11.16.4.7.1

EE24BTECH11003 - Akshara Sarma Chennubhatla

Question:

$$P(A) = 0.54 \tag{1}$$
$$P(B) = 0.69 \tag{2}$$
$$P(A \cap B) = 0.35 \tag{3}$$

Find $P(A \cup B)$

Solution:

Theoretical Solution:

For 2 Boolean variables A and B, the axioms of Boolean Algebra are defined as:

$$A + A' = 1 \tag{4}$$
$$A + A = A \tag{5}$$
$$AB = BA \tag{6}$$
$$A + B = B + A \tag{7}$$
$$AA' = 0 \tag{8}$$
$$P(1) = 1 \tag{9}$$
$$P(A + B) = P(A) + P(B), \text{ if } P(AB) = 0 \tag{10}$$

Using these axioms, we will try to prove that

$$P(A + B) = P(A) + P(B) - P(AB) \tag{11}$$

We will start by representing A and B as:

$$A = AB + AB' \tag{12}$$
$$B = AB + A'B \tag{13}$$
$$P(A) = P(AB) + P(AB') \tag{14}$$
$$P(B) = P(AB) + P(A'B) \tag{15}$$

On adding (12) and (13),

$$A + B = AB + AB + AB' + A'B \tag{16}$$

$$A + B = AB + AB' + A'B \tag{17}$$

$$P(A + B) = P(AB + AB' + A'B) \tag{18}$$

$$P(A + B) = P(AB) + P(AB') + P(A'B) \tag{19}$$

$$P(A + B) = P(AB) + P(A) - P(AB) + P(B) - P(AB) \tag{20}$$

$$\implies P(A + B) = P(A) + P(B) - P(AB) \tag{21}$$

Using the given values of $P(A), P(B)$ and $P(AB)$,

$$P(A + B) = 0.54 + 0.69 - 0.35 \tag{22}$$

$$P(A + B) = 0.88 \tag{23}$$

Therefore, the value of $P(A + B)$ is 0.88.

Simulated Solution:
Let X_1 be an indicator random variable of the event A.
X_1 is defined as:

$$X_1 = \begin{cases} 1, & A \\ 0, & A' \end{cases} \tag{24}$$

Let X_2 be the indicator random variable of the event B.
X_2 is defined as:

$$X_2 = \begin{cases} 1, & B \\ 0, & B' \end{cases} \tag{25}$$

Let X_3 be the indicator random variable of the event AB.
X_3 is defined as:

$$X_3 = \begin{cases} 1, & AB \\ 0, & (AB)' \end{cases} \tag{26}$$

The PMF of the random variable X_1 is:

$$p_{X_1}(n) = \begin{cases} p_1, & n = 1 \\ 1 - p_1, & n = 0 \end{cases} \tag{27}$$

The PMF of the random variable X_2 is:

$$p_{X_2}(n) = \begin{cases} p_2, & n = 1 \\ 1 - p_2, & n = 0 \end{cases} \tag{28}$$

The PMF of the random variable X_3 is:

$$p_{X_3}(n) = \begin{cases} p_3, & n = 1 \\ 1 - p_3, & n = 0 \end{cases} \tag{29}$$

where,

$$p_1 = 0.54 \tag{30}$$
$$p_2 = 0.69 \tag{31}$$
$$p_3 = 0.35 \tag{32}$$
$$\tag{33}$$

Let Y be the random variable which is defined as follows:

$$Y = X_1 + X_2 - X_3 \tag{34}$$

But we know that X_3 can never be 0 when X_1 and X_2 are 1 and vice versa.
So, Y is another Indicator Random variable whose PMF is defined as:

$$p_Y(n) = \begin{cases} p, & n = 1 \\ 1 - p, & n = 0 \end{cases} \tag{35}$$

From (34),

$$E(Y) = E(X_1 + X_2 - X_3) \tag{36}$$
$$E(Y) = E(X_1) + E(X_2) - E(X_3) \tag{37}$$
$$1.(p) + 0.(1-p) = 1.(p_1) + 0.(1-p_1) + 1.(p_2) + 0.(1-p_2) - 1.(p_3) - 0.(1-p_3) \tag{38}$$
$$p = p_1 + p_2 - p_3 \tag{39}$$

Through our definition, we know that,

$$P(A) = p_1 \tag{40}$$
$$P(B) = p_2 \tag{41}$$
$$P(AB) = p_3 \tag{42}$$

Therefore, by comparison of the axiom

$$P(A + B) = P(A) + P(B) - P(AB) \tag{43}$$

and the equation (39),

$$p = P(A + B) \tag{44}$$
$$P(A + B) = 0.54 + 0.69 - 0.35 \tag{45}$$
$$\implies P(A + B) = 0.88 \tag{46}$$

Below is the plot for the simulation of the probabilities, where the grey stems represent the theretical probabilities and the coloured stems represent the simulated ones. Through observation in the last stem, we have proved through the code that

$$P(A+B) = P(A) + P(B) - P(AB) \tag{47}$$

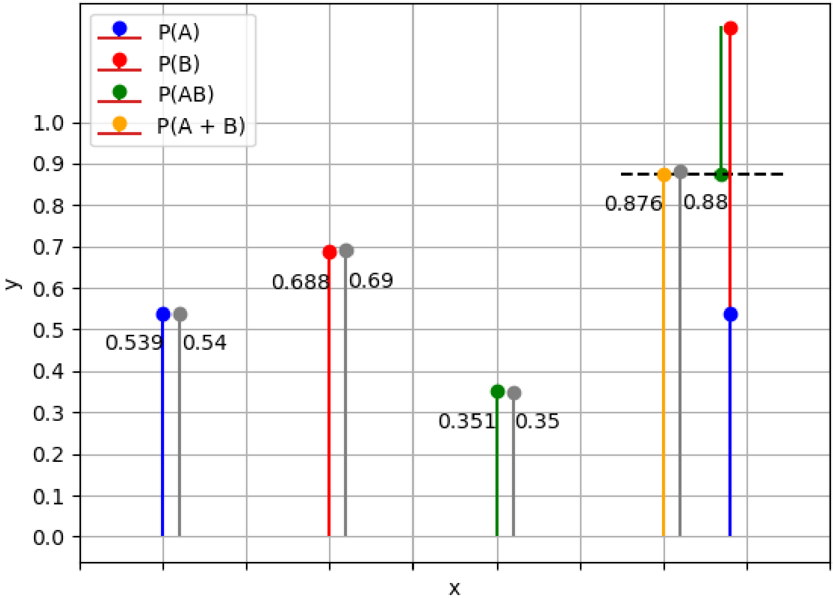

Fig. 1: Plot of the probabilities