INNER PRODUCT SPACES :-

(a) Inner Product in R^n :-

Let $\underline{X}, \underline{Y} \in R^n$

$(\underline{X}, \underline{Y}) = x_1 y_1 + x_2 y_2 + \text{———} + x_n y_n = \underline{Y}^T X$

Inner product

(b) Inner product in C^n :-

Let $\underline{X}, \underline{Y} \in C^n$

then $(\underline{X}, \underline{Y}) = x_1 \bar{y}_1 + x_2 \bar{y}_2 + \text{———} + x_n \bar{y}_n = \underline{Y}^* \underline{X}$

where $\bar{y}_i$ is complex conjugate of y_i

(c) **Norm in R^n** :- Length of a vector

Let $X \in R^n$

$\|X\| = \sqrt{x_1^2 + \text{———} + x_n^2} = \sqrt{(\underline{X}, \underline{X})}$

(d) **Norm in C^n** :-

Let $\underline{X} \in C^n$, then

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$$\|\underline{X}\| = \sqrt{|x_1|^2 + \_\_\_\_ + |x_n|^2}$$

$$= \sqrt{x_1 \bar{x}_1 + \_\_\_\_ + x_n \bar{x}_n}$$

$$= \sqrt{(\underline{X}, \underline{X})}$$

let $\underline{X} = \begin{bmatrix} x_1 \\ x_2 \\ | \\ | \\ x_n \end{bmatrix} = \begin{bmatrix} a_1 + ib_1 \\ | \\ | \\ a_n + ib_n \end{bmatrix}$

than

$$\|\underline{X}\| = \sqrt{a_1^2 + b_1^2 + a_2^2 + b_2^2 + \_\_\_\_ + a_n^2 + b_n^2}$$

$$\|\underline{X}\| = \sqrt{\sum_{k=1}^{n} (a_k^2 + b_k^2)}$$

* Hermition ~~Matrix~~ Adjoint of a Matrix A:-

$A^* = (\bar{A})^T$ (Conjugate + Transpose)

Properties of Inner Product :-

(1) (Conjugate) Symmetry

$$(\underline{X}, \underline{Y}) = \overline{(\underline{Y}, \underline{X})}$$

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$$(\underline{Y}, \underline{X}) = \{ y_1 \bar{x}_1 + y_2 \bar{x}_2 + \cancel{x_1 x} \text{———} + y_n \bar{x}_n$$

$$\overline{(\underline{Y}, \underline{X})} = \bar{y}_1 x_1 + x_2 \bar{y}_2 + \text{———} + x_n \bar{y}_n$$

(2) Linearity :- $(\alpha \underline{x} + \beta \underline{Y}, \underline{z})$

$$= \alpha(\underline{x}, \underline{z}) + \beta(\underline{Y}, \underline{z})$$

$\forall\ \underline{x}, \underline{y}, \underline{z} \in V$ and

$\alpha, \beta \in f$

(3) Non-~~Linearity~~ Negative :-

$$(\underline{X}, \underline{X}) \geq 0 \quad \forall\ X \in V$$

(4) Non-degeneracy

$(\underline{X}, \underline{X}) = 0$ if and only if $\underline{X} = 0$

Example :- Let $V = P_n(C)$

Let $f, g \in V$

$$(f, g) = \int_0 f(t)\ \overline{g(t)}\ dt$$

Example 3 Let $V = F^{m \times n}$

Let $A, B \in V$

FROBENIUS INNER PRODUCT :-

$$(A, B) = \text{trace}(B^* A)$$

$$= \text{trace}(\bar{B}^T A)$$

$$= \sum_{k=1}^{n} (\bar{B}^T A)_{k,k}$$

$$= \sum_{k=1}^{n} \sum_{i=1}^{m} \bar{B}^T_{ki} A_{ik}$$

$$= \sum_{k=1}^{n} \sum_{i=1}^{m} \bar{B}_{ik} A_{ik}$$

HW :- Verify that Frob. Inner product satisfies all 4 prop. of inner product.

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One more property which can be derived from 1^{st} and 2^{nd}

(5) $\left(\underline{X}, \alpha \underline{Y} + \beta \underline{Z}\right) = \bar{\alpha}\left(\underline{X}, \underline{Y}\right) + \bar{\beta}\left(\underline{X}, \underline{Z}\right.$

(2) $\left(\alpha \underline{Y} + \beta \underline{Z}, \underline{X}\right) = \alpha\left(\underline{Y}, \underline{X}\right) + \beta\left(\underline{Z}, \underline{X}\right.$

Proof

$\left(\underline{X}, \alpha \underline{Y} + \beta \underline{Z}\right) = \overline{\left(\alpha \underline{Y} + \beta \underline{Z}, \underline{X}\right)}$ (By I pr

$\Rightarrow \bar{\alpha}\,\overline{(\underline{Y},\underline{X})} + \bar{\beta}\,\overline{(\underline{Z},\underline{X})}$

$\Rightarrow \bar{\alpha}(\underline{X},\underline{Y}) + \bar{\beta}(\underline{X},\underline{Z})$.

Hence proved

$\underline{X},\underline{Y},\underline{Z} \in V$

$\alpha,\beta \in F$

LEMMA (1):- Let $\underline{X} \in V$, then $\underline{X} = \underline{0}$ iff $(\underline{X},\underline{Y}) = 0 \ \forall \ \underline{Y} \in V$

(1) $X = 0$

$(\underline{0},\underline{Y}) = 0$

(2) let $(\underline{X},\underline{Y}) = 0$

As, $\underline{Y} \in V$

So, take $\underline{Y} = \underline{X}$

$\Rightarrow (\underline{X},\underline{X}) = 0 \Rightarrow \underline{X} = 0$

IMPLICATIONS OF LEMMA 1:-

(a) let $\underline{X}, \underline{Y} \in V$, then $(\underline{X},\underline{Z}) = (\underline{Y},\underline{Z}) \ \forall \underline{Z} \in$

iff $\underline{X} = \underline{Y}$

As, $(\underline{X}, \underline{Z}) = (\underline{Y}, \underline{Z})$

$\Rightarrow (\underline{X} - \underline{Y}, \underline{Z}) = 0$ Using

$\Rightarrow \cancel{(\alpha \underline{X} + \beta \underline{Y}, \underline{Z}) = \alpha(\underline{X}, \underline{Z})}$

$\Rightarrow \cancel{(\underline{X}, \underline{Z})} \Rightarrow \underline{X} - \underline{Y} = 0$ from

$\Rightarrow \underline{X} = \underline{Y}$

Let $A, B \in L(U, V)$

$(A\underline{X}, \underline{Y}) = (B\underline{X}, \underline{Y}) \quad \forall \underline{X} \in U$
$\forall \underline{Y} \in V$

iff $\underline{A} = \underline{B}$

$(A\underline{X}, \underline{Y}) = (B\underline{X}, \underline{Y})$

$\Rightarrow ((A-B)\underline{X}, \underline{Y}) = 0$

$\Rightarrow (A-B)\underline{X} = 0 \quad \forall X \in \underline{U}$

So, $A - B = 0$

$A = B$

Cauchy - Scwarz Inequality :-

Let V be a vector space with inner product $(\underline{X}, \underline{Y})$ $\underline{X}, \underline{Y} \in V$

$$(\underline{X}, \underline{Y}) \leq ||X|| \cdot ||Y||$$

Proof :-

(i) If any one of the $\underline{X}$ and $\underline{Y}$ is a zero vector, than the results hold trivially.

as, $(X, Y) = 0$

$||X|| \cdot ||Y|| = 0$

(ii) Consider the vector $(\underline{X} - t\underline{Y})$

$$\Rightarrow \underbrace{(\underline{X} - t\underline{Y}, \underline{X} - t\underline{Y})}_{} \geq 0$$

β β'

$$\Rightarrow (X, (X - tY)) - (tY, (X - tY)) \geq 0$$

By 5^{th} prop.

$$\Rightarrow \cancel{\bar{t}(X, Y)} \; (X, X) - \bar{t}(X, Y) - t(Y, X) + t\bar{t}(Y, Y)$$

$$||X||^2 + |t|^2 ||Y||^2 - \bar{t}(\underline{X}, \underline{Y}) - t(\underline{Y}, \underline{X}) \geq 0$$

$\Rightarrow$ If this Eq^n is diff. and equated to zero to get $t = \dfrac{(\underline{X}, \underline{Y})}{||Y||^2}$

then

$$(\underline{x} - t\underline{y},\ \underline{x} - t\underline{y})$$

$$= \|x\|^2 + \frac{|(x,y)|^2}{\|\underline{y}\|^2} - \frac{(x,y)\overline{(x,y)}}{\|y\|^2}$$

$$- \frac{\overline{(x,y)}(x,y)}{\|y\|^2}$$

$$= \|\underline{x}\|^2 - \frac{|(\underline{x},\underline{y})|^2}{\|y\|^2} \geq 0$$

$$\Rightarrow |(\underline{x},\underline{y})|^2 \leq \|\underline{x}\|^2\|y\|^2$$

$$\Rightarrow |(x,y)| \leq \|\underline{x}\|\|\underline{y}\|$$

OTHER EASY PROOF :-

$$|\cos\theta| = \left|\frac{(\underline{x},\underline{y})}{\|\underline{x}\|.\|\underline{y}\|}\right| \leq 1$$

$$\Rightarrow |(\underline{x},\underline{y})| \leq \|\underline{x}\|.\|\underline{y}\|$$

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(4) $\langle c\underline{\alpha}_1 + d\underline{\alpha}_2, \underline{\beta} \rangle = c\langle \alpha_1, \beta\rangle + d\langle \alpha_2, \beta\rangle$

$$\langle \alpha, c\beta_1 + d\beta_2 \rangle = \overline{(c\beta_1 + d\beta_2, \alpha)}$$
$$= \bar{c}\,\overline{(\beta_1, \alpha)} + \bar{d}\,\overline{(\beta_2, \alpha)}$$
$$= \bar{c}(\alpha, \beta_1) + \bar{d}(\alpha, \beta_2)$$

$$\|\alpha\| = \sqrt{(\alpha \cdot \alpha)}$$

Cauchy Schwartz

$$|\langle \alpha, \beta \rangle| \leq \|\alpha\| \, \|\beta\|$$

Triangular Inequality

$$\|\alpha + \beta\| \leq \|\alpha\| + \|\beta\|$$ → Do It: PROOF

ORTHOGONAL PROJECTION Of Some $\underline{v} \in V$ a subspace W

→ ~~Orthogonality~~ $(v \perp w)$ perpendicular

v is said to be orthogonal to w if $\langle v, w \rangle = 0$

Orthogonal Bunch

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Lemma :-

If $\{v_1 - - - v_n\}$ are mutually orthogonal

Than $v_1 - - - v_n$ are linear Independent

C. D. Its Proof

Proof :- let they are l. Dependent

$$v_j = \sum_{\substack{i=1 \\ i \neq j}}^{n} c_i v_i$$

Take inner product with v_j

$$\langle v_j, v_j \rangle = \sum_{\substack{i=1 \\ i \neq j}}^{n} c_i \cancel{\langle v_i, v_j \rangle}$$

$= 0$

$v_j = 0$

Show that

$\langle v, 0 \rangle = 0 \quad \forall v$

Proof → $\langle v, 0\rangle = \langle v, v - v\rangle$

Apply linearity

$1\langle v, v\rangle - 1\langle v, v\rangle = 0$

(*) Suppose $\dim(V) = n$

Than say set of n mutually ortho. non-zero vectors form a basis

ORTHOGONAL PROJECT^n OF some $\underline{v} \subset V$ on a subspace W

$$= \frac{\langle v, w\rangle}{\langle w, w\rangle} \underline{w} \in \operatorname{span}(\underline{w})$$

Def^n :- Orthogonal projection of v on W is defined as a vector w s.t

(i) $w \in W$

(ii) $(v - w) \perp W$ $\left((v - w) \perp w' \text{ for } \forall w' \in W \right)$

$\Rightarrow \underline{v} = (\underline{v} - \underline{w}) + \underline{w}$

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Suppose $n = \dim(V)$ & $\{v_1 --- v_n\}$

is an orthogonal basis

for any $v \in V$

$$\underline{v} = \sum_{i=1}^{n} c_i v_i$$

We are interested in finding out c_i

So, we gonna do inner product with v_j on both sides

$$\langle v . v_j \rangle = c_j \langle v_j , v_j \rangle$$

So, $c_j = \dfrac{\langle v, v_j \rangle}{\langle v_j , v_j \rangle}$

Theorem :- Any basis $\{v_1 --- v_n\}$ of V can be used to obtain orthogonal basis $\{w_1 -- w_n\}$ s.t

$$\text{span}(\{v_1 ----- v_i\}) = \text{span}(\{w_1 -- w_i\}) \quad \forall\, i = 1 -- n$$

Proof :- let $\underline{w}_1 = v_1$

$w_2 = v_2 -$ proj of v_2 on span$(\{w_1\})$

NEXT ON ENGLISH PAGE

* $w_2 \neq 0$

as proj of v_2 on span (w_1) = span (v_1)

Can't be v_2 if it happens

v_2 and v_1 lie on same line

but they are l.i, not possible

$$\text{span}(\{w_1, w_2\}) = \text{span}(\{v_1, v_2\})$$

* $w_3 = v_3 -$ proj of v_3 on span $(\{w_1, w_2\})$

sim — — — — do this you get orthogonal basis

* As $w_2 = v_2 - \underbrace{\text{proj of } v_2 \text{ on span}(\{w_1\})}_{\alpha w_1}$

$w_2 = v_2 - \alpha w_1$ It is orthogonal to span(w_1)

Computing the proj. of $\underline{v}$ on W

5TH SEPT. 2018

ABATE → TO REDUCE

Pick an orthogonal basis for W

$= \{w_1 \text{ --- } w_n\}$ then if

If $\underline{w}$ is th proj. of v on W $(w_1 - - w_n)$

(1) $\underline{w} \in \text{Span}(w_1 - - w_n)$

(2) $(\underline{v} - \underline{w}) \perp \underline{w}$

$$w = \sum_{i=1}^{r} \frac{\langle v, w_i \rangle}{\langle w_i, w_i \rangle} w_i$$

Check (1) & (2) are satisfied

Orthonormal bases

$\{w_1 \text{ --- } w_n\}$

(i) $\langle w_i, w_j \rangle = 0$ iff $i \neq j$

$\|w_i\| = \sqrt{\langle w_i, w_i \rangle} = 1$

~~let~~ $\sum_{i=1}^{m} d_i = n$ (Becau. of ass. (b) is true)

Thus

$V = E(\lambda_1, T) + \_\_ \_\_ + E(\lambda_m, T)$

now, we have to show that it is infact a direct sum

let $\beta_i \in E(\lambda_i, T)$

Then $\bar{\beta}_1 + \_\_ \_\_ + \bar{\beta}_m = \bar{0}$

implies $\beta_i = \bar{0}$

because $\beta_1 - - \beta_m$ are lin. ind. (as they are eigenvectors corresponding to distinct eigen values)

hence $V = E(\lambda_1, T) \oplus \_\_ \oplus E(\lambda_m, T)$

(d) $\Rightarrow$ (e) (Exercise)

(c) $\Rightarrow$ (b)

$\dim V = \dim \mathcal{E}(\lambda_1, T) + \_\_\_ + \dim \mathcal{E}(\lambda_m, T)$

Let $\dim \mathcal{E}(\lambda_i, T) = d_i$

$\sum_{i=1}^{m} d_i = n$

Let $(\bar{\alpha}_{i1} - - - \bar{\alpha}_{id_i}) \rightarrow B_i$ is a bases of $\mathcal{E}(\lambda_i, T)$

$T \bar{\alpha}_{ij} = d_i \bar{\alpha}_{ij}$

$i = 1, - - - m$

$j = 1 - - - d_i$

we have to show that

$B_1 \cup B_2 \_\_\_ \cup B_m$

form a bases of V

(i) Show they are lin. ind.

$\sum_{j=1}^{d_i} c_{ij} \alpha_{ij} + \_\_\_\_ + \sum_{j=1}^{d_m} c_{mj} \alpha_{mj} = \bar{0}$ (1)

Let $\gamma_i = \sum_{l=1}^{T} c_{il} \alpha_{il}$

ε_e — (1) — —

$\gamma_1 + \text{———} \text{———} + \gamma_m = 0$

As, γ_1 — — γ_m are linearly ind.

$\gamma_1 = \text{———} \gamma_m = 0$

Here $\gamma_i \in \mathcal{E}(\lambda_i, T)$ which is a subspace

So, γ_i can be 0. (γ_i → not a Eigen vector)

Thus $\sum_{l=1}^{d_i} c_{il} \alpha_{il} = 0$

$c_{il} = 0 \ \forall i = 1 \ldots$

$\Rightarrow c_{il} = 0 \ \forall i = 1 \text{——} k$

$l = 1 \text{——} d_i$

Let $V = \mathbb{C}^{n\times 1}$ (or $\mathbb{R}^{n\times 1}$)

Std. inner product $\langle \underline{\alpha}, \underline{\beta} \rangle = \sum_{i=1}^{n} \alpha_i \bar{\beta}_i$

$$\underline{\alpha} = \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix}, \quad \underline{\beta} = \begin{bmatrix} \beta_1 \\ \vdots \\ \beta_n \end{bmatrix}$$

$$\sum_{i=1}^{n} \alpha_i \bar{\beta}_i = \begin{bmatrix} \bar{\beta}_1 & \cdots & \bar{\beta}_n \end{bmatrix} \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_n \end{bmatrix}$$

Std. inner product

$\langle \underline{\alpha}, \underline{\beta} \rangle = \underline{\beta}^H \underline{\alpha}$

$= \bar{\underline{\beta}}^T \underline{\alpha}$

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$= \beta^H \alpha$ (Hermitian (adjoint))

Suppose $\{u_1 \text{———} u_n\}$ is an orthogo. basis

$\Rightarrow u_i^H \cdot u_j = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases}$

$$\Rightarrow \underbrace{\begin{bmatrix} u_1^H \\ \vdots \\ u_n^H \end{bmatrix}}_{U^H} \underbrace{\begin{bmatrix} u_1 & \text{———} & u_n \end{bmatrix}}_{U} = I$$

$\Rightarrow \underline{U^H U = I}$ → U is called unitary Matrix (square)

⇒ HERMITIAN MATRIX OR SELF-ADJOINT Matrix

A matrix A is said to be Hermitian if $A = A^H$ (for over $\mathbb{R}$ they are also symmetric)

Symmetric matrices over $\mathbb{R}$ are not necessary Hermitian over $\mathbb{C}$

LEMMA :-

The eigen values of Hermitian matrices are real

Consider $\langle \, , \, \rangle$ std inner product

$$\langle \underline{x}, Ay \rangle = (Ay)^H \cdot x_{n\times 1}$$

$$= y^H_{1\times n} A^H_{n\times n} \cdot x_{n\times 1}$$

$$= \langle A^H x, y \rangle \quad \text{--- (1)}$$

Given $A = A^H$ Let λ be ~~the~~ an eigen value (x be an eigen vector of)

LHS

$$\langle \underline{x}, A\underline{x} \rangle = \langle \underline{x}, \lambda \underline{x} \rangle$$

$$= \bar{\lambda} \langle \underline{x}, \underline{x} \rangle = \bar{\lambda} \|x\|^2$$

BY (1)

LHS

$$= \langle A^H x, \underline{x} \rangle = \langle Ax, \underline{x} \rangle = \langle \lambda \underline{x}, x \rangle$$

$$= \lambda \langle \underline{x}, \underline{x} \rangle$$

$$= (\lambda) \|x\|^2$$

So, $(\lambda - \bar{\lambda}) \|x\|^2 = 0$

$\Rightarrow$ $\lambda = \bar{\lambda}$ since x is an eigen vector $\neq 0$

Theorem:- Any Hermitian matrix is diagonalizable using same orthonormal basis.

In other words if $A = A^H$,

$$A = U^H D U$$