CT303 - Digital Communications
Autumn 2019/20

In-Sem 1 — 6^{th} September 2019

Duration: 90 minutes — Total Marks 20

Note: You may use any result/example derived/stated in class/lab without proof.

1. Assume that you are sitting at a point on the rim of a Ferris wheel, as you did in Lab-1. The only difference is that, the Ferris wheel is a *nested* one, i.e., a second wheel rotates about its center c_1 which is fixed on the rim of the first one (and moves with the first wheel), and a third wheel rotates about its center c_2 which is fixed on the rim of the second wheel (and moves with the first and second wheel). The wheels are each of radius r_0, r_1, r_2, each completing one rotation in $T, \frac{T}{2}, \frac{T}{4}$ seconds respectively with constant velocity, and center of the first wheel c_0 is fixed at the point $(0, h)$. Each wheel rotates in the direction indicated in Figure 1, and you are seated on the rim of the third wheel. **[3+1]**

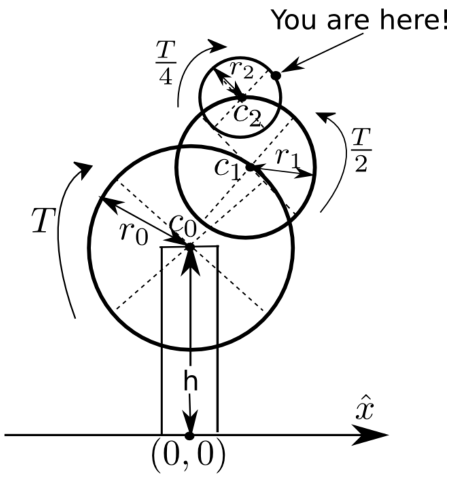


Figure 1: Nested Ferris wheel. Assume $\hat{y}$ along the height of the Ferris wheel, and $h > r_0 + r_1 + r_2$.

   (a) Assuming any starting position you want (draw it), describe your (x, y) position as a function of time.

   (b) What is the Nyquist sampling rate for sampling these functions.

2. Short Questions (and short answers):

   (a) Let $X(t)$ be a stochastic process with ACF $R_X(\tau) = k\delta(\tau)$, $\tau \in \mathbb{R}$, where $k \in \mathbb{R}$ is some positive constant. Find an LTI system (i.e. specify its impulse response h) such that if X is given as an input, the output stochastic process Y has a cross-correlation with the input X given by $R_{YX}(\tau) = \frac{1}{2}\cos(\tau), \tau \in \mathbb{R}$. **[2]**

   (b) Let $X(t)$ be a zero-mean White Gaussian stationary process with PSD N_0. Compute the variance of $X(t)$. **[2]**

   (c) Let $\eta(t)$ be a stationary zero-mean stochastic process that models noise. Let it contaminate a known but arbitrary deterministic signal $s(t)$ to yield the noisy signal (stochastic process) $Z(t) = s(t) + \eta(t)$. Find whether $Z(t)$ is stationary or not. **[1]**

(d) Verify if the following functions are valid autocorrelation functions defined on $\mathbb{R}$ with justification. (a) $R_x(\tau) = \exp(-\tau)$, (b) $R_x(\tau) = (sinc(\tau T))^2$, (c) $R_x(\tau) = rect(\frac{\tau}{T})$. **[3]**

(e) Given a noiseless channel with bandwidth W Hz, state with justification the theoretical maximum symbol rate in symbols/sec that the channel can support? What about the maximum data rate in bits/sec? **[2]**

3. Consider a binary PCM signalling scheme with waveforms s_1 (for bit 0) and either s_2 or s_3 (for bit 1) as shown in Figure 2 below. For each waveform, determine the expression of the matched filter impulse response. On the receiver end, the bit transmitted is estimated based on which of the two matched filter produces a higher output every T seconds. Assume same prior probabilities, AWGN noise with PSD $\frac{N_0}{2}$ Watts/Hz and no inter-symbol interference. Compute the probability of error of the receiver with the matched filters (maybe in terms of the Q function, if possible), and decide which of the two waveforms s_2 and s_3 will you choose to represent bit 1. **[6]**

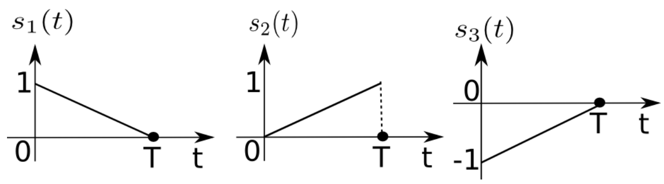
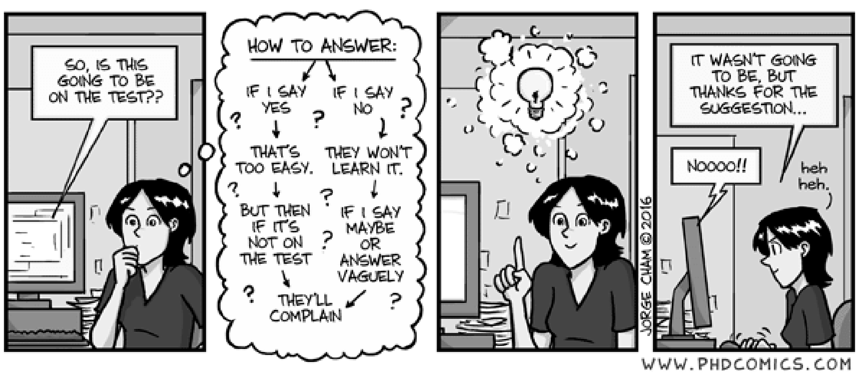


Figure 2: Binary PCM signaling. What will you prefer, $\{s_1, s_2\}$ or $\{s_1, s_3\}$?



Brief Solutions

1. Ferris Wheel:

   (a) Let $c_i(t), i = 0,1,2$ denote the coordinated of the center of the i^{th} Ferris wheel. Coordinates of your position $p(t)$ is then given by $p(t) = r_2(\cos\left(\frac{8\pi t}{T}\right), -\sin\left(\frac{8\pi t}{T}\right)) + c_2(t)$. Similarly, $c_2(t) = r_1(\cos\left(\frac{4\pi t}{T}\right), \sin\left(\frac{4\pi t}{T}\right)) + c_1(t)$, and $c_1(t) = r_0(\cos\left(\frac{2\pi t}{T}\right), -\sin\left(\frac{2\pi t}{T}\right)) + c_0$. Thus,

   $$\begin{aligned} p(t) =& r_2(\cos\left(\frac{8\pi t}{T}\right), -\sin\left(\frac{8\pi t}{T}\right)) \\ &+ r_1(\cos\left(\frac{4\pi t}{T}\right), \sin\left(\frac{4\pi t}{T}\right)) \\ &+ r_0(\cos\left(\frac{2\pi t}{T}\right), -\sin\left(\frac{2\pi t}{T}\right)) + (0,h). \end{aligned}$$

   (b) Nyquist sampling rate: $f_s = \frac{8}{T}$ Hz.

2. Short Questions.

   (a) $R_{YX}(\tau) = (h * R_X)(\tau)$. Thus $h(t) = \frac{1}{2k}\cos(\tau)$.

   (b) $\sigma^2 = \mathbb{E}[X_t X_t] = R_X(0) = \int_{-\infty}^{\infty} |G_x(f)|\, df$. Given that $G_x(f) \equiv N_0$, $\sigma^2 = \infty$.

   (c) $\mathbb{E}[Z(t_1)] = \mathbb{E}[s(t_1) + \eta(t_1)] = s(t_1)$, while $\mathbb{E}[Z(t_2)] = \mathbb{E}[s(t_2) + \eta(t_2)] = s(t_2)$. Since $s(t_1) \neq s(t_2)$, $Z(t)$ is not a stationary stochastic process.

   (d) Valid autocorrelation functions. (a) Invalid, since $R_X(-1) > R_X(0)$. (b) Valid, since $R_X(0) \geq RX(\tau), \forall \tau$, $R_X(\tau) = R_X(-tau)$, $G_X(f)$ is the triangle hat/tent function, which is non-negative and real. (c) Invalid, since $G_X(f)$ is the sinc function, which is negative at several frequencies.

   (e) As already shown in class, given a noiseless channel with bandwidth W Hz, it can support $2W$ symbols/sec using sinc pulses. Since no constraint if given on the receiver, each symbol can encode infinitely many bits, and thus there is no upper bound for the bit rate.

   (f) Matched filter:
   $h_1(t) = s_1(T-t) = s_2(t) = \frac{t}{T}, \forall t \in [0,T]$, $h_2(t) = s_2(T-t) = s_1(t) = 1 - \frac{t}{T}, \forall t \in [0,T]$, $h_3(t) = s_3(T-t) = -s_1(t) = \frac{t}{T} - 1, \forall t \in [0,T]$. Note that $|H_1(f)| = |H_2(f)| = |H_3(f)|$. Thus noise power at the output of the filter is the same for all three matched filters. Let z_1, z_2, z_3 represent the output of the three corresponding filters sampled at time T. The distributions of the three random variables are: $z_1 \sim \mathcal{N}(a_{11}, \sigma_0) + \mathcal{N}(a_{12}, \sigma_0)$ or $z_1 \sim \mathcal{N}(a_{11}, \sigma_0) + \mathcal{N}(a_{13}, \sigma_0)$, where σ_0 is the noise output power, and a_{ij} is the output of filter i when the received signal is purely $s_i(t)$. Since energy of the three symbols is the same $a_{11} = a_{22} = a_{33}$. Similarly, $z_2 \sim \mathcal{N}(a_{21}, \sigma_0) + \mathcal{N}(a_{22}, \sigma_0)$, and $z_3 \sim \mathcal{N}(a_{31}, \sigma_0) + \mathcal{N}(a_{33}, \sigma_0)$. For symbols s_1, s_2 the probability of error is $p(e) = 0.5p(z_1 < z_2|s_1) + 0.5p(z_1 > z_2|s_2) = p(z_1 < z_2|s_1)$, which in turn will depend on the difference between a_{11} and a_{21}. Similarly, for the symbol set s_1, s_3, the probability of error will depend on the difference between a_{33} and a_{31}, which is greater than the difference between a_{22} and a_{21}. Thus the symbol set s_1, s_3 will have a lesser probability of error.