INDIAN INSTITUTE OF TECHNOLOGY KANPUR

DEPARTMENT OF AEROSPACE ENGINEERING

AE-675A INTRODUCTION TO FINITE ELEMENT METHOD

One-dimensional hp Code

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One-dimensional hp Code plots

Figure 1 shows an elastic bar under traction load and constrained at the ends A and B. Develop a generic finite element code to get the approximate solution to the resulting governing differential equation for the bar shown in Figure 1.

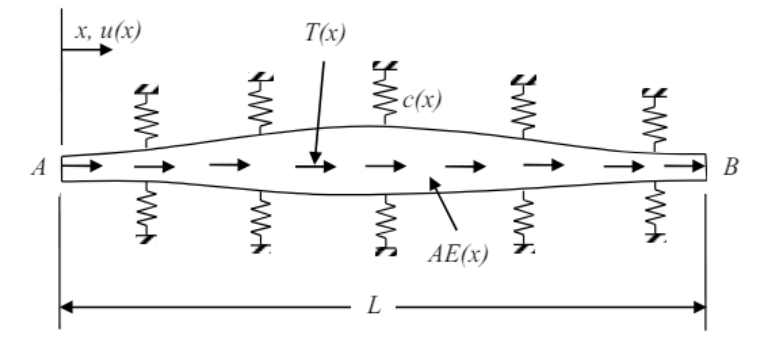


Figure 1: Elastic bar loaded in traction and constrained at ends A and B.

The code should have the following capabilities:

1. Boundary conditions/End Constraints: Both ends can be constrained by specifying (a) primary variable (Dirichlet/Displacement/Essential), (b) secondary variable or force (Force/Neumann/Natural) and (c) springs (Mixed/Robin)
2. The variables $T(x), c(x)$ and $AE(x)$ can vary from a constant to a quadratic function.
3. The length L and the number of elements will be input values. Discretize the domain into given number of elements with equal lengths.
4. There should be a provision to put at least one concentrated load at any given location (excluding the ends).
5. Use of either Lagrange interpolation or hierarchic shape functions upto quartic order should be possible.
6. Postprocessing must be able to represent the primary, secondary and other variables over the domain either continuously or discretely as required.

Solution

1. Do the patch test for the following cases:

   $AE(x) = 1$ and $c(x) = 0$ with $u(x)|_{x=0} = 0$ and $\left.\frac{du}{dx}\right|_{x=1} = 0$. When $T(x) = 1$, use 1, 2, 5, 10 and 100 number of linear and quadratic elements and when $T(x) = x$ use 1, 2, 5, 10 and 100 number of linear, quadratic and cubic elements and superimpose your solutions with the respective exact solutions. Plot the error in the solution. Also plot the derivative of the exact solution and finite element solutions. Discuss the results.

2. For the problem in Point 1, plot the strain energy of the exact and finite element solution against the number of elements in the mesh for all the cases. Also plot the strain energy of the solution. Discuss the results.

For AE(x) = 1, C(x) = 0, T(x)=1

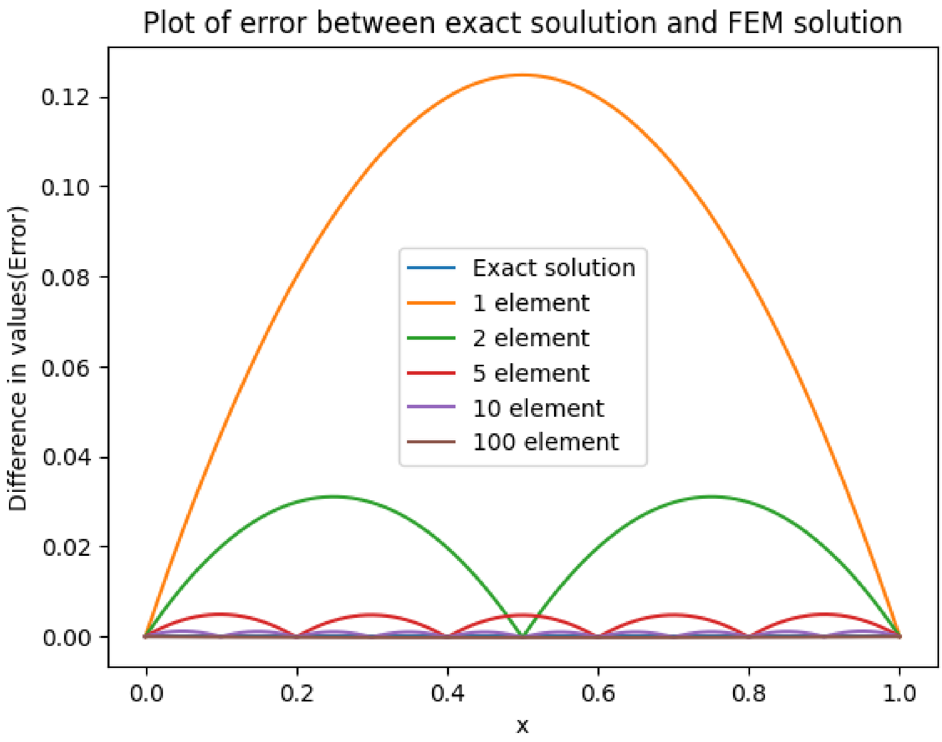


Figura 1: Linear Approximation

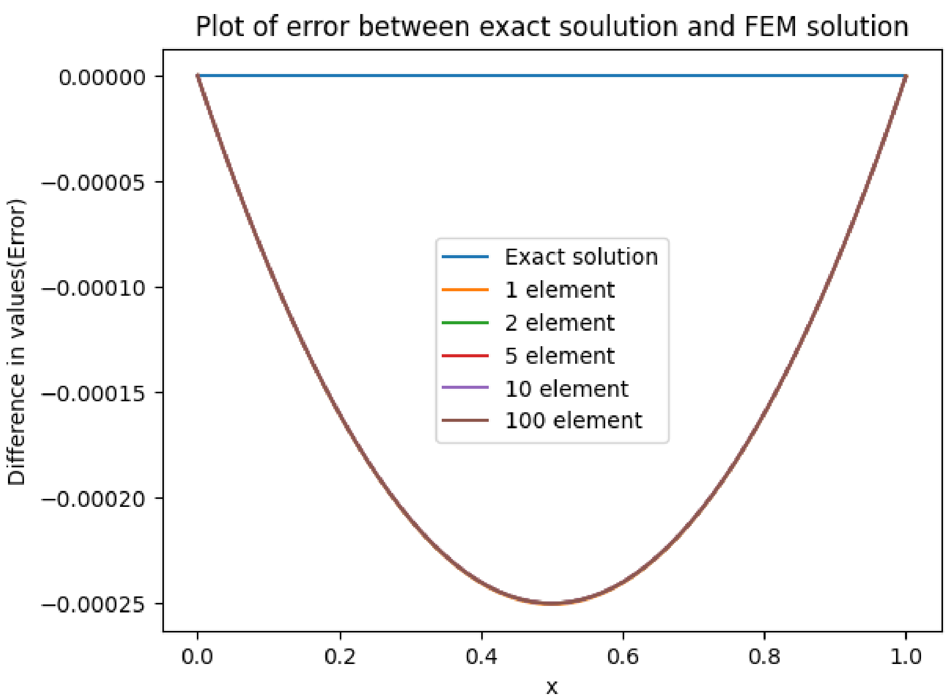


Figura 2: Quadratic Approximation

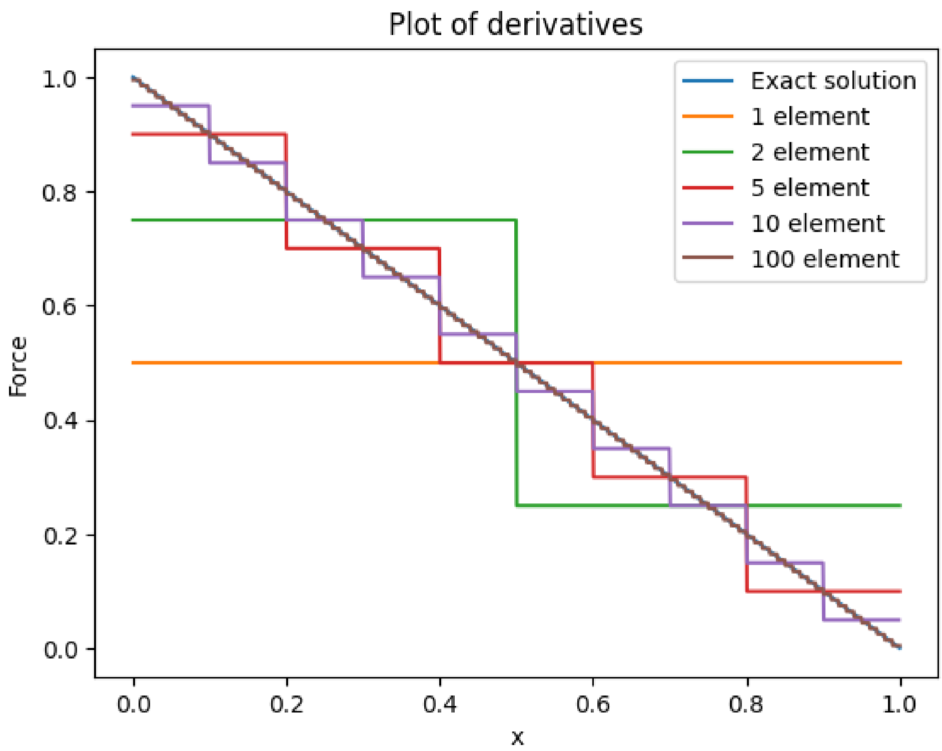


Figura 3: Linear Approximation

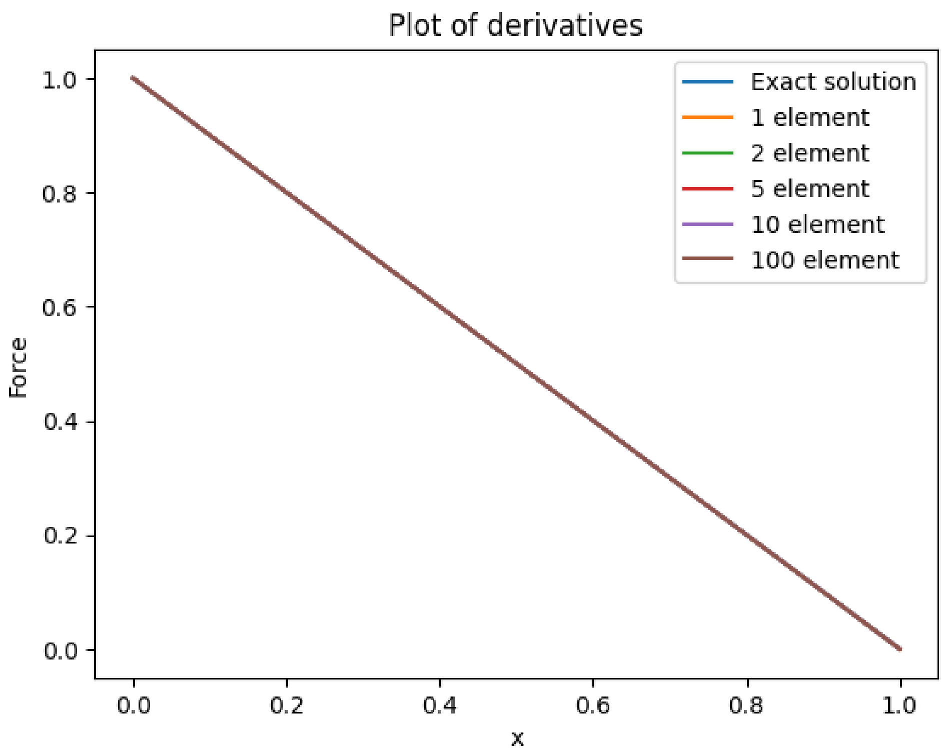


Figura 4: Quadratic Approximation

For AE(x) = 1, C(x) = 0, T(x)=x

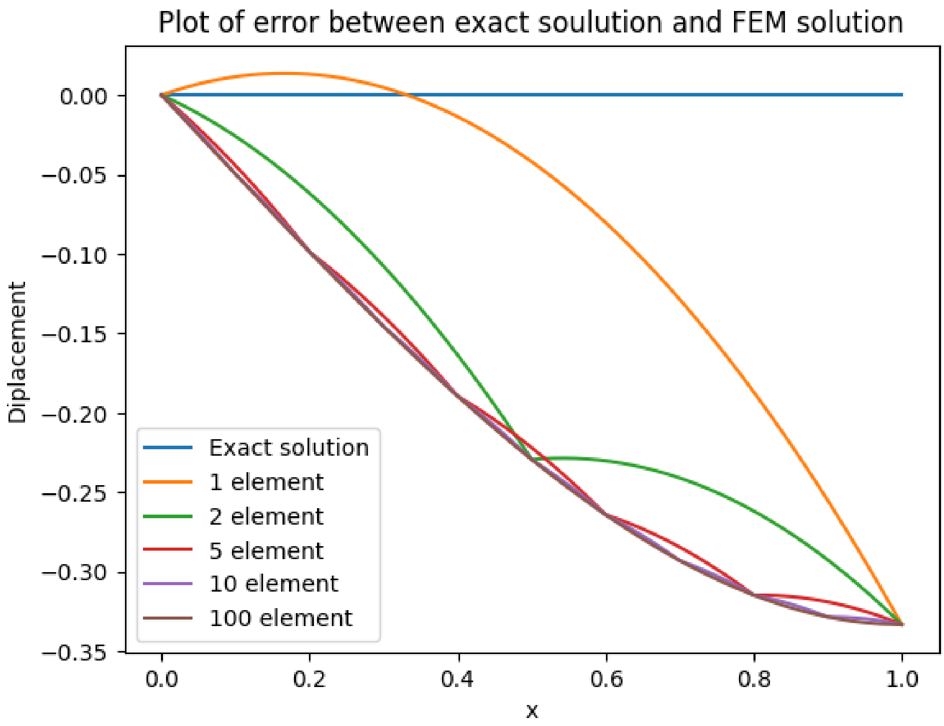


Figura 5: Linear Approximation

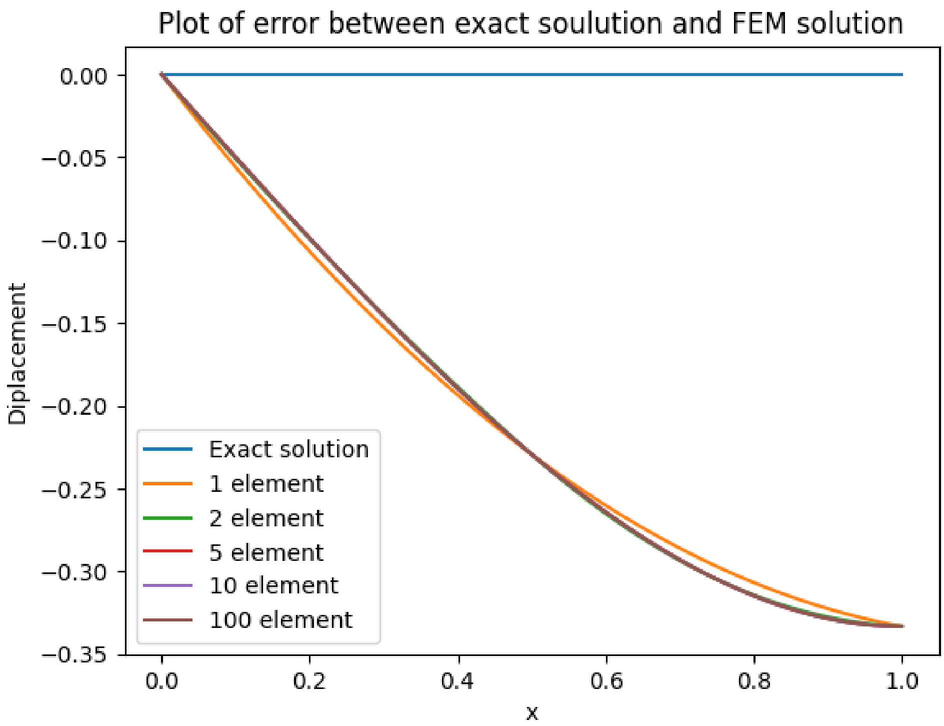


Figura 6: Quadratic Approximation

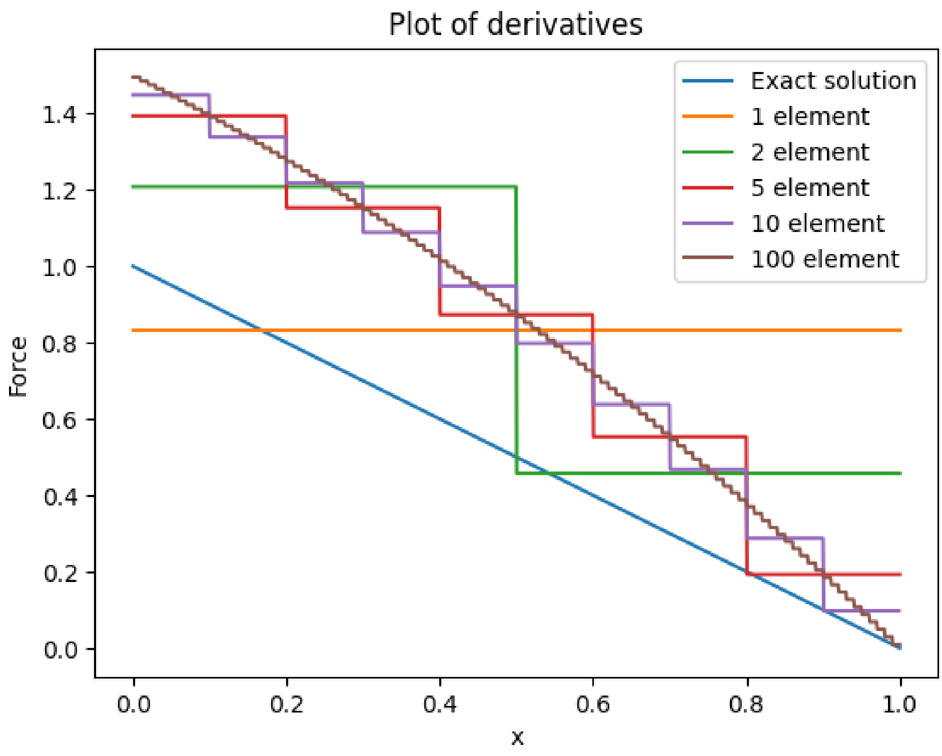


Figura 7: Linear Approximation

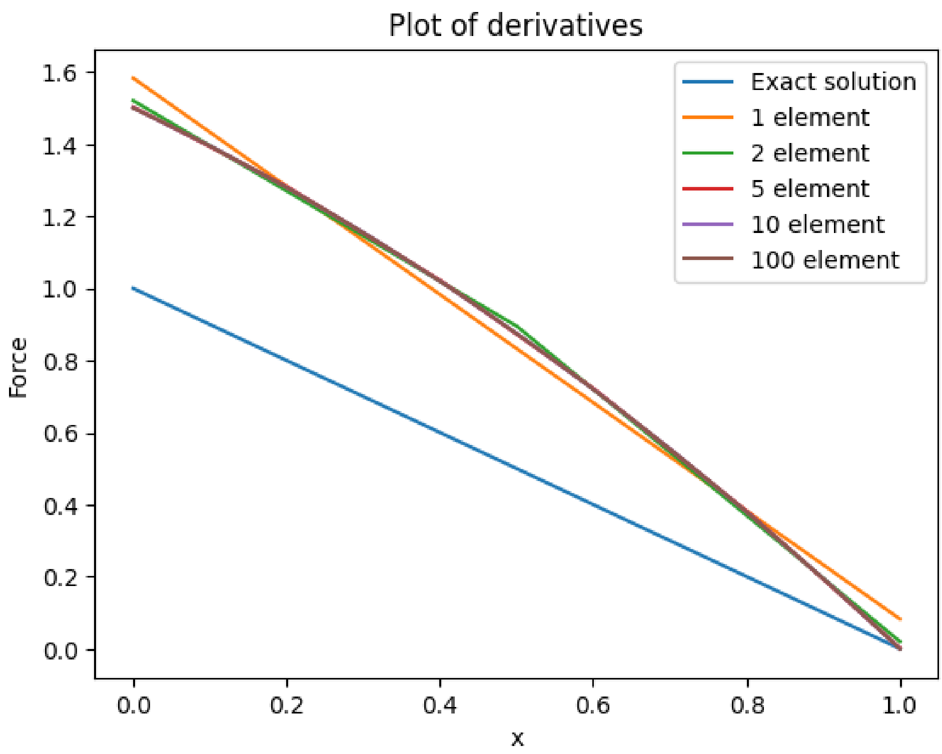
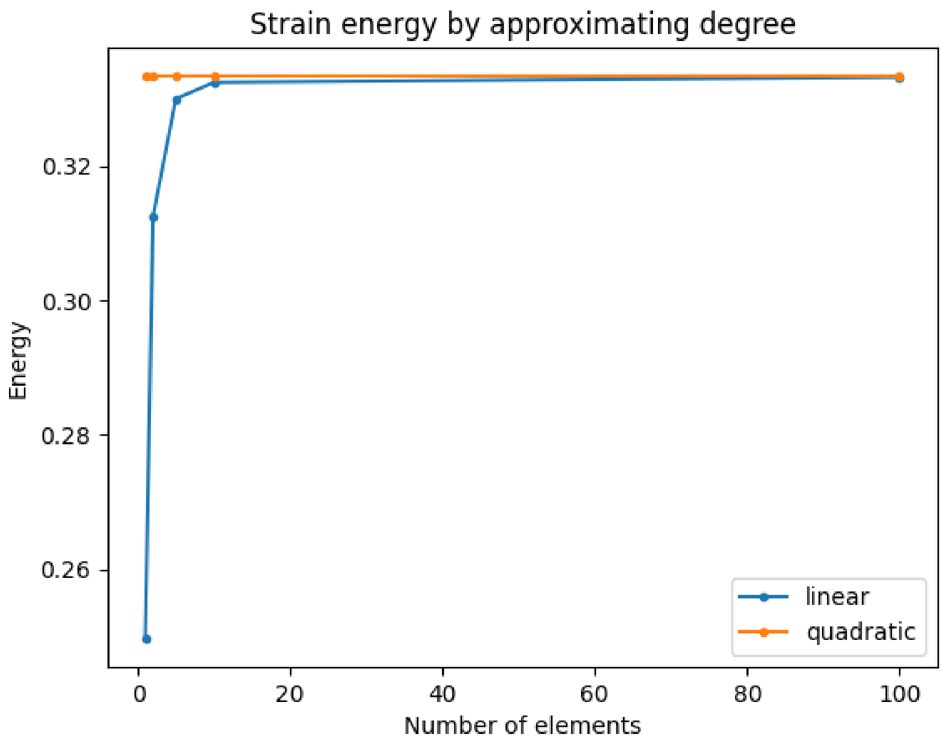
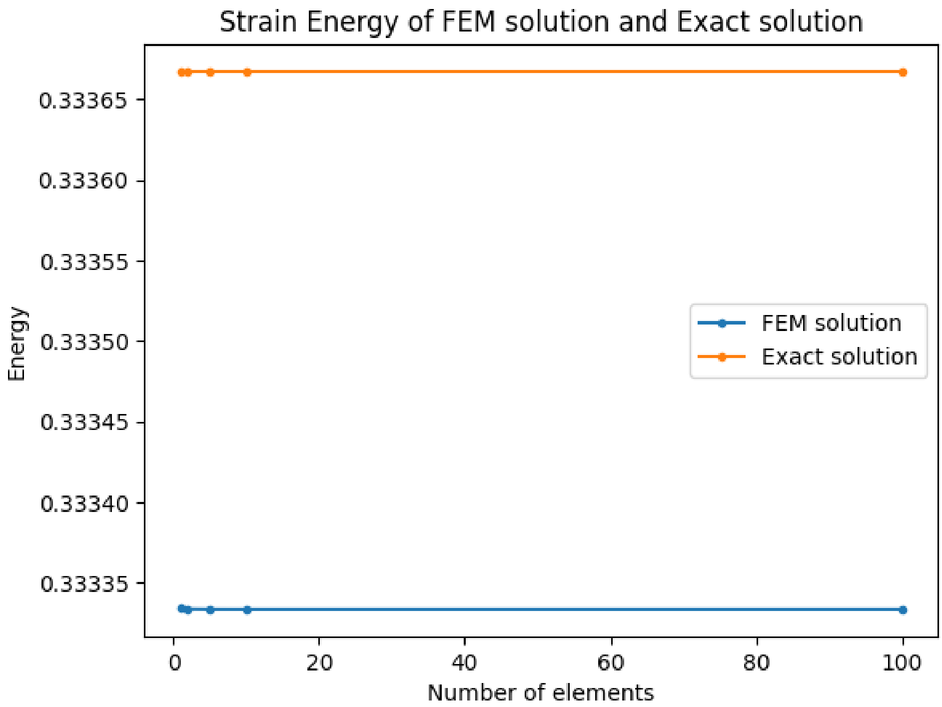


Figura 8: Quadratic Approximation


Strain energy by approximating degree
Energy
0.32
0.30
0.28
0.26
0
20
40
60
80
100
Number of elements
linear
quadratic


Strain Energy of FEM solution and Exact solution
Energy
0.33365
0.33360
0.33355
0.33350
0.33345
0.33340
0.33335
0
20
40
60
80
100
Number of elements
FEM solution
Exact solution

3. Take $AE(x) = 1$, $c(x) = 1$ and $T(x) = 1$ with $u(x)|_{x=0} = 0$ and $\left.\frac{du}{dx}\right|_{x=1} = 0$. Obtain the finite element solution with linear, quadratic, cubic and quartic elements respectively for 1, 10, 20, 40, 80 and 100 number of elements. For these cases:
   a) Plot the exact and finite element solutions together for these cases.
   b) Plot the error in the solution for these cases.
   c) Plot the strain energy of the finite element and exact solution as a function of number of elements.
   d) Plot the strain energy of the error as a function of number of elements.
   e) Plot the log of the relative error in the energy norm versus the log of number of elements.
   f) Try to estimate the convergence rate.

   Discuss the results.

The code solves the weak form equation using finite element analysis and gives us primary and secondary variables, which we can plot to analyze the accuracy of our model with varying numbers of elements starting from 1 and going all the way to 100. We can observe that the more the number of elements and the higher the order of approximation of function, the more accurate the result will be, as shown in the plots. Also, here we can edit boundary conditions and values of traction to achieve all the plots required for part 2 of the question.

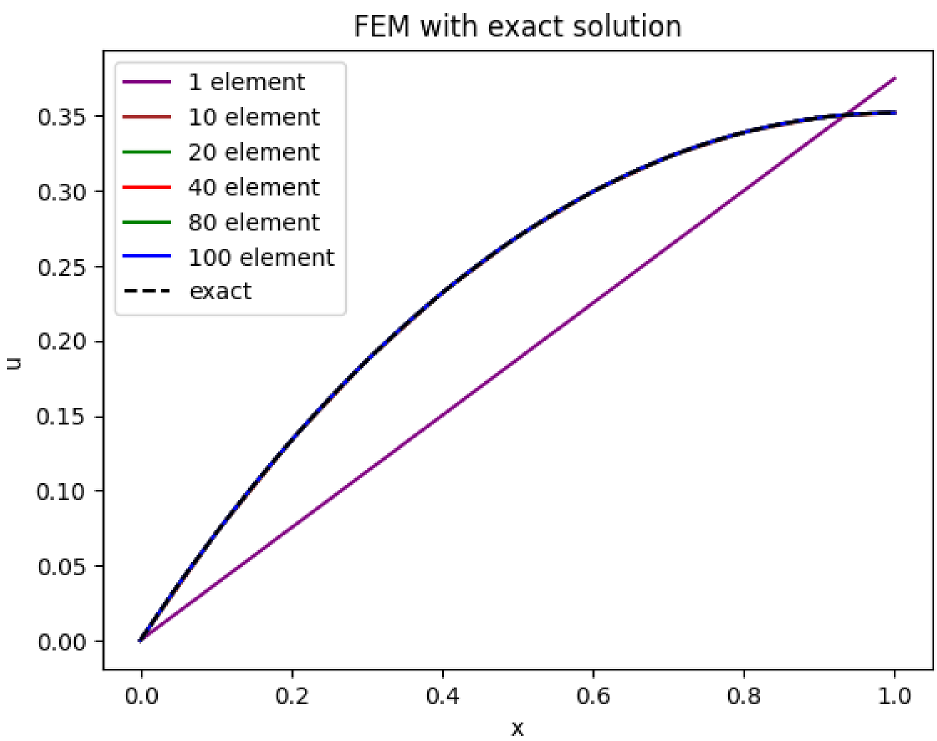


Figura 9: Linear Element

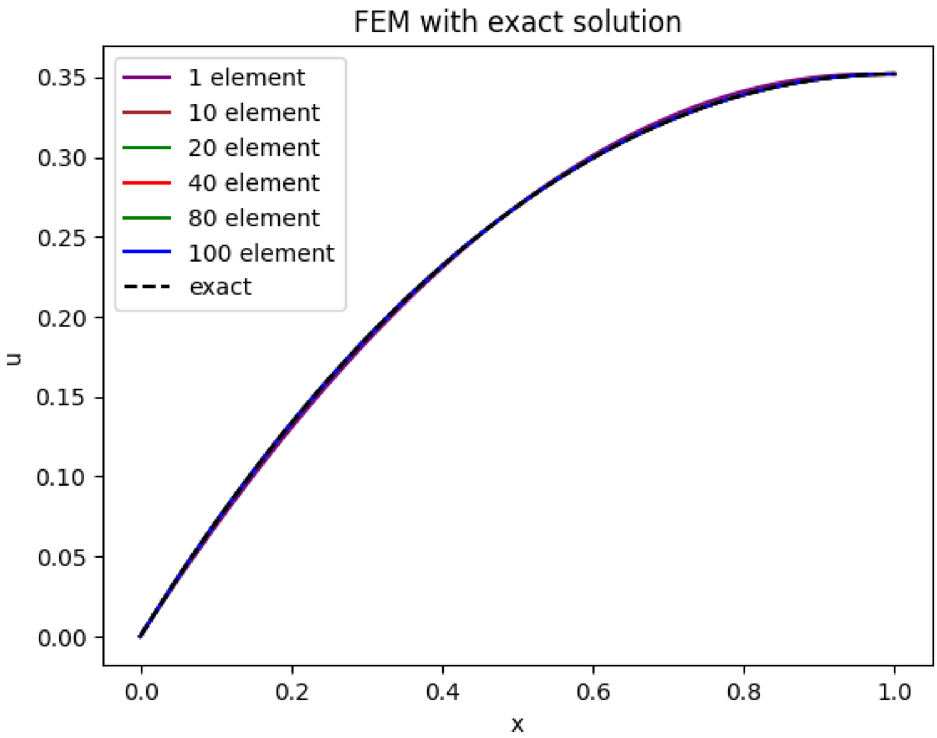


Figura 10: Quadratic Element

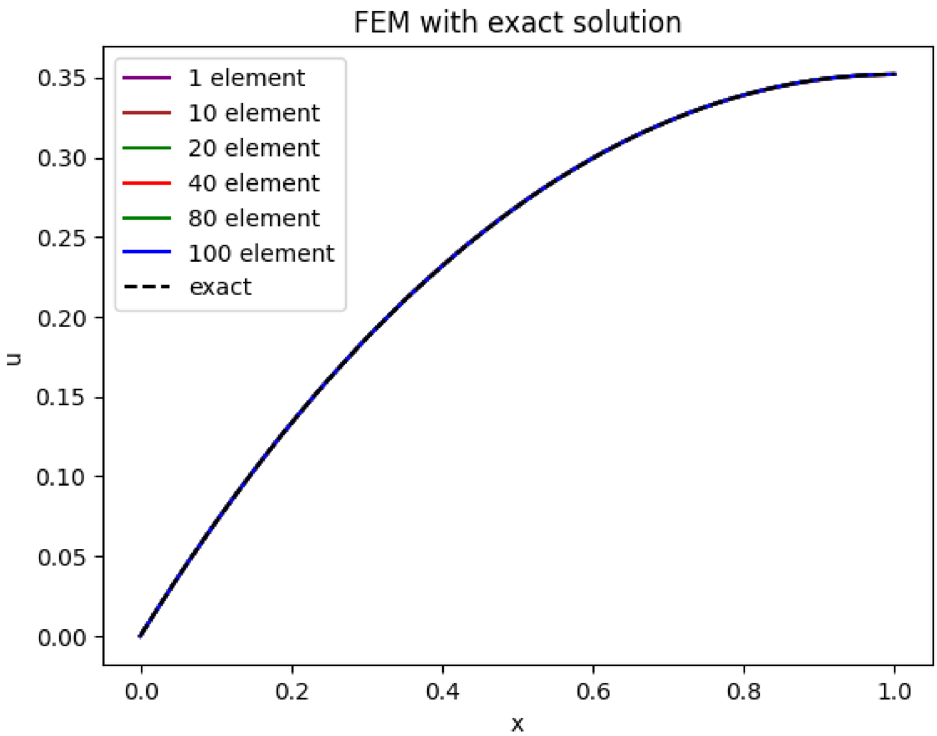


Figura 11: Cubic Element

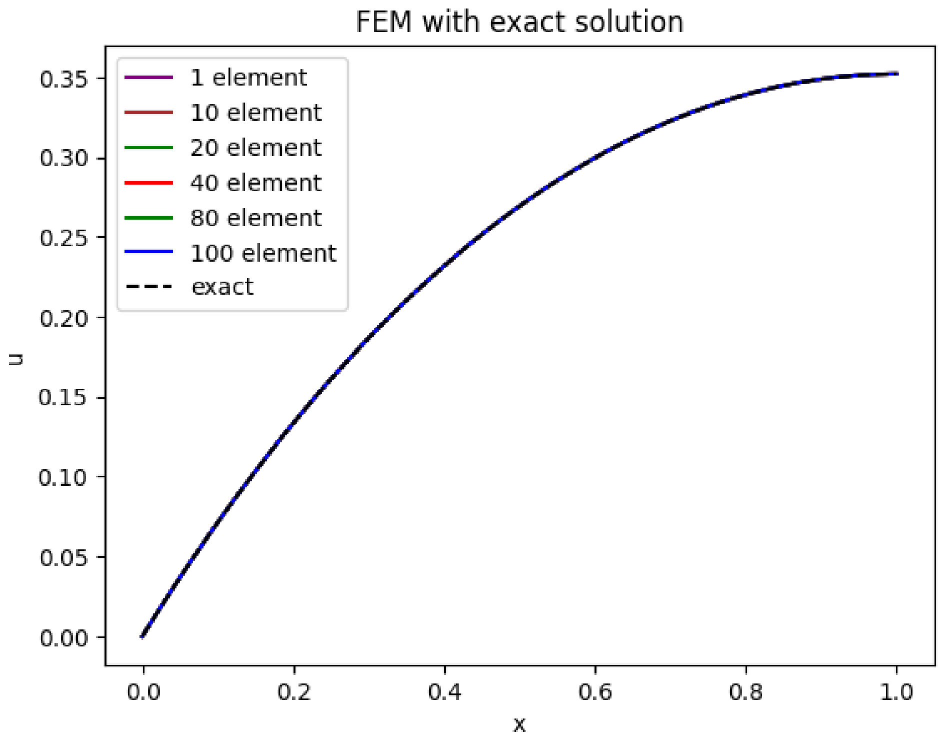


Figura 12: Quartic Element

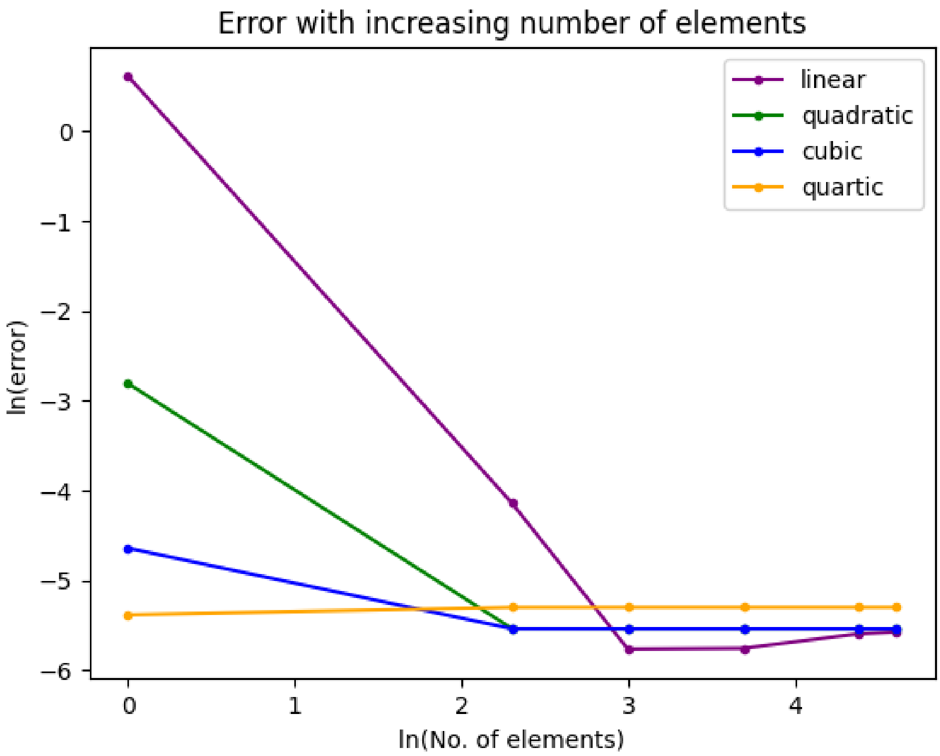


Figura 13: Error

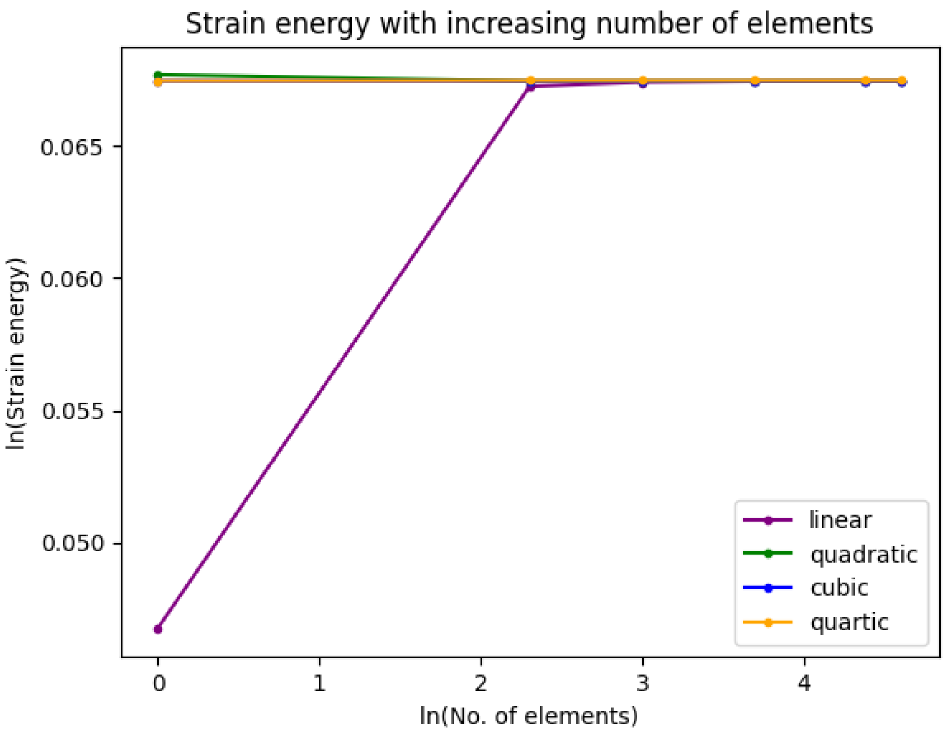


Figura 14: Strain Energy

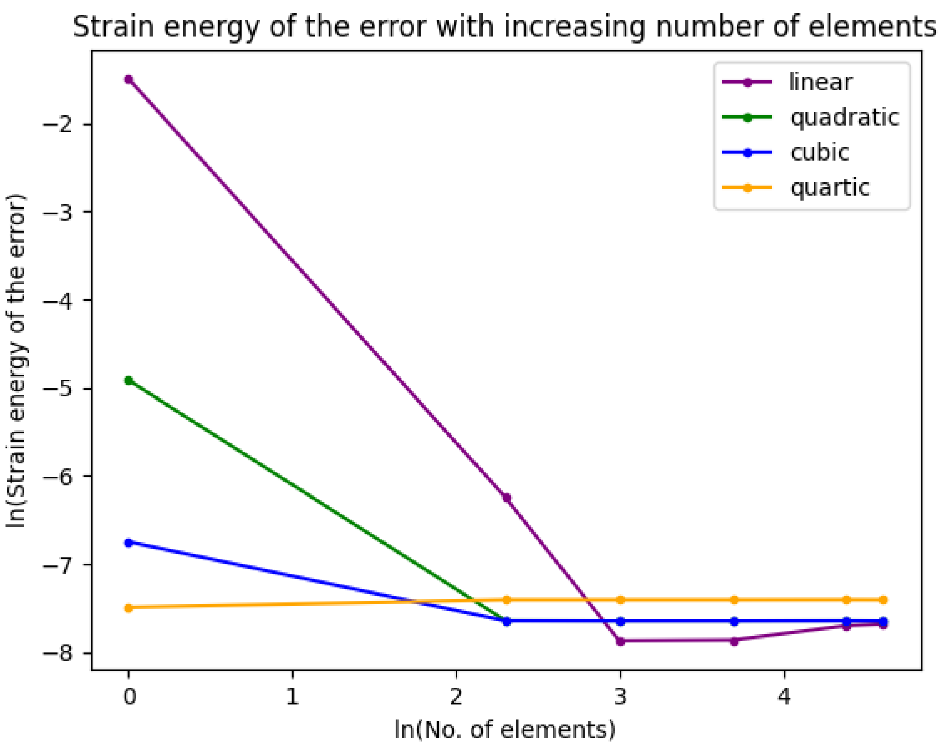


Figura 15: Strain Energy of the error

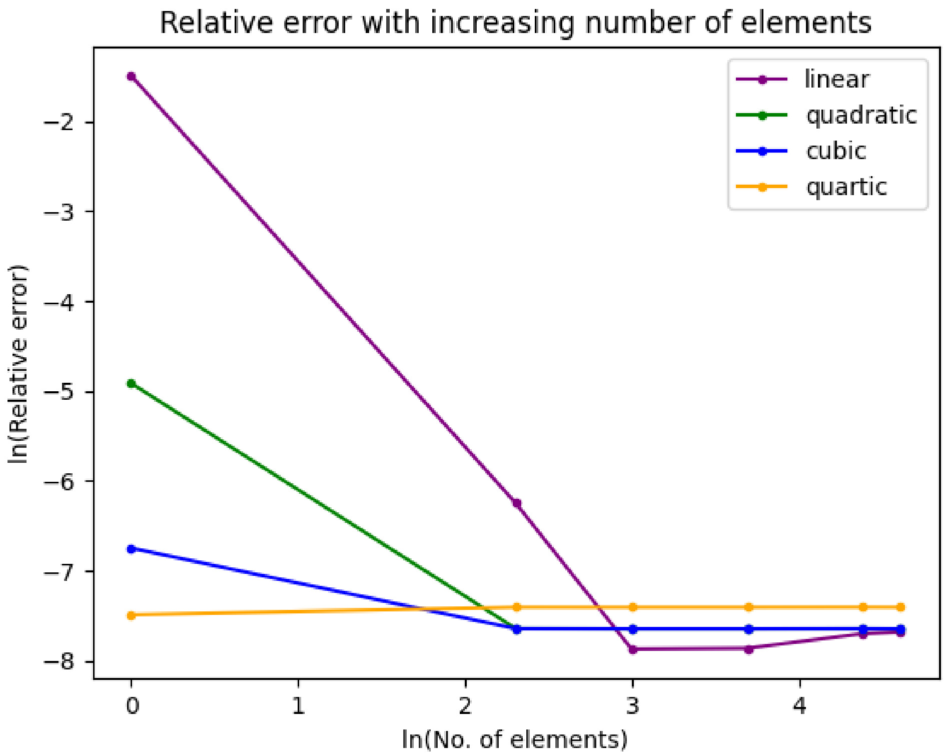


Figura 16: Relative Error

Inference from the above plots

As the discretization of the domain increases, the Finite Element Method (FEM) solution gradually converges towards the exact solution. This convergence is attributed to the finer mesh allowing for a more accurate representation of the exact solution. With a finer mesh, each element captures local variations more effectively, and as the mesh refinement progresses, these variations combine to yield a more faithful representation of the overall solution.

Furthermore, the choice of the order of approximation functions within each element significantly influences the accuracy of the FEM solution. Higher-order approximation functions are capable of better capturing complex variations in the solution. They excel in representing phenomena such as steep gradients or sharp changes in behavior. By increasing the order of approximation functions, the FEM can provide a closer approximation to the exact solution.

When employing quartic functions for approximation, the impact of the 6-point integration errors becomes more pronounced in the approximated solution. This is evident as we increase the number of elements in the discretization. The dominance of these errors underscores the importance of carefully managing integration schemes and considering higher-order approximations to improve the accuracy of the FEM solution.

4. Take $AE(x) = 1$, $c(x) = 0$ and $T(x) = \sin\frac{\pi}{L}x$ with $AE\frac{du}{dx}\Big|_{x=0} = \frac{1}{\pi}$ and $AE\frac{du}{dx}\Big|_{x=1} = k_L(\delta_L - u(L))$ with $k_L = 10$ and $\delta_L = 0$. Then repeat the exercise given a) through f) in Point 3.

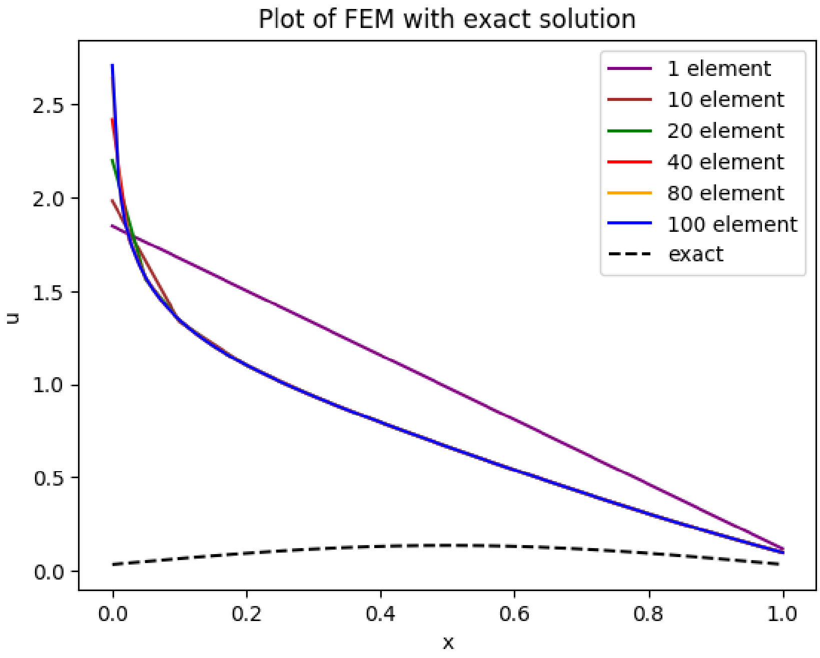


Figura 17: Linear Element

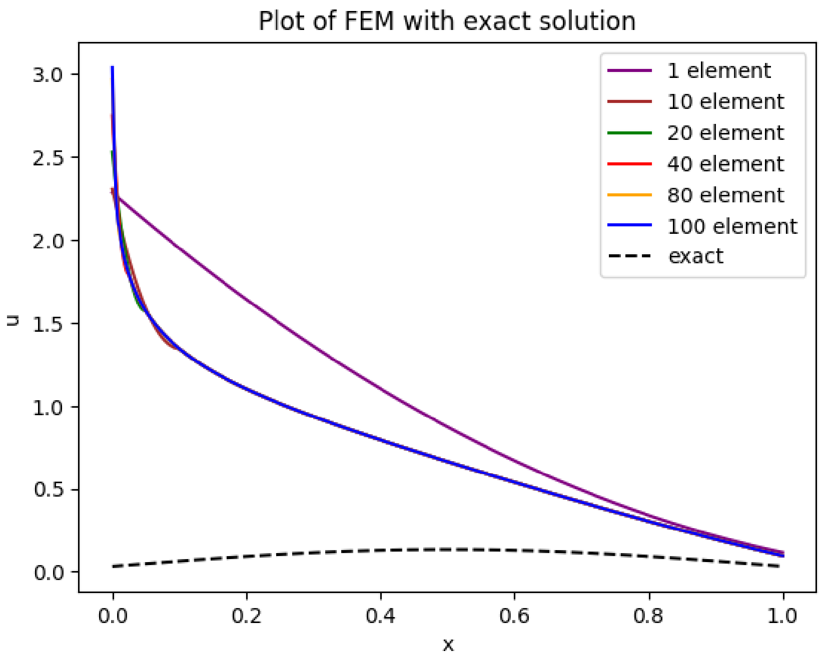


Figura 18: Quadratic Element

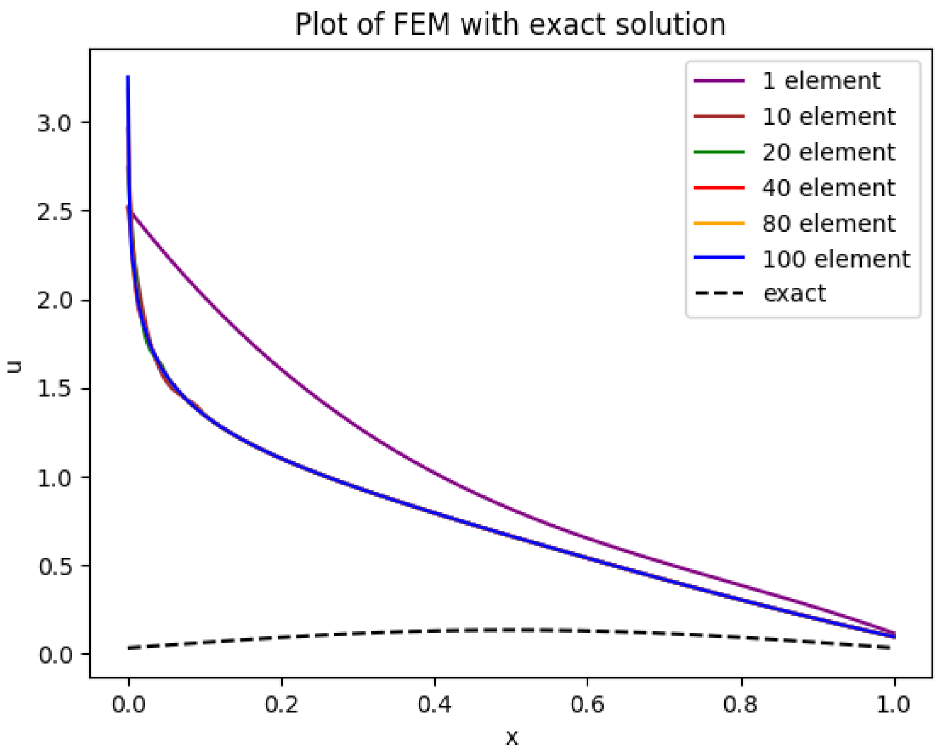


Figura 19: Cubic Element

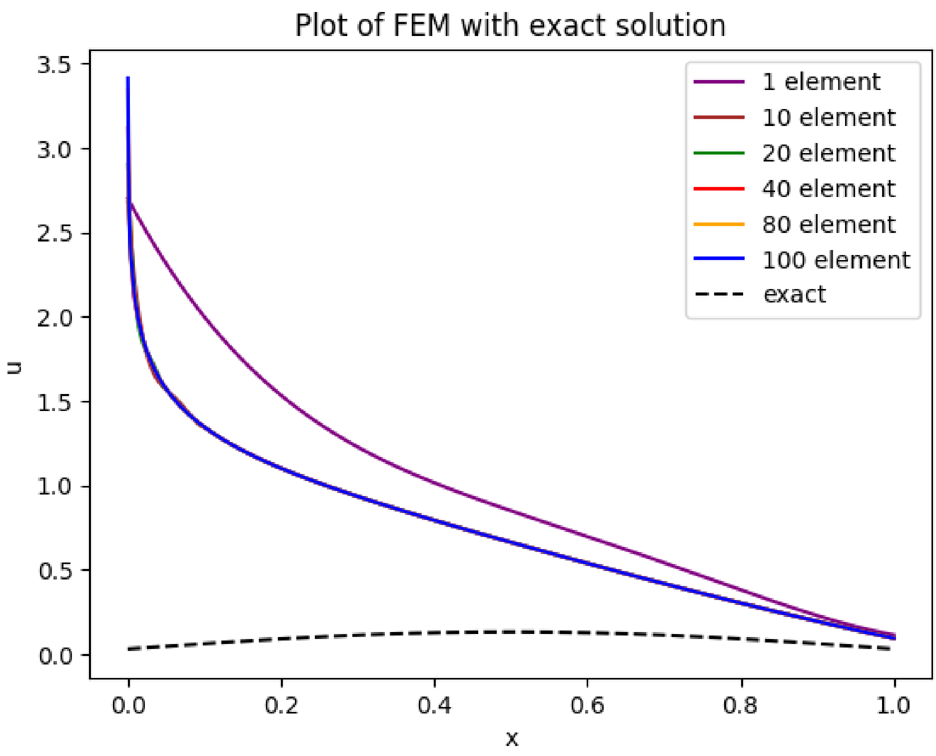


Figura 20: Quartic Element

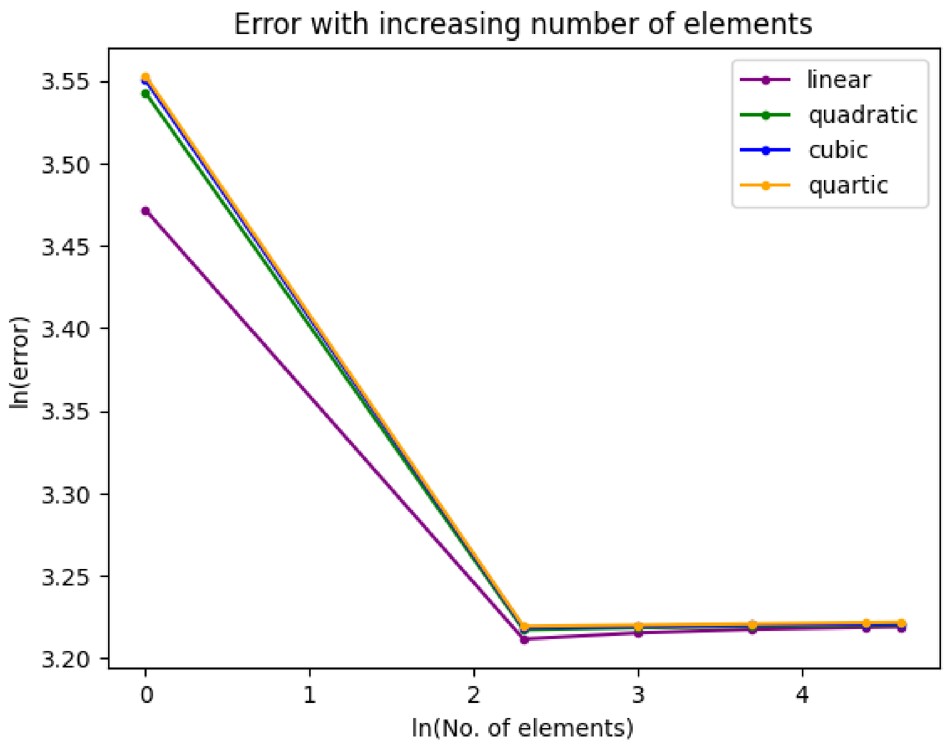


Figura 21: Error

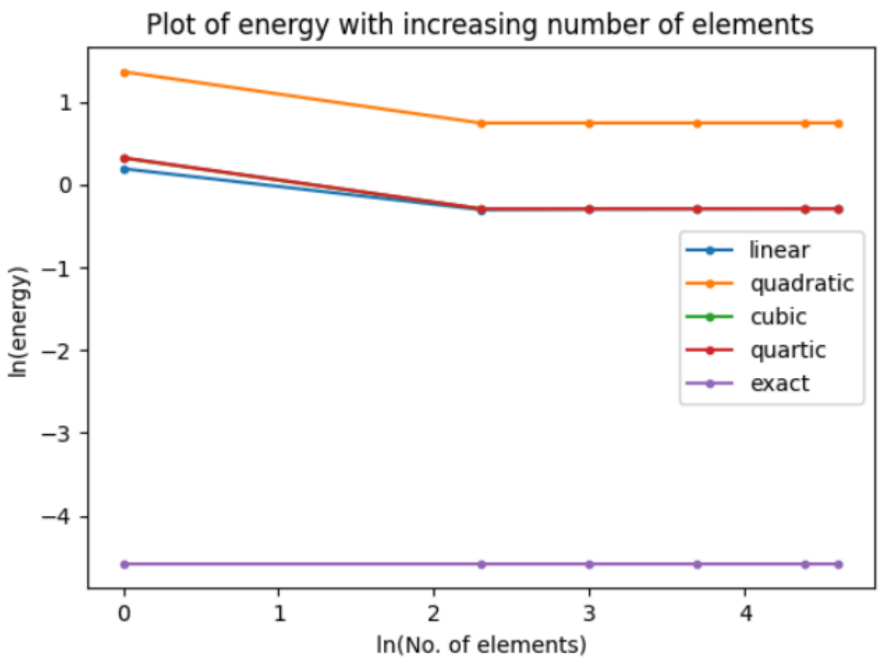


Figura 22: Strain Energy

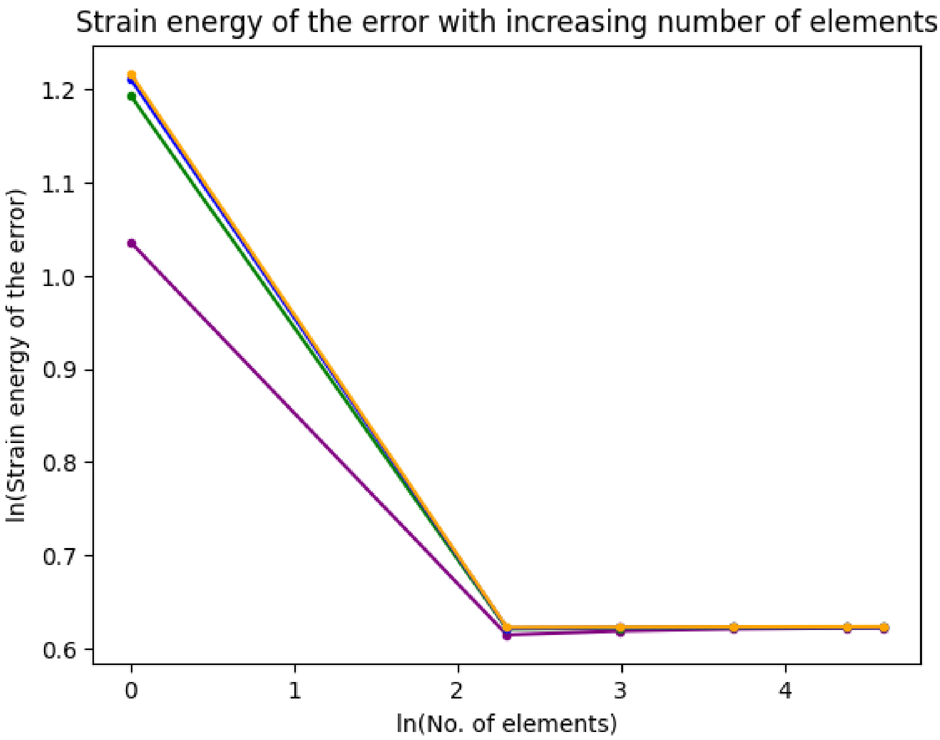


Figura 23: Strain Energy of the Error

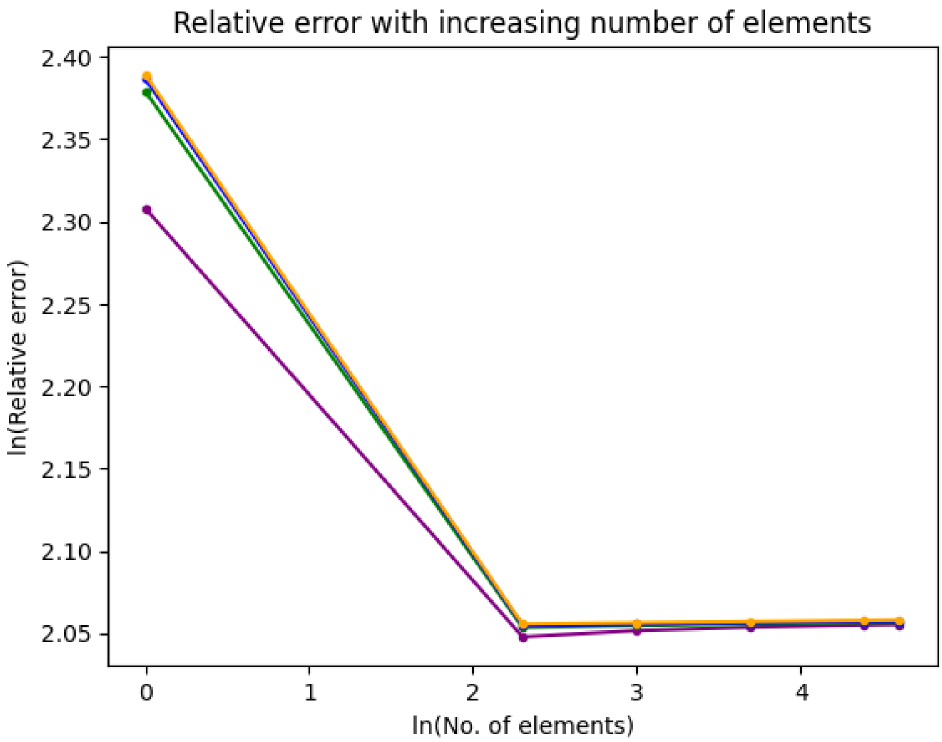


Figura 24: Relative Error

Inference from the above plots

Incorporating a forcing term as $\sin(\pi x)$ necessitates a Taylor series approximation extended up to at least the 9th power. Failing to extend the series adequately, such as truncating it at the 3rd power, results in significant discrepancies in matching the boundary conditions. This discrepancy stems from the inherent limitations of our code, which is not explicitly designed to handle sinusoidal forces. It's worth noting that employing sinusoidal approximating functions could potentially yield solutions closer to the exact solution.

By utilizing 9th-order polynomials for approximating the forcing term, the error in boundary conditions is notably diminished. However, due to our reliance on a 6-point integration scheme over the master element, errors escalate as we progress from linear to quartic approximations. Unfortunately, this error is inherent and cannot be entirely mitigated, thus leading to solutions that appear increasingly divergent from the exact solution.