Minimum Spanning Trees

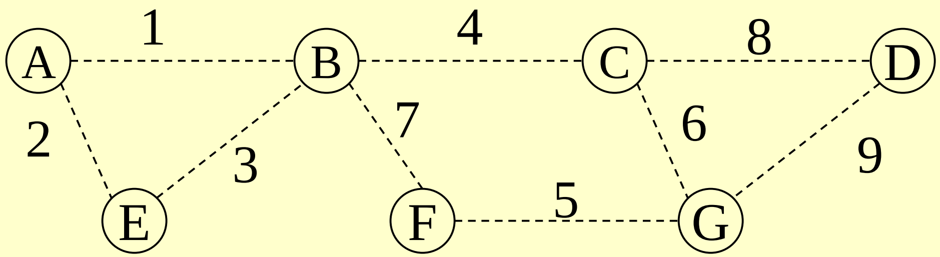
*www.ucl.ac.uk/~ucahbtw/docs/d1lesson1/**Kruskals_Prims.ppt***
*www.cse.msu.edu/~torng/Classes/Archives/cse830.../**Lecture15.ppt***
*https://courses.cs.washington.edu/courses/.../**Lecture10/Lecture10.ppt***
*https://courses.cs.washington.edu/courses/cse326/.../**part7-unionfind.ppt***

Problem Definition

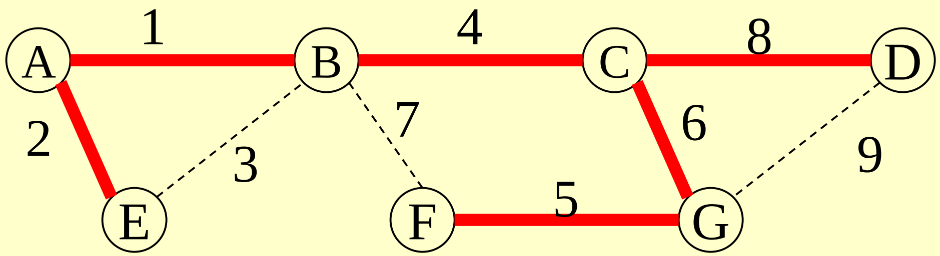
- Input
  - Weighted, connected undirected graph G=(V,E)
    - Weight (length) function w on each edge e in E
- Task
  - Compute a spanning tree of G of minimum total weight
- Spanning tree
  - If there are n nodes in G, a spanning tree consists of n-1 edges such that no cycles are formed

Example

Input:



Output:

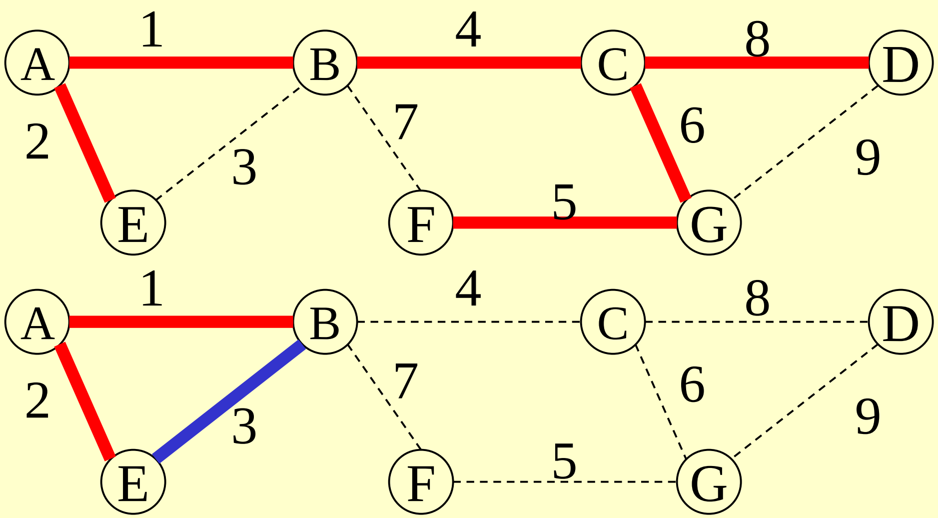


Two Properties of MST's

- **Cycle Property:** For any cycle C in a graph, the heaviest edge in C does not appear in the minimum spanning tree
  - Used to rule edges out

- **Cut Property**: For any proper non-empty subset X of the vertices, the lightest edge with exactly one endpoint in X belongs to the minimum spanning forest
  - Used to rule edges in

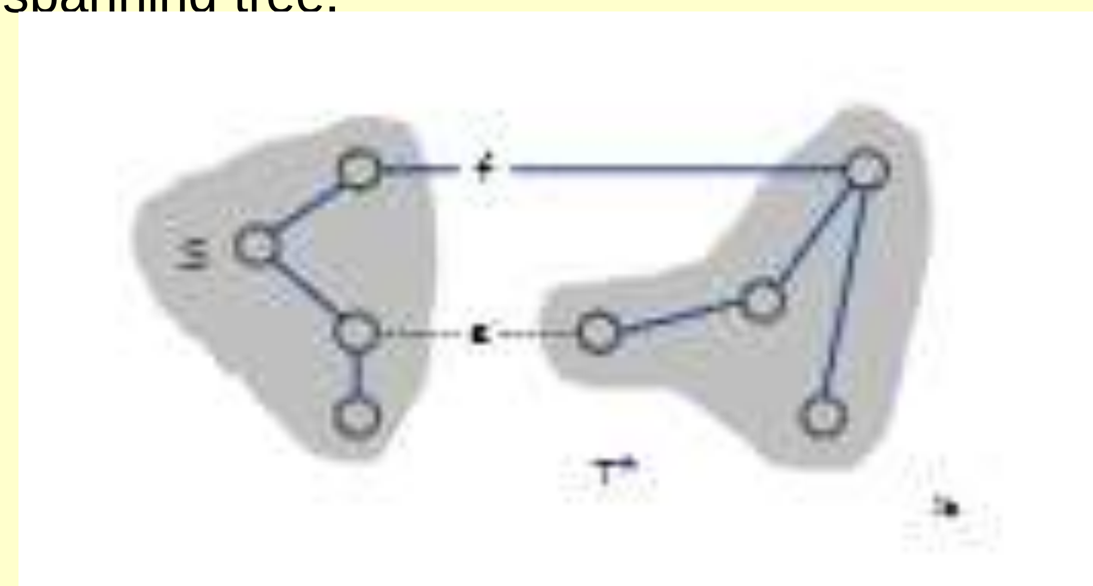
Cycle Property Illustration/Proof



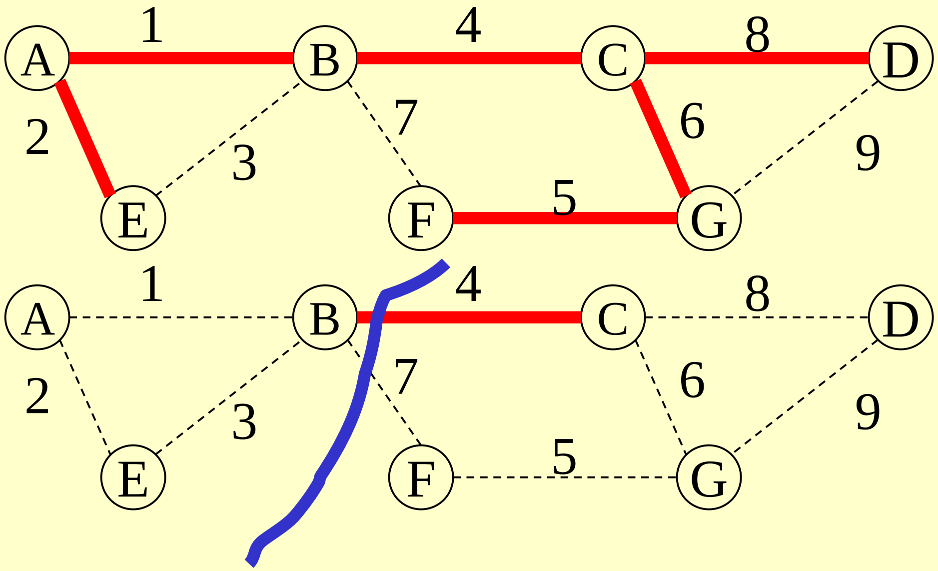
- Proof by contradiction:
  - Suppose T is an MST with such an edge e.
  - Derive a contradiction showing that T is not an MST

- Pf[by contradiction]

- Suppose f the heaviest edge belongs to T*. Let's see what happens.
- Deleting f from T* disconnects T*. Let S be one side of the cut.
- Some other edge in C,say e, has exactly one endpoint in S.
- T = T*+ {e} - {f} is also a spanning tree.
- Since c(e)<c(f),
- cost(T) < cost(T*).
- Thus T* not a MST!!



Cut Property Illustration/Proof



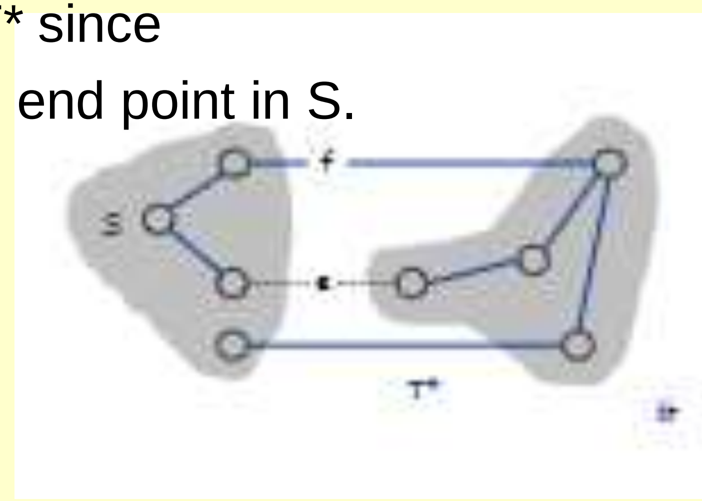
- Proof by contradiction:
  - Suppose T is an MST without such an edge e.
  - Derive a contradiction showing that T is not an MST

- Simplifying assumption : All edge costs are distinct.

- Let S be any subset of vertices, and let e be the min cost edge with exactly one endpoint in S.
- Then the MST T* contains e.

Pf. [by contradiction]!

- Suppose e does not belong to T*.
- Let's see what happens.
- Adding e to T* creates a (unique) cycle C in T* since
- Some other edge in T*, say f, has exactly one end point in S.
- T = T* + {e} - {f} is also a spanning tree.
  Since $c(e)<c(f)$, $cost(T) < cost(T^*)$.
- This is a contradiction.



Minimum Connector Algorithms

Kruskal's algorithm

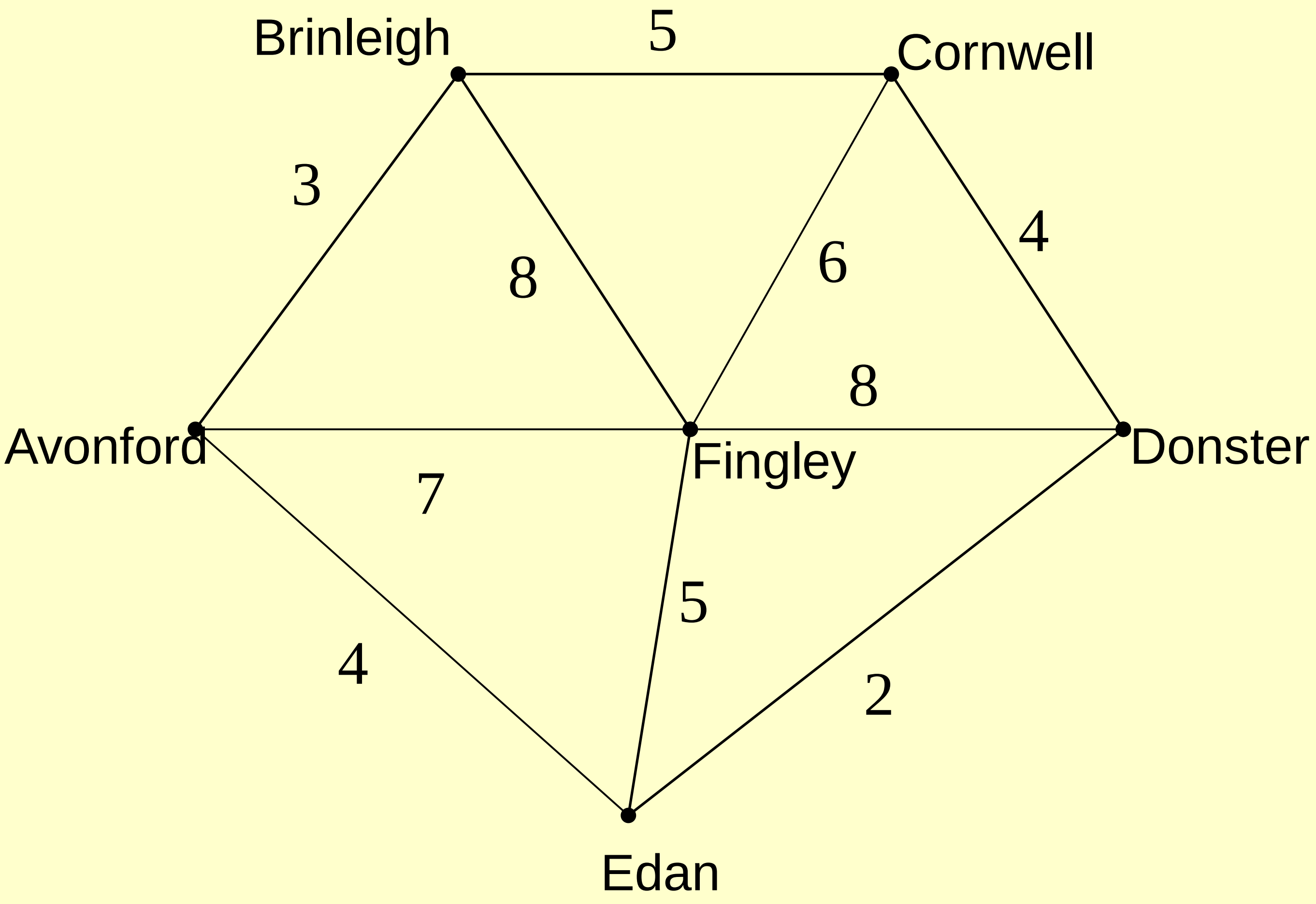
1. Select the shortest edge in a network
2. Select the next shortest edge which does not create a cycle
3. Repeat step 2 until all vertices have been connected

Prim's algorithm

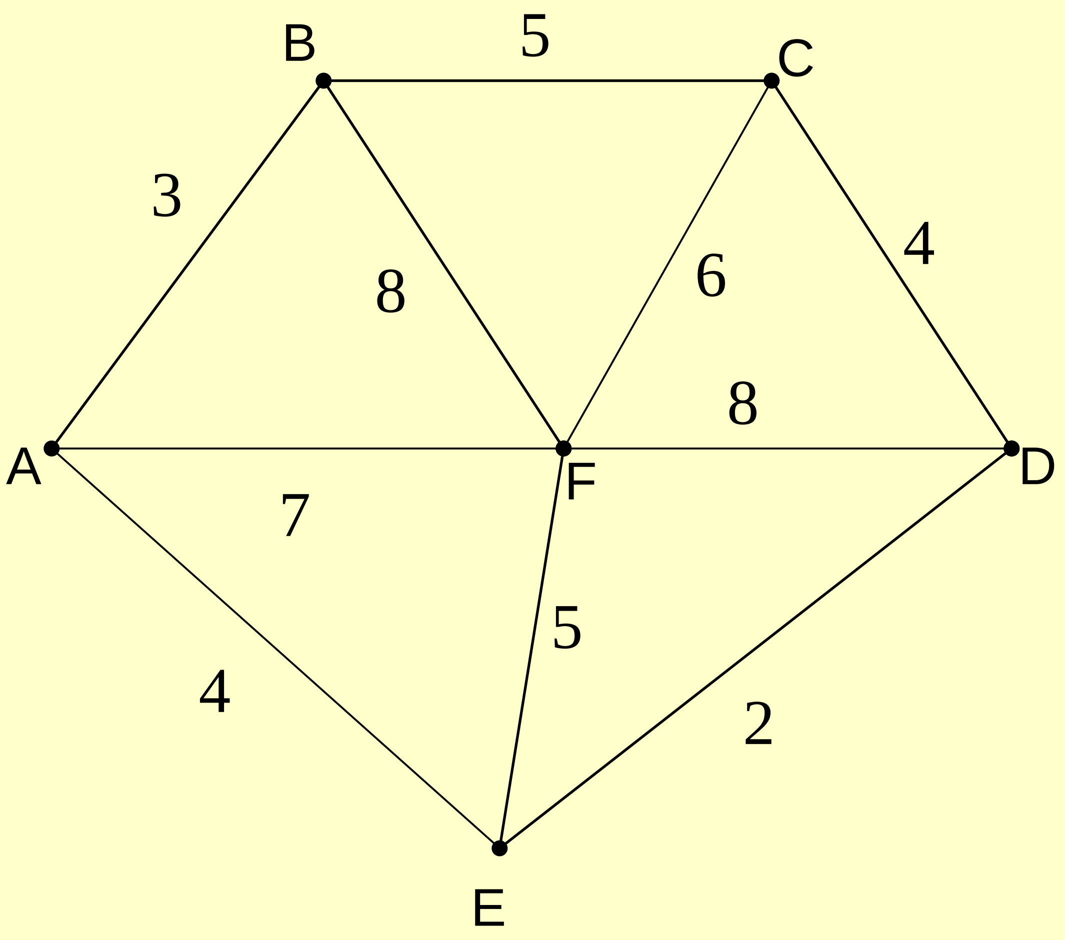
1. Select any vertex
2. Select the shortest edge connected to that vertex
3. Select the shortest edge connected to any vertex already connected
4. Repeat step 3 until all vertices have been connected

Example

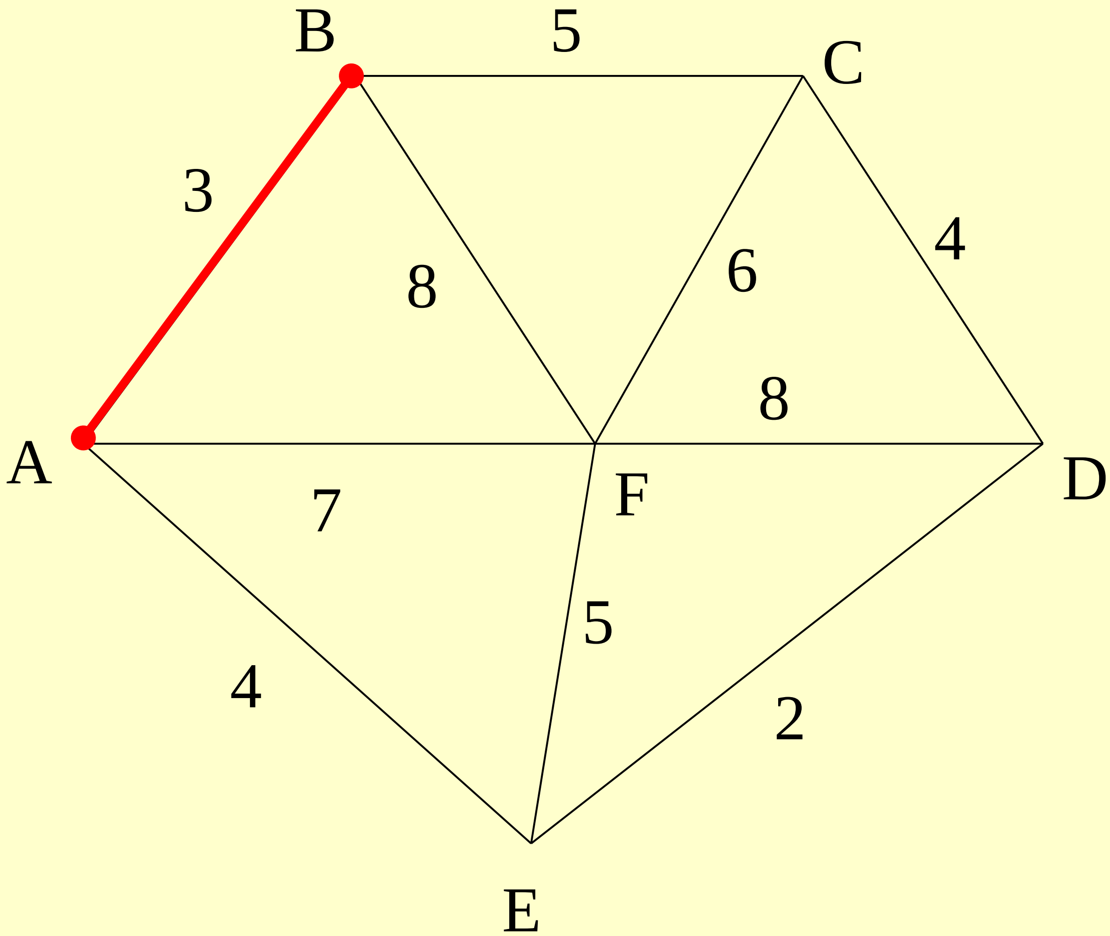
A cable company want to connect five villages to their network which currently extends to the market town of Avonford. What is the minimum length of cable needed?



We model the situation as a network, then the problem is to find the minimum connector for the network



Prim's Algorithm



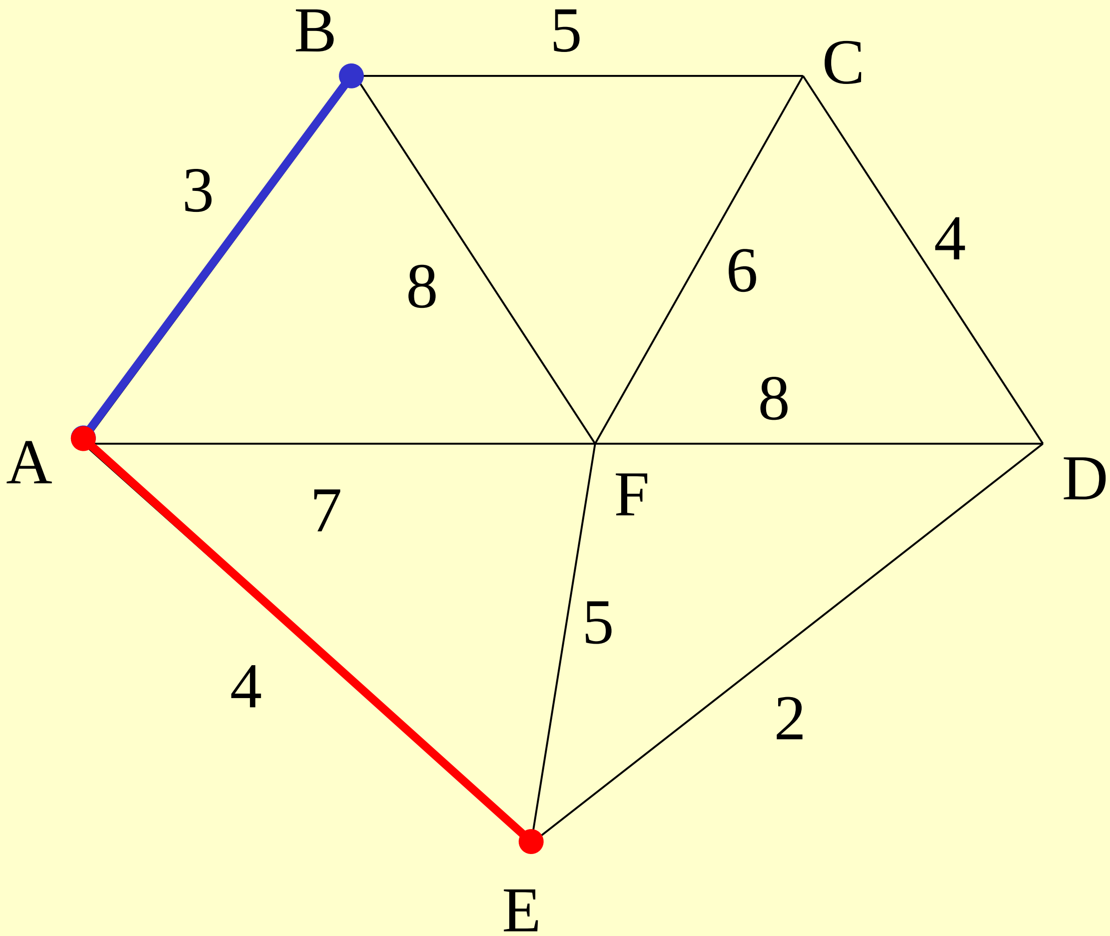
Select any vertex

A

Select the shortest edge connected to that vertex

AB 3

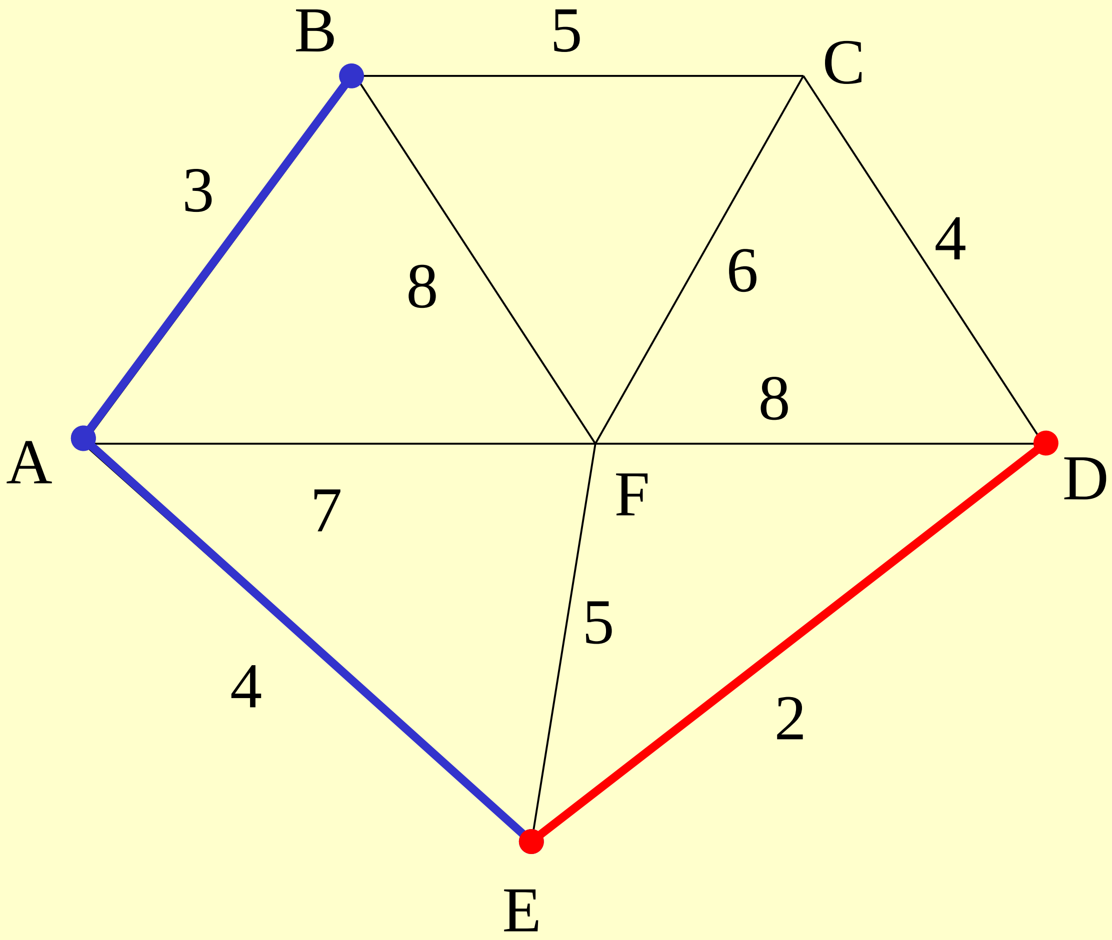
Prim’s Algorithm



Select the shortest edge connected to any vertex already connected.

AE 4

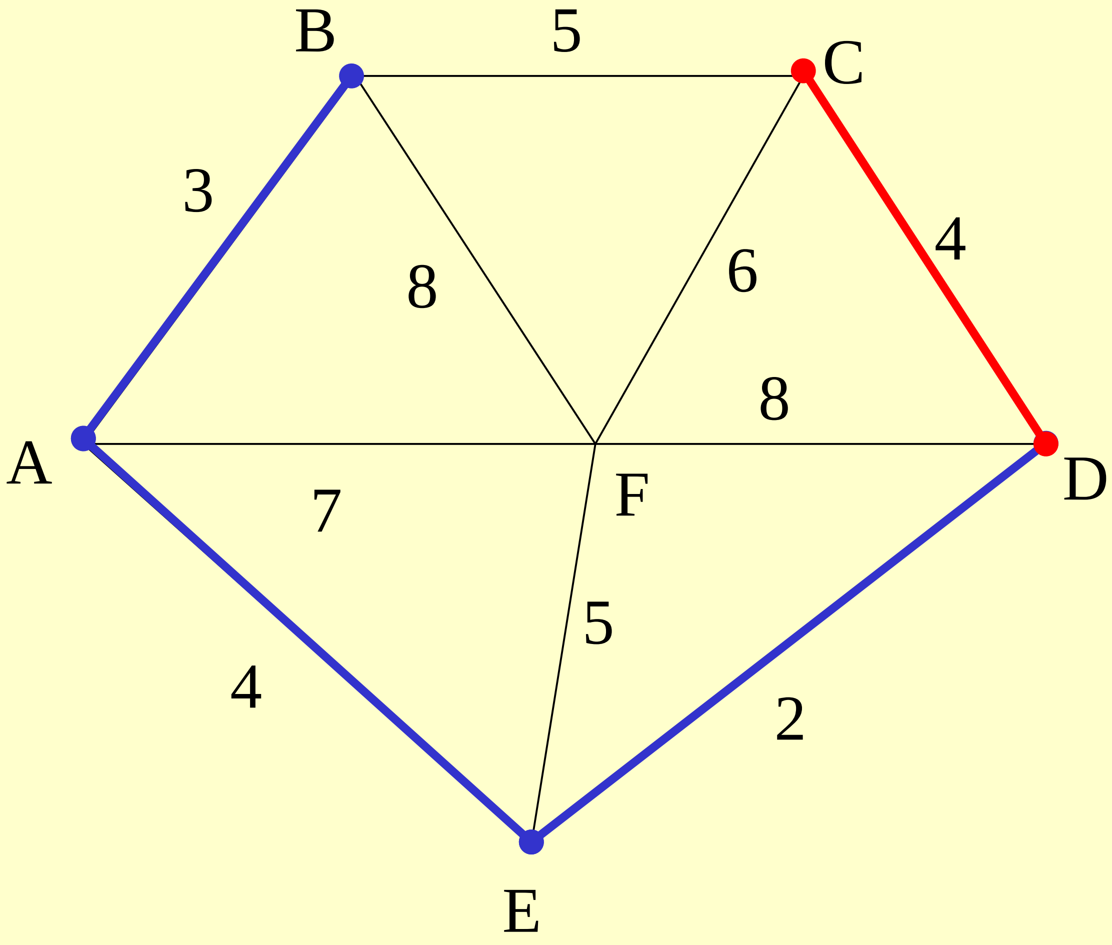
Prim's Algorithm



Select the shortest edge connected to any vertex already connected.

ED 2

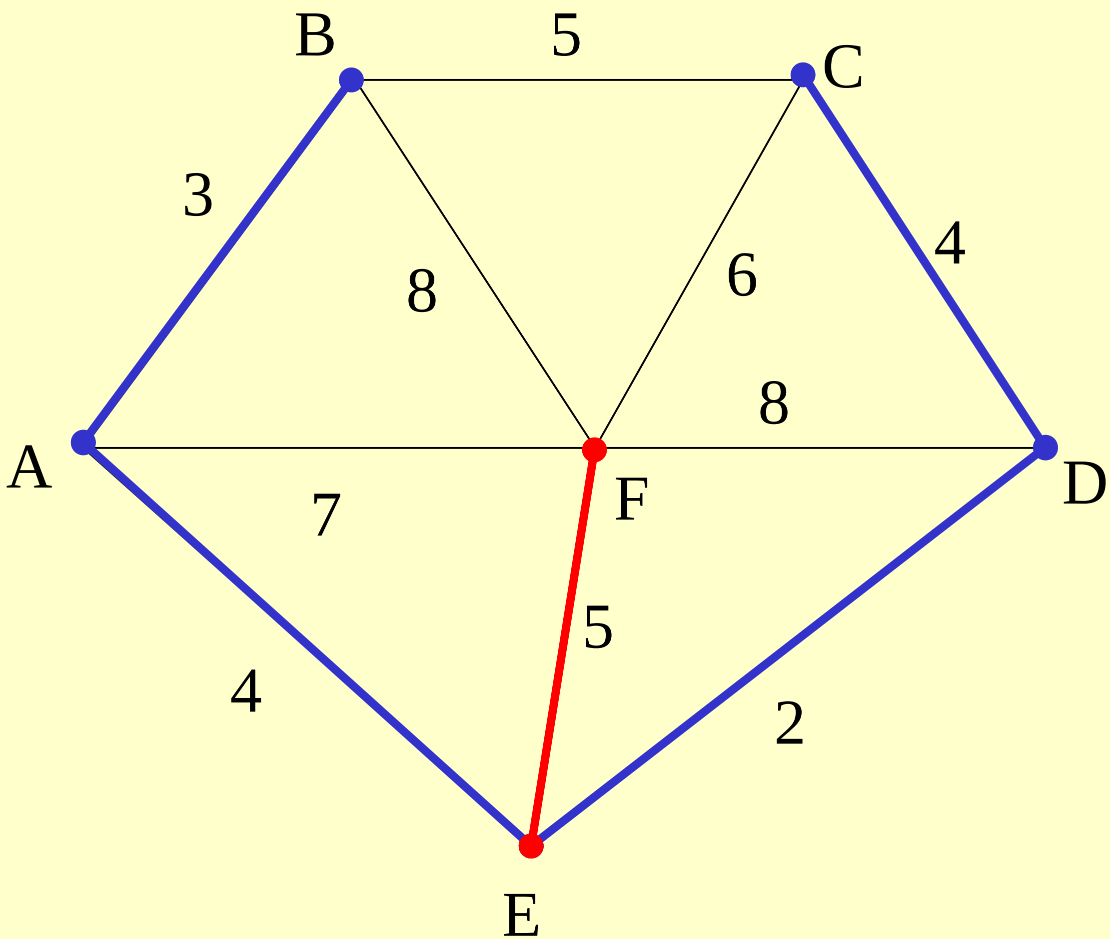
Prim's Algorithm



Select the shortest edge connected to any vertex already connected.

DC 4

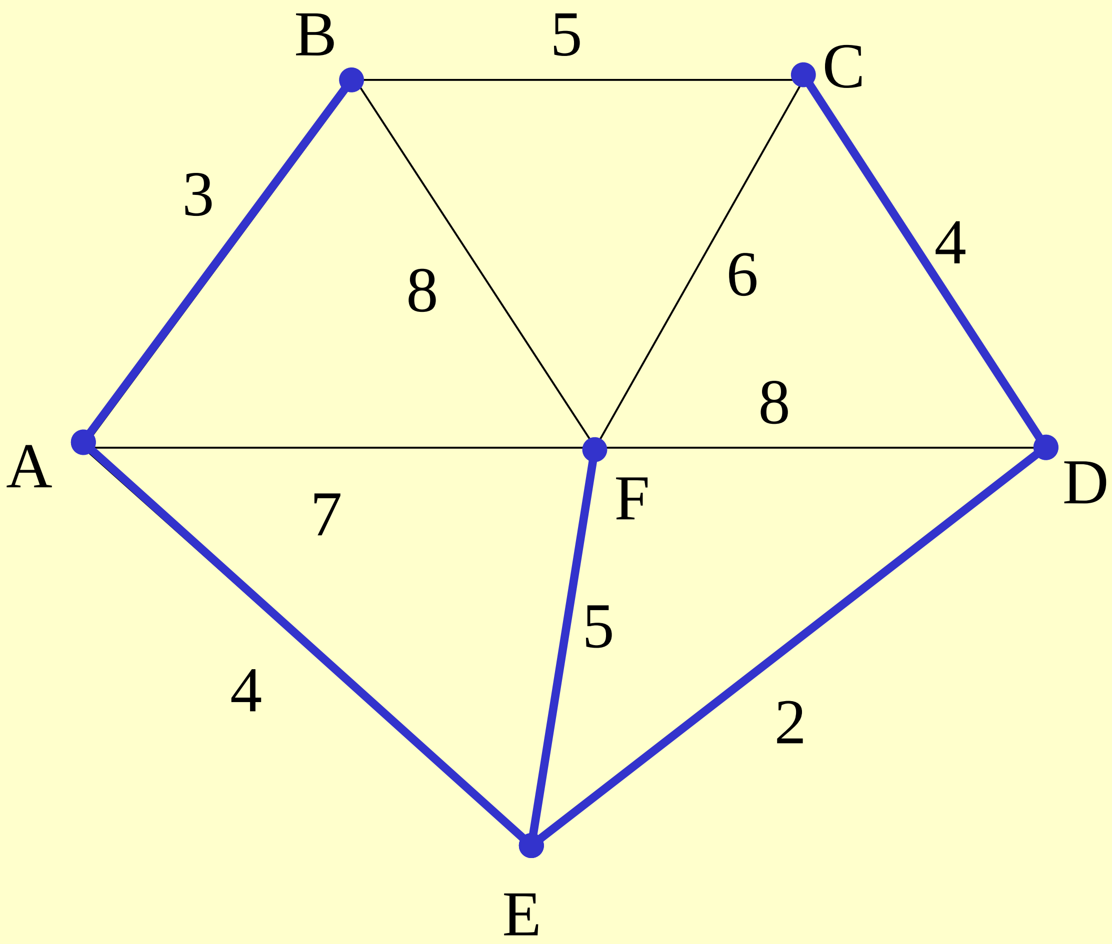
Prim's Algorithm



Select the shortest edge connected to any vertex already connected.

EF 5

Prim's Algorithm



All vertices have been connected.

The solution is

AB 3
AE 4
ED 2
DC 4
EF 5

Total weight of tree: 18

Correctness

By cut property, the lightest cut is always part of the MST. In each iteration Prims picks the lightest cut between the nodes selected And the complimentary set.

H.W

In case of an edge with multiple cycles with common edges between the cycles, does the cycle property hold ?

Is the minimum spanning tree and the shortest path tree of a graph always the same ?

Prove: For a graph of distinct edges, the minimum spanning tree is unique while for a graph with non distinct edges, there can be several minimum spanning trees