The Theory of Speculation | Bachelier

The 1938 Monograph on "Speculation and the Calculus of Probability" by Louis Bachelier.

1/. Introduces time and absolute continuity. Movement of probabilities, their "radiation, reflection and refraction."

2/. Use of infinitesimal Calculus.

3/. The market behaves randomly — Constantly subjected to infinitely many varying influences acting along diverse directions, Such a market must ultimately behave as if no single cause came to play but as if randomness acted alone.

The results of the theory would be Contradicted only if a single cause would be constantly contributing in the same direction; in general, the diversity of Causes allows their elimination; the

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incoherence of the market is itself its method; and it is because it does not obey any law that it fatally follows the law of randomness.

4/. We consider the variations of the price of a security (a goverment bond, for example) in a large market.

5/. The price variations are random and the problem is in seeking the probability that at a given time, the price differs from the actual price by a given quantity.

6/. Due to the excessive complexity of the causes of these variations, everything happens as if by chance.

7/. At a given instant, a price quoted has as many purchasers as sellers; purchasers believe in a rise and sellers in a drop.

8/. The MARKET, a collection of all speculators, does not believe in either rise or drop. Since for the quoted price there are as many sellers as purchasers. Thus, the quoted price represents the real value of the bond under consideration.

Mathematical Formulation of the Theory of Speculation — Bachelier

1/ The price at $\boxed{t=0}$ is <u>the actual quoted price</u>.

2/. We are looking for the probability that at time $\boxed{t\ (t>0)}$ the price differs by a given amount from the actual quoted price.

3/. Consider the actual price as <u>zero</u>.

4/. The relative price that "spreads" from the actual price is x.

5/. The price is positive when it corresponds to a rise, and negative for a fall.

6/. The probability for a price x to be quoted at a time t is the probability that the price is between x and $x+dx$.

7/. We consider this probability to be represented as a function, $\boxed{f(x,t)\,dx}$ (with x and t being continuous).

8/. This function is positive and over all x adds up to unity.

9/. The function can be plotted with positive ordinates, and with a total area equal to unity, because this area corresponds to the sum of all probabilities.

10/. The price variations taking place at any instant are independent of previous variations and also of the price quoted at that instant. (The principle of randomness).

11/. The mathematical expectation of an eventual gain is the product of that gain by the probability of its occurring. $\longrightarrow (x_i P_i) \longrightarrow \boxed{\sum_i x_i P_i}$

12/ The mathematical expectation of a loss is negative.

13/ The total mathematical expectation is the sum of the products of all uncertain gains/losses by their corresponding probabilities.

14/ When the total mathematical expectation vanishes the speculator is neither at an advantage nor at a disadvantage. The game is FAIR.

15/ If a game is fair in each round, then it is fair as a whole.

16/ Transactions of the exchange are subject to the law of supply and demand.

17/ As every speculator is free to perform a given transaction, a priori a speculative transaction does not favour or penalise any one of the parties.

18/ A priori, there is no advantage or

18/. (continued)

disadvantage in a transaction.

"The mathematical expectation of any transaction is zero (vanishes)".

19/. There is no useful information in historical price variations.

20/. The probability for a price x to be quoted at a time t, is included between x and $x+dx$. It is

$$\boxed{P(x,t)\,dx = \frac{1}{\sqrt{\pi}\,\sqrt{\phi(t)}}\, e^{-\frac{x^2}{\phi(t)}}\, dx},$$

in which $\phi(t)$ is positive and increasing. $\phi(t) \to$ Instability function.

21/. As, $t \to \infty$, $P(x,t) \longrightarrow 0$.

22/. $P(x,t)$ is a continuous function of both x and t. It is even in x. Hence, prices $+x$ and $-x$ have the same probability.

$P(x,t)$ is maximum at $x=0$ and has two inflection points at $\boxed{x = \pm\sqrt{\frac{\phi(t)}{2}}}$.

<u>Inflection Points:</u> $\boxed{P(x,t) = \frac{1}{\sqrt{\pi\phi}}\, e^{-\frac{x^2}{\phi}}}$.

$$\boxed{\frac{\partial P}{\partial x} = \frac{1}{\sqrt{\pi\phi}}\, e^{-x^2/\phi} \times \left(-\frac{2x}{\phi}\right)} \rightarrow \text{First derivative.}$$

$$\Rightarrow \boxed{\frac{\partial P}{\partial x} = P\left(-\frac{2x}{\phi}\right)} \;.\; (\phi \text{ does } \underline{\text{not}} \text{ depend on } x)$$

$$\therefore \boxed{\frac{\partial^2 P}{\partial x^2} = \frac{\partial P}{\partial x}\cdot\left(-\frac{2x}{\phi}\right) + P\left(-\frac{2}{\phi}\right)} \rightarrow \text{Second Derivative}$$

$$\Rightarrow \boxed{\frac{\partial^2 P}{\partial x^2} = P\left(-\frac{2x}{\phi}\right)^2 + P\left(-\frac{2}{\phi}\right) = 0} \;\leftarrow \text{Condition for inflection}$$

$\phi \rightarrow$ Instability function

$$\Rightarrow P\left(-\frac{2}{\phi}\right)\left[\left(-\frac{2}{\phi}\right)x^2 + 1\right] = 0 .$$

$$\Rightarrow \boxed{x^2 = \frac{\phi}{2}} \Rightarrow \boxed{x = \frac{\pm\sqrt{\phi(t)}}{\sqrt{2}}} .$$

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23). $\boxed{\phi(t) = 4\pi k^2 t}$ (<u>Wiener Process</u>). This <u>ϕ in $P(x,t)$</u> is to be <u>compared</u> with the <u>Point-Source Solution</u> of the <u>diffusion equation</u>:

$$\boxed{\psi(x,t) = \frac{1}{\sqrt{2\pi}\sqrt{2Dt}} \cdot \exp\left[-\frac{x^2}{2(2Dt)}\right]}$$

and the <u>Gaussian probability function</u>.

$$\boxed{P(x) = \frac{1}{\sqrt{2\pi}\,\sigma} \cdot \exp\left[-\frac{(x-\mu)^2}{2\sigma^2}\right]} .\; \text{Consider } \boxed{\mu = 0} .$$

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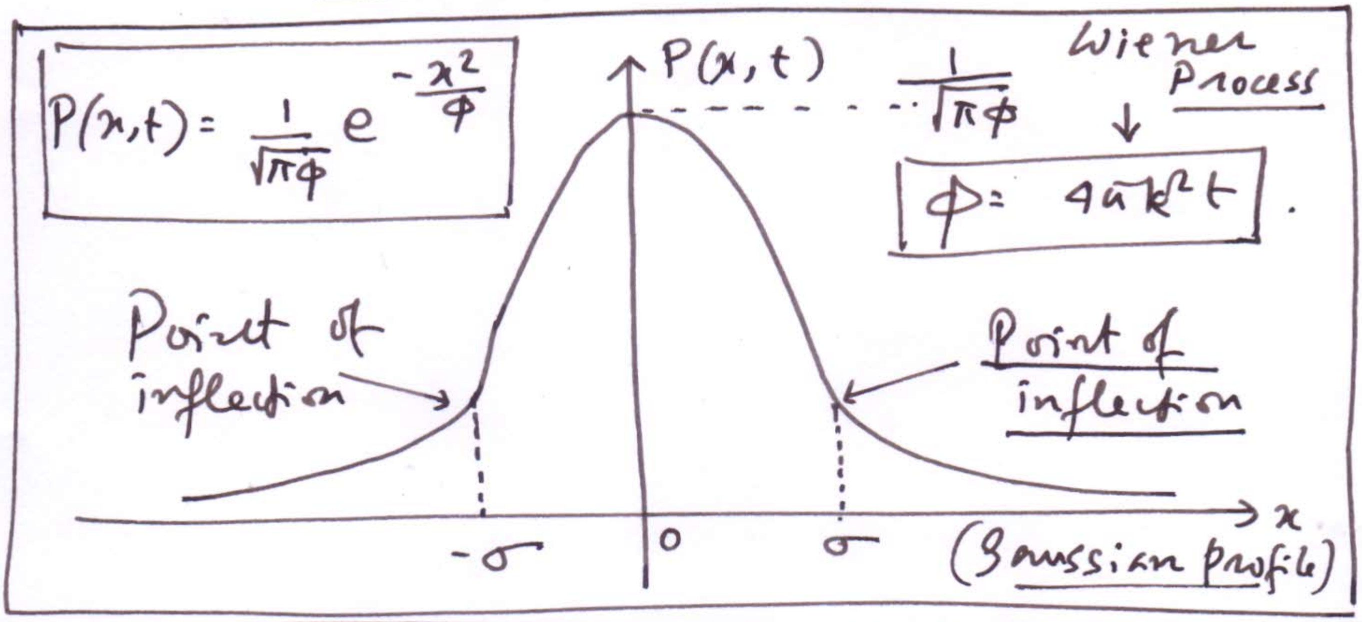
$\therefore \boxed{2\sigma^2 \equiv 2(2Dt) \equiv \phi \equiv 4\pi k^2 t}$. | Setting equivalence

$\Rightarrow \quad 2\sigma^2 = \phi \Rightarrow \boxed{\sigma^2 = \phi/2} \Rightarrow \boxed{\sigma = \pm\sqrt{\phi/2}}$

$\Rightarrow$ The points of inflection occur for $\boxed{x = \sigma}$.
(at standard deviation)

Further $\boxed{4Dt = 4\pi k^2 t} \Rightarrow \boxed{D = \pi k^2}$



24/. A speculator buys a bond at a current price, to sell it at a later time t. His (at $t=0$) positive mathematical expectation is $\boxed{\langle x\rangle_+ = \int_0^\infty \frac{x\, e^{-x^2/\phi}}{\sqrt{\pi\phi}}\, dx}$.

Write $\boxed{u = x^2/\phi}$.

$\therefore \; du = \frac{2x\,dx}{\phi} \Rightarrow \langle x\rangle_+ = \int_0^\infty \frac{\phi}{2\sqrt{\pi\phi}} e^{-u}\, du$.

$\Rightarrow \langle x\rangle_+ = \frac{1}{2}\sqrt{\frac{\phi}{\pi}}\left[-1\, e^{-u}\Big|_0^\infty\right]$ Now $-\left[e^{-\infty} - e^{0}\right] = 1$.

$\therefore \boxed{\langle x\rangle_+ = \frac{1}{2}\sqrt{\frac{\phi}{\pi}}}$. Since $\boxed{\phi = 4\pi k^2 t}$, $\boxed{\langle x\rangle_+ = k t^{1/2}}$