Assignment 1

Student: Aditya Tripathy, ee24btech11001@iith.ac.in

Problem 1

(a) The fields due to the charges at a point $z\hat{\mathbf{k}}$ can be written as follows:

$$\vec{E_1} = \frac{1}{4\pi\epsilon_0}\frac{q\left(z\hat{\mathbf{k}} - \frac{d}{2}\hat{\mathbf{k}}\right)}{\left\|z\hat{\mathbf{k}} - \frac{d}{2}\hat{\mathbf{k}}\right\|^3}$$
$$\vec{E_2} = \frac{1}{4\pi\epsilon_0}\frac{q\left(z\hat{\mathbf{k}} + \frac{d}{2}\hat{\mathbf{k}}\right)}{\left\|z\hat{\mathbf{k}} + \frac{d}{2}\hat{\mathbf{k}}\right\|^3}$$

Now using superposition of electric field vectors, net field at the point $z\hat{\mathbf{k}}$ is written as,

$$\vec{E} = \vec{E_1} + \vec{E_2}$$
$$\implies \vec{E} = \begin{cases} \frac{q\hat{\mathbf{k}}}{4\pi\epsilon_0}\left(\frac{1}{\left(z-\frac{d}{2}\right)^2} + \frac{1}{\left(z+\frac{d}{2}\right)^2}\right) & z > \frac{d}{2} \\ \frac{q\hat{\mathbf{k}}}{4\pi\epsilon_0}\left(\frac{1}{\left(z+\frac{d}{2}\right)^2} - \frac{1}{\left(z-\frac{d}{2}\right)^2}\right) & \frac{-d}{2} < z < \frac{d}{2} \\ \frac{-q\hat{\mathbf{k}}}{4\pi\epsilon_0}\left(\frac{1}{\left(z-\frac{d}{2}\right)^2} + \frac{1}{\left(z+\frac{d}{2}\right)^2}\right) & z < \frac{-d}{2} \end{cases}$$

On simplification,

$$\vec{E} = \begin{cases} \frac{2q\hat{\mathbf{k}}}{4\pi\epsilon_0}\left(\frac{z^2+\frac{d^2}{4}}{\left(z^2-\frac{d^2}{4}\right)^2}\right) & z > \frac{d}{2} \\ \frac{-2q\hat{\mathbf{k}}}{4\pi\epsilon_0}\left(\frac{zd}{\left(z^2-\frac{d^2}{4}\right)^2}\right) & \frac{-d}{2} < z < \frac{d}{2} \\ \frac{-2q\hat{\mathbf{k}}}{4\pi\epsilon_0}\left(\frac{z^2+\frac{d^2}{4}}{\left(z^2-\frac{d^2}{4}\right)^2}\right) & z < \frac{-d}{2} \end{cases}$$

(b) The fields due to the charges at a point $x\hat{\mathbf{i}}$ can be written as follows,

$$\vec{E_1} = \frac{1}{4\pi\epsilon_0}\frac{q\left(x\hat{\mathbf{i}} - \frac{d}{2}\hat{\mathbf{k}}\right)}{\left\|x\hat{\mathbf{i}} - \frac{d}{2}\hat{\mathbf{k}}\right\|^3} = \frac{q}{4\pi\epsilon_0}\frac{x\hat{\mathbf{i}} - \frac{d}{2}\hat{\mathbf{k}}}{\left(x^2 + \frac{d^2}{4}\right)^{\frac{3}{2}}}$$
$$\vec{E_2} = \frac{1}{4\pi\epsilon_0}\frac{q\left(x\hat{\mathbf{i}} + \frac{d}{2}\hat{\mathbf{k}}\right)}{\left\|x\hat{\mathbf{i}} + \frac{d}{2}\hat{\mathbf{k}}\right\|^3} = \frac{q}{4\pi\epsilon_0}\frac{x\hat{\mathbf{i}} + \frac{d}{2}\hat{\mathbf{k}}}{\left(x^2 + \frac{d^2}{4}\right)^{\frac{3}{2}}}$$

Again, using superposition, the net field at the point $x\hat{\mathbf{i}}$ can be written as,

$$\vec{E} = \vec{E_1} + \vec{E_2}$$
$$\implies \vec{E} = \frac{q}{4\pi\epsilon_0}\frac{x\hat{\mathbf{i}} - \frac{d}{2}\hat{\mathbf{k}}}{\left(x^2 + \frac{d^2}{4}\right)^{\frac{3}{2}}} + \frac{q}{4\pi\epsilon_0}\frac{x\hat{\mathbf{i}} + \frac{d}{2}\hat{\mathbf{k}}}{\left(x^2 + \frac{d^2}{4}\right)^{\frac{3}{2}}}$$
$$\implies \vec{E} = \frac{2q}{4\pi\epsilon_0}\frac{x\hat{\mathbf{i}}}{\left(x^2 + \frac{d^2}{4}\right)^{\frac{3}{2}}}$$

(c) If the sign of the charge at $\frac{-d}{2}\hat{\mathbf{k}}$ is flipped, we get

(a)

$$\vec{E} = \vec{E_1} - \vec{E_2}$$
$$\implies \vec{E} = \begin{cases} \frac{q\hat{\mathbf{k}}}{4\pi\epsilon_0}\left(\frac{1}{\left(z-\frac{d}{2}\right)^2} - \frac{1}{\left(z+\frac{d}{2}\right)^2}\right) & z > \frac{d}{2} \\ \frac{-q\hat{\mathbf{k}}}{4\pi\epsilon_0}\left(\frac{1}{\left(z+\frac{d}{2}\right)^2} + \frac{1}{\left(z-\frac{d}{2}\right)^2}\right) & \frac{-d}{2} < z < \frac{d}{2} \\ \frac{q\hat{\mathbf{k}}}{4\pi\epsilon_0}\left(\frac{1}{\left(z+\frac{d}{2}\right)^2} - \frac{1}{\left(z-\frac{d}{2}\right)^2}\right) & z < \frac{-d}{2} \end{cases}$$

On simplification,

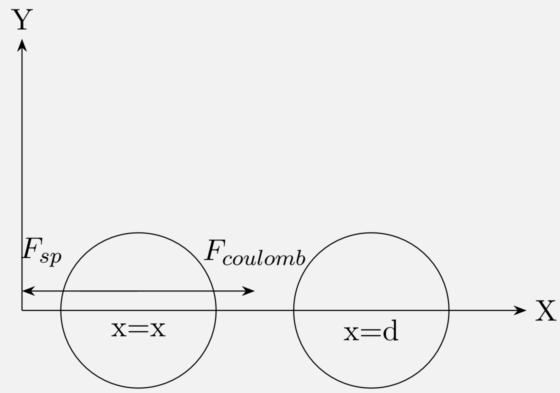
$$\vec{E} = \begin{cases} \frac{2q\hat{\mathbf{k}}}{4\pi\epsilon_0}\left(\frac{zd}{\left(z^2-\frac{d^2}{4}\right)^2}\right) & z > \frac{d}{2} \\ \frac{-2q\hat{\mathbf{k}}}{4\pi\epsilon_0}\left(\frac{z^2+\frac{d^2}{4}}{\left(z^2-\frac{d^2}{4}\right)^2}\right) & \frac{-d}{2} < z < \frac{d}{2} \\ \frac{-2q\hat{\mathbf{k}}}{4\pi\epsilon_0}\left(\frac{zd}{\left(z^2-\frac{d^2}{4}\right)^2}\right) & \frac{-d}{2} < z < \frac{d}{2} \end{cases} \tag{1}$$

(b)

$$\vec{E} = \vec{E_1} - \vec{E_2}$$
$$\implies \vec{E} = \frac{-q}{4\pi\epsilon_0}\frac{d\hat{\mathbf{i}}}{\left(x^2 + \frac{d^2}{4}\right)^{\frac{3}{2}}}$$

Problem 2

(a) The situation described in the problem can be represented in the following free-body diagram,



For the sphere at $x = x$ to be at equilibrium,

$$\begin{aligned} \sum \vec{F_x} &= 0 \\ \implies k(x)(-\hat{\mathbf{i}}) + \frac{1}{4\pi\epsilon_0}\frac{(-Q^2)(x-d)\hat{\mathbf{i}}}{\left\|(x-d)\hat{\mathbf{i}}\right\|^3} &= 0 \\ \implies \frac{1}{4\pi\epsilon_0}\frac{(Q^2)}{(d-x)^2} &= kx \\ \implies Q &= \sqrt{4\pi k\epsilon_0 x(d-x)^2} \end{aligned}$$

(b) If we observe the expression for Q, for $x \in [0, d]$, the maximum value of Q is achieved when,

$$\begin{aligned} \frac{d}{dx}\left(x(d-x)^2\right) &= 0 \\ \implies (d-x)^2 - 2x(d-x) &= 0 \\ \implies d^2 + x^2 - 2xd - 2xd + 2x^2 &= 0 \\ \implies 3x^2 + d^2 - 4xd &= 0 \\ \implies x &= \frac{d}{3}, d \end{aligned}$$

Since both the charges cannot be at $x = d$, because of infinite coulombic force but finite spring force. Therefore the maximum charge that can be measured using the given method is therefore,

$$Q_{max} = \sqrt{\frac{16}{27}\pi k\epsilon_0 d^3}$$

If the charge is any larger than Q_{max} then the system is never at equilibrium and the charged sphere initially at origin will collapse towards the fixed sphere.

Problem 3

(a)

$$\begin{aligned}\text{Flux} &= \iint_{\mathbb{R}} (\vec{F_1} \cdot \hat{\mathbf{n}}) dS \\ &= \int_{\phi=0}^{\phi=\frac{\pi}{2}} \int_{\theta=0}^{\theta=2\pi} \left(\left(5 \left\| \vec{a_z} \right\| \hat{\mathbf{k}} \right) \cdot \hat{\boldsymbol{\rho}} \right) \rho \sin(\phi) d\theta d\phi\end{aligned}$$

We know,

$$\hat{\mathbf{k}} = \cos(\phi)\hat{\boldsymbol{\rho}} - \sin(\phi)\hat{\boldsymbol{\phi}}$$

Substituting,

$$\begin{aligned}\text{Flux} &= \int_{\phi=0}^{\phi=\frac{\pi}{2}} \int_{\theta=0}^{\theta=2\pi} \left(\left(5 \left\| \vec{a_z} \right\| \left(\cos(\phi)\hat{\boldsymbol{\rho}} - \sin(\phi)\hat{\boldsymbol{\phi}} \right) \right) \cdot \hat{\boldsymbol{\rho}} \right) \rho^2 \sin(\phi) d\theta d\phi \\ &= \int_{\phi=0}^{\phi=\frac{\pi}{2}} \int_{\theta=0}^{\theta=2\pi} 5 \left\| \vec{a_z} \right\| \rho^2 \sin(\phi) \cos\left(\phi\right) d\theta d\phi \\ &= \int_{\phi=0}^{\phi=\frac{\pi}{2}} \int_{\theta=0}^{\theta=2\pi} 5 \left\| \vec{a_z} \right\| \rho^2 \sin(\phi) \cos\left(\phi\right) d\theta d\phi \\ &= \int_{\phi=0}^{\phi=\frac{\pi}{2}} 10\pi \left\| \vec{a_z} \right\| \rho^2 \sin(\phi) \cos\left(\phi\right) d\theta d\phi \\ &= 5\pi \left\| \vec{a_z} \right\| \rho^2\end{aligned}$$

A simple observation would have saved us a lot of trouble,

Observation 1. *Flux of a divergence-free field through a closed surface is zero.*

If we consider the hemispherical surface along with the circular base(surface normal in the $-\hat{\mathbf{k}}$ direction), the flux through it will be zero, therefore

$$\begin{aligned}\Phi_1 + \Phi_2 &= 0 \\ \implies \Phi_1 &= -\Phi_2 \\ \implies \Phi_1 &= -\left(\pi\rho^2 \left(\textbf{-}\hat{\mathbf{k}} \right) \right) \cdot \left(5 \left\| a_z \right\| \hat{\mathbf{k}} \right) = 5\pi \left\| \vec{a_z} \right\| \rho^2\end{aligned}$$

(b)

$$\text{Flux} = \iint_{\mathbb{R}_1} (\vec{F_1} \cdot \hat{\mathbf{n}}) dS + \iint_{\mathbb{R}_2} (\vec{F_1} \cdot \hat{\mathbf{n}}) dS$$

where $\mathbb{R}_1$ is the hemispherical surface and $\mathbb{R}_2$ is the circular base.

$$\begin{aligned}\iint_{\mathbb{R}_2}\left(\vec{F_1}\cdot\hat{\mathbf{n}}\right)dS &= \iint_{\mathbb{R}_2}\left(\left(5z\left\|\vec{a_z}\right\|\hat{\mathbf{k}}\right)\cdot\left(\text{-}\hat{\mathbf{k}}\right)\right)dS\\ &= \iint_{\mathbb{R}_2}\left(\left(5\,(0)\left\|\vec{a_z}\right\|\hat{\mathbf{k}}\right)\cdot\left(\text{-}\hat{\mathbf{k}}\right)\right)dS\\ &= 0\\ \iint_{\mathbb{R}_1}\left(\vec{F_1}\cdot\hat{\mathbf{n}}\right)dS &= \iint_{\mathbb{R}}(\vec{F_1}\cdot\hat{\mathbf{n}})dS\\ &= \int_{\phi=0}^{\phi=\frac{\pi}{2}}\int_{\theta=0}^{\theta=2\pi}\left(\left(5z\left\|\vec{a_z}\right\|\hat{\mathbf{k}}\right)\cdot\hat{\boldsymbol{\rho}}\right)\rho\sin(\phi)d\theta d\phi\end{aligned}$$

Using,

$$\begin{aligned}\hat{\mathbf{k}} &= \cos(\phi)\hat{\boldsymbol{\rho}}-\sin(\phi)\hat{\boldsymbol{\phi}}\\ z &= \rho\sin(\phi)\end{aligned}$$

We get,

$$\begin{aligned}\iint_{\mathbb{R}_1}\left(\vec{F_1}\cdot\hat{\mathbf{n}}\right)dS &= \int_{\phi=0}^{\phi=\frac{\pi}{2}}\int_{\theta=0}^{\theta=2\pi}\left(\left(5\rho\sin(\phi)\left\|\vec{a_z}\right\|\left(\cos(\phi)\hat{\boldsymbol{\rho}}-\sin(\phi)\hat{\boldsymbol{\phi}}\right)\right)\cdot\hat{\boldsymbol{\rho}}\right)\rho^2\sin(\phi)d\theta d\phi\\ &= \int_{\phi=0}^{\phi=\frac{\pi}{2}}\int_{\theta=0}^{\theta=2\pi}5\rho\sin(\phi)\left\|\vec{a_z}\right\|\rho\sin(\phi)\cos(\phi)\,d\theta d\phi\\ &= \int_{\phi=0}^{\phi=\frac{\pi}{2}}\int_{\theta=0}^{\theta=2\pi}5\left\|\vec{a_z}\right\|\rho^3\sin^2(\phi)\cos(\phi)\,d\theta d\phi\\ &= \int_{\phi=0}^{\phi=\frac{\pi}{2}}10\pi\left\|\vec{a_z}\right\|\rho^3\sin^2(\phi)\cos(\phi)\,d\theta d\phi\\ &= \frac{10}{3}\pi\left\|\vec{a_z}\right\|\rho^3\end{aligned}$$

Therefore, total flux is given by,

$$\text{Flux} = \frac{10}{3}\pi\left\|\vec{a_z}\right\|\rho^3$$

(c) We can solve part (b) using Divergence Theorem

$$\begin{aligned}\nabla\cdot\vec{F_1} &= \frac{\partial F_{1x}}{\partial x}+\frac{\partial F_{1y}}{\partial y}+\frac{\partial F_{1z}}{\partial z}\\ &= \frac{\partial F_{1z}}{\partial z}\\ &= 5\left\|\vec{a_z}\right\|\end{aligned}$$

Applying Divergence Theorem,

$$\begin{aligned}\text{Flux} = \iint_{\mathbb{R}}\left(\vec{F_1}\cdot\hat{\mathbf{n}}\right)dS = \iiint_{\mathbb{V}}\nabla\cdot\vec{F_1}\,dV &= 5\left\|\vec{a_z}\right\|\iiint_{\mathbb{V}}dV\\ &= 5\left\|\vec{a_z}\right\|\left(\frac{2\pi\rho^3}{3}\right)\\ &= \left\|\vec{a_z}\right\|\left(\frac{10\pi\rho^3}{3}\right)\end{aligned}$$

Problem 4

Owing to the cylindrically symmetric nature of the charge distribution, the electric field must be radially emerging outwards. Exploiting this symmetry, we can use Gauss' theorem,

$$\oiint \vec{E} \cdot d\vec{s} = \frac{Q_{enc}}{\epsilon_0}$$

Taking the Gaussian surface to be a cylinder with radius $0 < r < b$ and length l (going into the page in the following figure),

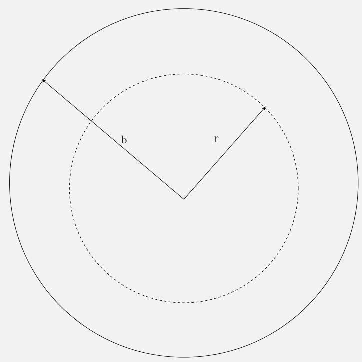


Figure 1: Top view of the cylindrical dielectric

Q_{enc} can be calculated as,

$$\begin{aligned}
Q_{enc} &= \int_{z=0}^{z=l} \int_{\theta=0}^{\theta=2\pi} \int_{\rho=0}^{\rho=r} \rho_v \rho \, d\rho \, dz \, d\theta \\
&= \int_{z=0}^{z=l} \int_{\theta=0}^{\theta=2\pi} \int_{\rho=0}^{\rho=r} a\rho^3 \, d\rho \, dz \, d\theta \\
&= \frac{\pi l r^4}{2}
\end{aligned}$$

Now, using this in Gauss' Law,

$$\begin{aligned}
\left\|\vec{E}\right\| \cdot (2\pi r l) &= \frac{\pi l r^4}{2\epsilon_0} \\
\implies \vec{E} &= \frac{r^3}{4\epsilon_0}\hat{\boldsymbol{\rho}}, \quad 0 < r < a
\end{aligned}$$

For $r \geq a$,

$$Q_{enc} = \frac{\pi l a^4}{2}$$

Again using Gauss' Law,

$$\begin{aligned}
\left\|\vec{E}\right\| \cdot (2\pi r l) &= \frac{\pi l a^4}{2\epsilon_0} \\
\implies \vec{E} &= \frac{a^4}{4r\epsilon_0}\hat{\boldsymbol{\rho}}
\end{aligned}$$

Problem 5

(a)

$$
\begin{aligned}
\left\|\vec{F}\right\| &= \sqrt{\left\|\vec{F_s}\right\|^2 + \left\|\vec{F_\phi}\right\|^2 + \left\|\vec{F_z}\right\|^2} \\
&= \sqrt{(4 + (3\left(\cos\phi + \sin\phi\right))^2 + 9\left(\cos\phi - \sin\phi\right)^2 + 4} \\
&= \sqrt{(16 + 9\left(1 + 2\sin\phi\cos\phi\right) + 12\left(\cos\phi + \sin\phi\right)) + 9\left(1 - 2\cos\phi\sin\phi\right) + 4} \\
&= \sqrt{(25 + 18\sin\phi\cos\phi) + 12\left(\cos\phi + \sin\phi\right) + 9 - 18\cos\phi\sin\phi + 4} \\
&= \sqrt{38 + 12\left(\cos\phi + \sin\phi\right)}
\end{aligned}
$$

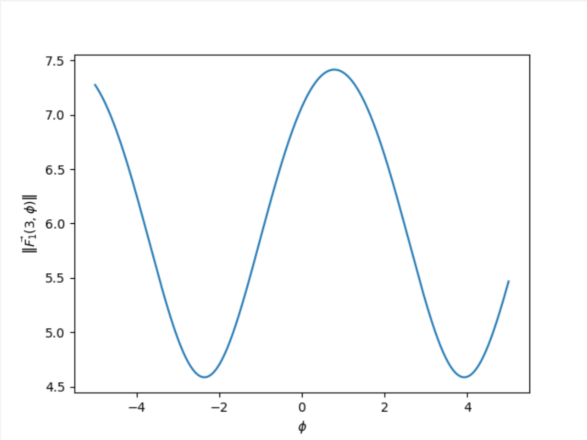


Figure 2: Field with $s = 3$ as a function of ϕ

(b)

$$
\begin{aligned}
\left\|\vec{F}\right\| &= \sqrt{\left\|\vec{F_s}\right\|^2 + \left\|\vec{F_\phi}\right\|^2 + \left\|\vec{F_z}\right\|^2} \\
&= \sqrt{\left(\frac{40}{s^2+1} + 3\sqrt{2}\right)^2 + 4}
\end{aligned}
$$

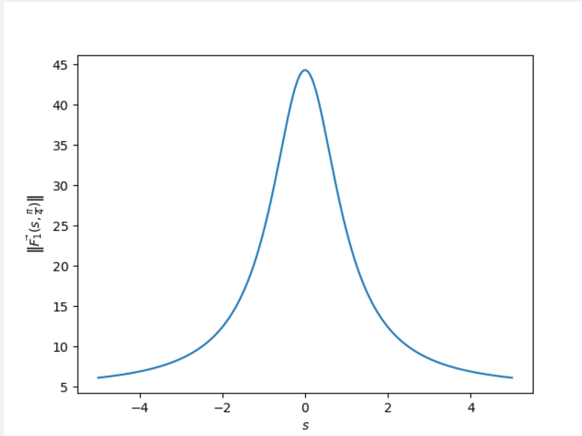


Figure 3: Field with $\phi = \frac{\pi}{4}$ as function of s

(c) Divergence in cylindrical coordinates is given by,

$$\nabla \cdot \vec{F} = \frac{1}{s}\frac{\partial (sF_s)}{\partial s} + \frac{1}{s}\frac{\partial A_\phi}{\partial \phi} + \frac{\partial A_z}{\partial z}$$

$$\begin{aligned}
\frac{\partial (sF_s)}{\partial s} &= \frac{d}{ds}\left(\frac{40s}{s^2+1} + 3s(\cos\phi + \sin\phi)\right) \\
&= \frac{40\left(s^2+1-2s^2\right)}{\left(s^2+1\right)^2} + 3(\cos\phi + \sin\phi) \\
&= \frac{40\left(1-s^2\right)}{\left(s^2+1\right)^2} + 3(\cos\phi + \sin\phi) \\
\frac{1}{s}\frac{\partial (sF_s)}{\partial s} &= \frac{40\left(1-s^2\right)}{s\left(s^2+1\right)^2} + \frac{3(\cos\phi + \sin\phi)}{s} \\
\frac{\partial A_\phi}{\partial \phi} &= -3(\sin\phi + \cos\phi) \\
\frac{1}{s}\frac{\partial A_\phi}{\partial \phi} &= \frac{-3(\sin\phi + \cos\phi)}{s} \\
\frac{\partial A_z}{\partial z} &= 0
\end{aligned}$$

Finally, the divergence is given by,

$$\nabla \cdot \vec{F} = \frac{40\left(1-s^2\right)}{s\left(s^2+1\right)^2}$$

(d) The curl of a vector field $\mathbf{F} = F_s\hat{s} + F_\phi\hat{\phi} + F_z\hat{z}$ in cylindrical coordinates (s, ϕ, z) is given by:

$$\nabla \times \mathbf{F} = \begin{vmatrix} \hat{s} & \hat{\phi} & \hat{z} \\ \frac{\partial}{\partial s} & \frac{1}{s}\frac{\partial}{\partial \phi} & \frac{\partial}{\partial z} \\ F_s & F_\phi & F_z \end{vmatrix}$$

Expanding this determinant, the components of the curl are:

$$(\nabla \times \mathbf{F})_s = \frac{1}{s}\left(\frac{\partial F_z}{\partial \phi} - \frac{\partial}{\partial z}(sF_\phi)\right)$$

$$(\nabla \times \mathbf{F})_\phi = \frac{\partial F_s}{\partial z} - \frac{\partial F_z}{\partial s}$$

$$(\nabla \times \mathbf{F})_z = \frac{1}{s}\left[\frac{\partial}{\partial s}(sF_\phi) - \frac{\partial F_s}{\partial \phi}\right]$$

$F_z = -2z \implies \frac{\partial F_z}{\partial \phi} = 0$. $F_\phi = 3(\cos\phi - \sin\phi)$, so:

$$\frac{\partial F_\phi}{\partial z} = 0, \quad sF_\phi = 3s(\cos\phi - \sin\phi)$$

$$\frac{\partial}{\partial z}(sF_\phi) = 0$$

Thus, $(\nabla \times \mathbf{F})_s = 0$.
$F_s = \frac{40}{s^2+1} + 3(\cos\phi + \sin\phi)$, so:

$$\frac{\partial F_s}{\partial z} = 0$$

$\frac{\partial F_z}{\partial s} = \frac{\partial(-2z)}{\partial s} = 0$
Thus, $(\nabla \times \mathbf{F})_\phi = 0$.
We already have $sF_\phi = 3s(\cos\phi - \sin\phi)$, so:

$$\frac{\partial}{\partial s}(sF_\phi) = \frac{\partial}{\partial s}[3s(\cos\phi - \sin\phi)] = 3(\cos\phi - \sin\phi)$$

For $\frac{\partial F_s}{\partial \phi}$:

$$\frac{\partial F_s}{\partial \phi} = \frac{\partial}{\partial \phi}\left[\frac{40}{s^2+1} + 3(\cos\phi + \sin\phi)\right]$$

$$= 3(-\sin\phi + \cos\phi)$$

Thus,

$$(\nabla \times \mathbf{F})_z = \frac{1}{s}\left[3(\cos\phi - \sin\phi) - 3(-\sin\phi + \cos\phi)\right] = 0.$$

Since $\nabla \cdot \vec{F} = 0$ field is conservative.

Problem 6

(a)

$$\vec{E_1} = \frac{1}{4\pi\epsilon_0}\left(\frac{x\hat{\mathbf{i}} + y\hat{\mathbf{j}} - 1\hat{\mathbf{j}}}{\left\|x\hat{\mathbf{i}} + y\hat{\mathbf{j}} - 1\hat{\mathbf{j}}\right\|^3}\right) = \frac{1}{4\pi\epsilon_0}\left(\frac{x\hat{\mathbf{i}} + y\hat{\mathbf{j}} - 1\hat{\mathbf{j}}}{\left(x^2 + (y-1)^2\right)^{\frac{3}{2}}}\right)$$

$$\vec{E_2} = \frac{1}{4\pi\epsilon_0}\left(\frac{x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + 1\hat{\mathbf{j}}}{\left\|x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + 1\hat{\mathbf{j}}\right\|^3}\right) = \frac{1}{4\pi\epsilon_0}\left(\frac{x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + 1\hat{\mathbf{j}}}{\left(x^2 + (y+1)^2\right)^{\frac{3}{2}}}\right)$$

$$\vec{E_3} = \frac{1}{4\pi\epsilon_0}\left(\frac{x\hat{\mathbf{i}} + y\hat{\mathbf{j}} - 1\hat{\mathbf{i}}}{\left\|x\hat{\mathbf{i}} + y\hat{\mathbf{j}} - 1\hat{\mathbf{i}}\right\|^3}\right) = \frac{1}{4\pi\epsilon_0}\left(\frac{x\hat{\mathbf{i}} + y\hat{\mathbf{j}} - 1\hat{\mathbf{i}}}{\left(y^2 + (x-1)^2\right)^{\frac{3}{2}}}\right)$$

$$\vec{E_4} = \frac{1}{4\pi\epsilon_0}\left(\frac{x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + 1\hat{\mathbf{i}}}{\left\|x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + 1\hat{\mathbf{i}}\right\|^3}\right) = \frac{1}{4\pi\epsilon_0}\left(\frac{x\hat{\mathbf{i}} + y\hat{\mathbf{j}} + 1\hat{\mathbf{i}}}{\left(y^2 + (x+1)^2\right)^{\frac{3}{2}}}\right)$$

Net field is given by,

$$\vec{E}_{net} = \vec{E_1} + \vec{E_2} + \vec{E_3} + \vec{E_4}$$

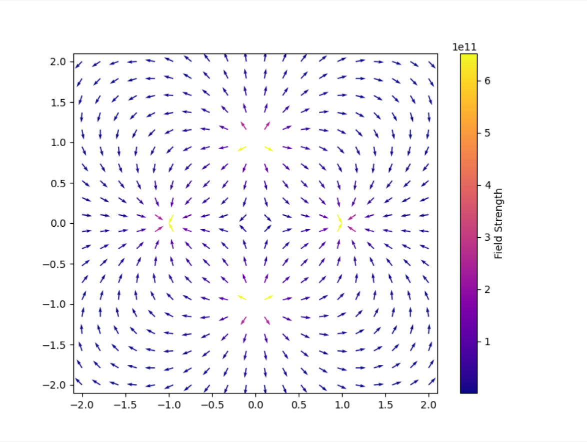


Figure 4: Field Map

(b)

$$\vec{E_1} = \frac{1}{4\pi\epsilon_0}\left(\frac{1}{\left(x^2+(y-1)^2\right)^{\frac{1}{2}}}\right)$$

$$\vec{E_2} = \frac{1}{4\pi\epsilon_0}\left(\frac{1}{\left(x^2+(y+1)^2\right)^{\frac{1}{2}}}\right)$$

$$\vec{E_3} = \frac{1}{4\pi\epsilon_0}\left(\frac{1}{\left(y^2+(x-1)^2\right)^{\frac{3}{2}}}\right)$$

$$\vec{E_4} = \frac{1}{4\pi\epsilon_0}\left(\frac{1}{\left(y^2+(x+1)^2\right)^{\frac{3}{2}}}\right)$$

Net potential is given by,

$$V_{net} = V_1 + V_2 + V_3 + V_4$$

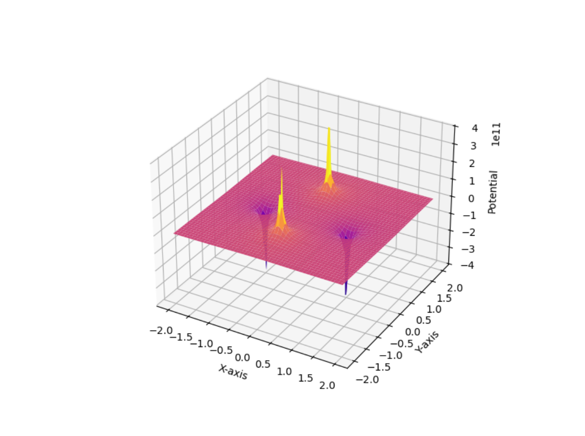


Figure 5: Potential Map

(c) The potential energy of a system of point charges is given by:

$$U = \frac{1}{4\pi\epsilon_0}\sum_{i<j}\frac{q_i q_j}{r_{ij}}.$$

Now summing up the contributions:

$$U = \frac{1}{4\pi\epsilon_0}\left[\frac{(+q)(-q)}{\sqrt{2}} + \frac{(+q)(-q)}{\sqrt{2}} + \frac{(+q)(-q)}{\sqrt{2}} + \frac{(+q)(-q)}{\sqrt{2}} + \frac{(+q)(+q)}{2} + \frac{(-q)(-q)}{2}\right].$$

Simplifying:

$$\begin{aligned} U &= \frac{1}{4\pi\epsilon_0}\left[-\frac{4q^2}{\sqrt{2}} + \frac{q^2}{2} + \frac{q^2}{2}\right] \\ &= \frac{1}{4\pi\epsilon_0}\left[-2\sqrt{2}q^2 + q^2\right] \\ &= \frac{q^2}{4\pi\epsilon_0}\left[1 - 2\sqrt{2}\right]. \end{aligned}$$

Thus, the potential energy of the configuration is:

$$U = \frac{q^2}{4\pi\epsilon_0}(1 - 2\sqrt{2}).$$

(d) The electric field $\vec{E}$ due to a point charge q at position $\mathbf{r}_0$ is given by:

$$\vec{E} = \frac{1}{4\pi\varepsilon_0}\frac{q(\vec{r} - \vec{r}_0)}{|\vec{r} - \vec{r}_0|^3}.$$

By the superposition principle, the total electric field $\mathbf{E}$ is the sum of the contributions from all four charges.

The divergence of the electric field is given by Gauss's law:

$$\nabla \cdot \vec{E} = \frac{\rho}{\varepsilon_0},$$

where $\rho(\mathbf{r})$ is the charge density.

Since we have point charges, we use the Dirac delta function representation of charge density:

$$\rho(\mathbf{r}) = q\delta(x)\delta(y-1) + q\delta(x)\delta(y+1) - q\delta(x-1)\delta(y) - q\delta(x+1)\delta(y).$$

Thus, applying Gauss's law:

$$\nabla \cdot \vec{E} = \frac{q}{\varepsilon_0}\left[\delta(x)\delta(y-1) + \delta(x)\delta(y+1) - \delta(x-1)\delta(y) - \delta(x+1)\delta(y)\right].$$

This expression shows that the divergence of $\vec{E}$ is nonzero only at the locations of the charges, reinforcing the fact that electric fields for point charges satisfy Gauss's law in differential form.

(e) The curl of a vector field $\vec{E}$ is given by:

$$\nabla \times \vec{E} = \begin{bmatrix} \frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} \\ \frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x} \\ \frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} \end{bmatrix}.$$

For a single point charge, the electric field components in Cartesian coordinates are:

$$E_x = \frac{q}{4\pi\varepsilon_0}\frac{x - x_0}{[(x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2]^{3/2}},$$

$$E_y = \frac{q}{4\pi\varepsilon_0} \frac{y - y_0}{[(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2]^{3/2}},$$

$$E_z = \frac{q}{4\pi\varepsilon_0} \frac{z - z_0}{[(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2]^{3/2}}.$$

Computing the partial derivatives:

$$\frac{\partial E_x}{\partial y} = \frac{q}{4\pi\varepsilon_0} \left[\frac{-3(x - x_0)(y - y_0)}{(r^2)^{5/2}} \right],$$

$$\frac{\partial E_x}{\partial z} = \frac{q}{4\pi\varepsilon_0} \left[\frac{-3(x - x_0)(z - z_0)}{(r^2)^{5/2}} \right],$$

$$\frac{\partial E_y}{\partial x} = \frac{q}{4\pi\varepsilon_0} \left[\frac{-3(y - y_0)(x - x_0)}{(r^2)^{5/2}} \right],$$

$$\frac{\partial E_y}{\partial z} = \frac{q}{4\pi\varepsilon_0} \left[\frac{-3(y - y_0)(z - z_0)}{(r^2)^{5/2}} \right],$$

$$\frac{\partial E_z}{\partial x} = \frac{q}{4\pi\varepsilon_0} \left[\frac{-3(z - z_0)(x - x_0)}{(r^2)^{5/2}} \right],$$

$$\frac{\partial E_z}{\partial y} = \frac{q}{4\pi\varepsilon_0} \left[\frac{-3(z - z_0)(y - y_0)}{(r^2)^{5/2}} \right].$$

Substituting these into the curl expression:

$$\frac{\partial E_z}{\partial y} - \frac{\partial E_y}{\partial z} = 0,$$

$$\frac{\partial E_x}{\partial z} - \frac{\partial E_z}{\partial x} = 0,$$

$$\frac{\partial E_y}{\partial x} - \frac{\partial E_x}{\partial y} = 0.$$

Thus, for a single charge:

$$\nabla \times \vec{E} = \vec{0}.$$

Since the curl of the field due to a single point charge is zero, and the curl operator is linear, the total electric field, being a superposition of individual point charge fields, also has zero curl:

$$\nabla \times \vec{E} = \vec{0}.$$

Problem 7

(a) From Poisson's Equation we know,

$$\nabla^2 V = -\frac{\rho}{\epsilon_0}$$

Writing the Laplacian in spherical coordinates,

$$\nabla^2 f = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial f}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial f}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 f}{\partial\phi^2}.$$

We can see that

$$\frac{\partial V}{\partial\theta} = \frac{\partial V}{\partial\phi} = 0$$

And,

$$\begin{aligned}
\frac{\partial V}{\partial r} &= -\frac{V_0}{a}e^{\frac{-r}{a}} \\
r^2\frac{\partial V}{\partial r} &= -\frac{V_0 r^2}{a}e^{\frac{-r}{a}} \\
\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right) &= \frac{-V_0}{a}e^{\frac{-r}{a}}\left(2r - \frac{r^3}{a}\right) \\
\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right) &= \frac{-V_0}{a}e^{\frac{-r}{a}}\left(\frac{2}{r} - \frac{r}{a}\right)
\end{aligned}$$

Therefore,

$$\begin{aligned}
\rho_v(r) &= \frac{V_0\epsilon_0}{a}e^{\frac{-r}{a}}\left(\frac{2}{r} - \frac{r}{a}\right) \\
\rho_v(a) &= \frac{V_0\epsilon_0}{a}e^{-1}\left(\frac{2}{a} - 1\right)
\end{aligned}$$

(b) We know,

$$\begin{aligned}
\vec{E} &= -\nabla V \\
&= -\left(\frac{\partial V}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial V}{\partial\theta}\hat{\theta} + \frac{1}{r\sin\theta}\frac{\partial V}{\partial\phi}\hat{\phi}\right) \\
&= \frac{V_0}{a}e^{\frac{-r}{a}}\hat{r}
\end{aligned}$$

(c) To calculate the total charge,

$$\begin{aligned}
Q_{tot} &= \epsilon_0\lim_{r\to\infty}\left(4\pi r^2\right)\left(\frac{V_0}{a}e^{\frac{-r}{a}}\right) \\
&= 0
\end{aligned}$$

Problem 8

Due to the infinite size of the parallel plates in the x-y plane, the electric potential and electric field will vary only with z as all points in the x-y plane are identical in terms of field and potential.

(a) With this information, we write Poisson's Equation,

$$\nabla^2 V = \frac{-\rho}{\epsilon_0}$$
$$\implies \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = \frac{-\rho}{\epsilon_0}$$

With,

$$\frac{\partial^2 V}{\partial x^2} = 0$$
$$\frac{\partial^2 V}{\partial y^2} = 0$$

Now,

$$\frac{\partial^2 V}{\partial z^2} = \frac{-\rho_0}{\epsilon_0}$$
$$\implies V(x, y, z) = \frac{-\rho_0 z^2}{2\epsilon_0} + C_1 z + C_2$$

Using the boundary conditions,

$$V(x, y, 0) = 0$$
$$V(x, y, d) = 0$$

$$\implies C_2 = 0$$
$$\text{and} \quad C_1 = \frac{\rho_0 d}{2\epsilon_0}$$

Finally,

$$V(x, y, z) = \frac{-\rho_0 z^2}{2\epsilon_0} + \frac{\rho_0 dz}{2\epsilon_0}$$

(b) We know,

$$\vec{E} = -\nabla V = -\frac{\partial V}{\partial z}\hat{\mathbf{k}}$$
$$= \hat{\mathbf{k}}\frac{\partial}{\partial z}\left(\frac{\rho_0 z^2}{2\epsilon_0} - \frac{\rho_0 dz}{2\epsilon_0}\right)$$
$$= \frac{\rho_0}{\epsilon_0}\left(z - \frac{d}{2}\right)\hat{\mathbf{k}}$$

(c) (a) In this problem we just change one of the boundary condition, namely,

$$V(x, y, d) = V_0$$

So we will still get

$$C_2 = 0$$

due to the previous boundary condition. Using the new boundary condition, we get

$$\begin{aligned} V_0 &= \frac{-\rho_0 d^2}{2\epsilon_0} + C_1 d \\ \implies C_1 &= \frac{1}{d}\left(V_0 + \frac{\rho_0 d^2}{2\epsilon_0}\right) \end{aligned}$$

So the new expression for the potential is,

$$V(x, y, z) = \frac{-\rho_0 z^2}{2\epsilon_0} + \frac{z}{d}\left(V_0 + \frac{\rho_0 d^2}{2\epsilon_0}\right)$$

(b) Again

$$\begin{aligned} \vec{E} = -\nabla V &= -\frac{\partial V}{\partial z}\hat{\mathbf{k}} \\ &= \hat{\mathbf{k}}\frac{\partial}{\partial z}\left(\frac{-\rho_0 z^2}{2\epsilon_0} + \frac{z}{d}\left(V_0 + \frac{\rho_0 d^2}{2\epsilon_0}\right)\right) \\ &= \left(\frac{-\rho_0 z}{\epsilon_0} + \frac{1}{d}\left(V_0 + \frac{\rho_0 d^2}{2\epsilon_0}\right)\right)\hat{\mathbf{k}} \end{aligned}$$

The following are the source codes for the simulations and plots,

---

```
 # Code for field map simulation
import numpy as np
import matplotlib.pyplot as plt

# Grid size
N = 20

x_values = np.linspace(-2, 2, N)
y_values = np.linspace(-2, 2, N)
X, Y = np.meshgrid(x_values, y_values)

# Charges positions and Coulomb constant
charges = [
    (1, 0, 1),  # positive charge at (1,0)
    (-1, 0, 1), # positive charge at (-1,0)
    (0, 1, -1), # negative charge at (0,1)
    (0, -1, -1) # negative charge at (0,-1)
```

```
]

k = 8.99e9 # Coulomb constant

# Initialize field components
Ex, Ey = np.zeros(X.shape), np.zeros(Y.shape)

for x_q, y_q, q in charges:
    dx = X - x_q
    dy = Y - y_q
    r_squared = dx**2 + dy**2
    r_squared[r_squared == 0] = np.nan # Avoid division by zero
    r_power = np.sqrt(r_squared) ** 3

    Ex += k * q * dx / r_power
    Ey += k * q * dy / r_power

# Calculate field strength and normalize for quiver plot
E_strength = np.sqrt(Ex**2 + Ey**2)
Ex /= E_strength
Ey /= E_strength

fig, ax = plt.subplots(figsize=(8, 6))
quiver = ax.quiver(X, Y, Ex, Ey, E_strength, cmap=plt.cm.plasma,
    norm=plt.Normalize(vmin=np.nanmin(E_strength), vmax=np.nanmax(E_strength)))

# Colorbar
cbar = plt.colorbar(quiver, ax=ax)
cbar.set_label("Field Strength")

plt.xlim(-2.1, 2.1)
plt.ylim(-2.1, 2.1)
plt.savefig("../figs/field_map.png")
plt.show()
```

```
# Code for Potential Simulation
import numpy as np
import matplotlib.pyplot as plt

# Constants
k = 8.99e9 # Coulomb constant

# Charge positions and values
charges = [
    (0, 1, 1),   # +1C at (0, +1)
    (0, -1, 1),  # +1C at (0, -1)
    (1, 0, -1),  # -1C at (+1, 0)
    (-1, 0, -1)  # -1C at (-1, 0)
]

# Define grid for the x-y plane
x = np.linspace(-2, 2, 100)
y = np.linspace(-2, 2, 100)
```

```
X, Y = np.meshgrid(x, y)

# Compute potential at each point on the grid
V = np.zeros_like(X)
for charge in charges:
    x_c, y_c, q = charge
    R = np.sqrt((X - x_c)**2 + (Y - y_c)**2)
    R[R == 0] = 1e-9
    V += k * q / R

# Create 3D plot
fig = plt.figure(figsize=(8, 6))
ax = fig.add_subplot(111, projection="3d")
ax.plot_surface(X, Y, V, cmap="plasma", edgecolor="none")
ax.set_xlabel("X-axis")
ax.set_ylabel("Y-axis")
ax.set_zlabel("Potential")
plt.savefig("../figs/potential.png")
plt.show()
```

```
# Code for Problem 4a
import matplotlib.pyplot as plt
import numpy as np

x = np.arange(-5, 5, 1e-5)
y = np.sqrt(38 + 12 * (np.cos(x) + np.sin(x)))

plt.plot(x, y)
plt.xlabel("$\phi$")
plt.ylabel(r"$\left\Vert\vec{F_1}(3, \phi)\right\Vert$")
plt.savefig("../figs/4a.png")

plt.show()
```

```
# Code for Problem 4b
import matplotlib.pyplot as plt
import numpy as np

x = np.arange(-5, 5, 1e-5)
y = np.sqrt(((40)/(x**2+1) + 3*np.sqrt(2))**2 + 4)

plt.plot(x, y)
plt.xlabel("$s$")
plt.ylabel(r"$\left\Vert\vec{F_1}(s, \frac{\pi}{4})\right\Vert$")
plt.savefig("../figs/4b.png")

plt.show()
```