Second Order Systems: Examples and Applications

Richardson's Theory of Conflict (Lewis Fry Richardson)

(The British Journal of Psychology: Generalised Foreign Politics, 1939)

$x(t) \rightarrow$	War potential or armaments of a nation.
$y(t) \rightarrow$	War potential of an enemy nation.

1/. For x: $\boxed{\frac{dx}{dt} = ky + g - \alpha x}$ in which, $\boxed{k, g, \alpha > 0}$. k indicates the war-readiness of y, g indicates the grievance x feels towards y, and α is the cost of armaments incurred by x (which restrains growth of x).

2/. For y: $\boxed{\frac{dy}{dt} = lx + h - \beta y}$ in which $\boxed{l, h, \beta > 0}$. l indicates the war readiness of x, h indicates the grievance y feels towards x, and β is the cost of armaments incurred by y.

$k, g, l, h \rightarrow$ hawk parameters,	$\alpha\, \beta \rightarrow$ dove parameters
k, l (Thucydides, Grey),	g, h (Leo Amery)

Case I: Mutual Disarmament without Animosity and Grievance.

$\boxed{g = h = 0}$ $\Rightarrow$ $\boxed{\frac{dx}{dt} = -\alpha x + ky}$ and

$\boxed{\frac{dy}{dt} = lx - \beta y}$ $\Rightarrow$ Equilibrium $\boxed{\frac{dx}{dt} = \frac{dy}{dt} = 0}$

$\Rightarrow$ $\boxed{x_e = 0}$ and $\boxed{y_e = 0}$ are equilibrium solutions.

For a system $\boxed{\frac{dx}{dt} = Ax + By}$ and $\boxed{\frac{dy}{dt} = Cx + Dy}$,

we can get $\boxed{\frac{d^2x}{dt^2} - (A+D)\frac{dx}{dt} + (AD - BC)x = 0}$.

(The same applies for y). Here $\boxed{\tau = A + D}$ and $\boxed{\Delta = AD - BC}$.

[Initial value $x_0 = x(0)$]

Use a solution $\boxed{x = x_0 e^{\omega t}}$, to get,

$\boxed{\frac{dx}{dt} = \omega x}$ and $\boxed{\frac{d^2x}{dt^2} = \omega^2 x}$. From these

we can write $\boxed{\omega^2 - \tau\omega + \Delta = 0}$, which

implies $\boxed{\omega_{1,2} = \frac{\tau \pm \sqrt{\tau^2 - 4\Delta}}{2}}$. $\boxed{\text{Now } A = -\alpha,\ B = k,\ C = l,\ D = -\beta}$

Hence, $\boxed{\omega_{1,2} = \frac{-(\alpha+\beta) \pm \sqrt{(\alpha+\beta)^2 - 4(\alpha\beta - kl)}}{2}}$ $\therefore$ Starting with $x(0) \neq 0$ and $y(0) \neq 0$,

if $\boxed{\alpha\beta > kl}$, then the discriminant of the quadratic above will be less than $(\alpha+\beta)$. Hence both roots of ω will be negative, i.e. $\boxed{x_e = 0}$ and $\boxed{y_e = 0}$ will be stable equilibrium solutions. This state represents mutual disarmament.

(P.T.O.)

(continued) ($x_c, y_c \rightarrow$ equilibrium values)

Hence with $\boxed{x_c = 0}$ and $\boxed{y_c = 0}$ (mutual disarmament) and with both roots of $\boxed{\omega_{1,2} < 0}$, peace prevails for all time.

Example: Canada / U.S., Norway / Sweden.

Case II: Mutual Disarmament without Satisfaction of Grievance.

Initially $\boxed{x = y = 0}$ (mutual disarmament)

but $\boxed{g, h \neq 0}$ (Grievance continues)

Hence $\boxed{\frac{dx}{dt} = g}$ and $\boxed{\frac{dy}{dt} = h}$. Since

both $\boxed{g, h > 0}$, x and y will grow in time.

Case III: Unilateral disarmament.

Initially $\boxed{y = 0}$ but $\boxed{x \neq 0}$ (Unilateral disarmament)

Hence, $\boxed{\frac{dy}{dt} = lx + h}$ Since, x, l and h are all positive.

$\boxed{\frac{dy}{dt} > 0}$ $\Rightarrow$ y will grow again in time.

Example: German rearmament before World War II

Can be reduced by reducing grievance and building confidence. Eg. Germany and Japan after World War II.

Case IV : Arms Race

$\boxed{\Delta = -kl < 0} \Rightarrow$ fixed point is a saddle point

Initially set $\boxed{\alpha = \beta = 0}$ $\Rightarrow$ No restraint on armament.

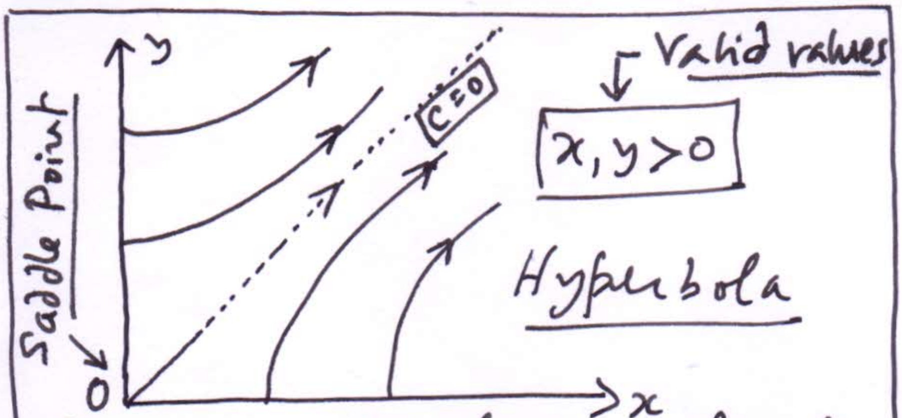
Also $\boxed{g = h = 0}$ $\Rightarrow$ No history of animosity.

$\Rightarrow$ $\boxed{\dfrac{dx}{dt} = ky}$ and $\boxed{\dfrac{dy}{dt} = lx}$. Equilibrium is obtained for $\boxed{\dfrac{dx}{dt} = \dfrac{dy}{dt} = 0}$ $\Rightarrow$ $\boxed{x_c = y_c = 0}$.

Now $\boxed{\dfrac{dy/dt}{dx/dt} = \dfrac{dy}{dx} = \dfrac{lx}{ky}}$ $\Rightarrow$ $\boxed{\int ky\,dy = \int lx\,dx}$

$\Rightarrow$ $\boxed{\dfrac{lx^2}{2} - \dfrac{ky^2}{2} = \underset{(c)}{\text{Constant}}}$ $\Rightarrow$ $\boxed{\dfrac{x^2}{2kc} - \dfrac{y^2}{2lc} = 1}$

A hyperbola



Curves in the first quadrant

Whenever x grows, y will also grow, and vice-versa

$\Rightarrow$ an arms race

Example: USA/Soviet Union

$\boxed{k, l > 0}$ Now $\boxed{\dfrac{d^2x}{dt^2} = k\dfrac{dy}{dt} = klx = \omega^2 x}$ $\boxed{\text{where } \omega = \sqrt{kl}}$

$\Rightarrow$ $\boxed{x = Ae^{\sqrt{kl}\,t} + Be^{-\sqrt{kl}\,t}}$. Since $\boxed{y = \dfrac{1}{k}\dfrac{dx}{dt}}$,

we get $\boxed{y = A\sqrt{\dfrac{l}{k}}\,e^{\sqrt{kl}\,t} - B\sqrt{\dfrac{l}{k}}\,e^{-\sqrt{kl}\,t}}$ after differentiating.

As $\boxed{t \to \infty}$, $\boxed{x \to \infty \text{ and } y \to \infty}$ (Uncontrolled Growth).

The General Condition ($\alpha, \beta, k, l, h, g > 0$ are all non-zero)

$$\boxed{\frac{dx}{dt} = -\alpha x + ky + g} \text{ and } \boxed{\frac{dy}{dt} = lx - \beta y + h}$$

Equilibrium is obtained for $\boxed{\frac{dx}{dt} = \frac{dy}{dt} = 0}$.

$$\Rightarrow \boxed{\begin{array}{l} -\alpha x_c + k y_c + g = 0 \\ l x_c - \beta y_c + h = 0 \end{array}} \text{ and } \Rightarrow \boxed{\begin{array}{l} -l\alpha x_c + kl y_c + lg = 0 \\ l\alpha x_c - \alpha\beta y_c + \alpha h = 0 \end{array}} \text{ and}$$

$$\Rightarrow \boxed{y_c = \frac{\alpha h + lg}{\alpha\beta - lk}}.$$

Similarly we also get

$$\boxed{\begin{array}{l} -\alpha\beta x_c + \beta k y_c + \beta g = 0 \\ kl x_c - k\beta y_c + kh = 0 \end{array}} \text{ and } \Rightarrow \boxed{x_c = \frac{kh + \beta g}{\alpha\beta - lk}}.$$

If $\boxed{\alpha\beta > lk}$, then $\boxed{x_c, y_c > 0}$. This is a permanent fixed state of war preparedness.

Example: India / Pakistan, North Korea / South Korea

Estimation of the Parameters:

1/. α, β, k, l all have the dimension of inverse time.

2/. α^{-1} and β^{-1} $\rightarrow$ Life time of policy implementation. (Example is life time of the parliament $\simeq$ 5 years).

$\Rightarrow \alpha^{-1} \simeq 5$ yrs $\therefore$ $\alpha = 0.2$ unit.

3/. k and l $\longrightarrow$ Depends on the industrial capacity.

4/. g and h $\longrightarrow$ Historical grievances are not constant in time but can change suddenly.