Pricing Exotic Options using Monte Carlo simulations

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Abstract:

Certain types of exotic options are cheaper than standard vanilla options, because a zero payoff may occur before expiry. The pricing of path dependent exotic options such as barrier options is not straight forward using analytical formulae. The objective of this project is to compare pricing of the barrier options using standard Black Scholes model and Monte Carlo simulations. This project compares the two variance reduction techniques such as Antithetic Variates and Control Variates with crude Monte Carlo simulation in pricing the barrier options and the convergence with the analytical solution as we increase the number of simulations. Also, we look at the impact of barrier price on the different variance reduction methods. From the computational results, we find that the antithetic variates method substantially improves the precision of Monte Carlo estimates. Our results also show that using the ordinary put option as a control variate significantly reduces standard errors.

Keywords: barrier option, Black-Scholes model, Monte Carlo simulations, variance reduction

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1. Introduction

Options are one of the highly traded financial derivatives in the market. Exchange traded options are mostly plain vanilla options. The more complex path dependent options which are also commonly known as exotic options are traded over the counter (OTC). The exotic options have non- standard features and are designed to cater the needs of a particular investor or client. Barrier options are a type of exotic options which depends on the price of the underlying asset reaching a pre-specified barrier level. Barrier options behave like European options apart from the fact that the options vanish or comes into existence once the price of the underlying asset hits the barrier level. The path dependency and the barrier level features of a barrier option gives us more options than the standard options for hedging.

Various approaches for pricing barrier options have been developed. One such method is Monte Carlo simulation. Under the Black-Scholes model, barrier options can be considered that the asset price follows the Geometric Brownian Motion. Monte Carlo simulation has been proven to be an effective and simple tool in pricing options. The efficiency and accuracy of estimating option prices can be improved by introducing variance reduction techniques such as antithetic variates, control variates.

The aim of this paper is to price the down-and-out put option using standard BS model and Monte Carlo simulations using four different variance reduction methods. The remainder of this paper is organized as follows: Section 2 presents the Empirical Methodologies whereby the concepts of pricing using BS model, Monte Carlo simulations, and variance reduction methods are analyzed. While the results are discussed in Section 3 and Section 4 concludes.

2. Empirical Methodology

Derivative securities are financial contracts, or financial instruments, whose values are derived from the value of underlying assets. The underlying asset can be a stock, a bond, a foreign currency, an index portfolio, or another derivative security. Derivatives can be used by individuals or financial institutions to hedge risks. There are many types of derivatives such as options, forwards, futures and swaps.

Options are categorized into mainly two types based on the exercise time or time to maturity: European and American. A European option can only be exercised at the time to maturity whereas an American option can be exercised at any point of time until maturity. The price of a European call option, denoted by c, at time of initiation of the contract is,

$$c = E[e^{-rT}(S_T - K)^+]$$

Where S_T is the price of the underlying at maturity T, K is the exercise price and r is the rate of interest. Similarly, the price of a European put option, denoted by p, at time of initiation of the contract is,

$$p = E[e^{-rT}(K - S_T)^+]$$

2.1 Black-Scholes Model

The benchmark model to price the options where the underlying asset is a stock is developed by Black-Scholes (1973). The BS model is based on the assumption that underlying stock prices follows a Geometric Brownian Motion (GBM) diffusion process and under real-world probability measure P, it is defined as:

$$dS_t = \mu S_t dt + \sigma S_t dW_t$$

where μ is the constant drift and σ is the constant volatility of the diffusion process. The BS model assumes market is complete and there are no arbitrage opportunities. Since the market is complete and there are no arbitrage opportunities, there exists a risk neutral measure. The GBM diffusion process for stock price under risk neutral probability measure $\widetilde{P}$ is defined as:

$$dS_t = rS_t dt + \sigma S_t dW_t$$

where r is the risk-free interest rate. The BS model price of a plain vanilla European call option at initiation of the contract is:

$$C = S_0 N(d_1) - Ke^{-rT} N(d_2) \qquad (1)$$

where $d_1 = \frac{[\ln\left(\frac{S_0}{K}\right) + \left(r - 1/2\sigma^2 T\right)]}{\sigma\sqrt{T}}$ and $d_2 = d_1 - \sigma\sqrt{T}$

N(.) is the cumulative normal distribution function, and S_0 is the price of the underlying stock at initiation of the contract.

An extension of the Black-Scholes formula is the put-call parity. From this equation, we can easily calculate the price of the put option, knowing the price of the call option.

$$S_0 + P - C = Ke^{-rT}$$

Where C is the call price and P is the put price.

2.2 Barrier Options

A barrier option is a type of path-dependent option where the payoff is determined by whether or not the price of the stock crosses a certain level B during its life. There are two general types of barrier options, 'in' and 'out' options. In knock-out options, the contract is canceled if the barrier is crossed throughout the whole life. Knock-in options on the other hand are activated only if the barrier is crossed. The relationship between the barrier B and the current asset price S_0 indicates whether the option is an up or down option. If B> S_0, we have an up option; if B< S_0, we have a down option. Combining these features with the payoffs of call and put options, we can define an array of barrier options.

For example, a down-and-out put option is a put option that becomes worthless if the asset price falls below the barrier B Therefore, the risk for the option writer is reduced. It is reasonable to expect that a down-and-out put option is cheaper than a vanilla one, since it may expire worthless if the barrier is hit while the vanilla option would have paid off. For a given set of parameters, we can combine an in and an out option of the same type to replicate an ordinary vanilla option. This is due to the fact that when one option gets knocked out, the other is knocked in. Therefore, holding both a down-and-out and a down-and-in put option is equivalent to holding a vanilla put option. The parity relationship can be described as:

$$P = P_{di} + P_{do} \quad (2),$$

where P is the price of the vanilla put, and P_{di} is the price for the down-and-in option and P_{do} is the price of the down-and-out options.

It is worth pointing out the relationship between the option price and the barrier level. For a knock-out option, increasing the absolute difference between the barrier level and the initial spot price has a positive effect on the option value, since the probability of knocking out tends to zero as the absolute difference increases. The value of the option converges to the value of an ordinary vanilla option. It is opposite for the knock-in option: increasing the absolute difference reduces the option value since the probability of knocking in approaches zero.

The price of a down-and-out put option for the case where the underlying stock price follows a GBM diffusion process, is:

$$\mathrm{P_{do}} = \{\mathrm{K}e^{-\mathrm{rT}}[\mathrm{N}(d_4) - \mathrm{N}(d_2)] - \mathrm{a}[\mathrm{N}(d_7) - \mathrm{N}(d_5)\} - \mathrm{S_0}\{[\mathrm{N}(d_3) - \mathrm{N}(d_1)] - \mathrm{b}[\mathrm{N}(d_8) - \mathrm{N}(d_6)]\}$$

where

$$\mathrm{a} = (\mathrm{B}/S_0)^{-1+2\mathrm{r}/\sigma^2}$$

$$\mathrm{b} = (\mathrm{B}/S_0)^{1+2\mathrm{r}/\sigma^2}$$

$$\mathrm{d1} = [\ln\left(\frac{\mathrm{S_0}}{\mathrm{K}}\right) + \left(\mathrm{r} + \left(\frac{1}{2}\right)\sigma^2\right)\mathrm{T}]\,/\sigma\sqrt{\mathrm{T}}$$

$$\mathrm{d2} = [\ln\left(\frac{\mathrm{S_0}}{\mathrm{K}}\right) + \left(\mathrm{r} - \left(\frac{1}{2}\right)\sigma^2\right)\mathrm{T}]\,/\sigma\sqrt{\mathrm{T}}$$

$$\mathrm{d3} = [\ln\left(\frac{\mathrm{S_0}}{\mathrm{B}}\right) + \left(\mathrm{r} + \left(\frac{1}{2}\right)\sigma^2\right)\mathrm{T}]\,/\sigma\sqrt{\mathrm{T}}$$

$$\mathrm{d4} = [\ln\left(\frac{\mathrm{S_0}}{\mathrm{B}}\right) + \left(\mathrm{r} - \left(\frac{1}{2}\right)\sigma^2\right)\mathrm{T}]\,/\sigma\sqrt{\mathrm{T}}$$

$$\mathrm{d5} = [\ln\left(\frac{\mathrm{S_0}}{\mathrm{B}}\right) - \left(\mathrm{r} - \left(\frac{1}{2}\right)\sigma^2\right)\mathrm{T}]\,/\sigma\sqrt{\mathrm{T}}$$

$$\mathrm{d6} = [\ln\left(\frac{\mathrm{S_0}}{\mathrm{B}}\right) - \left(\mathrm{r} + \left(\frac{1}{2}\right)\sigma^2\right)\mathrm{T}]\,/\sigma\sqrt{\mathrm{T}}$$

$$\mathrm{d7} = [\ln\left(\frac{\mathrm{S_0K}}{\mathrm{B^2}}\right) - \left(\mathrm{r} - \left(\frac{1}{2}\right)\sigma^2\right)\mathrm{T}]\,/\sigma\sqrt{\mathrm{T}}$$

$$d8 = [\ln\left(\frac{S_0 K}{B^2}\right) - \left(r + \left(\frac{1}{2}\right)\sigma^2\right) T] / \sigma\sqrt{T}$$

2.3 Monte Carlo simulations

Monte Carlo simulations is a commonly used numerical method to price options when there isn't any closed form solution for a particular model. In order to estimate the price of any option using Monte Carlo simulations, we have to generate simulations of price of the underlying stock price at maturity using its probability density function. Then, the option payoff for each simulation is calculated. Finally, the price of the option is calculated by averaging the discounted payoffs of all simulations. Under the BS model assumption that stock prices follow GBM, the integral form of the price of the underlying stock price at time t is given by

$$dS_t = \mu S_t d_t + \sigma S_t dW_t$$

Using Euler scheme, we have

$$S_{t+\Delta t} = (1 + \mu\Delta t)\, S_t + \sigma S_t \sqrt{\Delta t}\delta$$

By applying Ito's lemma, we receive following expression:

$$dlogS_t = \left(\mu - \frac{1}{2}\sigma^2\right) dt + \sigma dW_t \qquad (3)$$

S_t is log-normally distributed and thereby we have

$$E[\log(S(t)/S(0))] = \vartheta t$$

$$Var[\log(S(t)/S(0))] = \sigma^2 t$$

$$E[S(t)/S(0)] = e^{\mu t}$$

$$Var[S(t)/S(0)] = e^{2\mu t}(e^{\sigma^2 t} - 1)$$

Where $\vartheta = \mu - \sigma^2/2$. Integrating equation 3 we get:

$$S_t = S_0 \exp\left(\vartheta t + \sigma \int_0^t dW(\tau)\right)$$

After discretizing the time interval (0,T) with a time step Δt, we obtain

$$S_{t+\Delta t} = S_0 \exp\left((\mu - 0.5\sigma^2)\Delta t + \sigma\sqrt{\Delta t}\,\delta\right) \qquad (4)$$

where $\delta \sim N(0,1)$, is a standard normal random variable. Since Monte Carlo simulations is a discrete time valuation, we use the equation (4) to simulate the price of the underlying stock. The price of the barrier option depends on the path of the underlying path, so we generate the simulations of the path of underlying stock price in order to capture the time at which the path of underlying stock price hits the barrier level. Then we can calculate the price of the barrier option by averaging the discounted payoffs from all simulations.

2.4 Variance Reduction methods

The accuracy of the pricing barrier option using Monte Carlo simulations can be increased by increasing the number of time steps over the time interval (0, T). Another way to improve the accuracy is by reducing error or variance of the Monte Carlo simulation using variance reduction methods.

2.4.1 Antithetic Variates

In Monte Carlo simulations, increasing the sample size is computationally costly. So, we may focus on reducing the sample variance. The method of antithetic variates works by replacing independent *X*'s with negatively correlated random variables.

Suppose we want to estimate $E(g(X_1, X_2, \ldots, X_n))$, where $X_1, X_2, \ldots, X_n$ are independent random variables. Let Y_1 and Y_2 be random variables with the same distribution as $g(X_1, X_2, \ldots, X_n)$. Then

$$\mathrm{Var}\left(\frac{Y_1 + Y_2}{2}\right) = \frac{\mathrm{Var}(Y_1) + \mathrm{Var}(Y_2) + 2\mathrm{Cov}(Y_1, Y_2)}{4}$$

Since our assumption is Y_1 and Y_2 are negatively correlated, the covariance will be negative, and the variance will be lower compared to the case if Y_1 and Y_2 are positively correlated.
Following the idea outlined above, we generate a random draw from the $N(0, 1)$ distribution. By symmetry $Z \sim -Z$ where $Z \sim N(0, 1)$, we set Y_1 from $X_1, \ldots X_n \sim N(0, 1)$ and Y_2 from $-X_1, \ldots -X_n \sim N(0,$

1). Thus, we generate a pair of random variables (Y1, Y2) where $Y_1 = X_1$ and $Y_2 = -X_2$. X_1 *and* $-X_2$are the so called antithetic variables.

The pair of standard normal random variables generate the simulations for paths of price of underlying stock price and then the average of discounted payoffs from both the simulations is averaged to get the estimate of the price of the barrier option. Since the payoff function of a put option is monotonically decreasing then the covariance of payoffs generated using negatively correlated BMs will be negative always. Hence, the variance of the Monte Carlo simulation will be reduced using this method.

2.4.2 Control Variates

The control variates variance reduction method makes use of a control variate. Let X be a random variable and Y be another random variable with known expected value E(Y), which is correlated with X. In this case, Y is the control variate. Let Z be another random variable defined such that the expected value of Z has to be equal to expected value of X,

$$\mathrm{Z} = \mathrm{X} + \mathrm{c}\big(\mathrm{Y} - \mathrm{E(Y)}\big)$$

where c is some arbitrary value. The variance of Z is,

$$\mathrm{Var(Z)} = \mathrm{Var(X)} + 2\mathrm{cCov(X,Y)} + \mathrm{c}^2\mathrm{Var(Y)} \qquad (5)$$

We will use Z as our random variable to run Monte Carlo simulations. The value of c to minimize the variance of Z is,

$$\mathrm{c_{min}} = -\frac{\mathrm{Cov(X,Y)}}{\mathrm{Var(Y)}} \qquad (6)$$

The unknown Cov (X, Y) and Var (Y) can be estimated using Monte Carlo simulations.

In order to calculate the price of the barrier option, we need to calculate the Monte Carlo simulations of the payoff function and then average the discounted payoffs from all the simulations. In order to use control variates reduction method, we need to find the covariance between the plain vanilla put option and the down-and-out put option. This is done by running a set of pilot replications. Then we can calculate the optimal value of $c_{\min}$ using the equation (6). By utilizing equation (5), we can get a reduced variance for our Monte Carlo simulation of price of the down-and-out put option by incorporating $c_{\min}$.

3. Results

The price of the down-and-out put option is calculated using the BS model and Monte Carlo using simulations. All the simulations were done by varying barrier price B and the number of steps over the time interval (0, T). Other variables are fixed at constant values such as $S_0 = 50$, K = 50, sigma = 0.2, $r = \sigma =$0.01, number of replications (NRepl) = 100, number of pilot runs (NPilot) = 100, time to maturity, T = 1 year.

Table 1: Summary of Monte Carlo simulation and BS model prices for down-and-out put option

<table>
<tr><th>Initial value (S0)</th><th>Barrier level (B)</th><th># Steps (n)</th><th>BS model price</th><th>Monte Carlo price</th><th>Standard error</th></tr>
<tr><td rowspan="9">50</td><td rowspan="3">40</td><td>100</td><td rowspan="3">0.6264</td><td>0.8060</td><td>0.1862</td></tr>
<tr><td>1000</td><td>0.7126</td><td>0.0556</td></tr>
<tr><td>10000</td><td>0.7101</td><td>0.0171</td></tr>
<tr><td rowspan="3">35</td><td>100</td><td rowspan="3">1.4404</td><td>1.5444</td><td>0.2808</td></tr>
<tr><td>1000</td><td>1.5423</td><td>0.0936</td></tr>
<tr><td>10000</td><td>1.5063</td><td>0.0288</td></tr>
<tr><td rowspan="3">30</td><td>100</td><td rowspan="3">1.8136</td><td>1.6537</td><td>0.3344</td></tr>
<tr><td>1000</td><td>1.7509</td><td>0.1024</td></tr>
<tr><td>10000</td><td>1.8689</td><td>0.0343</td></tr>
</table>

$K = 50$, $r = 0.1$,$\sigma = 0.2$, $T = 1$, $m = 100$, $n = 100,1000,10000$

BS model prices for different initial values and barrier levels is calculated. It is evident that as the barrier level is decreased, the price of the down-and-out put option is increased. Since the probability of reaching the barrier decreases when the barrier level is decreased, so the increase in the price of the option. Thus, the relationship between the option price and barrier level is in agreement with what we have discussed.

For a knock-out option, increasing the absolute difference between the barrier level and the initial spot price has a positive effect on the option value. The probability of knocking out approaches to zero as the absolute difference increases, and therefore making the value of the option converge to an ordinary vanilla put value.

The Monte Carlo simulations price gradually converges to the BS model price as the number of simulations are increased. Moreover, the standard error decreases slightly with the increase in number of simulations. Using Monte Carlo simulations, we can get a price very close to the BS model price as the number of simulations are increased but the computational time is always a concern.

Table 2: Comparison of Monte Carlo simulation and Antithetic variates prices for down-and-out put option

Initial value (S0)	Barrier level (B)	BS model price	# Steps (n)	Monte Carlo price	Standard error	Antithetic variates price	Standard error
50	40	0.6264	100	0.8060	0.1862	0.7457	0.1140
			1000	0.7126	0.0556	0.6947	0.0383
			10000	0.7101	0.0171	0.6927	0.0119
	35	1.4404	100	1.5444	0.2808	1.8714	0.2264
			1000	1.5423	0.0936	1.5276	0.0673
			10000	1.5063	0.0288	1.4966	0.0203
	30	1.8136	100	1.6537	0.3344	2.0965	0.2550
			1000	1.7509	0.1024	1.7871	0.0761
			10000	1.8689	0.0343	1.7969	0.0237

$K = 50$, $r = 0.1$, $\sigma = 0.2$, $T = 1$, $m = 100$, $n = 100,1000,10000$

Table 2 compares the results by the crude Monte Carlo estimator and the antithetic variates estimator. Note that, the use of antithetic variables has given smaller standard errors for prices, in most case they were reduced significantly. Using the antithetic variates method, the number of simulations required to obtain the same accuracy as the crude Monte Carlo method is reduced. Variance reduction techniques indeed help to cut the need for increasing the number of simulations. Thus, Monte Carlo simulations using Antithetic variates variance method works better than simple Monte Carlo and gives better results even at fewer number of simulations.

Table 3: Comparison of Monte Carlo simulation and Control variates prices for down-and-out put option

Initial value (S0)	Barrier level (B)	BS model price	# Steps (n)	Monte Carlo price	Standard error	Control variates price	Standard error
50	40	0.6264	100	0.8060	0.1862	0.7928	0.1761
			1000	0.7126	0.0556	0.7419	0.0512
			10000	0.7101	0.0171	0.6953	0.0161
	35	1.4404	100	1.5444	0.2808	1.5301	0.1708
			1000	1.5423	0.0936	1.4889	0.0633
			10000	1.5063	0.0288	1.4832	0.0188
	30	1.8136	100	1.6537	0.3344	1.8927	0.0252
			1000	1.7509	0.1024	1.8640	0.0229
			10000	1.8689	0.0343	1.8207	0.0096

$K = 50$, $r = 0.1$, $\sigma = 0.2$, $T = 1$, $m = 100$, $n = 100,1000,10000$

Table 3 compares the Monte Carlo simulation prices with the prices calculated using the Control variates variance reduction method. The results are kind of mixed, but it performs better than simple Monte Carlo when compared at same number of simulations for both. Overall, the error of Monte Carlo simulations using control variates is reduced compared to simple Monte Carlo and the prices converges to BS prices as number of simulations are increased.

4 8. Conclusions

In this paper, an overview of how to use Monte Carlo simulation for the price estimation for barrier options is provided. The Monte Carlo simulation, which is rather easy to perform, comes with the cost of computational efficiency. We describe two different variance reduction techniques to eliminate standard errors as much as possible. The variance reduction methods can significantly increase the accuracy of the estimates and reduce the number of simulations needed. We find that the results of simulations for the down-and-out barrier option very well agree with the analytical formula prices. Furthermore, for a knock-out option, increasing the absolute difference between the barrier level and the initial spot price has a positive effect on option value. The probability of knocking out approaches to zero as the absolute difference increases, and therefore making the value of the option converges to an ordinary vanilla put value

The variance reduction methods such as Antithetic variates and Control variates reduces error with very few number of simulations compared to simple Monte Carlo simulation. We find that the antithetic variates method substantially improves the precision of Monte Carlo estimates. Our results also show that using the ordinary put option as a control variate significantly reduces standard errors.

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