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Modelling the Orbit of Mars

IB SL Mathematics AA Internal Assessment

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Introduction and Aim

I have always wondered ever since I learned about our solar system how the mechanics of the universe allows planets to orbit around giant celestial bodies without being sent off into the abyss of outer space. I became even more fascinated with planetary orbit after I received an apple watch for my 15th birthday, which has a watch setting that displays the exact position of every planet in our solar system in their orbit relative to the sun at any given time. How have physicists managed to determine a planets position so accurately, I wondered? The secret of it lies within Kepler's Laws of Planetary motion, founded by Johannes Kepler.[1] The law that is crucial to this investigation is his First Law, which states that planets orbit the sun elliptically. This is a seemingly reasonable claim, however, the math behind this statement is quite complex. <u>My goal is to derive this law in a very simple and understandable way, and then use it to formulate both polar and rectangular equations to model the orbit of Mars.</u> The reason Mars was chosen is because of its importance right now. Serkan Seydam from the University of New South Wales stated that the start of human colonization of Mars is possible by 2050, only 28 years from this point.[2] Knowing the equation of the ellipse that governs Mars' movement around the sun, as well as the equation for Earths movement is crucial since it can give us a date when the Earth is as close as possible to Mars. This is a significant factor to consider because it reduces the amount of fuel needed for a trip to Mars.[3] Additionally, the formula can help us determine the period of Mars' orbit (as well as other planets), which will be done in the applications section. Overall, this exploration will require an understanding of differential calculus, vectors, polar and rectangular equations, conics, and trigonometry.

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[1] *13.5 Kepler's laws of planetary motion - university physics volume 1*. OpenStax. (n.d.). Retrieved November 10, 2022, from https://openstax.org/books/university-physics-volume-1/pages/13-5-keplers-laws-of-planetary motion#:~:text=Kepler%27s%20first%20law%20states%20that,two%20foci%20is%20a%20constant.

[2] Wales, U. of N. S. (2021, March 21). *Mars settlement likely by 2050 says expert – but not at levels predicted by Elon Musk*. SciTechDaily. Retrieved November 10, 2022, from https://scitechdaily.com/mars-settlement-likely-by-2050-says-expert-but-not-at-levels-predicted-by-elon-musk/#:~:text=Robotic%20mining%20that%20can%20provide,quickly%20become%20more%20commercially%20viable.

[3] "Keplers Laws and the Motion of Planets (13.5)." *UNIVERSITY PHYSICS*, by Hugh D. Young and Roger A. Freedman, PEARSON, 2016, pp. 409-415.

Background Information

There are a few key terms and methodologies that should be discussed before the investigation to ensure there is no confusion later. Firstly, the term **perihelion** is the point in a planets orbit where it is closest to its primary body (the sun), and **aphelion** is the furthest point from the sun in a planets orbit.[4] Moreover, for any vector z, it will be defined as $\vec{\boldsymbol{z}}$. Finally, note that the cross product of two vectors $\vec{\boldsymbol{a}}$ and $\vec{\boldsymbol{b}}$ is $(\vec{\boldsymbol{a}} \times \vec{\boldsymbol{b}}) = \|\boldsymbol{a}\|\|\boldsymbol{b}\|\boldsymbol{sin}\ \boldsymbol{\theta}$ and the dot product of the same vectors is $(\vec{\boldsymbol{a}} \cdot \vec{\boldsymbol{b}}) = \|\vec{\boldsymbol{a}}\|\|\vec{\boldsymbol{b}}\|\boldsymbol{cos}\ \boldsymbol{\theta}$.[5]

Derivation of Kepler's First Law

Kepler's first law tells us that all planets have an elliptical orbit, however, even Kepler himself did not understand why the planets followed this law; he was simply basing it off his observations.[6] It was not until a century later when Isaac Newton came along with a rigorous derivation of Kepler's law after he noticed it was a direct consequence of his Law of Gravitation.[7] Our goal is not to copy Newtons derivation, rather, it is to produce our own simplified derivation that is easy to understand, which can ultimately be used to determine an orbit equation and model the orbit of Mars.

Let's first define a few vectors. Let $\vec{\boldsymbol{r}}$ be the position vector of the planet with magnitude $\|\vec{\boldsymbol{r}}\| = \boldsymbol{r}$, at any given time (t). Therefore, this position vector is everchanging as a function of time. This is because it changes with respect to the angle $\boldsymbol{\theta}$ between the vector and the plane, which itself changes with time. This vector $\vec{\boldsymbol{r}}$ is a *radial vector* since it points directly away from the origin at any given time. $\vec{\boldsymbol{v}}$ is the velocity vector, and it is equal to the derivative of $\vec{\boldsymbol{r}}$ with respect to time, based on our already known knowledge that the derivative of displacement with respect to time is velocity. This gives us

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[4] *Perihelion, aphelion and the solstices*. 2022 / 2023. (n.d.). Retrieved November 10, 2022, from https://www.timeanddate.com/astronomy/perihelion-aphelion-solstice.html

[5] *Dot and cross products on vectors*. GeeksforGeeks. (2022, September 19). Retrieved November 10, 2022, from https://www.geeksforgeeks.org/dot-and-cross-products-on-vectors/

[6] "Keplers Laws and the Motion of Planets (13.5)." *UNIVERSITY PHYSICS*, by Hugh D. Young and Roger A. Freedman, PEARSON, 2016, pp. 409-415.

[7] Libretexts. (2020, November 5). *5.6: Kepler's laws*. Physics LibreTexts. Retrieved November 10, 2022, from https://phys.libretexts.org/Bookshelves/University_Physics/Book%3A_Physics_(Boundless)/5%3A_Uniform_Circular_Motion_and_Gravitation/5.6%3A_Keplers_Laws#:~:text=Kepler%27s%20third%20law%20can%20be,2a3GM.

$\vec{v} = \frac{d\vec{r}}{dt}$. This velocity vector is the *tangential velocity* of the planet, which simply means the vector is at a tangent to the elliptical orbit at a given time. What this also means is that our vector $\vec{v}$ is always orthogonal (i.e. perpendicular) to $\vec{r}$**.** Additionally, $\vec{a}$ the acceleration vector, is $\vec{a} = \frac{d\vec{v}}{dt} = \frac{d^2\vec{r}}{dt^2}$ (based on the fact that the derivative of velocity with respect to time is acceleration).[8] $\vec{a}$ is specifically the centripetal acceleration, whose vector points directly towards the origin, meaning it points in the exact opposite direction as $\vec{r}$. It is the centripetal acceleration caused by the Suns force of gravity that keeps the planet in orbit.[9] The final vector is $\vec{u}$**,** the unit vector of $\vec{r}$**,** meaning $\vec{u} = \frac{\vec{r}}{r}$**.** [10]

With these defined vectors, we can use Newtons Law of Gravitation and second law of motion to determine an equation that we can manipulate into the ellipse formula to prove Keplers first Law.

Law of Gravitation: $\vec{F} = -\frac{GMm}{r^2}\vec{u}$ [11]

$\vec{F}$ = Gravitational force the sun exerts on the planet

G = Gravitational Constant

M = Mass of the sun

m = Mass of the planet (Mars)

r = radius between the objects (which is also the magnitude of $\vec{r}$)

$\vec{u}$ = unit vector of $\vec{r}$

Second Law of Motion: $\vec{F} = \mathbf{m}\vec{a}$ [12]

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[8] Libretexts. (2020, December 21). *2.5: Velocity and acceleration*. Mathematics LibreTexts. Retrieved November 10, 2022, from https://math.libretexts.org/Bookshelves/Calculus/Supplemental_Modules_(Calculus)/Vector_Calculus/2%3A_Vector-Valued_Functions_and_Motion_in_Space/2.5%3A_Velocity_and_Acceleration

[9] NASA. (n.d.). *Basics of space flight - solar system exploration: NASA science*. NASA. Retrieved November 10, 2022, from https://solarsystem.nasa.gov/basics/chapter3-3/#:~:text=Acceleration%20in%20Orbit,-Newton%27s%20first%20law&text=To%20move%20in%20a%20curved,the%20sun%20and%20the%20planet.

[10] *Unit vector - formula, definition, caculate, notation*. Cuemath. (n.d.). Retrieved November 10, 2022, from https://www.cuemath.com/calculus/unit-vector/

[11] "Keplers Laws and the Motion of Planets (13.5)." *UNIVERSITY PHYSICS*, by Hugh D. Young and Roger A. Freedman, PEARSON, 2016, pp. 409-415 .

[12] "Keplers Laws and the Motion of Planets (13.5)." *UNIVERSITY PHYSICS*, by Hugh D. Young and Roger A. Freedman, PEARSON, 2016, pp. 409-415 .

The first step in deriving Kepler's First Law is proving that the planets orbit the sun in one plane (*xy*, *xz* or *yz*). Let's start by setting the second law of motion equal to the law of gravitation:

$$\mathbf{m}\vec{\boldsymbol{a}} = -\frac{\boldsymbol{GMm}}{\boldsymbol{r^2}}\vec{\boldsymbol{u}}$$

$$\vec{\boldsymbol{a}} = -\frac{\boldsymbol{GM}}{\boldsymbol{r^2}}\vec{\boldsymbol{u}}$$

This tells us that the centripetal acceleration is equal to a scalar times the unit vector, but since *G*, *M* and *r* are always positive, the centripetal acceleration has a direction directly opposite to the radial vector. This makes sense since the radial vector points directly away from the origin, so the centripetal acceleration vector points directly towards the origin, as expected.[13]

The methodology to prove that the planet orbits in only one plane is quite elegant. We can come to this conclusion by taking the cross product of $\vec{\boldsymbol{r}}$ and $\vec{\boldsymbol{v}}$, the radial and tangential vectors, and then taking their derivative with respect to time. Logically speaking, $\vec{\boldsymbol{r}} \perp \vec{\boldsymbol{v}}$, since $\vec{\boldsymbol{v}}$ is tangent to any point on the elliptical orbit described by the position vector $\vec{\boldsymbol{r}}$. Therefore, since the cross product produces a vector orthogonal to both vectors involved in the product, the vector yielded, $\vec{\boldsymbol{c}}$ ($\vec{\boldsymbol{c}} = \vec{\boldsymbol{r}} \times \vec{\boldsymbol{v}}$), will be directly on the *z* axis (assume that $\vec{\boldsymbol{r}}$ and $\vec{\boldsymbol{v}}$ exist on the *xy* plane). The calculation is shown below:

$$\frac{d}{dt}(\vec{\boldsymbol{r}} \times \vec{\boldsymbol{v}}) = \vec{\boldsymbol{r}} \times \frac{d\vec{v}}{dt} + \vec{\boldsymbol{v}} \times \frac{d\vec{r}}{dt}$$

$$\frac{d}{dt}(\vec{\boldsymbol{r}} \times \vec{\boldsymbol{v}}) = \vec{\boldsymbol{r}} \times \vec{\boldsymbol{a}} + \vec{\boldsymbol{v}} \times \vec{\boldsymbol{v}}$$

$$\frac{d}{dt}(\vec{\boldsymbol{r}} \times \vec{\boldsymbol{v}}) = \mathbf{0} + \mathbf{0}$$

$$\frac{d}{dt}(\vec{\boldsymbol{r}} \times \vec{\boldsymbol{v}}) = \mathbf{0}$$

[13] OpenStax. (2016, August 3). *University physics volume 1*. 4.4 Uniform Circular Motion | University Physics Volume 1. Retrieved November 10, 2022, from https://courses.lumenlearning.com/suny-osuniversityphysics/chapter/4-4-uniform-circular-motion/

Since the derivative of the cross product equals 0, we can deduce using our calculus knowledge that the cross product itself was a constant value. Similar to how when the derivative of a function is zero, its rate of change is zero, when the derivative of $(\vec{\boldsymbol{r}} \times \vec{\boldsymbol{v}})$, and therefore the derivative of $\vec{\boldsymbol{c}}$ is zero, it also has a rate of change equal to zero.[14] This means that $\vec{\boldsymbol{c}}$ is a *constant vector* (it does not change with respect to time unlike $\vec{\boldsymbol{r}}, \vec{\boldsymbol{v}}$ and $\vec{\boldsymbol{a}}$). Since $\vec{\boldsymbol{c}}$ is constant, for $\vec{\boldsymbol{r}}$ and $\vec{\boldsymbol{v}}$ to be orthogonal to $\vec{\boldsymbol{c}}$, they both must be in only one plane, otherwise $\vec{\boldsymbol{c}}$ would not remain constant. For simplicity's sake, the $\vec{\boldsymbol{r}}$ and $\vec{\boldsymbol{v}}$ vectors will be on the *xy* plane, meaning that $\vec{\boldsymbol{c}}$ is directly on the *z* axis. Let's also take a quick look at why both cross products in the second line of calculation equal to 0. Firstly, $\vec{\boldsymbol{r}} \times \vec{\boldsymbol{a}}$, is equal to 0 because the angle between these vectors is exactly 180°. Therefore, $\vec{\boldsymbol{r}} \times \vec{\boldsymbol{a}} = \|\vec{r}\|\|\vec{a}\|sin(180°) = \|\vec{r}\|\|\vec{a}\|(0) = \mathbf{0}$.

Secondly, $\vec{\boldsymbol{v}} \times \vec{\boldsymbol{v}} = 0$ because the angle between these two vectors is 0° since they are the same vector. Thus, $\|\vec{\boldsymbol{v}}\|\|\vec{\boldsymbol{v}}\|sin(0°) = \|\vec{\boldsymbol{v}}\|\|\vec{\boldsymbol{v}}\|(0) = \mathbf{0}$. Let's now use this information to come to our general equation of planetary motion.

Firstly, here is a visual of all the information gathered.

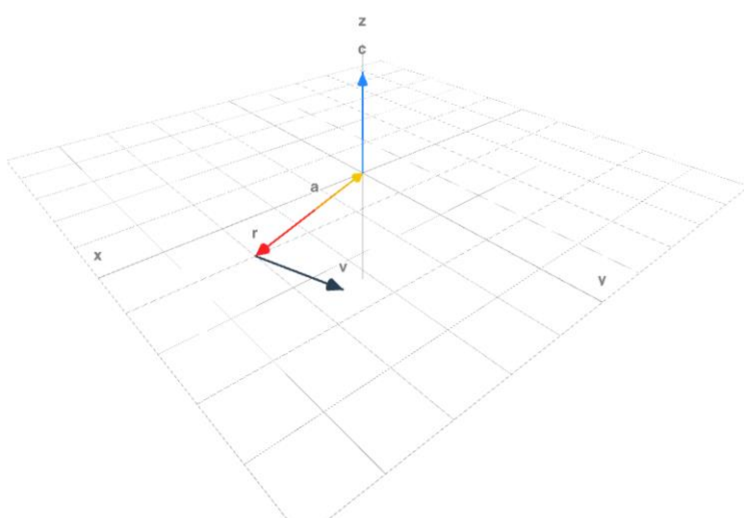

Figure 1: *Visualization of all vectors using Math3d.org* (Note that: a = $\vec{\boldsymbol{a}}$, r = $\vec{\boldsymbol{r}}$, v = $\vec{\boldsymbol{v}}$, c = $\vec{\boldsymbol{c}}$)

[14] Libretexts. (2020, November 10). *13.2: Derivatives and integrals of vector functions*. Mathematics LibreTexts. Retrieved November 10, 2022, from https://math.libretexts.org/Bookshelves/Calculus/Map%3A_Calculus__Early_Transcendentals_(Stewart)/13%3A_Vector_Functions/13.02%3A_Derivatives_and_Integrals_of_Vector_Functions

What is the next step to deriving Kepler's law? Recall that the radial position vector $\vec{\boldsymbol{r}}$ is orthogonal to both $\vec{\boldsymbol{c}}$ and $\vec{\boldsymbol{v}}$. We want to be able to get the cross product of $\vec{\boldsymbol{c}}$ and $\vec{\boldsymbol{v}}$ in a form where the terms *G* and *M* from Newtons Law of Gravitation are present. This is necessary since it would result in a vector that is in the same direction as $\vec{\boldsymbol{r}}$ and on the *xy* plane. To start solving for $(\vec{\boldsymbol{c}} \times \vec{\boldsymbol{v}})$**,** we first need to take the derivative of the cross product: $\frac{d}{dt}(\vec{\boldsymbol{c}} \times \vec{\boldsymbol{v}}) = \vec{\boldsymbol{c}} \times \frac{d\vec{\boldsymbol{v}}}{dt} = \vec{\boldsymbol{c}} \times \vec{\boldsymbol{a}}$. The reason I took the derivative of the cross product is because we can find $\vec{\boldsymbol{c}}$ in terms of $\vec{\boldsymbol{r}}$ and $\vec{\boldsymbol{u}}$ using a bit of mathematical manipulation. Moreover, we previously derived that $\vec{\boldsymbol{a}} = -\frac{\boldsymbol{GM}}{\boldsymbol{r^2}}\vec{\boldsymbol{u}}$, which has both **r** and $\vec{\boldsymbol{u}}$ in it. The idea is to cancel out the **r** and $\vec{\boldsymbol{u}}$ in the cross product $\vec{\boldsymbol{c}} \times \vec{\boldsymbol{a}}$, and then rewrite $\vec{\boldsymbol{a}}$ as the derivative of $\vec{\boldsymbol{v}}$. This is expected to result in two derivatives with respect to time on opposite sides of the equation, which cancel out, ultimately leaving $\vec{\boldsymbol{c}} \times \vec{\boldsymbol{v}}$ on one side of the equation, and the result of the cross product on the other side. Firstly, we know that $\vec{\boldsymbol{r}} = \boldsymbol{r}\vec{\boldsymbol{u}}$, and using this, we can infer that:

$$\vec{\boldsymbol{v}} = \frac{d}{dt}(\boldsymbol{r}\vec{\boldsymbol{u}})$$

$$\vec{\boldsymbol{v}} = \boldsymbol{r}\frac{d\vec{\boldsymbol{u}}}{dt} + \vec{\boldsymbol{u}}\frac{dr}{dt} \qquad \text{(product rule)}$$

Moreover, we know that $\vec{\boldsymbol{c}} = \vec{\boldsymbol{v}} \times \vec{\boldsymbol{r}}$. Substituting $\boldsymbol{r}\frac{d\vec{\boldsymbol{u}}}{dt} + \vec{\boldsymbol{u}}\frac{dr}{dt}$ for $\vec{\boldsymbol{v}}$ and $\boldsymbol{r}\vec{\boldsymbol{u}}$ for $\vec{\boldsymbol{r}}$:

$$\vec{\boldsymbol{c}} = \boldsymbol{r}\vec{\boldsymbol{u}} \times \left(\boldsymbol{r}\frac{d\vec{\boldsymbol{u}}}{dt} + \vec{\boldsymbol{u}}\frac{dr}{dt} \right)$$

$$\vec{\boldsymbol{c}} = \boldsymbol{r}^2 \left(\vec{\boldsymbol{u}} \times \frac{d\vec{\boldsymbol{u}}}{dt}\right) + \boldsymbol{r}\frac{dr}{dt}(\vec{\boldsymbol{u}} \times \vec{\boldsymbol{u}})$$

$$\vec{\boldsymbol{c}} = \boldsymbol{r}^2 \left(\vec{\boldsymbol{u}} \times \frac{d\vec{\boldsymbol{u}}}{dt}\right)$$

Note that the second term, $\boldsymbol{r}\frac{dr}{dt}(\vec{\boldsymbol{u}} \times \vec{\boldsymbol{u}})$, evaluates to 0 since the cross product of a vector with itself is 0 as we found earlier in this exploration by using the cross product formula.

Finally, we can substitute our expressions for $\vec{\boldsymbol{c}}$ and $\vec{\boldsymbol{a}}$, and then simplify the results to get $\vec{\boldsymbol{c}} \times \vec{\boldsymbol{v}}$**.**

$$\vec{\boldsymbol{a}} \times \vec{\boldsymbol{c}} = -\frac{\boldsymbol{GM}}{\boldsymbol{r}^2}\vec{\boldsymbol{u}} \times \boldsymbol{r}^2 (\vec{\boldsymbol{u}} \times \frac{d\vec{\boldsymbol{u}}}{dt})$$

$$\vec{\boldsymbol{a}} \times \vec{\boldsymbol{c}} = -\boldsymbol{GM}\vec{\boldsymbol{u}} \times (\vec{\boldsymbol{u}} \times \frac{d\vec{\boldsymbol{u}}}{dt}) \qquad (\boldsymbol{r}^2 \text{ terms cancel out})$$

$$\vec{\boldsymbol{a}} \times \vec{\boldsymbol{c}} = -\boldsymbol{GM}[\vec{\boldsymbol{u}} \times (\vec{\boldsymbol{u}} \times \frac{d\vec{\boldsymbol{u}}}{dt})]$$

$$\vec{\boldsymbol{a}} \times \vec{\boldsymbol{c}} = -\boldsymbol{GM}[(\vec{\boldsymbol{u}} \cdot \frac{d\vec{\boldsymbol{u}}}{dt})\,\vec{\boldsymbol{u}} - (\vec{\boldsymbol{u}} \cdot \vec{\boldsymbol{u}})\,\frac{d\vec{\boldsymbol{u}}}{dt}]$$

Note that for a vector A: A × (A × A’) = (A · A’)A – (A · A)A’ [15]

Using this identity, we turned step 3 in the calculation to step 4. Let’s now simplify the final step of the calculation we got. Firstly, $\vec{\boldsymbol{u}} \cdot \vec{\boldsymbol{u}} = \|\vec{\boldsymbol{u}}\|\|\vec{\boldsymbol{u}}\|\boldsymbol{\cos\theta}$ = (1)(1)cos(0) = (1)(1)(1) = 1. Moreover, $\vec{\boldsymbol{u}} \cdot \frac{d\vec{\boldsymbol{u}}}{dt} = 0$. We can proof this using a chain of logic based on information we already know. The unit vector $\vec{\boldsymbol{u}}$ is in the same direction as $\vec{\boldsymbol{r}}$. Additionally, $\frac{d\vec{\boldsymbol{r}}}{dt}$ is simply $\vec{\boldsymbol{v}}$ which is perpendicular to $\vec{\boldsymbol{r}}$ as stated and proven earlier when we determined that the cross product of $\vec{\boldsymbol{v}}$ and $\vec{\boldsymbol{r}}$ is 0. Also, $\frac{d\vec{\boldsymbol{u}}}{dt}$ must be in the same direction as $\frac{d\vec{\boldsymbol{r}}}{dt}$ since $\vec{\boldsymbol{u}}$ and $\vec{\boldsymbol{r}}$ are in the same direction. Therefore, connecting the logic, if $\frac{d\vec{\boldsymbol{r}}}{dt}$ is perpendicular to $\vec{\boldsymbol{r}}$, then $\frac{d\vec{\boldsymbol{u}}}{dt}$ is perpendicular to $\vec{\boldsymbol{u}}$. Since $\frac{d\vec{\boldsymbol{u}}}{dt}$ and $\vec{\boldsymbol{u}}$ are perpendicular, their dot product must be 0 because:

$$\frac{d\vec{\boldsymbol{u}}}{dt} \cdot \vec{\boldsymbol{u}} = \left\|\frac{d\vec{\boldsymbol{u}}}{dt}\right\| \|\vec{\boldsymbol{u}}\|\boldsymbol{\cos\theta} = \left\|\frac{d\vec{\boldsymbol{u}}}{dt}\right\| \|\vec{\boldsymbol{u}}\|\boldsymbol{\cos}\,(90°) = \left\|\frac{d\vec{\boldsymbol{u}}}{dt}\right\| \|\vec{\boldsymbol{u}}\|\,(0) = 0.$$

This further simplification gives us:

$$\vec{\boldsymbol{a}} \times \vec{\boldsymbol{c}} = \boldsymbol{GM}\,\frac{d\vec{\boldsymbol{u}}}{dt}$$

Finally, we can rewrite $\vec{\boldsymbol{a}}$ as the derivative of velocity with respect to time and place $\boldsymbol{GM}$ inside of the derivative with $\vec{\boldsymbol{u}}$ since it is a constant.

[15] *Cross product*. from Wolfram MathWorld. (n.d.). Retrieved November 10, 2022, from https://mathworld.wolfram.com/CrossProduct.html

$$\frac{d\vec{v}}{dt} \times \vec{c} = GM\frac{d\vec{u}}{dt}$$

$$\frac{d(\vec{v} \times \vec{c})}{dt} = \frac{d(GM\vec{u})}{dt}$$

$$\int \frac{d(\vec{v} \times \vec{c})}{dt}\, dt = \int \frac{d(GM\vec{u})}{dt}\, dt$$

integrating the equation allows us to remove the $\frac{d}{dt}$ from both sides, leaving us with:

$$\vec{v} \times \vec{c} = GM\vec{u} + \vec{f}$$

As we can see, manipulating already known information in a very clever way allowed us to determine the value of the cross product we wanted. We can now implement one last step to reach our equation. But wait, why did we add the $(+\vec{f})$ term to our solution? An integral is just an antiderivative. This means that if we take the derivative of a function, and then take its integral, we end up back to the original function. However, when we take the derivative of a function, we lose any constants that were present in it, so to make up for that, we add a "+ c" (the integration constant) when taking the integral of a function.[16] When taking the integral involving vectors, we add a constant vector term instead, in this case defined as $\vec{f}$. This constant vector lies directly on the *x* axis as it is orthogonal to $\vec{c}$.

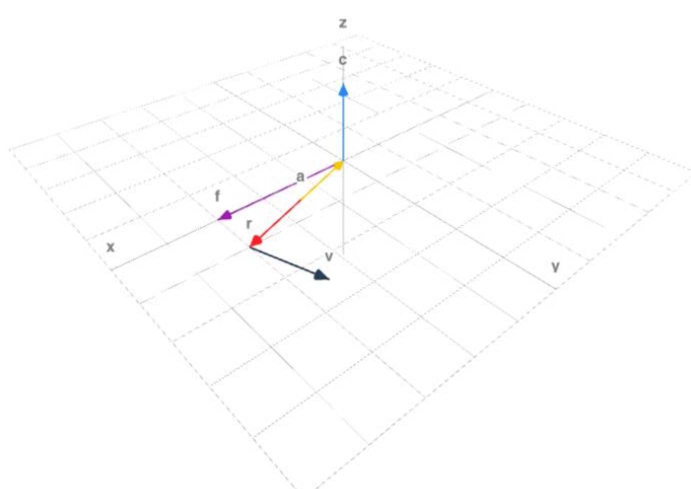

Figure 2: *Visualization of all vectors* (*f* = $\vec{f}$ and all other variables remain the same as in *Figure 1*)

[16] *13.2 derivatives and integrals of vector functions - USNA*. (n.d.). Retrieved November 11, 2022, from https://www.usna.edu/Users/oceano/raylee/SM223/Ch13_2_Stewart(2016).pdf

What we just discovered is crucial to reaching the solution we want. Now is a good time to introduce the polar equation of an ellipse. The equation is shown below:[17]

$$\boldsymbol{r} = \frac{\boldsymbol{ed}}{(\mathbf{1}+\boldsymbol{e}\ \boldsymbol{cos\theta})}$$

Where:

e = eccentricity of the ellipse (a measure of how much the ellipse deviates from a circle)

d = directrix of the ellipse

- Line parallel to the latus rectum, a vertical line through the focus of the ellipse. Since an ellipse has two foci, there are two latus recta.
- Line perpendicular to the major axis of the ellipse
- Since there are two latus recta, an ellipse has two directrices, both an equidistance away from the centre of the ellipse

r = radius of the ellipse at an angle $\boldsymbol{\theta}$

$\boldsymbol{\theta}$ = angle between the major axis and ***r***

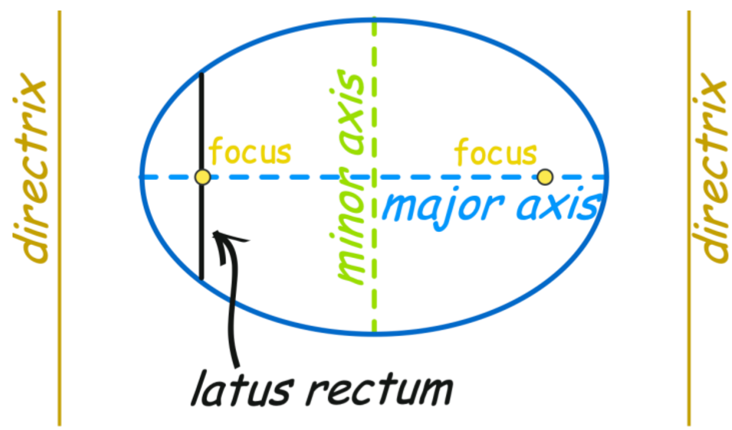

Figure 3: *Diagram of the components of an ellipse*

Let's now intuitively think about how we can use our known cross products and dot products to derive the equation of an ellipse, to finally prove the elliptical nature of Mars' orbit. Firstly, notice that we need to isolate ***r*** on the left-hand side of the equation. This can be done using our cross product ($\vec{\boldsymbol{v}} \times \vec{\boldsymbol{c}}$), and our previous cross product:

[17] *Ellipses in polar form*. Ellipses. (n.d.). Retrieved November 10, 2022, from https://ellipsesconicsections.weebly.com/ellipses-in-polar-form.html

$\vec{\boldsymbol{c}} = \vec{\boldsymbol{r}} \times \vec{\boldsymbol{v}}$. Since $\vec{\boldsymbol{c}}$ is a constant, we can say that:

$$(\vec{\boldsymbol{r}} \times \vec{\boldsymbol{v}}) \cdot \vec{\boldsymbol{c}} = \vec{\boldsymbol{c}}^{\,2}$$

Which also yields a constant, $\vec{\boldsymbol{c}}^{\,2}$. Now, using a cross and dot product identity, we can rearrange our equation to:

$$\vec{\boldsymbol{c}}^{\,2} = \vec{\boldsymbol{r}} \cdot (\vec{\boldsymbol{c}} \times \vec{\boldsymbol{v}})$$

As planned, we have already evaluated the $\vec{\boldsymbol{c}} \times \vec{\boldsymbol{v}}$ cross product, so let's go ahead and substitute it back into the equation:

$$\vec{\boldsymbol{c}}^{\,2} = \vec{\boldsymbol{r}} \cdot (\boldsymbol{GM}\vec{\boldsymbol{u}} + \vec{\boldsymbol{f}})$$

A little bit of critical thinking is needed here to get to the next step. A key part of the polar ellipse equation is the cos$\boldsymbol{\theta}$ term, so how can we go about implementing it into this equation. First, lets expand the dot product, since we know that dot products can be distributed across two added terms in brackets.

$$\vec{\boldsymbol{c}}^{\,2} = \vec{\boldsymbol{r}} \cdot \boldsymbol{GM}\vec{\boldsymbol{u}} + \vec{\boldsymbol{r}} \cdot \vec{\boldsymbol{f}}$$

The next step is to simplify what we have right now. The first term can be simplified very easily since the dot product of $\vec{\boldsymbol{r}}$ and $\vec{\boldsymbol{u}}$ is simply $\boldsymbol{r}$. Recall that $\vec{\boldsymbol{u}}$ is in the same direction as $\vec{\boldsymbol{r}}$ meaning that the angle between them is exactly 0°. Therefore:

$$\vec{\boldsymbol{r}} \cdot \vec{\boldsymbol{u}} = \|\vec{\boldsymbol{r}}\|\|\vec{\boldsymbol{u}}\|\boldsymbol{cos}(0°)$$

$$\vec{\boldsymbol{r}} \cdot \vec{\boldsymbol{u}} = \|\vec{\boldsymbol{r}}\|(1)(1)$$

$$\vec{\boldsymbol{r}} \cdot \vec{\boldsymbol{u}} = \boldsymbol{r}$$

(Recall that we defined $\|\vec{\boldsymbol{r}}\| = \boldsymbol{r}$ earlier)

Moreover, we mentioned that we needed to implement a cos$\boldsymbol{\theta}$ into the equation. We can do this by writing out the expression for $\vec{\boldsymbol{r}} \cdot \vec{\boldsymbol{f}}$.

Overall, this converts our equation into:

$$\vec{c}^{\,2} = GMr + \|\vec{r}\|\|\vec{f}\|\cos\theta$$

$$\vec{c}^{\,2} = GMr + r\|\vec{f}\|\cos\theta$$

$$\vec{c}^{\,2} = GMr + rf\cos\theta \qquad (\text{let } \|\vec{f}\| = f)$$

Finally, we just need to rearrange the equation in terms of r.

$$\vec{c}^{\,2} = r(GM + f\cos\theta)$$

$$r = \frac{\vec{c}^{\,2}}{(GM + f\cos\theta)}$$

We did get an expression in terms of r, but we are just not there quite yet. We have these terms $\vec{c}^{\,2}$, GM, and f that we need to convert into ed, 1, and d respectively. We actually need to do a bit of *unit analysis* to determine how to remove out the unwanted terms and replace them with needed terms. Firstly, notice that GM needs to become 1. This can of course be done by dividing everything by GM.

$$r = \frac{\frac{\vec{c}^{\,2}}{GM}}{\left(\frac{GM}{GM} + \frac{f\cos\theta}{GM}\right)}$$

As mentioned previously, the eccentricity, e, of an ellipse is a dimensionless quantity. Recall that when we were finding $\vec{v} \times \vec{c}$, the result was $GM\vec{u} + \vec{f}$. What this tells us is that GM and f have the same units, meaning in $\frac{f}{GM}$, the units cancel out. Therefore, we can deduce that:

$$e = \frac{f}{GM}$$

What we have so far is:

$$r = \frac{\frac{\vec{c}^{\,2}}{GM}}{(1 + e\cos\theta)}$$

We know that $\frac{\vec{c}^{\,2}}{GM}$ will equal $\boldsymbol{ed}$, however, it is not satisfactory nor reflective to just state that it works, so let's think about why $\frac{\vec{c}^{\,2}}{GM} = \boldsymbol{ed}$. Firstly, we can rearrange our equation for $\boldsymbol{e}$ such that:

$$GM = \frac{f}{e}$$

Using this, we can substitute in $\frac{f}{e}$ for $\boldsymbol{GM}$ into our formula for $\boldsymbol{ed}$:

$$\frac{\vec{c}^{\,2}}{\frac{f}{e}} = ed$$

$$\frac{\vec{c}^{\,2}e}{f} = ed$$

$$\frac{\vec{c}^{\,2}}{f} = d$$

Finally, we know that $\boldsymbol{f} = \boldsymbol{eGM}$. Doing one more substitution:

$$\frac{\vec{c}^{\,2}}{eGM} = d$$

$$\frac{\vec{c}^{\,2}}{GM} = ed$$

Even though we showed mathematically that $\frac{\vec{c}^{\,2}}{GM} = \boldsymbol{ed}$, we can also make sense of it using our mathematical intuition. In the polar ellipse equation, all the terms on the right-hand side involving $\boldsymbol{e}$ are dimensionless, but the value $\boldsymbol{r}$, which we are solving for has units associated with it.[18] Therefore, $\boldsymbol{d}$ must be a value that is not dimensionless. The term $\frac{\vec{c}^{\,2}}{GM}$ also has units of measure, since if $\boldsymbol{f}$ and $\boldsymbol{GM}$ have the same units, and $\vec{f}$ is also orthogonal to $\vec{c}$ ($\vec{f}$ is on the *x* axis and $\vec{c}$ is on the *z* axis), then they cannot have the same units, thus, $\frac{\vec{c}^{\,2}}{GM}$ has units as they do not cancel out. Since $\frac{\vec{c}^{\,2}}{GM}$ has units, then that allows $\boldsymbol{d}$ to be a value with dimensions which is what we were looking for.

[18] *Eccentricity*. Definition of eccentricity in Algebra. (n.d.). Retrieved November 10, 2022, from http://kolibri.teacherinabox.org.au/modules/en-boundless/www.boundless.com/algebra/definition/eccentricity/index.html

We have reached the desired polar equation of an ellipse, proving that planets orbit elliptically!

$$\boldsymbol{r} = \frac{\boldsymbol{ed}}{(\boldsymbol{1} + \boldsymbol{e}\boldsymbol{cos}\boldsymbol{\theta})}$$

Note that the eccentricity value for an ellipse is always $0 < \boldsymbol{e} < 1$. For a parabola it is $\boldsymbol{e} = 1$, and for a hyperbola it is $\boldsymbol{e} > 1$.[19] When using data sheets to find out $\boldsymbol{e}$ values for our planet's orbits, it is important to ensure that the value falls in the range for an ellipse, otherwise, you may be looking at an incorrect value.

Applying the Polar Ellipse Equation for Mars' Orbit

Three decimal places of accuracy have been used for this analysis as it is the suggested value by NASA. The Nasa.gov planetary fact sheet for Mars states the following variables:[20]

eccentricity	0.0935
Aphelion	$249.261 \cdot 10^6$ km
Perihelion	$206.650 \cdot 10^6$ km

All we need to do now is to substitute the values to find the directrix, and then we will have an equation that represents Mars's orbit around the sun! Recall that at Perihelion, the angle $\boldsymbol{\theta}$ is 0° and at Aphelion the angle $\boldsymbol{\theta}$ 180°.

<u>First, lets find the directrix using aphelion:</u>

$$\boldsymbol{r} = \frac{\boldsymbol{ed}}{(\boldsymbol{1} + \boldsymbol{e}\boldsymbol{cos}\boldsymbol{\theta})}$$

$$249.261 \cdot 10^6 = \frac{(0.0935)d}{(1 + (0.0935)\cos(180°))}$$

$$249.261 \cdot 10^6 = \frac{(0.0935)d}{(1 + (0.0935)(-1))}$$

$$d_{aphelion} = \frac{249.261 \cdot 10^6 \cdot 0.9065}{0.0935} = 2{,}406{,}936{,}860.96 \ km$$

$$d_{aphelion} = \boldsymbol{2.41 \cdot 10^6\ km} \ \textit{(3 significant digits)}$$

[19] *Eccentricity*. Math is Fun. (n.d.). Retrieved November 10, 2022, from https://www.mathsisfun.com/geometry/eccentricity.html

[20] NASA. (n.d.). *Mars fact sheet*. NASA. Retrieved November 10, 2022, from https://nssdc.gsfc.nasa.gov/planetary/factsheet/marsfact.html

Next, let find the directrix using perihelion:

$$\boldsymbol{r} = \frac{\boldsymbol{ed}}{(\mathbf{1} + \boldsymbol{e\cos\theta})}$$

$$206.650 \cdot 10^6 = \frac{(0.0935)d}{(1 + (0.0935)\cos(0°))}$$

$$206.650 \cdot 10^6 = \frac{(0.0935)d}{(1 + (0.0935)(1))}$$

$$d_{perihelion} = \frac{206.650 \cdot 10^6 \cdot 1.0935}{0.0935} = 2{,}416{,}810{,}427.80\ km$$

$$d_{\text{perihelion}} = \boldsymbol{2.42 \cdot 10^6\ km}\ \textit{(3 significant digits)}$$

Just a quick note, logically, the reason the directrix from perihelion and aphelion are different is because the perihelion and aphelion exist in the first place. This means that the sun is not directly in the centre of Mars' orbit, which leads to the foci not being an equidistance away from the sun itself. As a result, the directrices are slightly different. To determine our final directrix value, *d*, we can simply take the average of $\boldsymbol{d}_{\text{aphelion}}$ and $\boldsymbol{d}_{\text{perihelion}}$:

$$\boldsymbol{d} = \frac{2{,}406{,}936{,}860.96\ km + 2{,}416{,}810{,}427.80\ km}{2} = 2.411 \cdot 10^9 \text{ km} \text{ (1 extra significant digit kept)}$$

There we have it! Our polar conic ellipse equation for Mars is:

$$\boldsymbol{r(\theta)} = \frac{(0.0935)(2.411 \cdot 10^9)}{(1 + 0.0935\ \boldsymbol{cos\theta})}$$

This is quite an elegant solution. We converted a complex physics and mathematics problem into a simple polar equation that represents the orbit of Mars around the sun. However, it is difficult to visualize the actual path Mars takes around the sun if our equation is in polar form. Therefore, the final part of this investigation is to convert our polar equation into a rectangular ellipse equation, which we can then graph to see the orbit of Mars.

Converting the Polar Equation into the Rectangular Equation

The equation of an ellipse in rectangular coordinates is:[21]

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$$

What we need to do is convert our polar equation into this form by using two conversions:

$x = r \cos \theta$
$r^2 = x^2 + y^2$

Let's reorganize the polar equation for an ellipse as such:

$$r(1 + e\cos\theta) = ed$$

$$r + er\cos\theta = ed$$

Now we can replace $r \cos \theta$ with x:

$$r + ex = ed$$

$$r = e(d - x)$$

One of the conversions listed requires an r^2 term, so lets square both sides of the equation so that we can replace r in terms of x and y. This brings us one step closer to the rectangular ellipse equation form since we eliminated the r and θ terms:

$$r^2 = e^2(d - x)^2$$

$$x^2 + y^2 = e^2(d - x)^2$$

$$x^2 + y^2 = e^2 (d^2 - 2dx + x^2)$$

$$x^2 + y^2 = e^2d^2 - 2e^2dx + e^2x^2$$

Notice that the ellipse formula only has one term on the right-hand side. Let's bring over the $2e^2dx + e^2x^2$ term to the left-hand side since it has our coordinate variable, x, in it:

$$x^2 + 2e^2dx - e^2x^2 + y^2 = e^2d^2$$

$$x^2(1 - e^2) + 2e^2dx + y^2 = e^2d^2$$

[21] Ellipses. (n.d.). Retrieved November 10, 2022, from https://web.ma.utexas.edu/users/m408m/Display10-5-3.shtml#:~:text=The%20standard%20formula%20for%20an,b))%20are%20called%20vertices.

$$x^2 + \frac{2e^2dx}{(1-e^2)} + \frac{y^2}{(1-e^2)} = \frac{e^2d^2}{(1-e^2)}$$

Now we can implement a really clever strategy to reach the form that we want. It is possible to complete the square of the quadratic relative to *x* and leave *y* alone since it is already in the state that we desire (y over a constant value). A quadratic is in the form $ax^2 + bx + c$. For our quadratic: $a = 1$ and $b = \frac{2e^2d}{(1-e^2)}$.

For completing the square, we need to add $(\frac{b}{2})^2$ to both sides, which equals to:

$$\left(\frac{2e^2d}{(1-e^2)} \cdot \frac{1}{2}\right)^2 = \left(\frac{e^2d}{(1-e^2)}\right)^2$$

Using this, we can now complete the square:

$$x^2 + \frac{2e^2dx}{(1-e^2)} + \left(\frac{e^2d}{(1-e^2)}\right)^2 \frac{y^2}{(1-e^2)} = \frac{e^2d^2}{(1-e^2)} + \left(\frac{e^2d}{(1-e^2)}\right)^2$$

$$\left(x + \frac{e^2d}{(1-e^2)}\right)^2 + \frac{y^2}{(1-e^2)} = \frac{e^2d^2}{(1-e^2)} + \frac{e^4d^2}{(1-e^2)^2}$$

$$\left(x + \frac{e^2d}{(1-e^2)}\right)^2 + \frac{y^2}{(1-e^2)} = \frac{e^2d^2(1-e^2)}{(1-e^2)(1-e^2)} + \frac{e^4d^2}{(1-e^2)^2}$$

$$\left(x + \frac{e^2d}{(1-e^2)}\right)^2 + \frac{y^2}{(1-e^2)} = \frac{e^2d^2(1-e^2) + e^4d^2}{(1-e^2)^2}$$

$$\left(x + \frac{e^2d}{(1-e^2)}\right)^2 + \frac{y^2}{(1-e^2)} = \frac{e^2d^2}{(1-e^2)^2}$$

Amazing! Our formula is starting to take shape. In the original formula, the right-hand side is equated to 1. Dividing both sides by $\frac{e^2d^2}{(1-e^2)^2}$ will give us our final rectangular equation!

Dividing by the left-hand side:

$$\frac{\left(x + \frac{e^2d}{(1-e^2)}\right)^2}{\frac{e^2d^2}{(1-e^2)^2}} + \frac{\frac{y^2}{(1-e^2)}}{\frac{e^2d^2}{(1-e^2)^2}} = 1$$

$$\frac{\left(x + \frac{e^2d}{(1-e^2)}\right)^2}{\frac{e^2d^2}{(1-e^2)^2}} + \frac{y^2}{\frac{e^2d^2}{(1-e^2)}} = 1$$

There we have it! We already found the universal polar elliptical orbit equation for all planets in our solar system, and now we have the rectangular version of it as well. As you can see:

$$h = -\frac{e^2 d}{(1-e^2)}, \quad k = 0, \quad a = \frac{e^2 d^2}{(1-e^2)^2}, \quad b = \frac{e^2 d^2}{(1-e^2)}$$

Let’s substitute our values for *e* and *d* to get the rectangular equation for Mars’ elliptical orbit.

$$\frac{\left(x + \frac{(0.0935)^2(2.411 \cdot 10^9)}{(1-(0.0935)^2)}\right)^2}{\frac{(0.0935)^2(2.411 \cdot 10^9)^2}{(1-(0.0935)^2)^2}} + \frac{y^2}{\frac{(0.0935)^2(2.411 \cdot 10^9)^2}{(1-(0.0935)^2)}} = 1$$

$$\frac{(x + 2.13 \cdot 10^7)^2}{5.17 \cdot 10^{16}} + \frac{y^2}{5.12 \cdot 10^{16}} = 1$$

After all this hard work, here are the two equations we came up with for the orbit of mars

$$\textit{Polar equation: } r(\theta) = \frac{(0.0935)(2.411 \cdot 10^9)}{(1 + 0.0935\, cos\boldsymbol{\theta})}$$

$$\textit{Rectangular Equation: } \frac{(x + 2.13 \cdot 10^7)^2}{5.17 \cdot 10^{16}} + \frac{y^2}{5.12 \cdot 10^{16}} = 1$$

Visual Representation of the Rectangular Equation

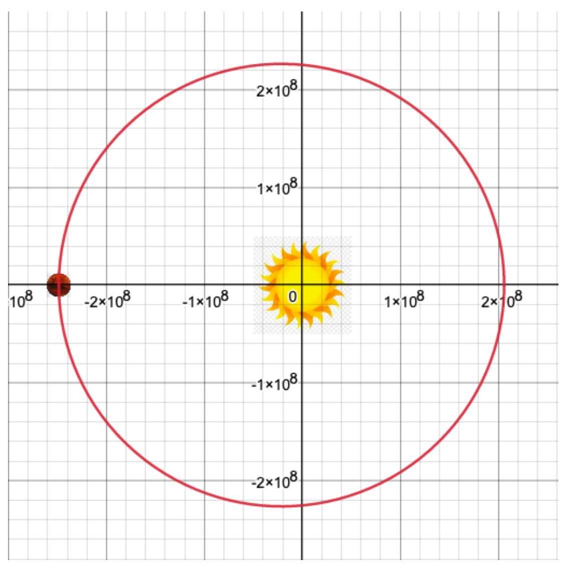

Figure 4: *Visual Representation of Mar’s orbit (This graph was produced using desmos.com)*

What a beautiful graph! It shows Mars at aphelion, and we can clearly see the elliptical nature of the orbit, as well as the fact that the sun is indeed not in the centre of the orbit, which is why there are aphelion and perihelion points in the first place. With that being said, what are the applications of being able to produce an equation for the orbit of a planet using Kepler’s first law, and how is this useful is this to astronomers as well as humanity?

Applications of The Derived Formulas

One very nice application of the equation we derived from Kepler's First Law is that we can find the period of Mars' orbit, or in other words, how long it takes for Mars to orbit the sun. First, we need to find the perimeter Mars' orbit, using the perimeter of an ellipse formula:[22]

$$Perimeter = 2\pi\sqrt{\frac{a^2+b^2}{2}}$$

$$Perimeter = 2\pi\sqrt{\frac{5.17 \cdot 10^{16} + 5.12 \cdot 10^{16}}{2}}$$

$$Perimeter = 1.425 \cdot 10^9\ km$$ (One extra significant digit kept)

Moreover, using data from Nasa.gov, the average speed at which Mars orbits the sun is 24.07 km/s. Utilizing a bit of basic math, we can determine the length of Mars' orbit:

$$Orbit\ Time = \frac{1.425 \cdot 10^9\ km}{24.07\,\frac{km}{s}}$$

$$Orbit\ Time = 59{,}210{,}191.76\ s$$

$$Orbit\ Time = 59{,}210{,}191.76\ s \cdot \frac{1\ day}{86{,}400\ s}$$

$$Orbit\ Time = 685.3 \text{ days}$$

There we have it! The orbit period for Mars is 685.3 days, and this value demonstrates the accuracy of our equation since it only deviates by 1.7 days from the actual value calculated by Nasa, which is 687.0 earth days. Other possible applications are determining the distance between Mars and the Sun at any point in the orbit (using the polar equation) or evaluating when two planets would be closest to each other. This can be done by creating an excel sheet function that first determines the coordinates of each planet using the rectangular equation, and then using another function that finds the distance between the two points (distance formula). Finally, the last thing we need to consider is accuracy and limitations; why and by how much do our values differ from database values.

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[22] *Perimeter of an ellipse*. Math is Fun Advanced. (n.d.). Retrieved November 10, 2022, from https://www.mathsisfun.com/geometry/ellipse-perimeter.html

Accuracy and Limitations of the Ellipse Equation

Firstly, lets take a look at a few limitations of our ellipse equation. The main limitation is that we assume the velocity of the orbiting planet stays the same at all points on the orbit. This is definitely false since the strength of gravity is a function of the distance between the centres of the two objects. This means that at perihelion, Mars is moving faster than it would be at aphelion. It is Kepler's second law that describes this phenomenon and is a great idea for a future extension of this exploration. Another limitation is the number of significant digits we keep. The eccentricity value that was given by NASA.gov was only to 3 significant digits, whereas the actual number was calculated to an extremely large number of significant digits since precision is very important for anything that may impact space travel. Due to the eccentricity having only 3 significant digits, all values in the ellipse equation were also limited to 3 significant digits since they were determined using the eccentricity. This loss of data could explain the 1.7 day difference in the calculated orbit period and actual orbit period for Mars. The table below determines the percent error and therefore accuracy of our final equations.[23]

Value	**Calculated**	**Actual**	**Percent Error**
Length of Mars' Orbit	$1.425 \cdot 10^9\ km$	$1.429 \cdot 10^9\ km$	0.280%
Period of Mars' Orbit	$685.3\ days$	$687.0\ days$	0.247%

Percent Error $= \frac{|Actual - Calculated|}{Actual} \times 100\%$

Our average percent error based on the two calculations done using the orbit equation comes out to 0.264%, which is an absurdly low percent error, showing not only the accuracy and precision with which we derived the polar equation, but also confirms that we correctly transformed the polar ellipse equation into the rectangular ellipse equation. Personally, I am very pleased with this result since it really demonstrates how nicely math can be used to put our ridiculously complex world into perspective, and how mathematics can allow us to accurately describe the world!

[23] NASA. (n.d.). *Mars fact sheet*. NASA. Retrieved November 10, 2022, from https://nssdc.gsfc.nasa.gov/planetary/factsheet/marsfact.html

Conclusion

Overall, we can clearly see the useful implications of this exploration and its significance to physics. I chose this topic because it corresponded with my level and understanding of mathematics and physics yet introduced new concepts that challenged my critical thinking. Through rigorous mathematical derivation, I was able to accurately model the orbit Mars takes around the sun. Considering the elliptical nature of the orbit, and the fact that it is not centred around the sun, this exploration helped me develop a deeper understanding of conics that I have never had before. Something I genuinely struggled with at the start of this exploration was being able to intuitively make sense of the formulas I was writing out; they worked theoretically, but I could not understand what they truly represented in the physical world. Through hours of research, and with the help of the physics textbook by Hugh D. Young, I was able to finally develop a clear comprehension of how the mathematics I was doing represented the real world around us. Another remarkable aspect of this exploration was the amount of critical thinking demanded, especially in the dimension analysis section where I had to derive the ellipse formula. Wanting to pursue a career in computer science, the courses are extremely heavy on proofs and logic, so I can happily say that this exploration will also help me with my future academic endeavors. Moreover, I was able to satiate my curiosity that led me to this exploration in the first place; I now understand how my Apple watch and other astronomers can model the orbits of planets. Lastly, this exploration allowed me to improve my skills with vectors, differentiation, integration, conics, polar functions, rectangular functions, complex algebraic manipulation and using modelling software.

Before concluding this exploration, I would like to critically reflect on the process that led me to my results. First, I proved that all planets orbit in a flat plane by deriving a constant vector irrespective to time. Then, through a long derivation and dimensional analysis, I demonstrated that the path of the orbit is elliptical in nature and has both a perihelion and aphelion, suggesting that the sun is not at the centre of the orbit. Finally, I developed a polar a formula for the orbit of Mars, which was then converted into a rectangular one with rigorous algebraic manipulation. This was applied to find the orbit period and orbit length of Mars, and comparison to known values allowed me to determine the accuracy of the investigation to be within an error of 0.247% and 0.280%. The equations, despite having a few limitations such as not being able to represent the changing acceleration of planets, are an immensely useful tool in the field of astronomy, and can serve as the rough groundwork for humanities eventual landing on Mars.

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