Logistic modelling of economic dynamics

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We demonstrate the effectiveness of the logistic function to model the evolution of two economic systems. The first is the GDP and trade growth of the USA, and the second is the revenue and human resource growth of IBM. Our modelling is based on the World Bank data in the case of the USA, and on the company data in the case of IBM. The coupled dynamics of the two relevant variables in both systems — GDP and trade for the USA, and revenue and human resource for IBM — follows a power-law behaviour.

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I. THE LOGISTIC FUNCTION

The logistic equation is a standard example of a first-order autonomous nonlinear dynamical system [1]. Introduced originally to study population dynamics [1, 2], it was later applied to various problems of socio-economic [2–6] and scientific interest [1]. The growth of many natural systems is modelled accurately by the logistic equation, the growth of species being a case in point [2]. Modelling with the logistic equation is thus compatible with natural evolution itself. This principle can be extended to the free evolution of economic systems as well, a view that is supported by the successful logistic modelling of the GDP-trade dynamics of national economies [6] and industrial dynamics [5].

First-order autonomous dynamical systems have the general form of $\dot{x} \equiv \mathrm{d}x/\mathrm{d}t = f(x)$ where $x \equiv x(t)$, with t being time [1]. Such a system may be linear or nonlinear, depending on $f(x)$ being, respectively, a linear or a nonlinear function of x [1]. A basic model of a nonlinear function is given by $f(x) = ax - bx^2$, with a and b being fixed parameters. This leads to the well-known logistic equation,

$$\dot{x} \equiv \frac{\mathrm{d}x}{\mathrm{d}t} = f(x) = ax - bx^2. \tag{1}$$

Under the initial condition of $x(0) = x_0$, and with the definition of $k = a/b$, the integral solution of Eq. (1) is

$$x(t) = \frac{kx_0 e^{at}}{k + x_0(e^{at} - 1)}, \tag{2}$$

which is the logistic function. From Eq. (2) we see that x converges to the limiting value of k when $t \longrightarrow \infty$. This limit is known as the carrying capacity in studies of population dynamics, and it is also a fixed point of the dynamical system [1]. This becomes clear when we set the fixed point condition $\dot{x} = f(x) = 0$ [1]. The two fixed points that result from Eq. (1) are $x = 0$ and $x = k = a/b$.

On early time scales, when $t \ll a^{-1}$, the growth of x can be approximated to be exponential, i.e. $x \simeq x_0 \exp(at)$. This gives $\ln x \sim at$, which is a linear relation on a linear-log plot. Furthermore, we can interpret $a \simeq \dot{x}/x$ as the relative (or fractional) growth rate in the early exponential regime. However, this exponential growth is not indefinite, and on times scales of $t \gg a^{-1}$ (or $t \longrightarrow \infty$) there is a convergence to $x = k$. Clearly, the transition from the exponential regime to the saturation regime occurs when $t \sim a^{-1}$. This time scale corresponds to the time when the nonlinear term in Eq. (1) becomes significant compared to the linear term. The precise time for the nonlinear effect to start asserting itself can be determined from the condition $\ddot{x} = f'(x)\dot{x} = 0$ when $\dot{x} \neq 0$, with the prime indicating a derivative with respect to x. This requires solving $f'(x) = a - 2bx = 0$ to get $x = a/2b = k/2$. Using $x = k/2$ in Eq. (2) gives the nonlinear time scale as

$$t_{\mathrm{nl}} = \frac{1}{a} \ln\left(\frac{k}{x_0} - 1\right), \tag{3}$$

which, we stress again, is the maximum duration over which a robust exponential growth can be sustained. Hereafter, we shall use Eqs. (2) and (3) to model two different economic systems. The first is the GDP-trade dynamics of the USA, whose national economy leads the world. The second is the revenue and human resource growth of the company, IBM.

II. THE COUPLED DYNAMICS OF GDP AND TRADE

The GDP (Gross Domestic Product) of a country is the market value of goods and services produced by the country in a year [7–9]. GDP thus quantifies the aggregate outcome of the economic activities of a country that are performed all round the year. As such, the GDP of a national economy is a dynamic quantity and its evolution (commonly implying growth) can be followed through time.

Contribution to the GDP of a country comes from another dynamic quantity — the annual trade in which the country engages itself [9]. The global trade network among countries exhibits some typical properties of a complex network, namely, a scale-free degree distribution and small-world clusters [10]. If countries are to be treated as vertices in this network, then global trade can be viewed as the exchange of wealth among the vertices [11]. The fitness of a vertex (a country) is measured by its GDP, which also stands for the potential ability of a vertex to grow trading relations

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with other vertices [11]. Moreover, GDP itself follows its own power-law distribution [9, 11], which in turn determines the topology of the global trade network [11]. In qualitative terms, these networks-based perspectives of the interrelation between GDP and trade are in agreement with the Gravity Model of trade, which mathematically formulates the trade between two countries to be proportional to the GDP of both [12] (also see [13, 14] for subsequent reviews). Considering all of the foregoing facts together, it is quite evident that GDP and trade are intimately correlated. Both form a coupled system, in which the dynamics of the one reinforces the dynamics of the other.

We look at the coupled dynamics of GDP and trade within the mathematical framework of the logistic equation [1, 2]. This is in line with a study carried out on countries that are ranked high globally in terms of their national GDPs [6]. The temporal evolution of the total GDP of the world economy (measured in US dollars) from 1870 to 2000 does indicate a logistic trend [8]. Empirical evidence also exists for a power-law feature in the interdependent growth of GDP and trade [15]. We unify these observations in a theoretical model based on World Bank data that specifically pertain to the annual GDP and trade growth of the USA [16, 17].

We quantify GDP by the variable $G \equiv G(t)$, with G measured in US dollars and t in years. To model the annual growth of $G(t)$ with the logistic equation, as in Eq. (1), we write

$$\dot{G} \equiv \frac{\mathrm{d}G}{\mathrm{d}t} = \mathcal{G}(G) = \gamma_1 G - \gamma_2 G^2. \tag{4}$$

Noting that x, a and b in Eq. (1) translate, respectively, to G, γ_1 and γ_2 in Eq. (4), we can write the integral solution of $G(t)$ in the same form as Eq. (2). It then follows that when $t \longrightarrow \infty$, $G(t)$ converges to a limiting value, i.e. $G \longrightarrow k_G = \gamma_1/\gamma_2$. The early exponential growth of the GDP of the USA and its later convergence to a finite limit are modelled in the upper linear-log plot in Fig. 1. The smooth dotted curve tracks the GDP data [16] with the integral solution of Eq. (4), which will be in the mathematical form of Eq. (2).

As with GDP, the growth of trade can also be modelled with the logistic equation [6]. The annual trade of a country accounts for the total import and export of goods and services. The World Bank data on the annual trade of the USA are given as a percentage of the annual GDP [16, 17]. Knowing the annual GDP, the trade percentage can be expressed explicitly in terms of US dollars, which we denote by the variable $T \equiv T(t)$, with t continuing to be measured in years. We model the dynamics of $T(t)$ with the logistic equation, as done in Eq. (4), and write

$$\dot{T} \equiv \frac{\mathrm{d}T}{\mathrm{d}t} = \mathcal{T}(T) = \tau_1 T - \tau_2 T^2. \tag{5}$$

Comparing Eq. (5) with Eq. (1), we note that x, a and b translate, respectively, to T, τ_1 and τ_2. Hence, from the integral solution of $T(t)$, which will be in the same form as Eq. (2), we will get a convergence of $T \longrightarrow k_T = \tau_1/\tau_2$, when $t \longrightarrow \infty$. The integral solution of Eq. (5) fits the trade data of the USA [17] in the lower linear-log plot in Fig. 1.

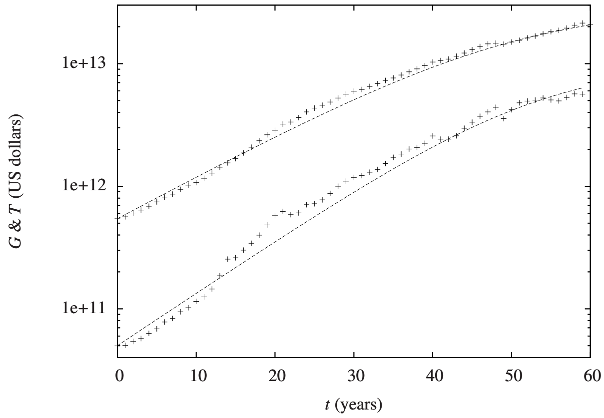

FIG. 1. Modelling the dynamics of GDP (the upper plot) and trade (the lower plot) using World Bank data for the USA [16, 17]. The zero year of both plots is 1960. The GDP plot ends in 2020, but the trade plot ends in 2019. The two smooth dotted curves follow the logistic function, as given by Eq. (2). The parameter values to fit the GDP growth are $\gamma_1 = 0.080\,\mathrm{year}^{-1}$ and $k_G = \$30$ trillion (the predicted maximum value of the GDP). With respect to the logistic function, the yearly relative variation of the GDP data has a mean $\mu_G = 0.0492$ and a standard deviation $\sigma_G = 0.0873$. The parameter values to fit the trade growth are $\tau_1 = 0.099\,\mathrm{year}^{-1}$ and $k_T = \$10$ trillion (the predicated maximum value of trade). With respect to the logistic function, the yearly relative variation of the trade data has a mean $\mu_T = 0.1160$ and a standard deviation $\sigma_T = 0.2040$.

We note further in Fig. 1 that the logistic modelling of trade growth closely resembles the logistic modelling of the GDP growth. The similarity between the two plots is captured by a correlation coefficient of 0.992 between the GDP and the trade of the USA [6]. This high correlation is expected, because GDP and trade are dynamically connected to each other [8, 9, 11]. As such, the coupled dynamics of GDP and trade must be governed by an autonomous system of the second order, given as $\dot{T} = \mathcal{T}(T, G)$ and $\dot{G} = \mathcal{G}(T, G)$. The T-G phase solutions are determined by integrating

$$\frac{\mathrm{d}G}{\mathrm{d}T} = \frac{\dot{G}}{\dot{T}} = \frac{\mathcal{G}(T, G)}{\mathcal{T}(T, G)} \tag{6}$$

for various initial values of the (T, G) coordinates [1]. Since the functions $\mathcal{G}(T, G)$ and $\mathcal{T}(T, G)$ are not known a priori, we apply a linear ansatz of $\mathcal{G} \simeq \gamma_1 G$ in Eq. (4) and $\mathcal{T} \simeq \tau_1 T$ in Eq. (5).[1] This approach agrees with the multiplicative character of GDP and trade, whereby the revenue generated in one year is reinvested in the economic cycle of the next year [9]. The linearization gives a scaling formula (with $\alpha = \gamma_1/\tau_1$)

$$G(T) \sim T^{\alpha}. \tag{7}$$

Empirical evidence to support the power law implied by Eq. (7) was found from 1948 to 2000, in a survey of nearly

[1] For the coupled growth of G and T, a second-order dynamical system like $\dot{G} \sim T$ and $\dot{T} \sim G$ may appear apt. This, however, gives phase solutions like $G^2 \sim T^2$, which is not borne out by a study of GDP and trade growth [15].

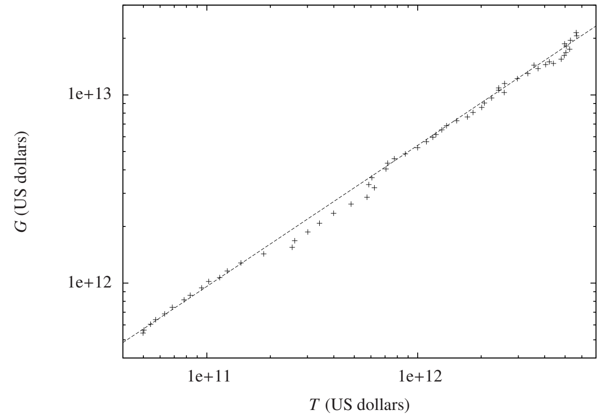

FIG. 2. Plotting GDP against trade using World Bank data for the USA [16, 17]. The plot begins in 1960 and ends in 2019. The straight dotted line follows the power-law function in Eq. (7) with $\alpha = 0.75$, a value that is close to $\gamma_1/\tau_1 \simeq 0.81$ (as given in Fig. 1). With respect to the logarithm of the power-law function, the yearly variation of the logarithm of the GDP data has a mean $\mu_\alpha = -0.0012$ and a standard deviation $\sigma_\alpha = 0.0024$.

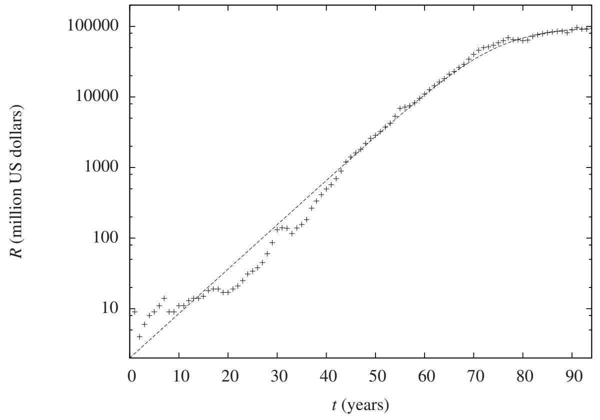

FIG. 3. Modelling the annual revenue growth of IBM, using the company data from 1914 to 2006 [19]. The smooth dotted curve is the logistic function, as given by Eq. (2). The parameter values to fit the revenue growth are $\rho_1 = 0.145\,\mathrm{year}^{-1}$ and $k_R = \$\,100$ billion (the predicted maximum revenue that IBM can earn). With respect to the logistic function, the yearly relative variation of the revenue has a mean $\mu_R = 0.025$ and a standard deviation $\sigma_R = 0.4897$. Saturation of the revenue growth starts on the time scale of 75-80 years.

two dozen countries of varying economic strength (high, middle and low-income economies) [15].

The power-law function in Eq. (7) becomes linear in a log-log plot. This is indeed what we see in Fig. 2 which models the coupled growth of the GDP and trade of the USA. The power-law exponent α is given by the slope of the linear fit, within the range of $0 < \alpha < 1$ [6]. Keeping only the linear terms in Eqs. (4) and (5), which lead to the phase solutions in Eq. (7), we find that $\alpha = \gamma_1/\tau_1$. The values of γ_1, τ_1 and α, required for plotting Figs. 1 and 2, do show that α is quite close to γ_1/τ_1. This independently validates our modelling of GDP and trade growth with the logistic equation.

Looking at Fig. 2, we realize that the power-law scaling of G with respect to T holds true over nearly three orders of magnitude. For high values of T and G, deviation from this scaling behaviour is possible due to the nonlinear effects in the real data [6]. However, we have not considered nonlinearity in the coupled autonomous functions $\mathcal{G}(T, G)$ and $\mathcal{T}(T, G)$ to derive Eq. (7) [6]. We also note that $\mathrm{d}^2G/\mathrm{d}T^2 < 0$ for $\alpha < 1$, i.e. G increases with T at a decreasing rate as time progresses. This explains the steady reduction of the gap between the GDP and the trade plots in Fig. 1 on long time scales.

III. THE LOGISTIC DYNAMICS OF A COMPANY

The growth (and the health) of a company is gauged by the annual revenue that it generates and the human resource that it employs. Regular monitoring of these two variables is necessary for a precise understanding of the patterns of industrial growth. Even when a company shows noticeable growth in the early stages, a saturation in its growth occurs on later time scales [18]. Clearly, as the size of an organization increases, its growth rate becomes progressively inhibited. Therefore, to explain saturation in industrial growth, an effective mathematical model has to study the growth of a company that operates on the largest possible scale, which can only be the global scale. What is more, when a company operates on the global scale, its overall growth pattern becomes free of local inhomogeneities. This itself affords an advantage for the mathematical modelling. In view of this, we analyze the growth of the annual revenue and the human resource strength of the multi-national company, IBM. Data about its annual revenue generation, the net annual earnings and the cumulative human resource strength, dating from 1914 to 2006, are available on the company website [19].

As we have done for the GDP and trade dynamics of the USA in Sec. II, we posit the logistic equation to model the annual revenue and human resource growth of IBM. If the revenue is $R \equiv R(t)$, with R measured in US dollars and t in years, then the logistic model for the revenue growth is

$$\dot{R} \equiv \frac{\mathrm{d}R}{\mathrm{d}t} = \mathcal{R}(R) = \rho_1 R - \rho_2 R^2. \tag{8}$$

Since x, a and b in Eq. (1) translate, respectively, to R, ρ_1 and ρ_2 in Eq. (8), the integral solution for $R(t)$ will be in the same form as Eq. (2). And so when $t \longrightarrow \infty$, $R(t)$ will converge to a limiting value of $R \longrightarrow k_R = \rho_1/\rho_2$. The early exponential growth of the revenue of IBM and its later saturation to a finite limit are modelled in the linear-log plot in Fig. 3. The smooth dotted curve fits the revenue data [19] according to the logistic function, in the form of Eq. (2). The most noteworthy feature in Fig. 3 is the saturation in the revenue growth of IBM around 75-80 years. This time scale can be also obtained from the formula of the nonlinear time scale in Eq. (3), by equating the parameter values as $a = \rho_1$ and $k = k_R$. From this it is clear that the revenue growth of IBM entered the nonlinear regime around the time of 75-80 years (the initial years of the 1990s).

Now we write the human resource of IBM as $H \equiv H(t)$,

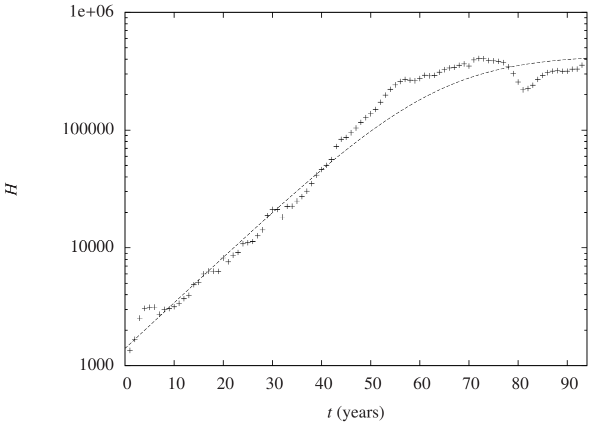

FIG. 4. Modelling the human resource growth of IBM, using the company data from 1914 to 2006 [19]. The smooth dotted curve is the logistic function, as given by Eq. (2). The parameter values to fit the human resource growth are $\eta_1 = 0.09\,\text{year}^{-1}$ and $k_H = 500000$ (the predicted maximum employees of IBM). With respect to the logistic function, the yearly relative variation of the human resource has a mean $\mu_H = 0.106$ and a standard deviation $\sigma_H = 0.2999$. The human resource graph declines around 75-80 years.

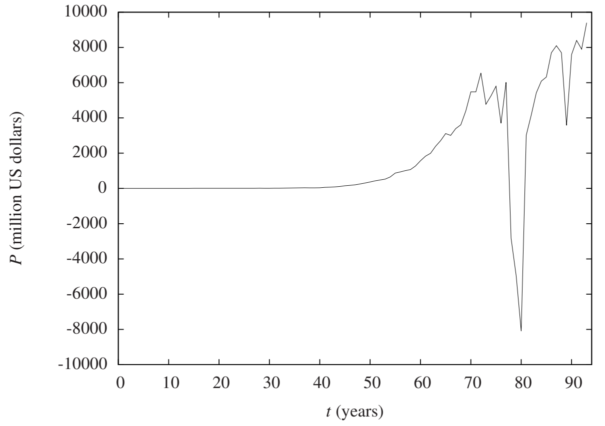

FIG. 5. The net annual earnings (the profit P) of IBM grow steadily till about 75-80 years (the early years of the 1990s). Around this time IBM suffered major losses in its net earnings ($ 8 billion in 1993), and this time scale corresponds closely to the time scale for the onset of nonlinear saturation in revenue growth, which is also 75-80 years.

with t in years as usual. Then translating x, a and b in Eq. (1), respectively, to H, η_1 and η_2, the logistic equation for the human resource growth becomes

$$\dot{H} \equiv \frac{\mathrm{d}H}{\mathrm{d}t} = \mathcal{H}(H) = \eta_1 H - \eta_2 H^2, \tag{9}$$

whose integral solution will have the form of Eq. (2). With $t \longrightarrow \infty$, a convergence to a limiting value occurs for $H(t)$, which goes as $H \longrightarrow k_H = \eta_1/\eta_2$. The early exponential growth of the human resource of IBM and its later convergence to a finite limit are modelled in the linear-log plot in Fig. 4. The smooth dotted curve is the model logistic function and it follows the human resource growth of IBM [16].

A point to note in Fig. 4 is the depletion of the human resource of IBM on the time scale of 75-80 years. This is evidently correlated with the saturation of the revenue growth of IBM on the same time scale, as Fig. 3 shows. That the revenue and the human resource of a company are correlated is entirely to be expected. If a company generates enough revenue, it becomes financially viable for it to maintain a functioning human resource pool, which in turn generates further revenue. In this manner both the revenue and the human resource of a company sustain the growth of each other. Whenever one of the variables is affected adversely, there is an equally adverse impact on the other variable. In the case of IBM, the saturation of its revenue growth around 75-80 years resulted in a loss of human resource on the same time scale. An additional confirmation of this argument comes from Fig. 5, which plots the net annual earnings (the profit P) of IBM against time. The company suffered major financial losses in the early 1990s (upto $ 8 billion in 1993), which matches our estimate of the nonlinear time scale of 75-80 years.

To analyze the correlated growth of R and H, we set down a coupled autonomous dynamical system as $\dot{R} = \mathcal{R}(H,R)$ and

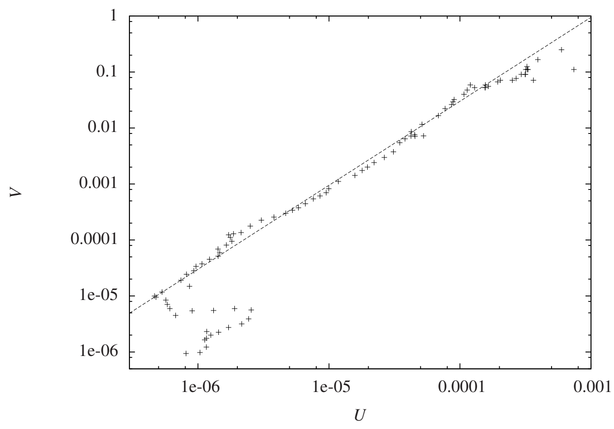

FIG. 6. Fitting Eq. (10) to the correlated growth of H and R, with $\beta = 1.5$ (close to $\beta = \rho_1/\eta_1 \simeq 1.6$). The cusp in the data points at the bottom left is due to human resource loss around 75-80 years.

$\dot{H} = \mathcal{H}(H,R)$. As in the case of the derivation of Eq. (7), the coupled autonomous functions $\mathcal{R}(H,R)$ and $\mathcal{H}(H,R)$ are not known a priori. Therefore, in a basic approach, we assign to these functions the uncoupled logistic forms in Eqs. (8) and (9), respectively. This simplifies to $\dot{R} = \mathcal{R}(R)$ and $\dot{H} = \mathcal{H}(H)$. The variable t, which is implicit in this set of equations, can be eliminated to obtain the H-R phase solutions for initial values of the (H,R) coordinates [1]. Defining $V = R^{-1} - k_R^{-1}$, $U = H^{-1} - k_H^{-1}$ and $\beta = \rho_1/\eta_1$, the H-R phase solutions are transformed to a compact power-law form as

$$V \sim U^{\beta}. \tag{10}$$

The power-law in Eq. (10) fits the data well in the log-log plot in Fig. 6, except for the cusp at the bottom left. However, the lower arm of the cusp has nearly the same positive slope as the extended straight-line fit in Fig. 6, which shows that intermittent deviations do not affect the overall growth too much [3].

IV. CONCLUDING REMARKS

The fact that the logistic equation shows a saturation in growth on long time scales implies long-term economic stagnation. Reasons for this are dwindling natural resources, natural calamities, pandemics, obsolescence of technology, military conflicts, etc. The decisive reasons are often unforeseen. Nevertheless, the logistic equation continues to be a favoured mathematical tool for modelling the evolution of socio-economic systems [2, 3]. For example, our use of the logistic equation and the power-law correlation function in the phase plot was equally effective in modelling the GDP-trade dynamics of six top national economies at present [6] and industrial growth [5]. This analogy between national economies and companies is of interest because studies point to universal mechanisms that underlie the economic dynamics of countries and companies [20, 21]. This commonality can help in understanding the dynamics of large companies, whose stock values grow to the scale of national economies.

The economy of the USA is suited well for our logistic modelling because of its balanced GDP growth, as we can see from the closeness between the theoretical logistic function and the actual GDP data in Figs. 1 and 2. Moreover, from a macroeconomic perspective, GDP is a reliable yardstick of the state of a national economy, and in a global comparison of national economies, the USA has the highest GDP in the world at present. The balanced and robust growth of the US economy has been possible because of democratic values in internal politics, the absence of military conflicts on the national borders, and the promotion of free economic growth [6].

Global economic recessions from time to time make it imperative to devise accurate mathematical models for understanding economic stagnation and for predicting correct outcomes. The logistic equation proves effective in both respects, as has been demonstrated by a recent study on the GDP competitiveness among some leading national economies [22]. That said, we also have to remember that studies of this type come under the general category of social systems, and consequently their predictive power depends on socio-economic factors. Unforeseen natural, social and political events can compromise the forecasts made in these studies, and force mathematical models to be recalibrated.

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