

9.6.6.10

EE24BTECH11003 - Akshara Sarma Chennubhatla

Question: The sum of the perimeter of a circle and square is k , where k is some constant. Prove that the sum of their areas is least when the side of square is double the radius of the circle.

Solution:

The method used to solve this question is the gradient descent.
Taking the radius of the circle as r and the side of the square as a , through given information,

$$2\pi r + 4a = k \quad (1)$$

$$(2)$$

We need to minimize

$$\pi r^2 + a^2 \quad (3)$$

$$(4)$$

and prove that at the minimum sum of areas point, $a = 2r$.
Substituting (1) in (2),

$$A(r) = \pi r^2 + \left(\frac{k - 2\pi r}{4}\right)^2 \quad (5)$$

$$A(r) = \left(\frac{\pi^2}{4} + \pi\right)r^2 - \frac{\pi k}{4}r + \frac{k^2}{16} \quad (6)$$

Applying Gradient Descent to this function,

$$r_{n+1} = r_n - \mu A'(r_n) \quad (7)$$

$$A'(x) = 2\left(\frac{\pi^2}{4} + \pi\right)r - \frac{\pi k}{4} \quad (8)$$

$$r_{n+1} = r_n \left(1 - 2\mu\left(\frac{\pi^2}{4} + \pi\right)\right) + \frac{\mu\pi k}{4} \quad (9)$$

Applying unilateral Z-transform,

$$zR(z) - zr_0 = R(z) \left(1 - 2\mu \left(\frac{\pi^2}{4} + \pi \right) \right) + \frac{\mu\pi k}{4} \frac{1}{1 - z^{-1}} \quad (10)$$

$$R(z) = \frac{zr_0}{z - \left(1 - 2\mu \left(\frac{\pi^2}{4} + \pi \right) \right)} + \frac{\mu\pi k}{4} \frac{1}{1 - z^{-1}} \quad (11)$$

$$R(z) = \frac{zr_0}{z - \left(1 - 2\mu \left(\frac{\pi^2}{4} + \pi \right) \right)} + \frac{k}{10\pi(z - 1)} - \frac{k \left(1 - 2\mu \left(\frac{\pi^2}{4} + \pi \right) \right)}{10\pi \left(z - \left(1 - 2\mu \left(\frac{\pi^2}{4} + \pi \right) \right) \right)} \quad (12)$$

$$R(z) = \frac{r_0}{1 - z^{-1} \left(1 - 2\mu \left(\frac{\pi^2}{4} + \pi \right) \right)} + \frac{kz^{-1}}{10\pi(1 - z^{-1})} - \frac{z^{-1}k \left(1 - 2\mu \left(\frac{\pi^2}{4} + \pi \right) \right)}{10\pi \left(1 - z^{-1} \left(1 - 2\mu \left(\frac{\pi^2}{4} + \pi \right) \right) \right)} \quad (13)$$

(14)

From this equation, the ROC is,

$$|z| > \max \left(\left| 1 - 2\mu \left(\frac{\pi^2}{4} + \pi \right) \right|, 1 \right) \quad (15)$$

$$\Rightarrow |z| > \begin{cases} 1 & , \mu > 0 \\ 1 - 2\mu \left(\frac{\pi^2}{4} + \pi \right) & , \mu < 0 \end{cases} \quad (16)$$

Assuming μ satisfies the above condition,

$$\lim_{n \rightarrow \infty} \|r_{n+1} - r_n\| = 0 \quad (17)$$

$$\lim_{n \rightarrow \infty} \left\| \frac{\mu k \pi}{4} - r_n 2\mu \left(\frac{\pi^2}{4} + \pi \right) \right\| = 0 \quad (18)$$

$$\Rightarrow \lim_{n \rightarrow \infty} \|r_n\| = \frac{k}{8 \left(\frac{\pi}{4} + 1 \right)} \quad (19)$$

But, we know that,

$$2\pi r_n + 4a_n = k \quad (20)$$

$$2\pi \left(\frac{k}{2(\pi + 4)} \right) + 4a_n = k \quad (21)$$

$$\Rightarrow a_n = \frac{k}{\pi + 4} \quad (22)$$

$$\Rightarrow a_n = 2r_n \quad (23)$$

Thus, it is proved that at minimum condition, the length of the side of the square is double the radius of the circle.

Below is the plot for the area vs radius graph with initial conditions as follows,

$$r_0 = -10 \quad (24)$$

$$\mu = 0.001 \quad (25)$$

$$\text{tolerance of convergence} = 10^{-6} \quad (26)$$

$$k = 10\pi \quad (27)$$

$$(28)$$

The point of minima we get is,

$$x_{min} = 2.199416... \quad (29)$$

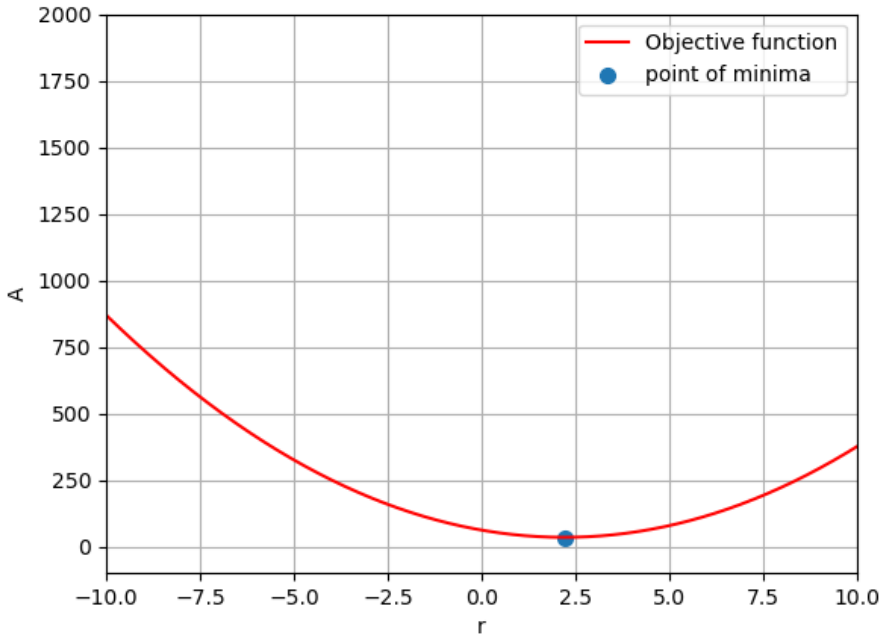


Fig. 1: Plot of the area vs radius