Sprog Presentation

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Problem Statement

A train travels $360 km$ at a uniform speed. If the speed had been $5 km/h$ more, it would have taken 1 hour less for the same journey. Find the speed of the train.

Theoretical Solution

Let s be the speed of the train, then,

$$\frac{360}{s} - 1 = \frac{360}{s+5} \tag{0.1}$$

$$\implies s^2 + 5s = 1800 \tag{0.2}$$

Using the quadratic formula,

$$s = \frac{-5 \pm \sqrt{5^2 - 4\left(1\right)\left(-1800\right)}}{2\left(1\right)} \tag{0.3}$$

$$s = -45, 40 \tag{0.4}$$

Simulated Solution - 1

By Newton-Ralphson method,

Take initial guess s_0, then run the following loop,

$$s_{n+1} = s_n - \frac{f\left(s_n\right)}{f'\left(s_n\right)} \tag{0.5}$$

$$f\left(s\right) = s^2 + 5s - 1800 \tag{0.6}$$

$$f'\left(s\right) = 2s + 5 \tag{0.7}$$

$$s_{n+1} = s_n - \frac{s_n^2 + 5s_n - 1800}{2s_n + 5} \tag{0.8}$$

Simulated Solution - 1

The values of s got through this method are,

$$s = -45 \tag{0.9}$$

$$s = 40 \tag{0.10}$$

Simulated Solution - 2

Alternatively, we can solve the question by using the eigen values of the companion matrix.
For a polynomial equation of form
$x_n + c_{n-1}x^{n-1} + \cdots + c_2x^2 + c_1x + c_0 = 0$ the companion matrix if of the form

$$\begin{pmatrix} 0 & 0 & \cdots & 0 & -c_0 \\ 1 & 0 & \cdots & 0 & -c_1 \\ 0 & 1 & \cdots & 0 & -c_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & -c_{n-1} \end{pmatrix} \tag{0.11}$$

Simulated Solution - 2

The eigen values of this matrix are the roots of the polynomial equation. For this question,

$$n = 2 \tag{0.12}$$

$$c_0 = -1800 \tag{0.13}$$

$$c_1 = 5 \tag{0.14}$$

$$C = \begin{pmatrix} 0 & 1800 \\ 1 & -5 \end{pmatrix} \tag{0.15}$$

Simulated Solution - 2

To find the eigen values of the matrix, we use the method of QR decomposition of the matrix.
The QR algorithm decomposes a matrix A into the product of an orthogonal matrix Q and an upper triangular matrix R, such that

$$A = QR \tag{0.16}$$

The matrix is then updated iteratively as:

$$A_{new} = RQ \tag{0.17}$$

This process is repeated until A converges to an upper triangular form.

Simulated Solution - 2

A square matrix A of order $n \times n$ is said to be in upper Hessenberg form if all the entries below the first subdiagonal are zero. For example:

$$H = \begin{bmatrix} h_{11} & h_{12} & h_{13} & \cdots & h_{1n} \\ h_{21} & h_{22} & h_{23} & \cdots & h_{2n} \\ 0 & h_{32} & h_{33} & \cdots & h_{3n} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & 0 & h_{n-1,n-1} & h_{n-1,n} \\ 0 & \cdots & \cdots & 0 & h_{nn} \end{bmatrix} \tag{0.18}$$

Simulated Solution - 2

Steps to apply Householder transformation:

1. Select a Subvector **x**:

$$x = \begin{bmatrix} A_{2,1} \\ A_{3,1} \\ \vdots \\ A_{n,1} \end{bmatrix}. \tag{0.19}$$

2. Define the Target Vector: The goal is to transform **x** into a new vector **y** where only the first element is non-zero, and all the other elements are zero. First, compute $\|x\|$:

$$\mathbf{y} = \pm\|x\|e_1, \tag{0.20}$$

3. Construct the Householder Vector **v**: To generate a reflection that transforms x to y, the Householder vector v is defined as:

$$v = x - \text{sign}(x_1)\|x\|e_1 \tag{0.21}$$

$$\text{sign}(x_1) = \frac{x_1}{|x_1|}, \tag{0.22}$$

After defining v, it is normalized to a unit vector:

4. Construct the Householder Matrix H_k: The Householder matrix H_k is constructed as:

$$H_k = I - 2\frac{vv^*}{v^*v}, \tag{0.23}$$

Simulated Solution - 2

5. Apply the Householder Transformation: The matrix H_k is applied to A as:

$$A' = H_k A H_k^*, \tag{0.24}$$

This will reduce the matrix to Hessenberg form by eliminating the sub-diagonal elements of the first column.

6. Repeat for Subsequent Columns: This Householder transformation approach ensures that the matrix is gradually transformed to a Hessenberg form, where all elements below the first sub-diagonal are zero.

Simulated Solution - 2

Givens rotations:
A Givens rotation matrix ($G(i,j,\theta)$) zeroes out the element a_{ij}. It is defined as:

$$G(i,j,\theta) = \begin{bmatrix} 1 & \cdots & 0 & \cdots & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & c & \cdots & s & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & -\overline{s} & \cdots & \overline{c} & \cdots & 0 \\ \vdots & \ddots & \vdots & \ddots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & \cdots & 0 & \cdots & 1 \end{bmatrix}$$

$$c = \frac{\overline{a_i}}{\sqrt{a_i^2 + a_j^2}}, \quad s = \frac{-\overline{a_j}}{\sqrt{a_i^2 + a_j^2}} \tag{0.25}$$

Simulated Solution - 2

Visualizing the process,

$$\begin{bmatrix} \times & \times & \times & \times \\ \times & \times & \times & \times \\ 0 & \times & \times & \times \\ 0 & 0 & \times & \times \end{bmatrix} \xrightarrow{G(3,2,\theta_1)} \begin{bmatrix} \times & \times & \times & \times \\ \times & \times & \times & \times \\ 0 & 0 & \times & \times \\ 0 & 0 & \times & \times \end{bmatrix} \xrightarrow{G(4,3,\theta_2)} \begin{bmatrix} \times & \times & \times & \times \\ \times & \times & \times & \times \\ 0 & 0 & \times & \times \\ 0 & 0 & 0 & \times \end{bmatrix}. \tag{0.26}$$

Simulated Solution - 2

After all Givens rotations, the resulting matrix is upper triangular:

$$R = \begin{bmatrix} \times & \times & \times & \times \\ 0 & \times & \times & \times \\ 0 & 0 & \times & \times \\ 0 & 0 & 0 & \times \end{bmatrix}. \tag{0.27}$$

Simulated Solution - 2

The sequence of Givens rotations $G_1, G_2, \ldots, G_m$ satisfies:

$$G_m \cdots G_2 G_1 A = R, \tag{0.28}$$

where R is upper triangular. The QR decomposition is obtained by combining the transposes of the Givens rotations into Q:

$$A = QR, \quad Q = G_1^\top G_2^\top \cdots G_m^\top. \tag{0.29}$$

$$\begin{aligned}
A_{k+1} &= R_k Q_k && (0.30)\\
&= (G_n \ldots G_2 G_1) A_k (G_1^\top G_2^\top \ldots G_n^\top) && (0.31)\\
&= (G_n \ldots G_2 G_1) A_k (G_n \ldots G_2 G_1)^\top && (0.32)
\end{aligned}$$

Iteratively repeating this process causes the matrix to converge to upper triangular.

Simulated Solution - 2

Handling Jordan Blocks:

Jordan blocks pose challenges in eigenvalue computation because the matrix cannot be diagonalized. A Jordan block for eigenvalue λ appears as:

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \tag{0.33}$$

where a and b are the diagonal elements and c is a non zero sub-diagonal element.

To handle Jordan blocks, the QR algorithm implemented here solves for the eigenvalues directly using the characteristic polynomial of the block.

Simulated Solution - 2

For a 2×2 Jordan block, the eigenvalues are roots of:

$$\lambda^2 - (\text{trace})\lambda + \det = 0. \tag{0.34}$$

In this case, the eigen values of the matrix computed are,

$$\lambda_1 = -45 \tag{0.35}$$

$$\lambda_2 = 40 \tag{0.36}$$

Below is the plot for given quadratic equation, obtained by iterating through the values of x with step size of h

Simulated Solution - 2