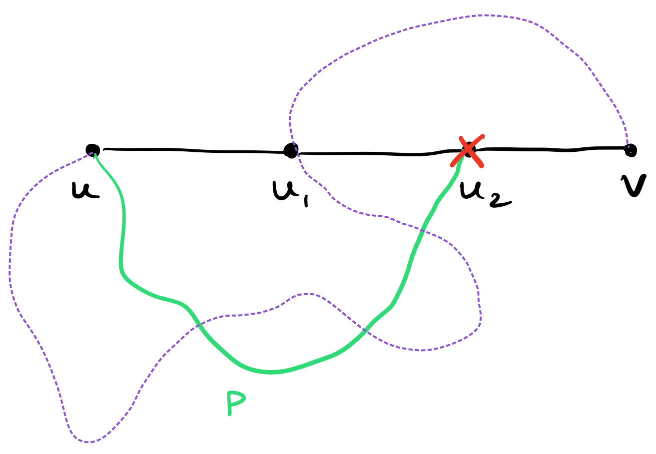
Idea : we know that there's a path P from u to u_2 that doesn't involve u_1.



Suppose u_2 is removed. Then we know there is a path P' from u to v. This path may involve u_1, it may intersect with the path P.

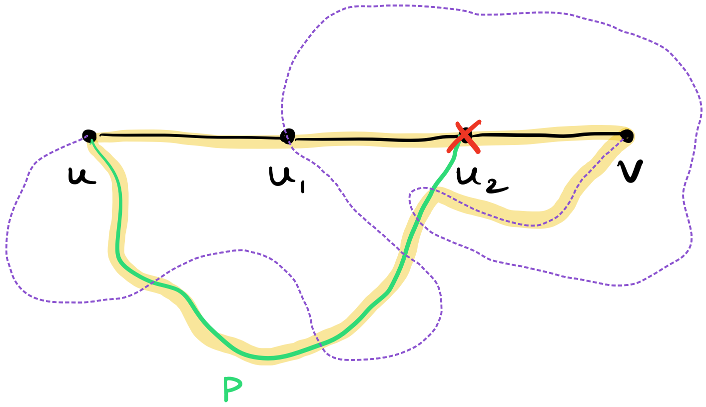


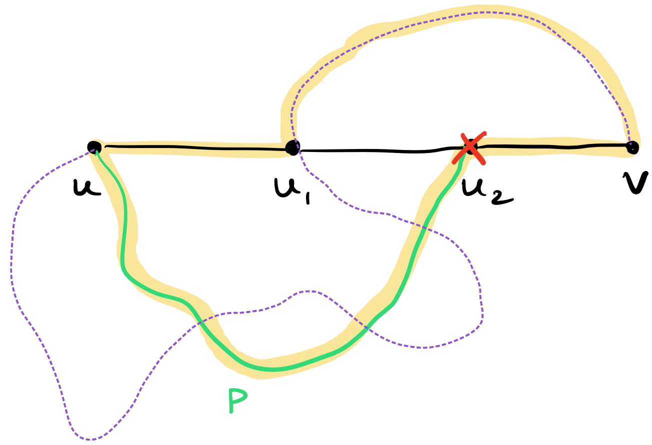
P': path from u to v avoiding u_2



Look at the last point where P' intersects with $V(P) \cup \{u_1\}$. This last intersection pt. can either be on path P, or it can be the vertex u_1.

In both these cases, we get a cycle as shown below.





The above solution suggests how to proceed for the general case : perform induction on the distance between u and v.

Predicate $Q(k)$: For any 2-connected graph $G = (V, E)$ for any $u, v \in V$ s.t. $dist_G(u,v) = k$, there exists a cycle in G containing u and v.

To prove : $Q(k)$ holds for all $k \geq 1$. Proof by regular induction.

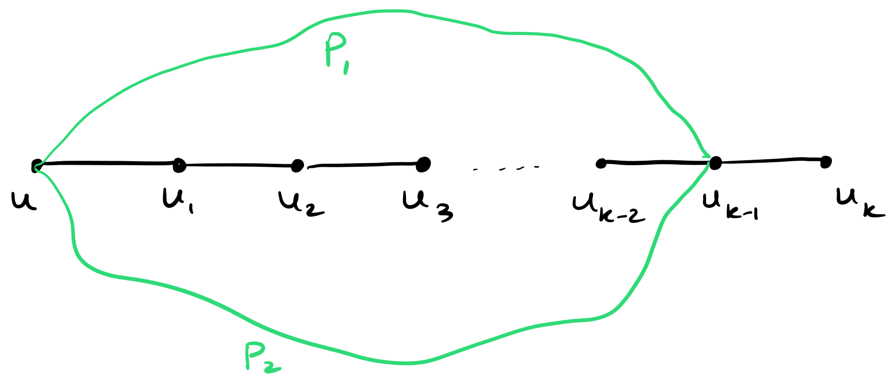
<u>Base cases</u> : We have proven $Q(1)$, $Q(2)$, $Q(3)$.

<u>Induction step</u> : Suppose $Q(k-1)$ holds. To prove : $Q(k)$.

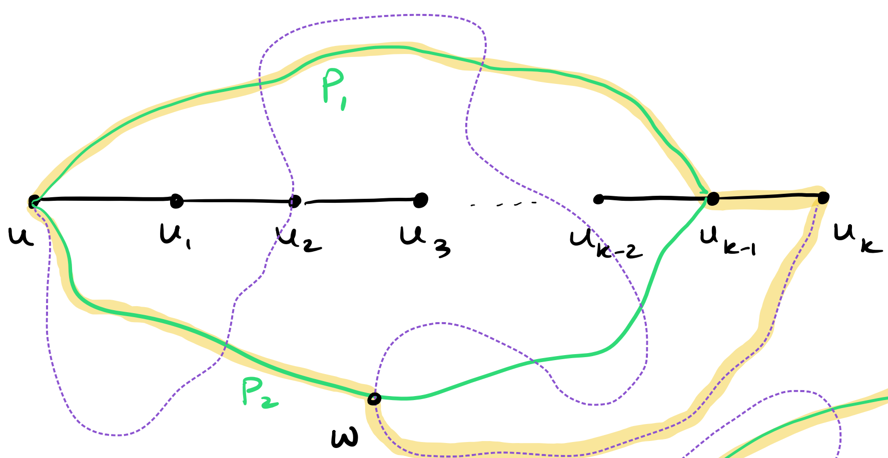
Consider any 2-connected graph, and any two vertices u, v s.t. $dist(u,v) = k$. Let $P = (u_0 = u, u_1, u_2, \ldots, u_k = v)$ be a shortest path from u to v.

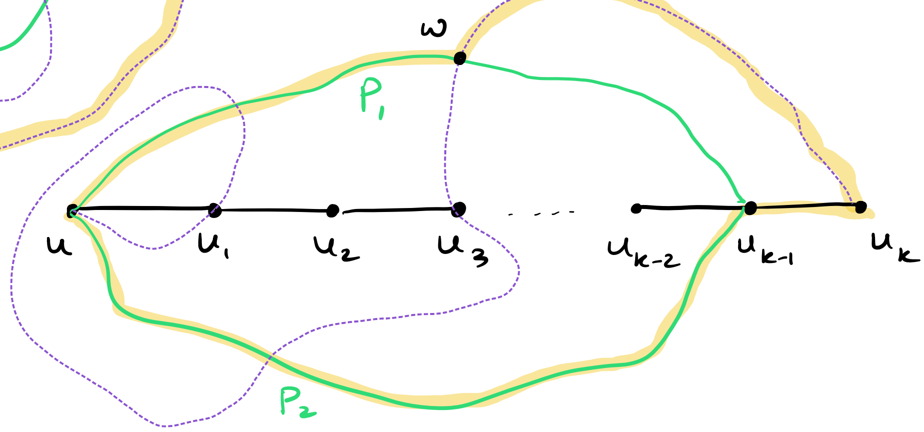
<u>Obs</u>: $\text{dist}(u, u_{k-1}) = k-1$.

Using induction hypothesis, we know there is a cycle containing u and u_{k-1}. Therefore, there are two vertex-disjoint paths from u to u_{k-1}.



Suppose u_{k-1} is removed. By the 2-connectedness of G, there is a path $\hat{P}$ from u to v that doesn't involve u_{k-1}. Let w be the last vertex where $\hat{P}$ intersects $V(P_1) \cup V(P_2)$.





There are two cases : $w \in V(P_1)$, and $w \in V(P_2)$.

In both cases, we have a cycle containing u and v.

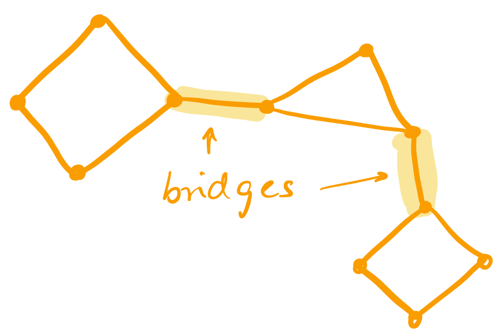
■

Note one subtlety : the cycle containing u & v may not contain the shortest path $P = (u, u_1, \ldots, v)$. Why ? Construct a 2-connected graph which has two vertices u, v s.t. any cycle containing u and v does not include the shortest path from u to v.

Similar to vertex-connectivity, we can also consider the effects of edge removal / failure.

Def [2-edge-connectivity] Graph $G = (V, E)$ is 2-edge-connected if for all $e \in E$, $G \setminus \{e\}$ is a connected graph.

Removal of one edge does not disconnect the graph. Edges whose removal disconnects the graph are called <u>bridges</u>.



Obs: 2-vertex-conn. $\Rightarrow$ 2-edge-conn.

This is easy to prove, left as an exercise.

Recall, in Thm 25.1, we proved that a graph is 2-vertex-connected (or simply, 2-connected) if and only if, for every pair of vertices u, v, there exist two vertex-disjoint paths P_1 and P_2 from u to v.

We can prove a similar result for 2-edge-connectivity.

Thm 25.2: Graph $G = (V, E)$ is 2-edge-connected if and only if, for every pair of vertices u, v there exist two edge-disjoint paths P_1 and P_2 from u to v.

Proof of Theorem 25.2 will be similar to proof of Thm 25.1, it is left as an exercise.

A (SURPRISING) CHARACTERIZATION OF 2-VERTEX CONNECTED / 2-EDGE-CONNECTED GRAPHS

So far, we have seen a few proofs of the following form:

If a graph as property $\mathcal{P}$, — even deg. for every vertex

then it has property $\mathcal{Q}$. — has an Eulerian walk.

Whenever we used induction for such theorems, we went from a larger graph having property $\mathcal{P}$ to a smaller graph having property $\mathcal{P}$.

Why did we do it this way? It's because, generally, when you take a larger graph G with property $\mathcal{P}$, and produce a smaller graph G', you only need to prove $\mathcal{P}$ for G'. If you want to go from a smaller graph with property $\mathcal{P}$ and produce a larger graph of size t (size can either be number of vertices, edges, or some other measure) you need to show that all size t graphs with property $\mathcal{P}$ can be produced using your incremental approach. This is generally more challenging. However, there are some exceptions.

An incremental approach for generating all 2-vertex-connected graphs

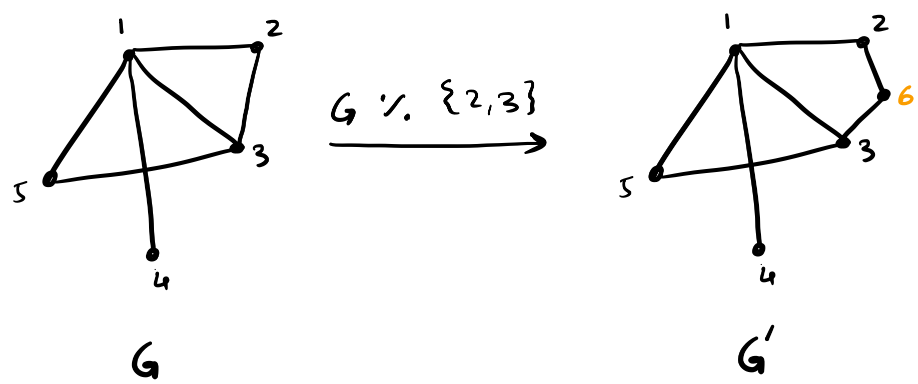
You can view this as a recursive description of the set of all 2-vertex-connected graphs.

Define two operations on graphs:

$G + e$: add edge e to $G = (V, E)$
(if $e \in E$, do nothing

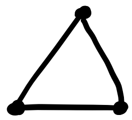
$G \,\%\, e$: valid only if $e \in E$.
'split the edge' by adding a vertex

example :



Note: we've removed the edge {2,3}, added a new vertex '6', and added edges {2,6} and {3,6}.

Set $\mathcal{G}$ of graphs is recursively defined as follows:

Base case :  $\in \mathcal{G}$

Recursive step :

(1) If $G = (V, E) \in \mathcal{G}$, and $e \notin E$, then $G + e \in \mathcal{G}$

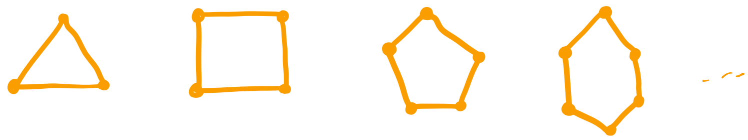
(2) If $G = (V, E) \in \mathcal{G}$ and $e \in E$, then $G \% e \in \mathcal{G}$.

Theorem 25.3 : $\mathcal{G}$ = set of all 2-vertex-cktd. graphs

We will see a related, but perhaps less-surprising, characterization of 2-vertex-connected graphs. This is popularly called the "ear decomposition" of 2-vertex-connected graphs.

Set $\mathcal{G}'$ of graphs is recursively defined as follows:

Base case: all connected cycles are in $\mathcal{G}'$.



Recursive step:

Rule 1: If $G = (V, E) \in \mathcal{G}'$, and $u, v \in V$

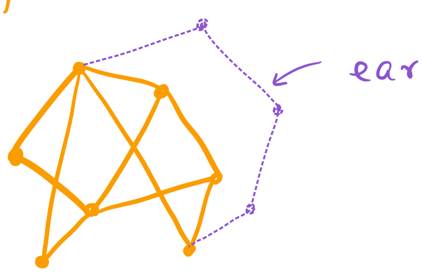
- if $\{u,v\} \notin E$, then $G + \{u,v\} \in \mathcal{G}$.

Rule 2: If $G = (V, E) \in \mathcal{G}'$ and $u, v \in V$, for all $k \in \mathbb{N}$, let $V' = V \cup \underbrace{\{v_1, \dots v_k\}}_{\text{new vertices}}$

$E' = E \cup \{\{u, v_1\}, \{v_1, v_2\} \dots \{v_k, v\}\}$

$G' = (V', E') \in \mathcal{G}'$.

Adding an "ear" to the graph



Theorem 25.4: $\mathcal{G}'$ = set of all 2-vertex-connected graphs.

Why can the 'ear decomposition' be useful? Suppose you want to prove that all 2-vertex-connected graphs have some property Q. One way to prove this is using structural induction - prove Q for all cycles, then prove that Q is preserved by adding edges, and by adding ears.

Proof of Theorem 25.4: It is easy to see that all graphs in $\mathcal{G}'$ are 2-vertex-connected.

To show: every 2-vertex-connected graph is in $\mathcal{G}'$.

Take any 2-vertex-connected graph $G = (V, E)$.

Obs 1: G has a cycle. In other words, $\exists$ $V_0 \subseteq V$, $E_0 \subseteq E$ s.t. $G_0 = (V_0, E_0)$ is a cycle.

Follows from 2-vertex-connectivity.

We will keep adding edges/ears to G_0, thereby getting incremental graphs $G_1, G_2, \ldots$ For each i, $V_{i-1} \subseteq V_i \subseteq V$, $E_{i-1} \subseteq E_i \subseteq E$, and $|E_{i+1}| > E_i$. As a result, this process terminates in finite steps.

Claim : $\exists$ $V_0 \subseteq V_1 \subseteq \cdots \subseteq V_i = V$,
$E_0 \subset E_1 \subseteq \cdots \subseteq E_i = E$
s.t. each $G_j = (V_j, E_j)$ is produced
by adding an edge / ear to G_{j-1}.

Process terminates in $\leq |E| - 3$ steps.

We already have G_0 from Obs. 1. Suppose we have $G_j = (V_j, E_j)$ s.t. $V_j \subsetneq V$, $E_j \subsetneq E$
and G_j is 2-vertex-connected.

If $V_j = V$, then we can keep adding edges until $E_j = E$.

Obs 2 : $\exists$ $e = \{u, v\} \in E \setminus E_j$ s.t. either $u \in V_j$
or $v \in V_j$.

Follows from the fact that G is connected.

Obs 3 : If $u, v \in V_j$, then set $V_{j+1} = V_j$,
$E_{j+1} = E_j \cup \{e\}$.

Obs 4 : If $u \in V_j$, $v \notin V_j$, then there exist
vertices $v = v_1 \cdots v_{k-1} \in V \setminus V_j$, and $v_k \in V_j$
s.t. $\{v_\ell, v_{\ell+1}\} \in E$, $\{v_{k-1}, v_k\} \in E$.

Idea : remove u, consider $\mathrm{dist}_{G\setminus\{u\}}(v, y)$ for all $y \in V_j \setminus \{u\}$.

Note that $\mathrm{dist}_{G\setminus\{u\}}(v, y) < \infty$ for all $y \in V_j \setminus \{u\}$ (since G is 2-vertex-connected).

Let $w \in V_j \setminus \{u\}$ s.t. $\mathrm{dist}_{G\setminus\{u\}}(v, w)$

$$\leq \mathrm{dist}_{G\setminus\{u\}}(v, y) \text{ for all } y \in V_j \setminus \{u\}.$$

Consider the shortest path P from v to w in $G\setminus\{u\}$.

Observe that this path gives us an 'ear'. If $P = (v_1 = v, v_2, \dots, v_{k-1}, v_k = w)$, then $(u, v, v_2, \dots, v_{k-1}, w)$ is an ear that can be added to G_j.

This completes our proof of Obs. 4. Note that we have seen how to go from G_j to $G_{j+1} = G_j + \mathrm{ear}(u, v_1, \dots, v_{k-1}, w)$.

Hence, we have shown that $\mathcal{G}'$ is equal to the set of all 2-vertex-connected graphs. ■

Proof of Theorem 25.3 :

It is easy to observe that any graph in $\mathcal{G}$ is 2-vertex-connected.

It is also easy to show that $\mathcal{G}' \subseteq \mathcal{G}$ (just need to show that any cycle is in $\mathcal{G}$, and "ear addition" can be simulated by edge addition + edge splitting).

Finally, we showed that

$$\left\{ \begin{array}{c} \text{2-vertex-connected} \\ \text{graphs} \end{array} \right\} \subseteq \mathcal{G}'.$$

Therefore, $\mathcal{G} \equiv \left\{ \begin{array}{c} \text{2-vertex-connected} \\ \text{graphs} \end{array} \right\}$