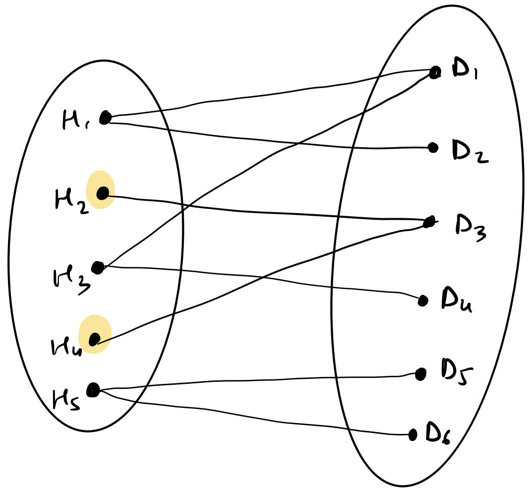
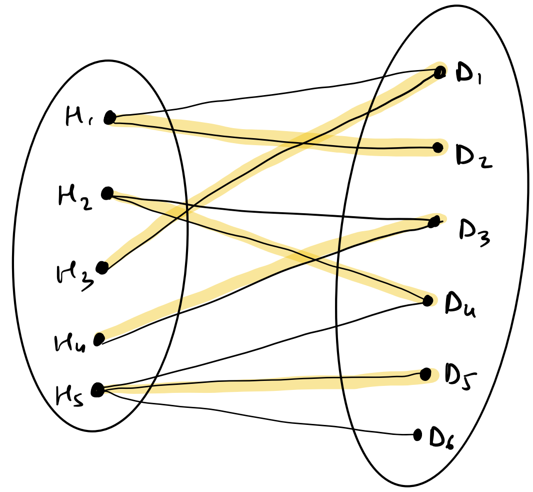
Recap :

n hospitals , m doctors , $m \geq n$.
Each doctor has applied to at least one hospital.

Goal : Can we assign at least one doctor to each hospital?



no assignment possible
to all hospitals since
the number of doctors
applying to $\{H_2, H_4\}$ is 1.



Such an assignment is called a MATCHING.

Def: Let $G = (X \cup Y, E)$ be an undirected bipartite graph. A matching M in G is a subset of edges such that for all $e, e' \in M$, $e \cap e' = \phi$. We say G has an X-matching if $\exists$ a matching M where $\forall x \in X$, $\exists$ $e \in M$ s.t. x is an endpoint of e.

QN: Necessary and sufficient conditions for existence of X- matching ?

Necessary condition: For any $S \subseteq X$, let

$$\mathrm{Nbr}_G(S) = \{ y \in Y \text{ s.t. } \{x, y\} \in E \text{ for some } x \in S \}$$

Then $|\mathrm{Nbr}_G(S)| \geq |S|$.

Surprisingly, the above necessary condⁿ is also sufficient ! This was shown by Philip Hall in 1935, and is referred to as Hall's Theorem.

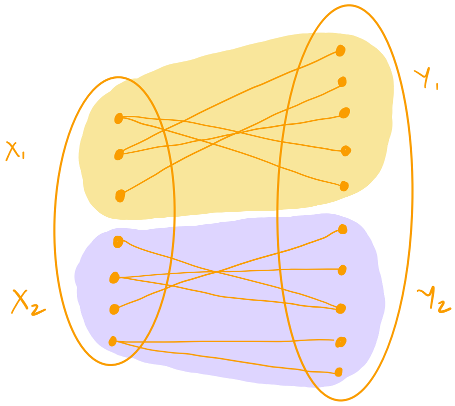
Theorem: Let $G = (X \cup Y, E)$ be an undirected bipartite graph. Then G has an X - matching if and only if,

$$\boxed{\text{for any } S \subseteq X, \ |\mathrm{Nbr}_G(S)| \geq |S|.}$$

If G satisfies this condition, then we say that Hall's condition holds.

Proof: only if : this is clear from the definition of X- matching.

The formal proof for the 'if' direction uses strong induction on $|X|$. Before looking at the formal proof, let us build some intuition using a few examples.



In this example, we are able to partition the vertex sets $X = X_1 \cup X_2$, $Y = Y_1 \cup Y_2$, and the edge set $E = E_1 \cup E_2$ s.t. X-matching exists in G iff X_1-matching exists in $G_1 = (X_1 \cup Y_1, E_1)$ and X_2-matching exists in $G_2 = (X_2 \cup Y_2, E_2)$.

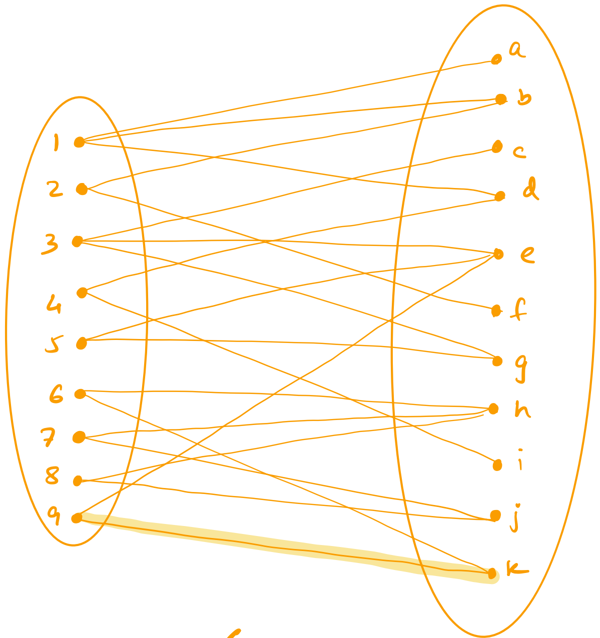
Looking ahead, in the full proof, we will partition $X = X_1 \cup X_2$, $Y = Y_1 \cup Y_2$, and find non-overlapping edge sets $E_1 \subseteq E$, $E_2 \subseteq E$, $E_1 \cap E_2 = \phi$ s.t.

- edges in E_i have one end point in X_i, another in Y_i
- X-matching in G exists iff
  [X_1-matching exists in $G_1 = (X_1 \cup Y_1, E_1)$ and
  X_2-matching exists in $G_2 = (X_2 \cup Y_2, E_2)$]

Note: E_1 & E_2 may not be a partitioning of E

When applying induction, we go from a large graph G, to one/more smaller graphs. How to apply this strategy here?

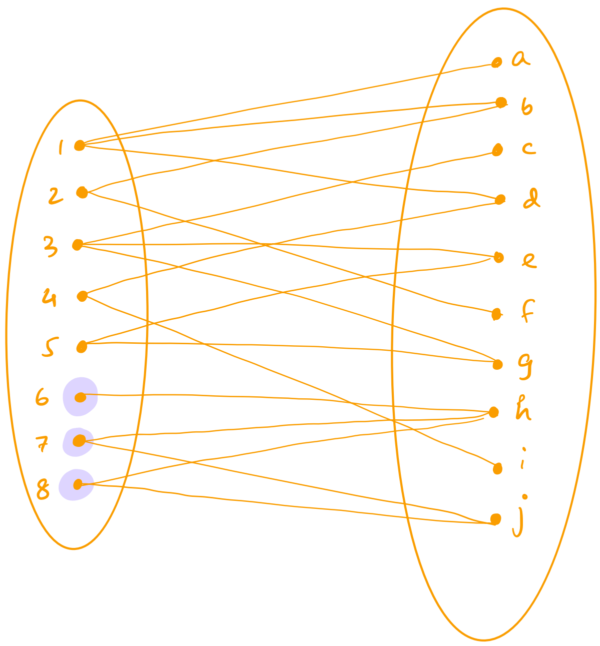
Given : $G = (X \cup Y, E)$ satisfies Hall's condition
[for every $S \subseteq X$, $|Nbr_G(S)| \geq |S|$]



G

Consider this graph. Hall's condition holds on this graph.

Pick the last vertex in X ("9" here), and pick an arbitrary neighbour of "9" ("k" here).



G'

Remove 9, k from the graph (also remove all the vertices adjacent to "9" and "k"). Let us call this graph G'.

2 cases :

(a) Hall's condition holds in G'. This case is easy. We find a matching M' in G', and $M' \cup \{\{\text{"9"}, \text{"k"}\}\}$ is a matching in G'.

(b) Hall's condition does not hold in G'. Unfortunately, our example is in this case. Check that

$$Nbr_{G'}(\{"6", "7", "8"\}) = \{"h", "j"\}.$$

But then, consider $Nbr_{G}(\{"6", "7", "8"\})$

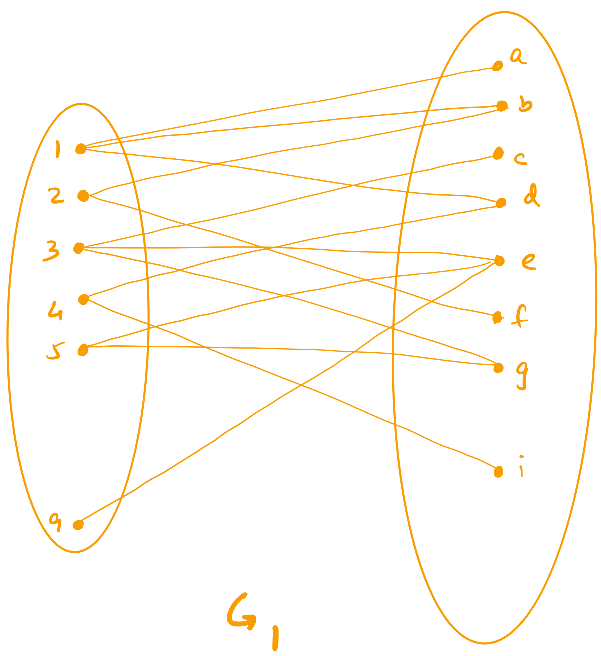
Note: we are now looking at the neighbourhood of this set in G.

Check that $Nbr_G(\{"6", "7", "8"\}) = \{"h", "j", "k"\}$.

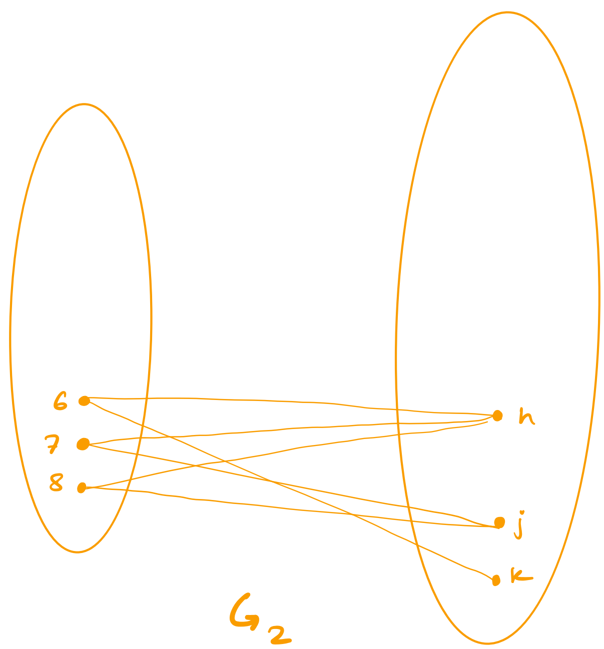
It is not a coincidence that $|Nbr_G(\{"6", "7", "8"\})| = |\{"6", "7", "8"\}|$. Think why this is guaranteed.

As a result, there exists an X-matching in G if and only if $\{"6", "7", "8"\}$ can be matched with $\{"h", "j", "k"\}$ and $\{"1", "2", "3", "4", "5", "9"\}$ can be matched with $\{"a", "b", "c", "d", "e", "f", "g", "i"\}$.

Therefore, we now have two smaller graphs.



G_1



G_2

An X-matching in G exists if and only if an X_1-matching exists in G_1, and an X_2 matching exists in G_2.

All that remains to show is that Hall's condition holds in G_1 and G_2. If it does, then we have a matching M_1 in G_1, and a matching M_2 in G_2. The final matching is $M_1 \cup M_2$.

We will prove this formally. Think why this is the case.

FORMAL PROOF FOR THE SUFFICIENT COND^{n}

$G = (X \cup Y, E)$. If G satifies Hall's condition

Hall's condition : $\forall S \subseteq X$, $|Nbr_G(S)| \geq |S|$.

then G has an X-matching.

proof by strong induction on $|X|$.

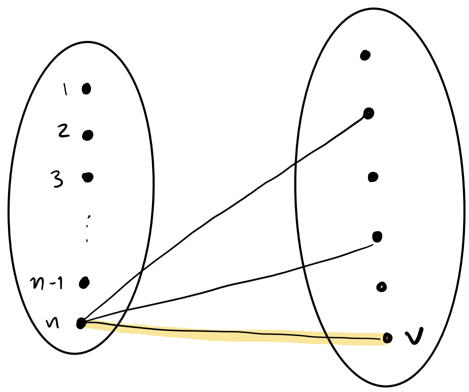
$P(n)$: for any bipartite graph $G = (X \cup Y, E)$, if $|X| = n$, and for all $S \subseteq X$, $|Nbr_G(S)| \geq |S|$, then G has an X-matching.

<u>Base case</u> : $P(1)$ ✓

If $X = \{x\}$, and $Nbr_G(x) = \{y_1, \dots\}$,
Therefore, we can find an X-matching.

<u>Induction Step</u> : to show $P(1), P(2), \dots, P(n-1) \Rightarrow P(n)$

Consider any $G = (X \cup Y, E)$ where $X = \{1, \dots, n\}$,
and Hall's cond^n holds for G.



$|Nbr_G(\{\text{"}n\text{"}\})| \geq 1$
by Hall's cond^n.

Pick an arbitrary neighbour of "n", say "v".

Consider the graph G' obtained by deleting "n" $\in X$, "v" $\in Y$.
Let $X' = X \setminus \{n\}$, $Y' = Y \setminus \{v\}$.

2 cases :

for defining this case (a), we are using neighbourhood in G'

(a) $\forall\ S \subseteq X'$, $|Nbr_{G'}(S)| = |Nbr_G(S) \cap Y'| \geq |S|$
Then, using $P(n-1)$, we have an
X'-matching M' in G'.
$M' \cup \{\{n, v\}\}$ is an X-matching in G.

(b) $\exists\ S \subseteq X'$ s.t. $|Nbr_{G'}(S)| = |Nbr_G(S) \cap Y'| < |S|$.

Since Hall's condition holds on G, $|Nbr_G(S)| \geq |S|$.

Finally, note that $|Nbr_{G'}(S)| \geq |Nbr_G(S)| - 1$.

As a result, $|Nbr_{G'}(S)| = |S| - 1$, and $|Nbr_G(S)| = |S|$.

From this point onwards, we can forget about $X' = X \setminus \{u\}$, we only need to remember that $S \subseteq X'$, $|Nbr_G(S)| = |S|$.

Let $G_1 = (S \cup Nbr(S), E_1)$

$G_2 = (X \setminus S \cup Y \setminus Nbr(S), E_2)$.

where $E_1 = \{\{x, y\} \in E \text{ s.t. } x \in S\}$

$E_2 = \{\{x, y\} \in E \text{ s.t. } y \in Y \setminus Nbr(S)\}$

claim : If Hall's condition holds on G, then Hall's cond^n. also holds on G_1 and G_2.

Proof: Let us focus on G_1.

Hall's condition holds for G
$\Rightarrow$ Hall's condition holds for G_1.

For any $T \subseteq S$, $Nbr_{G_1}(T) = Nbr_G(T)$ since all neighbours of T in G are present in G_1.

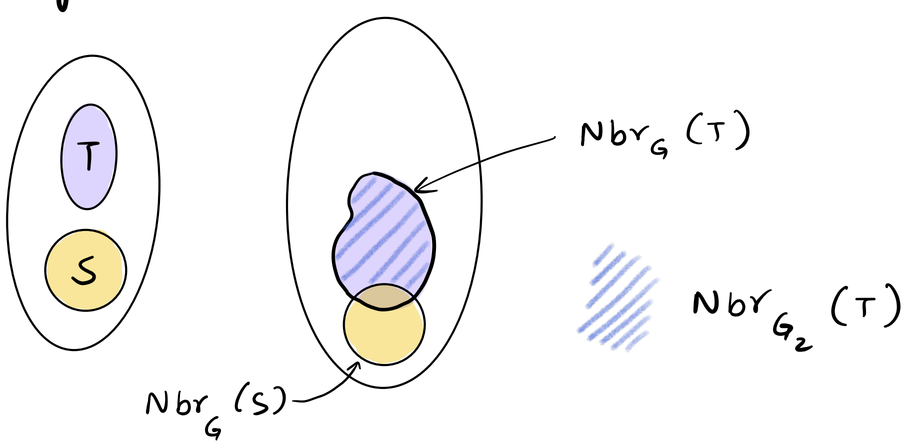
We know $|Nbr_G(T)| \geq |T|$ since Hall's condition holds for G.

As a result, $|Nbr_{G_1}(T)| \geq |T|$.

Using induction hypothesis $P(|S|)$, we know that G_1 has an S-matching M_1.

Next, let us consider G_2.

Take any $T \subseteq X \setminus S$



Suppose $|Nbr_{G_2}(T)| < |T|$.

Then $|Nbr_G(S \cup T)| = |Nbr_{G_2}(T)| + |Nbr_G(S)|$

$= |Nbr_{G_2}(T)| + |S|$

$< |T| + |S| = |T \cup S|$

Here, we have used $|Nbr_G(S)| = |S|$.

$T \subseteq X \setminus S$, hence $T \cap S = \phi$, $|T \cup S| = |T| + |S|$.

Hence, we have a contradiction.

$\blacksquare$

We have shown that G_1 and G_2 satisfy Hall's cond^n. Using the induction hypothesis $P(|S|)$ and $P(n - |S|)$, we get matchings M_1 and M_2 s.t. M_1 is an S-matching on G_1, M_2 is an $X \setminus S$-matching on G_2.

$\Rightarrow$ $M_1 \cup M_2$ is an X-matching on G.

$\blacksquare$

Def : Perfect Matchings

Let $|X| = |Y|$.

$G = (X \cup Y, E)$: bipartite graph.

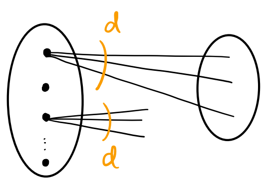
$M \subseteq E$ is a perfect matching if M is a matching and $|M| = |X|$.

When are we guaranteed to have a perfect matching?

Claim : Let $|X| = |Y|$, $G = (X \cup Y, E)$ s.t. every vertex has degree d. Then G has a perfect matching.

Proof : Fix any $S \subseteq X$.

Is it possible that $|Nbr_G(S)| < |S|$?



E_1 = set of edges incident on S

$|E_1| = d|S|$.

E_2 = set of edges incident on $Nbr_G(S)$

$|E_2| = d\,|Nbr_G(S)|$

Some of the neighbours of $Nbr(S)$ may be outside S.

Therefore, $E_1 \subseteq E_2$.

$\therefore \quad |E_1| \le |E_2| \Rightarrow d|S| \le d\,|Nbr_G(S)|$

Since Hall's condition holds, $\exists$ X- matching.

Since $|X| = |Y|$, all vertices in Y are also matched.

$\blacksquare$

The above claim holds even if G is a multigraph.

——— End of lecture and course ———