Van Meegeren Art Forgery Case

Radioactivity : $\boxed{\text{Rate} \propto \text{State}}$

$\boxed{\dfrac{dN}{dt} = -\lambda N}$ $\lambda \rightarrow$ Decay constant

$\boxed{\lambda > 0}$, radioactive DECAY

Integrate : $\Rightarrow$ $\boxed{\ln N = -\lambda t + c}$.

Initial condition is when $\boxed{t = t_0, N = N_0}$.

$\therefore$ $\boxed{c = \ln N_0 + \lambda t_0}$ $\Rightarrow$ $\boxed{\ln N - \ln N_0 = -\lambda(t - t_0)}$

$\Rightarrow$ $\boxed{N = N_0 e^{-\lambda(t-t_0)}}$ $\rightarrow$ Exponential decay.

Half - life : $\Rightarrow$ $\boxed{N = N_0/2}$

Time taken to decay to half the initial amount.

Write $t - t_0 = T_{1/2}$

$\Rightarrow$ $\dfrac{N}{N_0} = 2^{-1} = e^{-\lambda(t-t_0)}$

$\Rightarrow$ $-\lambda(t - t_0) = -\ln 2$

$\Rightarrow$ $\boxed{t - t_0 = T_{1/2} = \dfrac{\ln 2}{\lambda} = \dfrac{0.693}{\lambda}}$

Eg. $T_{1/2}\left(\underset{C_{14}}{\text{Carbon}}\right) = 5568$ years, $T_{1/2}\left(\underset{U_{238}}{\text{Uranium}}\right) = 4.5 \times 10^9$ years

Actual Age : $\boxed{t - t_0 = \dfrac{1}{\lambda} \ln (N_0/N)}$

OR $\boxed{t - t_0 = \dfrac{T_{1/2}}{\ln 2} \ln (N_0/N)}$

1/. N and λ can be measured.

2/. The difficulty is in knowing N_0 (the initial amount).

All paints contain white lead (lead oxide).
White lead contains radioactive Pb-210, with a half life of approximately $\boxed{22 \text{ years}}$, in which time it decays to Pb-206 (non-radioactive).
Let $\boxed{x_0 = x(t_0)}$ be the amount of Pb-210 ~~per~~ contained per gram of white lead, at the time of manufacture of the pigment.
The decay rate of Pb-210 is given by

$$\boxed{\frac{dx}{dt} = -\lambda x + s(t)}$$

, in which $s(t)$ is the rate at which Pb 210 is replenished due to the radioactive decay of Ra-226 per minute per gram of white lead. If R is the amount of radium (Ra-226) at time t, with a half life of $\boxed{T_{R1/2} = 1600 \text{ years}}$, we write the decay equation of Ra-226 as $\boxed{R = R_0\, e^{-\lambda_R (t - t_0)}}$.
We expand this as $\boxed{R = R_0\left[1 - \lambda_R (t - t_0) + \cdots\right]}$.
Now, $\boxed{t - t_0 = 300 \text{ years}}$ at most, which is the age of the original painting. Further $\boxed{\lambda_R = \frac{\ln 2}{T_{R1/2}}}$

Hence, $\boxed{\lambda_R (t - t_0) = \frac{\ln 2}{T_{R1/2}} (t - t_0) \simeq 0.13 \ll 1}$.

Therefore, we neglect all the higher powers in the expansion and retain only,

$$\boxed{R \simeq R_0\left[1-\frac{\ln 2}{T_{R1/2}}(t-t_0)\right]}$$

. The decay rate of Ra-226 is

$$\boxed{\frac{dR}{dt} \simeq -\frac{R_0 \ln 2}{T_{R1/2}} = -r(t)}$$

, which is constant. Hence, the rate of replenishment of Pb 210, $r(t)$ is also constant.

$$\Rightarrow \boxed{r(t) = \frac{R_0 \ln 2}{T_{R1/2}}}$$

. The decay rate of Pb 210 is given now as

$$\boxed{\frac{dx}{dt} = r - \lambda x}$$

, which, with $\boxed{r, \lambda > 0}$, is now in the form $\boxed{\frac{dx}{dt} = a - bx}$.

Integration: $$\boxed{\frac{dx}{r - \lambda x} = dt}$$ Separation of variables.

$$\Rightarrow \boxed{\int \frac{d(-\lambda x)}{r - \lambda x} = -\lambda \int dt} \Rightarrow \boxed{\ln(r - \lambda x) = -\lambda t + c}$$

The initial condition is when $\boxed{t = t_0, \quad x = x_0}$.

$$\Rightarrow \boxed{c = \lambda t_0 + \ln(r - \lambda x_0)}$$

. Using this we get

$$\boxed{\ln\left(\frac{r - \lambda x}{r - \lambda x_0}\right) = -\lambda(t - t_0)}$$

$$\Rightarrow r - \lambda x = (r - \lambda x_0)\, e^{-\lambda(t - t_0)}$$

$$\Rightarrow r - \lambda x_0 = (r - \lambda x)\, e^{+\lambda(t - t_0)}$$

Only x and t are variables.

$$\Rightarrow \boxed{x_0 = \frac{r}{\lambda} - \left(\frac{r}{\lambda} - x\right) e^{\lambda(t - t_0)}}.$$

$$\boxed{x_0 = \frac{1}{\lambda} + \left(x - \frac{1}{\lambda}\right) e^{\lambda(t-t_0)}}$$

. In this equation, both λ and 1 are fixed known quantities. x can be measured. For a new painting x is large and $t - t_0$ is small, and for an old painting, x is small and $t - t_0$ is large. x_0 is ALWAYS fixed.

i/. When $t - t_0 = 300$ years, $\lambda(t - t_0) = 9.45$

ii/. When $t - t_0 = 20$ years, $\lambda(t - t_0) = 0.63$.

For measured values of x, using $\boxed{t - t_0 = 300 \text{ yrs}}$ makes the value of x_0 absurdly high. x_0 is acceptably small when $\boxed{t - t_0 = 20 \text{ years}}$.

Hence, the painting is a forgery.

Radio-Carbon Dating: Age of Ancient Cultures.

$$\boxed{{}^{1}n + {}^{14}N \longrightarrow {}^{14}C + {}^{1}p} \rightarrow \text{Willard Libby}$$

$$\boxed{N = N_0 e^{-\lambda(t-t_0)}} \Rightarrow \boxed{\frac{N_0}{N} = e^{\lambda(t-t_0)}}.$$

$$\frac{dN}{dt} = \dot{N} = N_0 e^{-\lambda(t-t_0)} \times -\lambda = -\lambda N. \text{ (late \& state)}$$

At $\boxed{t = t_0}$, $\boxed{\frac{dN}{dt} = \dot{N}(t_0) = -\lambda N_0}$. $(N_0 = N(t_0))$

$$\Rightarrow t - t_0 = \frac{1}{\lambda} \ln\left(\frac{N_0}{N}\right) = \frac{1}{\lambda} \ln\left[\frac{\dot{N}(t_0)}{\dot{N}(t)}\right]$$

$$\Rightarrow \boxed{t - t_0 = \frac{T_{1/2}}{\ln 2} \ln\left[\frac{\dot{N}(t_0)}{\dot{N}(t)}\right]} \quad \boxed{T_{1/2} = 5568 \text{ years}}$$