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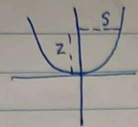
HW#2 D4A

Q1

a)



$z = \frac{\cancel{x}^2}{a}$ $\quad 0 \le \phi \le 2\pi, \; 0 \le s \le R$

$\vec{r}(s,\phi) = \cancel{\langle s, \phi, \tfrac{z^2}{a} \rangle}$

$\vec{r}(s,\phi) = s\hat{s} + \frac{s^2}{a}\hat{z}$

b) $d\vec{A} = \partial_\phi \times \partial_s$

$\partial_\phi = \frac{\partial\, s\hat{s}}{\partial\phi}\partial\phi + \frac{\partial\, s^2\hat{z}/a}{\partial\phi}\partial\phi$

$= \left[\frac{\partial s}{\partial\phi}\hat{s} + \frac{\partial\hat{s}}{\partial\phi}s\right]\partial\phi + \left[\hat{z}\frac{\partial \frac{s^2}{a}}{\partial\phi} + \frac{s^2}{a}\frac{\partial\hat{z}}{\partial\phi}\right]\partial\phi$

$= 0 + s\hat{\phi}\,\partial\phi + 0 + 0$

$\partial_s = \cancel{\frac{\partial s\hat{s}}{\partial s}\partial\phi} + \cancel{\frac{\partial \frac{s^2\hat{z}}{a}}{\partial s}\partial\phi}\quad \frac{\partial\, s\hat{s}}{\partial s}\partial s + \frac{\partial\,\frac{s^2}{a}\hat{z}}{\partial s}\partial s$

$= \cancel{\partial\phi[}$ $\quad = \partial s\left[\frac{\partial s}{\partial s}\hat{s} + \frac{s\,\partial\hat{s}}{\partial s}\right] +$

$\partial s\left[\frac{\partial s^2/a}{\partial s}\hat{z} + \frac{s^2}{a}\frac{\partial\hat{z}}{\partial s}\right]$

$= \partial s\left(\hat{s} + 0\right) + \partial s\left(\frac{2s}{a}\hat{z} + 0\right)$

$s \le R$

$\frac{\partial \hat{z}}{\partial \phi} \Big] \partial \phi$

$\frac{s^2}{a} \hat{z} \, \partial s$

s

$\frac{\partial \hat{z}}{\partial s} \Big]$

$\cancel{dA = s\,\partial\phi\,\hat{s}}$

$\partial \phi = s \partial \phi \, \hat{\phi}$

$\partial s = \partial s \, \hat{s} + \frac{2s}{a} \partial s \, \hat{z}$

$$\vec{dA} = \partial \phi \times \partial s = \begin{vmatrix} \hat{s} & \hat{\phi} & \hat{z} \\ 0 & s\partial\phi & 0 \\ \partial s & 0 & \frac{2s}{a}\partial s \end{vmatrix}$$

$$= s\partial\phi \left(\frac{2s}{a} \partial s \, \hat{s} - \partial s \, \hat{z} \right)$$

$$= \frac{2s^2}{a} \partial\phi \, \partial s \, \hat{s} - s\partial s \partial \phi \, \hat{z}$$

c) $E = \frac{\lambda}{2\pi \varepsilon_0 s} \hat{s}$, $\Phi = \oint E \cdot dA$

$$\Phi = \int E \cdot dA = \iint \frac{2s^2}{a} \frac{\lambda}{2\pi \varepsilon_0 s} \partial\phi \, \partial s = \int_0^R \int_0^{2\pi} \frac{2s\lambda}{a 2\pi \varepsilon_0} \partial\phi \, \partial s$$

$$= \int_0^R \frac{s\lambda \, 2\cancel{\pi}}{a \cancel{\pi} \varepsilon_0} \partial s = \frac{R^2 \lambda}{\varepsilon_0 a} \rightarrow \frac{Q_{enclosed}}{\varepsilon_0}$$

Q2

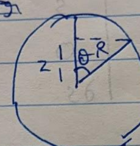
a)

$z(t) = R - v_z t$

$\phi(t) = \omega t$

$\cancel{\theta(t)}\ R(t) = R$

$\cancel{r(t) = R\hat{r}}$



$\cos\theta = \frac{z}{R}$

$\theta = \cos^{-1}\left[\frac{z}{R}\right]$

$0 \le \theta \le \pi$

$0 \le \phi \le \frac{R\omega(1-\cos\theta)}{v_z}$

$R = R$

$z = R\cos\theta$

$R - v_z t = R\cos\theta$

$t = \frac{R - R\cos\theta}{v_z}$

$\vec{r}(\theta) = R\hat{r}$

$\cancel{d\theta =}\ dl = \frac{\partial R\hat{r}}{\partial\theta}\partial\theta$

$= \left[\frac{\partial R}{\partial\theta}\hat{r} + R\frac{\partial\hat{r}}{\partial\theta}\right]\partial\theta$

$\partial l = 0 + R\hat{\theta}\,\partial\theta$

$D = \int_0^{\pi} |dl| =$

$\vec{dl} = dr\,\hat{r} + r\,d\theta\,\hat{\theta} + r\sin\theta\,d\phi\,\hat{\phi}$

$\partial l_\theta = \frac{\partial l}{\partial\theta}\,d\theta = d\theta\left[\frac{dr}{d\theta}\hat{r} + r\frac{d\theta}{d\theta}\hat{\theta} + r\sin\theta\frac{d\phi}{d\theta}\hat{\phi}\right]$

$= d\theta\left[0 + R\hat{\theta} + \hat{\phi}\frac{R^2\sin\theta}{V_2}\omega\sin\theta\right]$

$dl_\theta = d\theta\,R\hat{\theta} + \frac{R^2\sin^2\theta\,\omega\,d\theta}{V_2}\hat{\phi}$

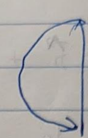
$\text{Path} = \int_l \|\vec{dl_\theta}\| \Rightarrow$

$|dl| = \sqrt{R^2 d\theta^2|\hat{\theta}|^2 + \left(\frac{R^2\sin^2\theta\,\omega\,d\theta}{V_2}\hat{\phi}\right)^2}$

$|dl| = R\sqrt{1 + R^2\sin^4\theta\frac{\omega^2}{V_2^2}}\,d\theta$

$\text{Path distance} = \int_0^\pi R\sqrt{1 + \frac{R^2\omega^2}{V_2^2}\sin^4\theta}\,d\theta$

b) if ω is $<< 1$



$$\text{Path} = \int_0^{\pi} R\sqrt{1 + \frac{R^2\omega^2}{v_2^2}\sin^2\theta}\, d\theta$$

$$= \int_0^{\pi} R\left(1 + \frac{1}{2}\frac{R^2\omega^2}{v_2^2}\sin^2\theta\right) d\theta$$

$$= \pi R + \frac{1}{2}\frac{\omega^2}{v_2^2}\frac{3\pi}{8} \quad , \text{as } \omega \approx 0$$

$$= \pi R$$

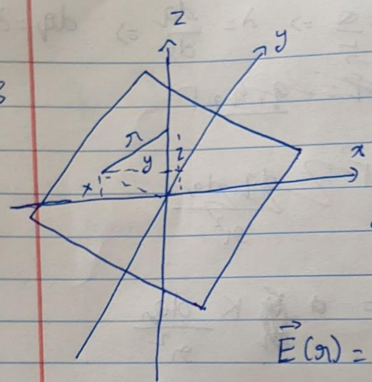
c) $$W = \int \vec{F}\cdot d\vec{l}$$

$$= \int -F\hat{\phi}\cdot\left(R\,d\theta\,\hat{\theta} + \frac{R^2\sin^2\theta}{v_2}\,\omega\, d\theta\,\hat{\phi}\right)$$

$$= \int_0^{\pi} -\frac{F R^2\sin^2\theta}{v_2}\,\omega\, d\theta$$

$$W = -\frac{F R^2}{v_2}\,\omega\,\frac{\pi}{2}\ \mathrm{J}$$

Q3



$$\cancel{E = \int K \frac{dq}{r^2}}$$

$$E \neq \int K q$$

$$\vec{E}(r) = Kq \frac{\vec{r} - \vec{r_q}}{|\vec{r} - \vec{r_q}|^3}$$

$$\Rightarrow \vec{E}(r) = K \int dq \frac{\vec{r} - \vec{r_q}}{|\vec{r} - \vec{r_q}|^3}$$

$$\vec{r} = x\hat{x} + y\hat{y} + z\hat{z}$$
$$\vec{r_q} = 0 + 0 + z_q\hat{z}$$

$$\vec{E} = K \int \frac{x\hat{x} + y\hat{y} + (z - z_q)\hat{z}}{\left[x^2 + y^2 + (z - z_q)^2\right]^{3/2}} dq$$

$$\frac{dq}{dl} = \lambda_0$$

$$dl = dz$$

$$\Rightarrow \lambda_0 = \frac{dq}{dz}$$

$$\Rightarrow dq = \lambda_0 dz$$

$$E = K\int \frac{x\hat{x} + y\hat{y} + (z - z_q)\hat{z}}{(x^2+y^2+(z-z_q)^2)^{3/2}}\, \lambda_0\, dz$$

$$z=0 \Rightarrow E = K\int_{-L}^{L} \lambda_0 \frac{x\hat{x} + y\hat{y} - z_q\hat{z}}{(x^2+y^2+z_q^2)^{3/2}}\, dz$$

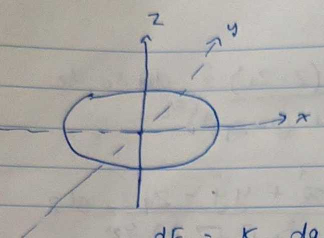
$\cancel{x\hat{x} + y\hat{y} = S \Rightarrow E = K\lambda_0 \int_{-L}^{L} S - z_q\hat{z}}$

$$E = \left\langle K\lambda_0 \int_{-L}^{L} \frac{x}{(x^2+y^2+z_q^2)^{3/2}}\, dz_q,\ K\lambda_0 \int_{-L}^{L} \frac{y}{(x^2+y^2+z_q^2)^{3/2}}\, dz_q,\ K\lambda_0 \int_{-L}^{L} \frac{z_q}{(x^2+y^2+z_q^2)^{3/2}}\, dz_q \right\rangle$$

Via <u>wolfram alpha</u>

$$E = \left\langle \frac{2K\lambda_0 L x}{(y^2+x^2)\sqrt{y^2+x^2+L^2}},\ \frac{2K\lambda_0 L y}{(y^2+x^2)\sqrt{y^2+x^2+L^2}},\ 0 \right\rangle$$

Q4



$z \in [0, \infty)$

$\lambda = \lambda_0 \sin\phi$

$$E(\vec{r}) = K\, q\, \frac{\vec{r} - \vec{r}_q}{|\vec{r} - \vec{r}_q|^3}$$

$$dE = K\, \frac{dq\, (\vec{r} - \vec{r}_q)}{|\vec{r} - \vec{r}_q|^3}$$

Now $\vec{r}$ only has a z component, no x or y component and $\vec{r}_q$ only has x and y component.

$$\Rightarrow \vec{r} = z\,\hat{z} \quad \text{and} \quad \vec{r}_q = x_q\,\hat{x} + y_q\,\hat{y}$$

$$\boxed{\cancel{E(\vec{r}) = K \int \frac{z\,\hat{z} - x_q\,\hat{x} - y_q\,\hat{y}}{|z\,\hat{z} - x_q\,\hat{x} - y_q\,\hat{y}|^3}\, dq}}$$

$$\lambda = \frac{dq}{dl}$$

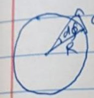
$$\Rightarrow \quad dq = \lambda\, dl$$

$$dq = \lambda_0 \sin\phi\, dl$$

In cylindrical coords:

$$E(\vec{r}) = K \int \frac{z\,\hat{z} - s_q\,\hat{s}_q}{|z\,\hat{z} - s_q\,\hat{s}_q|^3}\, \lambda_0 \sin\phi\, dl$$

$$E = K \int \left[\frac{z\hat{z} - S_q \hat{S_q}}{(z^2 + S_q^2)^{3/2}} \right] \lambda_0 \sin\phi \, dl$$



$$d\phi = \frac{dl}{R}, \quad dl = R\,d\phi$$

$$E = K \int_0^{2\pi} \frac{z\hat{z} - R\hat{S_q}}{(z^2 + R^2)^{3/2}} \lambda_0 \sin\phi \, R\,d\phi$$

for integration }
something constant,

$\hat{S_q} = \cos\phi\,\hat{x} + \sin\phi\,\hat{y}$ } as $\hat{S_q}$ changes with $d\phi$, we need,

$$E = \int_0^{2\pi} \frac{(z\hat{z} - R\cos\phi\,\hat{x} - R\sin\phi\,\hat{y})}{(z^2 + R^2)^{3/2}} \lambda_0 \sin\phi \, R\,d\phi$$

$$\cancel{E = \int_0^{2\pi} \frac{\lambda_0 z \sin\phi \, R\,d\phi}{(z^2 + R^2)^{3/2}} \hat{z} \; (\ldots) \int_0^{2\pi}}$$

$$E = \int_0^{2\pi} K \left[\frac{z\hat{z} - R\cos\phi\,\hat{x} - R\sin\phi\,\hat{y}}{(z^2 + R^2)^{3/2}} \right] \lambda_0 R \sin\phi \, d\phi$$

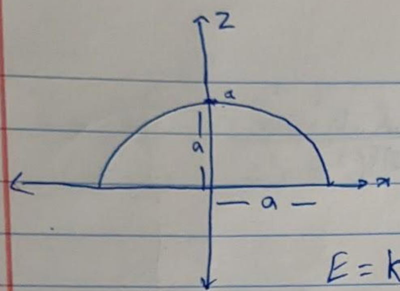
$$E_x = \int_0^{2\pi} K \frac{-\lambda_0 R \sin\phi \, R\cos\phi}{(z^2 + R^2)^{3/2}} \, d\phi \quad = 0$$

$$E_y = k\int_0^{2\pi} -\frac{R^2 \sin^2\phi\ \lambda_0}{(z^2+R^2)^{3/2}}\, d\phi = \frac{-k\pi R^2 \lambda_0}{(z^2+R^2)^{3/2}}$$

$$E_z = k\int_0^{2\pi} \frac{z\ R \sin\phi\ \lambda_0}{(z^2+R^2)^{3/2}}\, d\phi = 0$$

$$E = \left\langle 0,\ k\frac{-\pi R^2 \lambda_0}{(z^2+R^2)^{3/2}},\ 0 \right\rangle$$

Q5



$$E(r) = K\,q\,\frac{\vec{r} - \vec{r}_q}{|\vec{r} - \vec{r}_q|^3}$$

$$E = k\int dq\,\frac{\vec{r} - \vec{r}_q}{|\vec{r} - \vec{r}_q|^3}$$

$$\sigma = \frac{dq}{|dA|}$$

$$dq = \sigma\,|dA|,$$

Parameterize the region -> $r = a$

$0 \le \phi \le 2\pi$

$0 \le \theta \le \pi/2$

$$\vec{dA} = \partial l_\phi \times \partial l_\theta$$

$$\partial l_\phi = \frac{\partial l}{\partial\phi}\,d\phi = \left[\frac{dr}{d\phi}\,\hat{r} + \frac{r\,d\theta}{d\phi}\,\hat{\theta} + r\sin\theta\,\frac{d\phi}{d\phi}\,\hat{\phi}\right]d\phi$$

$$\partial l_\phi = r\sin\theta\,\hat{\phi}\,d\phi$$

$$\partial l_\theta = \frac{\partial l}{\partial\theta}\,\partial\theta = \left[\frac{dr}{d\theta}\,\hat{r} + \frac{r\,d\theta}{d\theta}\,\hat{\theta} + r\sin\theta\,\frac{d\phi}{d\theta}\,\hat{\phi}\right]d\theta$$

$$\partial l_\theta = r\,\hat{\theta}\,d\theta$$

$$\vec{dA} = \partial l_\phi \times \partial l_\theta = \begin{vmatrix} \hat{r} & \hat{\theta} & \hat{\phi} \\ 0 & 0 & r\sin\theta\,d\phi \\ 0 & r\,d\theta & 0 \end{vmatrix}$$

$$\vec{dA} = \hat{r}\,(r^2\sin\theta\,d\phi\,d\theta)$$

$$\cancel{E = K \int z\hat{z} - a\cos\theta\cos\phi\,\hat{x} - a\cos}$$

$$E = K \int \frac{z\hat{z} - a\sin\theta\cos\phi\,\hat{x} - a\sin\theta\sin\phi\,\hat{y} - a\cos\theta\,\hat{z}}{|z\hat{z} - a\sin\theta\cos\phi\,\hat{x} - a\sin\theta\sin\phi\,\hat{y} - a\cos\theta\,\hat{z}|^3}\,\sigma\,dA$$

$$E = K \int_0^{\pi/2}\int_0^{2\pi}\left[\frac{(z - a\cos\theta)\hat{z} - a\sin\theta\cos\phi\,\hat{x} - a\sin\theta\sin\phi\,\hat{y}}{\left((z - a\cos\theta)^2 + a^2\sin^2\theta\cos^2\phi + a^2\sin^2\theta\sin^2\phi\right)^{3/2}}\right]\sigma a^2\sin\theta\,d\phi\,d\theta$$

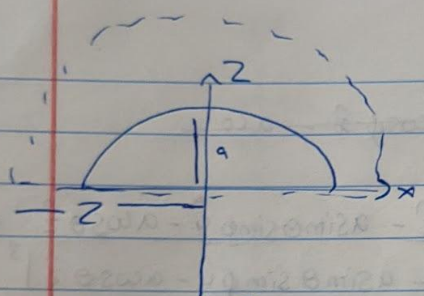
$$E = K \int_0^{\pi/2}\int_0^{2\pi}\left[\frac{(z - a\cos\theta)\hat{z} - a\sin\theta\cos\phi\,\hat{x} - a\sin\theta\sin\phi\,\hat{y}}{\left((z - a\cos\theta)^2 + a^2\sin^2\theta\right)^{3/2}}\right]\sigma a^2\sin\theta\,d\phi\,d\theta$$

$$(z - a\cos\theta)^2 = z^2 + a^2\cos^2\theta - 2za\cos\theta$$

$\Rightarrow$

$$E_z = K \int_0^{\pi/2}\int_0^{2\pi}\frac{(z - a\cos\theta)\,\sigma a^2\sin\theta\,d\phi\,d\theta}{(z^2 + a^2 - 2za\cos\theta)^{3/2}}$$

$$E_y = K \int_0^{\pi/2}\int_0^{2\pi}\frac{-(a\sin\theta\,\cancel{\cos}^{\sin}\phi)\,\sigma a^2\sin\theta\,d\phi\,d\theta}{(z^2 + a^2 - 2za\cos\theta)^{3/2}}$$



$z < a$, gauss law

$q_{enclosed} = 0$

$\Rightarrow E(\vec{z}) = 0$

$$\oint E \cdot dA = \frac{q_{enc}}{\varepsilon_0}$$ (hemisphere)

draw gaussian surface with radius z

$E(z) \Rightarrow$

$$E(z) \cdot A = \frac{q}{\varepsilon_0}$$

$$\sigma = \frac{q}{A}$$

$$q = \sigma \cdot A$$

$$= \sigma \cdot 2\pi a^2$$

$$E = \frac{\sigma \cdot 2\pi a^2}{2 4\pi z^2} = \frac{\sigma a^2}{2 z^2}$$