Assignment 2
MA3.101: Linear Algebra (Spring 2019)
Submission Deadline: 8th March, 2019
Total Marks: 50

Instructions

1. Attempting all questions is mandatory.

2. Each question carries 5 marks.

Question 1

Let $F = \left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}$. Thus, F is the standard basis of $\mathbb{R}^{2\times 2}$. Let $B = \begin{pmatrix} -2 & 1 \\ 3 & 4 \end{pmatrix}$. Define $L : \mathbb{R}^{2\times 2} \to \mathbb{R}^{2\times 2}$ by $L(M) = BM$, $\forall M \in \mathbb{R}^{2\times 2}$. Find the matrix representation of L with respect to the standard basis F of $\mathbb{R}^{2\times 2}$.

Question 2

Let L be the linear transformation that rotates the plane through an angle of θ degrees. Let A be the matrix representation of L. Then matrix representation of L with respect to the standard basis of $\mathbb{R}^2$ is

$$A = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}$$

Find the matrix representations of $L^2, L^3, \ldots, L^n$. (Hint: What is L^2 geometrically?)

Question 3

Let V be an n-dimensional vector space over $\mathbb{R}$ and W be an m-dimensional vector space over $\mathbb{R}$ with $n > m$. Let $T : V \to W$ and $U : W \to V$ be 2 linear transformations. Prove that $UT : V \to V$ is not invertible.

Question 4

Let $V = \{\sum_{j=1}^{2}(a_j \cos jx + b_j \sin jx) : a_j, b_j \in \mathbb{R}\}$ be a vector space over $\mathbb{R}$ (x is some indeterminate). Define $L : V \to V$ by

$$L\left(\sum_{j=1}^{2}(a_j \cos jx + b_j \sin jx)\right) = \sum_{j=1}^{2}(-ja_j \sin jx + jb_j \cos jx)$$

1. Find a basis F for the vector space V (you have to *prove* that your chosen set is in fact a basis, don't just state).

2. Find the matrix representation A of L with respect to your basis.

Question 5

Let $A = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ and $B = \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix}$. Prove that matrices A and B are similar over the field of complex numbers.

Question 6

Let V be a finite-dimensional vector space and let T be a linear operator on V. Suppose that $rank(T^2) = rank(T)$. Prove that the range and null space of T intersect trivially, i.e., have only the zero vector in common.

Question 7

Let p, m and n be integers and $\mathbb{F}$ be a field. Let V be the vector space of $m \times n$ matrices over $\mathbb{F}$ and W be the space of $p \times n$ matrices over $\mathbb{F}$. Let B be a fixed $p \times m$ matrix and let T be the linear transformation from V into W defined by $T(A) = BA$. Prove that T is invertible if and only if $p = m$ and B is an invertible $m \times m$ matrix.

Question 8

Let V be an n-dimensional vector space over the field $\mathbb{F}$ and let T be a linear transformation from V into V such that the range and null space of T are identical. Prove that n is even. (Can you give an example of such a linear transformation T?)

Question 9

Show that the set $L(V, \mathbb{F})$ of all linear transformations from a vector space V over $\mathbb{F}$ to $\mathbb{F}$ (domain is V, codomain is $\mathbb{F}$) forms a vector space.

1. Show that the dimension of $L(V, \mathbb{F})$ is the same as that of V.
2. Consider that $V = \mathbb{F}^n$. Consider the map $T_i : (x_1, ..., x_n) \to x_i$. Show that $\{T_i : i = 1, .., n\}$ form a basis for $L(\mathbb{F}^n, \mathbb{F})$.

The vector space $L(V, \mathbb{F})$ is called the *dual space* of V.

Question 10

Let T be a linear operator over finite dimensional space V. A subspace W is called an T-invariant subspace if $T\boldsymbol{v} \in W$ for all $\boldsymbol{v} \in W$. Suppose $W_i, i = 1, .., r$ are T-invariant subspaces such that $V = \oplus_{i=1}^{r} W_i$. Let $dim(W_i) = n_i, \forall i$. Show that T has a matrix representation according to some chosen basis B for V, in the form

$$A = \begin{bmatrix} A_1 & 0 & \dots & 0 \\ 0 & A_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & A_r \end{bmatrix},$$

where A_i is a matrix of dimension $n_i \times n_i$.