Assignment 1
MA3.101: Linear Algebra (Spring 2019)
Submission Deadline: 2nd Feb, 2019
Total Marks: 50

January 26, 2019

Question 1

Let $\mathbb{R}^\infty$ denote the vector space of all sequences of real numbers. (Addition and scalar multiplication are defined coordinate-wise.) In each of the following, a subset of $\mathbb{R}^\infty$ is described. Verify whether the set is a subspace of $\mathbb{R}^\infty$ or not.

1. Sequences that are absolutely summable. (A sequence (x_k) is absolutely summable if $\sum_{k=1}^{\infty} |x_k| < \infty$).
2. Bounded sequences. (A sequence (x_k) is bounded if there is a positive number M such that $|x_k| \leq M$ for every k).
3. Arithmetic progressions. (A sequence (x_k) is arithmetic if it is of the form $(a, a+k, a+2k, a+3k, \ldots)$ for some constant k).
4. Geometric progressions. (A sequence (x_k) is geometric if it is of the form $(a, ka, k^2a, k^3a, \ldots)$ for some constant k).

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Question 2

Let V be the set of real numbers. Regard V as a vector space over the field of rational numbers, with the usual operations. Prove that this vector space is not finite dimensional.

Question 3

In the space $C[0,1]$ define the vectors f, g, and h by $f(x) = x$, $g(x) = e^x$ and $h(x) = e^{-x}$ for $0 \leq x \leq 1$. Use the definition of linear independence to show that the functions f, g, and h are linearly independent.

Question 4

Let V be the vector space of all 2×2 matrices over $\mathbb{R}$. Let W_1 be the set of matrices of the form

$$\begin{pmatrix} x & -x \\ y & z \end{pmatrix}$$

and let W_2 be the set of matrices of the form

$$\begin{pmatrix} a & b \\ -a & c \end{pmatrix}$$

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1. Prove that W_1 and W_2 are subspaces of V.
2. Find the dimensions of W_1, W_2, $W_1 + W_2$ and $W_1 \cap W_2$. Also exhibit a basis for each of them.

Question 5

Let W_1 and W_2 be subspaces of vector space V such that $W_1 + W_2 = V$ and $W_1 \cap W_2 = \{\mathbf{0}\}$. Prove that for each vector $\mathbf{a} \in V$, there are unique vector $\mathbf{a}_1 \in W_1$ and $\mathbf{a}_2 \in W_2$ such that $\mathbf{a} = \mathbf{a}_1 + \mathbf{a}_2$.

Question 6

Suppose that W_1, W_2, W_3 are subspaces of a vector space.

1. Is it always true that $W_1 \cap (W_2 + W_3) = W_1 \cap W_2 + W_1 \cap W_3$? Prove the statement, or disprove using a counterexample.
2. Prove that $W_1 \cap (W_2 + (W_1 \cap W_3)) = (W_1 \cap W_2) + (W_1 \cap W_3)$.

Question 7

1. Under what conditions on the scalar x are the vectors $(1 + x, 1 - x)$ and $(1 - x, 1 + x)$ in $\mathbb{C}^2$ linearly dependent ?
2. Under what conditions on the scalar x are the vectors $(x, 1, 0), (1, x, 1)$, and $(0, 1, x)$ in $\mathbb{R}^3$ linearly dependent?
3. What is the answer to (2) for $\mathbb{Q}^3$ in the place of $\mathbb{R}^3$?

Question 8

Let $\mathbb{Q}(\sqrt{2})$ be the set of all real numbers of the form $\alpha + \beta\sqrt{2}$, where α and β are rational.

1. Is $\mathbb{Q}(\sqrt{2})$ a field?
2. Answer (1) if α and β are taken from integers only.
3. Is $\mathbb{Q}(\sqrt{2})$ a vector space over $\mathbb{Q}$? If so, describe a basis for $\mathbb{Q}(\sqrt{2})$.

Question 9

1. Let $\mathbb{Z}_p$ denote the set of all integers modulo p, with the operations addition and multiplication *mod* p. Show that $\mathbb{Z}_p$ is a field if and only if p is prime. (*Hint:* The only trouble lies perhaps in showing that multiplicative inverse exists for all non-zero elements of $\mathbb{Z}_p$ if p is prime. To show the multipicative inverse use the fact that for any two integers a, b, there exists two integers l, s such that

$$la + sb = gcd(a, b).$$

Now take $a = p$, the prime number, and b to be the element in $\mathbb{Z}_p$ for which you want to find an inverse. Do *mod* p on both sides and see what happens.)

2. Consider the set $\mathbb{Z}_p[x]$ consisting of all polynomials with coefficients coming from $\mathbb{Z}_p$, p being a prime. A polynomial $g(x) \in \mathbb{Z}_p[x]$ is said to be *irreducible* if $g(x)$ has no nontrivial factors (i.e., $r(x)$ divides $g(x)$ if and only if $r(x) = cg(x)$ or $r(x) = c$, for some non-zero constant c). Let $\mathbb{Z}_{p^m}$ denote the set of polynomials in $\mathbb{Z}_p[x]$(modulo $g(x)$) (where $g(x)$ is a degree m irreducible polynomial). In other words, we take all the polynomials in $\mathbb{Z}_p[x]$ and divide each of them by $g(x)$, and take only the remainders in $\mathbb{Z}_{p^m}$. Show that $\mathbb{Z}_{p^m}$ is a field under addition and multiplication modulo $g(x)$. *Hint:* If you could do part (1) then this follows similarly. Again, you may want to use the following fact for the multiplicative inverse.

   - For any two polynomials $a(x)$ and $b(x)$ in $\mathbb{Z}_p[x]$, there exists polynomials $l(x)$ and $s(x)$ in $\mathbb{Z}_p[x]$ such that

$$l(x)a(x) + s(x)b(x) = gcd(a(x), b(x)).$$

Question 10

1. Show that a basis of a subspace W is (a) a maximal independent subset of W (b) a minimal spanning set of W.
2. Let W be a subspace of V with a basis $\{\boldsymbol{\alpha_i} : i = 1, .., m\}$. Let $\boldsymbol{\beta} \in V \backslash W$ (in V but not in W). Show that the set $\{\boldsymbol{\alpha_i} + \boldsymbol{\beta} : i = 1, .., m\}$ spans an m dimensional subspace of V.