Bacteria versus Toxin: (A non-autonomous system)

$\boxed{\begin{array}{l} x(t) \rightarrow \text{Number of bacteria at time, } t. \\ T(t) \rightarrow \text{Amount of toxin at time, } t. \end{array}}$

i.) In the ~~presence~~ absence of toxins bacteria grow, $\boxed{\frac{dx}{dt} = bx}$. $(b > 0)$.

ii.) In the presence of toxins, bacteria die out. $\boxed{\frac{dx}{dt} = -axT}$ $(a > 0)$.

iii.) Growth rate of toxins. $\boxed{\frac{dT}{dt} = c}$. $(c > 0)$.

$\Rightarrow \boxed{T = ct + k}$ (linear function) ← Initial condition: When $t = 0$, $T = 0$ $\Rightarrow \boxed{k = 0}$ $\Rightarrow \boxed{T = ct}$.

$\therefore \boxed{\frac{dx}{dt} = -axct}$ in the presence of toxins.

Combined Equation: $\boxed{\frac{dx}{dt} = bx - axct}$

$\Rightarrow \boxed{\frac{dx}{dt} = f(x,t) = x(b - act)}$ Non-autonomous equation

Integral Solution: $\boxed{\int \frac{dx}{x} = \int (b - act)\,dt}$

$\Rightarrow \boxed{\ln x = \ln x_0 + bt - \frac{act^2}{2}}$ | x_0 is an integration constant.

$\Rightarrow \boxed{x = x_0 \exp\left[bt - \frac{act^2}{2}\right]}$. From this

Solution we see that when $\boxed{t = 0,\ x = x_0}$ (initial condition). Further when $t \to \infty$, the square power dominates and $x \to 0$ (the limiting condition).

Now we write $\boxed{bt - \frac{act^2}{2} = \frac{2bt - act^2}{2}}$.

This is $-\frac{1}{2}\left[\left(\sqrt{ac}\,t\right)^2 - 2\frac{b}{\sqrt{ac}}\sqrt{ac}\,t + \frac{b^2}{ac} - \frac{b^2}{ac}\right]$

which can be written as a square,

$$\boxed{-\frac{1}{2}\left[\left(\sqrt{ac}\,t - \frac{b}{\sqrt{ac}}\right)^2 - \frac{b^2}{ac}\right] = \frac{b^2}{2ac} - \frac{ac}{2}\left(t - \frac{b}{ac}\right)^2}$$

Hence $$\boxed{x = x_0\, e^{b^2/2ac} \times \exp\left[-\frac{ac}{2}\left(t - \frac{b}{ac}\right)^2\right]}$$

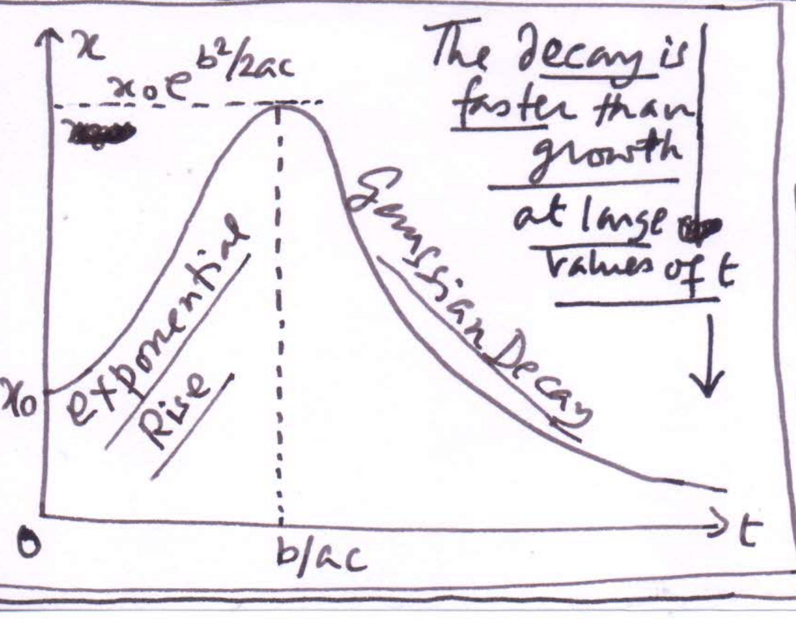
Clearly, i.) When $\boxed{t = 0,\ x = x_0}$, ii.) When $\boxed{t \to \infty,\ x \to 0}$, and iii.) When $\boxed{t = \frac{b}{ac}}$, $\boxed{x = x_0 e^{b^2/2ac} > x_0}$

Looking at $\boxed{\frac{dx}{dt} = x(b - act)}$, we see that when $\boxed{t = b/ac}$, $\boxed{\frac{dx}{dt} = 0}$.

Hence, $\boxed{t = b/ac}$ gives a turning point for $x(t)$.

The Second Derivative: $\boxed{\frac{d^2x}{dt^2} = \frac{dx}{dt}(b - act) + x(-ac)}$

When $\boxed{t = b/ac}$, $\boxed{\frac{d^2x}{dt^2} = -acx < 0}$. This is the condition for a maximum value of $x(t)$.



Rescale: $\boxed{X = x/x_0}$ and $\boxed{T = t/(b/ac)}$. This gives

$$\boxed{X = e^{b^2/2ac} \times \exp\left[-\frac{b^2}{2ac} \times (T-1)^2\right]}.$$

i.) For $\boxed{T < 1}$, early growth is exponential.

ii.) For $\boxed{T > 1}$, the decay is Gaussian.