The Spread of Technological Innovations

Agricultural Innovation among Farmers

i.) Total number of farmers in a farming Community is N (fixed value).

ii.) $x(t) \rightarrow$ Number of farmers who have adopted an innovation.

iii.) $N - x(t) \rightarrow$ Number of farmers who have not adopted the innovation.

$\boxed{\Delta x \propto \Delta t}$, $\boxed{\Delta x \propto x}$ and $\boxed{\Delta x \propto (N-x)}$

$\therefore$ $\boxed{\Delta x \propto x(N-x)\Delta t}$ $\Rightarrow$ $\boxed{\frac{\Delta x}{\Delta t} \equiv \frac{dx}{dt} = c\,x(N-x)}$.

(jointly proportional) $\quad$ $\underbrace{(c>0)}_{\text{(proportional constant)}}$

The initial condition is $\boxed{x(0) = 1}$

Rescale : $\boxed{\frac{d(x/N)}{dt} = cN\frac{x}{N}\left(1-\frac{x}{N}\right)}$ $\Rightarrow$ the logistic equation,

define $\boxed{X = x/N}$ and $\boxed{T = cNt}$ to get

$\boxed{\frac{dX}{dT} = X(1-X)}$ whose solution is $\boxed{X = \frac{1}{1+A^{-1}e^{-T}}}$.

Hence $\boxed{x = \frac{N}{1+A^{-1}e^{-cNt}}}$. When $\boxed{t=0,\ x=1}$.

$\Rightarrow 1 = \frac{N}{1+A^{-1}} \Rightarrow \boxed{A^{-1} = N-1}$.

$$\Rightarrow \quad \boxed{x = \frac{N}{1+(N-1)e^{-cNt}} = \frac{Ne^{cNt}}{(N-1)+e^{cNt}}}.$$

i.) A discrepancy arises due to not accounting for information obtained through the mass media.

ii.) The ~~[illegible]~~ slowing of the growth rate happens later than expected.

Modification: $\boxed{\Delta x = c'(N-x)\Delta t}$ ($c' > 0$) proportional constant

(Correction due to impersonal communication.)

The total effect is $\boxed{\Delta x = cx(N-x)\Delta t + c'(N-x)\Delta t}$.

$$\Rightarrow \boxed{\frac{\Delta x}{\Delta t} = (cx + c')(N-x)} \Rightarrow \boxed{\frac{dx}{dt} = N(cx+c')\left(1-\frac{x}{N}\right)}.$$

$$\Rightarrow \boxed{\frac{dx}{dt} = Nc\left(x + \frac{c'}{c}\right)\left(1 - \frac{x}{N}\right)}, \quad \boxed{\frac{c'}{c} > 0}.$$

Early growth: When $\boxed{x \ll N}$, $\boxed{\frac{dx}{dt} \simeq Nc\left(x + \frac{c'}{c}\right)}$.

i.) Quicker than exponential, if $c'/c > 0$.
ii.) Slower than exponential, if $c'/c < 0$.

Since $c', c > 0$, the non-human intervention boosts early growth of the function, $x(t)$.

$$\Rightarrow \boxed{\frac{dx}{dt} = Nc\left(x + \frac{c'}{c} - \frac{x^2}{N} - \frac{c'}{c}\frac{x}{N}\right)}$$

Accounting for both human intervention and the mass media

$$\Rightarrow \frac{dx}{dt} = Ncx + Nc' - cx^2 - c'x$$

$$\Rightarrow \boxed{\frac{dx}{dt} = -\left[cx^2 - (Nc - c')x - Nc'\right]}$$

$$\Rightarrow \frac{dx}{dt} = -\left[(\sqrt{c}\,x)^2 - 2\,\frac{1}{2}\cdot\frac{\sqrt{c}\,x}{\sqrt{c}}(Nc-c') + \frac{(Nc-c')^2}{4c} - \frac{(Nc-c')^2}{4c} - Nc'\right]$$

$$\Rightarrow \frac{dx}{dt} = -\left[\sqrt{c}\,x - \frac{(Nc-c')}{2\sqrt{c}}\right]^2 + \left[Nc' + \frac{(Nc-c')^2}{4c}\right]$$

$$\Rightarrow \boxed{\frac{dx}{dt} = -c\left[x - \frac{(Nc-c')}{2c}\right]^2 + \left[Nc' + \frac{(Nc-c')^2}{4c}\right]}$$

Define $\boxed{y = x - \frac{(Nc - c')}{2c}}$ and $\boxed{\alpha^2 = Nc' + \frac{(Nc-c')^2}{4c}}$

Hence $\boxed{\frac{dy}{dt} = \frac{dx}{dt}}$ $\Rightarrow$ $\boxed{\frac{dy}{dt} = \alpha^2 - cy^2}$

$\Rightarrow$ $\boxed{\frac{1}{\alpha^2}\frac{dy}{dt} = 1 - \frac{y^2}{\alpha^2/c}}$. Rescaling now $\boxed{X = \frac{y}{\alpha/\sqrt{c}}}$ and $\boxed{T = \alpha\sqrt{c}\,t}$,

we get $\boxed{\frac{dX}{dT} = 1 - X^2}$, whose solution is known to be $\boxed{\left(\frac{1+X}{1-X}\right) = Ae^{2T}}$. The initial condition is when $\boxed{t = 0,\ x = 0}$. $\Rightarrow$ $\boxed{y_0 = -\frac{Nc - c'}{2c} = \frac{1}{2}\left(\frac{c'}{c} - N\right)}$

Hence, $\boxed{X_0 = \frac{y_0}{\alpha/\sqrt{c}}}$. Making X the subject of T,

we get, $\boxed{X = \frac{Ae^{2T} - 1}{Ae^{2T} + 1}}$ $\Rightarrow$ $\boxed{y = \frac{\alpha}{\sqrt{c}}\left(\frac{Ae^{2\alpha\sqrt{c}t} - 1}{Ae^{2\alpha\sqrt{c}t} + 1}\right)}$.

Now $4c\alpha^2 = 4Ncc' + (Nc - c')^2$ (from the definition of α)

$\Rightarrow$ $4c\alpha^2 = 4Ncc' + N^2c^2 - 2Ncc' + c'^2 = (Nc + c')^2$

$\Rightarrow$ $\boxed{2\alpha\sqrt{c} = (Nc + c')}$ & $\frac{\alpha}{\sqrt{c}} = \frac{2\alpha\sqrt{c}}{2c} = \frac{Nc + c'}{2c}$.

$\Rightarrow$ $\boxed{\frac{\alpha}{\sqrt{c}} = \frac{1}{2}\left(N + \frac{c'}{c}\right)}$. Further, when $t = 0$ $(T = 0)$ and $x = 0$ $(X = X_0)$, $\boxed{A = \frac{1 + X_0}{1 - X_0}}$.

Therefore, $\boxed{A = \frac{1 + y_0/\alpha\sqrt{c}}{1 - y_0/\alpha\sqrt{c}} = \frac{\alpha/\sqrt{c} + y_0}{\alpha/\sqrt{c} - y_0}}$

Now, $\boxed{y_0 + \frac{\alpha}{\sqrt{c}} = \frac{1}{2}\frac{c'}{c} - \cancel{\frac{N}{2}} + \cancel{\frac{N}{2}} + \frac{c'}{2c} = \frac{c'}{c}}$

Also,

$$\boxed{\frac{\alpha}{\sqrt{c}} - y_0 = \frac{N}{2} + \cancel{\frac{c'}{2c}} - \cancel{\frac{1}{2}\frac{c'}{c}} + \frac{N}{2} = N}$$

Hence, $\boxed{A = \dfrac{\alpha/\sqrt{c} + y_0}{\alpha/\sqrt{c} - y_0} = \dfrac{c'}{cN}}$. Hence we write the full integral solutions as $y = x - \dfrac{(Nc-c')}{2c} = \dfrac{\alpha}{\sqrt{c}}\left(\dfrac{Ae^{2\alpha\sqrt{c}t} - 1}{Ae^{2\alpha\sqrt{c}t} + 1}\right)$

Substituting for A, $2\alpha\sqrt{c}$, and $\alpha/\sqrt{c}$, we get,

$$\boxed{x = \frac{Nc - c'}{2c} + \frac{1}{2}\left(N + \frac{c'}{c}\right)\cdot\frac{(c'/Nc)e^{(Nc+c')t} - 1}{(c'/Nc)e^{(Nc+c')t} + 1}}.$$

$$\Rightarrow x = \frac{Nc - c'}{2c} + \frac{Nc + c'}{2c}\cdot\frac{c'e^{(Nc+c')t} - Nc}{c'e^{(Nc+c')t} + Nc}$$

$$\Rightarrow x = \frac{(Nc-c')\left[c'e^{(Nc+c')t} + Nc\right] + (Nc+c')\left[c'e^{(Nc+c')t} - Nc\right]}{2c\left[c'e^{(Nc+c')t} + Nc\right]}$$

$$\Rightarrow x = \frac{Ncc'e^{(Nc+c')t} - \cancel{c'^2 e^{(Nc+c')t}} + \cancel{(Nc)^2} - Ncc' + Ncc'e^{(Nc+c')t} + \cancel{c'^2 e^{(Nc+c')t}} - \cancel{(Nc)^2} - Ncc'}{2c\left[c'e^{(Nc+c')t} + Nc\right]}$$

$$\Rightarrow x = \frac{\cancel{2}N\cancel{c}c'e^{(Nc+c')t} - \cancel{2}N\cancel{c}c'}{\cancel{2}\cancel{c}\left[c'e^{(Nc+c')t} + Nc\right]} = \frac{Nc'e^{(Nc+c')t} - Nc'}{Nc + c'e^{(Nc+c')t}}$$

$$\Rightarrow \boxed{x = \frac{Nc'\left[e^{(Nc+c')t} - 1\right]}{cN + c'e^{(Nc+c')t}}}$$ → The integral solution of $\boxed{\dfrac{dx}{dt} = Nc\left(x + \dfrac{c'}{c}\right)\left(1 - \dfrac{x}{N}\right)}$.

The above solution is recast as

$$\boxed{x = \frac{Nc'\left[1 - e^{-(Nc+c')t}\right]}{c' + cNe^{-(Nc+c')t}}}$$

∴ When $t \to \infty$, $\boxed{x \to \dfrac{Nc'}{c'} = N}$ | The maximum value.