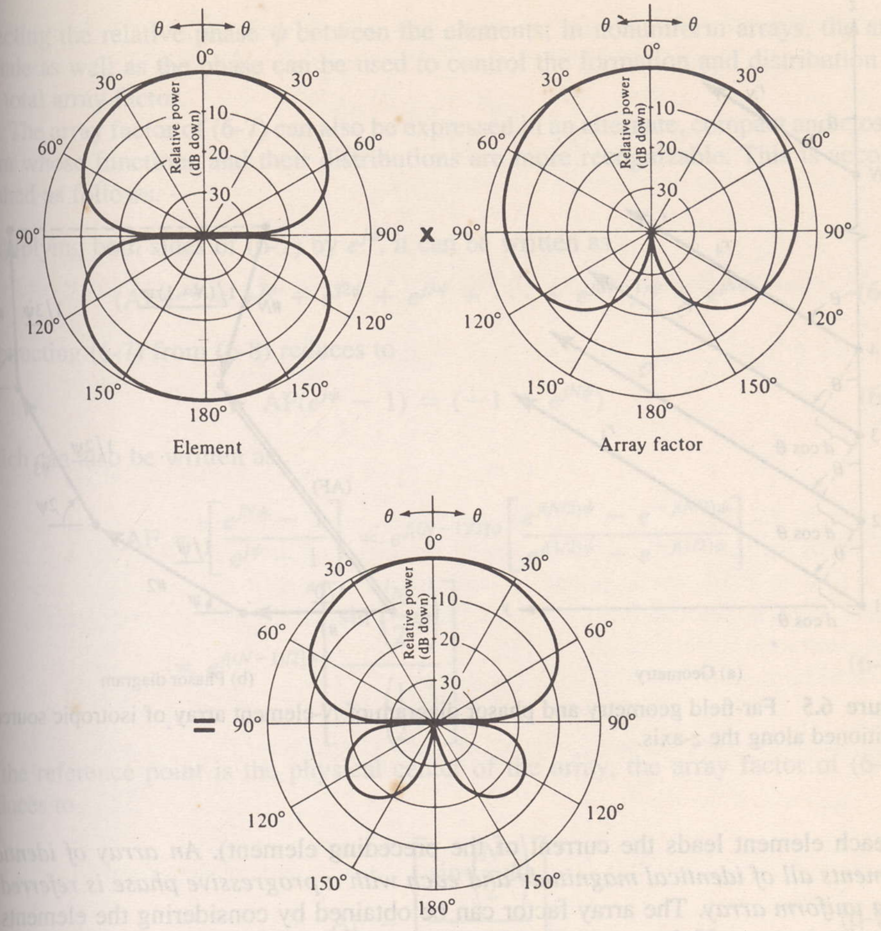

Figure 6.4 (b) Continued ($\beta = -90°$, $d = \lambda/4$).

The null at $\theta = 90°$ is attributed to the pattern of the individual elements of the array while the remaining ones are due to the formation of the array. For no phase difference between the elements ($\beta = 0$), the separation d must be equal or greater than half a wavelength ($d \geq \lambda/2$) in order for at least one null, due to the formation of the array, to occur.

6.3 *N*-ELEMENT LINEAR ARRAY: UNIFORM AMPLITUDE AND SPACING

Now that the arraying of elements has been introduced and it was illustrated by the two-element array, let us generalize the method to include N elements. Referring to the geometry of Figure 6.5(a), let us assume that all the elements have identical amplitudes but each succeeding element has a β progressive phase lead current excitation relative to the preceding one (β represents the phase by which the current

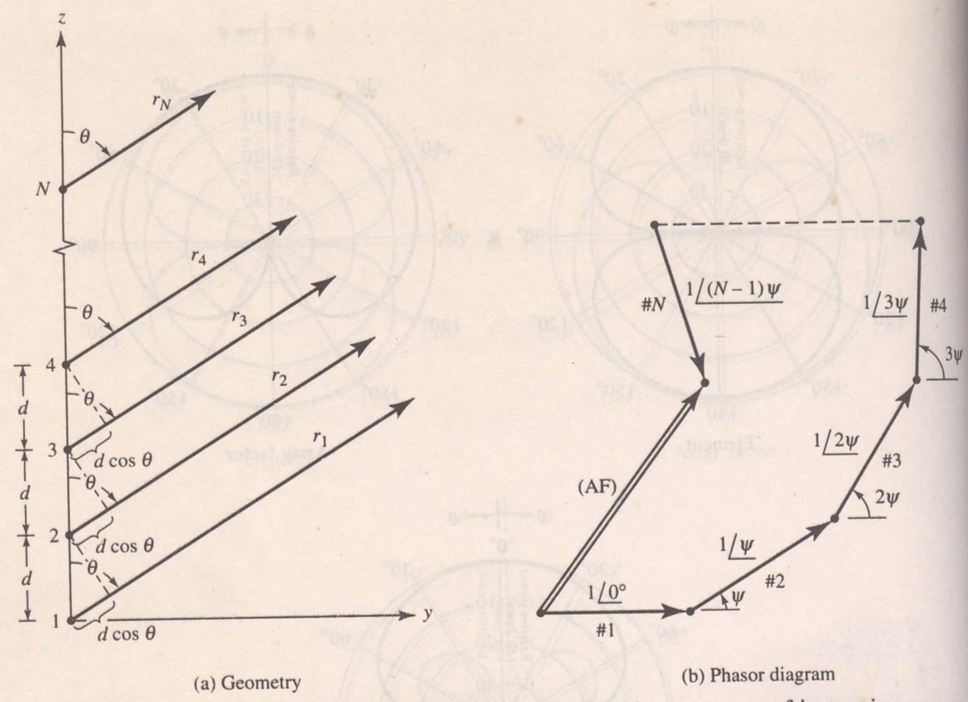

Figure 6.5 Far-field geometry and phasor diagram of N-element array of isotropic sources positioned along the z-axis.

in each element leads the current of the preceding element). *An array of identical elements all of identical magnitude and each with a progressive phase is referred to as a uniform array.* The array factor can be obtained by considering the elements to be point sources. If the actual elements are not isotropic sources, the total field can be formed by multiplying the array factor of the isotropic sources by the field of a single element. This is the pattern multiplication rule of (6-5), and it applies only for arrays of identical elements. The array factor is given by

$$\text{AF} = 1 + e^{+j(kd\cos\theta+\beta)} + e^{+j2(kd\cos\theta+\beta)} + \cdots + e^{j(N-1)(kd\cos\theta+\beta)}$$
$$\text{AF} = \sum_{n=1}^{N} e^{j(n-1)(kd\cos\theta+\beta)} \tag{6-6}$$

which can be written as

$$\boxed{\text{AF} = \sum_{n=1}^{N} e^{j(n-1)\psi} \qquad (6\text{-}7) \\ \text{where} \quad \psi = kd\cos\theta + \beta \qquad (6\text{-}7a)}$$

Since the total array factor for the uniform array is a summation of exponentials, it can be represented by the vector sum of N phasors each of unit amplitude and progressive phase ψ relative to the previous one. Graphically this is illustrated by the phasor diagram in Figure 6.5(b). It is apparent from the phasor diagram that the amplitude and phase of the AF can be controlled in uniform arrays by properly selecting the relative phase ψ between the elements; in nonuniform arrays, the amplitude as well as the phase can be used to control the formation and distribution of the total array factor.

The array factor of (6-7) can also be expressed in an alternate, compact and closed form whose functions and their distributions are more recognizable. This is accomplished as follows.

Multiplying both sides of (6-7) by $e^{j\psi}$, it can be written as

$$(\text{AF})e^{j\psi} = e^{j\psi} + e^{j2\psi} + e^{j3\psi} + \cdots + e^{j(N-1)\psi} + e^{jN\psi} \tag{6-8}$$

Subtracting (6-7) from (6-8) reduces to

$$\text{AF}(e^{j\psi} - 1) = (-1 + e^{jN\psi}) \tag{6-9}$$

which can also be written as

$$\begin{aligned} \text{AF} &= \left[\frac{e^{jN\psi} - 1}{e^{j\psi} - 1}\right] = e^{j[(N-1)/2]\psi}\left[\frac{e^{j(N/2)\psi} - e^{-j(N/2)\psi}}{e^{j(1/2)\psi} - e^{-j(1/2)\psi}}\right] \\ &= e^{j[(N-1)/2]\psi}\left[\frac{\sin\left(\frac{N}{2}\psi\right)}{\sin\left(\frac{1}{2}\psi\right)}\right] \end{aligned} \tag{6-10}$$

If the reference point is the physical center of the array, the array factor of (6-10) reduces to

$$\text{AF} = \left[\frac{\sin\left(\frac{N}{2}\psi\right)}{\sin\left(\frac{1}{2}\psi\right)}\right] \tag{6-10a}$$

For small values of ψ, the above expression can be approximated by

$$\text{AF} \simeq \left[\frac{\sin\left(\frac{N}{2}\psi\right)}{\frac{\psi}{2}}\right] \tag{6-10b}$$

The maximum value of (6-10a) or (6-10b) is equal to N. To normalize the array factors so that the maximum value of each is equal to unity, (6-10a) and (6-10b) are written in normalized form as (see Appendix II)

$$\boxed{(\text{AF})_n = \frac{1}{N}\left[\frac{\sin\left(\frac{N}{2}\psi\right)}{\sin\left(\frac{1}{2}\psi\right)}\right]} \tag{6-10c}$$

and (see Appendix I)

$$\boxed{(\mathrm{AF})_n \simeq \left[\frac{\sin\left(\frac{N}{2}\psi\right)}{\frac{N}{2}\psi}\right]} \tag{6-10d}$$

To find the nulls of the array, (6-10c) or (6-10d) are set equal to zero. That is,

$$\sin\left(\frac{N}{2}\psi\right) = 0 \Rightarrow \frac{N}{2}\psi\Big|_{\theta=\theta_n} = \pm n\pi \Rightarrow \theta_n = \cos^{-1}\left[\frac{\lambda}{2\pi d}\left(-\beta \pm \frac{2n}{N}\pi\right)\right] \tag{6-11}$$

$$n = 1, 2, 3, \ldots$$
$$n \neq N, 2N, 3N, \ldots \text{ with (6-10c)}$$

For $n = N, 2N, 3N, \ldots$, (6-10c) attains its maximum values because it reduces to a $\sin(0)/0$ form. The values of n determine the order of the nulls (first, second, etc.). For a zero to exist, the argument of the arccosine cannot exceed unity. Thus the number of nulls that can exist will be a function of the element separation d and the phase excitation difference β.

The maximum values of (6-10c) occur when

$$\frac{\psi}{2} = \frac{1}{2}(kd\cos\theta + \beta)|_{\theta=\theta_m} = \pm m\pi \Rightarrow \theta_m = \cos^{-1}\left[\frac{\lambda}{2\pi d}(-\beta \pm 2m\pi)\right]$$

$$m = 0, 1, 2, \ldots \tag{6-12}$$

The array factor of (6-10d) has only one maximum and occurs when $m = 0$ in (6-12). That is,

$$\theta_m = \cos^{-1}\left(\frac{\lambda\beta}{2\pi d}\right) \tag{6-13}$$

which is the observation angle that makes $\psi = 0$.

The 3-dB point for the array factor of (6-10d) occurs when (see Appendix I)

$$\frac{N}{2}\psi = \frac{N}{2}(kd\cos\theta + \beta)|_{\theta=\theta_h} = \pm 1.391$$

$$\Rightarrow \theta_h = \cos^{-1}\left[\frac{\lambda}{2\pi d}\left(-\beta \pm \frac{2.782}{N}\right)\right] \tag{6-14}$$

which can also be written as

$$\theta_h = \frac{\pi}{2} - \sin^{-1}\left[\frac{\lambda}{2\pi d}\left(-\beta \pm \frac{2.782}{N}\right)\right] \tag{6-14a}$$

For large values of $d(d \gg \lambda)$, it reduces to

$$\theta_h \simeq \left[\frac{\pi}{2} - \frac{\lambda}{2\pi d}\left(-\beta \pm \frac{2.782}{N}\right)\right] \tag{6-14b}$$