EE1204: Electromagnetics Assignment-1

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Aim:
Solving the given
questions and preparing
a document of solutions
with simulations if needed

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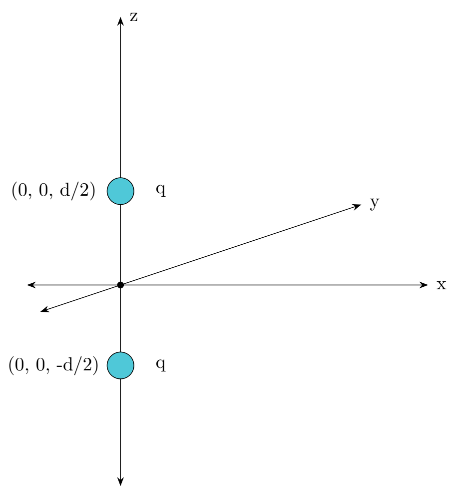
Questions:

1. Two point charges of equal magnitude q are positioned at z = ±d/2. We aim to determine:
   (a) The electric field everywhere on the z axis.
   (b) The electric field everywhere on the x axis.
   (c) The results of (a) and (b) if the charge at z = d / 2 is -q instead of +q.

Solution:



(a) Electric Field on the z-axis

The electric field due to a point charge is given by Coulomb's Law:

$$E = \frac{1}{4\pi\varepsilon_0} \frac{q}{r^2}, \tag{1}$$

where r is the distance between the charge and the field point.

The charges are located at $z = \pm d/2$. Consider a point at z on the z-axis. The distances from the charges are:

$$r_+ = \left| z - \frac{d}{2} \right|, \quad r_- = \left| z + \frac{d}{2} \right|. \tag{2}$$

The electric field due to each charge at z is:

$$E_+ = \frac{1}{4\pi\varepsilon_0} \frac{q}{r_+^2}, \quad E_- = \frac{1}{4\pi\varepsilon_0} \frac{q}{r_-^2}. \tag{3}$$

Since both charges lie on the z axis, the fields resulting from them on any point on the z axis lies on the z axis itself.
Case 1:

$$z > +\frac{d}{2} \tag{4}$$

The total electric field on the z-axis is:

$$E_z = \frac{q}{4\pi\varepsilon_0}\left(\frac{1}{(z-d/2)^2} + \frac{1}{(z+d/2)^2}\right)\hat{k} \tag{5}$$

Case 2:

$$-\frac{d}{2} < z < +\frac{d}{2} \tag{6}$$

The total electric field on the z-axis is:

$$E_z = \frac{q}{4\pi\varepsilon_0}\left(-\frac{1}{(z-d/2)^2} + \frac{1}{(z+d/2)^2}\right)\hat{k} \tag{7}$$

Case 3:

$$z < -\frac{d}{2} \tag{8}$$

The total electric field on the z-axis is:

$$E_z = -\frac{q}{4\pi\varepsilon_0}\left(\frac{1}{(z-d/2)^2} + \frac{1}{(z+d/2)^2}\right)\hat{k} \tag{9}$$

(b) Electric Field on the x-axis

Consider a point on the x-axis at $(x, 0, 0)$. The distances from the charges are:

$$r_+ = \sqrt{x^2 + (d/2)^2}, \quad r_- = \sqrt{x^2 + (d/2)^2}. \tag{10}$$

Since both charges are positive, their electric fields point radially outward. By symmetry, the z-components cancel, leaving only the x-component:

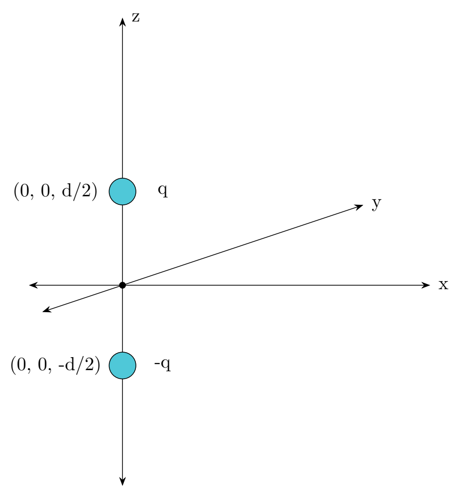
$$E_x = 2E_+ \cos\theta, \tag{11}$$

where

$$\cos\theta = \frac{x}{r_+}. \tag{12}$$

Substituting,

$$E_x = 2 \times \frac{1}{4\pi\varepsilon_0}\frac{q}{(x^2 + (d/2)^2)} \times \frac{x}{\sqrt{x^2 + (d/2)^2}}\hat{\imath} \tag{13}$$



$$E_x = \frac{2qx}{4\pi\varepsilon_0} \frac{1}{(x^2 + (d/2)^2)^{\frac{3}{2}}} \hat{i} \tag{14}$$

(c) Case of Opposite Charges

If the charge at $z = -d/2$ is $-q$,

(a) Electric Field on the z-axis:

The electric field due to each charge at z is:

$$E_+ = \frac{1}{4\pi\varepsilon_0} \frac{q}{r_+^2}, \quad E_- = -\frac{1}{4\pi\varepsilon_0} \frac{q}{r_-^2}. \tag{15}$$

Case 1:

$$z > +\frac{d}{2} \tag{16}$$

The total electric field on the z-axis is:

$$E_z = \frac{q}{4\pi\varepsilon_0} \left(\frac{1}{(z - d/2)^2} - \frac{1}{(z + d/2)^2} \right) \hat{k} \tag{17}$$

Case 2:

$$-\frac{d}{2} < z < +\frac{d}{2} \tag{18}$$

The total electric field on the z-axis is:

$$E_z = -\frac{q}{4\pi\varepsilon_0} \left(\frac{1}{(z - d/2)^2} + \frac{1}{(z + d/2)^2} \right) \hat{k} \tag{19}$$

Case 3:

$$z < -\frac{d}{2} \tag{20}$$

The total electric field on the z-axis is:

$$E_z = \frac{q}{4\pi\varepsilon_0}\left(-\frac{1}{(z-d/2)^2} + \frac{1}{(z+d/2)^2}\right)\hat{k} \tag{21}$$

(b) Electric Field on the x-axis:

Consider a point on the x-axis at $(x, 0, 0)$. The distances from the charges are:

$$r_+ = \sqrt{x^2 + (d/2)^2}, \quad r_- = \sqrt{x^2 + (d/2)^2}. \tag{22}$$

Since one charge are positive and the other is negative, by symmetry, the x-components cancel, leaving only the z-component in the negative z direction.

$$E_x = 2E_+ \sin\theta, \tag{23}$$

where

$$\sin\theta = \frac{d/2}{r_+}. \tag{24}$$

Substituting,

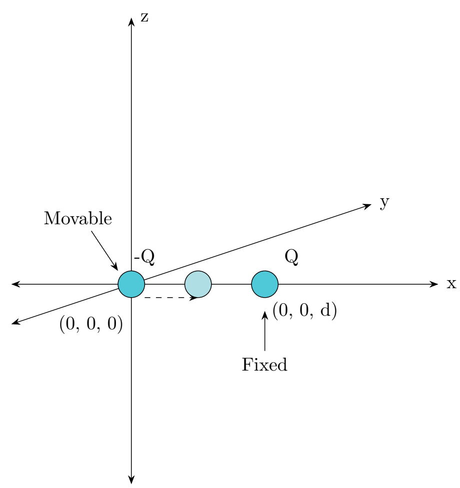
$$E_x = -2 \times \frac{1}{4\pi\varepsilon_0}\frac{q}{(x^2 + (d/2)^2)} \times \frac{d/2}{\sqrt{x^2 + (d/2)^2}}\hat{k} \tag{25}$$

$$E_x = -\frac{qd}{4\pi\varepsilon_0}\frac{1}{(x^2 + (d/2)^2)^{\frac{3}{2}}}\hat{k} \tag{26}$$

2. A crude device for measuring charge consists of two small insulating spheres of radius a, one of which is fixed in position. The other is movable along the x-axis and is subject to a restraining force kx, where k is a spring constant. The uncharged spheres are centered at $x = 0$ and $x = d$, the latter fixed. If the spheres are given equal and opposite charges of Q/C, obtain the expression by which Q may be found as a function of x. Determine the maximum charge that can be measured in terms of ϵ_0, k, and d, and then state the separation of the spheres. What happens if a larger charge is applied?

Solution:

A movable insulating sphere of radius a is restrained by a spring with force $F_s = kx$. A fixed sphere at $x = d$ and the movable sphere at $x = 0$ are given equal and opposite charges Q and $-Q$. We need to determine



the charge Q as a function of displacement x, and find the maximum measurable charge.

The forces acting on the movable sphere are:

1. Electrostatic Force due to the Coulomb interaction:

$$F_e = \frac{1}{4\pi\epsilon_0} \frac{Q^2}{(d-x)^2} \tag{27}$$

2. Spring Force due to Hooke's Law:

$$F_s = -kx \tag{28}$$

At equilibrium, these forces balance:

$$\frac{1}{4\pi\epsilon_0} \frac{Q^2}{(d-x)^2} = kx \tag{29}$$

$$Q = \sqrt{4\pi\epsilon_0 kx(d-x)^2} \tag{30}$$

To maximize Q, we have to differentiate the above expression with respect to x:

$$\frac{d}{dx}\left[x(d-x)^2\right] = (d-x)^2 + x \cdot 2(d-x)(-1) \tag{31}$$

Setting the derivative to zero:

$$(d-x)(d-3x) = 0 \tag{32}$$

$$x = \frac{d}{3} \text{ or } d \tag{33}$$

On applying the values of x in the equation for charge, we get 0 at $x = d$ and some positive value at $x = \frac{d}{3}$. So maximum charge which can be measured between 0 to d without colliding with the other sphere is at $\frac{d}{3}$ Thus, the maximum charge measurable is:

$$Q_{\max} = \sqrt{4\pi\epsilon_0 k \left(\frac{d}{3}\right)\left(d - \frac{d}{3}\right)^2} \tag{34}$$

$$Q_{\max} = \sqrt{4\pi\epsilon_0 k \frac{d}{3}\frac{4d^2}{9}} \tag{35}$$

$$Q_{\max} = \sqrt{\frac{16\pi\epsilon_0 k d^3}{27}} \tag{36}$$

$$Q_{\max} = \frac{4}{3\sqrt{3}}\sqrt{\pi\epsilon_0 k d^3} \tag{37}$$

The equilibrium separation at maximum charge is:

$$d - x = \frac{2d}{3} \tag{38}$$

If $Q > Q_{\max}$, the electrostatic force exceeds the restoring force of the spring, leading to instability where the movable sphere accelerates uncontrollably toward the fixed sphere and collides with that sphere.

3. A flux density field is given as

$$F_1 = 5a_z$$

The task is to evaluate the outward flux of F_1 through the hemispherical surface defined by $r = a$, $0 < \theta < \frac{\pi}{2}$, and $0 < \phi < 2\pi$. Next, consider what simple observation would have saved a lot of work in the previous part. Identifying symmetries or using alternative methods can simplify the calculations significantly. Now suppose the field is given by

$$F_2 = 5za_z$$

Using the appropriate surface integrals, determine the net outward flux of F_2 through the closed surface consisting of the hemisphere from the first part and its circular base in the xy plane. Finally, repeat the previous calculation by applying the divergence theorem and evaluating an appropriate volume integral. This approach should confirm the result obtained through direct surface integration.

Solution:

The given flux density field is:

$$F_1 = 5a_z. \tag{39}$$

We have to evaluate the outward flux through the hemispherical surface. The flux is given by:

$$\Phi = \iint_S \mathbf{F} \cdot d\mathbf{S}, \tag{40}$$

where $d\mathbf{S} = \hat{r}dS$ with $dS = a^2 \sin\theta d\theta d\phi$.

Expressing F_1 in spherical coordinates:

$$a_z = \cos\theta a_r - \sin\theta a_\theta, \tag{41}$$

$$F_1 = 5(\cos\theta a_r - \sin\theta a_\theta). \tag{42}$$

Taking the dot product with $d\mathbf{S}$:

$$\mathbf{F_1} \cdot d\mathbf{S} = 5\cos\theta dS. \tag{43}$$

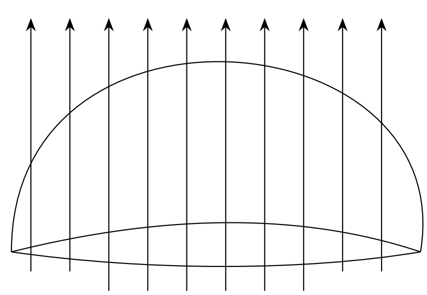
Evaluating the integral:

$$\begin{aligned}\Phi &= \int_0^{2\pi} \int_0^{\pi/2} 5\cos\theta a^2 \sin\theta d\theta d\phi. \\ &= 5a^2 \int_0^{\pi/2} \cos\theta \sin\theta d\theta \int_0^{2\pi} d\phi.\end{aligned}$$

We get:

$$\Phi = 5a^2 \cdot \frac{1}{2} \cdot 2\pi = 5\pi a^2. \tag{44}$$

We can see that the field is constant in the z direction, so all the field lines are going vertically upwards as shown in the figure below:



So, instead of integrating over the entire hemisphere, we can take the projection of the hemisphere on the xy plane, and now since the entire flux is perpendicular to the area, we can just multiply the magnitude of flux with the perpendicular area.

$$\Phi = 5 \times A_{\perp} \tag{45}$$
$$\Phi = 5\pi a^2. \tag{46}$$

Now, considering $\mathbf{F_2} = 5za_z$ and evaluating the net outward flux through the hemisphere and its circular base,

Flux Through the Hemisphere The differential surface element in spherical coordinates is:

$$dS = a^2 \sin\theta \; d\theta \; d\phi \; \hat{r}. \tag{47}$$

On the hemisphere, $z = a\cos\theta$, and the normal vector is radially outward:

$$\mathbf{F_2} \cdot dS = (5z\hat{a}_z) \cdot (\hat{r} \; dS). \tag{48}$$

Since $\hat{a}_z = \cos\theta\hat{r} + \sin\theta\hat{\theta}$, we get:

$$\mathbf{F_2} \cdot \hat{r} = 5z\cos\theta = 5a\cos^2\theta. \tag{49}$$

Thus, the flux integral over the hemisphere is:

$$\Phi_{\text{hemisphere}} = \int_0^{2\pi} \int_0^{\pi/2} 5a^3 \cos^2\theta \sin\theta \; d\theta \; d\phi. \tag{50}$$

Evaluating the integral:

$$\int_0^{\pi/2} \cos^2\theta \sin\theta \; d\theta = \frac{1}{3},$$
$$\int_0^{2\pi} d\phi = 2\pi.$$

Thus,

$$\Phi_{\text{hemisphere}} = 5a^3 \times \frac{1}{3} \times 2\pi = \frac{10}{3}\pi a^3. \tag{51}$$

For the circular base, the normal to the surface is $-a_z$, so:

$$\mathbf{F_2} \cdot d\mathbf{S} = (5za_z) \cdot (-a_z dS) = -5zdS. \tag{52}$$

Since $z = 0$ on the base, the flux contribution is:

$$\Phi_{\text{base}} = \int_0^a \int_0^{2\pi} -5(0)rdrd\phi = 0. \tag{53}$$

Thus, the total flux is:

$$\Phi_{\text{total}} = \Phi_{\text{hemisphere}} + \Phi_{\text{base}} = \frac{10}{3}\pi a^3. \tag{54}$$

Using the divergence theorem:

$$\iiint_V \nabla \cdot \mathbf{F} dV = \iint_S \mathbf{F} \cdot d\mathbf{S}. \tag{55}$$

Since $\nabla \cdot \mathbf{F_2} = 5$, the volume of the hemisphere is:

$$V = \frac{2}{3}\pi a^3. \tag{56}$$

Applying the divergence theorem:

$$\iiint_V 5dV = 5 \times \frac{2}{3}\pi a^3 = \frac{10}{3}\pi a^3. \tag{57}$$

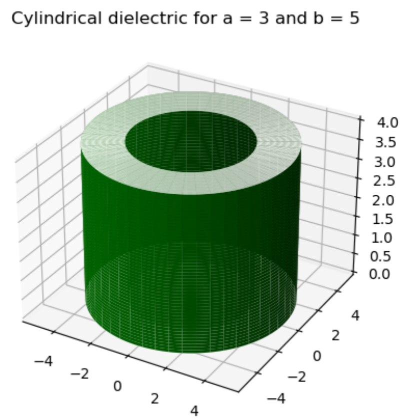
which confirms the surface integral result.

4. An infinitely long cylindrical dielectric of radius b contains charge within its volume with a charge density given by

$$\rho_v = a\rho^2 \tag{58}$$

where a is a constant. The goal is to determine the electric field strength **E** both inside and outside the cylinder.

Solution:



Electric Field Inside the Cylinder $(0 \leq \rho \leq b)$

Consider a Gaussian cylindrical surface of radius ρ and length L. The total enclosed charge within this surface is:

$$\begin{aligned} Q_{\text{enc}} &= L \int_0^{\rho} \int_0^{2\pi} \rho_v \rho d\rho d\phi \\ &= L \int_0^{\rho} \int_0^{2\pi} a\rho^2 \rho d\rho d\phi \\ &= aL \int_0^{\rho} 2\pi \rho^3 d\rho \\ &= aL2\pi \frac{\rho^4}{4} \\ &= \frac{2\pi aL\rho^4}{4}. \end{aligned}$$

By Gauss's law, since the electric field is constant at all points at a radius ρ from the center, the left-hand side simplifies to:

$$\mathbf{E} \cdot (2\pi\rho L) = \frac{2\pi aL\rho^4}{4\varepsilon}. \tag{59}$$

Solving for $\mathbf{E}$:

$$\mathbf{E} = \frac{a\rho^3}{4\varepsilon}, \quad 0 \leq \rho \leq b. \tag{60}$$

Electric Field Outside the Cylinder $(\rho \geq b)$

For $\rho > b$, the enclosed charge is:

$$\begin{aligned}
Q_{\text{enc}} &= L \int_0^b \int_0^{2\pi} a\rho^2 \rho d\rho d\phi \\
&= aL \int_0^b (2\pi)\rho^3 d\rho \\
&= \frac{2\pi a L b^4}{4}.
\end{aligned}$$

Applying Gauss's law for $\rho > b$:

$$\mathbf{E} \cdot (2\pi\rho L) = \frac{2\pi a L b^4}{4\varepsilon}. \tag{61}$$

Solving for $\mathbf{E}$:

$$\mathbf{E} = \frac{ab^4}{4\varepsilon\rho}, \quad \rho \geq b. \tag{62}$$

Final Expression for $E(\rho)$

$$\mathbf{E} = \begin{cases} \frac{a\rho^3}{4\varepsilon}, & 0 \leq \rho \leq b, \\ \\ \frac{ab^4}{4\varepsilon\rho}, & \rho \geq b. \end{cases} \tag{63}$$

5. Given the vector field in cylindrical coordinates:

$$\vec{F} = \left[\frac{40}{s^2+1} + 3(\cos\phi + \sin\phi)\right]\hat{s} + 3(\cos\phi - \sin\phi)\hat{\phi} - 2\hat{z}$$

(a) Compute and plot the magnitude $|\vec{F}|$as a function of ϕ for $s = 3$.
(b) Compute and plot the magnitude $|\vec{F}|$as a function of s for $\phi = 45^{\circ}$
(c) Calculate the divergence $\nabla \cdot \vec{F}$
(d) Calculate the curl $\nabla \times \vec{F}$ and verify whether the field is conservative.

Solution:

$|\vec{F}|$is given by,

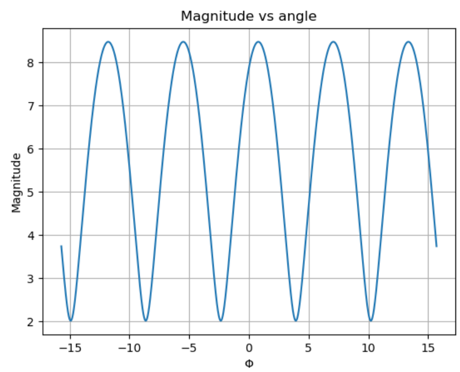
$$|\vec{F}| = \left(\frac{40}{s^2+1} + 3(\cos\phi + \sin\phi)\right)^2 + (3(\cos\phi - \sin\phi))^2 + 4$$

$$|\vec{F}| = \left(\left(\frac{40}{s^2+1}\right)^2 + \frac{240}{s^2+1}(\cos\phi + \sin\phi) + 22\right)^{\frac{1}{2}}$$

(a) Taking $s = 3$,

$$|\vec{F}| = (38 + 24(\cos\phi + \sin\phi))^{\frac{1}{2}}$$

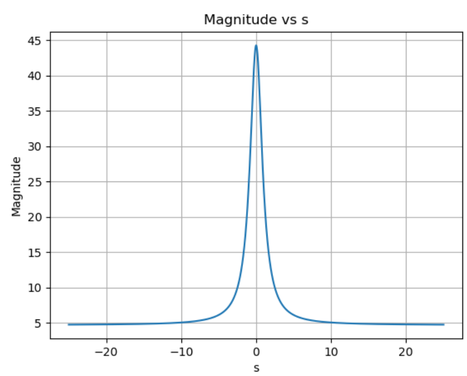
Plot of magnitude with varying ϕ:



(b) Taking $\phi = 45^{\circ}$,

$$\left(\left(\frac{40}{s^2+1}\right)^2 + \frac{240\sqrt{2}}{s^2+1} + 22\right)^{\frac{1}{2}}$$

Plot of magnitude with varying s:



(c) Compute the divergence $\nabla \cdot \vec{F}$

The divergence in cylindrical coordinates is given by:

$$\nabla \cdot \vec{F} = \frac{1}{s}\frac{\partial}{\partial s}(sF_s) + \frac{1}{s}\frac{\partial F_\phi}{\partial \phi} + \frac{\partial F_z}{\partial z}. \tag{64}$$

Substituting the components:

$$\begin{aligned}
F_s &= \frac{40}{s^2+1} + 3(\cos\phi + \sin\phi), \\
F_\phi &= 3(\cos\phi - \sin\phi), \\
F_z &= -2.
\end{aligned}$$

Computing each term:

$$\begin{aligned}
\frac{1}{s}\frac{\partial}{\partial s}\left(s\left(\frac{40}{s^2+1} + 3(\cos\phi + \sin\phi)\right)\right) &= \frac{1}{s}\left[\frac{\partial}{\partial s}\left(s\frac{40}{s^2+1}\right) + 3(\cos\phi + \sin\phi)\right], \\
\frac{1}{s}\frac{\partial}{\partial \phi}(3(\cos\phi - \sin\phi)) &= \frac{1}{s}(-3\sin\phi - 3\cos\phi), \\
\frac{\partial(-2)}{\partial z} &= 0.
\end{aligned}$$

On simplifying, the divergence is:

$$\nabla \cdot \vec{F} = \frac{40(1-s^2)}{s(s^2+1)^2} \tag{65}$$

(d) Compute the curl $\nabla \times \vec{F}$ and verify whether the field is conservative.

The curl in cylindrical coordinates is given by:

$$\nabla \times \mathbf{F} = \left(\frac{1}{s}\frac{\partial F_z}{\partial \phi} - \frac{\partial F_\phi}{\partial z}\right)\hat{s} + \left(\frac{\partial F_s}{\partial z} - \frac{\partial F_z}{\partial s}\right)\hat{\phi} + \frac{1}{s}\left[\frac{\partial}{\partial s}(sF_\phi) - \frac{\partial F_s}{\partial \phi}\right]\hat{z}. \tag{66}$$

Computing the components:

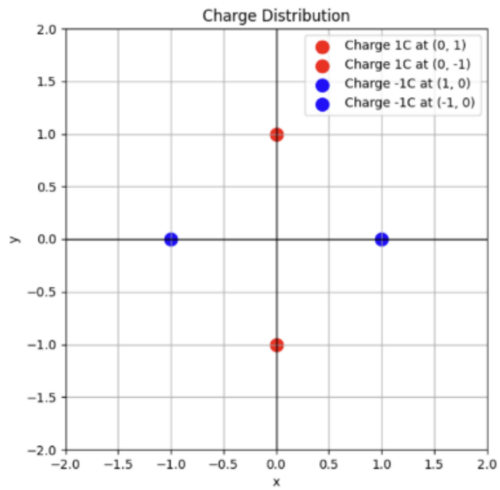
$$\begin{aligned}
\frac{1}{s}\frac{\partial(-2)}{\partial \phi} - \frac{\partial}{\partial z}(3(\cos\phi - \sin\phi)) &= 0, \\
\frac{\partial}{\partial z}\left(\frac{40}{s^2+1} + 3(\cos\phi + \sin\phi)\right) - \left(\frac{\partial(-2)}{\partial s}\right) &= 0 + 0 = 0, \\
\frac{1}{s}\left[\frac{\partial}{\partial s}\left(s(3(\cos\phi - \sin\phi))\right) - \frac{\partial}{\partial \phi}\left(\frac{40}{s^2+1} + 3(\cos\phi + \sin\phi)\right)\right] &= \\
\frac{1}{s}\left[3(\cos\phi - \sin\phi) - 3(\cos\phi - \sin\phi)\right] &= 0.
\end{aligned}$$

Thus, the curl is:

$$\nabla \times \vec{F} = 0. \tag{67}$$

Since $\nabla \times \vec{F} = 0$, the field is conservative.

6. Refer to the charge distribution below,



(a) Plot the Electric Field.
(b) Plot the potential with magnitude represented along the Z-axis.
(c) Compute the potential energy of the configuration.
(d) Use Gauss's Law to verify the divergence of the electric field.
(e) Verify whether the curl of the electric field is zero.

Solution:

(a) Electric Field as a Vector at Any Point (x, y)

The electric field due to a point charge at position (x_0, y_0) is given by:

$$\mathbf{E} = \frac{1}{4\pi\varepsilon_0} \frac{q(\mathbf{r} - \mathbf{r}_0)}{|\mathbf{r} - \mathbf{r}_0|^3}, \tag{68}$$

where $\mathbf{r} = x\hat{i} + y\hat{j}$ and $\mathbf{r}_0 = x_0\hat{i} + y_0\hat{j}$.
Using superposition, the total electric field due to the four charges is:

$$\mathbf{E} = E_x\hat{i} + E_y\hat{j}, \tag{69}$$

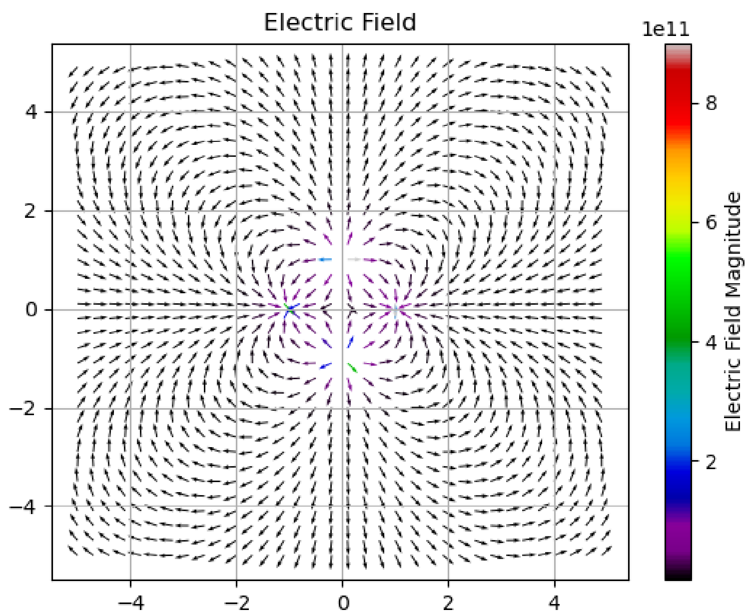
where the components are:

$$
\begin{aligned}
E_x = \frac{q}{4\pi\varepsilon_0} &\left[\frac{x}{(x^2+(y-1)^2)^{3/2}} + \frac{x}{(x^2+(y+1)^2)^{3/2}}\right] \\
&- \left[\frac{x-1}{((x-1)^2+y^2)^{3/2}} - \frac{x+1}{((x+1)^2+y^2)^{3/2}}\right], \\
E_y = \frac{q}{4\pi\varepsilon_0} &\left[\frac{y-1}{(x^2+(y-1)^2)^{3/2}} + \frac{y+1}{(x^2+(y+1)^2)^{3/2}}\right] \\
&- \left[\frac{y}{((x-1)^2+y^2)^{3/2}} - \frac{y}{((x+1)^2+y^2)^{3/2}}\right].
\end{aligned}
$$

Thus, the final vector form of the electric field is:

$$\mathbf{E} = (E_x)\,\hat{i} + (E_y)\,\hat{j}. \tag{70}$$

Plot of Electric Field:



(b) Electrostatic Potential at Any Point (x, y)

The electrostatic potential due to a point charge q at position (x_0, y_0) is given by:

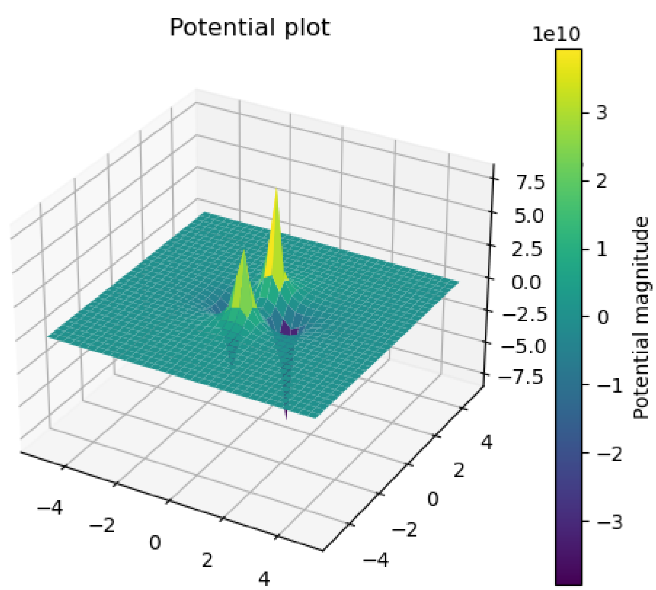
$$V = \frac{1}{4\pi\varepsilon_0} \frac{q}{|\mathbf{r} - \mathbf{r}_0|}, \tag{71}$$

where $\mathbf{r} = x\hat{i} + y\hat{j}$ and $\mathbf{r}_0 = x_0\hat{i} + y_0\hat{j}$.

Using the superposition principle, the total potential at any point (x, y) due to the four charges is:

$$V(x, y) = \frac{q}{4\pi\varepsilon_0} \left[\frac{1}{\sqrt{x^2 + (y-1)^2}} + \frac{1}{\sqrt{x^2 + (y+1)^2}} \right] - \left[\frac{1}{\sqrt{(x-1)^2 + y^2}} - \frac{1}{\sqrt{(x+1)^2 + y^2}} \right]$$

Plot of Potential as a function of position:



(c) Computing the the potential energy of the system of charges given,

The potential energy of a system of point charges is given by:

$$U = \frac{1}{4\pi\epsilon_0} \sum\sum_{0<i<j<5} \frac{q_i q_j}{r_{ij}}. \tag{72}$$

The given configuration consists of four charges:

- $q_1 = +q$ at $(0, 1)$
- $q_2 = +q$ at $(0, -1)$
- $q_3 = -q$ at $(1, 0)$
- $q_4 = -q$ at $(-1, 0)$

The distances between the charges are:

$$\begin{aligned} r_{(+q,-q)} &= \sqrt{(1-0)^2 + (0-1)^2} = \sqrt{2}, \\ r_{(+q,+q)} &= 2, \\ r_{(-q,-q)} &= 2. \end{aligned}$$

Now summing up the contributions:

$$U = [U_{1,2} + U_{1,3} + U_{1,4} + U_{2,3} + U_{2,4} + U_{3,4}]$$
$$U = \frac{1}{4\pi\epsilon_0} \left[\frac{(+q)(+q)}{2} + \frac{(+q)(-q)}{\sqrt{2}} + \frac{(+q)(-q)}{\sqrt{2}} + \frac{(+q)(-q)}{\sqrt{2}} + \frac{(+q)(+q)}{\sqrt{2}} + \frac{(-q)(-q)}{2} \right].$$

Simplifying:

$$\begin{aligned} U &= \frac{1}{4\pi\epsilon_0} \left[-\frac{4q^2}{\sqrt{2}} + \frac{q^2}{2} + \frac{q^2}{2} \right] \\ &= \frac{1}{4\pi\epsilon_0} \left[-2\sqrt{2}q^2 + q^2 \right] \\ &= \frac{q^2}{4\pi\epsilon_0} \left[1 - 2\sqrt{2} \right]. \end{aligned}$$

Thus, the potential energy of the configuration is:

$$U = \frac{q^2}{4\pi\epsilon_0}(1 - 2\sqrt{2}). \tag{73}$$

(d) Divergence of the electric field

The electric field $\mathbf{E}$ due to a point charge q at position $\mathbf{r}_0$ is given by:

$$\mathbf{E} = \frac{1}{4\pi\varepsilon_0} \frac{q(\mathbf{r} - \mathbf{r}_0)}{|\mathbf{r} - \mathbf{r}_0|^3}. \tag{74}$$

By the superposition principle, the total electric field $\mathbf{E}$ is the sum of the contributions from all four charges.

The divergence of a vector field $\mathbf{E} = (E_x, E_y, E_z)$ is given by:

$$\nabla \cdot \mathbf{E} = \frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z}. \tag{75}$$

For a single point charge, the electric field components in Cartesian coordinates are:

$$E_x = \frac{q}{4\pi\varepsilon_0} \frac{x - x_0}{[(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2]^{3/2}}, \tag{76}$$

$$E_y = \frac{q}{4\pi\varepsilon_0} \frac{y - y_0}{[(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2]^{3/2}}, \tag{77}$$

$$E_z = \frac{q}{4\pi\varepsilon_0} \frac{z - z_0}{[(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2]^{3/2}}. \tag{78}$$

Computing the partial derivatives:

$$\frac{\partial E_x}{\partial x} = \frac{q}{4\pi\varepsilon_0} \left[\frac{(r^2 - 3(x - x_0)^2)}{(r^2)^{5/2}} \right], \tag{79}$$

$$\frac{\partial E_y}{\partial y} = \frac{q}{4\pi\varepsilon_0} \left[\frac{(r^2 - 3(y - y_0)^2)}{(r^2)^{5/2}} \right], \tag{80}$$

$$\frac{\partial E_z}{\partial z} = \frac{q}{4\pi\varepsilon_0} \left[\frac{(r^2 - 3(z - z_0)^2)}{(r^2)^{5/2}} \right]. \tag{81}$$

Summing these up:

$$\nabla \cdot \mathbf{E} = \frac{q}{4\pi\varepsilon_0} \left[\frac{3r^2 - 3r^2}{(r^2)^{5/2}} \right] = \frac{q}{4\pi\varepsilon_0} \left[\frac{0}{r^5} \right] = 0, \quad \text{for } r \neq 0. \tag{82}$$

Thus, for a single charge, the divergence is zero everywhere except at the location of the charge itself. Using the property of the Dirac delta function, we write:

$$\nabla \cdot \mathbf{E} = \frac{q}{\varepsilon_0} \delta(\mathbf{r} - \mathbf{r}_0). \tag{83}$$

Since the total electric field is the sum of the fields due to the four charges, the total divergence is:

$$\nabla \cdot \mathbf{E} = \frac{q}{\varepsilon_0} [\delta(x, y - 1) + \delta(x, y + 1) - \delta(x - 1, y) - \delta(x + 1, y)]. \tag{84}$$

Verification Using the Differential Form of Gauss's Law:

The differential form of Gauss's law states:

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0}, \tag{85}$$

where ρ is the charge density.

In this case, the charge density $\rho(\mathbf{r})$ is given by a sum of Dirac delta functions corresponding to the four point charges:

$$\rho(\mathbf{r}) = q\delta(x, y-1) + q\delta(x, y+1) - q\delta(x-1, y) - q\delta(x+1, y). \tag{86}$$

Substituting this into Gauss's law:

$$\nabla \cdot \mathbf{E} = \frac{\rho}{\varepsilon_0} = \frac{q}{\varepsilon_0}[\delta(x, y-1) + \delta(x, y+1) - \delta(x-1, y) - \delta(x+1, y)]. \tag{87}$$

This result exactly matches our previous explicit calculation of $\nabla \cdot \mathbf{E}$, thus confirming the validity of our computations using the differential form of Gauss's law.

(e) Curl of the Electric Field Due to Four Charges

The curl of a vector field $\mathbf{E} = (E_x, E_y, E_z)$ is given by:

$$\nabla \times \mathbf{E} = \begin{bmatrix} \frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} \\ \frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x} \\ \frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} \end{bmatrix}. \tag{88}$$

Computing the partial derivatives:

$$\frac{\partial E_x}{\partial y} = \frac{q}{4\pi\varepsilon_0}\left[\frac{-3(x-x_0)(y-y_0)}{(r^2)^{5/2}}\right], \tag{89}$$

$$\frac{\partial E_x}{\partial z} = \frac{q}{4\pi\varepsilon_0}\left[\frac{-3(x-x_0)(z-z_0)}{(r^2)^{5/2}}\right], \tag{90}$$

$$\frac{\partial E_y}{\partial x} = \frac{q}{4\pi\varepsilon_0}\left[\frac{-3(y-y_0)(x-x_0)}{(r^2)^{5/2}}\right], \tag{91}$$

$$\frac{\partial E_y}{\partial z} = \frac{q}{4\pi\varepsilon_0}\left[\frac{-3(y-y_0)(z-z_0)}{(r^2)^{5/2}}\right], \tag{92}$$

$$\frac{\partial E_z}{\partial x} = \frac{q}{4\pi\varepsilon_0}\left[\frac{-3(z-z_0)(x-x_0)}{(r^2)^{5/2}}\right], \tag{93}$$

$$\frac{\partial E_z}{\partial y} = \frac{q}{4\pi\varepsilon_0}\left[\frac{-3(z-z_0)(y-y_0)}{(r^2)^{5/2}}\right]. \tag{94}$$

Substituting these into the curl expression:

$$\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} = 0, \tag{95}$$

$$\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x} = 0, \tag{96}$$

$$\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = 0. \tag{97}$$

Thus, for a single charge:

$$\nabla \times \mathbf{E} = \mathbf{0}. \tag{98}$$

Since the curl of the field due to a single point charge is zero, and the curl operator is linear, the total electric field, being a superposition of individual point charge fields, also has zero curl:

$$\nabla \times \mathbf{E} = 0. \tag{99}$$

Thus, the electric field in this charge configuration is irrotational, confirming that it is a conservative field.

7. Given the spherically symmetric potential field in free space,

$$V(r) = V_0 e^{-r/a},$$

determine the following:
(a) Find the charge density ρ_v at $r = a$
(b) Calculate the electric field $\mathbf{E}$ at $r = a$
(c) Compute the total charge.

Solution:

The given potential field is:

$$V(r) = V_0 e^{-r/a}. \tag{100}$$

(a) Charge Density ρ_v at $r = a$

Using Poisson's equation:

$$\nabla^2 V = -\frac{\rho_v}{\varepsilon_0}, \tag{101}$$

where ∇^2 in spherical coordinates for a radially symmetric function is:

$$\nabla^2 V = \frac{1}{r^2} \frac{d}{dr} \left(r^2 \frac{dV}{dr} \right). \tag{102}$$

Computing the first derivative:

$$\frac{dV}{dr} = -\frac{V_0}{a}e^{-r/a}. \tag{103}$$

Computing the second derivative:

$$\frac{d}{dr}\left(r^2\frac{dV}{dr}\right) = \frac{d}{dr}\left(-\frac{V_0}{a}r^2e^{-r/a}\right). \tag{104}$$

Using the product rule:

$$\frac{d}{dr}\left(-\frac{V_0}{a}r^2e^{-r/a}\right) = -\frac{V_0}{a}\left(2re^{-r/a} - \frac{r^2}{a}e^{-r/a}\right), \tag{105}$$

which simplifies to:

$$\nabla^2 V = \frac{V_0}{(ar)^2}e^{-r/a}(r^2 - 2ar). \tag{106}$$

Thus, the charge density at $r = a$ is:

$$\rho_v(a) = -\varepsilon_0\frac{V_0}{a^4}e^{-1}(a^2 - 2a^2) = \varepsilon_0\frac{V_0}{a^4}e^{-1}(2a^2 - a^2) = \frac{\varepsilon_0}{e}\frac{V_0}{a^2}. \tag{107}$$

(b) Electric Field E at $r = a$

The electric field is given by:

$$\mathbf{E} = -\nabla V. \tag{108}$$

Since V depends only on r, we have:

$$E_r = -\frac{dV}{dr} = \frac{V_0}{a}e^{-r/a}. \tag{109}$$

Evaluating at $r = a$:

$$E(a) = \frac{V_0}{a}e^{-1}. \tag{110}$$

(c) Total Charge

The total charge enclosed is given by:

$$Q_{enc} = \int \rho_v dV. \tag{111}$$

Using spherical volume element $dV = 4\pi r^2 dr$:

$$Q_{enc} = \int_0^\infty \rho_v(r) 4\pi r^2 dr. \tag{112}$$

Substituting ρ_v:

$$Q_{enc} = \int_0^\infty \left(-\varepsilon_0 \frac{V_0}{(ar)^2} e^{-r/a} (r - 2a) \right) 4\pi r^2 dr. \tag{113}$$

This integral when computed, leads to:

$$Q_{enc} = 0 \tag{114}$$

Therefore no net charge is enclosed in the free space.

8. A parallel-plate capacitor has plates located at $z = 0$ and $z = d$. The region between the plates is filled with a material that contains a uniform volume charge density ρ_0 C/m^3 and has permittivity ϵ. Both plates are held at ground potential.

   (a) Determine the potential field between the plates.
   (b) Determine the electric field intensity E between the plates.
   (c) Repeat parts (a) and (b) for the case where the plate at $z = d$ is raised to a potential V_0, with the plate at $z = 0$ grounded.

Solution:

(a) Potential Field for Both Plates at Ground Potential

Poisson's equation in one dimension is given by:

$$\frac{d^2V}{dz^2} = -\frac{\rho_0}{\epsilon}. \tag{115}$$

Integrating once:

$$\frac{dV}{dz} = -\frac{\rho_0}{\epsilon} z + C_1. \tag{116}$$

Integrating again:

$$V(z) = -\frac{\rho_0}{2\epsilon} z^2 + C_1 z + C_2. \tag{117}$$

Applying boundary conditions:

$$V(0) = 0 \Rightarrow C_2 = 0, \tag{118}$$

$$V(d) = 0 \Rightarrow -\frac{\rho_0}{2\epsilon} d^2 + C_1 d = 0. \tag{119}$$

Solving for C_1:

$$C_1 = \frac{\rho_0}{2\epsilon} d. \tag{120}$$

Thus, the potential field is:

$$V(z) = \frac{\rho_0 d}{2\epsilon} z - \frac{\rho_0}{2\epsilon} z^2. \tag{121}$$

(b) Electric Field Intensity Between the Plates

The electric field is given by:

$$E = -\frac{dV}{dz}. \tag{122}$$

Computing the derivative:

$$E(z) = -\left(\frac{\rho_0}{2\epsilon} d - \frac{\rho_0}{\epsilon} z\right) = \frac{\rho_0}{\epsilon} z - \frac{\rho_0}{2\epsilon} d. \tag{123}$$

(c) Case When the Plate at $z = d$ is Raised to a Potential V_0

For this case, we solve Poisson's equation again:

$$V(z) = -\frac{\rho_0}{2\epsilon} z^2 + C_1 z + C_2. \tag{124}$$

Applying boundary conditions:

$$V(0) = 0 \Rightarrow C_2 = 0, \tag{125}$$

$$V(d) = V_0 \Rightarrow -\frac{\rho_0}{2\epsilon} d^2 + C_1 d = V_0. \tag{126}$$

Solving for C_1:

$$C_1 = \frac{V_0}{d} + \frac{\rho_0}{2\epsilon} d. \tag{127}$$

Thus, the potential field is:

$$V(z) = \left(\frac{V_0}{d} + \frac{\rho_0}{2\epsilon} d\right) z - \frac{\rho_0}{2\epsilon} z^2. \tag{128}$$

The electric field is:

$$E(z) = -\frac{dV}{dz} = -\left(\frac{V_0}{d} + \frac{\rho_0}{2\epsilon} d - \frac{\rho_0}{\epsilon} z\right). \tag{129}$$

Simplifying:

$$E(z) = -\frac{V_0}{d} - \frac{\rho_0}{2\epsilon} d + \frac{\rho_0}{\epsilon} z. \tag{130}$$

Source Codes:

The source codes used in the following assignment are given below for reference

Code for question 4:

```
import numpy as np
import matplotlib.pyplot as plt

ax = plt.axes(projection = "3d")
theta = np.linspace(0, 2 * np.pi, 1000)

#for cylinder 1
z_inner = np.linspace(0, 4, 10000)
[Theta, Z_inner] = np.meshgrid(theta, z_inner)
x_inner = 3 * np.cos(Theta)
y_inner = 3 * np.sin(Theta)
plot_inner = ax.plot_surface(x_inner, y_inner,
      Z_inner, alpha = 0.4, color = "darkgreen")

#for cylinder 2
z_inner = np.linspace(0, 4, 10000)
[Theta, Z_inner] = np.meshgrid(theta, z_inner)
x_inner = 5 * np.cos(Theta)
y_inner = 5 * np.sin(Theta)
plot_inner = ax.plot_surface(x_inner, y_inner,
             Z_inner, alpha = 1, color = "darkgreen")

#middle surface to simulate filling
r_mid = np.linspace(3, 5, 50)
Theta_mid, R_mid = np.meshgrid(theta, r_mid)
X_mid = R_mid * np.cos(Theta_mid)
Y_mid = R_mid * np.sin(Theta_mid)

# Plot the "fill" surface
ax.plot_surface(X_mid, Y_mid, np.full_like(X_mid, 4), cmap="Greens", alpha=1)
ax.plot_surface(X_mid, Y_mid, np.full_like(X_mid, 0), cmap="Greens", alpha=1)


ax.set_title("Cylindrical dielectric for a = 3 and b = 5")
plt.savefig("figs/4.png")
plt.show()
```

Code for question 5:

```
import numpy as np
import matplotlib.pyplot as plt

def f(phi, s):
    mag = ((40/(s**2 + 1))**2 + (240 / (s**2 + 1))
       * (np.cos(phi) + np.sin(phi)) + 22)**0.5
    return mag

phi = np.linspace(-5*np.pi, 5*np.pi, 10000)
s = np.linspace(-25, 25, 10000)
mag_phi = f(phi, 3)
mag_s = f(np.pi / 4, s)

#plot for s = 3
plt.figure()
plt.plot(phi, mag_phi)
plt.title("Magnitude vs angle")
plt.xlabel("phi")
plt.ylabel("Magnitude")
plt.grid(True)
plt.savefig("figs/5a.png")

#plot for phi = 45
plt.figure()
plt.plot(s, mag_s)
plt.title("Magnitude vs s")
plt.xlabel("s")
plt.ylabel("Magnitude")
plt.grid(True)
plt.savefig("figs/5b.png")

plt.show()
```

Code for question 6(a):

```
import numpy as np
import matplotlib.pyplot as plt

#field along x axis at any point (x, y)
def Ex(x, y):
    k = 9e9
    E = (k * x) / ((x**2 + (y - 1)**2)**1.5) -
    (k * (x - 1)) / (((x - 1)**2 + (y)**2)**1.5) +
    (k * x) / ((x**2 + (y + 1)**2)**1.5) -
    (k * (x + 1)) / (((x + 1)**2 + y**2)**1.5)
    return E

#field along y axis at any point (x, y)
def Ey(x, y):
    k = 9e9
    E = (k * (y - 1)) / ((x**2 + (y - 1)**2)**1.5) -
    (k * y) / (((x - 1)**2 + (y)**2)**1.5) +
    (k * (y + 1)) / ((x**2 + (y + 1)**2)**1.5) -
    (k * y) / (((x + 1)**2 + y**2)**1.5)
    return E

#making set of (x, y) coordinates
u = np.arange(-5, 5, 0.3)
v = np.arange(-5, 5, 0.3)
[x, y] = np.meshgrid(u, v)

#calculating Ex and Ey at all the points
Ex_coord = Ex(x, y)
Ey_coord = Ey(x, y)
magnitude = np.sqrt(Ex_coord**2 + Ey_coord**2)

#plotting the quiver plot at all the points with varying
colours but same length to show different magnitudes
quiver = plt.quiver(x, y, Ex_coord / magnitude, Ey_coord / magnitude,
  magnitude, cmap = "nipy_spectral", scale = 40)
plt.colorbar(quiver, label = "Electric Field Magnitude")
plt.title("Electric Field")
plt.grid(True)
plt.savefig("figs/6a.png")
plt.show()
```

Code for question 6(b):

```
import numpy as np
import matplotlib.pyplot as plt

def potential(x, y):
    k = 9e9
    phi = k / ((x**2 + (y - 1)**2)**0.5) - k / (((x - 1)**2 + y**2)**0.5) +
    k / ((x**2 + (y + 1)**2)**0.5) - k / (((x + 1)**2 + y**2)**0.5)
    return phi

#plotting potential as a function of position
ax = plt.axes(projection = "3d")
u = np.arange(-5, 5, 0.3)
v = np.arange(-5, 5, 0.3)
[x, y] = np.meshgrid(u, v)
z = potential(x, y)
plot = ax.plot_surface(x, y, z, cmap = "viridis")
plt.colorbar(plot, label = "Potential magnitude")
ax.set_title("Potential plot")
plt.savefig("figs/6b.png")
plt.show()
```