# 10 Elementary Data Structures

In this chapter, we examine the representation of dynamic sets by simple data structures that use pointers. Although we can construct many complex data structures using pointers, we present only the rudimentary ones: stacks, queues, linked lists, and rooted trees. We also show ways to synthesize objects and pointers from arrays.

## 10.1 Stacks and queues

Stacks and queues are dynamic sets in which the element removed from the set by the DELETE operation is prespecified. In a ***stack***, the element deleted from the set is the one most recently inserted: the stack implements a ***last-in, first-out***, or ***LIFO***, policy. Similarly, in a ***queue***, the element deleted is always the one that has been in the set for the longest time: the queue implements a ***first-in, first-out***, or ***FIFO***, policy. There are several efficient ways to implement stacks and queues on a computer. In this section we show how to use a simple array to implement each.

### Stacks

The INSERT operation on a stack is often called PUSH, and the DELETE operation, which does not take an element argument, is often called POP. These names are allusions to physical stacks, such as the spring-loaded stacks of plates used in cafeterias. The order in which plates are popped from the stack is the reverse of the order in which they were pushed onto the stack, since only the top plate is accessible.

As Figure 10.1 shows, we can implement a stack of at most $n$ elements with an array $S[1..n]$. The array has an attribute $S.top$ that indexes the most recently

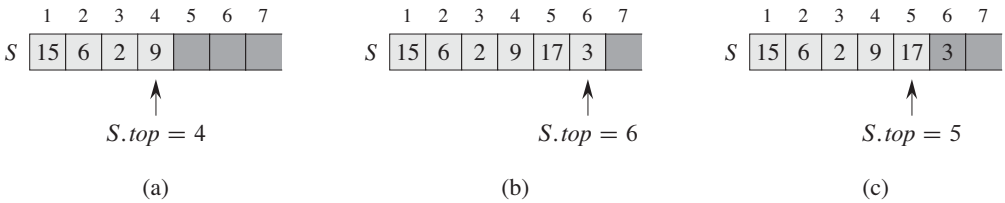


**Figure 10.1** An array implementation of a stack $S$. Stack elements appear only in the lightly shaded positions. **(a)** Stack $S$ has 4 elements. The top element is 9. **(b)** Stack $S$ after the calls PUSH$(S, 17)$ and PUSH$(S, 3)$. **(c)** Stack $S$ after the call POP$(S)$ has returned the element 3, which is the one most recently pushed. Although element 3 still appears in the array, it is no longer in the stack; the top is element 17.

inserted element. The stack consists of elements $S[1\,..\,S.top]$, where $S[1]$ is the element at the bottom of the stack and $S[S.top]$ is the element at the top.

When $S.top = 0$, the stack contains no elements and is ***empty***. We can test to see whether the stack is empty by query operation STACK-EMPTY. If we attempt to pop an empty stack, we say the stack ***underflows***, which is normally an error. If $S.top$ exceeds $n$, the stack ***overflows***. (In our pseudocode implementation, we don't worry about stack overflow.)

We can implement each of the stack operations with just a few lines of code:

```
STACK-EMPTY(S)
1  if S.top == 0
2       return TRUE
3  else return FALSE

PUSH(S, x)
1  S.top = S.top + 1
2  S[S.top] = x

POP(S)
1  if STACK-EMPTY(S)
2       error "underflow"
3  else S.top = S.top − 1
4       return S[S.top + 1]
```

Figure 10.1 shows the effects of the modifying operations PUSH and POP. Each of the three stack operations takes $O(1)$ time.

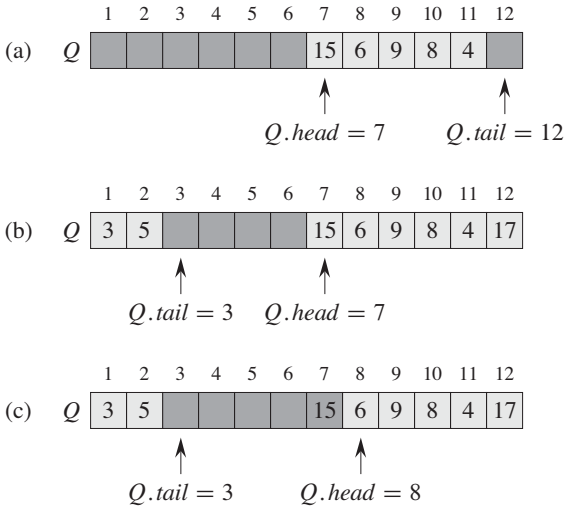


**Figure 10.2** A queue implemented using an array $Q[1..12]$. Queue elements appear only in the lightly shaded positions. **(a)** The queue has 5 elements, in locations $Q[7..11]$. **(b)** The configuration of the queue after the calls ENQUEUE$(Q, 17)$, ENQUEUE$(Q, 3)$, and ENQUEUE$(Q, 5)$. **(c)** The configuration of the queue after the call DEQUEUE$(Q)$ returns the key value 15 formerly at the head of the queue. The new head has key 6.

## Queues

We call the INSERT operation on a queue ENQUEUE, and we call the DELETE operation DEQUEUE; like the stack operation POP, DEQUEUE takes no element argument. The FIFO property of a queue causes it to operate like a line of customers waiting to pay a cashier. The queue has a ***head*** and a ***tail***. When an element is enqueued, it takes its place at the tail of the queue, just as a newly arriving customer takes a place at the end of the line. The element dequeued is always the one at the head of the queue, like the customer at the head of the line who has waited the longest.

Figure 10.2 shows one way to implement a queue of at most $n - 1$ elements using an array $Q[1..n]$. The queue has an attribute $Q.head$ that indexes, or points to, its head. The attribute $Q.tail$ indexes the next location at which a newly arriving element will be inserted into the queue. The elements in the queue reside in locations $Q.head, Q.head + 1, \ldots, Q.tail - 1$, where we "wrap around" in the sense that location 1 immediately follows location $n$ in a circular order. When $Q.head = Q.tail$, the queue is empty. Initially, we have $Q.head = Q.tail = 1$. If we attempt to dequeue an element from an empty queue, the queue underflows.

When $Q.head = Q.tail + 1$, the queue is full, and if we attempt to enqueue an element, then the queue overflows.

In our procedures ENQUEUE and DEQUEUE, we have omitted the error checking for underflow and overflow. (Exercise 10.1-4 asks you to supply code that checks for these two error conditions.) The pseudocode assumes that $n = Q.length$.

```
ENQUEUE(Q, x)
1  Q[Q.tail] = x
2  if Q.tail == Q.length
3       Q.tail = 1
4  else Q.tail = Q.tail + 1
```

```
DEQUEUE(Q)
1  x = Q[Q.head]
2  if Q.head == Q.length
3       Q.head = 1
4  else Q.head = Q.head + 1
5  return x
```

Figure 10.2 shows the effects of the ENQUEUE and DEQUEUE operations. Each operation takes $O(1)$ time.

## Exercises

### *10.1-1*

Using Figure 10.1 as a model, illustrate the result of each operation in the sequence PUSH$(S, 4)$, PUSH$(S, 1)$, PUSH$(S, 3)$, POP$(S)$, PUSH$(S, 8)$, and POP$(S)$ on an initially empty stack $S$ stored in array $S[1 \,..\, 6]$.

### *10.1-2*

Explain how to implement two stacks in one array $A[1 \,..\, n]$ in such a way that neither stack overflows unless the total number of elements in both stacks together is $n$. The PUSH and POP operations should run in $O(1)$ time.

### *10.1-3*

Using Figure 10.2 as a model, illustrate the result of each operation in the sequence ENQUEUE$(Q, 4)$, ENQUEUE$(Q, 1)$, ENQUEUE$(Q, 3)$, DEQUEUE$(Q)$, ENQUEUE$(Q, 8)$, and DEQUEUE$(Q)$ on an initially empty queue $Q$ stored in array $Q[1 \,..\, 6]$.

### *10.1-4*

Rewrite ENQUEUE and DEQUEUE to detect underflow and overflow of a queue.

***10.1-5***
Whereas a stack allows insertion and deletion of elements at only one end, and a queue allows insertion at one end and deletion at the other end, a ***deque*** (double-ended queue) allows insertion and deletion at both ends. Write four $O(1)$-time procedures to insert elements into and delete elements from both ends of a deque implemented by an array.

***10.1-6***
Show how to implement a queue using two stacks. Analyze the running time of the queue operations.

***10.1-7***
Show how to implement a stack using two queues. Analyze the running time of the stack operations.

## 10.2 Linked lists

A ***linked list*** is a data structure in which the objects are arranged in a linear order. Unlike an array, however, in which the linear order is determined by the array indices, the order in a linked list is determined by a pointer in each object. Linked lists provide a simple, flexible representation for dynamic sets, supporting (though not necessarily efficiently) all the operations listed on page 230.

As shown in Figure 10.3, each element of a ***doubly linked list*** $L$ is an object with an attribute *key* and two other pointer attributes: *next* and *prev*. The object may also contain other satellite data. Given an element $x$ in the list, $x.next$ points to its successor in the linked list, and $x.prev$ points to its predecessor. If $x.prev = \text{NIL}$, the element $x$ has no predecessor and is therefore the first element, or ***head***, of the list. If $x.next = \text{NIL}$, the element $x$ has no successor and is therefore the last element, or ***tail***, of the list. An attribute $L.head$ points to the first element of the list. If $L.head = \text{NIL}$, the list is empty.

A list may have one of several forms. It may be either singly linked or doubly linked, it may be sorted or not, and it may be circular or not. If a list is ***singly linked***, we omit the *prev* pointer in each element. If a list is ***sorted***, the linear order of the list corresponds to the linear order of keys stored in elements of the list; the minimum element is then the head of the list, and the maximum element is the tail. If the list is ***unsorted***, the elements can appear in any order. In a ***circular list***, the *prev* pointer of the head of the list points to the tail, and the *next* pointer of the tail of the list points to the head. We can think of a circular list as a ring of

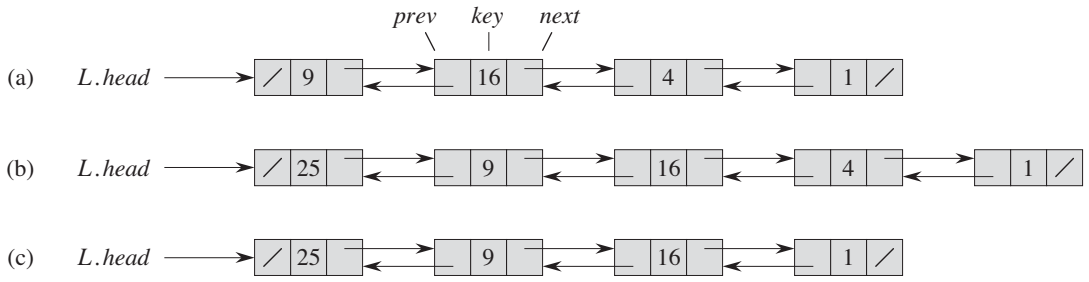


**Figure 10.3** **(a)** A doubly linked list $L$ representing the dynamic set $\{1, 4, 9, 16\}$. Each element in the list is an object with attributes for the key and pointers (shown by arrows) to the next and previous objects. The *next* attribute of the tail and the *prev* attribute of the head are NIL, indicated by a diagonal slash. The attribute $L.head$ points to the head. **(b)** Following the execution of LIST-INSERT$(L, x)$, where $x.key = 25$, the linked list has a new object with key 25 as the new head. This new object points to the old head with key 9. **(c)** The result of the subsequent call LIST-DELETE$(L, x)$, where $x$ points to the object with key 4.

elements. In the remainder of this section, we assume that the lists with which we are working are unsorted and doubly linked.

### Searching a linked list

The procedure LIST-SEARCH$(L, k)$ finds the first element with key $k$ in list $L$ by a simple linear search, returning a pointer to this element. If no object with key $k$ appears in the list, then the procedure returns NIL. For the linked list in Figure 10.3(a), the call LIST-SEARCH$(L, 4)$ returns a pointer to the third element, and the call LIST-SEARCH$(L, 7)$ returns NIL.

```
LIST-SEARCH(L, k)
1  x = L.head
2  while x ≠ NIL and x.key ≠ k
3       x = x.next
4  return x
```

To search a list of $n$ objects, the LIST-SEARCH procedure takes $\Theta(n)$ time in the worst case, since it may have to search the entire list.

### Inserting into a linked list

Given an element $x$ whose *key* attribute has already been set, the LIST-INSERT procedure "splices" $x$ onto the front of the linked list, as shown in Figure 10.3(b).

```
LIST-INSERT(L, x)
1  x.next = L.head
2  if L.head ≠ NIL
3       L.head.prev = x
4  L.head = x
5  x.prev = NIL
```

(Recall that our attribute notation can cascade, so that $L.head.prev$ denotes the $prev$ attribute of the object that $L.head$ points to.) The running time for LIST-INSERT on a list of $n$ elements is $O(1)$.

### Deleting from a linked list

The procedure LIST-DELETE removes an element $x$ from a linked list $L$. It must be given a pointer to $x$, and it then "splices" $x$ out of the list by updating pointers. If we wish to delete an element with a given key, we must first call LIST-SEARCH to retrieve a pointer to the element.

```
LIST-DELETE(L, x)
1  if x.prev ≠ NIL
2       x.prev.next = x.next
3  else L.head = x.next
4  if x.next ≠ NIL
5       x.next.prev = x.prev
```

Figure 10.3(c) shows how an element is deleted from a linked list. LIST-DELETE runs in $O(1)$ time, but if we wish to delete an element with a given key, $\Theta(n)$ time is required in the worst case because we must first call LIST-SEARCH to find the element.

### Sentinels

The code for LIST-DELETE would be simpler if we could ignore the boundary conditions at the head and tail of the list:

```
LIST-DELETE′(L, x)
1  x.prev.next = x.next
2  x.next.prev = x.prev
```

A ***sentinel*** is a dummy object that allows us to simplify boundary conditions. For example, suppose that we provide with list $L$ an object $L.nil$ that represents NIL

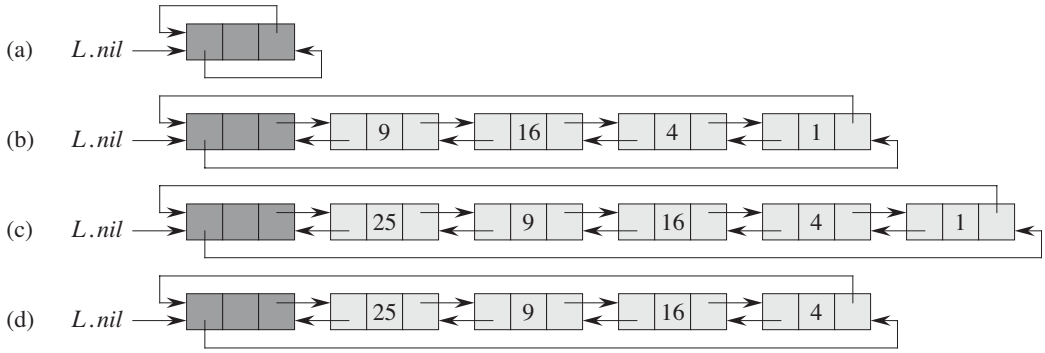


**Figure 10.4** A circular, doubly linked list with a sentinel. The sentinel $L.nil$ appears between the head and tail. The attribute $L.head$ is no longer needed, since we can access the head of the list by $L.nil.next$. **(a)** An empty list. **(b)** The linked list from Figure 10.3(a), with key 9 at the head and key 1 at the tail. **(c)** The list after executing LIST-INSERT$'(L, x)$, where $x.key = 25$. The new object becomes the head of the list. **(d)** The list after deleting the object with key 1. The new tail is the object with key 4.

but has all the attributes of the other objects in the list. Wherever we have a reference to NIL in list code, we replace it by a reference to the sentinel $L.nil$. As shown in Figure 10.4, this change turns a regular doubly linked list into a ***circular, doubly linked list with a sentinel***, in which the sentinel $L.nil$ lies between the head and tail. The attribute $L.nil.next$ points to the head of the list, and $L.nil.prev$ points to the tail. Similarly, both the *next* attribute of the tail and the *prev* attribute of the head point to $L.nil$. Since $L.nil.next$ points to the head, we can eliminate the attribute $L.head$ altogether, replacing references to it by references to $L.nil.next$. Figure 10.4(a) shows that an empty list consists of just the sentinel, and both $L.nil.next$ and $L.nil.prev$ point to $L.nil$.

The code for LIST-SEARCH remains the same as before, but with the references to NIL and $L.head$ changed as specified above:

```
LIST-SEARCH'(L, k)
1  x = L.nil.next
2  while x ≠ L.nil and x.key ≠ k
3       x = x.next
4  return x
```

We use the two-line procedure LIST-DELETE$'$ from before to delete an element from the list. The following procedure inserts an element into the list:

```
LIST-INSERT′(L, x)
1  x.next = L.nil.next
2  L.nil.next.prev = x
3  L.nil.next = x
4  x.prev = L.nil
```

Figure 10.4 shows the effects of LIST-INSERT′ and LIST-DELETE′ on a sample list.

Sentinels rarely reduce the asymptotic time bounds of data structure operations, but they can reduce constant factors. The gain from using sentinels within loops is usually a matter of clarity of code rather than speed; the linked list code, for example, becomes simpler when we use sentinels, but we save only $O(1)$ time in the LIST-INSERT′ and LIST-DELETE′ procedures. In other situations, however, the use of sentinels helps to tighten the code in a loop, thus reducing the coefficient of, say, $n$ or $n^2$ in the running time.

We should use sentinels judiciously. When there are many small lists, the extra storage used by their sentinels can represent significant wasted memory. In this book, we use sentinels only when they truly simplify the code.

## Exercises

### *10.2-1*

Can you implement the dynamic-set operation INSERT on a singly linked list in $O(1)$ time? How about DELETE?

### *10.2-2*

Implement a stack using a singly linked list $L$. The operations PUSH and POP should still take $O(1)$ time.

### *10.2-3*

Implement a queue by a singly linked list $L$. The operations ENQUEUE and DEQUEUE should still take $O(1)$ time.

### *10.2-4*

As written, each loop iteration in the LIST-SEARCH′ procedure requires two tests: one for $x \neq L.nil$ and one for $x.key \neq k$. Show how to eliminate the test for $x \neq L.nil$ in each iteration.

### *10.2-5*

Implement the dictionary operations INSERT, DELETE, and SEARCH using singly linked, circular lists. What are the running times of your procedures?

***10.2-6***
The dynamic-set operation UNION takes two disjoint sets $S_1$ and $S_2$ as input, and it returns a set $S = S_1 \cup S_2$ consisting of all the elements of $S_1$ and $S_2$. The sets $S_1$ and $S_2$ are usually destroyed by the operation. Show how to support UNION in $O(1)$ time using a suitable list data structure.

***10.2-7***
Give a $\Theta(n)$-time nonrecursive procedure that reverses a singly linked list of $n$ elements. The procedure should use no more than constant storage beyond that needed for the list itself.

***10.2-8*** ⋆
Explain how to implement doubly linked lists using only one pointer value $x.np$ per item instead of the usual two (*next* and *prev*). Assume that all pointer values can be interpreted as $k$-bit integers, and define $x.np$ to be $x.np = x.next$ XOR $x.prev$, the $k$-bit "exclusive-or" of $x.next$ and $x.prev$. (The value NIL is represented by 0.) Be sure to describe what information you need to access the head of the list. Show how to implement the SEARCH, INSERT, and DELETE operations on such a list. Also show how to reverse such a list in $O(1)$ time.

## 10.3 Implementing pointers and objects

How do we implement pointers and objects in languages that do not provide them? In this section, we shall see two ways of implementing linked data structures without an explicit pointer data type. We shall synthesize objects and pointers from arrays and array indices.

### A multiple-array representation of objects

We can represent a collection of objects that have the same attributes by using an array for each attribute. As an example, Figure 10.5 shows how we can implement the linked list of Figure 10.3(a) with three arrays. The array *key* holds the values of the keys currently in the dynamic set, and the pointers reside in the arrays *next* and *prev*. For a given array index $x$, the array entries $key[x]$, $next[x]$, and $prev[x]$ represent an object in the linked list. Under this interpretation, a pointer $x$ is simply a common index into the *key*, *next*, and *prev* arrays.

In Figure 10.3(a), the object with key 4 follows the object with key 16 in the linked list. In Figure 10.5, key 4 appears in $key[2]$, and key 16 appears in $key[5]$, and so $next[5] = 2$ and $prev[2] = 5$. Although the constant NIL appears in the *next*

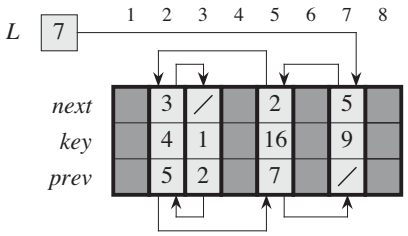


**Figure 10.5** The linked list of Figure 10.3(a) represented by the arrays *key*, *next*, and *prev*. Each vertical slice of the arrays represents a single object. Stored pointers correspond to the array indices shown at the top; the arrows show how to interpret them. Lightly shaded object positions contain list elements. The variable $L$ keeps the index of the head.

attribute of the tail and the *prev* attribute of the head, we usually use an integer (such as 0 or $-1$) that cannot possibly represent an actual index into the arrays. A variable $L$ holds the index of the head of the list.

### A single-array representation of objects

The words in a computer memory are typically addressed by integers from 0 to $M - 1$, where $M$ is a suitably large integer. In many programming languages, an object occupies a contiguous set of locations in the computer memory. A pointer is simply the address of the first memory location of the object, and we can address other memory locations within the object by adding an offset to the pointer.

We can use the same strategy for implementing objects in programming environments that do not provide explicit pointer data types. For example, Figure 10.6 shows how to use a single array $A$ to store the linked list from Figures 10.3(a) and 10.5. An object occupies a contiguous subarray $A[j\,..\,k]$. Each attribute of the object corresponds to an offset in the range from 0 to $k - j$, and a pointer to the object is the index $j$. In Figure 10.6, the offsets corresponding to *key*, *next*, and *prev* are 0, 1, and 2, respectively. To read the value of $i.prev$, given a pointer $i$, we add the value $i$ of the pointer to the offset 2, thus reading $A[i + 2]$.

The single-array representation is flexible in that it permits objects of different lengths to be stored in the same array. The problem of managing such a heterogeneous collection of objects is more difficult than the problem of managing a homogeneous collection, where all objects have the same attributes. Since most of the data structures we shall consider are composed of homogeneous elements, it will be sufficient for our purposes to use the multiple-array representation of objects.

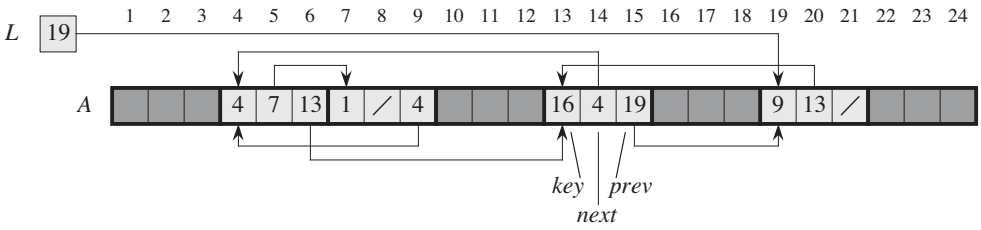


**Figure 10.6** The linked list of Figures 10.3(a) and 10.5 represented in a single array $A$. Each list element is an object that occupies a contiguous subarray of length 3 within the array. The three attributes *key*, *next*, and *prev* correspond to the offsets 0, 1, and 2, respectively, within each object. A pointer to an object is the index of the first element of the object. Objects containing list elements are lightly shaded, and arrows show the list ordering.

### Allocating and freeing objects

To insert a key into a dynamic set represented by a doubly linked list, we must allocate a pointer to a currently unused object in the linked-list representation. Thus, it is useful to manage the storage of objects not currently used in the linked-list representation so that one can be allocated. In some systems, a ***garbage collector*** is responsible for determining which objects are unused. Many applications, however, are simple enough that they can bear responsibility for returning an unused object to a storage manager. We shall now explore the problem of allocating and freeing (or deallocating) homogeneous objects using the example of a doubly linked list represented by multiple arrays.

Suppose that the arrays in the multiple-array representation have length $m$ and that at some moment the dynamic set contains $n \leq m$ elements. Then $n$ objects represent elements currently in the dynamic set, and the remaining $m-n$ objects are ***free***; the free objects are available to represent elements inserted into the dynamic set in the future.

We keep the free objects in a singly linked list, which we call the ***free list***. The free list uses only the *next* array, which stores the *next* pointers within the list. The head of the free list is held in the global variable *free*. When the dynamic set represented by linked list $L$ is nonempty, the free list may be intertwined with list $L$, as shown in Figure 10.7. Note that each object in the representation is either in list $L$ or in the free list, but not in both.

The free list acts like a stack: the next object allocated is the last one freed. We can use a list implementation of the stack operations PUSH and POP to implement the procedures for allocating and freeing objects, respectively. We assume that the global variable *free* used in the following procedures points to the first element of the free list.

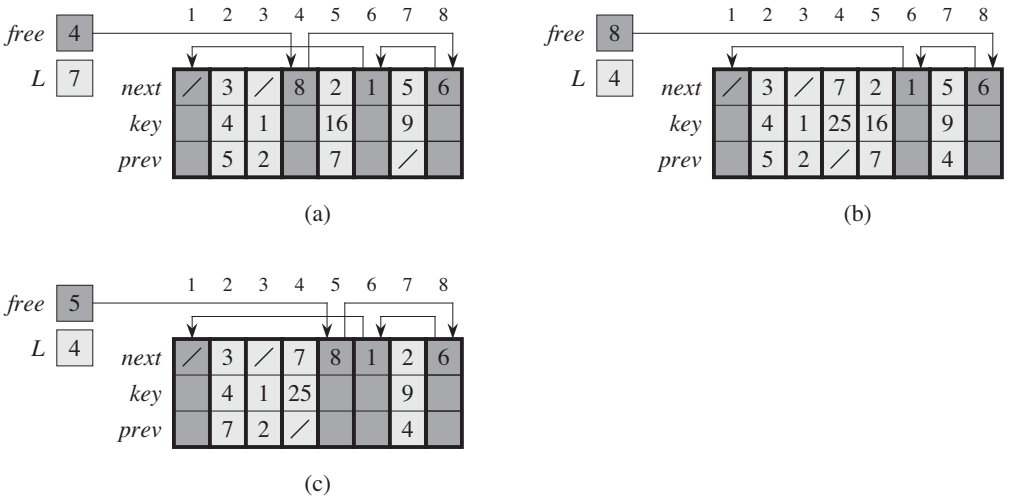


**Figure 10.7** The effect of the ALLOCATE-OBJECT and FREE-OBJECT procedures. **(a)** The list of Figure 10.5 (lightly shaded) and a free list (heavily shaded). Arrows show the free-list structure. **(b)** The result of calling ALLOCATE-OBJECT() (which returns index 4), setting *key*[4] to 25, and calling LIST-INSERT($L, 4$). The new free-list head is object 8, which had been *next*[4] on the free list. **(c)** After executing LIST-DELETE($L, 5$), we call FREE-OBJECT(5). Object 5 becomes the new free-list head, with object 8 following it on the free list.

```
ALLOCATE-OBJECT()
1  if free == NIL
2       error "out of space"
3  else x = free
4       free = x.next
5       return x
```

```
FREE-OBJECT(x)
1  x.next = free
2  free = x
```

The free list initially contains all $n$ unallocated objects. Once the free list has been exhausted, running the ALLOCATE-OBJECT procedure signals an error. We can even service several linked lists with just a single free list. Figure 10.8 shows two linked lists and a free list intertwined through *key*, *next*, and *prev* arrays.

The two procedures run in $O(1)$ time, which makes them quite practical. We can modify them to work for any homogeneous collection of objects by letting any one of the attributes in the object act like a *next* attribute in the free list.

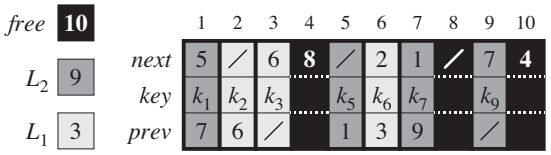


**Figure 10.8** Two linked lists, $L_1$ (lightly shaded) and $L_2$ (heavily shaded), and a free list (darkened) intertwined.

## Exercises

### *10.3-1*

Draw a picture of the sequence $\langle 13, 4, 8, 19, 5, 11 \rangle$ stored as a doubly linked list using the multiple-array representation. Do the same for the single-array representation.

### *10.3-2*

Write the procedures ALLOCATE-OBJECT and FREE-OBJECT for a homogeneous collection of objects implemented by the single-array representation.

### *10.3-3*

Why don't we need to set or reset the *prev* attributes of objects in the implementation of the ALLOCATE-OBJECT and FREE-OBJECT procedures?

### *10.3-4*

It is often desirable to keep all elements of a doubly linked list compact in storage, using, for example, the first $m$ index locations in the multiple-array representation. (This is the case in a paged, virtual-memory computing environment.) Explain how to implement the procedures ALLOCATE-OBJECT and FREE-OBJECT so that the representation is compact. Assume that there are no pointers to elements of the linked list outside the list itself. (*Hint:* Use the array implementation of a stack.)

### *10.3-5*

Let $L$ be a doubly linked list of length $n$ stored in arrays *key*, *prev*, and *next* of length $m$. Suppose that these arrays are managed by ALLOCATE-OBJECT and FREE-OBJECT procedures that keep a doubly linked free list $F$. Suppose further that of the $m$ items, exactly $n$ are on list $L$ and $m - n$ are on the free list. Write a procedure COMPACTIFY-LIST$(L, F)$ that, given the list $L$ and the free list $F$, moves the items in $L$ so that they occupy array positions $1, 2, \ldots, n$ and adjusts the free list $F$ so that it remains correct, occupying array positions $n+1, n+2, \ldots, m$. The running time of your procedure should be $\Theta(n)$, and it should use only a constant amount of extra space. Argue that your procedure is correct.

## 10.4 Representing rooted trees

The methods for representing lists given in the previous section extend to any homogeneous data structure. In this section, we look specifically at the problem of representing rooted trees by linked data structures. We first look at binary trees, and then we present a method for rooted trees in which nodes can have an arbitrary number of children.

We represent each node of a tree by an object. As with linked lists, we assume that each node contains a *key* attribute. The remaining attributes of interest are pointers to other nodes, and they vary according to the type of tree.

### Binary trees

Figure 10.9 shows how we use the attributes $p$, *left*, and *right* to store pointers to the parent, left child, and right child of each node in a binary tree $T$. If $x.p = \text{NIL}$, then $x$ is the root. If node $x$ has no left child, then $x.left = \text{NIL}$, and similarly for the right child. The root of the entire tree $T$ is pointed to by the attribute $T.root$. If $T.root = \text{NIL}$, then the tree is empty.

### Rooted trees with unbounded branching

We can extend the scheme for representing a binary tree to any class of trees in which the number of children of each node is at most some constant $k$: we replace the *left* and *right* attributes by $child_1, child_2, \ldots, child_k$. This scheme no longer works when the number of children of a node is unbounded, since we do not know how many attributes (arrays in the multiple-array representation) to allocate in advance. Moreover, even if the number of children $k$ is bounded by a large constant but most nodes have a small number of children, we may waste a lot of memory.

Fortunately, there is a clever scheme to represent trees with arbitrary numbers of children. It has the advantage of using only $O(n)$ space for any $n$-node rooted tree. The ***left-child, right-sibling representation*** appears in Figure 10.10. As before, each node contains a parent pointer $p$, and $T.root$ points to the root of tree $T$. Instead of having a pointer to each of its children, however, each node $x$ has only two pointers:

1. $x.left\text{-}child$ points to the leftmost child of node $x$, and
2. $x.right\text{-}sibling$ points to the sibling of $x$ immediately to its right.

If node $x$ has no children, then $x.left\text{-}child = \text{NIL}$, and if node $x$ is the rightmost child of its parent, then $x.right\text{-}sibling = \text{NIL}$.

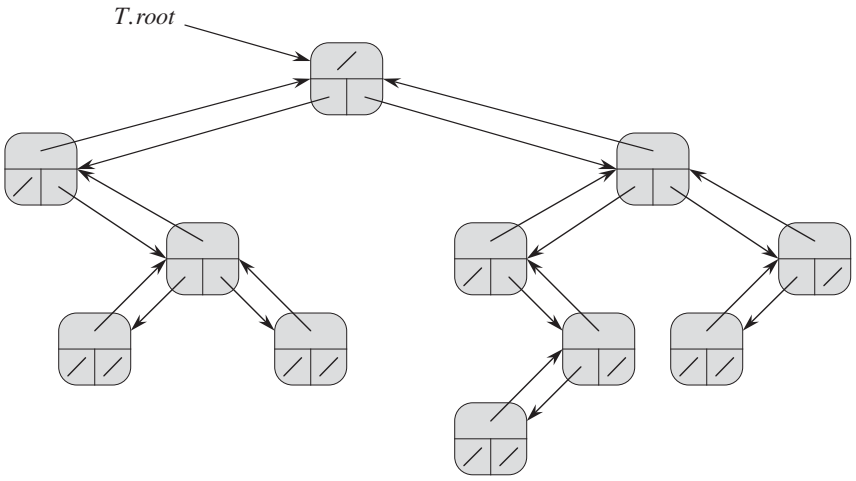


**Figure 10.9** The representation of a binary tree $T$. Each node $x$ has the attributes $x.p$ (top), $x.left$ (lower left), and $x.right$ (lower right). The *key* attributes are not shown.

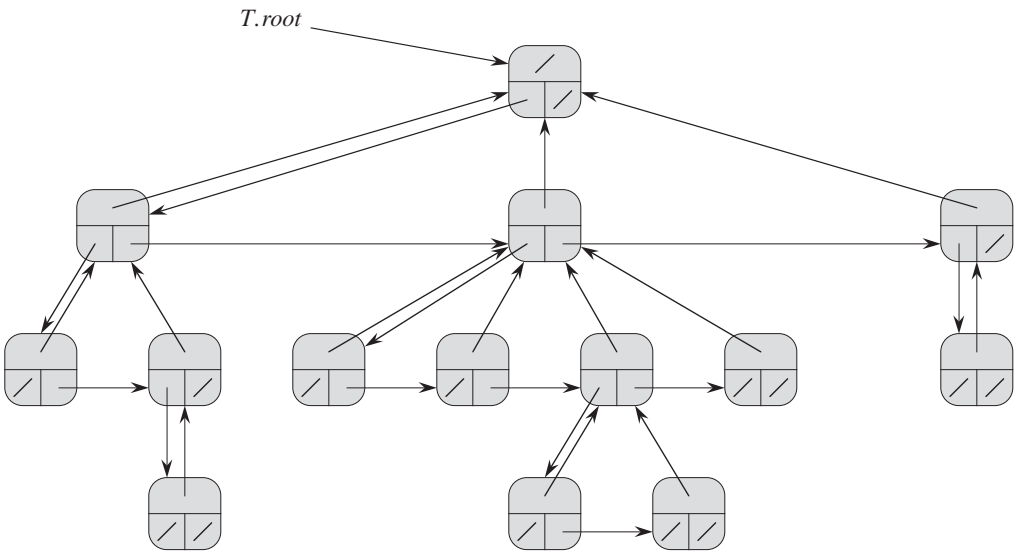


**Figure 10.10** The left-child, right-sibling representation of a tree $T$. Each node $x$ has attributes $x.p$ (top), $x.left\text{-}child$ (lower left), and $x.right\text{-}sibling$ (lower right). The *key* attributes are not shown.

### Other tree representations

We sometimes represent rooted trees in other ways. In Chapter 6, for example, we represented a heap, which is based on a complete binary tree, by a single array plus the index of the last node in the heap. The trees that appear in Chapter 21 are traversed only toward the root, and so only the parent pointers are present; there are no pointers to children. Many other schemes are possible. Which scheme is best depends on the application.

### Exercises

#### *10.4-1*

Draw the binary tree rooted at index 6 that is represented by the following attributes:

| index | *key* | *left* | *right* |
|---|---|---|---|
| 1 | 12 | 7 | 3 |
| 2 | 15 | 8 | NIL |
| 3 | 4 | 10 | NIL |
| 4 | 10 | 5 | 9 |
| 5 | 2 | NIL | NIL |
| 6 | 18 | 1 | 4 |
| 7 | 7 | NIL | NIL |
| 8 | 14 | 6 | 2 |
| 9 | 21 | NIL | NIL |
| 10 | 5 | NIL | NIL |

#### *10.4-2*

Write an $O(n)$-time recursive procedure that, given an $n$-node binary tree, prints out the key of each node in the tree.

#### *10.4-3*

Write an $O(n)$-time nonrecursive procedure that, given an $n$-node binary tree, prints out the key of each node in the tree. Use a stack as an auxiliary data structure.

#### *10.4-4*

Write an $O(n)$-time procedure that prints all the keys of an arbitrary rooted tree with $n$ nodes, where the tree is stored using the left-child, right-sibling representation.

#### *10.4-5* ⋆

Write an $O(n)$-time nonrecursive procedure that, given an $n$-node binary tree, prints out the key of each node. Use no more than constant extra space outside

of the tree itself and do not modify the tree, even temporarily, during the procedure.

***10.4-6*** ⋆

The left-child, right-sibling representation of an arbitrary rooted tree uses three pointers in each node: *left-child*, *right-sibling*, and *parent*. From any node, its parent can be reached and identified in constant time and all its children can be reached and identified in time linear in the number of children. Show how to use only two pointers and one boolean value in each node so that the parent of a node or all of its children can be reached and identified in time linear in the number of children.

## Problems

***10-1 Comparisons among lists***

For each of the four types of lists in the following table, what is the asymptotic worst-case running time for each dynamic-set operation listed?

| | unsorted, singly linked | sorted, singly linked | unsorted, doubly linked | sorted, doubly linked |
|---|---|---|---|---|
| SEARCH$(L, k)$ | | | | |
| INSERT$(L, x)$ | | | | |
| DELETE$(L, x)$ | | | | |
| SUCCESSOR$(L, x)$ | | | | |
| PREDECESSOR$(L, x)$ | | | | |
| MINIMUM$(L)$ | | | | |
| MAXIMUM$(L)$ | | | | |

### *10-2 Mergeable heaps using linked lists*

A ***mergeable heap*** supports the following operations: MAKE-HEAP (which creates an empty mergeable heap), INSERT, MINIMUM, EXTRACT-MIN, and UNION.[1] Show how to implement mergeable heaps using linked lists in each of the following cases. Try to make each operation as efficient as possible. Analyze the running time of each operation in terms of the size of the dynamic set(s) being operated on.

***a.*** Lists are sorted.

***b.*** Lists are unsorted.

***c.*** Lists are unsorted, and dynamic sets to be merged are disjoint.

### *10-3 Searching a sorted compact list*

Exercise 10.3-4 asked how we might maintain an $n$-element list compactly in the first $n$ positions of an array. We shall assume that all keys are distinct and that the compact list is also sorted, that is, $key[i] < key[next[i]]$ for all $i = 1, 2, \ldots, n$ such that $next[i] \neq \text{NIL}$. We will also assume that we have a variable $L$ that contains the index of the first element on the list. Under these assumptions, you will show that we can use the following randomized algorithm to search the list in $O(\sqrt{n})$ expected time.

```
COMPACT-LIST-SEARCH(L, n, k)
 1  i = L
 2  while i ≠ NIL and key[i] < k
 3       j = RANDOM(1, n)
 4       if key[i] < key[j] and key[j] ≤ k
 5            i = j
 6            if key[i] == k
 7                 return i
 8       i = next[i]
 9  if i == NIL or key[i] > k
10       return NIL
11  else return i
```

If we ignore lines 3–7 of the procedure, we have an ordinary algorithm for searching a sorted linked list, in which index $i$ points to each position of the list in

[1] Because we have defined a mergeable heap to support MINIMUM and EXTRACT-MIN, we can also refer to it as a ***mergeable min-heap***. Alternatively, if it supported MAXIMUM and EXTRACT-MAX, it would be a ***mergeable max-heap***.

turn. The search terminates once the index $i$ "falls off" the end of the list or once $key[i] \geq k$. In the latter case, if $key[i] = k$, clearly we have found a key with the value $k$. If, however, $key[i] > k$, then we will never find a key with the value $k$, and so terminating the search was the right thing to do.

Lines 3–7 attempt to skip ahead to a randomly chosen position $j$. Such a skip benefits us if $key[j]$ is larger than $key[i]$ and no larger than $k$; in such a case, $j$ marks a position in the list that $i$ would have to reach during an ordinary list search. Because the list is compact, we know that any choice of $j$ between 1 and $n$ indexes some object in the list rather than a slot on the free list.

Instead of analyzing the performance of COMPACT-LIST-SEARCH directly, we shall analyze a related algorithm, COMPACT-LIST-SEARCH$'$, which executes two separate loops. This algorithm takes an additional parameter $t$ which determines an upper bound on the number of iterations of the first loop.

```
COMPACT-LIST-SEARCH'(L, n, k, t)
 1  i = L
 2  for q = 1 to t
 3       j = RANDOM(1, n)
 4       if key[i] < key[j] and key[j] ≤ k
 5            i = j
 6            if key[i] == k
 7                 return i
 8  while i ≠ NIL and key[i] < k
 9       i = next[i]
10  if i == NIL or key[i] > k
11       return NIL
12  else return i
```

To compare the execution of the algorithms COMPACT-LIST-SEARCH$(L, n, k)$ and COMPACT-LIST-SEARCH$'(L, n, k, t)$, assume that the sequence of integers returned by the calls of RANDOM$(1, n)$ is the same for both algorithms.

***a.*** Suppose that COMPACT-LIST-SEARCH$(L, n, k)$ takes $t$ iterations of the **while** loop of lines 2–8. Argue that COMPACT-LIST-SEARCH$'(L, n, k, t)$ returns the same answer and that the total number of iterations of both the **for** and **while** loops within COMPACT-LIST-SEARCH$'$ is at least $t$.

In the call COMPACT-LIST-SEARCH$'(L, n, k, t)$, let $X_t$ be the random variable that describes the distance in the linked list (that is, through the chain of *next* pointers) from position $i$ to the desired key $k$ after $t$ iterations of the **for** loop of lines 2–7 have occurred.

***b.*** Argue that the expected running time of COMPACT-LIST-SEARCH$'(L, n, k, t)$ is $O(t + \mathrm{E}[X_t])$.

***c.*** Show that $\mathrm{E}[X_t] \le \sum_{r=1}^{n} (1 - r/n)^t$. (*Hint:* Use equation (C.25).)

***d.*** Show that $\sum_{r=0}^{n-1} r^t \le n^{t+1}/(t+1)$.

***e.*** Prove that $\mathrm{E}[X_t] \le n/(t+1)$.

***f.*** Show that COMPACT-LIST-SEARCH$'(L, n, k, t)$ runs in $O(t + n/t)$ expected time.

***g.*** Conclude that COMPACT-LIST-SEARCH runs in $O(\sqrt{n})$ expected time.

***h.*** Why do we assume that all keys are distinct in COMPACT-LIST-SEARCH? Argue that random skips do not necessarily help asymptotically when the list contains repeated key values.

## Chapter notes

Aho, Hopcroft, and Ullman [6] and Knuth [209] are excellent references for elementary data structures. Many other texts cover both basic data structures and their implementation in a particular programming language. Examples of these types of textbooks include Goodrich and Tamassia [147], Main [241], Shaffer [311], and Weiss [352, 353, 354]. Gonnet [145] provides experimental data on the performance of many data-structure operations.

The origin of stacks and queues as data structures in computer science is unclear, since corresponding notions already existed in mathematics and paper-based business practices before the introduction of digital computers. Knuth [209] cites A. M. Turing for the development of stacks for subroutine linkage in 1947.

Pointer-based data structures also seem to be a folk invention. According to Knuth, pointers were apparently used in early computers with drum memories. The A-1 language developed by G. M. Hopper in 1951 represented algebraic formulas as binary trees. Knuth credits the IPL-II language, developed in 1956 by A. Newell, J. C. Shaw, and H. A. Simon, for recognizing the importance and promoting the use of pointers. Their IPL-III language, developed in 1957, included explicit stack operations.