Drug Dosage : A Course of Medicine

1/. In a course of medication, a drug is ingested periodically (once a day or twice a day, etc.) in the GI tract.

2/. The equation for $x(t)$ (amount of drug in the GI tract) is modified as

$$\boxed{\frac{dx}{dt} = I - k_1 x}\;,\;\boxed{x(0)=0}\,.$$ $I \to$ Ingestion rate (constant)

3/. The equation for $y(t)$ (amount of drug in the blood) remains unchanged.

$$\therefore \boxed{\frac{dy}{dt} = k_1 x - k_2 y}\;,\;\boxed{y(0)=0}\,.\quad k_1, k_2 > 0$$

4/. Solution of $x(t)$: $\boxed{\dfrac{dx}{dt} = I - k_1 x}$ as in $\boxed{\begin{matrix}(\dot{x} = a - bx)\\ a \to I\\ b \to k_1\end{matrix}}$

$$\Rightarrow \boxed{\int \frac{dx}{I - k_1 x} = \int dt} \Rightarrow \boxed{\int \frac{d(-k_1 x)}{I - k_1 x} = -k_1 \int dt}$$

$$\Rightarrow \boxed{\ln(I - k_1 x) = -k_1 t + A}$$ $A \to$ Integration constant

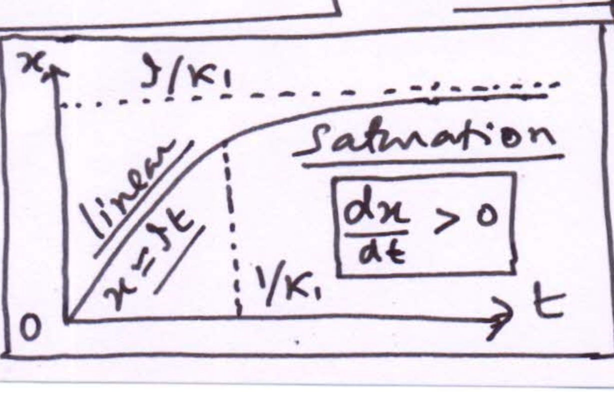
when $\boxed{t=0,\ x=0}$.

$$\boxed{A = \ln I}\,. \Rightarrow \boxed{\ln(I - k_1 x) = -k_1 t + \ln I}$$ (initial condition)

$$\Rightarrow I - k_1 x = I e^{-k_1 t}$$

$$\Rightarrow \boxed{x = \frac{I}{k_1}\left(1 - e^{-k_1 t}\right)}$$

when $\boxed{t \to \infty,\ x \to I/k_1}$



(P.T.O.)

5/. Solution of $y(t)$: $\boxed{\frac{dy}{dt} + k_2 y = k_1 x}$

$$\Rightarrow e^{k_2 t}\frac{dy}{dt} + k_2 y e^{k_2 t} = e^{k_2 t}\, \vartheta\left(1 - e^{-k_1 t}\right)$$

$$\Rightarrow \boxed{\frac{d}{dt}\left(y e^{k_2 t}\right) = \vartheta e^{k_2 t} - \vartheta e^{(k_2 - k_1)t}}$$

$$\Rightarrow y e^{k_2 t} = \vartheta \int e^{k_2 t}\,dt - \vartheta \int e^{(k_2 - k_1)t}\,dt$$

$$\Rightarrow \boxed{y e^{k_2 t} = \frac{\vartheta}{k_2} e^{k_2 t} - \frac{\vartheta\, e^{(k_2 - k_1)t}}{k_2 - k_1} + B}$$

When $\boxed{t = 0,\ y = 0}$ (Initial condition) $B \to$ Integration constant

$$\Rightarrow \boxed{0 = \frac{\vartheta}{k_2} - \frac{\vartheta}{k_2 - k_1} + B} \quad \Rightarrow \boxed{B = \frac{\vartheta}{k_2 - k_1} - \frac{\vartheta}{k_2}}$$

$$\Rightarrow \boxed{y(t) = \frac{\vartheta}{k_2}\left(1 - e^{-k_2 t}\right) - \frac{\vartheta}{k_2 - k_1}\left(e^{-k_1 t} - e^{-k_2 t}\right)}$$

When $\boxed{t \to \infty,\quad y(t) \longrightarrow \vartheta / k_2}$ (Non-zero limit)

Special Case: $\boxed{k_1 = k_2 = k}$

$$\boxed{y(t) = \frac{\vartheta}{k_2}\left(1 - e^{-k_2 t}\right) - \frac{\vartheta}{k_2 - k_1} e^{-k_1 t}\left[1 - e^{(k_1 - k_2)t}\right]}$$

$$1 - e^{(k_1 - k_2)t} = \cancel{1} - \left[\cancel{1} + (k_1 - k_2)t + \cdots\right] \simeq -(k_1 - k_2)t$$

(linear order when $k_1 \to k_2$)

$$\Rightarrow y \simeq \frac{\vartheta}{k_2}\left(1 - e^{-k_2 t}\right) - \frac{\vartheta}{\cancel{k_2 - k_1}} \times e^{-k_1 t} \times \cancel{(k_2 - k_1)}t$$

$$\Rightarrow \boxed{y \simeq \frac{\vartheta}{k_2}\left[1 - e^{-k_2 t} - k_2 t e^{-k_1 t}\right]}$$ since $k_1 \to k_2$, $k_1 = k_2 = k$.

$$\Rightarrow \boxed{y \simeq \frac{\vartheta}{k}\left[1 - (kt + 1)e^{-kt}\right]}$$

i) $t = 0,\ y = 0$.
ii) $t \to \infty,\ y \to \vartheta/k$

Solutions for Equal Rate Constants

$\boxed{k_1 = k_2 = k} \Rightarrow \boxed{\frac{dx}{dt} = \vartheta - kx}$ $\boxed{x(0) = 0}$

$\Rightarrow \boxed{x(t) = \frac{\vartheta}{k}\left(1 - e^{-kt}\right)}$ When $t \to \infty$, $x(t) \to \vartheta/k$.

$\boxed{\frac{dy}{dt} + ky = kx = \vartheta\left(1 - e^{-kt}\right)}$ By the method of integrating factors

$\boxed{e^{kt}\frac{dy}{dt} + ke^{kt}y = \vartheta e^{kt} - \vartheta} \Rightarrow \boxed{\frac{d}{dt}\left(ye^{kt}\right) = \vartheta\left(e^{kt} - 1\right)}$

$\Rightarrow \boxed{ye^{kt} = \vartheta\int\left(e^{kt} - 1\right)dt = \frac{\vartheta e^{kt}}{k} - \vartheta t + A}$ A is an integration constant

When $\boxed{t = 0,\ y = 0} \Rightarrow \boxed{A = -\vartheta/k}$

$\therefore\ y = \frac{\vartheta}{k} - \vartheta t e^{-kt} - \frac{\vartheta}{k}e^{-kt} = \frac{\vartheta}{k} - \frac{\vartheta}{k}kte^{-kt} - \frac{\vartheta}{k}e^{-kt}$

$\Rightarrow \boxed{y(t) = \frac{\vartheta}{k}\left[1 - (kt + 1)e^{-kt}\right]}$ As for the case of $k_1 \to k_2$

i.) when $\boxed{t = 0,\ y = 0}$. ii.) When $\boxed{t \to \infty,\ y \to \vartheta/k}$

iii.) ~~When~~ $\frac{dy}{dt} = \frac{\vartheta}{k}\left[-(kt+1)\times -ke^{-kt} - ke^{-kt}\right]$

$\Rightarrow \boxed{\frac{dy}{dt} = \frac{\vartheta}{k}\times -ke^{-kt}\left[-(kt+1) + 1\right] = 0} \Rightarrow \boxed{t = 0}$ (turning point)

iv.) $\boxed{\frac{d^2y}{dt^2} = k\frac{dx}{dt} - k\frac{dy}{dt}}$ At $t = 0$, $\boxed{\frac{d^2y}{dt^2} = k\frac{dx}{dt} > 0}$ (minimum)

v.) when $\boxed{kt \ll 1}$, $\boxed{y \simeq \frac{\vartheta}{k}\left[1 - (1 + kt)(1 - kt)\right]}$

$\Rightarrow y \simeq \frac{\vartheta}{k}\left[\cancel{1} - \left(\cancel{1} - k^2t^2\right)\right] \Rightarrow \boxed{y \simeq \vartheta k t^2}$ Early growth is parabolic.

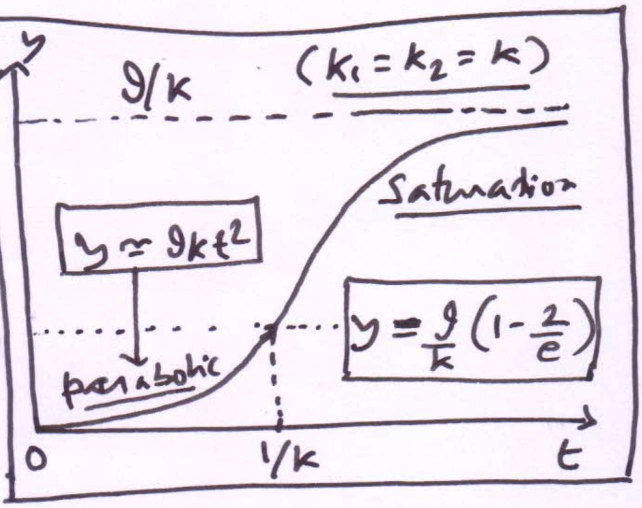
(continued)

$$\boxed{\frac{dy}{dt} = Jkte^{-kt} > 0}$$

(always growth)

$$\Rightarrow \frac{d^2y}{dt^2} = Jk\left[e^{-kt} + te^{-kt} \times -k\right]$$

$$\Rightarrow \boxed{\frac{d^2y}{dt^2} = Jke^{-kt}(1-kt)}$$



when $\boxed{\frac{d^2y}{dt^2} = 0}$ $\Rightarrow$ $\boxed{t = 1/k}$ (point of inflexion)

$$\Rightarrow \boxed{y(t) = \frac{J}{k}\left[1-(kt+1)e^{-kt}\right] = \frac{J}{k}\left(1-\frac{2}{e}\right)} \text{ at } \boxed{t = k^{-1}} \rightarrow \boxed{y \simeq 0.26\, J/k}$$

Early Growth when $\boxed{k_1 \neq k_2}$ (when $t \to 0$)

$$\boxed{y = \frac{J}{k_2}\left(1-e^{-k_2t}\right) - \frac{J}{k_2-k_1}\left(e^{-k_1t} - e^{-k_2t}\right)}$$

When $\boxed{t \to 0}$,

$$y \simeq \frac{J}{k_2}\left[\cancel{1} - \left(\cancel{1} - k_2t + \frac{k_2^2t^2}{2}\right)\right] - \frac{J}{k_2-k_1}\left[\left(\cancel{1} - k_1t + \frac{k_1^2t^2}{2}\right) - \left(\cancel{1} - k_2t + \frac{k_2^2t^2}{2}\right)\right]$$

$$\Rightarrow y \simeq \frac{J}{k_2}\left[k_2t - \frac{k_2^2t^2}{2}\right] - \frac{J}{k_2-k_1}\left[-k_1t + \frac{k_1^2t^2}{2} + k_2t - \frac{k_2^2t^2}{2}\right]$$

$$\Rightarrow y \simeq Jt - \frac{Jk_2t^2}{2} - \frac{J}{(k_2-k_1)}\left[(k_2-k_1)t - \frac{t^2}{2}(k_2^2-k_1^2)\right]$$

$$\Rightarrow \boxed{y \simeq \cancel{Jt} - \frac{Jk_2t^2}{2} - \cancel{Jt} + \frac{Jt^2}{2}(k_2+k_1) \simeq \frac{J}{2}k_1t^2} \leftarrow \text{parabolic}$$

Hence, i.) when $\boxed{t = 0,\ y = 0}$, ii.) when $\boxed{t \to 0,\ y \simeq \frac{Jk_1t^2}{2}}$

iii.) when $\boxed{t \to \infty,\ y \to J/k_2}$

1. Early growth is parabolic.
2. Saturates on long time.