

# 9.6.15

EE24BTECH11003 - Akshara Sarma Chennubhatla

**Question:** A train travels  $360\text{km}$  at a uniform speed. If the speed had been  $5\text{km/h}$  more, it would have taken 1 hour less for the same journey. Find the speed of the train.

**Solution:**

**Theoretical Solution:**

Let  $s$  be the speed of the train, then,

$$\frac{360}{s} - 1 = \frac{360}{s+5} \quad (1)$$

$$\implies s^2 + 5s = 1800 \quad (2)$$

This is a quadratic equation whose roots are the possible values of the speed. Using the quadratic formula,

$$s = \frac{-5 \pm \sqrt{5^2 - 4(1)(-1800)}}{2(1)} \quad (3)$$

$$s = -45, 40 \quad (4)$$

**Simulated Solution:**

By Newton-Raphson method,

Take initial guess  $s_0$ , then run the following loop,

$$s_{n+1} = s_n - \frac{f(s_n)}{f'(s_n)} \quad (5)$$

$$f(s) = s^2 + 5s - 1800 \quad (6)$$

$$f'(s) = 2s + 5 \quad (7)$$

$$s_{n+1} = s_n - \frac{s^2 + 5s - 1800}{2s + 5} \quad (8)$$

This method converges for real roots but when roots are complex, it can go to infinity as well. To avoid that, if our roots are complex, take initial guess as a complex number. The values of  $s$  got through this method are,

$$s = -45 \quad (9)$$

$$s = 40 \quad (10)$$

Alternatively, we can solve the question by using the eigen values of the companion matrix.

For a polynomial equation of form  $x_n + c_{n-1}x^{n-1} + \dots + c_2x^2 + c_1x + c_0 = 0$  the companion matrix is of the form

$$\begin{pmatrix} 0 & 0 & \dots & 0 & -c_0 \\ 1 & 0 & \dots & 0 & -c_1 \\ 0 & 1 & \dots & 0 & -c_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 1 & -c_{n-1} \end{pmatrix} \quad (11)$$

The eigen values of this matrix are the roots of the polynomial equation. For this question,

$$n = 2 \quad (12)$$

$$c_0 = -1800 \quad (13)$$

$$c_1 = 5 \quad (14)$$

$$C = \begin{pmatrix} 0 & 1800 \\ 1 & -5 \end{pmatrix} \quad (15)$$

To find the eigen values of the matrix, we use the method of  $QR$  decomposition of the matrix.

The  $QR$  algorithm decomposes a matrix  $A$  into the product of an orthogonal matrix  $Q$  and an upper triangular matrix  $R$ , such that

$$A = QR \quad (16)$$

The matrix is then updated iteratively as:

$$A_{new} = RQ \quad (17)$$

This process is repeated until  $A$  converges to an upper triangular form.

Steps to perform QR decomposition with accelerated convergence,

- 1) Convert to Upper Hessenberg form via Householder Reflections
- 2) Performing QR decomposition via Givens Rotations with shifts
- 3) Read off diagonal elements

### Householder Reflections:

A square matrix  $A$  of order  $n \times n$  is said to be in upper Hessenberg form if all the entries below the first subdiagonal are zero. For example:

$$H = \begin{bmatrix} h_{11} & h_{12} & h_{13} & \dots & h_{1n} \\ h_{21} & h_{22} & h_{23} & \dots & h_{2n} \\ 0 & h_{32} & h_{33} & \dots & h_{3n} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & h_{n-1,n-1} & h_{n-1,n} \\ 0 & \dots & \dots & 0 & h_{nn} \end{bmatrix} \quad (18)$$

- 1) Select a Subvector  $\mathbf{x}$ :

$$\mathbf{x} = \begin{bmatrix} A_{2,1} \\ A_{3,1} \\ \vdots \\ A_{n,1} \end{bmatrix}. \quad (19)$$

- 2) Define the Target Vector: The goal is to transform  $\mathbf{x}$  into a new vector  $\mathbf{y}$  where only the first element is non-zero, and all the other elements are zero. First, compute  $\|\mathbf{x}\|$ :

$$\mathbf{y} = \pm \|\mathbf{x}\| \mathbf{e}_1, \quad (20)$$

- 3) Construct the Householder Vector  $\mathbf{v}$ : To generate a reflection that transforms  $\mathbf{x}$  to  $\mathbf{y}$ , the Householder vector  $\mathbf{v}$  is defined as:

$$\mathbf{v} = \mathbf{x} - \text{sign}(x_1) \|\mathbf{x}\| \mathbf{e}_1 \quad (21)$$

$$\text{sign}(x_1) = \frac{x_1}{|x_1|}, \quad (22)$$

After defining  $\mathbf{v}$ , it is normalized to a unit vector:

- 4) Construct the Householder Matrix  $H_k$ : The Householder matrix  $H_k$  is constructed as:

$$H_k = I - 2 \frac{\mathbf{v}\mathbf{v}^*}{\mathbf{v}^*\mathbf{v}}, \quad (23)$$

- 5) Apply the Householder Transformation: The matrix  $H_k$  is applied to  $A$  as:

$$A' = H_k A H_k^*, \quad (24)$$

This will reduce the matrix to Hessenberg form by eliminating the sub-diagonal elements of the first column.

- 6) Repeat for Subsequent Columns: This Householder transformation approach ensures that the matrix is gradually transformed to a Hessenberg form, where all elements below the first sub-diagonal are zero.

### Givens Rotations:

$$\begin{bmatrix} h_{11} & h_{12} & h_{13} & \cdots & h_{1n} \\ h_{21} & \underline{h_{22}} & h_{23} & \cdots & h_{2n} \\ 0 & h_{32} & \underline{h_{33}} & \cdots & h_{3n} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & h_{n-1,n-1} & h_{n-1,n} \\ 0 & \cdots & \cdots & 0 & h_{nn} \end{bmatrix} \rightarrow \begin{bmatrix} r_{11} & r_{12} & r_{13} & \cdots & r_{1n} \\ 0 & r_{22} & r_{23} & \cdots & r_{2n} \\ 0 & 0 & r_{33} & \cdots & r_{3n} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & r_{n-1,n-1} & r_{n-1,n} \\ 0 & \cdots & \cdots & 0 & r_{nn} \end{bmatrix}$$

Each Givens rotation zeros out a specific subdiagonal element, progressively transforming the Hessenberg matrix into an upper triangular matrix.

To choose the values of  $c$  and  $s$  for the Givens rotation in QR decomposition, let  $a_j$  be the element we wish to null out (i.e. make 0). Pick an arbitrary non-zero pivot element  $a_i$  (on a different row). Usually, if we wish to null a particular sub-diagonal element, we pick the principal diagonal element above it as a pivot.

$$c = \frac{\bar{a}_i}{\sqrt{a_i^2 + a_j^2}}, \quad s = \frac{-\bar{a}_j}{\sqrt{a_i^2 + a_j^2}} \quad (25)$$

Givens rotation essentially rotates the two rows that  $a_i$  and  $a_j$  are on such that  $a_j = 0$  after rotation, other rows remain unaffected.

Visualizing the process,

$$\begin{bmatrix} \times & \times & \times & \times \\ \times & \times & \times & \times \\ 0 & \times & \times & \times \\ 0 & 0 & \times & \times \end{bmatrix} \xrightarrow{G(3,2,\theta_1)} \begin{bmatrix} \times & \times & \times & \times \\ \times & \times & \times & \times \\ 0 & 0 & \times & \times \\ 0 & 0 & \times & \times \end{bmatrix} \xrightarrow{G(4,3,\theta_2)} \begin{bmatrix} \times & \times & \times & \times \\ \times & \times & \times & \times \\ 0 & 0 & \times & \times \\ 0 & 0 & 0 & \times \end{bmatrix}. \quad (26)$$

After all Givens rotations, the resulting matrix is upper triangular:

$$R = \begin{bmatrix} \times & \times & \times & \times \\ 0 & \times & \times & \times \\ 0 & 0 & \times & \times \\ 0 & 0 & 0 & \times \end{bmatrix}. \quad (27)$$

The sequence of Givens rotations  $G_1, G_2, \dots, G_m$  satisfies:

$$G_m \cdots G_2 G_1 A = R, \quad (28)$$

where  $R$  is upper triangular. The QR decomposition is obtained by combining the

transposes of the Givens rotations into  $Q$ :

$$A = QR, \quad Q = G_1^\top G_2^\top \cdots G_m^\top. \quad (29)$$

$$A_{k+1} = R_k Q_k \quad (30)$$

$$= (G_n \cdots G_2 G_1) A_k (G_1^\top G_2^\top \cdots G_n^\top) \quad (31)$$

$$= (G_n \cdots G_2 G_1) A_k (G_n \cdots G_2 G_1)^\top \quad (32)$$

Iteratively repeating this process causes the matrix to converge to upper triangular.

### Handling Jordan Blocks:

Jordan blocks pose challenges in eigenvalue computation because the matrix cannot be diagonalized. A Jordan block for eigenvalue  $\lambda$  appears as:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad (33)$$

where  $a$  and  $b$  are the diagonal elements and  $c$  is a non zero sub-diagonal element. To handle Jordan blocks, the QR algorithm implemented here solves for the eigenvalues directly using the characteristic polynomial of the block. For a  $2 \times 2$  Jordan block, the eigenvalues are roots of:

$$\lambda^2 - (\text{trace})\lambda + \det = 0. \quad (34)$$

In this case, the eigen values of the matrix computed are,

$$\lambda_1 = -45 \quad (35)$$

$$\lambda_2 = 40 \quad (36)$$

$$(37)$$

Below is the plot for given quadratic equation, obtained by iterating through the values of  $x$  with step size of  $h$

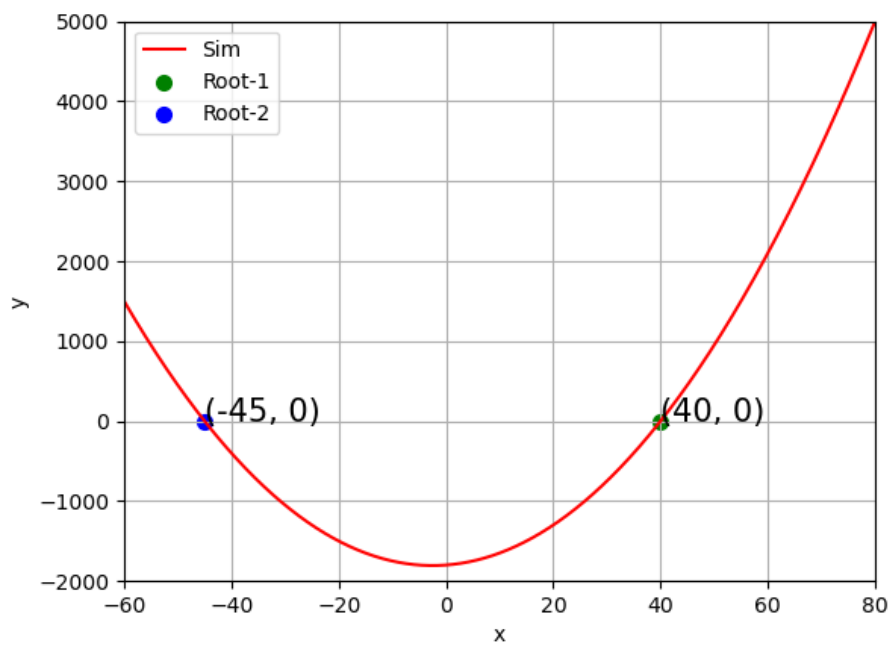


Fig. 1: Plot of  $s^2 + 5s = 1800$