# Power Laws in Non-Autonomous Systems

Consider a non-autonomous equation $\boxed{\frac{dx}{dt} = \alpha \frac{x}{t}}$.

Integral Solution: $\boxed{\int \frac{dx}{x} = \alpha \int \frac{dt}{t}} \Rightarrow \boxed{\ln x = \alpha \ln t - \alpha \ln c}$

$\therefore \boxed{x = \left(\frac{t}{c}\right)^{\alpha}}$ when $\alpha < 0$, for $t \to \infty$, $x \to 0$ and for $t \to 0$, $x \to \infty$.

To prevent this divergence we translate $\boxed{t \to t + t_0}$.

Hence $\boxed{T = t + t_0} \Rightarrow \boxed{dT = dt}$. We write an equation as $\boxed{\frac{dx}{dt} = \alpha \frac{x}{t+t_0}}$, which we transform as $\boxed{\frac{dx}{dT} = \alpha \frac{x}{T}}$. The integral solution of this equation is $\boxed{x = \left(\frac{t+t_0}{c}\right)^{\alpha}}$, in which when $t \to 0$ (for $\alpha < 0$), the divergence on $x$ is contained by $\boxed{x \to (t_0/c)^{\alpha}}$.

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A Nonlinear Generalisation: Consider now $\boxed{(t+t_0)\frac{dx}{dt} = \alpha x - b x^{\mu+1}}$, which is a nonlinear, non-autonomous equation.

Substitute $\boxed{T = t + t_0} \Rightarrow \boxed{dT = dt}$, and $\boxed{\xi = x^{\mu}}$.

$\therefore$ We get, $\boxed{T \frac{dx}{dT} = \alpha x \left(1 - \frac{x^{\mu}}{\alpha/b}\right)}$. $\boxed{k = \frac{\alpha}{b}}$

Now $\frac{d\xi}{dT} = \mu \frac{x^{\mu}}{x} \frac{dx}{dT} \Rightarrow \boxed{\frac{dx}{dT} = \frac{x}{\mu \xi} \frac{d\xi}{dT}}$

$$T\frac{dx}{dT} = \frac{T\cancel{x}}{\mu\,\mathcal{E}_s}\,\frac{d\mathcal{E}_s}{dT} = \alpha\cancel{x}\left(1-\frac{\mathcal{E}_s}{k}\right)$$

$$\Rightarrow \boxed{\frac{d\mathcal{E}_s}{dT} = \alpha\mu\,\frac{\mathcal{E}_s}{T}\left(1-\frac{\mathcal{E}_s}{k}\right)}$$

. Now rescale $\boxed{X = \mathcal{E}_s/K}$.

$$\Rightarrow \frac{d(\mathcal{E}_s/k)}{dT} = \alpha\mu\,\frac{(\mathcal{E}_s/k)}{T}\left(1-\frac{\mathcal{E}_s}{k}\right)$$

$$\Rightarrow \boxed{\frac{dX}{dT} = \alpha\mu\,\frac{X}{T}(1-X)}$$

. We integrate this equation by the method of separation of variables and partial fractions.

$$\Rightarrow \boxed{\int\frac{dX}{X(1-X)} = \alpha\mu\int\frac{dT}{T}}$$

. Now $\boxed{\frac{1}{X(1-X)} \equiv \frac{A}{X} + \frac{B}{1-X}}$

$$\Rightarrow \boxed{1 \equiv A(1-X) + BX}$$

Now when $X=0,\ A=1$. and when $X=1,\ B=1$.

$$\therefore \int\frac{dX}{X(1-X)} = \int\frac{dX}{X} - \int\frac{d(-X)}{1-X} = \alpha\mu\int\frac{dT}{T}$$

Integral constant $C>0$

$$\Rightarrow \boxed{\ln X - \ln(1-X) = \alpha\mu\ln T - \alpha\mu\ln c}$$

$$\Rightarrow \boxed{\ln\left(\frac{X}{1-X}\right) = \ln\left(\frac{T}{c}\right)^{\alpha\mu}} \Rightarrow \boxed{\frac{X}{1-X} = \left(\frac{T}{c}\right)^{\alpha\mu}}.$$

$$\Rightarrow X = \left(\frac{T}{c}\right)^{\alpha\mu} - X\left(\frac{T}{c}\right)^{\alpha\mu} \Rightarrow \boxed{X\left[1+\left(\frac{T}{c}\right)^{\alpha\mu}\right] = \left(\frac{T}{c}\right)^{\alpha\mu}}$$

$$\Rightarrow \boxed{X = \frac{(T/c)^{\alpha\mu}}{1+(T/c)^{\alpha\mu}}} \Rightarrow \boxed{X = \frac{1}{1+(T/c)^{-\alpha\mu}}}.$$

$$\boxed{X = \frac{x^{\mu}}{k}} \Rightarrow \boxed{x^{\mu} = \frac{k\,(T/c)^{\alpha\mu}}{1+(T/c)^{\alpha\mu}}}$$

in which $\boxed{T = t+t_0}$.

## Case I: $\boxed{\mu = 1}$ and $\boxed{\alpha > 0}$ and $\boxed{t_0 = 0}$.

$$\therefore \boxed{x = \frac{k(t/c)^{\alpha}}{1+(t/c)^{\alpha}}}$$

i.) When $\boxed{t \to 0}$, $\boxed{1 + \left(\frac{t}{c}\right)^{\alpha} \simeq 1}$

$\Rightarrow \boxed{x \simeq k(t/c)^{\alpha}}$ for small values of $t$.

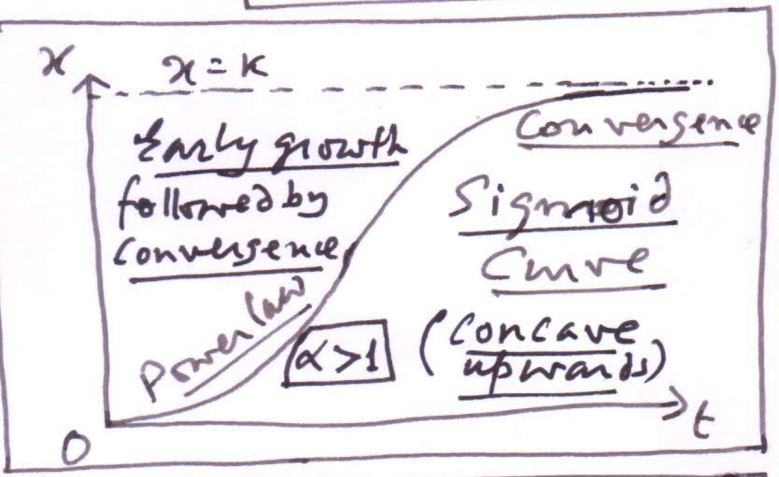
$\Rightarrow$ When $\boxed{t = 0, \ x = 0}$.

ii.) When $\boxed{t \to \infty}$,

$$\boxed{x = \frac{k}{1+(t/c)^{-\alpha}}}.$$

$\Rightarrow \boxed{x \longrightarrow k}$ (limiting value)

($x$ starts at $x = 0$)




## Case II: $\boxed{\mu = -1}$ and $\boxed{\alpha < 0}$ and $\boxed{t_0 \neq 0}$.

We write $\boxed{k^{-1} = \eta}$ in $\boxed{x^{\mu} = \frac{1}{k^{-1} + k^{-1}(T/c)^{-\alpha\mu}}}$

and $\boxed{\frac{1}{k} \cdot \frac{1}{c^{-\alpha\mu}} = \frac{1}{c_1^{-\alpha\mu}}}$, to get,

$$\boxed{x = \left[\frac{1}{\eta + \left(\frac{t+t_0}{c_1}\right)^{-\alpha\mu}}\right]^{1/\mu}} \Rightarrow \boxed{x = \left[\eta + \left(\frac{t+t_0}{c_1}\right)^{-\alpha\mu}\right]^{-1/\mu}}$$

When $\boxed{\mu = -1}$. $\boxed{x = \eta + \left(\frac{t+t_0}{c_1}\right)^{\alpha}}$

We know $\boxed{\alpha < 0}$. For the special case of $\boxed{\alpha = -2}$ (Zipf's law), (GEORGE KINGSLEY ZIPF)

$$\boxed{x = \eta + \left(\frac{c_1}{t+t_0}\right)^{2}}$$

When $\boxed{t \to \infty}$ $\boxed{x \longrightarrow \eta}$.