1. For elements a, b, c in a lattice, show that if $a \leq b$, then $a \vee (b \wedge c) \leq b \wedge (a \vee c)$.

Solution: We have

$$\begin{aligned} a &\leq b \\ b \wedge c &\leq b. \end{aligned}$$

Thus,

$$a \vee (b \wedge c) \leq b \vee b = b. \tag{1}$$

Also, we have

$$\begin{aligned} a &\leq a \vee c \\ b \wedge c &\leq c \leq a \vee c. \end{aligned}$$

Thus,

$$a \vee (b \wedge c) \leq (a \vee c) \vee (a \vee c) = (a \vee c). \tag{2}$$

Using (1) and (2), we have

$$a \vee (b \wedge c) = (a \vee (b \wedge c)) \wedge (a \vee (b \wedge c) \leq b \wedge (a \vee c).$$

2. For elements a, b, c in a lattice, show that, then $a \vee (b \wedge c) \leq (a \vee b) \wedge (a \vee c)$ and $(a \wedge b) \vee (a \wedge c) \leq a \wedge (b \vee c)$.

Solution: For the first inequality, note

$$\begin{aligned} a &\leq a \vee b \\ a &\leq a \vee c. \end{aligned}$$

Thus, we have

$$a = a \wedge a \leq (a \vee b) \wedge (a \vee c). \tag{3}$$

Also,

$$\begin{aligned} b \wedge c &\leq b \leq a \vee b \\ b \wedge c &\leq c \leq a \vee c. \end{aligned}$$

Thus, we have

$$b \wedge c = (b \wedge c) \wedge (b \wedge c) \leq (a \vee b) \wedge (a \vee c). \tag{4}$$

Thus, combining equations (3) and (4), we have

$$a \vee (b \wedge c) \leq ((a \vee b) \wedge (a \vee c)) \vee ((a \vee b) \wedge (a \vee c)) = (a \vee b) \wedge (a \vee c).$$

For the second inequality, $a \wedge b \leq a$ and $a \wedge c \leq a$. Thus, we have $(a \wedge b) \vee (a \wedge c) \leq a \vee a = a$. Further, $a \wedge b \leq b \leq b \vee c$ and $a \wedge c \leq c \leq b \vee c$. Therefore, we have $(a \wedge b) \vee (a \wedge c) \leq (b \vee c) \vee (b \vee c) = b \vee c$.

Now, combining the above, we have

$$(a \wedge b) \vee (a \wedge c) = ((a \wedge b) \vee (a \wedge c)) \wedge ((a \wedge b) \vee (a \wedge c)) \leq a \wedge (b \vee c).$$

3. There are 15 non-isomorphic lattices on six elements. List them in the form of Hasse diagrams. Among these, identify the seven lattices that are self-dual.

4. Given an associative, commutative, idempotent binary operation $\vee$ on a set P, define a relation $\leq$ on P as: for $x \leq y$ if and only if $x \vee y = y$ (for all $x, y \in P$). Show that $(P, \leq)$ is a poset.

   Solution: We just need to verify the conditions of Reflexivity, Antisymmetry and Transitivity.

   - **Reflexivity** : for $x \in P$, we have $x \vee x = x$ from the idempotent property of the binary operation $\vee$. Thus, $x \leq x$.
   - **Antisymmetry** : if $x \leq y$ and $y \leq x$, i.e., $x \vee y = y$ and $y \vee x = x$, we have $x = y \vee x = x \vee y = y$ where the second equality follows by commutative property of the binary operation $\vee$. Thus, $x = y$.
   - **Transitivity** : if $x \leq y$ and $y \leq z$, we have $x \vee y = y, y \vee z = z$. Then, $x \vee z = x \vee (y \vee z) = (x \vee y) \vee z = y \vee z = z$. Here we have used the associative property of the binary operation $\vee$ in Step 2. Thus, $x \leq z$.

5. For a fixed integer $n \geq 1$, let B be the set of all binary strings of length n. Define a relation $\leq$ on B as follows. For two strings $\mathbf{x} = x_1 x_2 \dots x_n$ and $\mathbf{y} = y_1 y_2 \dots y_n$ in B, $\mathbf{x} \leq \mathbf{y}$ if and only if for each i, whenever the bit $y_i = 0$, the bit $x_i = 0$ as well (Example: $0101 \leq 1101$). Show that $(B, \leq)$ is a lattice. Describe the join and meet operations of two strings in terms of the bits of the string.

   Solution: We use the following result:

   A poset $(L, <)$ is a lattice if $\sup\{a, b\}$ and $\inf\{a, b\}$ exist for all $a, b \in L$.

   First to $(B, \leq)$ is a poset, i.e., satisfy the conditions of Reflexivity, Antisymmetry and Transitivity.

   - **Reflexivity** : for $\mathbf{x} \in B$, we have $\mathbf{x} \leq \mathbf{x}$ by definition.
   - **Antisymmetry** : Suppose $\mathbf{x} \leq \mathbf{y}$ and $\mathbf{y} \leq \mathbf{x}$. We need to show, $\mathbf{x} = \mathbf{y}$, i.e., $x_i = y_i$ for all i. For any i, there are four possible cases: a) $x_i = 0, y_i = 0$, b) $x_i = 0, y_i = 1$, c) $x_i = 1, y_i = 0$ and d) $x_i = 1, y_i = 0$. Note that $x_i = 0$, since $\mathbf{y} \leq \mathbf{x}$, we must have $y_i = 0$. Similarly, if $y_i = 0$, since $\mathbf{x} \leq \mathbf{y}$, we have $x_i = 0$. Thus, case b) and case c) are not possible. In case a) and case d), we have $x_i = y_i$ as required.

- **Transitivity** : if $\mathbf{x} \leq \mathbf{y}$ and $\mathbf{y} \leq \mathbf{z}$, for any i, if $z_i = 0$, since $\mathbf{y} \leq \mathbf{z}$, we have $y_i = 0$. Again, since $y_i = 0$ and $\mathbf{x} \leq \mathbf{y}$, we have $x_i = 0$. Thus, $\mathbf{x} \leq \mathbf{z}$.

Next, let $\mathbf{x}, \mathbf{y} \in B$. Let $\mathbf{w} = w_1 w_2 \ldots w_n$ and $\mathbf{z} = z_1 z_2 \ldots z_n$ where $w_i = \max\{x_i, y_i\}$ and $z_i = \min\{x_i, y_i\}$. Observe that since $x_i, y_i \in \{0, 1\}$, we have $w_i, z_i \in \{0, 1\}$. Thus, $\mathbf{w}, \mathbf{z} \in B$. We claim that $\sup\{\mathbf{x}, \mathbf{y}\} = \mathbf{w}$ and $\inf\{\mathbf{x}, \mathbf{y}\} = \mathbf{z}$.

To show $\sup\{\mathbf{x}, \mathbf{y}\} = \mathbf{w}$ and $\inf\{\mathbf{x}, \mathbf{y}\} = \mathbf{z}$:

- $\mathbf{w}$ is an upper bound for both $\mathbf{x}$ and $\mathbf{y}$, i.e., $\mathbf{x} \leq \mathbf{w}$ and $\mathbf{y} \leq \mathbf{w}$. For any i, if $w_i = 0$, since $w_i = \max\{x_i, y_i\}$, we have $x_i = 0, y_i = 0$. Thus, $\mathbf{x} \leq \mathbf{w}$ and $\mathbf{y} \leq \mathbf{w}$.

  $\mathbf{z}$ is a lower bound for both $\mathbf{x}$ and $\mathbf{y}$, i.e., $\mathbf{z} \leq \mathbf{x}$ and $\mathbf{z} \leq \mathbf{y}$. For any i, if $x_i = 0$, since $z_i = \min\{x_i, y_i\}$, we have $z_i = 0$. Thus, $\mathbf{z} \leq \mathbf{x}$. Similarly if $y_i = 0$, since $z_i = \min\{x_i, y_i\}$, we have $z_i = 0$. Thus, $\mathbf{z} \leq \mathbf{y}$.
- Suppose $\mathbf{m}$ is an upper bound of both $\mathbf{x}$ and $\mathbf{y}$. To show $\mathbf{w} \leq \mathbf{m}$. For any i, if $m_i = 0$, since $\mathbf{x} \leq \mathbf{m}$ and $\mathbf{y} \leq \mathbf{m}$, we must have $x_i = 0 = y_i$. Thus, by definition, $w_i = \max\{x_i, y_i\} = 0$. Thus, $\mathbf{w} \leq \mathbf{m}$.

  Suppose $\mathbf{m}$ is a lower bound of both $\mathbf{x}$ and $\mathbf{y}$. To show $\mathbf{m} \leq \mathbf{z}$. For any i, if $z_i = 0$, since $z_i = \min\{x_i, y_i\}$, we must have either $x_i = 0$ or $y_i = 0$. If $x_i = 0$, since $\mathbf{m} \leq \mathbf{x}$, we must have $m_i = 0$. If $y_i = 0$, since $\mathbf{m} \leq \mathbf{y}$, we must have $m_i = 0$. Thus, in either case, $m_i = 0$. Thus, $\mathbf{m} \leq \mathbf{z}$.

Observe that $\sup\{\mathbf{x}, \mathbf{y}\} = \mathbf{w}$ represents the bitwise OR operation while $\inf\{\mathbf{x}, \mathbf{y}\} = \mathbf{z}$ represents the bitwise AND operation.

6. Show that for any two elements x, y in a distributive lattice, if there exists some element a such such that $a \vee x = a \vee y$ and $a \wedge x = a \wedge y$, then $x = y$.

   Solution: We need to show $x \leq y$ and $y \leq x$. Suppose that for some element a, we have $a \vee x = a \vee y$ and $a \wedge x = a \wedge y$. Now,

$$
\begin{aligned}
x &= x \wedge (x \vee a) && \text{[using absorption identity]} \\
&= x \wedge (y \vee a) && [\text{using } x \vee a = y \vee a] \\
&= (x \wedge y) \vee (x \wedge a) && \text{[using distributive property]} \\
&= (x \wedge y) \vee (y \wedge a) && [\text{using } x \wedge a = y \wedge a] \\
&= (y \wedge x) \vee (y \wedge a) && \text{[using commutativity property]} \\
&= y \wedge (x \vee a) && \text{[using distributive property]} \\
&= y \wedge (y \vee a) && [\text{using } x \vee a = y \vee a] \\
&= y && \text{[using absorption identity].}
\end{aligned}
$$

7. Prove that every chain is a distributive lattice.

   Solution : Let $(P, \leq)$ be any chain. Again, to show that it is a lattice, we need to show if $a \vee b = \sup\{a, b\}$ and $a \wedge b = \inf\{a, b\}$ exist for all $a, b \in P$. Since it is a chain, either $a \leq b$ or $b \leq a$. Let us assume $a \leq b$, the other case is similar. We claim $a \vee b = \sup\{a, b\} = b$ and $a \wedge b = \inf\{a, b\} = a$. Clearly, $a \leq b$, by assumption, and $b \leq b$ by reflexivity, thus b is an upper bound of both a and b. Now, if c is any upper bound of both a and b, by definition $b \leq c$. Thus, $b = \sup\{a, b\}$. Similarly, a is a lower bound of both a and b as $a \leq a$ by reflexivity and $a \leq b$ by assumption. If c is any lower bound of both a and b, then $c \leq a$ by definition. So, $a = \inf\{a, b\}$.

Now, to prove the distributive property, let $a, b, c \in P$. We need to show $(a \wedge b) \vee (a \wedge c) = a \wedge (b \vee c)$. Since P is a chain, either $b \leq c$ or $c \leq b$. Without loss of generality, we may assume that $b \leq c$. Now, either $a \leq b$ or $b \leq a$. If $a \leq b$, then $a \leq b \leq c$, so that $a \leq c$. Hence, LHS of the distributive property, $(a \wedge b) \vee (a \wedge c) = a \vee a = a$ and RHS of the distributive property, $a \wedge (b \vee c) = a \wedge b = a$. Now suppose $b \leq a$. Then there are two possible sub-cases: we must have either $a \leq c$ or $c \leq a$. So if $a \leq c$, LHS of the distributive property, $(a \wedge b) \vee (a \wedge c) = b \vee c = b$ and RHS of the distributive property, $a \wedge (b \vee c) = a \wedge b = b$. Finally, if $b \leq a$ and $c \leq a$, LHS of the distributive property, $(a \wedge b) \vee (a \wedge c) = b \vee c = b$ and RHS of the distributive property, $a \wedge (b \vee c) = a \wedge b = b$. Thus, we have verified the distributive property in all possible cases.

8. Show that for elements a and b in a Boolean algebra, $a \vee (\bar{a} \wedge b) = a \vee b$ and $a \wedge (\bar{a} \vee b) = a \wedge b$.

   Solution : We have

$$\begin{aligned} a \vee (\bar{a} \wedge b) &= (a \vee \bar{a}) \wedge (a \vee b) && \text{[using distributive property]} \\ &= 1 \wedge (a \vee b) && \text{[using } a \vee \bar{a} = 1\text{]} \\ &= a \vee b && \text{[using } 1 \wedge c = c, \text{ for } c \in L\text{].} \end{aligned}$$

   Similarly, we have

$$\begin{aligned} a \wedge (\bar{a} \vee b) &= (a \wedge \bar{a}) \vee (a \wedge b) && \text{[using distributive property]} \\ &= 0 \vee (a \wedge b) && \text{[using } a \wedge \bar{a} = 0\text{]} \\ &= a \wedge b && \text{[using } 0 \vee c = c, \text{ for } c \in L\text{].} \end{aligned}$$

9. If a and b are two elements of a Boolean algebra, show that $a = b$ if and only if $(a \wedge \bar{b}) \vee (\bar{a} \wedge b) = 0$.

   Solution : First assume that $(a \wedge \bar{b}) \vee (\bar{a} \wedge b) = 0$. Fact if $c \vee d = 0$, then $c = d = 0$. Indeed, we have $c = c \wedge (c \vee d) = c \wedge 0 = 0$ and similarly $d = d \wedge (c \vee d) = d \wedge 0 = 0$. Note we have used absorption property above. Thus, $a \wedge \bar{b} = \bar{a} \wedge b = 0$.

Now

$$
\begin{aligned}
a &= a \wedge 1 && [\text{using } 1 \wedge c = c, \text{ for } c \in L] \\
&= a \wedge (b \vee \bar{b}) && [\text{using } c \vee \bar{c} = 1, \text{ for } c \in L] \\
&= (a \wedge b) \vee (a \wedge \bar{b}) && [\text{using distributive property}] \\
&= (a \wedge b) \vee 0 && [\text{using given condition } a \wedge \bar{b} = 0] \\
&= (a \wedge b) \vee (\bar{a} \wedge b) && [\text{using given condition } \bar{a} \wedge b = 0] \\
&= (a \vee \bar{a}) \wedge b && [\text{using distributive property}] \\
&= 1 \wedge b && [\text{using } c \vee \bar{c} = 1, \text{ for } c \in L] \\
&= b && [\text{using } 1 \wedge c = c, \text{ for } c \in L].
\end{aligned}
$$

Conversely assume that $a = b$. Now,

$$
\begin{aligned}
b \wedge \bar{a} = a \wedge \bar{a} = 0 \\
\bar{a} \wedge b = \bar{a} \wedge a = 0.
\end{aligned}
$$

Hence, we have $(b \wedge \bar{a}) \vee (\bar{a} \wedge b) = \bar{a} \wedge b = 0$ using $b \wedge \bar{a} = 0$ in the first step.

10. For elements a and b in a Boolean algebra, show that $a \leq b$ if and only if $\bar{a} \vee b = 1$.

Solution : Suppose $\bar{a} \vee b = 1$. We have

$$
\begin{aligned}
a \vee b &= (a \wedge 1) \vee b && [\text{using } 1 \wedge c = c, \text{ for } c \in L] \\
&= (a \wedge (\bar{a} \vee b)) \vee b && [\text{using given condition } \bar{a} \vee b = 1] \\
&= ((a \wedge \bar{a}) \vee (a \wedge b)) \vee b && [\text{using distributive property}] \\
&= (0 \vee (a \wedge b)) \vee b && [\text{using } c \wedge \bar{c} = 0, \text{ for } c \in L] \\
&= (a \wedge b) \vee b && [\text{using } 0 \vee c = c, \text{ for } c \in L] \\
&= b && [\text{using absorption identity}].
\end{aligned}
$$

Thus, $a \leq b$.

Conversely suppose that $a \leq b$. Hence, $a = a \wedge b$. Now, we have

$$
\begin{aligned}
1 &= a \vee \bar{a} && [\text{using } c \vee \bar{c} = 1, \text{ for } c \in L] \\
&= (a \wedge b) \vee \bar{a} && [\text{using } a = a \wedge b] \\
&= (a \vee \bar{a}) \wedge (b \vee \bar{a}) && [\text{using distributive property}] \\
&= 1 \wedge (b \vee \bar{a}) && [\text{using } c \vee \bar{c} = 1, \text{ for } c \in L] \\
&= b \vee \bar{a} && [\text{using } 1 \wedge c = c, \text{ for } c \in L].
\end{aligned}
$$