DS313: Statistical Foundations of Data Science
Assignment No: 6

Course Instructor: Siddhartha Sarma *Date:* 26 Apr 2025
Submission deadline: 9 May 2025, 10 AM

Related topics:

- **Random vectors and their distributions**
- **Estimation: MMSE, LMSE, Maximum-likelihood and MAP**

Notations:

- $\mathbf{E}[X], \mathrm{Var}(X)$ denote the expectation and variance of random variable X, respectively.
- $N(\mu, \sigma)$ denotes Gaussian distribution with mean μ and standard deviation $\sigma > 0$.

Instructions & Information:

- You must include a table similar to the one below on the front page of your submission to indicate the page numbers of the answer to the assignment questions.

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1. For any two random variables X, Y, show that

$$\frac{\sqrt{\mathrm{Var}(X+Y)}}{\sqrt{\mathrm{Var}X} + \sqrt{\mathrm{Var}Y}} \leq 1.$$

   Can the above expression be generalised for more than two random variables?

2. Let $[X \quad Y]^T$ be a mean zero Gaussian vector with correlation matrix $\begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix}$, $|\rho| < 1$.

   (a) Express $\Pr(X \leq 1|Y)$ in terms of ρ, Y and the standard normal CDF.

   (b) Find $\mathbf{E}[(X-Y)^2|Y=y]$.

3. Let X, Y be zero-mean independent random variables with variances σ_X^2, σ_Y^2, respectively. Consider the sum $Z = aX + (1-a)Y$, $0 \le a \le 1$.

   (a) Find a that minimizes the variance of Z.

   (b) Find MMSE, LMSE of (i) Z given X, (ii) Z given Y, (iii) X given Z, and (iv) Y given Z.

4. For any two random variables X, Y with $\mathbf{E}[X^2] < \infty$, show that $\mathrm{Var}(X) = \mathrm{Var}(\mathbf{E}[X|Y]) + \mathbf{E}[\mathrm{Var}(X|Y)]$.

5. Show that if $\mathbf{\Sigma}$ is the correlation matrix of the random vector $\mathrm{X} = [X_1, X_2, \ldots, X_N]^T$ and its inverse exists, then

$$\mathbf{E}[\mathrm{X}^T \mathbf{\Sigma}^{-1} \mathrm{X}] = N,$$

   where $(.)^T$ denotes transpose of a vector. Recall that correlation matrix is

6. The best linear predictor of Y with respect to X_1 and X_2 is equal to $a + bX_1 + cX_2$, where a, b, and c are chosen to minimize $\mathbf{E}[(Y - (a + bX_1 + cX_2))^2]$. Determine a, b, and c.

7. Let $X = 1/(1+U)$, where U is uniformly distributed over the interval $[0, 1]$ Find the MMSE and LMSE of X and compare their mean-sqaure errors.

8. The best quadratic predictor of Y with respect to X is $a + bX + cX^2$, where a, b, and c are chosen to minimize $\mathbf{E}[(Y - (a + bX + cX^2))^2]$. Determine a, b, and c.

9. The time to failure of a bulb is an exponential random variable X with an unknown parameter λ. By testing 80 bulbs, it was observed that 62 of them lasted more than 200 hours. Find the ML estimate of λ.

10. Derive an expression to calculate the MAP estimate of the parameter μ if the samples are drawn from $N(\mu, 1)$ and μ is exponentially distributed with parameter λ.

Some questions were taken from the following books

- Papoulis & Pillai, "Probability, Random Variables and Stochastic Processes", McGraw Hill, 4th Ed
- Sheldon Ross, "A first course in Probability", 8th Ed
- Lecture notes of Prof. Hajek, University of Illinois Urbana-Champaign