The Log-Normal Distribution

$$\boxed{P_G(x)\,dx = \frac{1}{\sqrt{2\pi}\,\sigma} \exp\left[-\frac{(x-\mu)^2}{2\sigma^2}\right] dx}$$ The Gaussian probability Distribution

In the continuous Gaussian probability function the random variable is x. In the law of proportionate effect, the random variable is $\ln x$. $\therefore$ $\boxed{\text{Substitute } x \to \ln x}$.

$$\boxed{P_{LN}(x)\,dx = \frac{1}{\sqrt{2\pi}\,\sigma} \exp\left[-\frac{(\ln x-\mu)^2}{2\sigma^2}\right] d(\ln x)} \quad \boxed{d(\ln x) = \frac{dx}{x}}$$

$$\Rightarrow \boxed{P_{LN}(x)\,dx = \frac{1}{\sqrt{2\pi}\,\sigma}\,\frac{1}{x} \cdot \exp\left[-\frac{(\ln x-\mu)^2}{2\sigma^2}\right] dx}$$

- The log-normal probability distribution.

The Mode of the log-normal probability density distribution:

$$\boxed{P(x) = \frac{1}{\sqrt{2\pi}\,\sigma}\, x^{-1} \exp\left[-\frac{(\ln x-\mu)^2}{2\sigma^2}\right]}$$

$$\frac{dP}{dx} = \frac{1}{\sqrt{2\pi}\,\sigma} \times \frac{-1}{x^2} \exp\left[-\frac{(\ln x-\mu)^2}{2\sigma^2}\right]$$

$$+ \frac{1}{\sqrt{2\pi}\,\sigma} \cdot \frac{1}{x} \exp\left[-\frac{(\ln x-\mu)^2}{2\sigma^2}\right] \times -\frac{\cancel{2}(\ln x-\mu)\times\frac{1}{x}}{\cancel{2}\sigma^2}$$

$$\Rightarrow \boxed{\frac{dP}{dx} = -\frac{P}{x} - \frac{P}{x}\,\frac{(\ln x-\mu)}{\sigma^2}}$$ At the mode $\boxed{\frac{dP}{dx} = 0}$

[P.T.O.]

$$\Rightarrow \quad \frac{\ln x - \mu}{\sigma^2} = -1 \quad \Rightarrow \boxed{\ln x = \mu - \sigma^2}.$$

$$\Rightarrow \quad \boxed{x_{mode} = e^{\mu - \sigma^2}}$$ → Here $P_{LN}(x)$ is maximum.

$$P_{max} = \frac{1}{\sqrt{2\pi}\,\sigma} \cdot \frac{e^{\sigma^2}}{e^{\mu}} \cdot \exp\left[-\frac{(-\sigma^2)^2}{2\sigma^2}\right]$$

$$\Rightarrow P_{max} = \frac{1}{\sqrt{2\pi}\,\sigma} \cdot \frac{e^{\sigma^2}}{e^{\mu}} \cdot e^{-\sigma^2/2} = \frac{e^{\sigma^2/2 - \mu}}{\sqrt{2\pi}\,\sigma}.$$

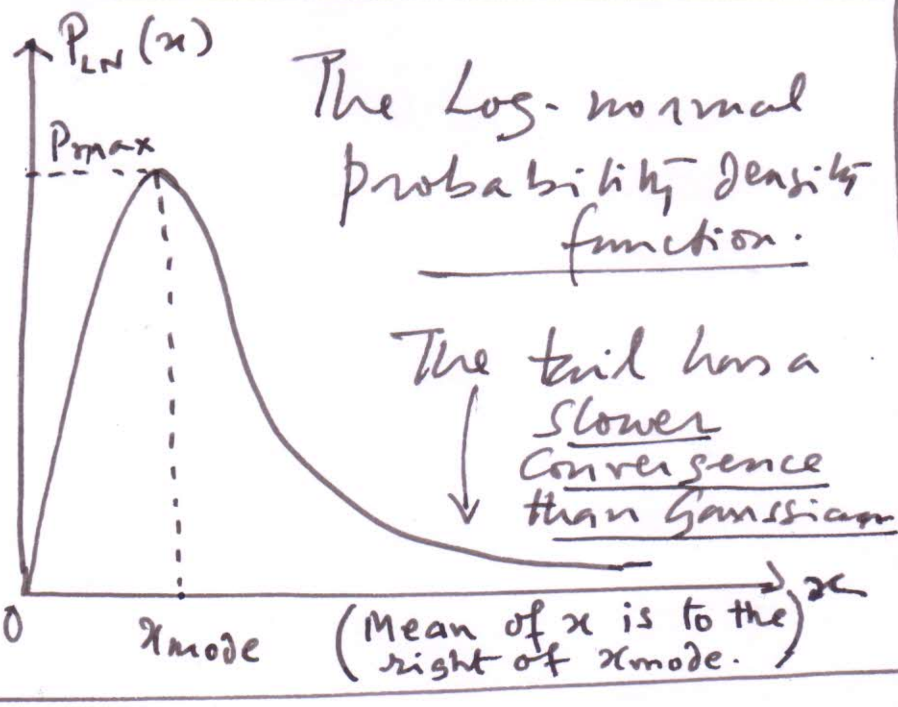
$$\therefore \boxed{P_{max} = \frac{1}{\sqrt{2\pi}\,\sigma} \exp\left(\frac{\sigma^2}{2} - \mu\right)}$$ → The Peak value.

i) When $x \to 0$, $\frac{1}{x} \to \infty$, $\ln x \to -\infty$.

$\therefore P \sim \frac{1}{x} e^{-(\ln x)^2} \to 0$ $\boxed{\because e^{-\infty^2} \text{ has a faster convergence than } x^{-1}.}$

ii) Also when $x \to \infty$, both $e^{-(\ln x)^2}$ and $x^{-1} \to 0 \Rightarrow P \to 0$

$\boxed{P_G(x) \sim e^{-x^2}}$



1/. The Gaussian tail is like $\sim e^{-x^2}$. The log-normal tail has much slower convergence.

2/. For $x \to \infty$, $P_{LN}(x) \gg P_G(x)$. Hence large x fluctuations are more probable.

The Log-Log Representation

$$\boxed{\ln P = -\ln\left(\sqrt{2\pi}\,\sigma\right) - \ln x - \frac{(\ln x - \mu)^2}{2\sigma^2}}$$

$$\Rightarrow \ln P = -\ln\left(\sqrt{2\pi}\,\sigma\right) - \ln x - \left[\frac{(\ln x)^2}{2\sigma^2} - \frac{2\mu \ln x}{2\sigma^2} + \frac{\mu^2}{2\sigma^2}\right]$$

$$\Rightarrow \boxed{\ln P = -\ln\left(\sqrt{2\pi}\,\sigma\right) - \frac{(\ln x)^2}{2\sigma^2} + \left(\frac{\mu}{\sigma^2} - 1\right)\ln x - \frac{\mu^2}{2\sigma^2}}$$

In a log-log plot $\ln P$ is a parabolic function of $\ln x$. For a small part of $\ln x$ the function will appear like

$$\boxed{\ln P \sim \left(\frac{\mu}{\sigma^2} - 1\right)\ln x}$$

giving the impression of a power law.

Moments of a Log-Normal variable x.

1/. $\boxed{E[x^n] = e^{n\mu + \frac{1}{2}n^2\sigma^2}}$ n-th moment of the variable x. (E → expectation)

2/. $\boxed{E[x] = e^{\mu + \frac{\sigma^2}{2}}}$ → The first moment (Mean).

3/. $\boxed{E[x^2] = e^{2\mu + 2\sigma^2}}$ → The second moment.

4/. $\mathrm{Var}[x] = E[x^2] - E[x]^2 = e^{2\mu + 2\sigma^2} - e^{2\mu + \sigma^2}$

$$= e^{2\mu + \sigma^2}\left(e^{\sigma^2} - 1\right)$$

5/. $\boxed{\sqrt{\mathrm{Var}[x]} = e^{\mu + \frac{\sigma^2}{2}}\sqrt{e^{\sigma^2} - 1}}$ → The standard deviation.