T-3 :

BV	x_1	x_2	S_1	S_2	a_1	a_2	Soln	
z^*	0	0	0	0	-1	-1	0	$\hat{R}_1 \to \bar{R}_1 - \bar{R}_2$
S_2	1/2	0	0	1	1/10	-1	1	$\hat{R}_2 \to \bar{R}_2/10$
S_1	1	0	1	0	0.	0	20	$\hat{R}_3 \to \bar{R}_3$
x_2	1/2	1	0	0	1/10.	0.	15	$\hat{R}_4 \to \bar{R}_4 + \bar{R}_2/10.$

Optimal Table

PHASE-II :- Remove the AVs from the optimal table of phase-I.

IMP:- Don't copy the values of z^* row.

Table of Phase-2:

T-1 :

	BV	x_1	x_2	S_1	S_2	Soln	
	z	2 ↓EV	0	0	0	120.	R_1
LV ←	S_2	1/2	0	0	1	1	R_2
	S_1	1	0	1	0	20	R_3
	x_2	1/2	1	0	0	15	R_4

T-2:

BV	x_1	x_2	S_1	S_2	Soln	
z	0	0	0	-4	116	$\bar{R}_1 \to R_1 - 4R_2$
x_1	1	0	0	2	2	$\bar{R}_2 \to 2R_2$
S_1	0	0	1	-2	18	$\bar{R}_3 \to R_3 - 2R_2$
x_2	0.	1	0	-1	14	$\bar{R}_4 \to R_4 - R_2$

Optimal Table.

$\boxed{z = 116 \quad ; \quad x_1 = 2,\ x_2 = 14}$

Question: MIN. $z = 2x_1 + x_2$

s.t. $3x_1 + x_2 = 3 \quad : C_1 \longrightarrow 3x_1 + x_2 = 3$

$4x_1 + 3x_2 \geq 6 \quad : C_2 \longrightarrow 4x_1 + 3x_2 - S_1 = 6$

$x_1 + 2x_2 \leq 3 \quad : C_3 \longrightarrow x_1 + 2x_2 + S_2 = 3.$

$$A = \begin{array}{c} \begin{array}{cccc} x_1 & x_2 & S_1 & S_2 \end{array} \\ \begin{bmatrix} 3 & 1 & 0 & 0 \\ 4 & 3 & -1 & 0 \\ 1 & 2 & 0 & 1 \end{bmatrix} \end{array} \qquad B = \not I_{3\times3}$$

Modified constraints:

$C_1: 3x_1 + x_2 + a_1 = 3$

$C_2: 4x_1 + 3x_2 - S_1 + a_2 = 6$

$C_3: x_1 + 2x_2 + S_2 = 3.$

$$\text{Revised } A = \begin{array}{c} \begin{array}{cccccc} x_1 & x_2 & S_1 & S_2 & a_1 & a_2 \end{array} \\ \begin{bmatrix} 3 & 1 & 0 & 0 & 1 & 0 \\ 4 & 3 & -1 & 0 & 0 & 1 \\ 1 & 2 & 0 & 1 & 0 & 0 \end{bmatrix}_{3\times6} \end{array}$$

$$\text{Revised } B = \begin{array}{c} \begin{array}{ccc} a_1 & a_2 & S_2 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{array} = I_{3\times3}$$

PHASE-I: MIN. $z^* = 1.a_1 + 1.a_2$

I-1

$C_b = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$

		$C_1=0$	$C_2=0$	$C_3=0$	$C_4=0$	$C_5=1$	$C_6=1$		
	BV	x_1	x_2	S_1	S_2	a_1	a_2	Sol^n	
	z^*	7 ↓EV	4	-1	0	0	0	9	R_1
LV ←	a_1	3	1	0	0	1	0	3	R_2
	a_2	4	3	-1	0	0	1	6	R_3
	S_2	1	2	0	1	0	0	3	R_4

T-2

BV	x_1	x_2	S_1	S_2	a_1	a_2	Sol^n	
Z^*	0	5/3 ↓EV	-1	0	$\cancel{0}$ -7/3	0	2	$\bar{R}_1 \to R_1 - \frac{7}{3}R_2$
x_1	1	1/3	0	0	$\cancel{0}$ 1/3	$\cancel{1}$ 0	1	$\bar{R}_2 \to R_2/3$
LV ← a_2	0	5/3	-1	0	-4/3	1	2	$\bar{R}_3 \to R_3 - \frac{4}{3}R_2$
S_2	0	5/3	0	1	$\cancel{0}$ -1/3	0	2	$\bar{R}_4 \to R_4 - R_2/3$

T-3

BV	x_1	x_2	S_1	S_2	a_1	a_2	Sol^n	
Z^*	0	0	0	0	-1	-1	0	$\hat{R}_1 \to \bar{R}_1 - \bar{R}_3$
x_1	1	0	1/5	0	3/5	-1/5	3/5	$\hat{R}_2 \to \bar{R}_2 - \bar{R}_3/5$
x_2	0	1	-3/5	0	-4/5	3/5	6/5	$\hat{R}_3 \to 3\bar{R}_3/5$
S_2	0	0	1	1	1	-1	0	$\hat{R}_4 \to \bar{R}_4 - \bar{R}_3$

Optimal Table.

Artificial variable column removal

PHASE-II :

BV	x_1	x_2	S_1	S_2	Sol^n
Z	0	0	-1/5	0	12/5
x_1	1	0	1/5	0	3/5
x_2	0	1	-3/5	0	6/5
S_2	0	0	1	1	0

Optimised

$$\boxed{Z = 12/5 \; ; \; x_1 = 3/5, \; x_2 = 6/5}$$

→ No graph paper & no calculator
→ 40 Marks. → 1.5 Hours.
→ 4-5 Questions

* Transportation Model.

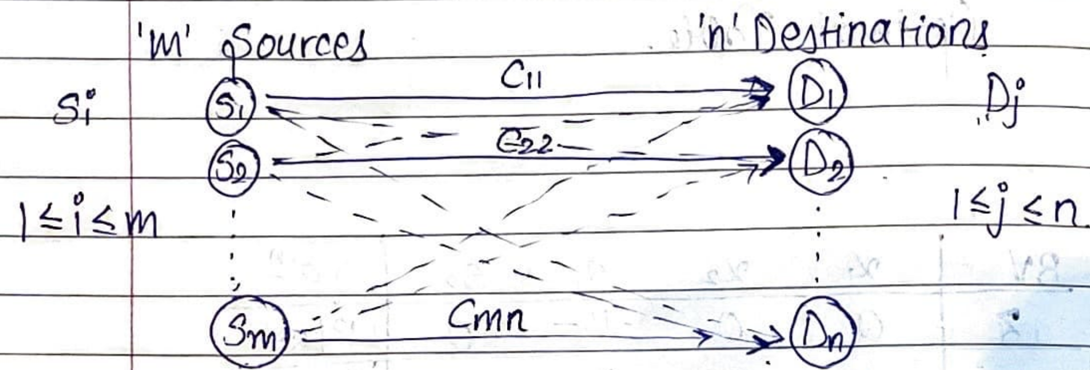
- cost & profit associated with transport of goods to any agency
- a practical model

* Transportation Problem:

Resource Allocat^n^ Problem

Source — Destination.

Essential keywords.



$S_i \longrightarrow D_j$
Source — Destination

- Cost associated in Transportation: C_{ij} → cost per unit
- Product transported from S_i to D_j: x_{ij} → no. of units.

~~COST~~.

COST: Transportation problem with cost mechanism

x_{ij}: Units of product transported from Source S_i to destination D_j

C_{ij}: cost per unit associated with product to be transported from source S_i to destination D_j.

SYSTEM MODEL:

	D_1	D_2	....	D_n	
S_1	x_{11}/C_{11}		....		a_1
S_2					a_2
$\vdots$					$\vdots$
S_m				x_{mn}/C_{mn}	a_m
	b_1	b_2		b_n	

$\rightarrow$ Supply (a_i)

$\rightarrow$ Demand (b_i)

→ Each source must have some data wrt supply. i.e. the amt. of product each source is able to produce. (Source → Supply)

→ Each destinat^n has some demand requirement which the sources take into consideration. (Destinat^n → Demand)

* Formulation of Transportation Problem:

- Objective function: $z = f(x_{ij}, C_{ij})$. ——→ Minimization

x_{ij} ↓ Resource $\quad C_{ij}$ ↓ cost associated with Resource Transportation.

$$\boxed{z = f(x_{ij}, C_{ij}) = \sum_{j=1}^{n} \sum_{i=1}^{m} x_{ij} \times C_{ij}}$$

$$f(x_{ij}, C_{ij}) = [x_{11}C_{11} + x_{12}C_{12} + \dots + x_{1n}C_{1n} +$$
$$x_{21}C_{21} + x_{22}C_{22} + \dots + x_{2n}C_{2n} +$$
$$\vdots$$
$$x_{m1}C_{m1} + x_{m2}C_{m2} + \dots + x_{mn}C_{mn}]$$

Total cost: m^{th} source to each of n^{th} dest.

Resource Allocation Optimisation

- Constraints:

1. Source constraints: $\sum_{j=1}^{n} x_{ij} \leq a_i \quad \forall\, i=1,2,\dots,m$

2. Destination constraints: $\sum_{i=1}^{m} x_{ij} \geq b_j \quad \forall\, j=1,2,\dots,n$

STANDARD TRANSPORTATION PROBLEM

→ Min. $z = f(x_{ij}, c_{ij}) = \sum_{j=1}^{n}\sum_{i=1}^{m} c_{ij} \times x_{ij}$

Source constraints: $\sum_{j=1}^{n} x_{ij} = a_i \quad \forall\, i=1,2,\dots,m$

Destination constraints: $\sum_{i=1}^{m} x_{ij} = b_j \quad \forall\, j=1,2,\dots,n$

$x_{ij} \geq 0$

Standard Transportation Problem

Similarity to LPP (Algebraic Method):-

→ $f(x_i, c_i) = c_1x_1 + c_2x_2 + \dots + c_nx_n = \bar{c}^T\bar{x}$

$A\bar{x} = \bar{b}$

where $\bar{c} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$ $\quad \bar{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$

- Here let $m=2, n=3$

So, here $A\bar{x} = \bar{b}$

$[A] \Rightarrow 5 \times 6$

$\boxed{\text{dim. of coefficient matrix} = (m+n) \times (mn)}$

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$$\bar{x} = \begin{bmatrix} x_{11} \\ x_{12} \\ x_{13} \\ x_{21} \\ x_{22} \\ x_{23} \end{bmatrix}$$

$$A = \begin{array}{c} \\ \\ \\ \\ \\ \end{array}\begin{array}{cccccc} x_{11} & x_{12} & x_{13} & x_{21} & x_{22} & x_{23} \end{array}$$

$$A = \left[\begin{array}{cccccc} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{array}\right]_{5\times 6} \begin{array}{l} \bar{a}_1 \\ \bar{a}_2 \\ \bar{a}_3 \\ \bar{a}_4 \\ \bar{a}_5 \end{array}$$

coefficient matrix

$$\bar{b} = \begin{bmatrix} a_1 \\ a_2 \\ b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

$\boxed{\text{Total no. of variables} = mn}$

So here,

$\boxed{\# BV = (m+n-1) \qquad \# NBV = .}$

$x_{11} + x_{12} + x_{13} = a_1$ —— (1)

$x_{21} + x_{22} + x_{23} = a_2$ —— (2)

$x_{11} + x_{21} = b_1$ —— (3)

$x_{12} + x_{22} = b_2$ —— (4)

$x_{13} + x_{23} = b_3$ —— (5).

(1) + (2) : $\boxed{A} \rightarrow x_{11} + x_{12} + x_{13} + x_{21} + x_{22} + x_{23} = a_2 + a_1$

(3) + (4) : $\boxed{B} \rightarrow x_{11} + x_{21} + x_{12} + x_{22} = b_2 + b_1$.

$\boxed{A} - \boxed{B} \rightarrow x_{13} + x_{23} = (a_1 + a_2) - (b_1 + b_2)$.

So. $(a_1 + a_2) - (b_1 + b_2) = b_3 \longrightarrow$ 'b_3' becms redundant.

$\hookrightarrow$ Supply = Demand.

Linear Combination of Vectors:

if $\lambda_1\bar{a}_1 + \lambda_2\bar{a}_2 + \lambda_3\bar{a}_3 + \lambda_4\bar{a}_4 + \lambda_5\bar{a}_5 = 0$

for any $\lambda_i \neq 0$.

Then ~~linearly~~ linear dependency exists.

- This shows that any combination of four rows will give the fifth row.

So, this reduces the no. of BVs.

→ For destination constraints, if we ~~supply~~ produce & supply more than what we need, it'll result in more cost.

So, modified destination constraints:

$$\sum_{i=1}^{m} x_{ij} = b_j \quad \forall j = 1, 2, \ldots, n.$$

→ Also $\underbrace{\sum_{i=1}^{m} a_i \geq \sum_{j=1}^{n} b_j}_{\text{Not Balanced TP.}}$ {Supply ≥ Demand.}

Theorem: A necessary & sufficient condition for existence of a feasible sol^n to Transportation Problem is:

Total Supply = Total Demand. {Balanced TP}

- We'll always discuss Balanced TP.

UnBalanced TP: Total Supply > Total Demand → gives infeasible sol^n
Total Supply < Total Demand.

↳ Transfer problem to Balanced TP.

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- We need to add dummy source or destinatⁿ to meet the requirements.

Eg. ①

	D_1	D_2	D_3	
S_1				20
S_2				15
	10	10	10	

Supply > Demand
↳ adding dummy destinatⁿ

c_{ij}: costs:

②

	D_1	D_2	D_3	
S_1	c_{11} 2	c_{12} 4	c_{13} 7	20
S_2	c_{21} 1	c_{22} 5	c_{23} 9	15
S_3	20	15	10	

Supply < Demand
↳ adding dummy source.

Supply

↳ $c_{31}=0,\ c_{32}=0,\ c_{33}=0$

→ $\boxed{x_{ij} \in [0, \min\{a_i, b_j\}]}$

→ $m=2$ sources $n=3$ destinations

Tabular representation:

$x_{ij} \rightarrow c_{ij}$ (association)

$1 \le i \le m$

$1 \le j \le n$

	D_1	D_2	D_3	
S_1	x_{11}	x_{12}	x_{13}	a_1
S_2	x_{21}	x_{22}	x_{23}	a_2
	b_1	b_2	b_3	

$1 \le i \le m$

$X_f = (x_{11}, x_{12}, x_{13}, x_{21}, x_{22}, x_{23})$

↓

If feasible solution

- A feasible solⁿ will be satisfying all the constraints.

- Source constraints: $x_{11} + x_{12} + x_{13} = a_1$: S_1 —— (A1)
$x_{21} + x_{22} + x_{23} = a_2$: S_2 —— (A2)

- Destinatⁿ constraints: $x_{11} + x_{21} = b_1$: D_1 —— (B1)
$x_{12} + x_{22} = b_2$: D_2 —— (B2)
$x_{13} + x_{23} = b_3$: D_3 —— (B3)

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Now A1 + A2 :

$$x_{11} + x_{12} + x_{13} + x_{21} + x_{22} + x_{23} = a_1 + a_2 \quad \text{--- (1)}$$

And B1 + B2 + B3 :

$$x_{11} + x_{21} + x_{12} + x_{22} + x_{13} + x_{23} = b_1 + b_2 + b_3 \quad \text{--- (2)}$$

From (1) & (2): $a_1 + a_2 = b_1 + b_2 + b_3$

$$\therefore \boxed{\sum_{\forall i} a_i = \sum_{\forall j} b_j}$$

For Balanced Transportation Problem

Corollary

- Now, let $\sum_{\forall i} a_i = \sum_{\forall j} b_j = \alpha$

To show: There exists a feasible solⁿ:

$\Downarrow$

There exists a solⁿ which satisfies all the constraints.

- Suppose the resource allocation mechanism we choose to be:

$$x_{ij} = \frac{1}{\alpha}(a_i \times b_j)$$

$$\sum_{\forall i} x_{ij} = \sum_{\forall i} \frac{1}{\alpha}(a_i \times b_j)$$

$$= b_j \sum_{\forall i} \frac{1}{\alpha} a_i$$

$$= b_j \times \frac{1}{\alpha} \sum_{\forall i} a_i$$

$$= b_j \times \frac{\alpha}{\alpha}$$

$$\sum_{\forall i} x_{ij} = b_j \longrightarrow \text{Destination constraints}$$

Similarly, $\sum_{\forall j} x_{ij} = a_i$ $\longrightarrow$ Source constraints.

So the given mechanism yields the constraints.

Hence, there exists a feasible sol^n.

Remark: In Transportation Problem, all x_{ij}'s are bounded in a limit $\{x_{ij} \in [0, \min\{a_i, b_j\}]\}$. So, the sol^n of this system model will also remain bounded.

Question:

outlets; c_{ij}'s

Manufacturing facility

	D_1	D_2	D_3	
S_1	x_{11} $\boxed{2}$	x_{12} $\boxed{4}$	x_{13} $\boxed{7}$	20
S_2	x_{21} $\boxed{1}$	x_{22} $\boxed{5}$	x_{23} $\boxed{9}$	25
	20	15	10	

Constraints: $x_{11} + x_{12} + x_{13} = 20$ $\qquad$ $x_{11} + x_{21} = 20$

$x_{21} + x_{22} + x_{23} = 25$ $\qquad$ $x_{12} + x_{22} = 15$

$x_{13} + x_{23} = 10$

Obj. fxn: $f(x_{ij}, c_{ij}) = (2x_{11} + 4x_{12} + 7x_{13}) + (x_{21} + 5x_{22} + 9x_{23})$

$\rightarrow$ Matrix representat^n: $A\bar{x} = \bar{b}$

$$\bar{x} = \begin{bmatrix} x_{11} \\ x_{12} \\ x_{13} \\ x_{21} \\ x_{22} \\ x_{23} \end{bmatrix}_{6\times 1} \qquad \bar{b} = \begin{bmatrix} 20 \\ 25 \\ 20 \\ 15 \\ 10 \end{bmatrix}_{5\times 1}$$

$(mn \times 1)$ $\qquad$ $((m+n)\times 1)$

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$$A = \begin{array}{c} \begin{array}{cccccc} x_{11} & x_{12} & x_{13} & x_{21} & x_{22} & x_{23} \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}_{5\times 6} \end{array}$$

Any one row can be written as linear combination of other remaining rows.
→ This is bcz of redundancy.

→ Solution of Transportation Problem:

Step 1: Find the initial basic feasible solution keeping cost minimisation & range of x_{ij}'s in mind.

Key metric associated with optimal resource allocation : $\boxed{x_{ij} \times c_{ij}}$

Step 2: We need to check if the solⁿ/ Resource allocatⁿ is Optimal ie a check for optimality needs to be incorporated.
cost problem: min. transportatⁿ prblm: $\boxed{z_{ij} - c_{ij} \leq 0}$

- Node & Arc Representation: Each source & destinatⁿ is represented by a Node. Arc (i,j) which represents "unit cost" from source i to destination j : $1 \leq i \leq m$, $1 \leq j \leq n$.

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Table Representation:

	D_1	D_2	....	D_n
S_1	$cell_{11}$	x_{12}	....	x_{1n}
S_2				
$\vdots$				
S_m				$cell_{mn}$

$$f(x_{ij}, c_{ij}) = \sum_{\forall j} \sum_{\forall i} c_{ij} \times x_{ij}$$

$$\left.\begin{aligned} \sum_{\forall j} x_{ij} &= a_i \quad \forall i \\ \sum_{\forall i} x_{ij} &= b_j \quad \forall j \end{aligned}\right\} \text{constraints for Balanced TP.}$$

- m sources → m constraints
  n destinat^n → n constraints

In a Balanced TP: There'll be $(m+n-1)$ independent constraints.

So, $\bar{c} = \begin{bmatrix} c_{11} \\ c_{12} \\ \vdots \\ c_{1n} \\ c_{21} \\ \vdots \\ c_{2n} \\ \vdots \\ c_{m1} \\ \vdots \\ c_{mn} \end{bmatrix}_{mn\times 1}$ $\quad \bar{x} = \begin{bmatrix} x_{11} \\ x_{12} \\ \vdots \\ x_{1n} \\ x_{21} \\ \vdots \\ x_{2n} \\ \vdots \\ x_{m1} \\ \vdots \\ x_{mn} \end{bmatrix}_{mn\times 1}$

- So objective fxn: $\boxed{f(\bar{c}; \bar{x}) = \bar{c}^T \bar{x}}$

  ↳ Optimize (min.)
  ↳ We only have $\bar{x}$ in our hand to formulate

- Allocating x_{ij} → b/w source i & destination j.

- Approaches of resource allocation in TP:
  1. Northwest corner Method (NWCM).
  2. Least cost Method (LCM)
  3. Vogel's Approximation Method (VAM)

~~Approach 3:~~ Steps for the given approaches.

~~Step 1~~: Using these approaches, we are finding initial basic ~~f~~ solution

Step 2: Check for optimality of resource allocation.

* Method 1 : NWCM:

let: $m = 2$, $n = 3$

	D_1	D_2	D_3
S_1	$cell_{11}$		
S_2			

Tabular Representation.

$\boxed{cell_{11} \rightarrow cell_{ij} \rightarrow \ldots \rightarrow cell_{mn}}$ → Path Traversal / Route Map.

Here $cell_{11} \rightarrow cell_{ij} \rightarrow \ldots \rightarrow cell_{23}$.

NWCM:

1. Allocate as much as possible to the selected cell (starting from NW cell), & adjust the associated amts of supply, demand by subtracting the allocated amount.

2. Cross out the row/column with zero supply/demand to indicate NO further assignments can be made

in that row/colum. If both row & column net to zero simultaneously, ~~crow~~ cross out only one & leave zero supply (or demand) in the uncrossed ~~one~~ row (or colum)

3. If exactly one row/column is left uncrossed out, stop! Otherwise move to the cell to the right if column has just been ~~cro~~ crossed out or below if row has been crossed out. Go to 1.

Ques-1

	D_1	D_2	D_3	D_4	Supply across S_i
S_1	5 [10]	10 [2]	X [20]	X [11]	~~15~~ ~~10~~.
S_2	X [12]	5 [7]	15 [9]	5 [20]	~~25~~ ~~20~~ ~~15~~
S_3	X [4]	X [14]	X [16]	10 [18]	10
Demand across D_j	~~5~~	~~15~~ ~~5~~	~~15~~	~~15~~ 10	

- Starting from $cell_{11}$. & giving the min. of demand & supply. crossing $cell_{21}$, $cell_{31}$ & changing demand & supply
- Moving to $cell_{12}$.

repeating same process

Movt:

$cell_{ij} \rightarrow cell_{i+1,j}$: Vertical movt.

$cell_{ij} \rightarrow cell_{i,j+1}$: Horizontal movt

Route / Path: $cell_{11} \rightarrow cell_{12} \rightarrow cell_{22} \rightarrow cell_{23} \rightarrow cell_{24} \rightarrow cell_{34}$.

- # cells = 12 = $\boxed{mn}$.
- # Resource allocated cells = 6 = $\boxed{m+n-1}$

↳ Same as no. of BV for Balanced TP.

So, # Basic cells = 6

So, resource is allocated across 6 cells

Q. What is the transportation cost?

→

$$\text{cost} = \sum_{\forall i} \sum_{\forall j} c_{ij} x_{ij} = f(\bar{x};\bar{c})$$

$$= (10 \times 5) + (10 \times 2) + (7 \times 5) + (15 \times 9) + (5 \times 20) + (10 \times 18)$$

$$\therefore f(\bar{x};\bar{c}) = 520$$

Ques-2 $m = 2, n = 3$

	D_1	D_2	D_3	
S_1	2 / 10	4 / 10	7	~~20~~ ~~10~~
S_2	1 / ×	5 / 5	9 / 10	~~25~~ ~~20~~, 10
	~~10~~	~~15~~ ~~5~~	~~10~~	

→ From Ques-1:
$x_{11} = 5,\ x_{12} = 10,\ x_{22} = 5,\ x_{23} = 15,\ x_{24} = 5,\ x_{34} = 10$

$f(\bar{x};\bar{c}) = 520$ (Total cost incurred)
This is initial basic solution. (We need to check for optimality).

Ques-3

	D_1	D_2	D_3	
S_1	2	1	4	10
S_2	6	3	1	20
S_3	4	2	3	10
	10	15	15	

NWCM
⇓
for finding unknowns (Resource Allocation).

NWCM → LCM → VAM.

Moving towards practicality / better initial sol^n;

⇒ After resource allocation, $x_{11} = 10$, both sources S_1 & destination D_1 are exhausted simultaneously. Since both exhausted simultaneously we can either move to cell$_{i,j+1}$ or cell$_{i+1,j}$. So, No matter go horizontal or vertical.

→ Let's go horizontal.

$$\begin{tikzcd} \text{cell}_{11} \arrow[r, bend left, "\text{NWCM}"] \arrow[r] \arrow[r, bend right] & \text{cell}_{mn} \end{tikzcd}$$

⇒ All supply and demand exhausted. So, proper resource allocation has been done.

- This is initial BFS / initial resource allocation.
  $\{x_{11} = 10,\ x_{12} = 0,\ x_{22} = 15,\ x_{23} = 5,\ x_{33} = 10\}$.

Total Incurred cost = \$(105)

⇒ # BV cells = $m+n-1 = 3+3-1 = 5$

- Non-zero resource allocation done in 4 cells.
  ↳ This is initial degenerate BFS.
  ~~[Example-1 was initial non-d BFS]~~

* Method 2: Least ~~Common~~ Cost Method (LCM):

- LCM finds a better starting / initial sol^n by concentrating on the cheapest route.
- The method assigns as much as possible to the cell with the smallest unit cost i.e. smallest C_{ij}.
  (This can be broken arbitrarily)
- Next the satisfied row / column is crossed out &

the amounts of supply & demand are adjusted accordingly.
- Next look for uncrossed cell with smallest unit cost & repeat until all supply & demands are exhausted.

Using LCM:
Question 4:

	D_1	D_2	D_3	D_4	Supply
S_1	× [10]	15 (1st) [2]	× [20]	× [11]	~~15~~
S_2	× [12]	× [7]	15 (3rd) [9]	10 (5th) [20]	~~25~~ ~~10~~
S_3	5 (2nd) [4]	× [14]	× [16]	5 (4th) [18]	~~10~~ ~~5~~
Demands	~~5~~	~~15~~	~~15~~	~~15~~ ~~10~~	

50 → Total supply
50 → Total demand
(Balanced)
(If unbalanced, add dummy source/destination with 0 cost)

⇒ $S_1 \rightarrow D_2$ paying $\boxed{\$2/\text{unit}}$: least unit cost.
So, first allocation to $cell_{12}$.

⇒ Next uncrossed cell with smallest unit cost : $cell_{31}$ $(S_3 \rightarrow D_1)$ (\$4/unit)
So, second allocation to $cell_{31}$

And allocation goes on till all supply & demand aren't exhausted.

⇒ Initial Solⁿ:
$x_{12} = 15$, $x_{23} = 15$, $x_{24} = 10$, $x_{31} = 5$, $x_{34} = 5$

BV cells = $m+n-1 = 6$
Total cost incurred = \$475 ✱
So cost (LCM) < cost (NWCM)
\$475 < \$520.

→ LCM is more optimal/feasible than NWCM.

* Idea of dummy source / destination:

	D_1	D_2	D_3	D_4	Supply avai.
S_1					200
S_2					180
S_3					110
(Dummy) S_4	0	0	0	0	50
Demand Req.	150	40	180	170.	

~~490~~ 540

540.

On adding dummy source, the total supply ↑ses from 490 to ~~540~~ 540. as (TS < TD)

→ Unbalanced Transportatⁿ Problem $\xrightarrow{\text{converts to}}$ Balanced Transportatⁿ Problem (by adding dummy variable).

(S_4 has zero contribution in Total cost).

Resource Allocation : $\bar{x}$ for all 3 methods would be different

Metrics → Variables

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Method 3 Vogel's Approximation Method (VAM):

- In ~~1~~ 45% of the cases, VAM is the most optimal method.

1) In each row, & each colⁿ, we find the cell having the smallest & next to smallest cost. Thereafter, we write the difference b/w these costs (penalties) along the side of table as row & colⁿ penalty.

2) The row/column with max. penalty is selected & then we find the cell that has least cost in selected row/colⁿ. Now allocate the resources x_{ij} in that cell.

3) In case of a tie in penalty, choose the least COST available. If there's a tie in least COST, in that case allocate max. possible x_{ij} [$x_{ij} = \min\{a_i, b_j\}$].
After adjusting supply & Demand, repeat 1 & 2

until all supply & demand are exhausted.

Question:

	D_1	D_2	D_3	D_4	Supply (34 units)	
S_1	19	x 30	50	10	7	19-10 = $\boxed{9}$ → least penalty
S_2	70	x 30	40	60	9	40-30 = 10
S_3	40	8 18 (I^{st})	70	20	~~18~~ 10.	20-8 = 12
Demand (34 units)	5	~~8~~	7	14		
	40-19 = 21	30-8 = $\boxed{22}$ max. penalty	50-40 = 10	20-10 = 10	← First level of penalty	

- Focusing on col^n D_2, least cost at cell$_{32}$. → I^{st} allocation

→ Moving to second level of penalty.

	D_1	D_2	D_3	D_4		
S_1	5 19 (II^{nd})	x 30	50	10	~~7~~ 2.	9
S_2	x 70	x 30	40	60	9	20 (60-40)
S_3	x 40	8 18 (I^{st})	70	20	10.	20.
	~~5~~	0	7	14		
	21	-	10	10	← Second level of penalty.	

no pt. in calculating penalty here

→ Third level of penalty.

	D_1	D_2	D_3	D_4		
S_1	5 [19] IInd	× [30]	[50]	[10]	2	40
S_2	× [70]	× [30]	[40]	[60]	9	20
S_3	× [40]	8 [8] Ist	× [70]	10 [20] IIIrd	10.	← 50.
	0	0	7	~~14~~ 4		
	–	–	10	10.	← Third level	

→ Fourth level of penalty:

	D_1	D_2	D_3	D_4		
S_1	5 [19] IInd	× [30]	× [50]	2 ~~4~~ [10] 4th	~~2~~	40
S_2	× [70]	× [30]	7 [40] 4th	2 [60] 5th	~~9~~ 2	20
S_3	× [40]	8 [8] Ist	× [70]	10 [20] IIIrd	~~10~~ 0	–
	0	0	~~7~~	~~4~~ 2		
	–	–	10	50	← Fourth level	

Fifth level of penalty is irrelevant.

So, $x_{11} = 5$, $x_{14} = 2$, $x_{23} = 7$, $x_{24} = 2$, $x_{32} = 8$, $x_{34} = 10$

↳ Initial solⁿ / resource allocation.

Total cost = \$779.

Here, $m = 3$, $n = 4$

\# BV / cell = $m + n - 1 = 6$

\# non-zero(+ve) resource allocation = 6

} nd-BFS

↳ $x_{11}, x_{14}, x_{23}, x_{24}, x_{32}, x_{34}$

Example:

	D_1	D_2	D_3	Supply	Level 1	Level 2	Level 3
S_1	10 [2] 1st	x [1]	x [4]	~~10~~ 0	1	~~3~~ –	–
S_2	x [6]	5 [3] 4th	15 [2] 2nd	~~20~~ ~~5~~ 0	1	1	–
S_3	x [4]	10 [2] 3rd	x [3]	~~10~~ 0	1	1	–
Demand	~~10~~ 0	~~15~~ ~~5~~ 0	~~15~~ 0				
Level 1	2	1	1				
Level 2	–	1	1				
Level 3	–	1	–				

- Initial Basic feasible solution (Resource Allocation):
  $x_{11} = 10,\ x_{22} = 5,\ x_{23} = 15,\ x_{32} = 10$

- \# Basic cells (variables) = $m+n-1 = 5$
But \# Resource Allocation = 4

} So this is d-BFS

→ NET EVALUATION:
<u>Min</u>: $z_j - c_j \le 0$. for optimality.

Here, we need to check: $z_{ij} - c_{ij} \le 0$ for non-basic cells. for optimality.
As for basic cells, $z_{ij} - c_{ij} = 0$.

⇒ <u>Transportation Model</u> { m sources, n destinations. → Resource Allocations (finding initial Basic Feasible Sol^n).

→ NWCM, LCM, VAM

[NWCM LCM VAM (Allocating x_{ij}'s in basic cells) → To find that: Is the Resource Allocation OPTIMAL?

→ In simplex → for optimal Table (let say, MIN.) prblm. → BV ($z_j - c_j = 0$) — ①

→ N-BV ($z_j - c_j \leq 0$) — ②

Looking at Duality:

PRIMAL TP:

- Min. : $z = f(\bar{x}; \bar{c}) = \sum_{j=1}^{n} \sum_{i=1}^{m} c_{ij} x_{ij}$

$$\text{s.t.} \quad \sum_{j=1}^{n} x_{ij} = a_i \quad ; 1 \leq i \leq m$$

$$\sum_{i=1}^{m} x_{ij} = b_j \quad ; 1 \leq j \leq n.$$

DUAL TP: [U-V Method, Modified Distribution (MODI) method]

Obj. fxn:

- Max. : $z_D = f_D(a_i, b_j, u_i, v_j) = \sum_{i=1}^{m} a_i u_i + \sum_{j=1}^{n} b_j v_j$

s.t. $u_i + v_j \leq c_{ij}$; u_i & v_j are unrestricted in sign.

$$\downarrow$$

$$\boxed{u_i + v_j = c_{ij}}$$

→ let $m = 2$, $n = 3$

In the Primal Method:

$$z = \sum_{j=1}^{3} \sum_{i=1}^{2} c_{ij} x_{ij}$$

	D_1	D_2	D_3	
S_1				a_1
S_2				a_2
	b_1	b_2	b_3	

s.t. $x_{11} + x_{12} + x_{13} = a_1$
$x_{21} + x_{22} + x_{23} = a_2$ } source constraints

$x_{11} + x_{21} = b_1$
$x_{12} + x_{22} = b_2$
$x_{13} + x_{23} = b_3$ } Destination constraints

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$\rightarrow \quad A\bar{x} = \bar{b}$

$$\begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_{11} \\ x_{12} \\ x_{13} \\ x_{21} \\ x_{22} \\ x_{23} \end{bmatrix} = \begin{bmatrix} a_1 \\ a_2 \\ b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

For dual TP: (for $m = 2$, $n = 3$)

Let $\bar{x}_D = \begin{bmatrix} u_1 \\ u_2 \\ v_1 \\ v_2 \\ v_3 \end{bmatrix}$

In terms of u_i, we'll have 'm' u_i's & 'n' v_j's.

$[A] = (m+n) \times (mn)$

$[\bar{x}] = (mn) \times 1$

$\cancel{[\bar{c}] = (m+n) \times 1}$

$[\bar{b}] = (m+n) \times 1$

$\bar{c} = \begin{bmatrix} c_{11} \\ c_{12} \\ c_{13} \\ c_{21} \\ c_{22} \\ c_{23} \end{bmatrix}$

$[\bar{x}_D] = (m+n) \times 1$

$[\bar{c}] = (mn) \times 1$

For Duality. $\rightarrow$ $\boxed{A^T \bar{x}_D = \bar{c}}$

$$\begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} c_{11} \\ c_{12} \\ c_{13} \\ c_{21} \\ c_{22} \\ c_{23} \end{bmatrix}$$

$\Rightarrow$ $u_1 + v_1 = c_{11}$

$u_1 + v_2 = c_{12}$

$u_1 + v_3 = c_{13}$

$u_2 + v_1 = c_{21}$

$u_2 + v_2 = c_{22}$

$u_2 + v_3 = c_{23}$

$\boxed{u_i + v_j = c_{ij}.}$

u_i's & v_j's → dual variables
↳ obtained from basic cells

For BV/cells: $z_{ij} - c_{ij} = 0$

For N-BV/cells: $z_{ij} - c_{ij} \leq 0$

→ In Simplex: $z_j - c_j = \bar{c}_B^T e_j - c_j$

→ In TP: $z_{ij} - c_{ij} = \bar{x}_D^T \cdot a_j - c_j$
↳ First row of (A^T)

Let $i = j = 1$:

$$z_{11} - c_{11} = [u_1 \; u_2 \; v_1 \; v_2 \; v_3] \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} - c_{11}$$

$$\boxed{z_{11} - c_{11} = u_1 + v_1 - c_{11}}$$ ⤳ For cell$_{11}$

Ily: $\boxed{z_{12} - c_{12} = u_1 + v_2 - c_{12}}$ ⤳ For cell$_{12}$

On generalization:

$$\boxed{z_{ij} - c_{ij} = u_i + v_j - c_{ij}} \quad \forall \text{ cells}$$

→ For Basic cells: $z_{ij} - c_{ij} = 0$

∴ $\boxed{u_i + v_j = c_{ij}}$ → Applicable only for basic cells.

IMP

→ For TP: other than Basic cells:
$\boxed{u_i + v_j \leq c_{ij}}$ or $\boxed{z_{ij} - c_{ij} \leq 0}$

→

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* Optimal Solution for a TP (Model):
[u-v Method / MoDi Method]

1) Balanced TP : If TP is unbalanced (UB-TP), then make it balanced TP by addition of dummy src/destination.

2) Find the initial BFS or Resource Allocation is to be made (NWCM, VAM, LCM).

3) No. of BFS = $m+n-1$ = # Basic cells
→ For degenerate ~~cell~~ sol^n : when no. of +ve valued cells ~~4~~ less than or not equal to no. of basic cells (i.e. $m+n-1$)
→ For non-degenerate sol^n : when no. of +ve valued cells = no. of basic cells (i.e. $m+n-1$)

4) Dual variable u_i & v_j are introduced $1 \le i \le m$, $1 \le j \le n$.

5) For BV's/cells : $u_i + v_j = c_{ij}$
For NBV's/cells : $z_{ij} - c_{ij} = u_i + v_j - c_{ij}$.

6) Check for optimality of resource allocation/Initial BFS:
Across Non-basic cells:

$$\boxed{\begin{array}{c} z_{ij} - c_{ij} \le 0 \quad \text{or} \\ u_i + v_j \le c_{ij} \end{array}}$$

- # constraints = $m+n$
- # independent constraints = $m+n-1$
- # of arbitrary or linearly ~~in~~dependent constraints = 1. (For Primal TP)

→ In Primal TP, one constraint is arbitrary corresponding in dual TP, one variable can choose any value arbitrarily.

Eg:

	D_1	D_2	D_3	D_4		I^st	II^nd	III^rd
S_1	7 $\boxed{14}$	x $\boxed{56}$	x $\boxed{48}$	6 $\boxed{27}$	~~13~~ ~~6~~ 0	13	21	—
S_2	x $\boxed{82}$	x $\boxed{35}$	19 $\boxed{21}$	x $\boxed{81}$	~~19~~ 0	14	14	14
S_3	x. $\boxed{99}$	14 $\boxed{31}$	2 $\boxed{71}$	x $\boxed{63}$	~~16~~ ~~14~~ 0	32	32	40
	~~7~~ 0	~~14~~ 0	~~21~~ 2	~~6~~ 0				

max (Penalty)

	D_1	D_2	D_3	D_4
I^st	↑68	4	27	36
II^nd	—	4	27	36↑
III^rd	—	4	↑50	—
IV^th	—	—	—	—

- When row & col. exhausted, then assign 0 to least cost (for non-assigned). In this case, $cell_{13}$ ($x_{13} = 0$). this cell isn't non-basic but basic cell.

BFS = $m+n-1 = 4+3-1 = 6$.

→ # +ve resource allocation = 5

→ # +ve resource allocation < # BFS.

So, the solⁿ is the degenerate solⁿ.

- Basic cells :- $u_i + v_j = c_{ij}$
- Dual variables :- $[m = 3 (u_1, u_2, u_3), n = 4 (v_1, v_2, v_3, v_4)]$.

i) $cell_{11} \Rightarrow u_1 + v_1 = 14$	$(v_1 = 37,\ u_1 = -23)$	end
ii) $cell_{13} \Rightarrow u_1 + v_3 = 48$	$(u_3 = 71,\ u_1 = -23)$	
iii) $cell_{14} \Rightarrow u_1 + v_4 = 27$	$(u_1 = -23,\ v_4 = 50)$	
iv) $cell_{23} \Rightarrow u_2 + v_3 = 21$	$(v_3 = 71,\ u_2 = -50)$	
v) $cell_{32} \Rightarrow u_3 + v_2 = 31$	$(v_2 = 31)$	
vi) $cell_{33} \Rightarrow u_3 + v_3 = 71$	$(u_3 = 0,\ v_3 = 71)$	start

- One dual variable value is chosen arbitrarily. So, assign one variable value of your choice. Let it be 0.

→ For non-basic cells, $z_{ij} - c_{ij} = u_i + v_j - c_{ij}$

* $cell_{12} \Rightarrow z_{12} - c_{12} = u_1 + v_2 - c_{12}$
  $= -23 + 31 - 56 = -48$

* $cell_{21} \Rightarrow z_{21} - c_{21} = u_2 + v_1 - c_{21} = -95$

* $cell_{22} \Rightarrow z_{22} - c_{22} = u_2 + v_2 - c_{22} = -54$

→ $z_{24} - c_{24} = u_2 + v_4 - c_{24} = -81$
→ $z_{31} - c_{31} = u_3 + v_1 - c_{31} = -62$
→ $z_{34} - c_{34} = u_3 + v_4 - c_{34} = -13$

* So, resource allocation using VAM is optimal.
  $\Rightarrow (z_{ij} - c_{ij} \leq 0)$.

* For all non-basic cell, $z_{ij} - c_{ij} \leq 0$, so this VAM method is optimal for the problem.

* Looping in TP:
(To be used when initial BFS/Resource Allocation isn't optimal):
(Its purpose is to facilitate optimal resource allocation).

1) An ordered set of 4 or more cells from a loop.
2) Any 2 adj. cells in the ordered set lie in the same row/column.
3) Any 3 or more adj. cells in the ordered set don't lie in same row/~~cell~~ column. 1st cell of set is set to follow the last cell. Each cell appears once in the loop.

Question:

	D_1	D_2	D_3	D_4	
S_1	7 [14]	6 [56]	x [48]	x [27]	~~18~~ 6
S_2	x [82]	8 [35]	11 [21]	x [81]	~~19~~ 11
S_3	x [99]	x [31]	10 [71]	6 [63]	~~16~~ 6
	~~7~~	~~14~~ 8	~~21~~ 10	~~6~~	

$[x_{11} = 7,\ x_{12} = 6,\ x_{22} = 8;\ x_{23} = 11;\ x_{33} = 10;\ x_{34} = 6]$

$\Rightarrow$ $m = 3$ Sources (u_1, u_2, u_3) } Check for
$n = 4$ Destinations (v_1, v_2, v_3, v_4) } optimality.

for Basic cell: $u_i + v_j = c_{ij}$ $\quad 1 \le i \le m,\ 1 \le j \le n$.

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cell$_{11}$ $u_1 + v_1 = 14$
cell$_{12}$ $u_1 + v_2 = 56$
cell$_{22}$ $u_2 + v_2 = 35$
cell$_{23}$ $u_2 + v_3 = 21$
cell$_{33}$ $u_3 + v_3 = 71$
cell$_{34}$ $u_3 + v_4 = 63$

⇒ Put $u_1 = 0$; $v_1 = 14$ $u_2 = -21$ $u_3 = 29$
$v_2 = 56$ $v_3 = 42$ $v_4 = 34$

choose that variable to be zero which is used most of the time in constraints.

(Using Basic cells found value of dual variable).
Value of u_is & v_js.

→ Check for optimality: $z_{ij} - c_{ij} = u_i + v_j - c_{ij}$

cell$_{13}$ $z_{13} - c_{13} = u_1 + v_3 - c_{13} = 0 + 42 - 48 = -6$
cell$_{14}$ $z_{14} - c_{14} = u_1 + v_4 - c_{14} = 0 + 34 - 27 = 7$
cell$_{21}$ $z_{21} - c_{21} = u_2 + v_1 - c_{21} = -89$
cell$_{24}$ $z_{24} - c_{24} = u_2 + v_4 - c_{24} = -68$
cell$_{31}$ $z_{31} - c_{31} = u_3 + v_1 - c_{31} = -56$
cell$_{32}$ $z_{32} - c_{32} = u_3 + v_2 - c_{32} = 54$

These are greater than 0, so, not optimal.

- cell$_{32}$ is max. +ve. So, we need to make some resource allocation here. (Like Simplex)

⇒

	D1	D2	D3	D4	
S1	7	6			13
S2		8	11		19
S3		α	10	6	16
	7	14	21	6	

cell$_{32}$: $z_{ij} - c_{ij} = 54$
(So, we do 'α' resource allocation at cell$_{32}$ for looping).

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Loop cells:

$\underset{(+)}{\overset{(\alpha)}{cell_{32}}} \longrightarrow \underset{(-)}{\overset{(8-\alpha)}{cell_{22}}} \longrightarrow \underset{(+)}{\overset{(11+\alpha)}{cell_{23}}} \longrightarrow \underset{(-)}{cell_{33}}\,(10-\alpha) \longrightarrow cell_{32}$

$\alpha = \min\{$All allocations in the basic cells of loop with −ve considerations$\}$

$\alpha = \min\{x_{22}, x_{33}\}$

$\alpha = \min\{8, 10\}$

So $\boxed{\alpha = 8}$

⇒

	D_1	D_2	D_3	D_4	
S_1					13
S_2		(0) $8-\alpha$	(19) $11+\alpha$		19
S_3		(8) α	(2) $10-\alpha$		16
	7	14	21	6	

→

	D_1	D_2	D_3	D_4
S_1	7	6		
S_2			19	
S_3		8	2	6

Rest of the cells outside the loop will be undisturbed.

$m = 3,\ n = 4$

Resource Allocation $= m+n-1 = 6$

$[x_{11} = 7,\ x_{12} = 6,\ x_{23} = 19,\ x_{32} = 8,\ x_{33} = 2,\ x_{34} = 6]$

↳ 6 allocations already made so no need to put $cell_{22}$ as 0.

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→ Now, we need to check for optimality:

⇒ Basic cells: $u_i + v_j = c_{ij}$

1) $u_1 + v_1 = 14$
2) $u_1 + v_2 = 56$
3) $u_2 + v_3 = 21$
4) $u_3 + v_2 = 31$
5) $u_3 + v_3 = 71$
6) $u_3 + v_4 = 63$

Put $u_3 = 0$ [Choose which is used most times]

⇓

$u_1 = 25$, $u_2 = -50$, $v_1 = -11$, $v_2 = 31$, $v_3 = 71$, $v_4 = 63$

⇒ Non-Basic cells:

cell$_{13}$ $z_{13} - c_{13} = u_1 + v_3 - c_{13} = 48$
cell$_{14}$ $z_{14} - c_{14} = 61$
cell$_{21}$ $z_{21} - c_{21} = -143$
cell$_{22}$ $z_{22} - c_{22} = -54$
cell$_{24}$ $z_{24} - c_{24} = -60$
cell$_{31}$ $z_{31} - c_{31} = -110$