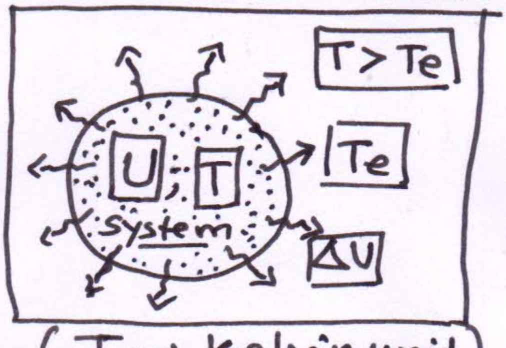
Newton's Law of Cooling

Statement: Rate of loss of heat is proportional to the excess temperature over the surroundings.

i.) The law is true for all excess temperatures in conditions of forced convection of air.
ii.) The law is approximately true in still air for a temperature excess of 20k - 30k.
iii.) For natural convection the proportional relation is $(\text{excess temperature})^{5/4}$ (Dulong and Petit)



1/. $\boxed{\Delta U}$ → Energy lost as heat

2/. $\boxed{(T - T_e)}$ → Excess temperature

3/. $\boxed{\Delta U/_{\Delta t} \propto (T - T_e)}$ Newton's law of Cooling ($T \to$ kelvin unit)

4/. $\boxed{\Delta U \to dU}$ (Infinitesimal loss).

5/. $\boxed{dU = mc\,dT}$ $m \to$ mass, $c \to$ Specific heat capacity.

6/. $\boxed{\dfrac{dU}{dt} = mc\dfrac{dT}{dt} \propto (T - T_e)} \Rightarrow \boxed{\dfrac{dU}{dt} = kS(T_e - T)}$

$k \to$ proportional constant, $S \to$ Surface area

7/. $\boxed{\dfrac{dT}{dt} = \dfrac{kS}{mc}(T_e - T)}$ $\boxed{\dfrac{dU}{dt}, \dfrac{dT}{dt} < 0}$ Since $T_e < T$.

(loss)

(P.T.O.)

8/. $\boxed{m = \rho V}$ $\rho \rightarrow$ density, $V \rightarrow$ Volume.

9/. $\boxed{\dfrac{dT}{dt} = \left(\dfrac{K}{\rho c}\right)\dfrac{S}{V}(T_e - T)} \Rightarrow \boxed{\dfrac{dT}{dt} \propto \dfrac{S}{V}}$

For bodies of similar shape, small bodies ~~coll~~ cool faster than large ones.

Example: A sphere of radius r. $\boxed{S = 4\pi r^2}$

$\boxed{V = \frac{4}{3}\pi r^3} \Rightarrow \boxed{\dfrac{S}{V} \propto \dfrac{1}{r}} \Rightarrow \boxed{\dfrac{dT}{dt} \propto \dfrac{1}{r}}$

10/. Integral Solution: $\boxed{\dfrac{dT}{dt} = \dfrac{kS}{mc}(T_e - T)}$ $\boxed{T > T_e}$ $\boxed{\dfrac{dT}{dt} < 0}$

$\Rightarrow \boxed{\displaystyle\int \frac{dT}{T - T_e} = -\frac{kS}{mc}\int dt}$ Write $\boxed{r = \dfrac{kS}{mc}}$.

$\Rightarrow \boxed{\ln(T - T_e) = -rt + A}$ when $\boxed{t = 0,\ T = T_0}$

$\Rightarrow \boxed{A = \ln(T_0 - T_e)}$ $\therefore \boxed{\ln(T - T_e) = \ln(T_0 - T_e) - rt}$

$\Rightarrow \boxed{T - T_e = (T_0 - T_e)\, e^{-rt}}$ i) when $t \rightarrow \infty$, $T \rightarrow T_e$

$\Rightarrow \boxed{T = T_e + (T_0 - T_e)e^{-rt}}$ ii) when $t = 0$, $T = T_0$.

Stefan - Boltzmann Law (in Radiation)

Radiation wattage: $\boxed{W = \epsilon \sigma S(T^4 - T_e^4)}$

$\boxed{W} \rightarrow$ Energy radiation/time (Joule/sec).

$\boxed{\sigma} \rightarrow$ Stefan's Constant, $\boxed{\epsilon = 1}$ (for a black body)

(joule $m^{-2} s^{-1} K^{-4}$) K → kelvin

(P.T.O.)

$$\therefore \boxed{W = \frac{dU}{dt} = \epsilon \sigma S \left(T^4 - T_e^4\right)} \qquad \boxed{\epsilon \le 1}$$

($\epsilon = 1$ for a black body)

$\boxed{T} \rightarrow$ Temperature of the body.

$\boxed{T_e} \rightarrow$ Environment temperature.

$\boxed{T = T_e + \Delta T}$

$$\Rightarrow T^4 - T_e^4 = \left(T_e + \Delta T\right)^4 - T_e^4 = T_e^4\left(1 + \frac{\Delta T}{T_e}\right)^4 - T_e^4$$

If $\boxed{\Delta T \ll T_e}$, $\boxed{\left(1 + \frac{\Delta T}{T_e}\right)^4 \simeq 1 + \frac{4\Delta T}{T_e}} \rightarrow$ Binomial expansion

$$\Rightarrow \boxed{T^4 - T_e^4 \simeq T_e^4\left(1 + \frac{4\Delta T}{T_e}\right) - T_e^4 \simeq 4 T_e^3 \Delta T}$$

(linear approximation)

Now $\boxed{\frac{dU}{dt} = mc\frac{dT}{dt} = \epsilon \sigma S\left(T^4 - T_e^4\right)}$

$$\Rightarrow \boxed{\frac{dT}{dt} = \frac{\epsilon \sigma S}{mc}\left(T^4 - T_e^4\right) \simeq \frac{\epsilon \sigma S}{mc} \cdot 4 T_e^3 \left(T - T_e\right)}$$

$$\Rightarrow \boxed{\frac{dT}{dt} \simeq \left(\frac{4\epsilon \sigma T_e^3}{mc}\right) S \left(T - T_e\right)}$$

Now $\boxed{T > T_e}$ but $\boxed{dT/dt < 0}$

$$\therefore \boxed{\frac{dT}{dt} \simeq \frac{\left(4 \epsilon \sigma T_e^3\right) S}{mc} \times \left(T_e - T\right)}$$

Forcing the negative sign.

Comparing with $\boxed{\frac{dT}{dt} = \frac{kS}{mc}\left(T_e - T\right)}$ (ignoring sign)

$\boxed{k \simeq 4 \epsilon \sigma T_e^3}$ (ϵ is dimensionless) ($\epsilon \le 1$)

$\Rightarrow$ $\boxed{\text{Unit of } k \text{ is } \frac{\text{joule}}{m^2 \sec \text{kelvin}^4} \times \text{kelvin}^3}$ In the S.I. System

$\Rightarrow$ $\boxed{\text{Unit of } k \rightarrow \frac{\text{joule}}{m^2 \times \sec \times \text{kelvin}} \equiv \text{Watt } m^{-2} K^{-1}}$

($K \rightarrow$ Kelvin)

Newton's Law of Cooling: An Application

Problem: A dead body has been found in a room whose temperature is $\boxed{\theta_e = 25\,°C}$. When discovered, the body temperature was $\boxed{\theta_1 = 28\,°C}$. One hour later the body temperature became $\boxed{\theta_2 = 27\,°C}$. Normal human body temperature is $\boxed{\theta_0 = 37\,°C}$. How long ago did the person die? $\boxed{t_2 - t_1 = \Delta t = 1 \text{ hour}}$

Solution: $\boxed{T(\text{kelvin}) = \theta(\text{celsius}) + 273}$

$\Rightarrow \boxed{\Delta T = \Delta\theta}$. $\boxed{\theta_1 = \theta_e + (\theta_0 - \theta_e)\exp(-\lambda t_1)}$

and $\boxed{\theta_2 = \theta_e + (\theta_0 - \theta_e)\exp(-\lambda t_2)}$ $\boxed{\lambda = \frac{kS}{mc}}$

$$\therefore \boxed{\ln\left(\frac{\theta_1 - \theta_e}{\theta_0 - \theta_e}\right) = -\lambda t_1} \text{ and } \boxed{\ln\left(\frac{\theta_2 - \theta_e}{\theta_0 - \theta_e}\right) = -\lambda t_2}$$

$$\Rightarrow \ln\left(\frac{\theta_1 - \theta_e}{\theta_0 - \theta_e}\right) - \ln\left(\frac{\theta_2 - \theta_e}{\theta_0 - \theta_e}\right) = -\cancel{\lambda t_1} + \cancel{\lambda t_1} + \lambda\Delta t$$

$$\Rightarrow \lambda\Delta t = \ln\left(\frac{28-25}{37-25}\right) - \ln\left(\frac{27-25}{37-25}\right) = \ln\left(\frac{3}{12}\right) - \ln\left(\frac{2}{12}\right)$$

$$\Rightarrow \lambda\Delta t = \ln\left(\frac{3}{\cancel{12}} \times \frac{\cancel{12}}{2}\right) = \ln\left(\frac{3}{2}\right) \Rightarrow \boxed{\lambda = \frac{\ln(1.5)}{\Delta t}}$$

$$\Rightarrow \boxed{\lambda t_1 = \ln\left(\frac{\theta_0 - \theta_e}{\theta_1 - \theta_e}\right)} \Rightarrow \boxed{t_1 = \frac{1}{\lambda}\ln\left(\frac{\cancel{12}^{\,4}}{\cancel{3}}\right)}$$

$$\Rightarrow \boxed{t_1 = \frac{\Delta t \times 2\ln 2}{\ln(3/2)}} \Rightarrow \boxed{t_1 = \frac{2\ln 2}{\ln(1.5)} \times 1 \text{ hour}}.$$

$$\Rightarrow \boxed{t_1 \simeq 3.42 \text{ hours} = 3 \text{ hours } 25 \text{ minutes}}$$

(Time of death)

Death took place 3.42 hours ago.