APPENDIX D
The Scattering Matrix

The scattering matrix is a useful analytical technique for studying multiport microwave networks. Its elements relate forward and reverse traveling waves at the various ports of the network. The technique is a logical extension of the interpretation of transmission-line phenomena in terms of incident and reflected waves. This appendix reviews the scattering matrix and the concept of power waves. Although the analysis that follows is based on a three-port, the results may be generalized to any N-port network.

Figure D–1 shows a linear three-port network with transmission lines connected to each port. Their characteristic impedances are denoted by Z_{01}, Z_{02}, and Z_{03} and are assumed to be real in all subsequent discussions. Also shown are the terminal voltages and currents, $\boldsymbol{V}_k$ and $\boldsymbol{I}_k$, where k indicates the port number. In classical circuit theory, the network is usually characterized by an impedance or admittance matrix that relates these terminal voltages and currents. This approach is presented in most texts on network theory and will not be repeated here. An alternate method

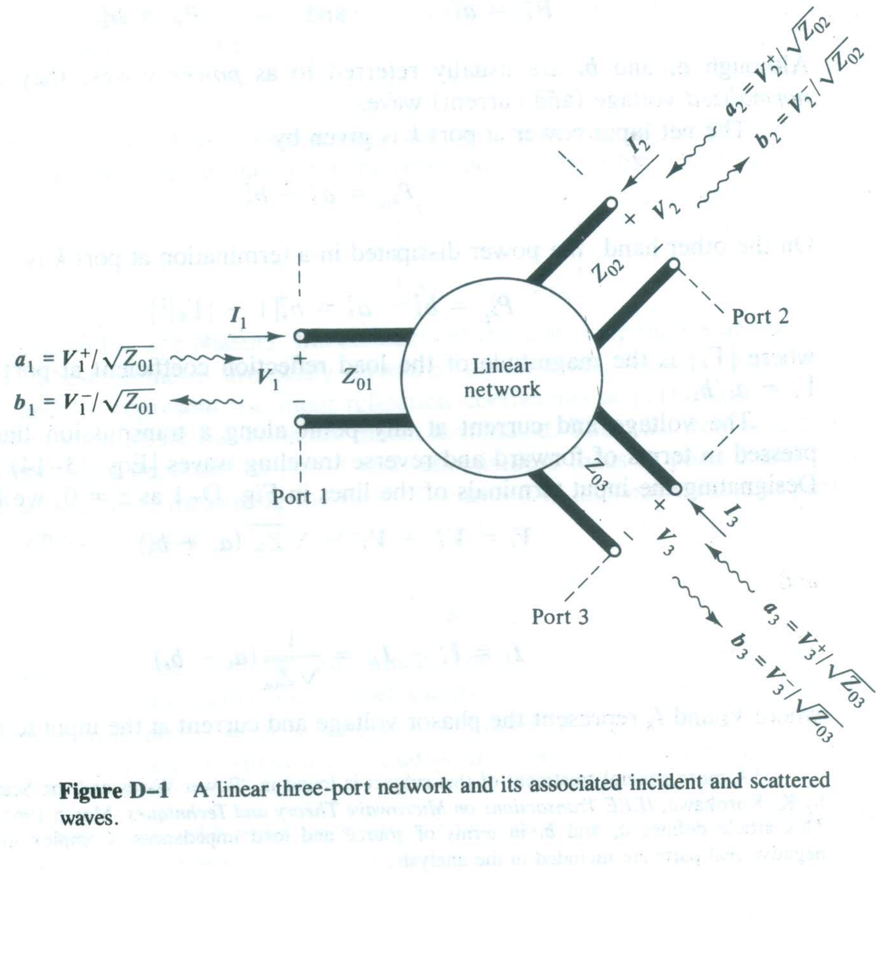

Figure D–1 A linear three-port network and its associated incident and scattered waves.

that is quite useful in microwave analysis is to describe the network behavior in terms of incident and scattered waves. These are shown in Fig. D–1 and are denoted by $\boldsymbol{a}_k$ and $\boldsymbol{b}_k$, respectively. Note that the outgoing waves ($\boldsymbol{b}_k$) are *not* labeled *reflected waves* since they are also associated with transmission from other ports. These phasor quantities are defined by the following equations.[1]

$$\boldsymbol{a}_k = \frac{\boldsymbol{V}_k^+}{\sqrt{Z_{0k}}} = \boldsymbol{I}_k^+\sqrt{Z_{0k}} \tag{D–1}$$

and

$$\boldsymbol{b}_k = \frac{\boldsymbol{V}_k^-}{\sqrt{Z_{0k}}} = \boldsymbol{I}_k^-\sqrt{Z_{0k}} \tag{D–2}$$

where $\boldsymbol{V}_k^+$, $\boldsymbol{V}_k^-$, $\boldsymbol{I}_k^+$, and $\boldsymbol{I}_k^-$ are the incident and scattered voltage and current waves and Z_{0k} is the characteristic impedance of the connecting line at port k. Since Z_{0k} is real, the phase angle of $\boldsymbol{a}_k$ is the same as $\boldsymbol{V}_k^+$ and $\boldsymbol{I}_k^+$. Similarly, the phase of $\boldsymbol{b}_k$ represents the phase of both the $\boldsymbol{V}_k^-$ and $\boldsymbol{I}_k^-$ waves. The square of their rms values equal the power flow associated with the incident and scattered waves. That is,

$$P_k^+ = a_k^2 \qquad \text{and} \qquad P_k^- = b_k^2 \tag{D–3}$$

Although $\boldsymbol{a}_k$ and $\boldsymbol{b}_k$ are usually referred to as *power* waves, they are in reality *normalized* voltage (and current) waves.

The net input power at port k is given by

$$P_{k_{\text{in}}} = a_k^2 - b_k^2 \tag{D–4}$$

On the other hand, the power dissipated in a termination at port k is

$$P_{k_L} = b_k^2 - a_k^2 = b_k^2[1 - |\Gamma_k|^2] \tag{D–5}$$

where $|\Gamma_k|$ is the magnitude of the load reflection coefficient at port k. Note that $\Gamma_k = \boldsymbol{a}_k/\boldsymbol{b}_k$.

The voltage and current at any point along a transmission line can be expressed in terms of forward and reverse traveling waves [Eqs. (3–14) and (3–16)]. Designating the input terminals of the lines in Fig. D–1 as $z = 0$, we have

$$\boldsymbol{V}_k = \boldsymbol{V}_k^+ + \boldsymbol{V}_k^- = \sqrt{Z_{0k}}\,(\boldsymbol{a}_k + \boldsymbol{b}_k)$$

and (D–6)

$$\boldsymbol{I}_k = \boldsymbol{I}_k^+ - \boldsymbol{I}_k^- = \frac{1}{\sqrt{Z_{0k}}}(\boldsymbol{a}_k - \boldsymbol{b}_k)$$

where $\boldsymbol{V}_k$ and $\boldsymbol{I}_k$ represent the phasor voltage and current at the input to the kth port.

[1] A more general treatment of this subject is found in "Power Waves and the Scattering Matrix" by K. Kurokawa, *IEEE Transactions on Microwave Theory and Techniques,* March 1965, pp. 194–202. This article defines $\boldsymbol{a}_k$ and $\boldsymbol{b}_k$ in terms of source and load impedances. Complex impedances with negative real parts are included in the analysis.

Solving the above equations for a_k and b_k in terms of the terminal voltage and current yields

$$a_k = \frac{1}{2}\left(\frac{V_k}{\sqrt{Z_{0k}}} + I_k\sqrt{Z_{0k}}\right)$$

and (D–7)

$$b_k = \frac{1}{2}\left(\frac{V_k}{\sqrt{Z_{0k}}} - I_k\sqrt{Z_{0k}}\right)$$

These equations describe the relationship between the power waves and the terminal voltages and currents.

Referring again to the three-port network in Fig. D–1, the incident and scattered waves are related to the network characteristics by the following equations.

$$\begin{aligned} b_1 &= S_{11}a_1 + S_{12}a_2 + S_{13}a_3 \\ b_2 &= S_{21}a_1 + S_{22}a_2 + S_{23}a_3 \\ b_3 &= S_{31}a_1 + S_{32}a_2 + S_{33}a_3 \end{aligned} \tag{D–8}$$

where the scattering coefficients (S_{11}, S_{12}, etc.) define the characteristics of the linear network.[2] Written in matrix form,

$$\{b\} = [S]\{a\}$$

where $\{a\}$ and $\{b\}$ are column matrices that represent the incident and scattered waves. The scattering matrix for the three-port is given by

$$[S] = \begin{bmatrix} S_{11} & S_{12} & S_{13} \\ S_{21} & S_{22} & S_{23} \\ S_{31} & S_{32} & S_{33} \end{bmatrix} \tag{D–9}$$

Since a_k and b_k are phasors, the elements of the scattering matrix are generally complex. These elements are easily measured for an actual network. For example, S_{11}, S_{22}, and S_{33} represent the input reflection coefficients at ports 1, 2, and 3, respectively, when all output ports are match terminated. Thus, S_{11} may be determined by connecting a generator to port 1 and reflectionless loads to ports 2 and 3. Since a_2 and a_3 are zero, measuring the ratio of reflected to incident voltage at port 1 yields S_{11}, where from Eqs. (D–1), (D–2), and (D–8)

$$S_{11} = \frac{b_1}{a_1} = \frac{V_1^-}{V_1^+}$$

S_{22} and S_{33} can be similarly determined. The off-diagonal elements of the scattering matrix represent transmission coefficients. For instance, $S_{21}a_1$ is the wave that emerges from port 2 when a generator is connected to port 1 and all other ports are terminated in reflectionless loads (that is, $a_2 = a_3 = 0$). Under these conditions S_{21}

[2] Since the a_k and b_k waves are defined in terms of Z_{0k}, the scattering coefficients are referenced to the characteristic impedance of the connecting lines.

is the transmission coefficient from port 1 to port 2. From Eqs. (D–1), (D–2), and (D–8),

$$S_{21} = \frac{b_2}{a_1} = \sqrt{\frac{Z_{01}}{Z_{02}}}\frac{V_2^-}{V_1^+} = \sqrt{\frac{Z_{02}}{Z_{01}}}\frac{I_2^-}{I_1^+}$$

Note that when $Z_{01} = Z_{02}$, S_{21} equals the transmission coefficient T described in Appendix C when $Z_L = Z_0$ [(Eq. (C–15)]. Similar conclusions may be drawn for the other off-diagonal scattering coefficients.

For reciprocal[3] networks, the scattering matrix is symmetrical. For the three-port, this means that $S_{12} = S_{21}$, $S_{13} = S_{31}$, and $S_{23} = S_{32}$. Since the scattering coefficients are complex, these equalities apply to both their magnitude and phase. Thus for a reciprocal three-port network, there are six independent coefficients and therefore twelve independent parameters in the scattering matrix. When the network is dissipationless, a set of conditions exist which further reduces the number of independent parameters. These conditions are helpful in analyzing multiport networks and hence are derived here.

Necessary conditions for a dissipationless network. Assume that a generator is connected to port 1 and reflectionless loads to ports 2 and 3 of the network shown in Fig. D–1. This means that $a_1 \neq 0$ and $a_2 = a_3 = 0$. If the network is dissipationless, the total power out of the network must equal the total input power. Thus from Eq. (D–3),

$$P_1^+ = a_1^2 = b_1^2 + b_2^2 + b_3^2$$

Taking the magnitude squared of both sides of Eqs. (D–8) with a_2 and a_3 equal to zero and adding yields

$$b_1^2 + b_2^2 + b_3^2 = \{|S_{11}|^2 + |S_{21}|^2 + |S_{31}|^2\}a_1^2$$

Since the left-hand side equals a_1^2,

$$|S_{11}|^2 + |S_{21}|^2 + |S_{31}|^2 = 1 \tag{D–10}$$

This equation states that the sum of the squares of the *magnitude* of the scattering coefficients in the first column equals unity. This statement holds true for all columns of the matrix. The following equation generalizes the result to an N-port dissipationless network. Namely,

$$\sum_{n=1}^{N} |S_{np}|^2 = \sum_{n=1}^{N} S_{np} S_{np}^* = 1 \tag{D–11}$$

for any column p from 1 to N. The asterisk (*) denotes the complex conjugate. This condition, as well as the one that follows, is valid for both reciprocal and nonreciprocal networks.

[3] The definition of reciprocity may be found in most texts on network theory. Essentially it means that interchanging the positions of a zero impedance source and a zero impedance ammeter in a network does not affect the ammeter reading.

The scattering coefficients of a dissipationless network are also constrained by the following relationship. Namely,

$$\sum_{n=1}^{N} S_{np} S_{nq}^{*} = 0 \qquad \text{for } p \neq q \tag{D–12}$$

where p and q represent two *different* columns of the matrix.

Let us now verify this equation for the first and second columns of a three-port network. Suppose $\boldsymbol{a}_1$ and $\boldsymbol{a}_2$ are finite, but $\boldsymbol{a}_3 = 0$. Taking the magnitude squared of both sides of Eqs. (D–8) and adding yields

$$b_1^2 + b_2^2 + b_3^2 = |S_{11}\boldsymbol{a}_1 + S_{12}\boldsymbol{a}_2|^2 + |S_{21}\boldsymbol{a}_1 + S_{22}\boldsymbol{a}_2|^2 + |S_{31}\boldsymbol{a}_1 + S_{32}\boldsymbol{a}_2|^2$$

Since the left-hand side is the power out of the network, it must equal the input power $(a_1^2 + a_2^2)$. Recognizing that for any complex quantity Q, $|Q|^2 = QQ^*$, the above expression may be multiplied out to give the following result.

$$\begin{aligned} a_1^2 + a_2^2 = {} & \{|S_{11}|^2 + |S_{21}|^2 + |S_{31}|^2\}a_1^2 + \{|S_{12}|^2 + |S_{22}|^2 + |S_{32}|^2\}a_2^2 \\ & + \{S_{11}S_{12}^* + S_{21}S_{22}^* + S_{31}S_{32}^*\}\boldsymbol{a}_1\boldsymbol{a}_2^* + \{S_{12}S_{11}^* + S_{22}S_{21}^* + S_{32}S_{31}^*\}\boldsymbol{a}_2\boldsymbol{a}_1^* \end{aligned}$$

Using Eq. (D–11) reduces the above to

$$\{S_{11}S_{12}^* + S_{21}S_{22}^* + S_{31}S_{32}^*\}\boldsymbol{a}_1\boldsymbol{a}_2^* + \{S_{12}S_{11}^* + S_{22}S_{21}^* + S_{32}S_{31}^*\}\boldsymbol{a}_2\boldsymbol{a}_1^* = 0$$

Since $\boldsymbol{a}_1$ and $\boldsymbol{a}_2$ are independent signals, they may be chosen in any convenient manner. Letting $\boldsymbol{a}_1 = \boldsymbol{a}_2$ and finite yields

$$\{S_{11}S_{12}^* + S_{21}S_{22}^* + S_{31}S_{32}^*\} + \{S_{12}S_{11}^* + S_{22}S_{21}^* + S_{32}S_{31}^*\} = 0$$

Next, letting $\boldsymbol{a}_1 = j\boldsymbol{a}_2$ and finite yields

$$\{S_{11}S_{12}^* + S_{21}S_{22}^* + S_{31}S_{32}^*\} - \{S_{12}S_{11}^* + S_{22}S_{21}^* + S_{32}S_{31}^*\} = 0$$

since $\boldsymbol{a}_1^* = (j\boldsymbol{a}_2)^* = -j\boldsymbol{a}_2^*$.

Combining the above expressions results in

$$S_{11}S_{12}^* + S_{21}S_{22}^* + S_{31}S_{32}^* = 0$$

and (D–13)

$$S_{12}S_{11}^* + S_{22}S_{21}^* + S_{32}S_{31}^* = 0$$

The first equation is exactly Eq. (D–12) when $p = 1$, $q = 2$, and $N = 3$. The second equation is the case when $p = 2$, $q = 1$, and $N = 3$. Thus for the first two columns of the scattering matrix, the sum of the products of the elements of one column with the complex conjugate of adjacent elements in the other column is zero. Equation (D–12) represents the generalization of this statement to any two columns of an N-port dissipationless network.

It should be noted that if the network is reciprocal, Eqs. (D–11) and (D–12) also apply to the *rows* of the scattering matrix since $S_{np} = S_{pn}$ and $S_{nq} = S_{qn}$.