mentioned in Sec. 1–2, this technique was introduced by Lord Kelvin and developed fully by Oliver Heaviside. Essentially, it is an extension of ac circuit theory to lines having distributed circuit elements. A disadvantage of this method is that it reveals little about the electromagnetic field pattern or other possible modes of propagation. However, it does describe the impedance and propagation characteristics of the line for the principal mode of transmission and hence is of considerable engineering value.

The following quantities may be defined for a uniform transmission line.

$R' \equiv$ Series resistance per unit length of line (ohm/m)
$G' \equiv$ Shunt conductance per unit length of line (mho/m)
$L' \equiv$ Series inductance per unit length of line (H/m)
$C' \equiv$ Shunt capacitance per unit length of line (F/m)

The quantity R' is related to the dimensions and conductivity of the metallic conductors. Because of skin effect, it is also a function of frequency. G' is related to the loss tangent of the insulating material between the conductors.[1] L' is associated with the magnetic flux linking the conductors, while C' is associated with the charge on the conductors. Expressions for the distributed elements of various transmission lines are given in Chapter 5.

With this concept of distributed elements, a uniform transmission line may be modeled by the circuit representation in Fig. 3–1. The line is pictured as a cascade of identical sections, each Δz long. Each section consists of series inductance and resistance ($L'\Delta z$ and $R'\Delta z$) as well as shunt capacitance and conductance ($C'\Delta z$ and $G'\Delta z$). Since Δz can always be chosen small compared to the operating wavelength, an individual section of line may be analyzed using ordinary ac circuit theory. In the derivation that follows, $\Delta z \rightarrow 0$ and hence the results are valid at all frequencies.

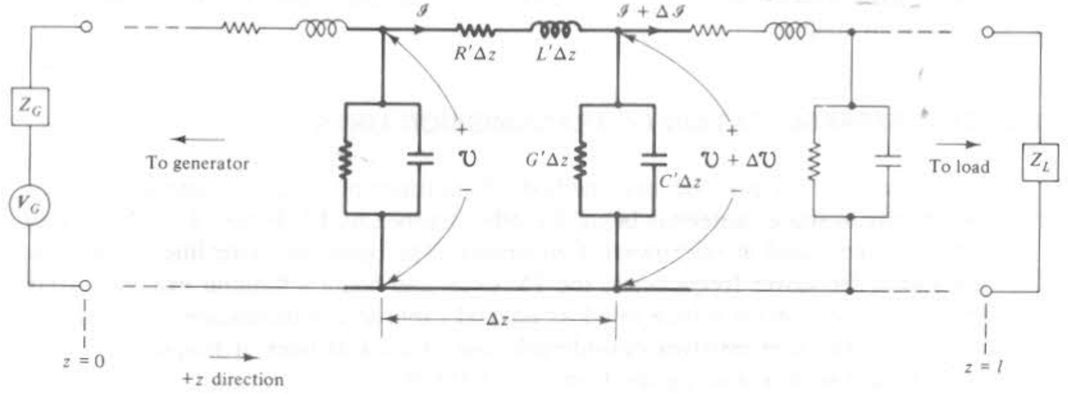


Figure 3–1 Circuit representation of a uniform transmission line.

[1] It is important to note that G' is *not* the reciprocal of R'. They are independent quantities, R' being related to the properties of the two conductors and G' to the characteristics of the insulating material between them.

In the figure, $\mathcal{V}$ and $\mathcal{I}$ represent the time-varying voltage and current at the input of a line section, while $\mathcal{V} + \Delta\mathcal{V}$ and $\mathcal{I} + \Delta\mathcal{I}$ represent the output values. The positive z direction is taken as horizontal and to the right, that is, from the generator toward the load. Also indicated are the assumed positive directions for the currents and voltages.

Applying Kirchhoff's voltage and current laws to the line section yields

$$\mathcal{V} = (R'\Delta z)\mathcal{I} + (L'\Delta z)\frac{\partial \mathcal{I}}{\partial t} + (\mathcal{V} + \Delta\mathcal{V})$$

and

$$\mathcal{I} = (G'\Delta z)(\mathcal{V} + \Delta\mathcal{V}) + (C'\Delta z)\frac{\partial}{\partial t}(\mathcal{V} + \Delta\mathcal{V}) + (\mathcal{I} + \Delta\mathcal{I})$$

Simplifying and recognizing that as $\Delta z \to 0$, $\mathcal{V} + \Delta\mathcal{V} \to \mathcal{V}$ results in the following partial differential equations.

$$-\frac{\partial \mathcal{V}}{\partial z} = R'\mathcal{I} + L'\frac{\partial \mathcal{I}}{\partial t} \qquad \text{and} \qquad -\frac{\partial \mathcal{I}}{\partial z} = G'\mathcal{V} + C'\frac{\partial \mathcal{V}}{\partial t} \tag{3-1}$$

By taking $\partial/\partial z$ of the first equation and $\partial/\partial t$ of the second equation and eliminating $\partial\mathcal{I}/\partial z$ and $\partial^2\mathcal{I}/\partial z\,\partial t$, a second-order differential equation for voltage is obtained.

$$\frac{\partial^2 \mathcal{V}}{\partial z^2} = L'C'\frac{\partial^2 \mathcal{V}}{\partial t^2} + (R'C' + G'L')\frac{\partial \mathcal{V}}{\partial t} + R'G'\mathcal{V} \tag{3-2}$$

Solving for current in a similar manner yields

$$\frac{\partial^2 \mathcal{I}}{\partial z^2} = L'C'\frac{\partial^2 \mathcal{I}}{\partial t^2} + (R'C' + G'L')\frac{\partial \mathcal{I}}{\partial t} + R'G'\mathcal{I} \tag{3-3}$$

The solution of either of these second-order equations and Eq. (3-1), together with the electrical properties of the generator and load, allow us to determine the instantaneous voltage and current at any time t and any place z along the uniform transmission line.

For the case of perfect conductors ($R' = 0$) and insulators ($G' = 0$), the above equations reduce to

$$\frac{\partial^2 \mathcal{V}}{\partial z^2} = L'C'\frac{\partial^2 \mathcal{V}}{\partial t^2} \qquad \text{and} \qquad \frac{\partial^2 \mathcal{I}}{\partial z^2} = L'C'\frac{\partial^2 \mathcal{I}}{\partial t^2} \tag{3-4}$$

while Eqs. (3-1) reduce to

$$-\frac{\partial \mathcal{V}}{\partial z} = L'\frac{\partial \mathcal{I}}{\partial t} \qquad \text{and} \qquad -\frac{\partial \mathcal{I}}{\partial z} = C'\frac{\partial \mathcal{V}}{\partial t} \tag{3-5}$$

Equations (3-4) and (3-5) represent the differential equations for a lossless line. Although real lines are never without loss, there are many in which it is sufficiently small that the lossless solution represents an excellent approximation.

Equations (3-4) are forms of the well-known wave equation of mathematical physics. We have already encountered it in Eq. (2-45). It was shown that the solution represented electromagnetic waves traveling in the plus and minus z directions with a velocity given by Eq. (2–53). The solutions of Eqs. (3–4) also represent traveling waves. In this case, they are voltage and current waves that travel with a velocity given by

$$v = \frac{1}{\sqrt{L'C'}} \tag{3-6}$$

In general, Eqs. (3–4) are satisfied by single-valued functions of the form $f(t \pm \sqrt{L'C'}\,z)$, where the plus sign indicates propagation in the *negative* z direction and the minus sign propagation in the *positive* z direction. To understand the meaning of these solutions, assume $\mathcal{V} = f(t - \sqrt{L'C'}\,z)$. At the point $z = 0$, the voltage versus time function is given by $\mathcal{V} = f(t)$. Further down the z axis at a point $z = z_1$, $\mathcal{V} = f(t - \sqrt{L'C'}\,z_1)$, which is exactly the same as $f(t)$ except that it has been time delayed by $t_d = \sqrt{L'C'}\,z_1$. Thus, it appears that the voltage versus time function at $z = 0$ has moved to $z = z_1$ with a velocity $v = z_1/t_d = 1/\sqrt{L'C'}$, which is exactly Eq. (3–6). By a similar argument, the $f(t + \sqrt{L'C'}\,z)$ solution represents a voltage function traveling in the negative z direction. In like manner, the solutions of the current equation may be interpreted as forward and reverse traveling current functions having the same velocity as the voltage. A similar conclusion was arrived at regarding the $\mathcal{E}$ and $\mathcal{H}$ waves discussed in Sec. 2–4, the explanation being that $\mathcal{E}$ generated $\mathcal{H}$ and vice versa. The voltage and current waves also travel with the same velocity since $\mathcal{V}$ and $\mathcal{I}$ generate each other. A physical explanation is presented in the next section to show the reasonableness of this conclusion.

Another result given in Sec. 2–4 is that the ratio of $\mathcal{E}$ to $\mathcal{H}$ for the traveling waves is a constant (η) which is a function of the electric and magnetic properties of the medium. Similarly, the ratio of $\mathcal{V}$ to $\mathcal{I}$ for a traveling wave on a transmission line is a constant. This constant is called the *characteristic impedance* (Z_0) of the line. For a lossless line, it is given by

$$Z_0 = \sqrt{\frac{L'}{C'}} \qquad \text{ohms} \tag{3-7}$$

To verify this expression, let $\mathcal{V} = f_1(u)$ and $\mathcal{I} = f_2(u)$, where $u \equiv t - \sqrt{L'C'}\,z$. Since

$$\frac{\partial \mathcal{V}}{\partial z} = \frac{\partial f_1}{\partial u}\cdot\frac{\partial u}{\partial z} = -\sqrt{L'C'}\,\frac{\partial f_1}{\partial u} \qquad \text{and} \qquad \frac{\partial \mathcal{I}}{\partial t} = \frac{\partial f_2}{\partial u}\cdot\frac{\partial u}{\partial t} = \frac{\partial f_2}{\partial u}$$

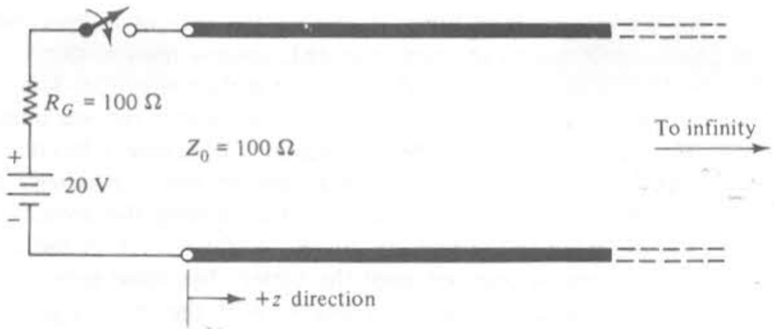
substitution into the first of Eqs. (3–5) yields $\sqrt{L'C'}\,\partial f_1/\partial u = L'\partial f_2/\partial u$. Integration with respect to u and simplifying results in $f_1/f_2 = \mathcal{V}/\mathcal{I} = \sqrt{L'/C'}$, which is Eq. (3–7).

It will be shown that Z_0 is a function of the cross-sectional dimensions of the line as well as the electrical properties of the insulating material between the conductors.

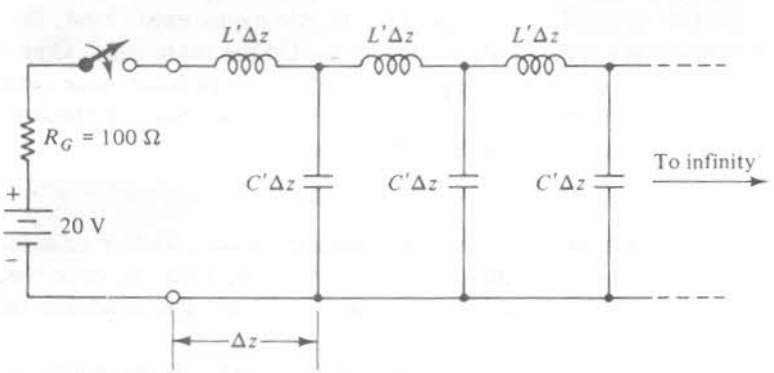
3-2 TRANSIENTS ON A TRANSMISSION LINE

We have seen that voltage and current waves travel along a transmission line with the same velocity. The physical argument presented here is intended to verify this fact while giving additional insight into the process of wave propagation along uniform lines.

Figure 3-2a shows a 20 V battery with an internal resistance of 100 ohms connected through a switch to an infinitely long transmission line with $Z_0 = 100$ ohms. Part *b* of the figure shows the same circuit with the transmission line replaced by its equivalent circuit representation. When the switch is closed, a voltage appears immediately at the input of the transmission line. However, it cannot appear instantaneously at other points along the line, since that would require a sudden change in voltage on all the capacitances. Furthermore, since it is current that delivers the electric charge to the capacitances, a sudden increase in current through the inductances would also be necessary. Since inductance opposes a current change and capacitance opposes a voltage change, the voltage and current require a finite time to propagate along the transmission line. The propagation process can be described in



(a) Transmission-line circuit



(b) Equivalent circuit representation

Figure 3–2 A dc source connected to an infinitely long, lossless line.

the following manner. When the switch is closed, the first inductance generates a back emf, in accordance with Lenz's law, to initially oppose an increase in current. Eventually, however, current flows through $L'\Delta z$ and charges the first shunt capacitance $C'\Delta z$ to a voltage V. The charged capacitor now acts like a voltage source and forces current through the next inductor. This charges the next capacitor and the process continues down the line. From this argument, it is apparent that voltage creates current and vice versa, thus requiring that they travel together along the transmission line. Since the line is infinitely long, only forward traveling waves of voltage and current exist and their ratio is given by Z_0 (100 ohms in this case). As time progresses, the battery continues to supply the current needed to charge the never-ending line of shunt capacitances. Thus, in the steady state, the infinite line presents an impedance of Z_0 to the battery. The current supplied by the battery is $20/(R_G + Z_0) = 0.10$ A. With half of the 20 V dropped across R_G, the voltage at the input to the line is 10 V. This 10 V voltage wave and its accompanying 0.10 A current wave travel in the positive z direction with a velocity given by Eq. (3–6).

Let us now look at some examples of finite length lines with various terminations.

An open-circuited line. Consider the case of a 20 V battery with $R_G =$ 100 ohms connected to an open-circuited, lossless transmission line. This situation, with specific values of l, v and Z_0, is shown at the top of Fig. 3–3. With the initially open switch closed at $t = 0$, the voltage at the input to the line immediately becomes 10 V. This occurs because at the first instant, the dc source has no indication that the line is *not* infinite in length and hence sees an input impedance $Z_0 = 100$ ohms.[2] Thus at $t = 0+$ (that is, immediately after closing the switch), the current and voltage at the input to the line are $20/(R_G + Z_0) = 0.10$ A and 10 V, respectively. These values remain constant until the battery has some indication (via a reflected wave) that the line is not infinite in length. With the velocity given as 2×10^8 m/s, it takes 10 ns for V and I to travel halfway down the 4 m line. This situation is shown in part a of Fig. 3–3. Part b shows the waves at $t = 20-$ ns (that is, slightly less than 20 ns). When the waves arrive at the open circuit, something must happen since two contradictory impedance requirements exist. First, the V/I ratio for the traveling wave must be $Z_0 = 100$ ohms. On the other hand, Ohm's law at the open-circuited end of the line requires an infinite impedance since current must be zero. The creation of reflected waves (V^-, I^-) allows both of these requirements to be satisfied. Thus at the load end ($z = 4$ m),

$$V_L = V^+ + V^- \qquad \text{and} \qquad I_L = I^+ - I^- = 0$$

where the + and − superscripts indicate forward and reverse traveling waves.[3] V_L and I_L represent the voltage and current at the load end once the forward traveling waves have arrived, which occurs at $t = 20$ ns. The condition that $I_L = 0$ requires

[2] Since Z_0 represents the impedance of the line when the switch is first closed, it is sometimes referred to as the *surge impedance* of the transmission line.

[3] The reason for the minus sign in the equation for I_L is that for the forward and reverse traveling waves, I^+ and I^- are oppositely directed.

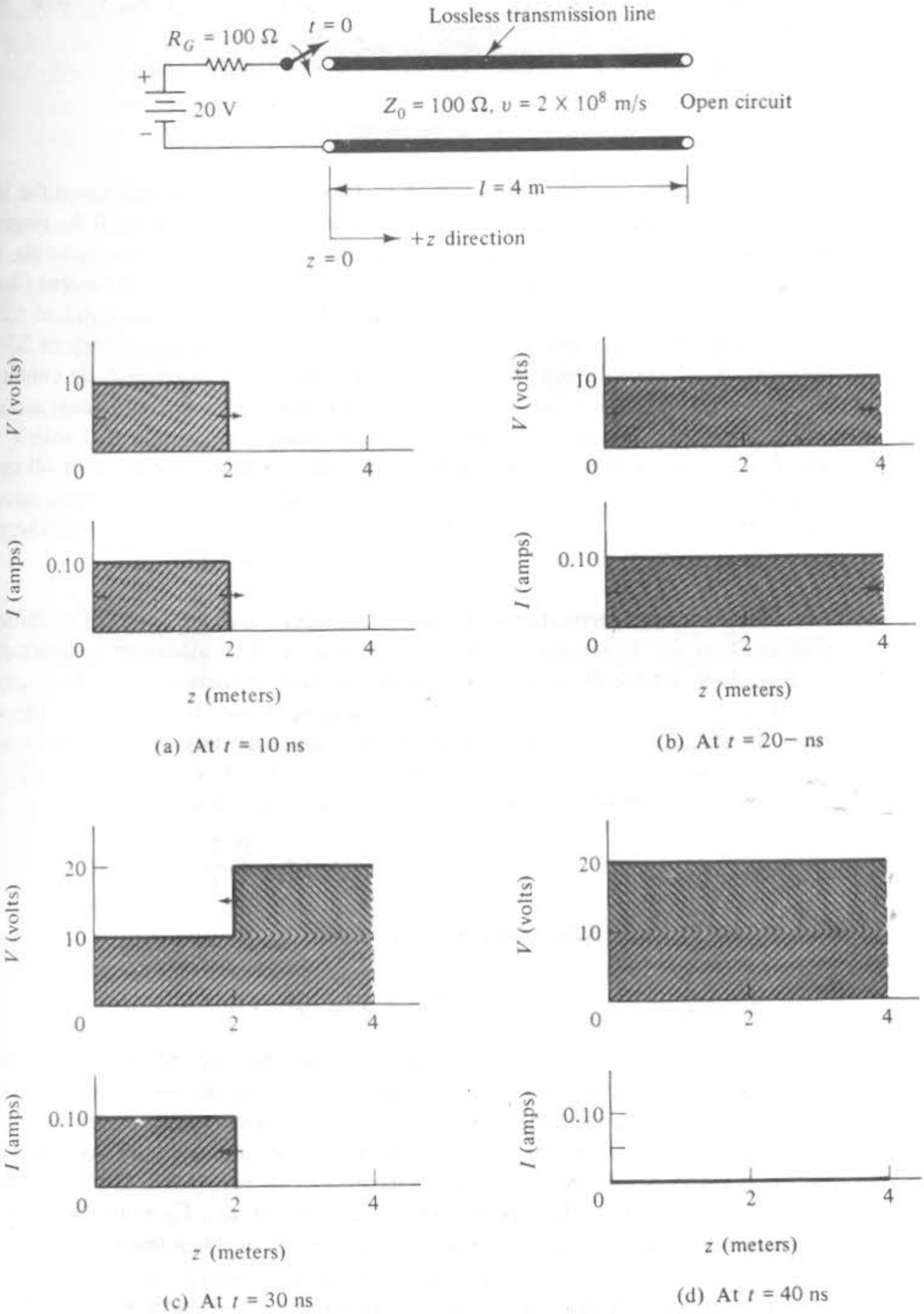


Figure 3–3 Voltage and current waves on an open-circuited transmission line. (The switch is closed at $t = 0$.)

that $I^- = I^+ = 0.10$ A. Also, with $V^+ = I^+Z_0$ and $V^- = I^-Z_0$, $V^- = V^+ = 10$ V. Therefore at $t = 20$ ns, $I_L = 0$ and $V_L = 20$ V.

If we define the reflection coefficient at the load as

$$\Gamma_L \equiv \frac{V^-}{V^+} = \frac{I^-}{I^+} \tag{3-8}$$

then for the open-circuit case, $\Gamma_L = +1$. The open-circuit condition at the load end thus creates reflected voltage and current waves of 10 V and 0.10 A, respectively. These waves travel in the negative z direction with the same velocity as the forward waves. Parts c and d of Fig. 3–3 show the resultant voltage and current (due to the sum of the + and − waves) at $t = 30$ and 40 ns. As the wavefront of the 10 V, 0.10 A reflected wave moves to the left, it leaves behind a net voltage of 20 V and a net current of zero. Since $R_G = 100$ ohms, both Ohm's law and the condition that $V^-/I^- = 100$ ohms are satisfied at $t = 40$ ns, and hence no reflections are required at the generator. The process thus ends and a steady state is achieved with $V = 20$ V and $I = 0$ everywhere on the transmission line. In other words, after 40 ns, the dc source finally *sees* the open circuit and behaves accordingly. If the open circuit were replaced by a short circuit, then for $t > 40$ ns, the conditions everywhere on the line would be $V = 0$ and $I = 20/R_G = 0.20$ A (Prob. 3–3).

Resistively terminated lines. Consider now the case of a finite length transmission line terminated with a pure resistance. This situation is shown at the top of Fig. 3–4, where R_L is the terminating or load resistance. As before, closing the switch initiates a 10 V, 0.10 A forward traveling wave. At $t = 20$ ns, the wave arrives at the load end. Since $R_L \neq Z_0$, Ohm's law can only be satisfied by assuming reflected waves. Thus at $z = 4$ m, $V_L = V^+ + V^-$ and $I_L = I^+ - I^- = (V^+ - V^-)/Z_0$. Ohm's law requires $V_L/I_L = R_L$ and hence

$$R_L = Z_0\frac{V^+ + V^-}{V^+ - V^-} = Z_0\frac{1 + \Gamma_L}{1 - \Gamma_L} \tag{3-9}$$

Solving for the load reflection coefficient yields

$$\Gamma_L = \frac{R_L - Z_0}{R_L + Z_0} \tag{3-10}$$

Note the similarity between this equation and Eq. (2–83) which describes the reflection coefficient for an electromagnetic wave. In the next section, Eqs. (3–9) and (3–10) are extended to include complex impedances when the source excitation is sinusoidal. For resistive terminations, Γ_L is real and can take on any value between −1 and +1. If $R_L = 0$ (short circuit), $\Gamma_L = -1$, while if $R_L = \infty$ (open circuit), $\Gamma_L = +1$. For the special case when $R_L = Z_0$, $\Gamma_L = 0$ and therefore no reflected waves are generated. What this means is that when the forward voltage and current waves arrive at the load, Ohm's law is automatically satisfied and reflections are not required. Referring to the circuit in Fig. 3–4 with $R_L = 100$ ohms, the steady-state condition is reached after 20 ns, namely, $V = 10$ V and $I = 0.10$ A

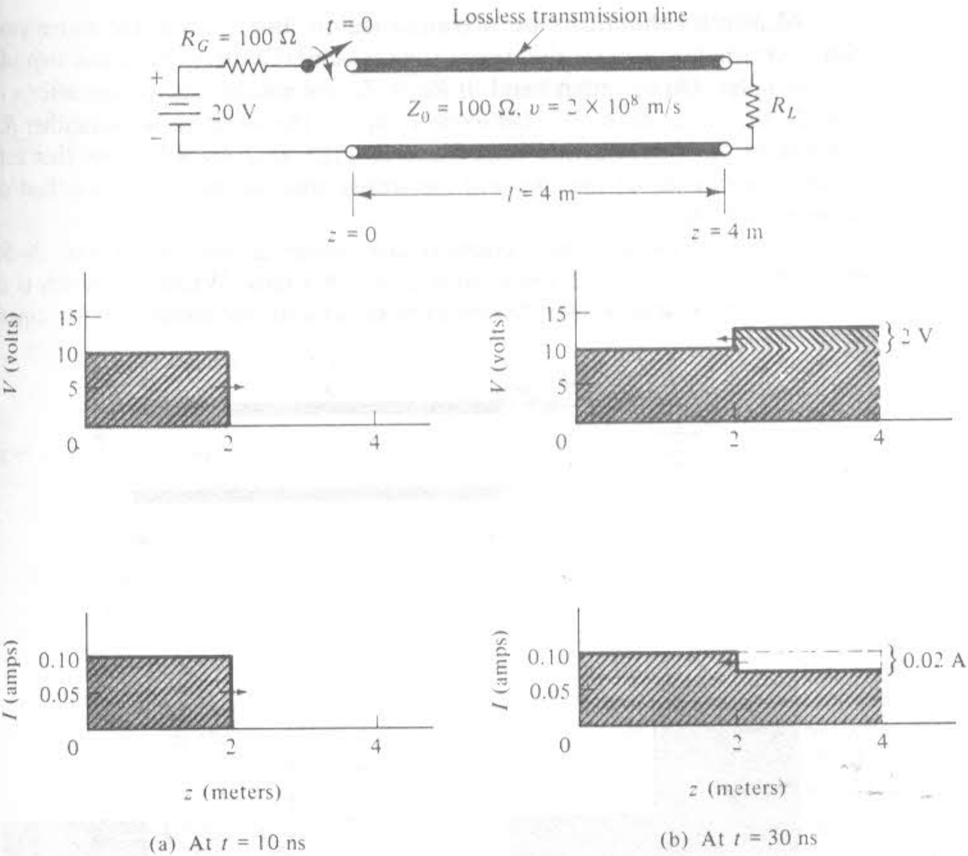


Figure 3–4 Voltage and current waves for the circuit shown when R_L = 150 ohms. (The switch is closed at t = 0.)

everywhere along the line. Note that in all these cases, the steady-state values of voltage and current are those expected from a dc analysis of the circuit.

Consider now the case where R_L = 150 ohms for the circuit shown at the top of Fig. 3–4. From Eq. (3–10), $\Gamma_L = 0.20$. With the forward wave again equal to 10 V and 0.10 A, the reflected voltage and current are 2 V and 0.02 A, respectively. Parts *a* and *b* of Fig. 3–4 shows the voltage and current along the line at t = 10 and 30 ns. At t = 10 ns, only the forward traveling waves exist, having arrived only at the halfway point of the 4 m line. At t = 30 ns, the reflected waves have been generated and have traveled halfway back toward the generator end of the line. At t = 40 ns (not shown), the reflected waves arrive at the input and the resultant voltage and current everywhere along the line become 12 V and 0.08 A. Since $R_G = Z_0$, no reflection is required at the generator end and the steady state is achieved after 40 ns. Again, the final values are those expected from a dc analysis of the circuit.

Multiple reflections on a transmission line. From the above cases, it is clear that when $R_G = Z_0$, the steady state is achieved after one *round trip* (40 ns, in our example). On the other hand, if $R_L = Z_0$, the steady state occurs after a *one-way trip* (20 ns, in our example). Let us now explore the situation when neither R_G nor R_L is equal to the characteristic impedance Z_0. The analysis will show that reflections occur at both ends of the line and the steady-state values are approached only as t becomes infinite.

As a specific example, consider the circuit at the top of Fig. 3–5, where $R_G = 200$ ohms, $R_L = 25$ ohms, and $Z_0 = 100$ ohms. When the switch is closed at $t = 0$, the 90 V source sees 200 ohms in series with the characteristic impedance of

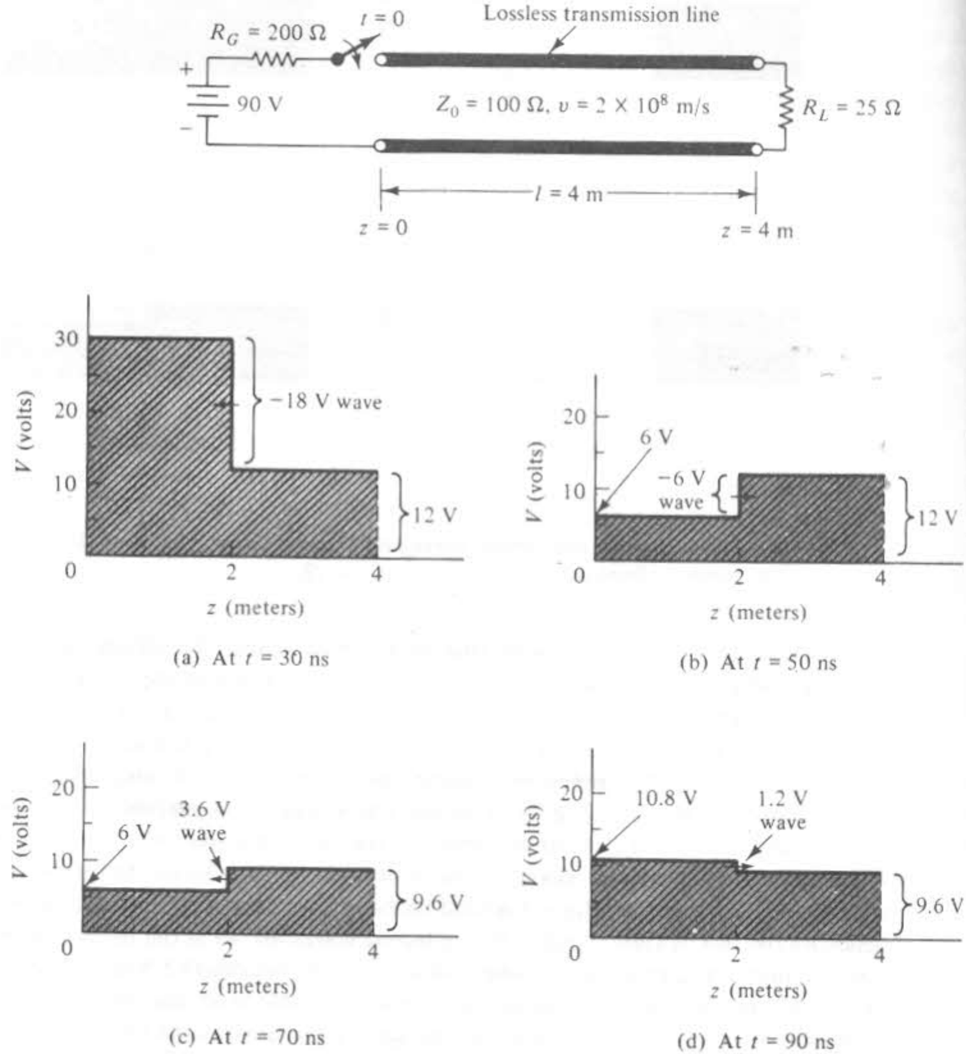


Figure 3–5 Multiple reflections on a resistively terminated transmission line. $Z_0 = 100$ ohms, $R_G = 200$ ohms, and $R_L = 25$ ohms. (The current waves are not shown.)

the line. Therefore, the current and voltage at the input end of the line ($z = 0$) are initially $I^+ = 90/300 = 0.3$ A and $V^+ = I^+ Z_0 = 30$ V. After 20 ns, the V^+ and I^+ waves arrive at the load end where the reflection coefficient $\Gamma_L = -75/125 = -0.6$ and hence $V^- = \Gamma_L V^+ = -18$ V and $I^- = \Gamma_L I^+ = -0.18$ A. At the end of 30 ns, the voltage between $z = 2$ m and $z = 4$ m is reduced to $30 - 18 = 12$ V, while the current has increased to $0.3 + 0.18 = 0.48$ A. The progress of the voltage wave along the line is shown in Fig. 3–5 for $t = 30$, 50, 70, and 90 ns. Let us observe the voltage wave as time marches on. At the end of 40 ns, the -18 V wave arrives at the input where it sees an impedance $R_G = 200$ ohms. Since $R_G \neq Z_0$, a reflection occurs at the generator end. By analogy with Γ_L, the generator reflection coefficient Γ_G is given by

$$\Gamma_G = \frac{R_G - Z_0}{R_G + Z_0} \tag{3–11}$$

For $R_G = 200$ ohms, $\Gamma_G = 1/3$ and hence a -6 V wave is rereflected toward the load end. At $t = 50$ ns, it has progressed halfway down the line, leaving behind it a voltage of $(30 - 18 - 6) = 6$ V. This is shown in part *b* of the figure. At $t = 60$ ns, the -6 V wave arrives at the load which generates a reflected wave of value $(-6)\Gamma_L = +3.6$ V. The situations at 70 and 90 ns are also shown in the figure. Note that at $t = 90$ ns, another forward traveling wave exists having a value $(+3.6)\Gamma_G = +1.2$ V. This process continues indefinitely with the amplitude of the rereflected waves getting smaller and smaller. A plot of voltage versus time at any fixed point on the line would show that, in the limit, the voltage becomes the expected dc value (namely, $90\, R_L/(R_G + R_L) = 10$ V). Such a plot at $z = 0$, the input, is shown in Fig. 3–6. Every step in voltage represents the arrival and generation of reflected waves at the input. After five *round trips* (200 ns), the voltage is within 0.10 percent of the steady-state value.

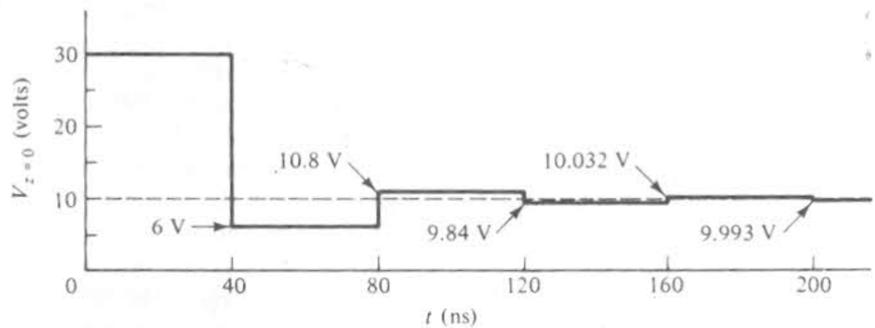


Figure 3–6 Input voltage versus time for the line shown in Fig. 3–5.

The space-time diagram developed by Bewley (Ref. 3–6) is a graphic aid in determining the voltage and current as a function of either time or position along the line. Figure 3–7 shows the diagram for the circuit conditions in Fig. 3–5. The abscissa indicates position along the line and the ordinate represents the time scale, $t = 0$ being the moment that the switch is closed. For reference, the values of Γ_G and Γ_L are given at the top of the diagram. The lines sloping downward and to the

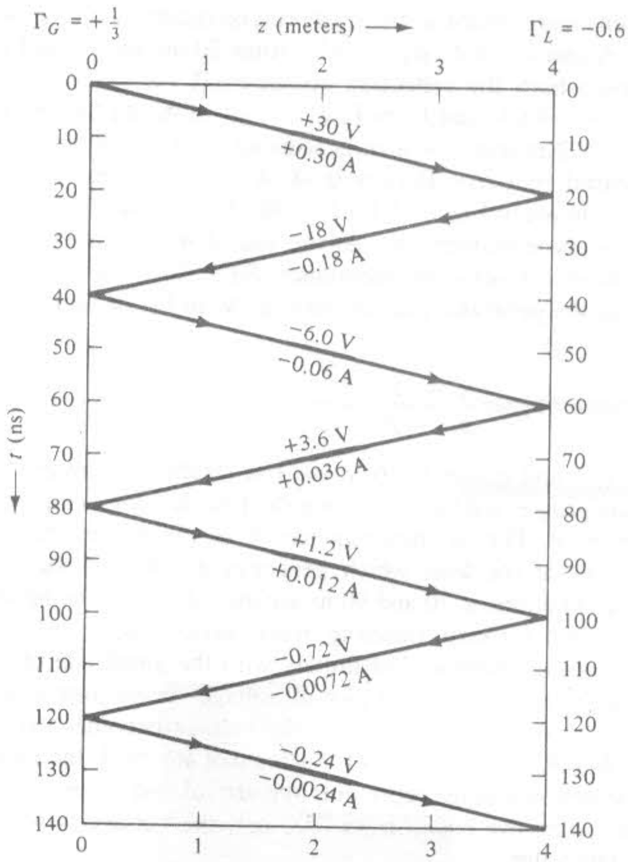


Figure 3–7 Space-time diagram for the transmission-line circuit shown in Fig. 3–5.

right represent forward traveling waves, while those sloping down and to the left represent reverse waves. The voltage and current values for the particular wave are shown above and below the sloping line. As explained, the load end creates reflections equal to Γ_L of the arriving wave. Generator reflections are equal to Γ_G times the value of the wave arriving at the generator end.

To illustrate, Fig. 3–7 will be used to determine the voltage and current at $z = 2$ m. Each intersection of a sloping line with the vertical $z = 2$ m line represents the arrival of a wavefront. For $t < 10$ ns, no intersection exists and hence both V and I are zero. For $10 < t < 30$ ns, there is one intersection which means $V = 30$ V and $I = 0.30$ A. For $t > 30$ ns, the voltage is the sum of all the forward and reverse waves that have passed the $z = 2$ m location. For example, at $t = 80$ ns, $V = 30 - 18 - 6 + 3.6 = 9.6$ V. The current may be determined in a similar manner except that current values associated with reverse waves must be subtracted from those associated with the forward waves. For example, at $t = 80$ ns, $I = 0.30 - (-0.18) + (-0.06) - (+0.036) = 0.384$ A. The diagram may also be used to determine voltage and current versus z for a fixed time by drawing a horizontal line corresponding to the particular value of time. The sum of the voltages above the line correspond to the voltage at that point on the line. The same applies to the current except that, as before, reverse-traveling current waves must be subtracted from forward-traveling current waves.

The space-time diagram may be extended to transmission lines having discontinuities and branches (Chapter 3 of Ref. 3–7).

It is interesting to note that the voltage shown in Fig. 3–6 is oscillatory as it approaches its final value. The period of this *ringing* effect is 80 ns (twice the *round-trip* time) and hence its reciprocal is the natural resonant frequency of the circuit, namely, 12.5 MHz. Since $v = 2 \times 10^8$ m/s, this means that the line is $\lambda/4$ long at the resonant frequency. Thus we see that by connecting a dc source to a transmission line, high frequency oscillations are possible. Granted, the oscillation is heavily damped in this example, but the damping can be reduced by increasing the magnitude of both reflection coefficients. In fact, if they are both unity, the oscillation will continue indefinitely (Prob. 3–6). In other words, a configuration consisting of two large reflections separated by a length of transmission line has the properties of a resonant circuit. Most of the microwave resonators described in Chapter 9 have exactly this configuration.

3-3 SINUSOIDAL EXCITATION OF TRANSMISSION LINES

Let us now turn to the important case of uniform transmission lines with sinusoidal excitation. Since our interest is in the steady-state solution, the rms-phasor method, reviewed in Sec. 1–4, will be employed.

Equations (3–1) resulted from a distributed circuit analysis of the uniform transmission line described in Fig. 3–1. Written in phasor form, they become

$$-\frac{d\mathbf{V}}{dz} = (R' + j\omega L')\mathbf{I} = Z'\mathbf{I} \quad \text{and} \quad -\frac{d\mathbf{I}}{dz} = (G' + j\omega C')\mathbf{V} = Y'\mathbf{V} \qquad (3\text{–}12)$$

where $Z' \equiv R' + j\omega L'$ is defined as the series impedance per unit length and $Y' \equiv G' + j\omega C'$ is defined as the shunt admittance per unit length.

Differentiating the first equation with respect to z and substituting $-Y'\mathbf{V}$ for $d\mathbf{I}/dz$ yields the following second-order differential equation.

$$\frac{d^2\mathbf{V}}{dz^2} = Z'Y'\mathbf{V} \qquad (3\text{–}13)$$

Its phasor solution may be written as

$$\mathbf{V} = \mathbf{V}_0^+ e^{-\gamma z} + \mathbf{V}_0^- e^{+\gamma z} = \mathbf{V}^+ + \mathbf{V}^- \qquad (3\text{–}14)$$

where γ is the propagation constant and given by

$$\gamma = \sqrt{Z'Y'} = \sqrt{(R' + j\omega L')(G' + j\omega C')} \qquad (3\text{–}15)$$

In general, γ is complex and may be written as $\gamma = \alpha + j\beta$, where as explained in Sec. 2–6 α is the attenuation constant (Np/length) and β is the phase constant (rad/length).

The phasor quantities $\mathbf{V}^+$ and $\mathbf{V}^-$ are functions of z and represent the forward and reverse voltage waves on the line. $\mathbf{V}_0^+$ and $\mathbf{V}_0^-$ represent their values at $z = 0$, the input end.[4] Substitution of Eq. (3–14) into the first of Eqs. (3–12) yields the accompanying solution for $\mathbf{I}$.

$$\mathbf{I} = \mathbf{I}_0^+ e^{-\gamma z} - \mathbf{I}_0^- e^{+\gamma z} = \mathbf{I}^+ - \mathbf{I}^- \tag{3–16}$$

where $\mathbf{I}_0^+ = \mathbf{V}_0^+/Z_0$ and $\mathbf{I}_0^- = \mathbf{V}_0^-/Z_0$.

The quantity Z_0 is known as the *characteristic impedance* of the transmission line and is given by

$$Z_0 \equiv \sqrt{\frac{Z'}{Y'}} = \sqrt{\frac{R' + j\omega L'}{G' + j\omega C'}} \quad \text{ohms} \tag{3–17}$$

It is the voltage-to-current ratio for the traveling waves. That is,

$$Z_0 = \frac{\mathbf{V}^+}{\mathbf{I}^+} = \frac{\mathbf{V}^-}{\mathbf{I}^-} \tag{3–18}$$

The reciprocal of Z_0 is defined as the characteristic admittance (Y_0) of the line and therefore

$$Y_0 \equiv \frac{1}{Z_0} = \sqrt{\frac{Y'}{Z'}} \quad \text{mhos} \tag{3–19}$$

Since the voltages and currents in Eqs. (3–14) and (3–16) are phasor quantities, they are generally complex and depend upon the specific conditions at the generator and load. The determination of their values will be discussed in the next section.

From ac theory, the net average power flow at any point on the line is related to $\mathbf{V}$ and $\mathbf{I}$ by

$$P = \mathrm{Re}(\mathbf{V}\,\mathbf{I}^*) = V\,I\cos\theta_{pf} \tag{3–20}$$

where * denotes the complex conjugate and θ_{pf} the power factor angle. V and I represent rms values.

Example 3–1:

A coaxial line has the following characteristics at 1000 MHz:
$R' = 4$ ohms/m, $L' = 450$ nH/m, $G' = 7 \times 10^{-4}$ mho/m, $C' = 50$ pF/m.
(a) Calculate Z_0, α, β, v, and λ at 1000 MHz.
(b) With $\mathbf{V}_0^+ = 10\underline{/0}$ V and $\mathbf{V}_0^- = 0$, calculate $\mathbf{V}$, $\mathbf{I}$, and P at $z = 4$ m.

Solution:
(a) At 1000 MHz,

$$Z' = 4 + j(2\pi \times 10^9)(450 \times 10^{-9}) = 4 + j2827 \quad \text{ohms/m}$$

and $$Y' = 7 \times 10^{-4} + j(2\pi \times 10^9)(50 \times 10^{-12}) = 10^{-4}(7 + j3142) \text{ mho/m.}$$

Therefore, $Z_0 = \sqrt{Z'/Y'} = 95\underline{/0.023^\circ}$ ohms

[4] It is important to understand the terminology being used. Quantities with + and − superscripts are associated with the traveling waves, while those without them represent the net value of the quantity at a particular point on the line.

and $\gamma = \alpha + j\beta = \sqrt{Z'Y'} = \sqrt{-888 + j3.24} = 0.054 + j29.8.$

Thus, $\alpha = 0.054$ Np/m or 0.47 dB/m
and $\beta = 29.8$ rad/m or 1707°/m.
From Eq. (2–56), $\beta = 2\pi/\lambda = \omega/v$ and hence

$$v = \frac{2\pi \times 10^9}{29.8} = 2.11 \times 10^8 \text{ m/s} \qquad \text{and} \qquad \lambda = \frac{2\pi}{29.8} = 0.21 \text{ m}$$

(b) Since $\mathbf{V}_0^- = 0$, only the forward wave exists. That is,

$$\mathbf{V} = \mathbf{V}^+ = \mathbf{V}_0^+ e^{-\alpha z} e^{-j\beta z} = 10e^{-\alpha z}\underline{/-\beta z}$$

For $z = 4$ m, $\alpha z = 0.22$ Np and $\beta z = 119.2$ rad.
Therefore at $z = 4$ m,

$$\mathbf{V} = 8.03\underline{/-119.2 \text{ rad}} \quad \text{V}$$

With $V_0^- = 0$, $I_0^- = V_0^-/Z_0 = 0$ and therefore at $z = 4$ m,

$$\mathbf{I} = \frac{8.03\underline{/-119.2 \text{ rad}}}{Z_0} = 0.084\underline{/-119.2 \text{ rad}} \quad \text{A}$$

Since Z_0 is practically real, the power factor angle is zero and the average power at $z = 4$ m is 0.677 W. This compares to an input power of 10(10/95) = 1.053 W, which translates to a 36 percent power loss over 4 m. This represents a medium quality coaxial line. Lines are commercially available with $\alpha < 0.10$ dB/m at 1000 MHz.

Low-loss lines. Certain conclusions can be drawn from the above example. First of all, for low-loss lines at high frequencies, $R' \ll \omega L'$ and $G' \ll \omega C'$. In the microwave range, these inequalities are almost always true. Applying these approximations to Eqs. (3–17) and (3–15) yields

$$Z_0 \approx \sqrt{\frac{L'}{C'}} \quad \text{ohms} \tag{3–21}$$

$$\alpha \approx \frac{R'}{2Z_0} + \frac{G'Z_0}{2} \quad \text{Np/length} \tag{3–22}$$

$$\beta \approx \omega\sqrt{L'C'} \quad \text{rad/length} \tag{3–23}$$

Since $\beta = \omega/v, f\lambda = v$ and $f\lambda_0 = c$,

$$v = \frac{1}{\sqrt{L'C'}} \tag{3–24}$$

and

$$\lambda = \lambda_0 \frac{v}{c} = \frac{\lambda_0}{c\sqrt{L'C'}} \tag{3–25}$$

The above results may be summarized as follows: For low-loss transmission lines at high frequencies,

1. The equations for Z_0, v, β, and λ are approximately the same as those for lossless lines.
2. Since Z_0 is practically real, the average power flow in a *traveling* wave at any point on the line is simply the product of the rms voltage and current at that point. That is, $P^+ = V^+I^+$ and $P^- = V^-I^-$.
3. For single frequency signals, the only appreciable effect of finite R' and G' is the introduction of some attenuation to the voltage and current waves as they propagate.[5] Their amplitudes are reduced by $e^{-\alpha\Delta z}$ when they travel a distance Δz, and hence the power flow is attenuated by $e^{-2\alpha\Delta z}$.

These conclusions are similar to those for electromagnetic waves in a lossy dielectric (Sec. 2–6a). The reason is that associated with the sinusoidal voltage and current waves is a sinusoidal electromagnetic wave. The only difference is that unlike those discussed in Chapter 2, this electromagnetic wave is *guided* by the transmission-line structure.

Wave propagation in coaxial lines. Consider the coaxial line whose cross section is shown in Fig. 3–8. Its inner conductor radius is denoted by a and the inner radius of the outer conductor by b. Also shown are the electric and magnetic field lines for the TEM mode.[6] Because of skin effect, practically all of the high-frequency currents are located near the surface of the conductors. The electric and magnetic fields are given by

$$E = \frac{V}{r\ln(b/a)} \quad \text{and} \quad H = \frac{I}{2\pi r} \tag{3–26}$$

where V is the voltage between the conductors and I is the current in each conductor. These results are derived from Exs. 2–1 and 2–2 in the previous chapter and are valid for $a \le r \le b$.

By using Eq. (2–60) and assuming sinusoidal fields, the average power flow for a traveling electromagnetic wave may be determined. In cylindrical coordinates, an element of surface in the transverse plane is $r\,d\phi\,dr$ and therefore for the coaxial line

$$P = \int_a^b \int_0^{2\pi} \frac{VI}{2\pi r^2 \ln(b/a)}\, r\,d\phi\,dr = VI$$

The meaning of this result is that the power flow associated with the voltage and current waves is merely another way of viewing electromagnetic propagation in the region *between* the conductors. Both points of view are equally valid and each is used where appropriate to describe various aspects of microwave transmission.

The reader may now suspect a relationship between the velocity given by Eq. (3–24) and that given by Eq. (2–53). For the coaxial line at high frequencies, $C = C'l$ and $L = L'l$ are given by Eqs. (2–21) and (2–31). Substitution of L' and

[5] Another consequence of finite R' and G' is that γ and Z_0 are frequency dependent. As a result, modulated signals and pulses become distorted as they propagate along the line.

[6] The transverse electromagnetic (TEM) mode of propagation is so designated because the direction of the electric and magnetic fields are transverse to the direction of propagation.

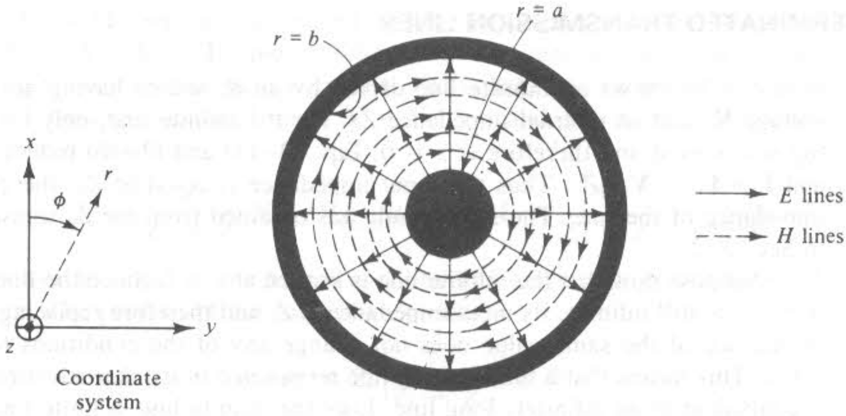


Figure 3–8 The TEM mode for a coaxial line at high frequencies. (Note: At high frequencies, the skin depth δ_s is much less than the inner conductor radius a and the thickness of the outer conductor. Therefore, the electromagnetic field is confined to the insulating region between the conductors.)

C' into Eq. (3–24) results in the following expression for the velocity of the voltage and current waves along a coaxial line.

$$v = \frac{1}{\sqrt{\mu_0 \mu_R \epsilon_0 \epsilon_R}} = \frac{c}{\sqrt{\mu_R \epsilon_R}} \tag{3–27}$$

This is exactly the expression for velocity of an electromagnetic wave in an unbounded insulating material. In Eq. (3–27), μ_R and ϵ_R represent the relative permeability and dielectric constant of the insulating material between the two conductors. Also since $f\lambda = v$, the wavelength in the coaxial line is given by

$$\lambda = \frac{c}{f\sqrt{\mu_R \epsilon_R}} = \frac{\lambda_0}{\sqrt{\mu_R \epsilon_R}} \tag{3–28}$$

which is the same as Eq. (2–55). With the phase constant $\beta = \omega/v$,

$$\beta = \frac{\omega}{c}\sqrt{\mu_R \epsilon_R} = \frac{2\pi}{\lambda} \quad \text{rad/length} \tag{3–29}$$

Let us now see if a relationship exists between the characteristic impedance of the coaxial line and the properties of the insulating material between the conductors. By using Eq. (3–21) and the above-mentioned relations for L' and C', the high-frequency characteristic impedance of a coaxial line may be written as

$$Z_0 = 60\sqrt{\frac{\mu_R}{\epsilon_R}}\ln\frac{b}{a} = 138\sqrt{\frac{\mu_R}{\epsilon_R}}\log\frac{b}{a} \quad \text{ohms} \tag{3–30}$$

where b and a are defined in Fig. 3–8. As expected, the characteristic impedance is indeed related to the properties of the insulating material.

Expressions for Z_0 of other type microwave lines are given in Chapter 5.

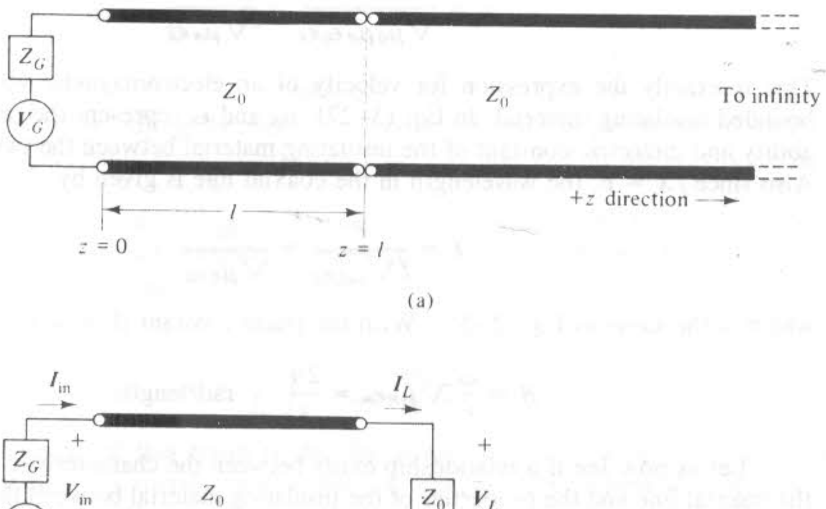
3-4 TERMINATED TRANSMISSION LINES

Figure 3–9a shows an infinite line driven by an ac source having an open-circuit voltage $\mathbf{V}_G$ and an internal impedance Z_G. For the infinite line, only forward traveling waves exist and therefore at $z = 0$, Eqs. (3–14) and (3–16) reduce to $\mathbf{V} = \mathbf{V}_0^+$ and $\mathbf{I} = \mathbf{I}_0^+ = \mathbf{V}_0^+/Z_0$. Thus the input impedance is equal to Z_0, the characteristic impedance of the line. The same result was obtained from the dc transient analysis in Sec. 3–2.

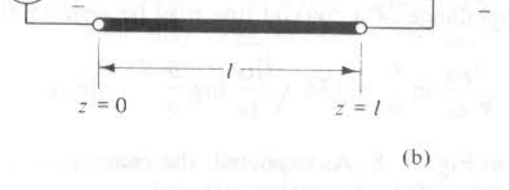
Suppose now that the infinite line is broken at $z = l$. Since the line to the right of $z = l$ is still infinite, its input impedance is Z_0 and therefore replacing it by a load impedance of the same value does not change any of the conditions to the left of $z = l$. This means that a finite length line *terminated in its characteristic impedance* is equivalent to an infinitely long line. Like the infinite line, a finite line terminated in Z_0 has no reflections. Also, its input impedance is equal to Z_0 and is independent of the line length l. As will be shown in the next section, this is *not* true for any other value of load impedance.

Referring to Fig. (3–9b), the input impedance is Z_0 and therefore

$$\mathbf{V}_{\text{in}} = \frac{Z_0}{Z_G + Z_0}\mathbf{V}_G \quad \text{and} \quad \mathbf{I}_{\text{in}} = \frac{\mathbf{V}_G}{Z_G + Z_0} \tag{3–31}$$



(a)



(b)

Figure 3–9 An infinite transmission line and its equivalent.

where $\mathbf{V}_{in}$ and $\mathbf{I}_{in}$ are the voltage and current phasors at $z = 0$.[7] For the reflectionless line, $\mathbf{V}_{in} = \mathbf{V}_0^+$, $\mathbf{I}_{in} = \mathbf{I}_0^+$ and therefore with the generator and line characteristics known, $\mathbf{V}_0^+$ and $\mathbf{I}_0^+$ may be determined. The voltage and current at any point on the line are readily calculated since $\mathbf{V} = \mathbf{V}_0^+ e^{-\gamma z}$ and $\mathbf{I} = \mathbf{I}_0^+ e^{-\gamma z}$. For example, at $z = l$, the load end

$$\mathbf{V}_L = \frac{Z_0}{Z_G + Z_0} \mathbf{V}_G\, e^{-\alpha l} \underline{/-\beta l} \quad \text{and} \quad \mathbf{I}_L = \frac{\mathbf{V}_G}{Z_G + Z_0} e^{-\alpha l} \underline{/-\beta l} \tag{3-32}$$

The power absorbed by the load is given by $V_L I_L$ since with Z_0 real the power factor is unity. The product of the rms values yields

$$P_L = Z_0 \left| \frac{V_G}{Z_G + Z_0} \right|^2 e^{-2\alpha l} \tag{3-33}$$

The magnitude sign is required since Z_G could be complex. Of course, if the line is lossless ($\alpha = 0$), P_L is exactly equal to P_{in}.

3-4a Lines Terminated in Z_L, the General Case

Figure 3-10 shows a transmission line terminated in an arbitrary load impedance Z_L. As explained in Sec. 3-2, reflections occur when $Z_L \neq Z_0$. The general expressions for voltage and current at any point on the line are given by Eqs. (3-14) and (3-16) and are repeated here.

$$\mathbf{V} = \mathbf{V}_0^+ e^{-\gamma z} + \mathbf{V}_0^- e^{+\gamma z} = \mathbf{V}^+ + \mathbf{V}^- \tag{3-34}$$

$$\mathbf{I} = \mathbf{I}_0^+ e^{-\gamma z} - \mathbf{I}_0^- e^{+\gamma z} = \mathbf{I}^+ - \mathbf{I}^- \tag{3-35}$$

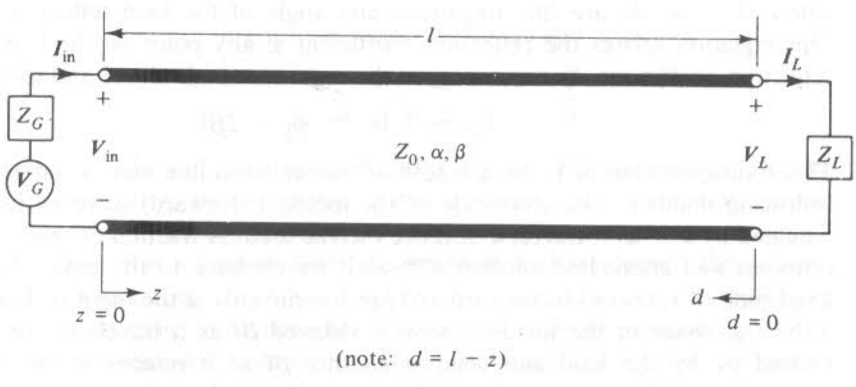


Figure 3-10 A uniform transmission line terminated in a load impedance Z_L and driven by an ac source.

[7] Subscripts denote electrical quantities at a specific point on the line, for example, *in* for input values and L for load values. The absence of subscripts indicate quantities at an arbitrary point on the line. Their values are generally a function of z.

The ratio of reflected to forward voltage (or current) is defined as the reflection coefficient Γ. That is,

$$\Gamma \equiv \frac{\text{Reflected Voltage (or current) at pt. } z}{\text{Forward Voltage (or current) at pt. } z} = \frac{\mathbf{V}^-}{\mathbf{V}^+} = \frac{\mathbf{I}^-}{\mathbf{I}^+} \tag{3–36}$$

Since the voltage and current phasors are functions of position, Γ is a function of z. For example, at the load end, $z = l$ and

$$\Gamma_L = \frac{\mathbf{V}_0^- e^{+\gamma l}}{\mathbf{V}_0^+ e^{-\gamma l}} = \frac{\mathbf{V}_0^-}{\mathbf{V}_0^+} e^{2\gamma l} \tag{3–37}$$

It is convenient to define a particular point on the line in terms of its distance from the load end. From Fig. 3–10, $d = l - z$ and therefore Eqs. (3–34) and (3–35) may be rewritten as

$$\mathbf{V} = \mathbf{V}_0^+ e^{-\gamma l}[e^{\gamma d} + \Gamma_L e^{-\gamma d}] \tag{3–38}$$

$$\mathbf{I} = \mathbf{I}_0^+ e^{-\gamma l}[e^{\gamma d} - \Gamma_L e^{-\gamma d}] \tag{3–39}$$

where use has been made of Eq. (3–37). Since $\mathbf{V}^+ = \mathbf{V}_0^+ e^{-\gamma l} e^{\gamma d}$ and $\mathbf{V}^- = \Gamma_L \mathbf{V}_0^+ e^{-\gamma l} e^{-\gamma d}$, the reflection coefficient at an arbitrary point d on the line is

$$\Gamma = \Gamma_L e^{-2\gamma d} \tag{3–40}$$

which may be rewritten as

$$\Gamma = \Gamma_L e^{-2\alpha d} \underline{/-2\beta d} \tag{3–41}$$

since $\gamma = \alpha + j\beta$. The reflection coefficient may be expressed in polar form as

$$\Gamma = |\Gamma| \underline{/\phi} = |\Gamma_L| e^{-2\alpha d} \underline{/\phi_L - 2\beta d} \tag{3–42}$$

where $|\Gamma_L|$ and ϕ_L are the magnitude and angle of the load reflection coefficient. This equation relates the reflection coefficient at any point on the line to the load reflection coefficient. For instance, at the input terminals $d = l$ and therefore

$$\Gamma_{\text{in}} = |\Gamma_L| e^{-2\alpha l} \underline{/\phi_L - 2\beta l} \tag{3–43}$$

This transformation of Γ_L by a length of transmission line may be interpreted in the following manner. The amplitude of the incident (forward) wave at the input is attenuated by $e^{-\alpha l}$ as it travels a distance l to the load. A fraction of this signal ($|\Gamma_L|$) is reflected and attenuated another $e^{-\alpha l}$ as it travels back to the input. Therefore the amplitude of reflected to forward voltage (or current) at the input is $|\Gamma_L| e^{-2\alpha l}$. Similarly, the phase of the incident wave is delayed βl as it travels to the load, phase shifted ϕ_L by the load and delayed another βl as it returns to the input. Thus, $\phi_{\text{in}} = \phi_L - 2\beta l$. If the line is lossless, $\alpha = 0$ and the magnitude of the reflection coefficient is the same at all points on the transmission line. The angle of the reflection coefficient ϕ, however, remains a function of position. Equation (3–42) is fundamental to the discussion of the impedance transformation and the Smith chart (Secs. 3–5 to 3–7).

When the value of Γ_L is known, Eq. (3–42) may be used to calculate the reflection coefficient at any other point on the line. The following analysis shows that Γ_L can be determined from a knowledge of Z_L and Z_0. From Eqs. (3–38) and (3–39) with $d = 0$, we have

$$\mathbf{V}_L = \mathbf{V}_0^+ e^{-\gamma l}[1 + \Gamma_L] \qquad \text{and} \qquad \mathbf{I}_L = \mathbf{I}_0^+ e^{-\gamma l}[1 - \Gamma_L]$$

Since $\mathbf{V}_0^+ = \mathbf{I}_0^+ Z_0$, the load impedance $(\mathbf{V}_L/\mathbf{I}_L)$ is

$$Z_L = Z_0 \frac{1 + \Gamma_L}{1 - \Gamma_L} \qquad \text{or} \qquad \bar{Z}_L = \frac{1 + \Gamma_L}{1 - \Gamma_L} \tag{3–44}$$

where the *bar* over the impedance denotes that it has been normalized with respect to the characteristic impedance of the line. Solving the above equation for Γ_L yields

$$\Gamma_L = \frac{Z_L - Z_0}{Z_L + Z_0} = \frac{\bar{Z}_L - 1}{\bar{Z}_L + 1} \tag{3–45}$$

Defining $Y_L \equiv 1/Z_L$, $Y_0 \equiv 1/Z_0$ and $\bar{Y}_L \equiv Y_L/Y_0$,

$$\Gamma_L = \frac{Y_0 - Y_L}{Y_0 + Y_L} = \frac{1 - \bar{Y}_L}{1 + \bar{Y}_L} \tag{3–46}$$

These equations are quite similar to Eqs. (3–9) and (3–10). In this case, they are based upon a steady-state ac analysis and hence apply to any complex impedance. This means that the reflection coefficient can be complex. The above equations show the relationship between the impedance concept of ac circuit theory and the reflection concept of wave theory.

If the impedance at any point on the line is defined as $Z \equiv \mathbf{V}/\mathbf{I}$, the above equations may be generalized in the following manner.[8]

$$Z = Z_0 \frac{1 + \Gamma}{1 - \Gamma} \qquad \text{and} \qquad Y = Y_0 \frac{1 - \Gamma}{1 + \Gamma} \tag{3–47}$$

where $Y \equiv 1/Z$. Solving for the reflection coefficient,

$$\Gamma = \frac{Z - Z_0}{Z + Z_0} = \frac{Y_0 - Y}{Y_0 + Y} \tag{3–48}$$

Equation (3–47) is derived by substituting Eq. (3–40) into Eqs. (3–38) and (3–39).

In Sec. 2–7, it was shown that a reflection causes standing waves. Thus if $Z_L \neq Z_0$, Γ_L is finite and standing waves of voltage and current exist along the transmission line. For a lossless line terminated in a short circuit, the patterns are the same as those in Fig. 2–24 and 2–25, where V replace E_x and I replaces H_y. As explained, successive minimums are $\lambda/2$ apart, while the distance between a maximum and an adjacent minimum is $\lambda/4$. Furthermore, the current pattern is shifted $\lambda/4$ relative to the voltage pattern. That is, wherever the rms voltage is maximum, the rms current is minimum (and vice versa).

[8] The impedance Z at any point on the line is the ratio of phasor voltage to phasor current at that point. Referring to Fig. 3–10, it represents the impedance of the circuit to the right of the point. To measure it, one would break the circuit at the desired point and connect an impedance bridge to the right hand part of the circuit. At $z = 0$, $Z = Z_{in}$, the input impedance of the complete transmission line-load impedance combination.

In general, the voltage at any point on the line is the phasor sum of the forward and reflected voltages ($\mathbf{V}^+ + \mathbf{V}^-$). In the lossless case, the change in $\mathbf{V}^+$ and $\mathbf{V}^-$ with position can be represented by two counterrotating vectors of fixed magnitudes. Figure 2–28 describes two such phasors (replace $\mathbf{E}_i$ and $\mathbf{E}_r$ by $\mathbf{V}^+$ and $\mathbf{V}^-$). At some point d, $\mathbf{V}^+$ and $\mathbf{V}^- = \Gamma\,\mathbf{V}^+$ are in phase, resulting in a voltage maximum. Its rms value is given by

$$V_{\max} = V^+ + |\Gamma|V^+ = \{1 + |\Gamma|\}V^+ \tag{3–49}$$

One quarter wavelength from the maximum, the two signals are 180° out-of-phase resulting in a voltage minimum. Its rms value is

$$V_{\min} = \{1 - |\Gamma|\}V^+ \tag{3–50}$$

For a lossless line, $|\Gamma|$ is the same for all values of d. As explained in Sec. 2–7, a standing-wave ratio (SWR) can be defined for the standing wave pattern.[9] That is,

$$\text{SWR} \equiv \frac{V_{\max}}{V_{\min}} \tag{3–51}$$

From the above expressions, the following useful equation is obtained.

$$\text{SWR} = \frac{1 + |\Gamma|}{1 - |\Gamma|} \tag{3–52}$$

By virtue of its definition, SWR is always equal to or greater than unity. If $|\Gamma| = 0$, no reflections exist and the SWR is unity, while for full reflection $|\Gamma| = 1.00$ and the SWR is infinite. Solving Eq. (3–52) for $|\Gamma|$ yields

$$|\Gamma| = \frac{\text{SWR} - 1}{\text{SWR} + 1} \tag{3–53}$$

Note that in both the above equations, we are talking about the *magnitude* of Γ, which is always between zero and unity when Z_0 is real.[10] On a lossless line, all the maximums are identical and likewise all the minimums. Therefore the SWR is the same everywhere along the line. For a lossy line, this is not true and consequently the meaning and value of SWR becomes questionable (see Fig. 5.7 in Ref. 3–1).

To bring together some of the ideas discussed thus far, consider the following illustrative example.

Example 3–2:

Referring to Fig. 3–10, a 600 MHz generator with $V_G = 10$ V and $Z_G = 0$ is connected to a load impedance $Z_L = 150 + j90$ ohms via an air-insulated coaxial line. The line has a characteristic impedance of 75 ohms and is 15 cm long.

(a) Assuming $\alpha = 0$, calculate Γ_L, Γ_{in} and the SWR on the line.

(b) What is the maximum rms voltage on the line?

(c) Determine $|\Gamma_{in}|$ if $\alpha = 2.0$ dB/m and the line is exactly one wavelength long.

[9] In some texts, SWR is written as VSWR (voltage standing-wave ratio). This can be misleading since SWR also equals $I_{\max}/I_{\min}$.

[10] An exception is when Re Z is negative. This situation occurs in negative-resistance devices such as tunnel diodes, parametric amplifiers, and masers.

Solution:

(a) $\bar{Z}_L = \dfrac{150 + j90}{75} = 2 + j1.2$ and from Eq. (3–45),

$$\Gamma_L = \frac{1 + j1.2}{3 + j1.2} = 0.48\underline{/28.4^\circ}$$

At 600 MHz, $\lambda_0 = 50$ cm and since both μ_R and ϵ_R are unity, $\lambda = 50$ cm. Thus,

$$\beta l = (2\pi/50)15 = 0.6\pi \text{ rad or } 108^\circ$$

and from Eq. (3–43) with $\alpha = 0$,

$$\Gamma_{\text{in}} = 0.48\underline{/28.4 - 2(108)} = 0.48\underline{/-187.6^\circ}$$

From Eq. (3–52),

$$\text{SWR} = \frac{1 + 0.48}{1 - 0.48} = 2.85$$

(b) At the input ($d = l$), Eq. (3–38) reduces to $\mathbf{V}_{\text{in}} = \mathbf{V}_0^+(1 + \Gamma_{\text{in}})$. With $Z_G = 0$, $\mathbf{V}_{\text{in}} = \mathbf{V}_G$, $\mathbf{V}_0^+ = \mathbf{V}_G/(1 + \Gamma_{\text{in}})$ and therefore

$$\mathbf{V}_0^+ = \frac{10\underline{/0}}{0.524 + j0.06} = 19\underline{/-6.5^\circ} \quad \text{V}$$

The maximum rms voltage is given by

$$V_{\max} = \{1 + |\Gamma|\}V^+$$

where for a lossless line $V^+ = V_0^+$ and $|\Gamma| = |\Gamma_{\text{in}}| = |\Gamma_L|$. Therefore,

$$V_{\max} = (1.48)(19) = 28.12 \text{ V}$$

(c) For $\alpha = 2.0$ dB/m $= 0.23$ Np/m and $l = \lambda = 0.5$ m, $\alpha l = 0.115$ Np. From Eq. (3–43),

$$|\Gamma_{\text{in}}| = |\Gamma_L|e^{-2\alpha l} = 0.48e^{-0.23} = 0.381$$

In this example, $\mathbf{V}_G = \mathbf{V}_{\text{in}}$ since $Z_G = 0$. Generally, $Z_G \neq 0$ which makes solving for $\mathbf{V}_{\text{in}}$ and $\mathbf{V}_L$ a little more involved. This case is treated in part *c* of this section.

3–4b Some Special Cases of Terminated Lines

Let us now study four special cases of load impedances terminating a lossless transmission line, namely, $Z_L = 0$ (short circuit), $Z_L = \infty$ (open circuit), $Z_L = jX$ (pure reactance) and $Z_L = R_L$ (pure resistance).

Line terminated in a short circuit. In this case $Z_L = 0$ and therefore $\Gamma_L = -1$ and the SWR is infinite. The standing wave patterns for rms voltage and current are the same as those in Fig. 2–24, where V replaced E_x and I replaces H_y. At $d = 0$ (the load end), $V = 0$, as it must be at a perfect short. Other voltage nulls occur at multiples of a half wavelength. Current, on the other hand, is a maximum at the load, its rms value being twice that of the incident current I_0^+. Because the voltage pattern is displaced with respect to the current pattern, the ratio of V to I is a function of d. For instance, at $d = \lambda/4$, V is a maximum and I is zero, which means that the impedance at that point is infinite. Thus, a quarter wavelength of line transforms the short circuit into an open circuit! This impedance transforming property of a transmission line is discussed in Secs. 3–5 and 3–6.

Line terminated in an open circuit. For an open-circuited line, $Y_L = 0$ and therefore $\Gamma_L = +1$ and the SWR is infinite. The voltage and current standing wave patterns are shown in Fig. 3–11. At the open circuit, $I = 0$, by definition, while the rms voltage is a maximum and equal to $2V_0^+$. Note that a quarter wavelength from the open, $V = 0$ and therefore the impedance at that point is zero, a short circuit. As always on a lossless line, a particular rms value of voltage or current reoccurs every half wavelength.

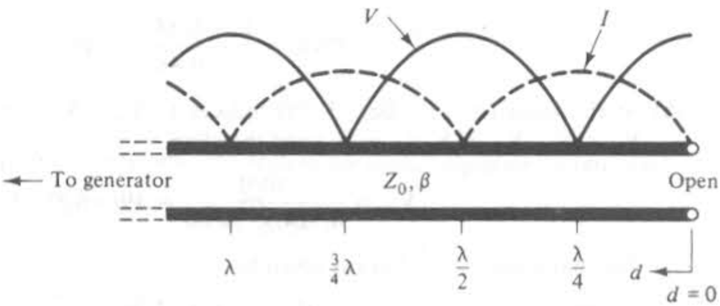


Figure 3–11 Standing-wave patterns on an open-circuited transmission line. ($\alpha = 0$.)

Reactively terminated lines. For a transmission line terminated in a purely reactive circuit, $Z_L = jX$ and therefore

$$\Gamma_L = \frac{j\bar{X} - 1}{j\bar{X} + 1} = 1\underline{/\pi - 2 \arctan \bar{X}} \tag{3–54}$$

where $\bar{X} \equiv X/Z_0$ is the normalized load reactance. For example, if $\bar{X} = +1$, $\Gamma_L = 1\underline{/90^\circ}$, while for $\bar{X} = -1$, $\Gamma_L = 1\underline{/-90^\circ}$.

Note that for a pure reactance, $|\Gamma_L| = 1$ and is independent of the value of $\bar{X}$. Thus the SWR is infinite. The physical interpretation is that in the steady state, a pure reactance cannot absorb power and hence all of the incident wave must be reflected. This same argument applies to the short and open-circuited cases. Consequently, $|\Gamma_L| < 1$ only when Z_L has a resistive component.

The standing wave patterns for $Z_L = jZ_0$ ($\bar{X} = +1$) are shown in Fig. 3–12. It is left to the reader to show that the first voltage null occurs at $d = 3\lambda/8$ (Prob. 3–17). This can be verified analytically or with the aid of a phasor diagram. The use of counterrotating phasors to determine the standing wave pattern was discussed in Sec. 2–7b and Fig. 2–28.

Resistively terminated lines. In many applications, the transmission line is terminated in a purely resistive network. That is, $Z_L = R_L$ and therefore $\Gamma_L = (R_L - Z_0)/(R_L + Z_0)$. Since R_L and Z_0 are real, Γ_L must be real. When $R_L = Z_0$, no

Substitution into Eq. (3–56) yields

$$P = \mathrm{Re}\left[(1 - |\Gamma|^2 + \Gamma - \Gamma^*)\frac{(V^+)^2}{Z_0^*}\right] \tag{3–58}$$

since $\Gamma\Gamma^* = |\Gamma|^2$ and $\mathbf{V}^+\mathbf{V}^{+*} = (V^+)^2$. For low-loss, high-frequency lines, Z_0 is real and the above expression reduces to

$$P = P^+(1 - |\Gamma|^2) = P^+ - P^- \tag{3–59}$$

since $\Gamma - \Gamma^*$ is imaginary. $P^+ = (V^+)^2/Z_0$ is the average power in the forward traveling wave and $P^- = |\Gamma|^2 P^+$ is the average power in the reflected wave. Equation (3–59) states that for Z_0 real, the net power flow at any point on the line is simply the difference between the power in the forward wave (P^+) and that in the reflected wave (P^-), where

$$P^- = |\Gamma|^2 P^+ \tag{3–60}$$

The same interpretation was given to the net power density of a uniform plane wave Eq. (2–85). This method of determining net power in a microwave circuit is very useful. It is the basis of the *power-wave* formulation described in Appendix D.

Equation (3–59) can be applied to any point along the line. For example, at the input and load terminals,

$$P_{\text{in}} = P_{\text{in}}^+(1 - |\Gamma_{\text{in}}|^2) \quad \text{and} \quad P_L = P_L^+(1 - |\Gamma_L|^2) \tag{3–61}$$

where Γ_{in} and Γ_L are defined in Eqs. (3–43) and (3–45). The reader is reminded that the + and − superscripts indicate quantities associated with the traveling waves. Thus

$$P_{\text{in}}^+ = \frac{(V_0^+)^2}{Z_0} \quad \text{and} \quad P_L^+ = \frac{(V_L^+)^2}{Z_0} = P_{\text{in}}^+ e^{-2\alpha l} \tag{3–62}$$

where P_L^+, the incident power at the load terminals, is simply P_{in}^+ attenuated by the line length l. If the line is lossless, $P_{\text{in}}^+ = P_L^+ = P^+$.

In order to calculate the *net* power flow at any point on the line, one must first determine P^+ at that point. With Z_0 real, $\mathbf{V}^+$ and $\mathbf{I}^+$ are in phase and therefore

$$P^+ = \frac{(V^+)^2}{Z_0} = \frac{(V_0^+)^2}{Z_0} e^{-2\alpha z} \tag{3–63}$$

$\mathbf{V}_0^+$ (and its rms value V_0^+) is obtained by applying Kirchhoff's voltage law at the input end. From Fig. 3–10,

$$\mathbf{V}_G = \mathbf{V}_{\text{in}} + \mathbf{I}_{\text{in}} Z_G = (\mathbf{V}_0^+ + \Gamma_{\text{in}}\mathbf{V}_0^+) + \frac{Z_G}{Z_0}(\mathbf{V}_0^+ - \Gamma_{\text{in}}\mathbf{V}_0^+)$$

Solving for $\mathbf{V}_0^+$, the forward traveling voltage wave at the input,

$$\mathbf{V}_0^+ = \frac{\mathbf{V}_G Z_0}{(Z_G + Z_0)(1 - \Gamma_G\Gamma_{\text{in}})} = \frac{\mathbf{V}_G Z_0}{(Z_G + Z_0)(1 - \Gamma_G\Gamma_L e^{-2\gamma l})} \tag{3–64}$$

where Γ_G, the reflection coefficient of the generator, is defined as

$$\Gamma_G = \frac{Z_G - Z_0}{Z_G + Z_0} \tag{3-65}$$

Thus, given the generator and load conditions ($\mathbf{V}_G$, Z_G, Z_L) and the line characteristics (Z_0, γ, l), $\mathbf{V}_0^+$ may be determined. An alternate form for Eq. (3–64) is

$$\mathbf{V}_0^+ = \frac{\mathbf{V}_G(1 - \Gamma_G)}{2(1 - \Gamma_G\Gamma_{\text{in}})} = \frac{\mathbf{V}_G(1 - \Gamma_G)}{2(1 - \Gamma_G\Gamma_L e^{-2\gamma l})} \tag{3-66}$$

Once $\mathbf{V}_0^+$ is known, the power flow can be determined at any point along the line.

Some insight into the meaning of $\mathbf{V}_0^+$ may be obtained by deriving Eq. (3–64) from a multiple reflection point of view. In Sec. 3–2, it was shown that the steady-state voltage and current values could be determined by considering the limiting process of multiple reflections. On the other hand, Eqs. (3–64) and (3–66) were derived from a steady-state ac analysis. We will now show that these two methods of analysis are merely alternate ways of viewing the same phenomena.

First, consider the special case when either Γ_G or Γ_L is zero (that is, no multiple reflections). Since the input impedance of the line *initially* appears to be Z_0, the voltage of the *initial* forward traveling wave is determined by the voltage-divider action between Z_G and Z_0. In the absence of multiple reflections, this is the only forward traveling wave and hence $\mathbf{V}_0^+ = (\mathbf{V}_G Z_0)/(Z_G + Z_0)$. This is exactly the result one obtains from Eq. (3–64) with either Γ_G or Γ_L equal to zero.

Let us now consider the situation in which neither Γ_G nor Γ_L is zero. In this case, multiple reflections occur and $\mathbf{V}_0^+$ *represents the phasor sum of all the forward traveling waves* at the input terminals. To verify this, Eq. (3–64) will now be derived from a multiple reflection point of view. As before, the *initial* forward traveling wave is $(\mathbf{V}_G Z_0)/(Z_G + Z_0)$. The next forward voltage wave is the portion of the initial wave that has been reflected at the load and rereflected at the generator. Its value is

$$(\Gamma_G\Gamma_L e^{-2\gamma l})(\mathbf{V}_G Z_0)/(Z_G + Z_0).$$

By continuing this process, the sum of all the forward traveling voltage waves at the input becomes

$$\mathbf{V}_0^+ = \frac{\mathbf{V}_G Z_0}{Z_G + Z_0}[1 + \Gamma_G\Gamma_L e^{-2\gamma l} + (\Gamma_G\Gamma_L e^{-2\gamma l})^2 + \cdots]$$

For $|\Gamma_G\Gamma_L e^{-2\gamma l}| < 1$, the infinite series converges to $(1 - \Gamma_G\Gamma_L e^{-2\gamma l})^{-1}$ and therefore the above expression reduces to Eq. (3–64). Thus the multiple reflection viewpoint is consistent with that based upon a steady-state ac analysis.

It is useful and informative to have expressions for P^+ and P in terms of available generator power P_A and reflection coefficients.[11] These may be derived with the

[11] From circuit theory, available power (P_A) is the maximum power that a source can deliver to a passive load. This occurs when the load impedance equals the complex conjugate of Z_G.

aid of Eqs. (3–59), (3–63), and (3–66). By substituting the rms value of $\mathbf{V}_0^+$ into Eq. (3–63), we obtain

$$P^+ = \frac{V_G^2}{4Z_0}\left|\frac{1-\Gamma_G}{1-\Gamma_G\Gamma_{in}}\right|^2 e^{-2\alpha z} \tag{3-67}$$

At the input $z = 0$,

$$P_{in}^+ = \frac{V_G^2}{4Z_0}\left|\frac{1-\Gamma_G}{1-\Gamma_G\Gamma_{in}}\right|^2 \tag{3-68}$$

where $\Gamma_{in} = \Gamma_L e^{-2\gamma l}$. When the line is lossless ($\alpha = 0$), $P^+ = P_{in}^+$ at any point along the line, including the load end.

The above equation may be rewritten in terms of the available power of the generator since $P_A = V_G^2/4R_G$, where R_G is the resistive portion of Z_G. Thus

$$P_{in}^+ = P_A \frac{1-|\Gamma_G|^2}{|1-\Gamma_G\Gamma_{in}|^2} \tag{3-69}$$

where use has been made of the following relations:

$$R_G = \frac{Z_G + Z_G^*}{2}, \quad Z_G = Z_0\frac{1+\Gamma_G}{1-\Gamma_G}, \quad Z_G^* = Z_0\frac{1+\Gamma_G^*}{1-\Gamma_G^*}$$

and the fact that $|1-\Gamma_G| = |1-\Gamma_G^*|$.

An expression for the *net* input power may be obtained by combining Eqs. (3–61) and (3–69).

$$P_{in} = P_A \frac{(1-|\Gamma_G|^2)(1-|\Gamma_{in}|^2)}{|1-\Gamma_G\Gamma_{in}|^2} \tag{3-70}$$

The net power delivered to the load is

$$P_L = P_A \frac{(1-|\Gamma_G|^2)(1-|\Gamma_L|^2)}{|1-\Gamma_G\Gamma_L e^{-2\gamma l}|^2} e^{-2\alpha l} \tag{3-71}$$

where use has been made of Eqs. (3–61), (3–62), and (3–69). When the line is lossless, $P_{in} = P_L$ and the above equations reduce to

$$P_{in} = P_L = P_A \frac{(1-|\Gamma_G|^2)(1-|\Gamma_L|^2)}{|1-\Gamma_G\Gamma_L e^{-j2\beta l}|^2} \tag{3-72}$$

since $|\Gamma_{in}| = |\Gamma_L|$ and $\Gamma_{in} = \Gamma_L e^{-j2\beta l}$.

These equations are very important in understanding power flow in a microwave circuit. For example, if $Z_G = Z_0$, $\Gamma_G = 0$ and therefore for the lossless line case, P_L is independent of the phase constant $\beta = \omega/v$ and the length of the transmission line l. This is a distinct practical advantage since it means that load power is insensitive to changes in generator frequency and line length. Designing the circuit so that $Z_G = Z_0$ is referred to as *matching the generator to the line*. Well-designed microwave systems usually satisfy this condition. With $\Gamma_G = 0$, Eqs. (3–66) and (3–69) reduce to

$$\mathbf{V}_0^+ = \frac{\mathbf{V}_G}{2} \quad \text{and} \quad P_{in}^+ = P_A = \frac{V_G^2}{4Z_0} \tag{3-73}$$

Thus, with the generator matched to the line, the power associated with the incident wave at the input is simply the available generator power. As a result, Eqs. (3–70) and (3–71) reduce to

$$P_{in} = P_A(1 - |\Gamma_{in}|^2) \quad \text{and} \quad P_L = P_A e^{-2\alpha l}(1 - |\Gamma_L|^2) \tag{3–74}$$

For a lossless line,

$$P_{in} = P_L = P_A(1 - |\Gamma_L|^2) \tag{3–75}$$

Note that if the line is lossless and the generator is matched, matching the load to the line ($\Gamma_L = 0$) results in all the available generator power being delivered to the load. However if $Z_G \neq Z_0$, matching the load to the line does *not* result in $P_L = P_A$. This can be seen from Eq. (3–72). In fact, maximizing P_L requires a standing wave (that is, $\Gamma_L \neq 0$) and proper adjustment of the line length l. Examples of this are the half-wave line and the quarter-wave transformer discussed in Sec. 3–6.

The following brief example shows the usefulness of the above equations.

Example 3–3:

The transmission line circuit in Fig. 3–10 has the following parameters: $V_G = 20$ V rms, $Z_G = 100$ ohms, $f = 500$ MHz, $Z_0 = 100$ ohms, $l = 4$ m and $Z_L = 150$ ohms. Calculate the input and load power if

(a) $\alpha = 0$ (b) $\alpha = 0.5$ dB/m.

Solution: The available power is $P_A = (20)^2/400 = 1.0$ W. From Eq. (3–45),

$$\Gamma_L = \frac{150 - 100}{150 + 100} = 0.20.$$

(a) With $Z_G = Z_0$ and $\alpha = 0$, P_{in} and P_L are given by Eq. (3–75). Therefore,

$$P_{in} = P_L = (1.0)[1 - (0.20)^2] = 0.96 \text{ W}.$$

(b) For $\alpha = 0.5$ dB/m, $\alpha l = 2.0$ dB or 0.23 Np. From Eq. (3–43),

$$|\Gamma_{in}| = 0.20\, e^{-2(0.23)} = 0.126.$$

P_{in} and P_L are determined from Eq. (3–74). Thus,

$$P_{in} = 0.984 \text{ W} \quad \text{and} \quad P_L = 0.605 \text{ W}.$$

The difference, 0.379 W, is dissipated in the lossy line.

Note that with $Z_G = Z_0$, the source frequency does not enter into the power calculations. However, if $Z_G \neq Z_0$, P_L is a function of βl as seen in Eq. (3–72). Problem 3–20 gives some indication of the variation P_L with βl.

Return and reflection losses. Two terms commonly used in conjunction with reflected signals are *return loss* and *reflection loss*. Return loss, denoted by L_R, is defined as

$$L_R \equiv 10 \log \frac{P^+}{P^-} \text{ dB} \tag{3–76}$$

From Eq. (3–60), $P^- = |\Gamma|^2 P^+$ and therefore

$$L_R = 10 \log \frac{1}{|\Gamma|^2} \text{ dB} \tag{3–77}$$

If the line is lossless, the return loss is the same everywhere along the line since $|\Gamma|$ is independent of position. For a lossy line, L_R is a function of position since from Eq. (3–42), $|\Gamma| = |\Gamma_L|e^{-2\alpha d}$, where αd is in nepers. By using Eqs. (3–43) and (3–77), a relation between the return loss at the input and its value at the load is obtained.

$$L_{R_{\text{in}}} = L_{R_{\text{load}}} + 2(8.686\alpha l) \tag{3–78}$$

where l is the length of the lossy line. This equation states that the input return loss equals the load return loss plus the *round-trip* attenuation (in dB) due to the lossy line.

The concept of reflection loss is generally used only when $Z_G = Z_0$. It is a measure of the reduction in load power due to the load impedance having a value other than Z_0. That is,

$$\text{REF. LOSS} \equiv 10 \log \frac{P_L \text{ if } Z_L = Z_0}{P_L \text{ in } Z_L} \quad \text{dB} \tag{3–79}$$

For $Z_G = Z_0$, the denominator is given by Eq. (3–74) while the numerator equals $P_A e^{-2\alpha l}$ and therefore

$$\text{REF. LOSS} = 10 \log \frac{1}{1 - |\Gamma_L|^2} = 10 \log \frac{(\text{SWR} + 1)^2}{4(\text{SWR})} \tag{3–80}$$

For example, if $|\Gamma_L| = 0.707$ (an SWR of 5.83), the reflection loss is 3 dB, which for a lossless line means P_L is half the available generator power. Reflection loss is sometimes referred to as *mismatch loss*. This concept is extended in Chapter 4 to include any linear two-port network.

3–5 THE IMPEDANCE TRANSFORMATION

In discussing the short-circuited line, it was observed that the impedance became infinite at a distance one quarter wavelength from the short. The ability to change impedance by adding a length of transmission line is very important to the microwave engineer. In this section, the general impedance transformation equation is derived. Examples of its effect and use are given in this and the following section. The equation involves hyperbolic functions with complex arguments. Appendix E contains a brief review of these functions.

Derivation of the impedance transformation equation. In the previous section, it was shown that the reflection coefficient is a function of position along the transmission line (Eq. 3–40). Since the reflection coefficient is related to impedance via Eq. (3–47), it is apparent that impedance must also be a function of position.[12]

[12] It is suggested that the reader review the definition of Z given in the footnote accompanying Eq. (3–47).

This relationship is called the impedance transformation equation and will now be derived. From Eqs. (3–47) and (3–40),

$$Z = Z_0 \frac{1 + \Gamma}{1 - \Gamma} = Z_0 \frac{1 + \Gamma_L e^{-2\gamma d}}{1 - \Gamma_L e^{-2\gamma d}}$$

Making use of Eq. (3–45),

$$Z = Z_0 \frac{(Z_L + Z_0)e^{\gamma d} + (Z_L - Z_0)e^{-\gamma d}}{(Z_L + Z_0)e^{\gamma d} - (Z_L - Z_0)e^{-\gamma d}}$$

Rearranging terms and using Eq. (E–3) in Appendix E yields the impedance transformation equation.

$$Z = Z_0 \frac{Z_L + Z_0 \tanh \gamma d}{Z_0 + Z_L \tanh \gamma d} \quad \text{or} \quad \bar{Z} = \frac{\bar{Z}_L + \tanh \gamma d}{1 + \bar{Z}_L \tanh \gamma d} \tag{3–81}$$

where

$$\gamma = \alpha + j\beta, \qquad \bar{Z} \equiv Z/Z_0 \qquad \text{and} \qquad \bar{Z}_L \equiv Z_L/Z_0.$$

The transformation equation can also be expressed in terms of admittance. Namely,

$$Y = Y_0 \frac{Y_L + Y_0 \tanh \gamma d}{Y_0 + Y_L \tanh \gamma d} \quad \text{or} \quad \bar{Y} = \frac{\bar{Y}_L + \tanh \gamma d}{1 + \bar{Y}_L \tanh \gamma d} \tag{3–82}$$

where

$$Y_0 \equiv 1/Z_0, \qquad \bar{Y} \equiv Y/Y_0 \qquad \text{and} \qquad \bar{Y}_L \equiv Y_L/Y_0.$$

Referring to Fig. 3–10, the input impedance or admittance is obtained by merely setting $d = l$.

In many practical situations, the line attenuation can be neglected and hence it is useful to write the above equations for the lossless case. For $\alpha = 0$, $\gamma d = j\beta d$ and from Eq. (E–6), $\tanh \gamma d$ becomes $j \tan \beta d$. Thus for the lossless line,

$$Z = Z_0 \frac{Z_L + jZ_0 \tan \beta d}{Z_0 + jZ_L \tan \beta d} \quad \text{or} \quad \bar{Z} = \frac{\bar{Z}_L + j \tan \beta d}{1 + j\bar{Z}_L \tan \beta d} \tag{3–83}$$

and

$$Y = Y_0 \frac{Y_L + jY_0 \tan \beta d}{Y_0 + jY_L \tan \beta d} \quad \text{or} \quad \bar{Y} = \frac{\bar{Y}_L + j \tan \beta d}{1 + j\bar{Y}_L \tan \beta d} \tag{3–84}$$

These transformation equations are quite remarkable. The following example shows how even a small length of low-loss line can produce a dramatic change in impedance.

Example 3–4:

A 15 cm length of air-insulated coaxial line is terminated in a short circuit. The characteristics of the line are $Z_0 = 75$ ohms and $\alpha = 0.4$ dB/m. Calculate the input impedance at 1500 MHz and 2000 MHz.

Solution: For the air-insulated line, $\mu_R = \epsilon_R = 1$ and therefore $\lambda = \lambda_0$. At 1500 MHz, $\lambda_0 = 20$ cm and therefore $\beta l = (2\pi/20)15 = 3\pi/2$ rad. Also, $\alpha l = (0.4)(0.15) = 0.06$ dB or 0.007 Np. Substituting $Z_L = 0$ and $d = l$ into Eq. (3–81) yields $Z_{in} = 75 \tanh (\alpha l + j\beta l)$. Using Eq. (E–6) and the fact that $\tanh \alpha l \approx \alpha l$,

$$Z_{in} = 75 \frac{0.007 + j \tan (3\pi/2)}{1 + j0.007 \tan (3\pi/2)}$$

Since $\tan (3\pi/2) = -\infty$, the input impedance equals $75/0.007 = 10{,}714$ ohms. Repeating the calculation at 2000 MHz ($\lambda_0 = 15$ cm), $\beta l = 2\pi$ rad and therefore $Z_{in} = 0.525$ ohms.

This example shows that a 15 cm section of low-loss line transforms a short circuit into 10,714 ohms at 1500 MHz! Furthermore, increasing the frequency by one-third reduces the input impedance to a mere 0.525 ohms. What is the explanation for these results? From a circuit point of view, one can say that because a transmission line contains distributed inductance and capacitance (Fig. 3–1), it is conceivable that their addition to the circuit will result in an impedance value radically different from the load impedance Z_L. In fact, impedance transformers consisting of series inductors and shunt capacitors are commonly used in radio engineering work.

An alternate way of explaining the impedance transformation effect is in terms of standing waves. For $Z_L \neq Z_0$, standing waves of voltage and current exist on the line. Because the current pattern is shifted relative to the voltage pattern, the ratio of $\mathbf{V}$ to $\mathbf{I}$ is a strong function of position d. Therefore, it *makes sense* that the impedance along the line (for example, at the input) can be significantly different from the load impedance. Conversely, without standing waves no impedance transformation can occur. To verify this, let $Z_L = Z_0$. In this case, $\Gamma_L = 0$, SWR = 1.00 and hence no standing waves exist. From Eq. (3–81), it is clear that with $Z_L = Z_0$, $Z = Z_0$, *independent* of d, which means no impedance transformation. The reason this transformation effect is not observed at low frequencies is that with $l \ll \lambda$, $\tan \beta l \approx 0$ and Eq. (3–81) reduces to $Z = Z_L$ for low-loss lines.

In the illustrative example given, an increase in frequency resulted in a large change in input impedance. The reason is that for a fixed length line, $\tan \beta l$ is a function of frequency since $\beta = \omega/v$. The fact that the input impedance is frequency dependent is a *two-edged* sword. It is an obstacle in designing broadband circuits, but is a very useful property in the design of microwave resonators and filters.

3–6 EXAMPLES OF THE IMPEDANCE TRANSFORMATION

In this section, several special cases of the impedance transformation are examined. All of the results obtained are directly applicable to a variety of microwave design problems. Situations in which the transmission line may be assumed lossless are considered first.

3-6a Impedance Transformations on a Lossless Line

For $\alpha = 0$, the impedance and admittance transformations are given by Eqs. (3–83) and (3–84). Let us now analyze some special cases of these equations.

The half-wavelength line. Figure 3–10 shows a transmission line connected to a load impedance Z_L. If the line length is a multiple of a half wavelength ($l = n\lambda/2$, n being any positive integer), the input impedance is equal to Z_L. This may be seen by substituting $\beta l = n\pi$ into Eq. (3–83). This result can be generalized to any two points that are separated by a multiple of a half wavelength. Since $\tan(\beta d + n\pi) = \tan \beta d$, impedance repeats every half wavelength along a lossless line.

An interesting case of the half-wavelength line occurs when $Z_G = Z_L$, both real. Suppose $Z_L = Z_G = 20$ ohms and $Z_0 = 100$ ohms. From Eq. (3–55), the SWR along the line is 5 to 1. With $l = n\lambda/2$, $Z_{in} = Z_L = 20$ ohms and therefore all the available generator power is delivered down the lossless line to the load impedance. Thus, despite the fact that standing waves exist on the line, maximum power is delivered to the load. The standing waves represent stored electric and magnetic energy, much like that in an L-C circuit. This situation was alluded to in Sec. 3–4c.

The quarter-wavelength line. Suppose now that the length of line shown in Fig. 3–10 is an odd multiple of a quarter wavelength. With $l = (2n - 1)\lambda/4$, n being a positive integer, $\tan \beta l \to \infty$ and Eq. (3–83) reduces to

$$Z_{in} = \frac{Z_0^2}{Z_L} \qquad \text{or} \qquad \bar{Z}_{in} = \frac{1}{\bar{Z}_L} \tag{3–85}$$

This equation shows that a quarter-wave line transforms a large (small) value of load impedance into a small (large) value at the input. Also, an inductive (capacitive) load is transformed into a capacitive (inductive) input impedance. Furthermore, one can show that if Z_L is a series resonant circuit, Z_{in} behaves like a parallel resonant circuit, and vice versa. The quarter-wavelength line is often called an impedance inverter since it inverts the normalized load impedance. It is also referred to as a quarter-wave transformer since for Z_L real it has all the properties of an ideal transformer. Figure 3–13 shows the equivalence for specific values of Z_0 and Z_L. For the values given, the SWR along the line is five (100/20). In this example, $Z_L < Z_0$ and hence a voltage minimum and current maximum exist at the load. A quarter wavelength away at the input, the voltage is maximum and the current minimum. The values are given in part *a* of the figure. From Eq. (3–85), the input impedance is 500 ohms. Therefore, the lossless quarter-wavelength line behaves like an ideal transformer with a turns ratio $n_t = 5$, the SWR value. For $Z_L > Z_0$, the line is equivalent to a transformer with $n_t < 1$ or $N_1 < N_2$.

The quarter-wave transformer is useful in matching a resistive load to a generator, which is a necessary condition for delivering all the available generator power to the load. If the generator impedance is real, then Z_{in} must equal R_G for maximum

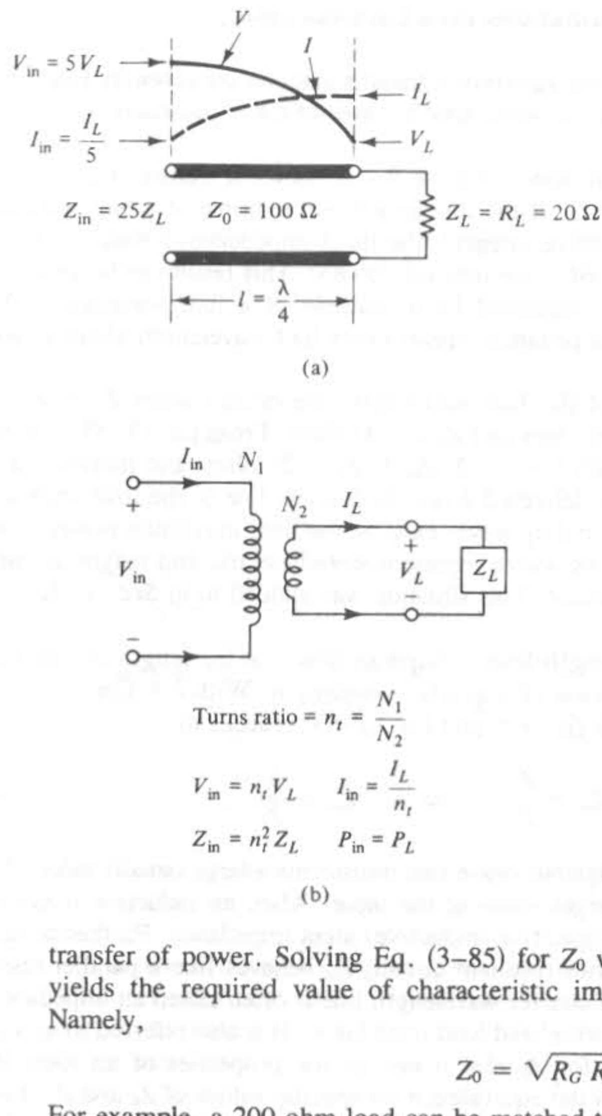


Figure 3–13 The quarter-wave transformer and its equivalent circuit when $Z_L = R_L$.

transfer of power. Solving Eq. (3–85) for Z_0 when the load is resistive ($Z_L = R_L$) yields the required value of characteristic impedance for the quarter-wave line. Namely,

$$Z_0 = \sqrt{R_G R_L} \tag{3–86}$$

For example, a 200 ohm load can be matched to a 50 ohm generator by inserting a quarter-wavelength section of 100 ohm line between them. If the line is lossless, $P_{in} = P_L$ and therefore the available generator power (P_A) is delivered to the load. Since the line can only be a quarter-wavelength long at one frequency, the transformer is frequency sensitive. The frequency characteristics of single and multisection transformers are discussed in Sec. 4–2.

The short-circuited line. When a lossless line is shorted at the load end, $Z_L = 0$ and from Eq. (3–83) the input impedance is

$$Z_{in} = jX_{in} = jZ_0 \tan \beta l \qquad \text{or} \qquad \bar{Z}_{in} = j\bar{X}_{in} = j \tan \beta l \tag{3–87}$$

A plot of normalized input reactance versus βl is shown in Fig. 3–14. It is simply a graph of the tangent function. For $\beta l < \pi/2$ ($l < \lambda/4$), the input impedance is inductive since X_{in} is positive. Thus $X_{\text{in}} = \omega L_{\text{eq}}$ and

$$L_{\text{eq}} = \frac{X_{\text{in}}}{\omega} = \frac{Z_0}{\omega}\tan\beta l \qquad \text{henries} \tag{3–88}$$

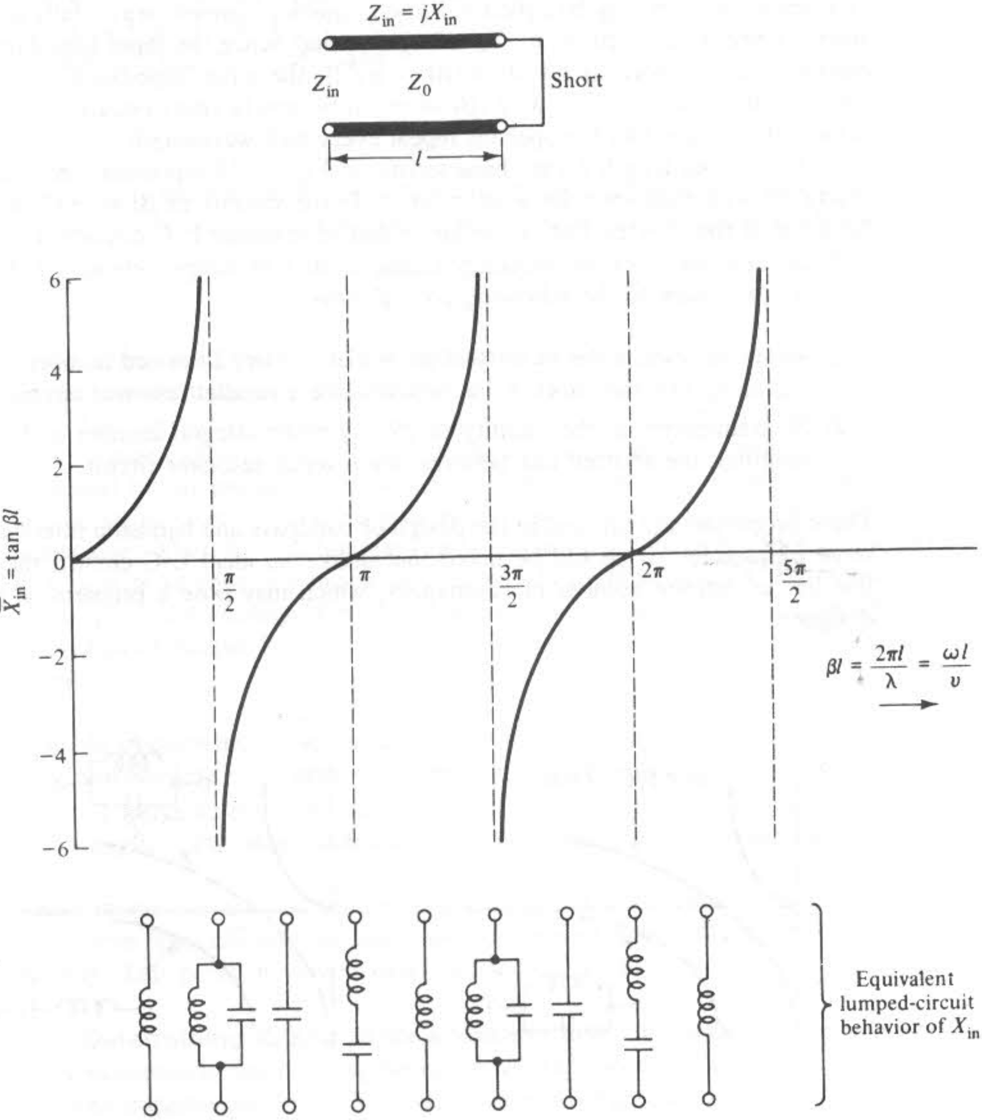


Figure 3–14 Normalized input reactance versus βl for a short-circuited, lossless transmission line.

where L_{eq} is the equivalent inductance of a shorted line with $l < \lambda/4$. For $\beta l <$ 0.5 rad ($l < 0.08\lambda$), $\tan \beta l \approx \beta l$ and the above equation reduces to

$$L_{eq} \approx \frac{Z_0 l}{v} = L'l \qquad \text{henries} \tag{3–89}$$

An inductive result for the shorted line appears logical since its configuration, shown in Fig. 3–14, is that of a one-turn loop or coil. For $\beta l \geq \pi/2$, however, our intuitive sense (or what may be called our *low-frequency common sense*) fails us. For example, when $\pi/2 < \beta l < \pi$, X_{in} is negative and hence the input impedance is capacitive. Furthermore, if $l = \lambda/4$ ($\beta l = \pi/2$), the input impedance is infinite (an open circuit), while for $l = \lambda/2$ ($\beta l = \pi$), it is zero (a short circuit). As discussed earlier, these impedance properties repeat every half wavelength.

For a fixed length l, the characteristic in Fig. 3–14 represents the variation of reactance with frequency since $\beta l = \omega l/v$. In the vicinity of $\beta l = \pi/2$ and π, the behavior of the shorted line is similar to that of resonant L-C circuits. Comparison with the reactance versus frequency characteristics of lumped-element L-C circuits (Fig. 3–15a) leads to the following equivalences:

1. At frequencies in the vicinity of $\beta l = (2n - 1)\pi/2$ (an odd number of quarter wavelengths), the shorted line behaves like a parallel resonant circuit.
2. At frequencies in the vicinity of $\beta l = n\pi$ (an integral number of half wavelengths), the shorted line behaves like a series resonant circuit.

These properties are utilized in the design of bandpass and bandstop filters at microwave frequencies. It should be noted that unlike an ideal L-C circuit, the shorted line has an infinite number of resonances, which may pose a problem to the filter designer.

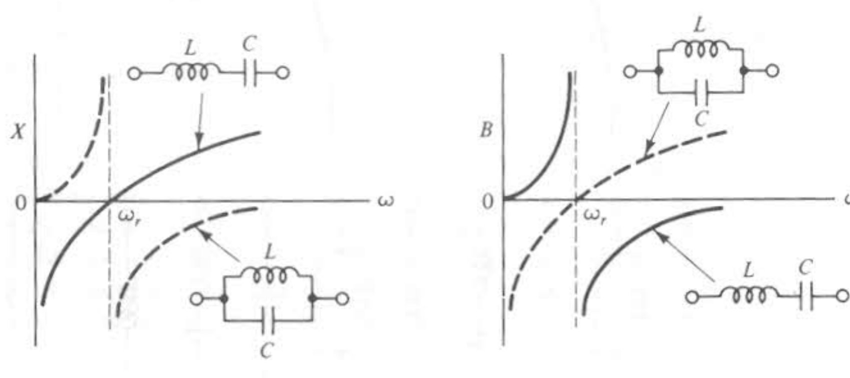


Figure 3–15 Frequency variation of reactance and susceptance for series and parallel resonant circuits (resonant frequency $\omega_r = 1\sqrt{LC}$).

The open-circuited line. When a lossless line is open circuited at the load end, $Y_L = 0$ and from Eq. (3–84) the input admittance is

$$Y_{in} = jB_{in} = jY_0 \tan \beta l \qquad \text{or} \qquad \overline{Y}_{in} = j\overline{B}_{in} = j \tan \beta l \tag{3–90}$$

Since the input impedance is the reciprocal of Y_{in},

$$Z_{in} = -jZ_0 \cot \beta l \tag{3–91}$$

A plot of normalized input susceptance versus βl is the tangent function and is shown in Fig. 3–16. For $\beta l < \pi/2$ ($l < \lambda/4$), the input is capacitive since B_{in} is positive. Thus $B_{in} = \omega C_{eq}$ and

$$C_{eq} = \frac{Y_0}{\omega} \tan \beta l \qquad \text{farads} \tag{3–92}$$

where C_{eq} is the equivalent capacitance of an open-circuited line with $l < \lambda/4$. For $\beta l < 0.5$ rad ($l < 0.08\lambda$), the above equation may be approximated by

$$C_{eq} \approx \frac{Y_0 l}{v} = \frac{l}{Z_0 v} = C'l \qquad \text{farads} \tag{3–93}$$

It should be noted that C_{eq} is independent of frequency only when $\beta l < 0.5$ rad. A similar comment applies to L_{eq} of a shorted line.

The capacitive result for the open-circuited line seems plausible since its configuration, shown in Fig. 3–16, looks like a capacitor (namely, two conductors separated by an insulator). However, as before, our *low-frequency common sense* fails us for $\beta l \geq \pi/2$. In the region $\pi/2 < \beta l < \pi$, the input is inductive since B_{in} is negative. Also, $Y_{in} = 0$ (an open circuit) when l is a multiple of a half wavelength and $Y_{in} = \infty$ (a short circuit) when l is an odd multiple of $\lambda/4$. By comparing B_{in} versus βl in the vicinity of $\pi/2$ and π with the susceptance versus frequency characteristics of simple L-C circuits (Fig. 3–15b), the following equivalences can be stated.

1. At frequencies in the vicinity of $\beta l = (2n - 1)\pi/2$ (an odd number of quarter wavelengths), the open-circuited line behaves like a series resonant circuit.
2. At frequencies in the vicinity of $\beta l = n\pi$ (an integral number of half wavelengths), the open-circuited line behaves like a parallel resonant circuit.

These properties are also utilized in the design of microwave filters.

Both open and shorted lines may be modeled as an infinite set of resonant L-C circuits. This point of view is described in Sec. 11.14 of Ref. 3–8.

Determining Z_0 and βl for a lossless line. Occasionally one is interested in experimentally determining the characteristic impedance and phase constant for a low-loss transmission line. Their values at a given frequency can be obtained from two measurements on a fixed length of line (l). With the load end shorted, measure the input impedance (denoted by Z_{sc}). Next, measure the input impedance with the

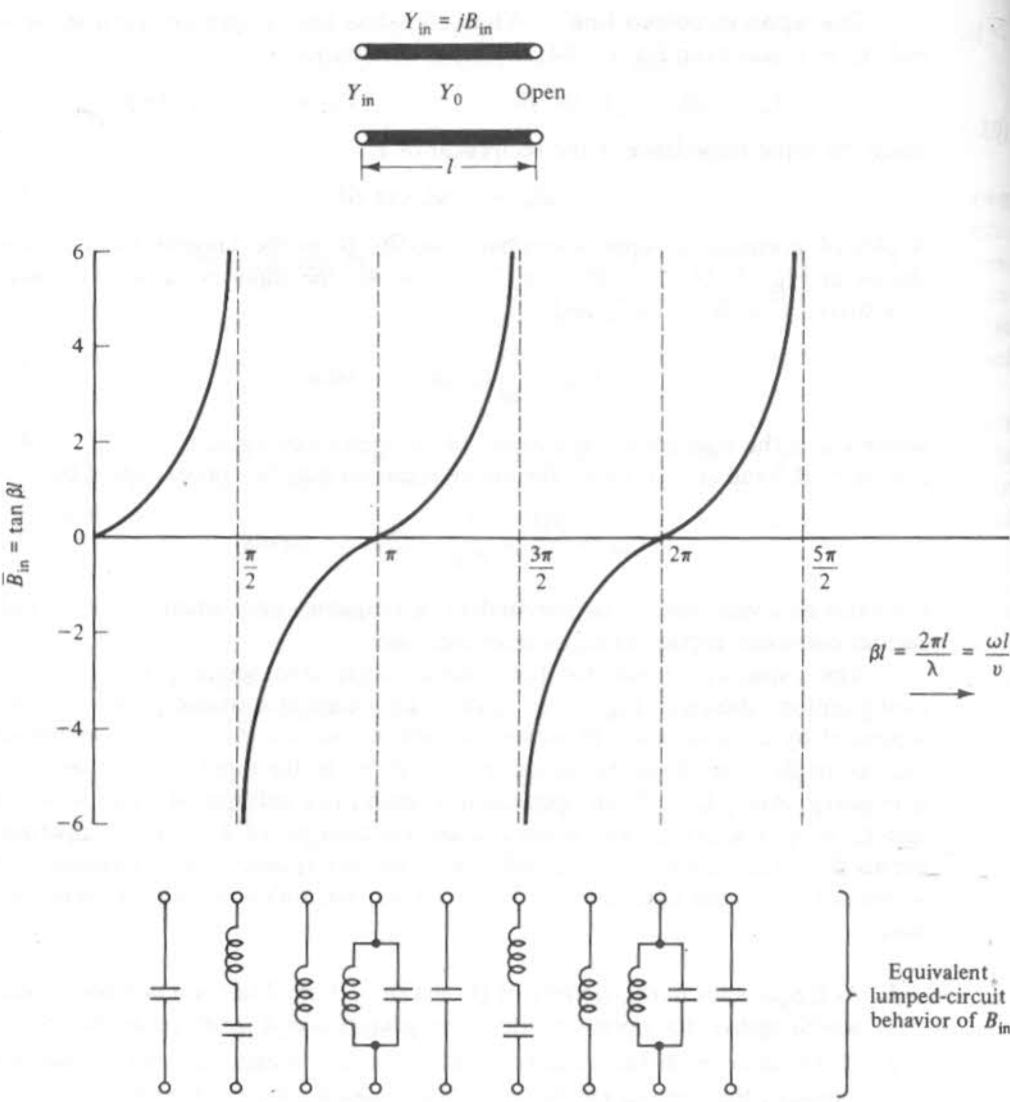


Figure 3–16 Normalized input susceptance versus βl for an open-circuited, lossless transmission line.

load end open circuited (denoted by Z_{oc}). Z_{sc} and Z_{oc} are given by Eqs. (3–87) and (3–91). Solving for Z_0 and βl yields

$$Z_0 = \sqrt{Z_{sc}\, Z_{oc}} \qquad \text{and} \qquad \beta l = \arctan \sqrt{-\frac{Z_{sc}}{Z_{oc}}} \tag{3–94}$$

For a lossless line Z_{sc} and Z_{oc} are reactive and of opposite sign. Since impedance repeats every half wavelength there are an infinite number of solutions for βl. In the absence of additional information, the primary solution ($0 \leq \beta l \leq \pi$) should be used. To determine whether the correct value is in the first or second quadrant, the two possible solutions should be checked against the given values of Z_{sc} and Z_{oc} (Prob. 3–38). This procedure gives the correct value of β if one knows that the line is less than a half wavelength long. If a longer specimen of the line is subsequently measured in the same manner, a still more accurate value of β can be determined after using the previous value to estimate the number of half wavelengths contained within the line.

The application of this technique to lossy lines is described in Sec. 4.7 of Ref. 3–2.

3–6b Impedance Transformations on a Lossy Line

For $\alpha \neq 0$, the impedance and admittance transformations are given by Eqs. (3–81) and (3–82). Three specific examples will now be considered.

The short-circuited line. For $Z_L = 0$, Eq. (3–81) reduces to $Z = Z_0 \tanh(\alpha d + j\beta d)$. By using Eq. (E–6) and letting $d = l$, the input impedance may be written as

$$Z_{in} = Z_0 \frac{\tanh \alpha l + j \tan \beta l}{1 + j \tanh \alpha l \tan \beta l} \tag{3–95}$$

If $\alpha = 0$ and $\beta l = n\pi$, then $Z_{in} = 0$, as discussed earlier. However, if the line is lossy, the input impedance is finite. For $\alpha \neq 0$ and the shorted line an integral number of half wavelengths,

$$Z_{in} = Z_0 \tanh \alpha l \approx Z_0 \alpha l \qquad \text{ohms} \tag{3–96}$$

If the line is an odd number of quarter wavelengths, $\tan \beta l \to \infty$. For a lossless line, this results in an open circuit at the input. For a lossy line, Eq. (3–95) reduces to

$$Z_{in} = \frac{Z_0}{\tanh \alpha l} \approx \frac{Z_0}{\alpha l} \qquad \text{ohms} \tag{3–97}$$

In both the above equations, the approximate expressions are valid for $\alpha l < 0.5$ Np. For the two cases considered here, the input impedance is finite and real.

The open-circuited line. Similar relations can be developed for a lossy, open-circuited line. In this case, $Y_L = 0$ and from Eq. (3–82) with $d = l$, the input admittance becomes $Y_{in} = Y_0 \tanh(\alpha l + j\beta l)$. For $\beta l = n\pi$, the input admittance is finite and real and is given by

$$Y_{in} = Y_0 \tanh \alpha l \approx Y_0 \alpha l = \frac{\alpha l}{Z_0} \qquad \text{mhos} \tag{3–98}$$

On the other hand, if $\beta l = (2n - 1)\pi/2$, the input admittance is given by

$$Y_{in} = \frac{Y_0}{\tanh \alpha l} \approx \frac{Y_0}{\alpha l} = \frac{1}{Z_0 \alpha l} \qquad \text{mhos} \tag{3–99}$$

Since $Z_{in} = 1/Y_{in}$, these results are the same as Eqs. (3–97) and (3–96) respectively.

The shorted line as an inductor. The input impedance of a lossless, short-circuited line is given by Eq. (3–87). For $l < \lambda/4$, the input is purely inductive. Any value of inductive reactance, no matter how large, is realizable by merely choosing βl so that $Z_0 \tan \beta l$ yields the desired value. Furthermore, the inductive reactance is achieved with no accompanying resistive component. One is tempted to say that, as long as a *low-loss line* is used, the resistive component will be negligible. The following example shows that this is not necessarily true.

Example 3–5:

The design of a microwave amplifier requires an inductor to series resonate a capacitive reactance of 15,000 ohms. Calculate the residual resistance if the inductor consists of a short-circuited line with $Z_0 = 75$ ohms, $\alpha = 0.002$ dB/cm, and $\lambda = 10$ cm.

Solution: To series resonate the $-j15{,}000$ ohms, the input impedance of the shorted line must be $+j15{,}000$ ohms. If the line were lossless, $Z_{in} = j15{,}000 = j75 \tan \beta l$ and therefore

$$\beta l = \arctan 200 \approx \frac{\pi}{2} - \frac{1}{200} \qquad \text{rad}$$

Since $\beta = 2\pi/\lambda$,

$$l = 2.5\left(1 - \frac{1}{100\pi}\right) \qquad \text{cm}$$

Thus a shorted lossless line about 0.3 percent less than a quarter wavelength would provide the desired reactance without adding any resistance.

All lines however have some loss. In this case, $\alpha l = 0.005$ dB or 5.76×10^{-4} Np. Since $\tanh \alpha l \approx \alpha l$, substituting $Z_0 = 75$ ohms and $\tan \beta l = 200$ into Eq. (3–95) yields

$$Z_{in} = 75\,\frac{5.76 \times 10^{-4} + j200}{1 + j0.115} = 1700 + j14{,}800 \qquad \text{ohms}$$

Note the effect of line loss! First of all, the reactive portion is reduced slightly. This presents no problem since a slight adjustment in line length can increase it to the required value of 15,000 ohms. Second, and much more important, a significant resistive component has been added. This means that series resonating the capacitive reactance with the shorted line leaves a residual resistance of 1700 ohms, surely not a negligible value. (With the line length adjusted for a reactance of 15,000 ohms, the residual resistance is acutally 1748 ohms.)

This example shows that despite the use of a low-loss line, the quality factor of the inductor is poor ($Q \equiv |X|/R \approx 8.7$). This is because the requirement for a large reactance necessitates that the shorted line be operated near resonance (that is, $l \approx \lambda/4$). A general expression for Q can be derived from Eq. (3–95). Assuming $\tanh \alpha l \approx \alpha l$, the quality factor of the shorted line when used as an inductor is

$$Q = \frac{\bar{X}}{\alpha l}\left[\frac{1 - (\alpha l)^2}{1 + \bar{X}^2}\right] \approx \frac{\bar{X}}{\alpha l\,(1 + \bar{X}^2)} \tag{3–100}$$

where $\bar{X} \equiv X/Z_0 = \tan \beta l$, the normalized value of required reactance. For $\alpha l < 10^{-3}$ Np, the quality factor will be greater than 100 if the required reactance is less than 10 Z_0 (that is, $X < 10$). The maximum realizable value of inductive reactance is also limited by the attenuation. One can show, using Eq. (3–95), that the maximum possible value of $X_L \approx Z_0/2\alpha l$. A similar problem occurs when attempting to realize large values of susceptance.

In general, the short terminating the line is never perfect. That is, it has finite resistance R, where $R \ll Z_0$. The maximum Q that can be attained in this case is also limited (Prob. 3–42), as is the maximum value of X_L.

3-7 THE SMITH CHART

A graphical procedure for solving impedance transformation problems is described in this section. The technique makes use of a special impedance/admittance chart developed by P. H. Smith (Refs. 3–9 to 3–11). As in any well-structured graphical method, the advantages are a reduction in the computational effort required and an improved intuitive understanding of how the individual variables affect the desired end result. The first reason is important in analysis, while the second is vital to good synthesis and design. In practice, high-frequency circuits often contain two or more transmission lines interspersed with series and shunt elements. The Smith chart technique can significantly reduce the numerical and algebraic manipulations required to solve such problems. Of course, one can also develop computer programs to solve transmission-line circuits. In these cases, a Smith chart analysis is useful in verifying the validity of the computer solutions.

3-7a The Basis of the Smith Chart

The Smith chart is a specially constructed impedance/admittance diagram for use in solving transmission-line problems. As such, it has a pair of coordinates for plotting complex values of impedance and admittance. The chart has several useful characteristics. First, all possible values of impedance and admittance can be plotted on the chart. Second, an easy method for converting impedance to admittance (and vice versa) is available. Third, and most important, the Smith chart provides a simple graphical method for determining the impedance transformation due to a length of transmission line. An ordinary rectangular chart with resistance plotted horizontally and reactance vertically has none of these advantages.

To understand the basis of the Smith chart, consider the polar coordinate system shown in Fig. 3–17. By convention, positive angles are plotted counterclockwise. For any passive impedance (that is, Re $Z \geq 0$) and Z_0 real, the magnitude of the reflection coefficient is less than or equal to one.[13] Thus all possible values of Γ can be plotted on a polar chart having a maximum radius value of unity. Such a

[13] These two conditions are assumed in all subsequent discussions. The case wherein Z_0 is complex is discussed briefly in Sec. 9.1 of Ref. 3–3.

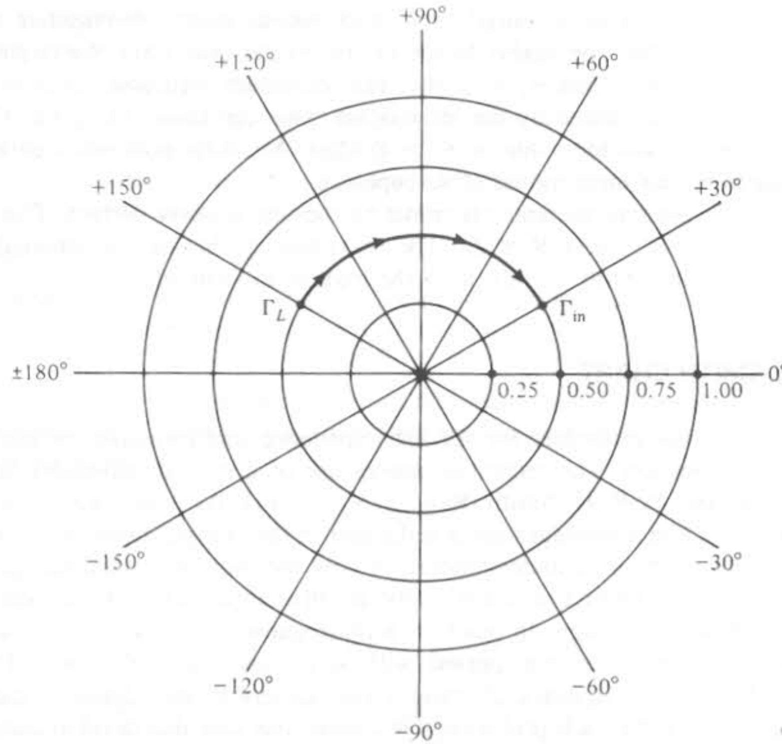


Figure 3–17 A polar chart for plotting complex reflection coefficients, $|\Gamma|\underline{/\phi}$.

chart is shown in Fig. 3–17. It can accommodate reflection coefficient values for all impedances from a short ($\Gamma = 1\underline{/180°}$) to an open ($\Gamma = 1\underline{/0}$). Furthermore, the change in reflection coefficient due to a length of transmission line (Eq. 3–43) is easily determined with the polar chart, particularly if the line is lossless.

Consider a load impedance $Z_L = 17.7 + j11.8$ ohms connected to a lossless 50 ohm line of length $l = \lambda/6$. From Eq. (3–45), $\Gamma_L = 0.5\underline{/150°}$, which is plotted in Fig. 3–17. The input reflection coefficient may be determined from Eq. (3–43). For $\alpha = 0$, $\Gamma_{in} = 0.5\underline{/150° - 120°}$, since $2\beta l = 2\pi/3$ rad or 120°. This same result can be obtained graphically using the polar chart. Since $|\Gamma_{in}| = |\Gamma_L|$ for a lossless line, merely rotate *clockwise* from the Γ_L point on the 0.5 radius circle an angular distance of 120° (namely, $2\beta l$), as indicated in Fig. 3–17. The reflection coefficient at any other point on the line is obtained by rotating clockwise $2\beta d$ where d is the distance from the load to the point.[14] Conversely, if Γ is known, Γ_L is obtained by rotating *counterclockwise* $2\beta d$.

[14] If the line is lossy ($\alpha \neq 0$), $|\Gamma| = |\Gamma_L|e^{-2\alpha d}$ and $|\Gamma_{in}| = |\Gamma_L|e^{-2\alpha l}$. In the example given, the Γ_{in} point would occur at the intersection of the 30° radial line and the $0.5e^{-2\alpha l}$ circle. Thus the locus of Γ points as a function of d would be a spiral of decreasing radius starting at Γ_L and ending at the Γ_{in} point. For low-loss lines, the approximation $\alpha = 0$ is usually valid.

The above example shows that the graphical solution of the reflection coefficient transformation equation is quite simple. All that is needed is a compass and a straight edge. This is very interesting, *except* that in most cases it is the *impedance* transformation that is required. In the example, Z_L and Z_0 were given and Eq. (3–45) was used to obtain Γ_L. With Γ at any other point determined graphically, its corresponding impedance can be calculated with the aid of Eq. (3–47). One might naturally ask "How can these calculations be avoided?" In other words, how can the impedance transformation be completely solved graphically? The answer is amazingly simple! Merely replace every point on the polar reflection coefficient chart by its equivalent normalized impedance. This is accomplished with Eq. (3–47) which may be rewritten as

$$\frac{Z}{Z_0} = \bar{Z} = \frac{1 + \Gamma}{1 - \Gamma} \qquad \text{and} \qquad \frac{Y}{Y_0} = \bar{Y} = \frac{1 - \Gamma}{1 + \Gamma} \tag{3–101}$$

For example, $\Gamma = 1\underline{/180^\circ}$ is equivalent to $\bar{Z} = 0$, while $\Gamma = 1\underline{/0}$ is equivalent to $\bar{Z} = \infty$, an open circuit. Also, $\Gamma = 0.5\underline{/180^\circ}$ converts to $\bar{Z} = 1/3$, while $\Gamma = 0.5\underline{/0}$ is equivalent to $\bar{Z} = 3$. Other normalized impedance values corresponding to certain values of Γ are shown in Fig. 3–18.

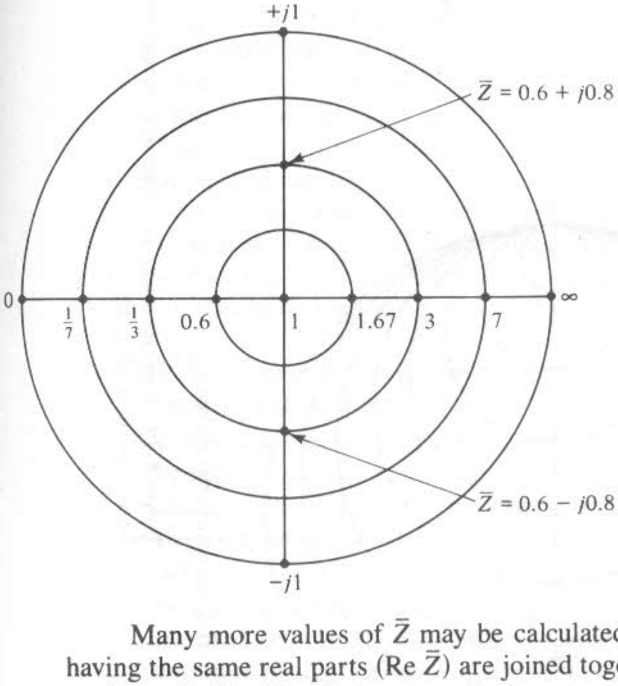
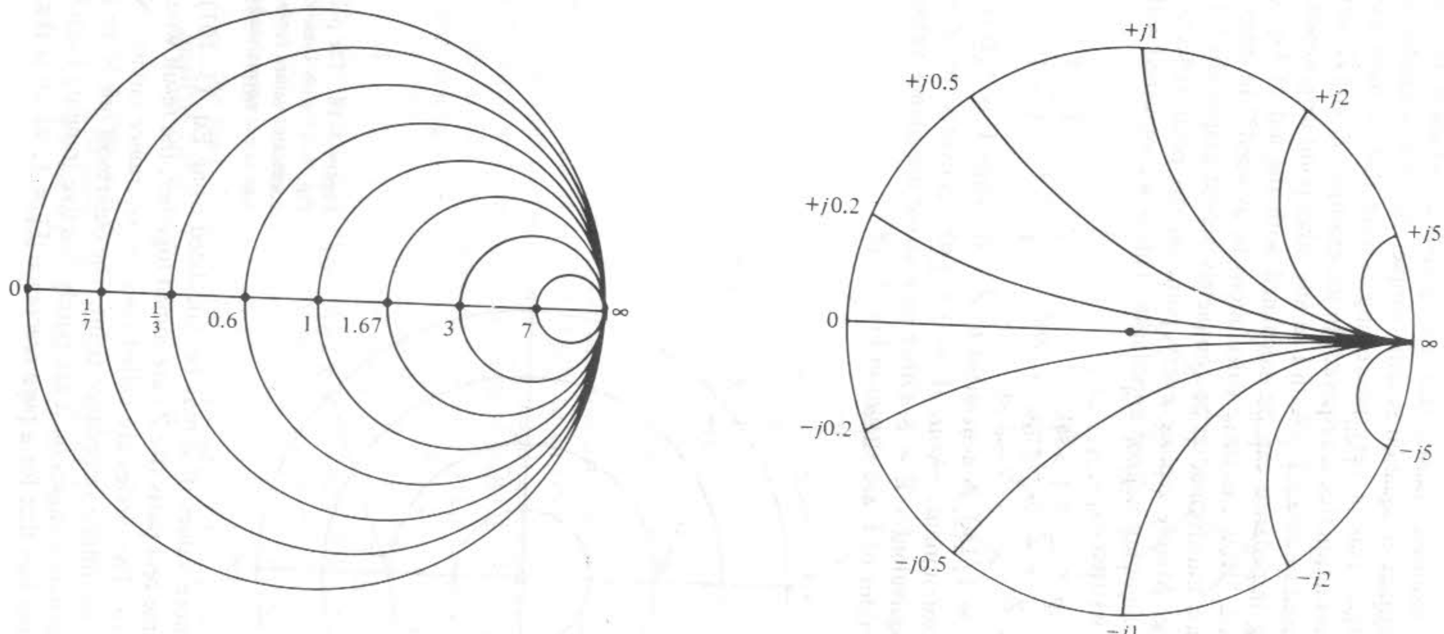


Figure 3–18 The polar chart of Fig. 3–17 with some of the reflection coefficient values replaced by their equivalent normalized impedances.

Many more values of $\bar{Z}$ may be calculated using Eq. (3–101). If all $\bar{Z}$ points having the same real parts (Re $\bar{Z}$) are joined together, the result would look like part *a* of Fig. 3–19. The circles are called *constant resistance* circles. Note that they all pass through the infinite impedance point. The outermost one is the $\bar{R} = 0$ circle and hence all impedance values on it are purely reactive (ranging from 0 to $\pm j\infty$). This agrees with the fact that for a pure reactance $|\Gamma| = 1$, which is the outermost circle



(a) Constant resistance circles

(b) Constant reactance lines

Figure 3–19 Constant resistance circles and constant reactance lines. (Note: The reactance lines are portions of circles.)

on the chart. Joining all points having the same imaginary values (Im $\bar{Z}$) results in the curved lines shown in part *b* of the figure. These are called *constant reactance* lines and in fact are portions of circles.[15] The $\bar{X} = 0$ line is the horizontal line between the zero and infinity points. It is the resistance axis of the impedance coordinates with values ranging from $\bar{R} = 0$ through $\bar{R} = 1$ (the center of the chart) to $\bar{R} = \infty$. The impedance grid formed by combining the constant resistance circles and the constant reactance lines is a Smith chart! Figure 3–20 shows a commercially available version used in many microwave laboratories. The chart was originally developed by P. H. Smith (Refs. 3–9 to 3–11). Basically, it is *a polar chart of reflection coefficient upon which a normalized impedance grid has been superimposed*. Equation (3–101) provides the conversion relationship from Γ coordinates to $\bar{Z}$ coordinates. Note that for the sake of clarity, the circles representing constant values of $|\Gamma|$ have been removed. However, the angle of reflection coefficient scale has been retained. It is the innermost of the three scales on the periphery of the chart. The two outermost scales allow us to perform the Γ transformation (and hence the Z transformation) without having to calculate $2\beta d$. The scales are given directly in fractions of a wavelength. Note that one complete rotation around the chart is a *half* wavelength since for $d = \lambda/2$, $2\beta d = 2\pi$ rad or 360°. This makes sense since it was shown in Sec. 3–6a that impedance repeats every half wavelength. Referring to the chart in Fig. 3–20, the upper half is the positive reactance region, which means that impedance values have an inductive component. The bottom half (negative reactance) denotes impedances with a capacitive component.

At the top of the chart are the words "Impedance and Admittance Coordinates." The reason for this is that the Smith chart can also be used with normalized admittance coordinates. For any value of $\bar{Z}$, its equivalent normalized admittance ($\bar{Y} \equiv Y/Y_0$) is 180° away on the same $|\Gamma|$ circle, since adding 180° to Γ (that is, changing its sign) converts $\bar{Z}$ to $\bar{Y}$. This can be seen by comparing the expressions for $\bar{Z}$ and $\bar{Y}$ in Eq. (3–101). When using the admittance coordinates, the following comments should be kept in mind.

1. The $\bar{Y} = 0$ point corresponds to an open circuit, while the $\bar{Y} = \infty$ point corresponds to a short circuit.
2. The resistance coordinates become conductance coordinates and the reactance coordinates become susceptance coordinates. Remember that a positive susceptance value (top half of the chart) represents a capacitive component, while a negative value denotes an inductive component. This is indicated on the chart.
3. When using admittance coordinates, the angle of the reflection coefficient scale must be rotated 180°. This can be accomplished by subtracting 180° from the values on the top half of the scale and adding 180° to the values on the bottom half.

[15] The proof that the locus of constant resistance and reactance values are circles and portions of circles is given in many texts. See, for example, Sec. 9.2 in Ref. 3–3 or Sec. 1.20 in Ref. 3–8.

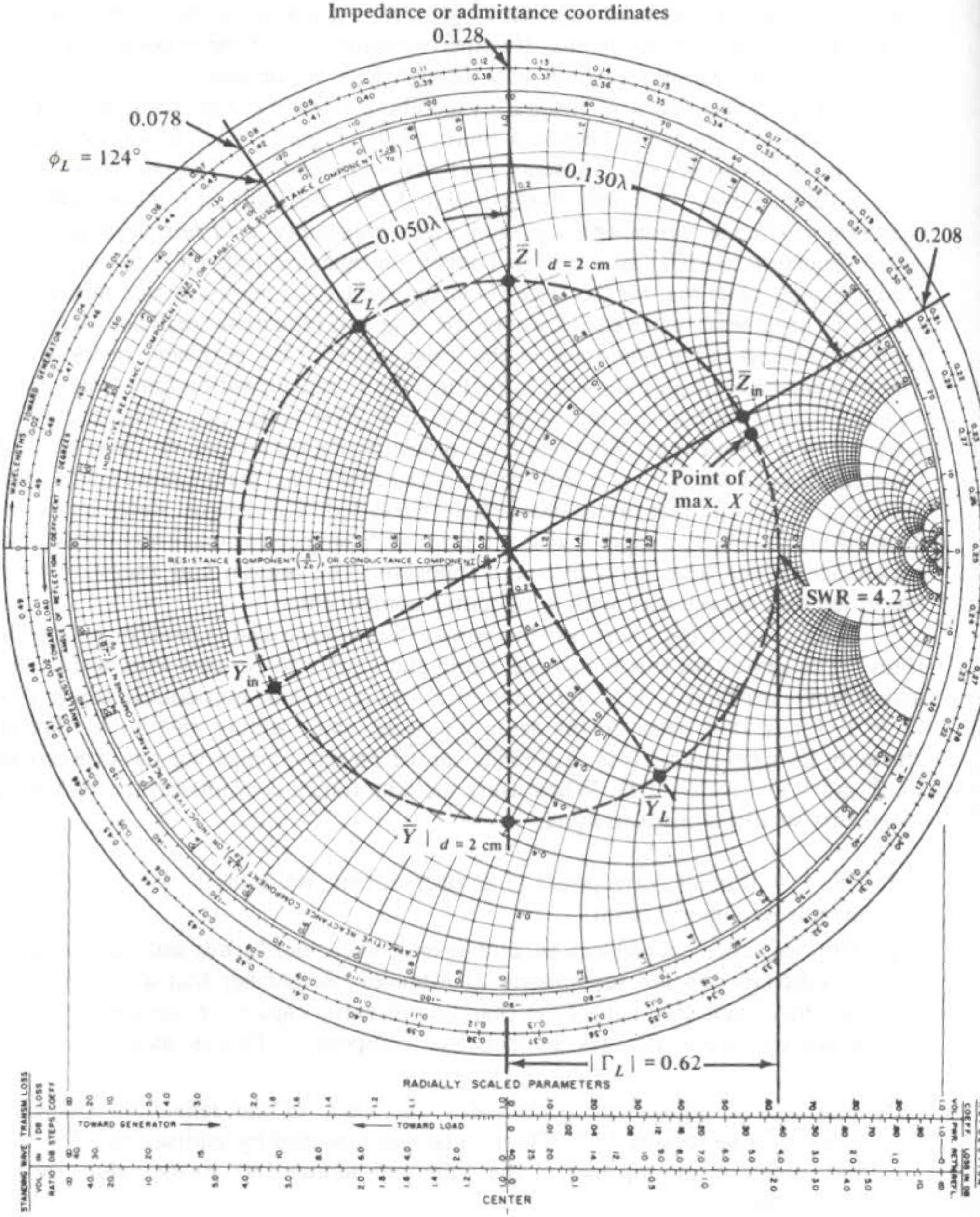


Figure 3–20 A commercially available form of the Smith chart. (Note: The data on the chart refer to Ex. 3–6.) Permission to reproduce Smith charts in this text has been granted by Phillip H. Smith, Murray Hill, N.J., under his renewal copyright issued in 1976.

Note that there are two wavelength scales on the periphery of the chart. One is labeled *Wavelengths toward Generator* and the other *Wavelengths toward Load*. The first one is used when determining the impedance at a point *nearer the input* than the known impedance. This clockwise rotation is referred to as *moving toward the generator*, the assumption being that a generator is connected to the input. The second scale is used in determining the impedance at a point *nearer the load* than the known impedance. This counterclockwise rotation is referred to as *moving toward the load*.

The following example illustrates the graphical procedure for determining the impedance and admittance transformation due to a length of transmission, line as well as other useful properties of the Smith chart.

Example 3–6:

A 5.2 cm length of lossless 100 ohm line is terminated in a load impedance $Z_L = 30 + j50$ ohms.

(a) Calculate $|\Gamma_L|$, ϕ_L, and the SWR along the line.

(b) Determine the impedance and admittance at the input and at a point 2.0 cm from the load end. The signal frequency is 750 MHz and $\lambda = \lambda_0$.

Solution:

(a) Plot the normalized load impedance $\bar{Z} \equiv Z_L/Z_0 = 0.30 + j0.50$ on the Smith chart in Fig. 3–20. This is done by starting from the 0.30 point on the resistance axis and moving up 0.50 reactance units along the constant resistance circle. Next, draw a circle with its center at $\bar{Z} = 1$ (the center of the chart) and a radius equal to the distance between $\bar{Z} = 1$ and $\bar{Z}_L$. This is shown in the figure and will hereafter be referred to as the SWR circle. It is, in fact, the constant $|\Gamma|$ circle for the given value of load impedance. The $\bar{Z}_L$ point on the Smith chart corresponds to its Γ_L value on the polar chart. Since the angle of the reflection coefficient scale has been retained, the value of ϕ_L (124° in this case) can be obtained from the chart as shown in the figure. The value of $|\Gamma_L|$ is obtained by measuring the radius of the SWR circle on the Reflection Coefficient-Vol. scale located at the bottom of the chart. In this example problem, $|\Gamma_L| = 0.62$.

One reason that the constant $|\Gamma|$ circle is called the *SWR circle* is that its intersection with the right half of the resistance axis (that is, between 1 and ∞) yields the SWR due to $\bar{Z}_L$. For this case, the SWR is 4.2. Since SWR is so easily obtained, many engineers prefer to calculate $|\Gamma|$ from Eq. (3–53) rather than using the Reflection Coefficient-Vol. scale. The reason that the resistance axis between 1 and ∞ serves as a SWR scale is that it corresponds to positive real values of Γ. As such it represents the magnitude of Γ for all normalized impedances on the SWR circle. Equation (3–101) transforms these values of Γ into $\bar{R} = (1 + |\Gamma|)/(1 - |\Gamma|)$, which is exactly the equation for SWR (Eq. 3–52). The unity SWR circle is simply a point at the center of the chart, while the infinite SWR circle is the periphery of the chart and is equivalent to $|\Gamma| = 1.00$.

(b) A graphical method of obtaining Γ_{in} when Γ_L and βl are known has been described. For a lossless line, it consists of rotating clockwise $2\beta l$ on the constant $|\Gamma|$ circle. The procedure for obtaining $\bar{Z}_{in}$ from $\bar{Z}_L$ is *exactly the same* except that the impedance coordinates of the Smith chart are used. The steps are as follows:

1. Plot $\bar{Z}_L$ (0.30 + j0.50 in this case) and draw its SWR circle (4.20 in this case).
2. Draw a radial line from the center of the chart through $\bar{Z}_L$ to the periphery. Read the value on the *Wavelengths toward Generator* scale (0.078 in this case).

This value in itself has no physical meaning. It is merely the starting point of the clockwise rotation in the next step.

3. Since $\lambda_0 = 40$ cm at 750 MHz and the input is 5.2 cm from the load, rotate clockwise *from* 0.078 a distance $l/\lambda = 5.2/40 = 0.130$. Draw a radial line from the center of the chart through the 0.208 point on the outer scale as shown in Fig. 3–20. The intersection of the radial line with the SWR circle represents $\overline{Z}_{in}$ since it corresponds to the Γ_{in} point on the polar reflection coefficient chart. In this case, $\overline{Z}_{in} = 2 + j2$ or $Z_{in} = 200 + j200$ ohms.

To obtain the impedance at $d = 2$ cm, start from $\overline{Z}_L$ and rotate clockwise $2/40 = 0.050$ and draw a radial line through the 0.128 point as shown. Its intersection with the SWR circle yields $\overline{Z} = 0.47 + j0.93$ or $Z = 47 + j93$ ohms.

This simple graphical method of determining the impedance transformation due to a length of lossless line is the most useful characteristic of the Smith chart. The procedure for determining the admittance transformation is *exactly the same* since all admittance points are directly opposite their corresponding impedance points on the SWR circle. Thus from the figure, $\overline{Y}_L = 0.88 - j1.47$, $\overline{Y}_{in} = 0.25 - j0.25$, and at $d = 2.0$ cm, $\overline{Y} = 0.43 - j0.87$. Multiplying these values by $Y_0 = 0.01$ mho gives the unnormalized admittance values.

Part *a* of this example problem illustrates how to determine $|\Gamma_L|$, ϕ_L, and SWR when Z_L and Z_0 are known. Since the line is lossless, the SWR and $|\Gamma|$ are the same at all other points on the line. The angle of the reflection coefficient, however, is a function of position and can be read on the periphery of the chart. In this example, $\phi_{in} = 30°$ while at $d = 2$ cm, ϕ is equal to 88°.

Part *b* describes the graphical solution to the impedance/admittance transformation equation for lossless lines.[16] Given $\overline{Z}_L$, the normalized impedance at any other point on the line is obtained by rotating clockwise on a fixed SWR circle the appropriate distance d/λ. Thus for a given load impedance, the SWR circle represents the locus of all possible impedance and admittance values available on the lossless line. Stated another way, given $\overline{Z}_L$, it is the locus of all possible values of $\overline{Z}_{in}$ and $\overline{Y}_{in}$ obtainable by varying the line length l. This is an example of how a good graphical procedure can show the effect of a variable (l) on the desired result (Z_{in}). For instance, it is obvious from Fig. 3–20 that if $\overline{Z}_L = 0.30 + j0.50$, varying the line length will never result in $\overline{Z}_{in} = 0.70 + j0.40$ since it is not on the SWR circle containing $\overline{Z}_L$. If the reader is ambitious, try proving this analytically. To further emphasize the chart's usefulness, consider the ease with which the following problem is solved. Given the impedance values in Ex. 3–6, what value of line length maximizes the reactive portion of the input impedance? With $\overline{Z}_L = 0.30 + j0.50$ plotted in Fig. 3–20, the SWR circle represents all possible values of $\overline{Z}_{in}$ that can be obtained by varying l. A brief look at the chart shows that the reactive portion is maximized at the point where the SWR circle is tangent to the reactance lines. A positive reactive maximum occurs when $\overline{Z}_{in} \approx 2.3 + j2.0$. A radial line through this point intersects the *Wavelengths toward Generator* scale at 0.216. Therefore, the 100 ohm line must be $(0.216 - 0.078)$ or 0.138λ long.

[16] For a lossy line, a second SWR circle is required. It is obtained by multiplying $|\Gamma_L|$ by $e^{-2\alpha d}$ and converting the resulting $|\Gamma|$ to SWR. The intersection of the radial line with the circle defined by the new SWR value yields $\overline{Z}$ at d units from the load.

3-7b Typical Smith Chart Computations

In practice, high-frequency circuits often contain both lumped circuit elements and transmission-line sections. The following example problems illustrate the use of the Smith chart in these cases.

Example 3-7:

A lossless 50 ohm line terminated in $Z_L = 100 + j75$ ohms is shown at the top of Fig. 3-21. The line is 0.18λ long. Calculate the input impedance if the L-C circuit shown is inserted at a point 0.12λ from the load end.

Solution: The bottom half of Fig. 3-21 shows the Smith chart solution of the problem. The subscripts A, B, and C are used to denote the impedance and admittance at various points along the line. For example, Z_B (Y_B) represents the impedance (admittance) of the circuit to the right of plane B. As such it includes Z_L, the 0.12λ line and the 30 ohm inductor.

To obtain the impedance at plane A, first plot $\bar{Z}_L \equiv Z_L/50 = 2 + j1.5$ and draw its SWR circle (SWR = 3.3). The impedance at plane A is obtained by rotating 0.12λ toward the generator on the 3.3 SWR circle, which yields $\bar{Z}_A = 1 - j1.3$. The impedance at plane B is $\bar{Z}_A$ plus the normalized impedance of the series inductance $(+j30/50)$. Thus, $\bar{Z}_B = 1 - j1.3 + j0.6 = 1 - j0.7$. Adding inductive reactance in series is equivalent to moving upward on a constant resistance circle. As indicated on the Smith chart, $\bar{Z}_B$ is +0.6 units above the $\bar{Z}_A$ point on the $\bar{R} = 1$ circle. Next, the impedance at plane C is determined, which requires that the effect of the $-j200$ ohm shunt capacitance be considered. Since impedances in parallel are not directly additive, the equivalent admittance values are needed to continue the graphical procedure. The $\bar{Y}_B$ point is directly opposite the $\bar{Z}_B$ point on *its* SWR circle (SWR = 2.0). From the chart, $\bar{Y}_B = 0.67 + j0.47$. The admittance at plane C is the sum of $\bar{Y}_B$ and the normalized admittance of the shunt capacitance. Since the admittance of the capacitor is the reciprocal of $-j200$ and $Y_0 = 0.02$ mho, $\bar{Y}_C = 0.67 + j0.47 + j0.25 = 0.67 + j0.72$. On the Smith chart, this is equivalent to moving *upward* (since capacitive susceptance is positive) 0.25 susceptance units on the $\bar{G} = 0.67$ circle.

With $\bar{Y}_C$ known, the input admittance is obtained by the same graphical procedure used for impedance transformations. Draw the SWR circle for $\bar{Y}_C$ (SWR = 2.6). Move from the $\bar{Y}_C$ point a distance of 0.06λ toward the generator. The result from the chart is $\bar{Y}_{in} = 1.3 + j1.1$. The normalized input impedance is diametrically opposite the $\bar{Y}_{in}$ point on the 2.6 SWR circle. Thus $\bar{Z}_{in} = 0.45 - j0.38$. With $Z_0 = 50$ ohms, $Z_{in} = 22.5 - j19$ ohms.

The Smith chart solution reveals additional useful information about the circuit. For example, the SWR along the 0.12λ line section is 3.30, while its value on the 0.06λ section is 2.60.

Often a microwave circuit contains two or more transmission lines with different characteristic impedances. This usually requires normalizing the chart to more than one value of Z_0. The following example problem illustrates this situation.

Example 3-8:

A 50 ohm line and a 90 ohm line are connected in tandem as shown at the top of Fig. 3-22. The 50 ohm line is terminated in a 20 ohm resistance. Both lines are lossless and their lengths are indicated in the figure. Calculate the input impedance and reflection coefficient if $\lambda = 20$ cm for both lines.

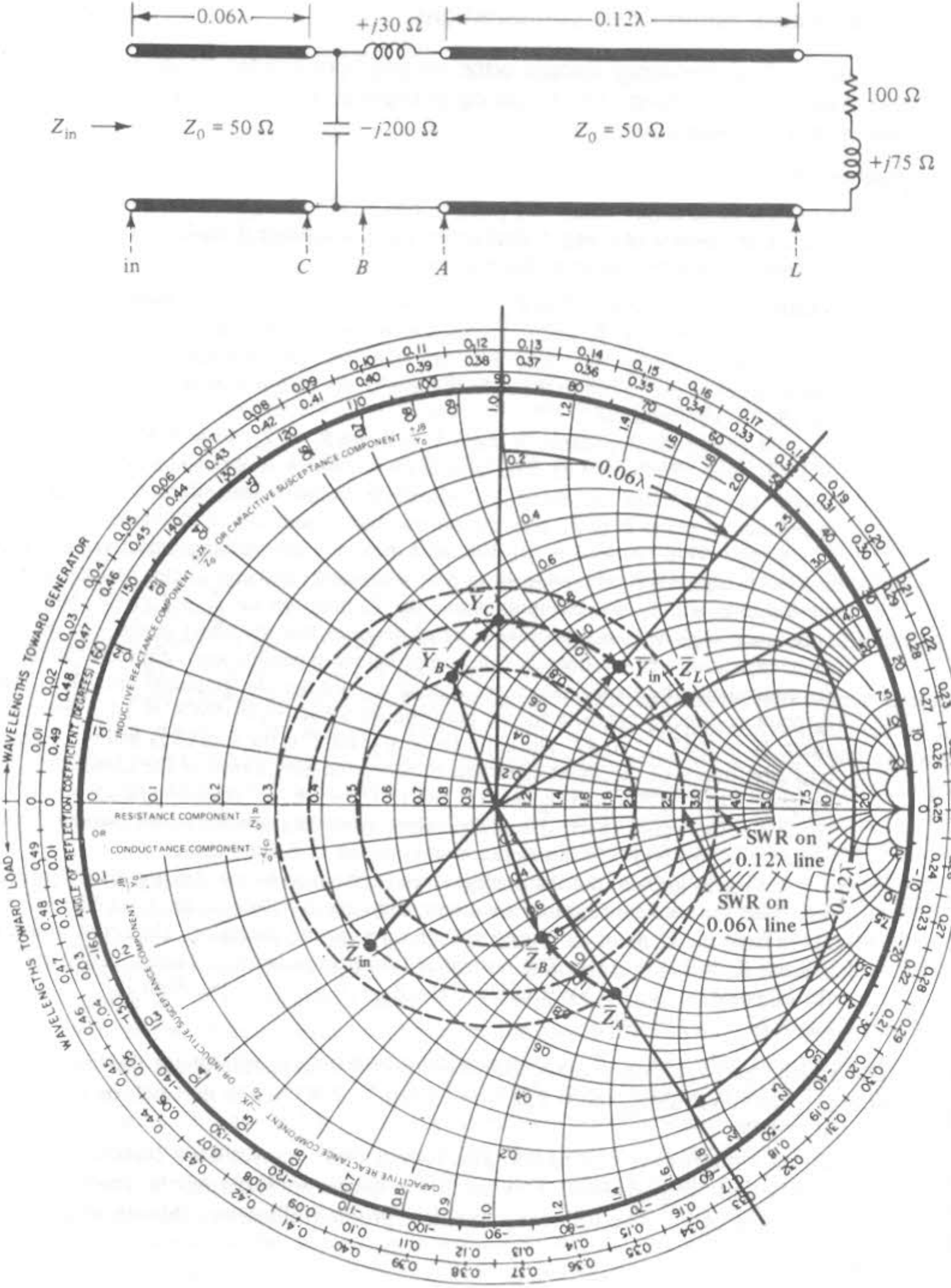


Figure 3–21 The transmission-line circuit and Smith chart solution for Ex. 3–7 in Sec. 3–7b.

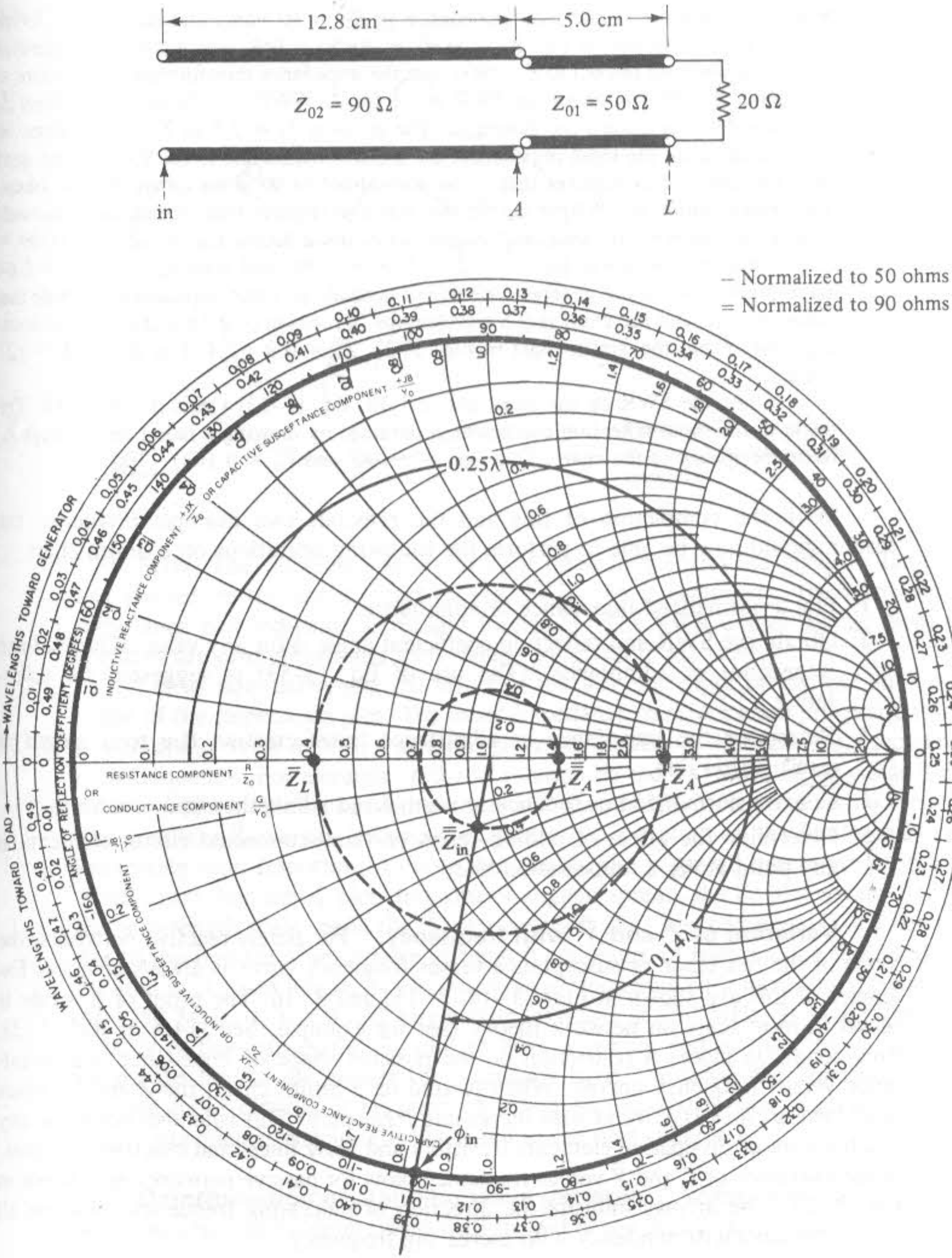


Figure 3–22 The transmission-line circuit and Smith chart solution for Ex. 3–8 in Sec. 3–7b.

Solution: Normalize the load impedance to $Z_{01} = 50$ ohms and plot on the Smith chart. This is indicated in Fig. 3–22 as $\bar{Z}_L = 20/50 = 0.4$, where the bar (–) denotes normalization with respect to Z_{01}. Next, use the impedance transformation procedure to determine Z_A. That is, draw the SWR circle for $\bar{Z}_L$ (SWR = 2.5) and rotate from $\bar{Z}_L$ 0.25 wavelengths toward the generator. The result is $\bar{Z}_A = 2.5$ or $Z_A = 125$ ohms. In order to calculate the input impedance, the transformation due to the 90 ohm line must be determined. This requires that Z_A be normalized to 90 ohms rather than 50 ohms. The determination of SWR on the 90 ohm line also requires that the data be so normalized. Denoting normalization with respect to Z_{02} by a double bar (=), $\bar{\bar{Z}}_A = 125/90 = 1.39$. Drawing the SWR circle for $\bar{\bar{Z}}_A$ (SWR = 1.39) and rotating $12.8/20 = 0.64$ wavelengths toward the generator yields the normalized input impedance $\bar{\bar{Z}}_{in}$. Note that since once around the chart is 0.5 wavelengths, a rotation of 0.14 is the same as rotating 0.64. From the Smith chart in Fig. 3–22, $\bar{\bar{Z}}_{in} = 0.9 - j0.3$ or $Z_{in} = 81 - j27$ ohms.

Since the SWR on the input line is 1.39, Eq. (3–53) yields $|\Gamma_{in}| = 0.163$. The angle of the input reflection coefficient is obtained by drawing a radial line through $\bar{\bar{Z}}_{in}$ to the periphery of the chart. Thus, $\phi_{in} = -100°$ and $\Gamma_{in} = 0.163\,\underline{/-100°}$.

With the completion of this and the previous two example problems, the reader should now be able to perform the following operations on a Smith chart.

1. Plot a normalized impedance or admittance.
2. Obtain the SWR and reflection coefficient angle ϕ for any value of normalized impedance or admittance. (The use of Eq. (3–53) is suggested for calculating $|\Gamma|$.)
3. Determine the impedance or admittance transformation due to a length of transmission line.
4. Convert normalized impedance to normalized admittance and vice versa.
5. Determine the effect of adding series or shunt-connected circuit elements at any point along a transmission line.

Variation of $\bar{Z}$ and $\bar{Y}$ with frequency. For purely reactive networks, the slope of the reactance or susceptance versus frequency curve is always positive. Examples of this are shown in Figs. 3–14, 3–15, and 3–16. The proof of this rule is found in most texts on network theory (see for example, Sec. 14.4 of Ref. 3–2). This condition places a restriction on the possible shape of impedance and admittance versus frequency curves. When plotted on a Smith chart, the $\bar{Z}$ and $\bar{Y}$ curves must have a *clockwise trend with increasing frequency*. This rule also holds for any combination of dissipative elements (resistors and lossy lines) and reactive elements. Some examples of $\bar{Z}$ and $\bar{Y}$ versus frequency plots for passive networks are shown in Fig. 3–23. The arrows indicate the direction of increasing frequency. Note in all cases, the clockwise tendency with increasing frequency.

An example of a frequency sensitive load impedance is shown at the top of Fig. 3–24. Normalizing to $Z_0 = 50$ ohms, $\bar{Z}_L = 0.50 + j\bar{X}$, where $\bar{X} = \{\omega L - (1/\omega C)\}/50$. For the given values of L and C, series resonance occurs at

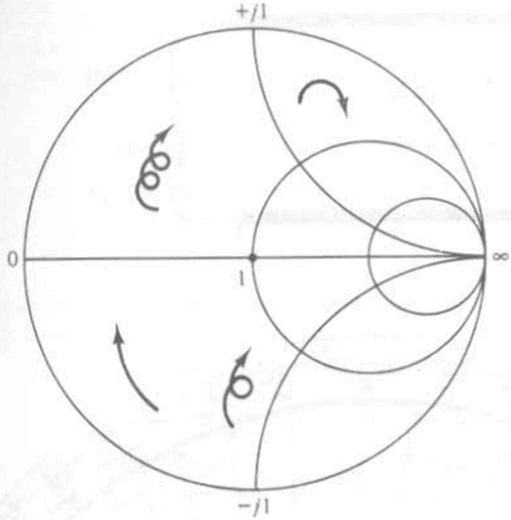


Figure 3-23 Typical Smith chart plots of $\bar{Z}$ and $\bar{Y}$ versus frequency for passive networks.

1000 MHz. The calculated values of $\bar{Z}_L$ at three frequencies are given in Table 3-1 and plotted on the Smith chart of Fig. 3-24. As is customary, the arrow on the impedance plot indicates increasing frequency. The SWR circles for the three frequencies are also shown and their values are given in the table.

Let us now determine the effect of the 4.0 cm length of lossless line on the shape of the impedance plot. The usual impedance transformation procedure is used to obtain $\bar{Z}_{in}$. With $\lambda = \lambda_0$, the table gives the amount of rotation (l/λ_0) required at each frequency. For example, a 0.133 rotation on the 2.0 SWR circle yields the $\bar{Z}_{in}$ value at 1000 MHz. This and the values at 750 and 1250 MHz are listed in the table. The resultant $\bar{Z}_{in}$ versus frequency characteristic is plotted on the Smith chart. It is interesting (and important) to note that the $\bar{Z}_{in}$ plot is spread over a greater portion of the chart than the $\bar{Z}_L$ plot. In terms of angular spread, the $\bar{Z}_{in}$ plot covers 0.151 on the wavelength scale while the $\bar{Z}_L$ plot covers 0.084. The additional 0.067 spread is due to the frequency sensitivity of the 4.0 cm length of transmission line. Although the line is of fixed length its *electrical length* increases with increasing frequency.[17] At 1250 MHz, l/λ is 0.067 larger than its value at 750 MHz. It is this frequency sensitivity that causes the $\bar{Z}_{in}$ plot to have a greater spread than the $\bar{Z}_L$ plot. The longer the line or the greater the frequency band, the larger the impedance spread and the more difficult it becomes to match an impedance over a given frequency range. Impedance matching is discussed in the next chapter.

Determination of standing wave patterns. Voltage and current standing wave patterns can be determined with the aid of the Smith chart. The phasor voltage

[17] The term *electrical length* denotes either βl or l/λ. For instance, if a line section is a quarter wavelength long, $\beta l = \pi/2$ rad or 90° and $l/\lambda = 0.25$.

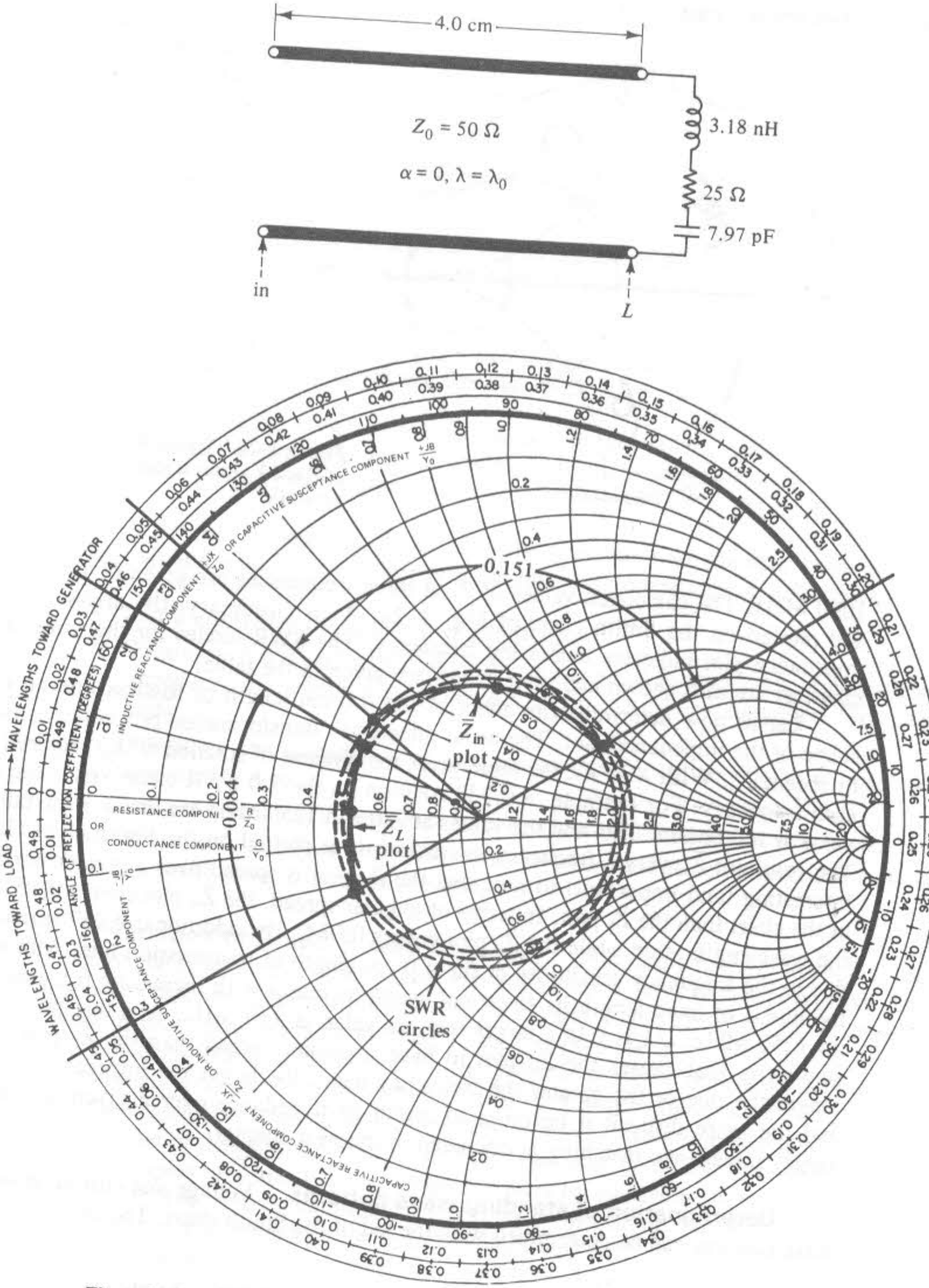


Figure 3–24 A transmission line terminated in a series resonant circuit and its corresponding Smith chart characteristics.

TABLE 3–1 Data for the circuit in Fig. 3–24.

Freq. (MHz)	λ_0 (cm)	l/λ_0	$\bar{Z}_L$	SWR	$\bar{Z}_{in}$
750	40	0.100	$0.50 - j0.23$	2.2	$0.50 + j0.27$
1000	30	0.133	0.50	2.0	$0.85 + j0.63$
1250	24	0.167	$0.50 + j0.18$	2.1	$1.60 + j0.70$

and current on a transmission line are given by Eqs. (3–34) and (3–35). Using the definition of Γ from Eq. (3–36), they may be rewritten as

$$\frac{\mathbf{V}}{\mathbf{V}^+} = 1 + \Gamma \qquad \text{and} \qquad \frac{\mathbf{I}}{\mathbf{I}^+} = 1 - \Gamma \tag{3–102}$$

The magnitude of these quantities are proportional to the rms voltage and current along the line. Thus the standing wave patterns are described by the following expressions.

$$\frac{V}{V^+} = |1 + \Gamma| \qquad \text{and} \qquad \frac{I}{I^+} = |1 - \Gamma| \tag{3–103}$$

where Γ, and hence V and I, are functions of position on the transmission line. When the line is lossless, V^+ and I^+ are independent of position.

The unity value in these expressions can be represented by the phasor $1\underline{/0}$. This is shown on the Smith chart in Fig. 3–25 as a fixed horizontal vector from the $\bar{Z} = 0$ point to $\bar{Z} = 1$, the center of the chart. Since a vector from the center of the chart to some impedance point $\bar{Z}$ is the reflection coefficient Γ, the *magnitude* of the vector sum of $1\underline{/0}$ and Γ represents the normalized rms voltage at that point on the line. The magnitude of the vector difference represents the normalized rms current at the same point. Figure 3–25 shows $\Gamma_L, -\Gamma_L$, V_L/V^+ and I_L/I^+ for the case when $\bar{Z}_L = 1 + j2$. When the line is lossless, the normalized voltage and current at any other position is obtained by rotating from the $\bar{Z}_L$ point on its SWR circle d/λ

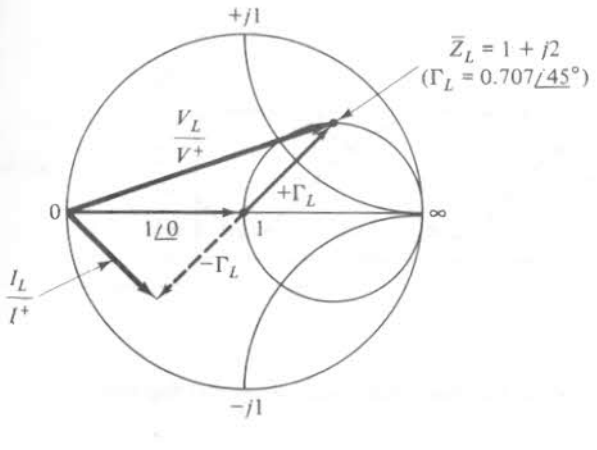


Figure 3–25 Determination of normalized load voltage and current using the Smith chart. In this example, $\bar{Z}_L = 1 + j2$.

units toward the generator and solving for $|1 + \Gamma|$ and $|1 - \Gamma|$. This is shown in Fig. 3–26 for $d = \lambda/16$, $3\lambda/16$, and $5\lambda/16$ when $\bar{Z}_L = 1 + j2$. The resultant standing wave patterns are shown at the bottom of the figure. For this case, a voltage maximum (current minimum) occurs $\lambda/16$ from the load. A quarter wavelength further at $d = 5\lambda/16$, a voltage minimum (current maximum) exists. This suggests the following procedure for determining the positions of maximum and minimum voltage and current on a transmission line. For a given load impedance, plot $\bar{Z}_L$ on the Smith chart. If $\bar{Y}_L$ is given, convert it to its equivalent impedance $\bar{Z}_L$. The clockwise angular distance from the $\bar{Z}_L$ point to the horizontal axis between $\bar{Z} = 1$ and $\bar{Z} = \infty$ is the distance to the first voltage maximum (current minimum). The distance in fractions of a wavelength is obtained by using the *Wavelength toward Generator* scale on the periphery of the chart. In like manner, the first voltage minimum (current maximum) is determined by measuring the clockwise angular distance from the $\bar{Z}_L$ point to the horizontal axis between $\bar{Z} = 0$ and $\bar{Z} = 1$. This is just another example of the usefulness of the Smith chart in solving transmission-line problems.

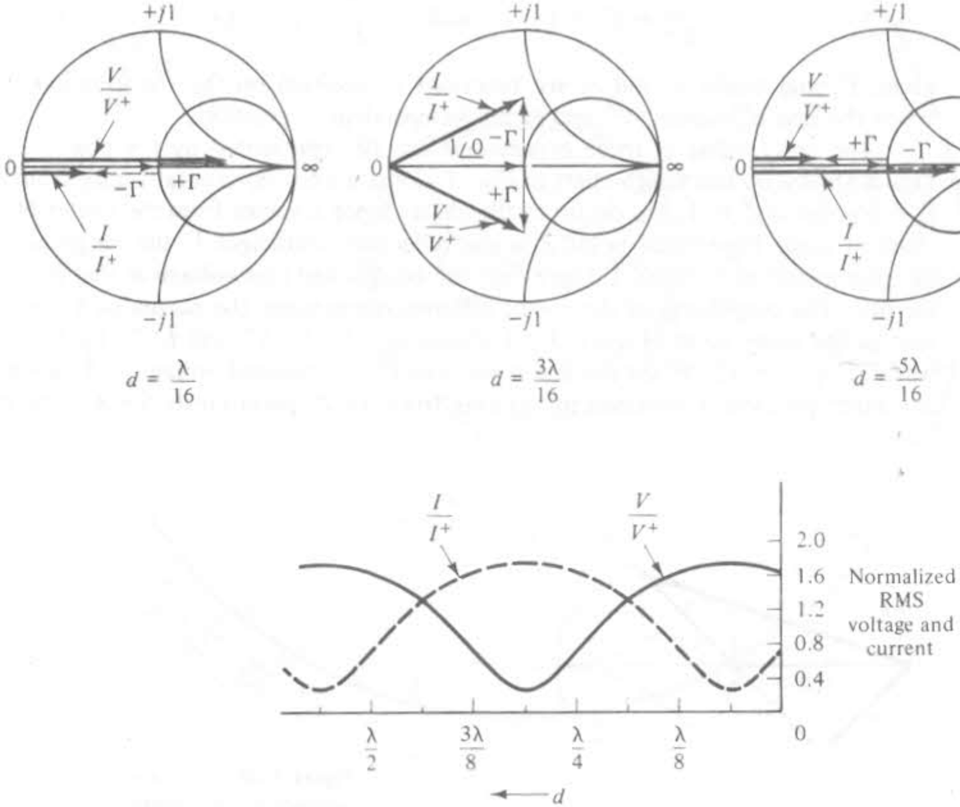


Figure 3–26 Use of the Smith chart to determine voltage and current standing-wave patterns. In this example, $\bar{Z}_L = 1 + j2$.

This chapter has reviewed the high-frequency aspects of transmission-line theory. The results are useful in the analysis and design of microwave components and systems. For a discussion of transmission-line characteristics at low frequencies, the reader is referred to any of the texts listed below.

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PROBLEMS

3–1. A lossless transmission line has an inductance per unit length of 1.35 μH/m and a capacitance per unit length of 15 pF/m. Calculate the wave velocity and characteristic impedance of the line.

3–2. A lossless line is terminated in a load resistance of 50 ohms. Calculate the two possible values of Z_0 if one-third of the incident voltage wave is reflected by the load.

3–3. The open circuit in Fig. 3–3 is replaced by a short. Show that for $t > 40$ ns, $V = 0$ and $I = 0.20$ A everywhere on the line.

3–4. Referring to the circuit at the top of Fig. 3–4, let R_L = 400 ohms. Find the voltage and current at z = 2 m when t = 5, 15, and 35 ns.

3–5. Plot the load current versus time (from 0 to 120 ns) for the circuit shown in Fig. 3–5. Use the space-time diagram.

3–6. Replace the values of R_G, R_L, and l in Fig. 3–5 by the following: $R_G = 0$, $R_L = 0$, and l = 60 cm.

(a) Plot the voltage at z = 30 cm versus time between 0 and 15 ns. Use the space-time diagram.

(b) What is the frequency of oscillation? Describe two ways of reducing its value.

3–7. A uniform transmission line has the following characteristics at 200 MHz: R' = 50 ohm/m, L' = 0.10 μH/m, G' = 0.06 mho/m, and C' = 200 pF/m. What is the percent error in using Eqs. (3–22) and (3–23) rather than Eq. (3–15) to calculate the attenuation and phase constants?

3–8. **(a)** Find the characteristic impedance of an air-filled coaxial line having inner and outer diameters of 0.25 cm and 0.75 cm, respectively.

(b) Determine the characteristic impedance and the wavelength at 3000 MHz when the line is teflon filled.

3–9. Referring to Fig. 3–9b, the input power is 80 W and the power absorbed by the load is 16 W. What is the power flow at a point halfway down the line?

3–10. Referring to Fig. 3–9b, a 30 V ac source is connected to a 50 ohm load via a 50 ohm lossless line. Calculate the phasor current at $z = 0.30\lambda$ when $Z_G = 25 + j25$ ohms. Determine the power absorbed by the load.

3–11. A load impedance of $30 - j75$ ohms is connected to a 75 ohm lossless line. Calculate Γ_L and Γ_{in} if the line is 0.15λ long. What is the SWR?

3–12. A 100 ohm air-insulated coaxial line is terminated by a parallel combination of an 80 ohm resistor and a 5.0 nH inductor. Calculate the input reflection coefficient at 2000 MHz if α = 1.5 dB/m and l = 40 cm.

3–13. The reflection coefficient at the input of a 50 ohm, air-insulated lossless line is $0.6 + j0.8$ at 1000 MHz. What type of circuit element is connected at the load end if the line is 8.0 cm long? What is its value?

3–14. A lossless line is terminated by an impedance that reflects 16 percent of the incident power. Calculate the SWR on the line.

3–15. The SWR on a lossless 75 ohm line is 4.0. Calculate the maximum and minimum values of voltage and current on the line when the incident voltage is 30 V.

3–16. A 300 ohm line is terminated in a pure inductance. The load reflection coefficient is $1.0\,\underline{/50^\circ}$ at 1500 MHz. Calculate the inductance (in nH).

3–17. The standing wave pattern in Fig. 3–12 is for the case $Z_L = +jZ_0$. Prove that the first voltage null occurs at $d = 3\lambda/8$. Verify the result by using the counterrotating phasor diagram.

3–18. A lossless line is terminated in $\Gamma_L = 0.7\underline{/45^\circ}$. Calculate the distance from the load to the first current minimum when λ = 20 cm. Is there a current maximum between the minimum and the load?

3–19. A 200 ohm lossless line is terminated in a pure resistance. What are the two possible values of load resistance if 25 percent of the incident power is reflected?

3–20. Referring to Fig. 3–10, V_G = 30 V, Z_G = 150 ohms, Z_0 = 50 ohms, Z_L = 100 ohms, and $\alpha = 0$. Find the change in load power when βl is increased from 1.5π to 2π rad. What would be the change in load power if Z_0 had been 150 ohms?

3–21. Referring to Fig. 3–10, $V_G = 30\underline{/0}$ V, $Z_G = 100$ ohms, $Z_L = 60 + j60$ ohms, $Z_0 = 50$ ohms, $l = 2.0$ m, $\alpha = 0$, and $\beta = 1.2\pi$ rad/m.

(a) Calculate the phasor voltage and current at the load ($\mathbf{V}_L$ and $\mathbf{I}_L$).

(b) Calculate the load and input powers.

3–22. Repeat part *b* of Prob. 3–21 with $\alpha = 0.40$ dB/m. Also calculate the power dissipated in the line.

3–23. Repeat Prob. 3–22 with $Z_0 = 100$ ohms.

3–24. A load impedance with a return loss of 6 dB is connected to a lossless line. Calculate the SWR on the line. Repeat for a return loss of 20 dB.

3–25. What fraction of the incident power is absorbed by the load when the load return loss is 0 dB? Repeat for an infinite return loss.

3–26. A resistive load reflects 5 percent of its incident power. Calculate the input return loss (in dB) when the line is 30 cm long and $\alpha = 0.20$ dB/cm.

3–27. Calculate the reflection loss (in dB) for the two cases in Prob. 3–24. Assume $Z_G = Z_0$.

3–28. A lossless 50 ohm line is terminated in $Z_L = 80 + j50$ ohms. Calculate the input impedance at 2000 MHz if the line is 8.4 cm long. Assume $\lambda = 0.8\lambda_0$.

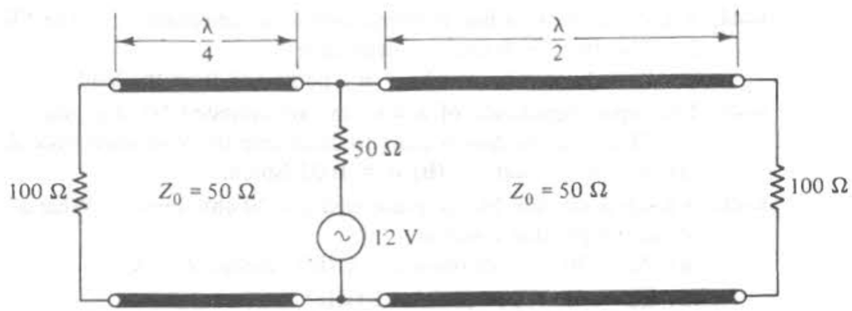
3–29. A lossless, dielectric-filled coaxial line with $Z_0 = 75$ ohms is terminated by a parallel combination of a 150 ohm resistor and a 4.0 nH inductor. Calculate the input admittance at 5000 MHz if the line is 0.8 cm long. Assume a nonmagnetic dielectric with $\epsilon_R = 2.25$.

3–30. A load impedance is connected to a lossless 50 ohm line. The impedance at $d = 0.4\lambda$ from the load is $30 - j20$ ohms. Calculate the value of load impedance.

3–31. A 60 cm length of 150 ohm transmission line is terminated in a reactive load having a value $-j150$ ohms. Calculate the real part of the input impedance when $\alpha = 3$ dB/m and $\lambda = 30$ cm.

3–32. A lossless 50 ohm line is terminated in $-j80$ ohms. What is the smallest value of l/λ that results in $Z_{in} = 0$?

3–33. A 12 V, 50 ohm generator is connected to a pair of lossless 50 ohm lines each terminated with a 100 ohm resistor as shown. One line is $\lambda/4$ long and the other is $\lambda/2$ in length. Calculate the power dissipated in each load resistor.



3–34. A 750 MHz, 12 V source has an internal resistance of 200 ohms.

(a) Design a quarter-wave transformer, using an air-insulated coaxial line, to deliver all the available generator power to an 8 ohm resistive load. Calculate the length and the *b/a* ratio for the coaxial line.

(b) What is the increase in load power (in dB) obtained by inserting the transformer between the generator and the load?

3–35. A one wavelength long, 90 ohm line is terminated in $Z_L = 20 - j30$ ohms. Its input terminals are connected to the load end of a 60 ohm, quarter-wavelength line. What is the impedance at the input to the 60 ohm line?

3–36. A length of 300 ohm TV twin-lead is short circuited at one end. Determine the minimum line length that results in an input impedance of $-j150$ ohms at 600 MHz. Assume $\lambda = 0.9\lambda_0$.

3–37. A length of open-circuited transmission line with $Z_0 = 90$ ohms provides the capacitive reactance for a microwave filter.

(a) Calculate the capacitive reactance at 400 MHz when $l = 5.0$ cm and $v = 2 \times 10^8$ m/s. What is the percent error if Eq. (3–93) is used to calculate the capacitive reactance?

(b) How much shorter would the line length be if a 30 ohm line were used to obtain the reactance value calculated in part *a?*

3–38. Determine the characteristic impedance and wave velocity of a 25 cm length of lossless transmission line from the following 300 MHz experimental data: $Z_{sc} = -j90$ ohms and $Z_{oc} = +j40$ ohms. Assume βl is less than π rad.

3–39. Given $\beta l = n\pi$ rad and $\alpha = 1.5$ dB/m, determine the minimum line length required for Z_{in} of either a shorted or open-circuited line to be within 2 percent of the characteristic impedance.

3–40. The input of a 1.4 m length of open-circuited line is connected in parallel with a 26 ohm resistor. The characteristics of the line are $Z_0 = 50$ ohms, $\alpha = 2.0$ dB/m, and $\lambda = 0.80$ m. What is the SWR when the parallel combination terminates a lossless 75 ohm line?

3–41. Derive Eq. (3–100). Calculate the Q at 2.5 GHz for a shorted, air-insulated 100 ohm line. The line is 2.90 cm long and its attenuation constant $\alpha = 0.20$ dB/m.

3–42. A lossless line of characteristic impedance Z_0 is terminated in an *imperfect* short having a resistance R, where $R \ll Z_0$. Determine the line length that results in maximum Q when the line is used as an inductor. Calculate X_L and Q if $Z_0 = 100$ ohms and $R = 1$ ohm.

3–43. A 75 ohm transmission line is terminated in a load impedance equal to $90 - j30$ ohms. Use the Smith chart to calculate the SWR and the load reflection coefficient.

3–44. A lossless 50 ohm line is terminated in an admittance Y_L. The SWR along the line is 3.0. Use the Smith chart to determine

(a) Y_L if $\phi_L = 60°$; **(b)** Y at a point $\lambda/8$ from the load.

3–45. The input impedance of a 4.8 cm, air-insulated 50 ohm line is $15 + j25$ ohms at 2.5 GHz. Use the Smith chart to determine the load impedance Z_L if

(a) $\alpha = 0$ and **(b)** $\alpha = 0.02$ Np/cm.

3–46. A lossless 40 ohm line is terminated in a 20 ohm resistor. What are the possible values of line length that result in

(a) $Z_{in} = 40 - j28$ ohms at 1 GHz? Assume $\lambda = \lambda_0$.

(b) $Z_{in} = 48 + j16$ ohms at 1 GHz?

Use the Smith chart rather than the impedance transformation equation.

3–47. A shorted, air-filled, 60 ohm coaxial line is used to provide inductive reactance in a microwave filter. Use the Smith chart to determine the line length required for $Z_{in} = +j30$ ohms at 4 GHz. Check your solution with Eq. (3–87). Will the line length be smaller if a 90 ohm line is used?

3–48. Given $Z_L = 25 - j20$ ohms, $Z_0 = 50$ ohms, and $\lambda = 10$ cm, use the Smith chart to determine the smallest line length that maximizes
(a) the input conductance; **(b)** the input inductive susceptance.

3–49. A 16 cm length of lossless 90 ohm line is terminated in a 135 ohm resistor. Assuming $\lambda = \lambda_0$, use the Smith chart to calculate the input impedance at 750 MHz if
(a) a 4.0 pF capacitance is connected in series at a point halfway along the line;
(b) the same capacitance is connected *across* the line at the halfway point.

3–50. A 50 ohm line is terminated in a 75 ohm resistor. Its input terminals are connected to the output of a 30 ohm line. Both lines are 0.12λ long.
(a) Find Z_{in} and Γ_{in} at the input to the 30 ohm line.
(b) What are the SWR values on the two lines?

3–51. A 2.5 cm length of teflon-filled coaxial line is terminated in a passive, frequency sensitive load. The normalized load admittance values at 4.0, 5.0, and 6.0 GHz are $0.50 + j0.60$, 0.80, and $0.50 - j0.60$, respectively. Plot $\overline{Y}_{in}$ at the three frequencies on the Smith chart. Does the admittance versus frequency plot contain a loop? Explain!

3–52. A lossless 100 ohm line is terminated in $Y_{in} = 0.03 - j0.02$ mhos. Determine the distances from the load to the first current minimum and to the first current maximum. Assume $\lambda = 15$ cm.