Ellipsoid Methods

Akshat Goyal, 2018101075

May 2, 2021

Table of Contents

1. SetUp for Ellipsoid Method
2. Ellipsoid Method and Feasibility Problem
3. Putting It All Together
4. Feasibility to Optimization
5. References

SetUp

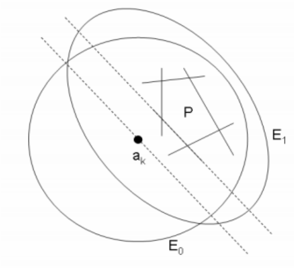
Given a convex set P with

1. $P \subseteq B(0,R) = \{x \mid \|x\| \leq R\}$ and promised that if $P \neq \emptyset$ then $P \geq B(a,r)$ for some a.
2. Separation Oracle that answer queries x by
    1. assert $x \in P$ OR
    2. assert $x \notin P$ and giving c such that $c^T y \leq c^T x \; \forall \; y \in P$.

Goal

- We are given a convex set P and we need to output any $x \in P$ or show that $P = \emptyset$. This is the Feasibility problem.
- We will first solve Feasibility problem using Ellipsoid Method and then reduce it to optimization problem in the later part.

Idea: Maintain ellipsoid $E_k \geq P$ and reduce its volume in each step.



What is Ellipsoid

An ellipsoid is defined as:

$$E(a, A) = \{x \mid (x - a)^T A^{-1}(x - a) \leq 1\}$$

where A is symmetric that is $A = A^T$ and positive definite ($x^T Ax > 0 \ \forall \ x \neq 0$)

.

A Ball in terms of Ellipsoid is defined as:

$$B(a, R) = E(a, R^2 I)$$

where a is the centre and R is the radius.

Ellipsoid Algorithm

Initialise $E_0 = B(0, R)$.
for $k = 0$ to M
  Let $E_k = (a_k, A_k)$ be current ellipsoid.
  Query separation oracle
  if $a_k \in P$
    Return a_k
  else
    Given c_k Find

$$E_{k+1} \supseteq E_k \cap \{x \mid c^T x \leq c^T a_k\}$$

Return $P = \emptyset$.

Now the main Q is that

1. What is the value of M.
2. How to find E_{k+1} satisfying above constraints.

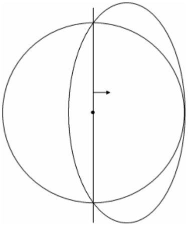
Special Case

Main Q: Is there a good choice for E_{k+1} and how do we find it?

Let's consider a special case.
$E = E(0,1)$ then wlog $c_k =$ -e_1.
Let's set

$$E^{`} = \{x \mid \left(\frac{n+1}{n}\right)^2 \left(x_1 - \frac{1}{n+1}\right)^2 + \left(\frac{n^2-1}{n^2}\right) \sum_{i=2}^{n} x_i^2 \leq 1\}$$

where n is the dimension of P.



Claim 1

As c_k = -e_1, therefore $x_1 \geq 0 \;\forall\; x \in P$. To prove that $E^{`}$ is the valid ellipsoid, we will prove that If $x \in E$ and $x_1 \geq 0$, then $x \in E^{`}$.

Proof:
Let's simplify first term of the Ellipsoid $E^{`}$

$$\left(\frac{n+1}{n}\right)^2 \left(x_1 - \frac{1}{n+1}\right)^2$$

$$= \left(\frac{n^2-1}{n^2} + \frac{2n+2}{n^2}\right) x_1^2 - 2\left(\frac{n+1}{n^2}\right) x_1 + \frac{1}{n^2}$$

If we add $\left(\frac{n^2-1}{n^2}\right) \sum_{i=2}^{n} x_i^2$ we get

$$\left(\frac{n^2-1}{n^2}\right) \sum_{i=1}^{n} x_i^2 + \frac{(2n+2)(x_1^2 - x_1)}{n^2} + \frac{1}{n^2}$$

Proof (Continued..)

1. $\sum_{i=2}^{n} x_i^2 \leq 1$ as $x \in E$.
2. Second term is less than or equal to 0 as $x_1 \geq 0$ and $x_1 \leq 1$. So $x_1^2 - x_1 \leq 0$.
3. Sum of first and third term is less than or equal to 1.

Hence

$$\left(\frac{n^2-1}{n^2}\right)\sum_{i=1}^{n} x_i^2 + \frac{(2n+2)(x_1^2 - x_1)}{n^2} + \frac{1}{n^2} \leq 1$$

for $x \in E$ and $x_1 \geq 0$.

Claim 2

$$\frac{Vol(E^{'})}{Vol(E)} \leq e^{-\dfrac{1}{2(n+1)}} \leq 1$$

Proof:
The volume of an ellipsoid is proportional to products of its side lengths. Using this we get

$$\frac{Vol(E^{'})}{Vol(E)} = \frac{\left(\dfrac{n}{n+1}\right)\left(\left(\dfrac{n^2}{n^2-1}\right)^{1/2}\right)^{n-1}}{1}$$

$$= \frac{\left(1-\dfrac{1}{n+1}\right)\left(\left(1+\dfrac{1}{n^2-1}\right)^{1/2}\right)^{n-1}}{1}$$

Proof (Continued..)

Using $e^{-x} \geq$ 1 - x $\forall$ x $\geq$ 0,

$$\leq e^{-\frac{1}{n+1}} e^{\frac{n-1}{2(n^2-1)}}$$

$$= e^{-\frac{1}{2(n+1)}}$$

After any $O(n)$ iterations, the volume of the ellipsoid will be reduced by a factor of ≈ 2.

General Case

We will now reduce the general case to special case.

1. Fact 1: A is positive definite iff A = B^TB for some invertible B.
2. Fact 2: Invertible linear transformations preserve volume ratios.

Now the plan is to transform E(a, A) to E(0, I) using a Transformation Matrix T. Get $E^{'}$ from E and use Transformation Matrix T^{-1} to get E_{k+1}.

Let y = T(x) = $(B^{-1})^T$(x - a) then

$$y^T y \leq 1$$

.

$$= (x-a)^T B^{-1}(B^{-1})^T(x-a) \leq 1$$

.

$$= (x-a)^T A^{-1}(x-a) \leq 1$$

.

As we can see, we are able to reduce general case to special case.

Putting It All Together

We now know how to find ellipsoid at each iteration. But there is still one question left which is what is the value of M. We take $M = O(n^2 ln(R/r))$ then

$$\frac{Vol(E_m)}{Vol(E_0)} \leq e^{\frac{-1}{2(n+1)}^{cn^2 ln(R/r)}} \leq \left(\frac{r}{R}\right)^{2n}$$

Therefore, if all the M iterations are covered, it means that $P = \emptyset$ as if P $\notin \emptyset$ then P $\supseteq$ B(a, r) which implies

$$\frac{Vol(E_m)}{Vol(E_0)} \geq \left(\frac{r}{R}\right)^{n}$$

This proves that Ellipsoid method solves feasibility problem in polynomial time.

Feasibility to Optimization

Now we will show that feasibility can be reduced to optimization in polynomial time.

1. Joint Feasibility
    1. Check feasibility for Primal. If primal is infeasible, we are done.
    2. Check feasibility for Dual. If dual is infeasible, we are done.
    3. Otherwise setup joint feasibility LP. e.g. $Ax \leq B$, $x \geq 0$. $A^T y \geq c$, $y \geq 0$, $c^T x = y^T b$.
2. Binary Search We can do binary search on the objective value and put the objective in the constraints. Binary search takes O(log) time. Therefore complexity of optimization still remains polynomial.

References

1. MIT 6.854 Spring 2016 Lecture 12: From Separation to Optimization and Back; Ellipsoid Method
2. ORIE 6300 Mathematical Programming I https://people.orie.cornell.edu/dpw/orie6300/Lectures/lec19.pdf