Exercise 2.1 : A, B finite sets.

Given : $f : A \to B$ inj , $g : B \to A$ inj.

Prove that $\exists$ bijection between A and B.

Pf : Since A, B are finite, if f is inj, then
$|A| \geq |B|$. (i)

Since A, B are finite, if g is inj, then
$|B| \leq |A|$. (ii)

(i) + (ii) $\Rightarrow$ $|A| = |B|$ (iii)

Since f is injective, and $|A| = |B|$,
it is also surjective. ▨

Qn : Infinite sets A, B. Can we compare different infinities ?

Def 1.8 : Let A, B be two infinite sets.
We say that A and B have same cardinality if $\exists$ bijective function $f : A \to B$.

Thm 2.1 : $\mathbb{Z}$: set of integers $\{\ldots -2, -1, 0, 1, 2, \ldots\}$
$\exists$ bijection $f : \mathbb{Z} \to \mathbb{N}$.

It suffices to show an injective fn. $g : \mathbb{Z} \to \mathbb{N}$, since we already have an injective fn. $h : \mathbb{N} \to \mathbb{Z}$ (namely $h(x) = x$), and we can use Schroder - Bernstein thm.

Proof Idea: assign a distinct natural number to $\mathbb{Z}$.

13	11	9	7	5	3	1	2	4	6	8	10	12	
-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	6	...

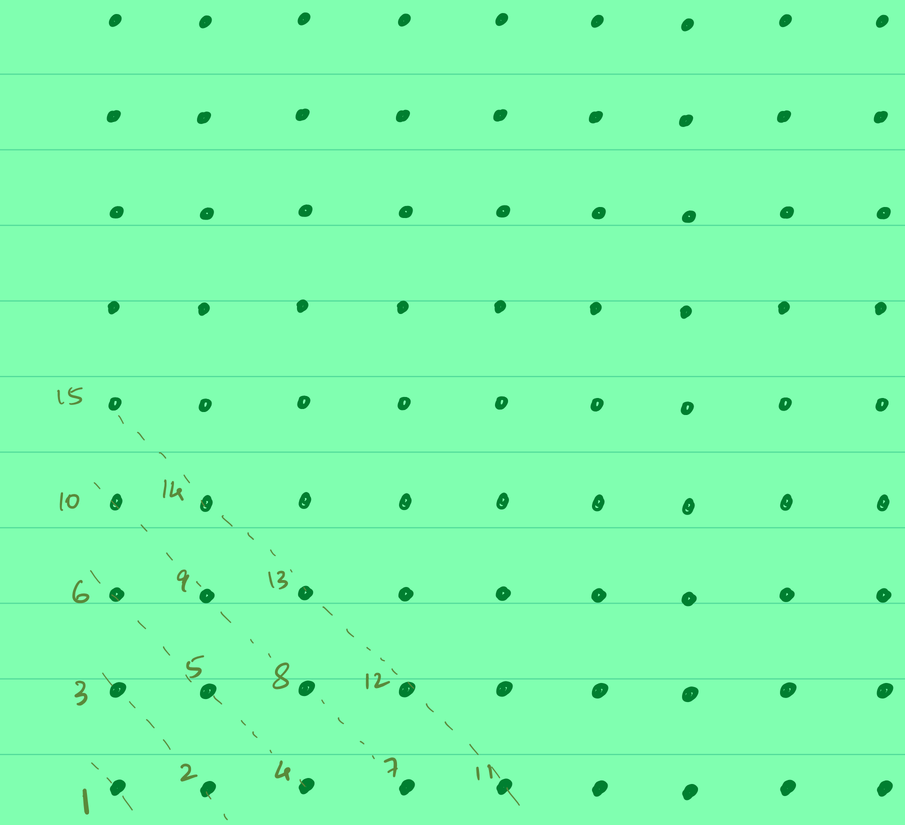
$$g(x) = \begin{cases} 2x & \text{if } x > 0 \\ 2(-x)+1 & \text{otherwise} \end{cases}$$

Easy to check its injective (and also surjective) ∎

Thm 1.5 : $\exists$ bijection $f : \mathbb{N} \to \mathbb{N} \times \mathbb{N}$.

suffices to show an injective fn. $g : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ (showing an injective fn. $h : \mathbb{N} \to \mathbb{N} \times \mathbb{N}$ is easy)

Proof Idea: Consider $\mathbb{N} \times \mathbb{N}$ as a grid. Assign a distinct number to each point in grid.



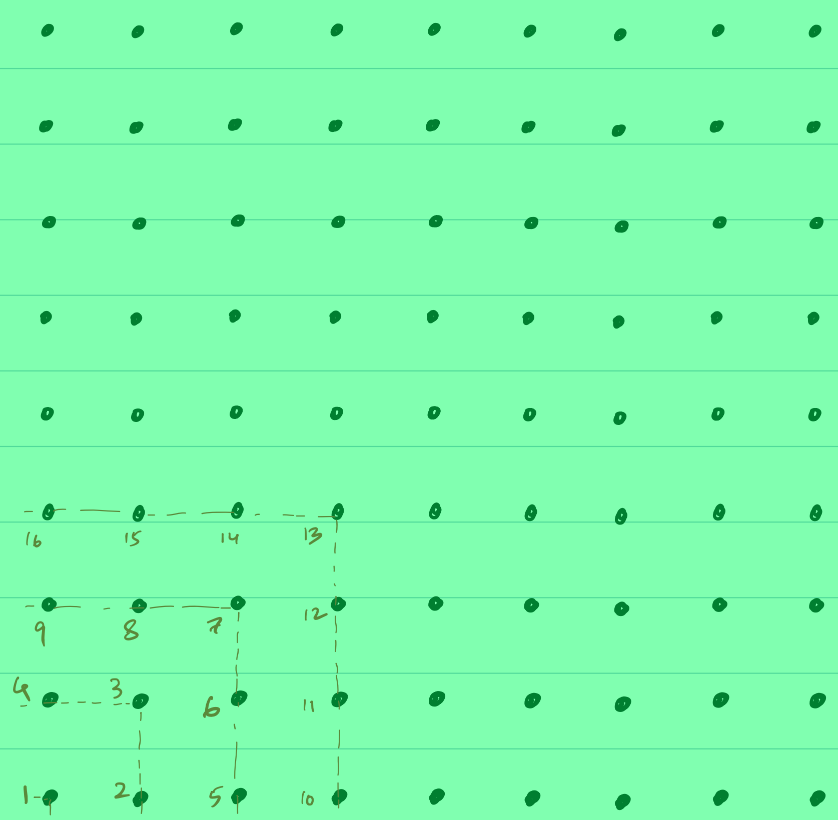
Solution 1:

Let $g(x,y)$ denote the number assigned to pt. (x,y). This is an injective mapping from $\mathbb{N} \times \mathbb{N} \to \mathbb{N}$.

Moreover, it is also surjective, every $z \in \mathbb{N}$ is assigned to some (x,y)

Solution 2:

Another possible numbering of grid.



Proof (using Solution 1): PROOF OF EXISTENCE BY EXPLICIT CONSTRUCTION

We will construct a function $h : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$,

and show that it is injective.

$$h(x,y) = \left(\sum_{i=1}^{s-2} i \right) + y \quad \text{where } s = x+y-2$$

$$= \frac{(x+y-1)(x+y-2)}{2} + y$$

complete the proof by showing h is injective. ∎

Key Observations:

1. To show that something exists, it suffices to show one example. Mention at start of proof that you're showing proof of existence via explicit construction.

2. To show a bijection betn A & B, it suffices to show inj. fns $g : A \to B$ and $h : B \to A$.

Other solutions proposed in class:

$h : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$

1. $h(x,y) = 2^{x-1}[2y-1]$ (both inj. and surj)

2. $h(x,y) = 2^x 3^y$ (inj)

3. $h(x,y) = (x^{th} \text{ prime number})^y$ (inj)

Qn: Is there any set thats 'bigger' than $\mathbb{N}$ (acc. to Def. 1.8)?

$\mathbb{N}, \mathbb{N}^2, \mathbb{N}^3, \mathbb{N}^4$ are all of same cardinality.

<u>Exercise 2.2</u> For any constant k, show that $\exists$ injective fn. $f : \mathbb{N}^k \to \mathbb{N}$.

Hint: use one of the alternate approaches described above. Note that you need to prove that its injective. What assumptions are you making about natural numbers in order to prove injectivity?

Exercise 2.3 : Let $\mathcal{F}(\mathbb{N})$ denote the set of all FINITE subsets of $\mathbb{N}$. Show that $\exists$ injective fn. $f : \mathcal{F}(\mathbb{N}) \to \mathbb{N}$.

QN: WHAT ABOUT THE SET OF ALL SUBSETS OF $\mathbb{N}$ (that is, the power set of $\mathbb{N}$ - $\mathcal{P}(\mathbb{N})$)?

We saw a few attempts to show that $\exists$ injective $f : \mathcal{P}(\mathbb{N}) \to \mathbb{N}$, as well as a few attempts to 'prove' that $\mathcal{P}(\mathbb{N})$ is strictly larger than $\mathbb{N}$.

Attempt 1 :

Claim : Cardinality of $\mathcal{P}(\mathbb{N})$ is strictly greater than $\mathbb{N}$.

"**Proof**" : Define the set $S_1 = \{ \{x\} : x \in \mathbb{N} \}$

✓ (i) There exists a bijection bet^{n} $\mathbb{N}$ and S_1.

Define the set $S_2 = \{\{x, y\} : x, y \in \mathbb{N},\ x \neq y\}$

✓ (ii) $S_1 \cap S_2 = \phi$

✓ (iii) $S_1 \subseteq \mathcal{P}(\mathbb{N})$, $S_2 \subseteq \mathcal{P}(\mathbb{N})$.

Since there exists a bijection betn $\mathbb{N}$ and S_1, and $S_1 \cap S_2 = \phi$, and $S_1 \subseteq \mathcal{P}(\mathbb{N})$, $S_2 \subseteq \mathcal{P}(\mathbb{N})$, therefore $\mathcal{P}(\mathbb{N})$ is strictly larger than $\mathbb{N}$.

■

This part, unfortunately, does not count as "formal reasoning". In fact, this is not even true! Consider the set of integers $\mathbb{Z}$. Let $S_1 = \mathbb{N}$, $S_2 = \{-x : x \in \mathbb{N}\}$. $S_1 \cap S_2 = \phi$, $S_1 \subseteq \mathbb{Z}$, $S_2 \subseteq \mathbb{Z}$, but we know (from Thm. 2.1) that $\exists$ bijection betn $\mathbb{N}$ and $\mathbb{Z}$.

This issue arises because we are using our intuition about finite sets, and trying to find an analogue for the same in infinite sets. This approach can lead to several fallacies, and therefore it is strongly recommended that you put aside this intuition, and work with the given definitions & axioms.

Claim : $\exists$ an injective fn. $f : \mathcal{P}(\mathbb{N}) \to \mathbb{N}$.

"Proof" : For any set $S = \{x_1, x_2, x_3, \dots\} \in \mathcal{P}(\mathbb{N})$,

define $f(S) = \prod_{i=1}^{\infty} p_i^{x_i}$ where p_i is the i^{th} prime.

This is not well defined, because if S is infinite, then this is a product of infinitely many numbers. The product of infinitely many numbers is not a natural number.

take this as a fact / axiom

$\blacksquare$

Again, we have tried to extend our intuition with finite sets (in this case, the product of finite set of numbers) to infinite sets.

There does not exist

Thm 2.2 : $\nexists$ a surjective $f : \mathbb{N} \to \mathcal{P}(\mathbb{N})$.

Before we prove Thm 2.2, let us prove a simpler result, that conveys the main idea. In this simpler result, we will show that there does not exist a surjective fn. f from $\mathbb{N}$ to the set of infinite binary strings

More formally, let $S = \{(b_1, b_2 \ldots) : b_i \in \{0,1\}\}$

Thm 2.3 : $\nexists$ a surjective $f : \mathbb{N} \to S$.

PROOF BY CONTRADICTION.

Suppose $\exists$ a surjective fn. $f : \mathbb{N} \longrightarrow S$.

Using this surjective fn, we can create an infinite table, where the i^{th} row contains $f(i)$.

Example

$f(1)$ =	1	0	0	1	1	0	...
$f(2)$ =	0	0	1	0	1	0	...
$f(3)$ =	0	0	0	1	1	0	...
$f(4)$ =	1	1	0	0	1	1	...
$f(5)$ =	0	0	1	1	0	1	...
$f(6)$ =	1	1	1	0	0	1	...

Consider the infinite string on the diagonal

$D = 1\ 0\ 0\ 0\ 0\ 1 \ldots$

Consider $\bar{D} = 0\ 1\ 1\ 1\ 1\ 0 \ldots$

Claim: $\forall i \in \mathbb{N}$, $f(i) \neq \bar{D}$ (where $\bar{D}$ is the string constructed above).

Proof: PROOF BY CONTRADICTION.

Suppose, on the contrary, $\exists\, i \in \mathbb{N}$ s.t. $f(i) = \bar{D}$. By construction, the i^{th} bit of D is equal to the i^{th} bit of $f(i)$, and therefore the i^{th} bit of $\bar{D}$ is not equal to i^{th} bit of $f(i)$. Hence, $f(i) \neq \bar{D}$.

$\blacksquare$

Using the above claim, we have a contradiction. We assumed f is surjective, but there exists a string $\bar{D}$ s.t. $\forall i \in \mathbb{N}$, $f(i) \neq \bar{D}$.

$\blacksquare$

The above result shows that $\nexists$ surjection from $\mathbb{N}$ to $\mathbb{R}$. Recall, earlier we showed a bijection between $\mathbb{N}$ and $\mathbb{N} \times \mathbb{N}$, which implies a bijection between $\mathbb{N}$ and $\mathbb{Q}$.

PROOF OF THEOREM 2.2 : PROOF BY CONTRADICTION

Suppose $\exists$ a surjection $g : \mathbb{N} \to \mathcal{P}(\mathbb{N})$.

Consider set $A_g = \{a : a \notin g(a)\}$

By definition of A_g, for any $a \in \mathbb{N}$,

$a \in A_g$ if and only if $a \notin g(a)$

<u>claim</u> : for any fn $g : \mathbb{N} \to \mathcal{P}(\mathbb{N})$,

$\nexists\ i$ s.t. $g(i) = A_g$.

PROOF OF CLAIM BY CONTRADICTION.

Suppose $g(i) = A_g$.

$i \in A_g$ if and only if $i \notin g(i) = A_g$

Contradiction ■

Using the claim, we can conclude that g is not a surjection as A_g has no preimage

■

Summary of sets, functions:

set : unordered collection of distinct objects
function : mapping from one set to another
└ injective, surjective, bijective.

A, B finite sets. $f : A \to B$
injective : $|A| \geq |B|$
surjective : $|A| \leq |B|$
bijective : $|A| = |B|$.

A, B finite sets. If $f : A \to B$ and $g : B \to A$ are injective, then $\exists$ bijection between A & B.

A, B finite sets. If $f : A \to B$ and $g : B \to A$ are surjective, then $\exists$ bijection between A & B.

same proof idea.

Comparing sizes of infinite sets:
If $\exists$ bijection between A and B, then we say A and B have same cardinality.

for infinite sets, these results are non-trivial
[Schröder - Bernstein]

Infinite sets with same cardinality: $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{N}^k$ for any k.

Infinite sets that are strictly bigger: $\mathcal{P}(\mathbb{N})$, $\mathbb{R}$

[Cantor]: $\nexists$ surjective $f: \mathbb{N} \to \mathcal{P}(\mathbb{N})$.

We have been somewhat informal with our description of sets. What is an "<u>unordered collection</u> of <u>distinct objects</u>"? All the underlined words are not formally defined.

Such description of sets, now referred to as "naive set theory" can lead to several paradoxes.

eg. Russell's paradox:

Let S be the set of all sets that dont contain itself.

$$W = \{ S : S \notin S \}$$

For any S, $S \in W \iff S \notin S$.

$$W \in W \iff W \notin W \ !!$$

Today, set theory is based on a few well-accepted axioms, proposed by Zermelo and Fraenkl. The set theory developed using ZF axioms avoid the above paradox.

We will not cover ZFC axioms in this course, and certainly wont try to prove anything using ZFC axioms.

PROOF TECHNIQUES FROM THIS PART:

1. Proof of Existence via explicit construction
2. Proof via contradiction