# CYCLE STRUCTURE OF RANDOM PERMUTATIONS
# WITH CYCLE WEIGHTS

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ABSTRACT. We investigate the typical cycle lengths, the total number of cycles, and the number of finite cycles in random permutations whose probability involves cycle weights. Typical cycle lengths and total number of cycles depend strongly on the parameters, while the distributions of finite cycles are usually independent Poisson random variables.

## CONTENTS

## 1. INTRODUCTION

Weighted random partitions and random permutations appear in mathematical biology and in theoretical physics. They are appealing because of their natural probabilistic structure and their combinatorial flavor. The sample space for random permutations is the set $\mathcal{S}_n$ of permutations of $n$ elements. Given $\sigma \in \mathcal{S}_n$, we let $R_j(\sigma)$ denote the number of cycles of length $j$ in $\sigma$. The probability of a permutation $\sigma$ is then defined as

$$\mathbb{P}_n(\sigma) = \frac{1}{n! h_n} \prod_{j \geqslant 1} \theta_j^{R_j(\sigma)}. \tag{1.1}$$

Here, $\theta_1, \theta_2, \ldots$ are nonnegative parameters and $h_n$ is the normalization that makes $\mathbb{P}_n$ a probability distribution, i.e.,

$$h_n = \frac{1}{n!} \sum_{\sigma \in \mathcal{S}_n} \prod_{j \geqslant 1} \theta_j^{R_j(\sigma)}. \tag{1.2}$$

Notice that $R_1, R_2, \ldots$ satisfy the following identity for all $\sigma \in \mathcal{S}_n$:

$$\sum_{j=1}^{n} j R_j(\sigma) = n. \tag{1.3}$$

The case $\theta_j \equiv 1$ corresponds to random permutations with uniform distribution and it has been studied e.g. in [11, 3, 10]. The case $\theta_j \equiv \theta$ is known as the Ewens distribution and it was introduced for the study of population dynamics in mathematical biology [12]. See [2, 15, 13] and references therein. Random permutations with restriction on the cycle lengths can be described with parameters $\theta_j \in \{0, 1\}$. Results have been obtained for random permutations restricted to finite cycles [18, 5] or to cycle lengths of given parity [16]. Another situation of interest is when we fix the asymptotic behavior of $\theta_j$ for large $j$. Such a setting was considered in [4], and it also appears in the study of the quantum Bose gas in statistical mechanics [7, 8]. The case where $\theta_j$ converges to a constant (i.e., the Ewens case, asymptotically) was considered in [16, 9]. Vanishing parameters $\theta_j \to 0$ were studied in [4, 9] where logarithmic cycle lengths were observed.

The random variables to be discussed in this article are the following:

- $L_1(\sigma)$ gives the length of the cycle that contains the index 1. Since our probability distribution is invariant under relabeling, we can interpret $L_1$ as giving the length of a "typical cycle", i.e. the length of the cycle that contains a random index.
- $K(\sigma) = \sum_{j=1}^{n} R_j(\sigma)$ gives the total number of cycles in the permutation $\sigma$.
- We also consider the distributions of $R_1, R_2, \ldots$ as $n \to \infty$, i.e., the number of finite cycles. In most cases, they will be seen to converge to independent Poisson random variables.

The probability distribution $\mathbb{P}_n$ depends only on the cycle structure $(R_1(\sigma), \ldots, R_n(\sigma))$ of $\sigma \in \mathcal{S}_n$. It can therefore be understood as a distribution on nonnegative integers $(r_1, \ldots, r_n)$ that satisfy $\sum j r_j = n$. As is well-known, these numbers are in one-to-one correspondence with integer partitions $(\lambda_1, \lambda_2, \ldots)$ with $\lambda_1 \geqslant \lambda_2 \geqslant \ldots$ and $\sum \lambda_i = n$, by defining $r_j = \#\{i : \lambda_i = j\}$. Taking into account the number of partitions that correspond to a given set $(r_1, \ldots, r_n)$, the probability of a weighted random partition is then

$$\mathbb{P}_n(\lambda_1, \lambda_2, \ldots) = \frac{1}{h_n} \prod_{j=1}^{n} \frac{1}{r_j!} \Big(\frac{\theta_j}{j}\Big)^{r_j}. \tag{1.4}$$

A random variable such as $L_1$ cannot be expressed in terms of $\{R_j\}$, but one can give an interpretation of its distribution in the context of random partitions. Given a random partition, pick $i \in \{1, \ldots, n\}$ uniformly at random, and take $j$ that satisfies

$$\lambda_1 + \cdots + \lambda_{j-1} < i \leqslant \lambda_1 + \cdots + \lambda_j. \tag{1.5}$$

The distribution of $\lambda_j$ is identical to that of $L_1$.

Let us discuss the heuristics behind the dependence of cycle lengths on parameters. It is tempting to think that, if $\theta_j$ is increasing, longer cycles are favored. But this turns out to be incorrect. For instance, typical cycle lengths are of order $n$ for $\theta_j \equiv 1$; they are of order $(\log n)^{1/\gamma}$ for $\theta_j = \mathrm{e}^{j^\gamma}$ with $0 < \gamma < 1$; they are again of order $n$ for $\theta_j = \mathrm{e}^j$. The reason for this apparently erratic behavior is best understood from the perspective of statistical mechanics, as already mentioned in [9]. We can assign to each index a weight that depends on the length of the cycle it belongs to. Let $L_i(\sigma)$ denote the length of the

| Parameters | Typical cycle lengths | Number of cycles | Finite cycles | Reference |
|---|---|---|---|---|
| $\theta_n = e^{n^\gamma}, \gamma > 1$ | $\mathbb{P}_n(L_1 = n) \to 1$ | $K \Rightarrow 1$ | $R_j \Rightarrow 0$ | Section 3 |
| $\theta_n \approx e^{n^\gamma}, 0 < \gamma < 1$ | $L_1/(\frac{1}{1-\gamma}\log n)^{1/\gamma} \Rightarrow 1$ | | $R_j \Rightarrow \text{Poisson}(\frac{\theta_j}{j})$ | Section 4 |
| $\theta_n \approx n^\gamma, \gamma > 0$ | $L_1/n^{\frac{1}{\gamma+1}} \Rightarrow \text{Gamma}(\gamma+1, \Gamma(\gamma+1)^{\frac{1}{\gamma+1}})$ | $\mathbb{E}_n(K) \approx An^{\frac{\gamma}{\gamma+1}}$ | $R_j \Rightarrow \text{Poisson}(\frac{\theta_j}{j})$ | Section 5 |
| $\theta_n \to \theta$ | $L_1/n \Rightarrow \text{Beta}(1, \theta)$ | $\mathbb{E}_n(K) \approx \theta \log n$ | $R_j \Rightarrow \text{Poisson}(\frac{\theta_j}{j})$ | Section 6 |
| $\theta_n = n^{-\gamma}, \gamma > 0$, or $\theta_n = e^{-n^\gamma}, 0 < \gamma < 1$ | $L_1/n \Rightarrow 1$ | $K \Rightarrow 1 + \text{Poisson}(\sum \frac{\theta_j}{j})$ | $R_j \Rightarrow \text{Poisson}(\frac{\theta_j}{j})$ | Section 7 |
| $\theta_n = e^{-n^\gamma}, \gamma > 1$ | $L_1/(\frac{1}{\gamma-1}\log n)^{1/\gamma} \Rightarrow 1$ | | $\mathbb{E}_n(R_j) \sim \exp\{j\gamma(\frac{\log n}{\gamma-1})^{\frac{\gamma-1}{\gamma}}\}$ | Section 8 |

TABLE 1. Overview of typical cycle lengths ($L_1$), total number of cycles ($K$), and number of finite cycles ($R_j$), for different parameters. As for the notation: '$\Rightarrow$' denotes convergence in distribution; '$a_n \approx b_n$' means that $a_n/b_n \to 1$ as $n \to \infty$; '$a_n \sim b_n$' is a much weaker relation.

cycle that contains $i$, and notice the identity

$$\sum_{i=1}^{n} a_{L_i(\sigma)} = \sum_{j=1}^{n} j a_j R_j(\sigma), \tag{1.6}$$

that holds for any $\sigma$ and any numbers $a_1, a_2, \ldots$. Choosing $a_j = \frac{1}{j} \log \theta_j$, the probability distribution $\mathbb{P}_n$ can be rewritten in the form of a "Gibbs state", namely

$$\mathbb{P}_n(\sigma) = \frac{1}{n! h_n} \exp\Big(\sum_{i=1}^{n} a_{L_i(\sigma)}\Big). \tag{1.7}$$

The weight $a_{L_i(\sigma)}$ plays the rôle of the negative of the energy. The heuristics become

- If $a_j = \frac{1}{j} \log \theta_j$ is increasing, indices prefer to be in longer cycles, and typical cycle lengths are longer.
- If $a_j = \frac{1}{j} \log \theta_j$ is decreasing, the converse happens.

Expression (1.7) also points to an important symmetry of the parameters: Adding a constant $c$ to each $a_j$ does not affect $\mathbb{P}_n$. But it amounts to changing the parameters from $\theta_j$ to $\theta_j \, \mathrm{e}^{cj}$. This also shows that purely exponential parameters are equivalent to uniform random permutations.

We have summarized the results about weighted random permutations in Table 1. Precise claims can be found in Sections 3–8. The behavior of $L_1$ is strikingly similar when the parameters grow sub-exponentially or decay super-exponentially. Notice that typical cycle lengths were obtained earlier in the asymptotic Ewens case [16, 9] and when $\theta_j$ decreases to 0 [9]. We complement these results with statements about finite cycles and total number of cycles (see also [4]). We also show that the joint distribution of $L_1, L_2, \ldots$ converges to the Poisson-Dirichlet distribution.

## 2. GENERALITIES

### 2.1. Generating functions and basic expressions for random variables.
Let us start with the exponential generating function of weighted cyclic permutations. There are $(n-1)!$ cyclic permutations of $n$ elements, and therefore

$$\sum_{\substack{n \geqslant 1}} \sum_{\substack{\sigma \in \mathcal{S}_n \\ \text{cyclic}}} \frac{1}{n!} \theta_n z^n = \sum_{n \geqslant 1} \frac{\theta_n}{n} z^n. \tag{2.1}$$

A general permutation can be viewed as a combinatorial set of cycles, so that the generating function of permutations is given by the exponential of the generating function of cyclic permutations (see Corollary 6.6 of [1]). We set $h_0 = 1$. Then

$$G_h(z) = \sum_{n \geqslant 0} h_n z^n = \sum_{n \geqslant 0} \frac{z^n}{n!} \sum_{\sigma \in \mathcal{S}_n} \prod_{j \geqslant 1} \theta_j^{R_j} = \exp \sum_{n \geqslant 1} \frac{\theta_n}{n} z^n. \tag{2.2}$$

We now obtain expressions that characterize the random variables $L_1, K, R_j$. For part (c), we use $r_{[k]}$ to denote the descending factorial,

$$r_{[k]} = r(r-1)\ldots(r-k+1). \tag{2.3}$$

**Proposition 2.1.**

(a) $\mathbb{P}_n(L_1 = j) = \dfrac{\theta_j h_{n-j}}{n h_n}.$

(b) $\mathbb{E}_n(K) = \displaystyle\sum_{j=1}^{n} \frac{\theta_j h_{n-j}}{j h_n}.$

(c) $\mathbb{E}_n\Big( \displaystyle\prod_{j \geqslant 1} (R_j)_{[k_j]} \Big) = \dfrac{h_{n-\sum_j j k_j}}{h_n} \displaystyle\prod_{j \geqslant 1} \Big( \dfrac{\theta_j}{j} \Big)^{k_j}$ for all integers $k_1, \ldots, k_n$ such that $\sum j k_j \leqslant n$.

Before proving this proposition, let us mention two useful consequences. Summing over all possible values for $j$ in the identity (a), we get a relation for the $h_n$s, namely

$$h_n = \frac{1}{n} \sum_{j=1}^{n} \theta_j h_{n-j}. \tag{2.4}$$

The following corollary will apply to all regimes of parameters that we consider, except super-exponential growth or decay.

**Corollary 2.2.** *When $h_{n-1}/h_n \to 1$ as $n \to \infty$, the joint distribution of the number of finite cycles, $R_1, R_2, R_3, \ldots$, converges weakly to independent Poisson with means $\theta_1, \frac{\theta_2}{2}, \frac{\theta_3}{3}, \ldots$.*

*Proof.* We have from Proposition 2.1 (c) that

$$\lim_{n \to \infty} \mathbb{E}_n\Big( \prod_{j \geqslant 1} (R_j)_{[k_j]} \Big) = \prod_{j \geqslant 1} \Big( \frac{\theta_j}{j} \Big)^{k_j} \tag{2.5}$$

for all $k_1, k_2, \ldots$ with finitely many nonzero terms. The result is then standard, see e.g. Lemma 2.8 of [19]. $\qquad\square$

*Proof of Proposition 2.1.* The sum over permutations with $L_1 = j$ can be done by first summing over the $(j-1)$ other indices that belong to the cycle that contains 1 (there are $(n-1) \ldots (n-j+1)$ possibilities), then by summing over permutations of the remaining $(n-j)$ indices. We get

$$\mathbb{P}_n(L_1 = j) = \frac{1}{n! h_n} (n-1) \ldots (n-j+1) \theta_j (n-j)! h_{n-j} = \frac{\theta_j h_{n-j}}{n h_n}. \tag{2.6}$$

We now prove the identity (c). We use the generating function $G_h(s)$. Let $k_1, k_2, \ldots$ be nonnegative integers such that $\sum_j j k_j \leqslant n$. Recall that $h_n$ and $G_h$ depend on $\theta_1, \theta_2, \ldots$ Using $h_n = [s^n] G_h(s)$, we have

$$
\begin{aligned}
\mathbb{E}_n\Big( \prod_{j \geqslant 1} (R_j)_{[k_j]} \Big) &= \frac{1}{h_n} \Big( \prod_{j \geqslant 1} \theta_j^{k_j} \frac{\mathrm{d}^{k_j}}{\mathrm{d}\theta_j^{k_j}} \Big) h_n \\
&= \frac{1}{h_n} [s^n] \Big( \prod_{j \geqslant 1} \theta_j^{k_j} \frac{\mathrm{d}^{k_j}}{\mathrm{d}\theta_j^{k_j}} \Big) G_h(s) \\
&= \frac{1}{h_n} \prod_{j \geqslant 1} \Big( \frac{\theta_j}{j} \Big)^{k_j} [s^{n-\sum_j j k_j}] G_h(s) \\
&= \frac{h_{n-\sum_j j k_j}}{h_n} \prod_{j \geqslant 1} \Big( \frac{\theta_j}{j} \Big)^{k_j}.
\end{aligned}
\tag{2.7}
$$

As a special case of (c) we have $\mathbb{E}_n(R_j) = \frac{\theta_j h_{n-j}}{j h_n}$, which yields the expression (b) for the expected number of cycles. $\qquad\square$

2.2. **Saddle point analysis.** Several asymptotic results will be derived using the method of steepest descent, which prompts us to introduce it here. We refer the reader unfamiliar with this method to [17] for appropriate background. For our particular application we will require a uniform extension of this method to a family of descent problems essentially indexed by $n$. This extension may more generally be referred to as *saddle point analysis*. What this entails and how it is justified will be discussed in the final section. We state here the main result based on this method and how it will apply to the cases of interest in this paper. More details can be found in Section 9.

Let $G(z)$ be a function that is analytic at the origin with Taylor series there having a finite, non-zero radius of convergence (which we will take to be 1 in all cases). For $r > 0$, let

$$\begin{aligned} \alpha(r) &= r(\log G(r))', \\ \beta(r) &= r\alpha'(r) \end{aligned} \tag{2.8}$$

and assume that

$$\lim_{r \to 1} \alpha(r) = \infty; \quad \lim_{r \to 1} \beta(r) = \infty.$$

Assume further that for $n$ sufficiently large there is an $\epsilon$ such that on $(1 - \epsilon, 1)$ there is a unique solution to the equation

$$\alpha(r) = n \tag{2.9}$$

and denote this root by $r_n$. Notice that in this range $r_n$ is increasing and $\lim_{n \to \infty} r_n = 1$. Then the main saddle point result we use states that

$$[z^n]G(z) = \frac{G(r_n)}{r_n^n \sqrt{2\pi\beta(r_n)}} \big(1 + o(1)\big). \tag{2.10}$$

We will apply this in two cases: (i) for the generating function $G_\theta$ of the parameters; (ii) for the generating function $G_h$ of the coefficients $h_n$.

It is convenient to introduce

$$I_\mu(z) = \sum_{n \geqslant 1} n^\mu \theta_n z^n. \tag{2.11}$$

These functions satisfy the following recursion relations

$$I_\mu(z) = z\, I'_{\mu-1}(z). \tag{2.12}$$

In the case of the parameter generating functions we take

$$G_\theta(z) = z^{-1}I_0(z) = \frac{d}{dz}I_{-1}(z). \tag{2.13}$$

One easily finds that

$$\alpha(z) = \frac{I_1(z)}{I_0(z)} - 1. \tag{2.14}$$

Define $\rho_n$ by the equation $\alpha(\rho_n) = n$. Then, applying (2.10), the parameters are asymptotically equal to

$$\theta_{n+1} = \frac{G_\theta(\rho_n)}{\rho_n^{n+\frac{1}{2}} \sqrt{2\pi\alpha'(\rho_n)}} \big(1 + o(1)\big). \tag{2.15}$$

Let us turn to the second case, i.e., the generating function of $h_n$ given by (2.2). We have the relations

$$
\begin{aligned}
G_h(z) &= \exp I_{-1}(z), \\
\alpha(r) &= I_0(r), \quad I_0(r_n) = n, \\
\beta(r) &= I_1(r).
\end{aligned}
\tag{2.16}
$$

Notice that $r_n$ is increasing in $n$ and that $\lim r_n$ is equal to the radius of convergence of $I_0$. The coefficients $h_n$ are asymptotically equal to

$$
h_n = \frac{e^{I_{-1}(r_n)}}{r_n^n \sqrt{2\pi I_1(r_n)}} \big(1 + o(1)\big).
\tag{2.17}
$$

We will actually deal with ratios of those numbers, and the following bounds will prove extremely useful.

**Proposition 2.3.** *Assume that $\theta_1, \theta_2, \ldots$ are such that the saddle point approximation involving $G_h$ is valid. Then*

$$
\sqrt{\frac{I_1(r_n)}{I_1(r_{n-j})}} r_{n-j}^j \;\leqslant\; \frac{h_{n-j}}{h_n}\big(1 + o(1)\big) \;\leqslant\; \sqrt{\frac{I_1(r_n)}{I_1(r_{n-j})}} r_n^j.
$$

*Proof.* The asymptotic approximation (2.17) for $h_n$ implies that

$$
\frac{h_{n-j}}{h_n} = \sqrt{\frac{I_1(r_n)}{I_1(r_{n-j})}}\, e^{\Delta(n,j)} \big(1 + o(1)\big),
\tag{2.18}
$$

where $\Delta(n,j)$ can be written as

$$
\Delta(n,j) = I_{-1}(r_{n-j}) - I_0(r_{n-j}) \log r_{n-j} - I_{-1}(r_n) + I_0(r_n) \log r_n.
\tag{2.19}
$$

By the fundamental theorem of calculus, using (2.12),

$$
\Delta(n,j) = \int_{r_n}^{r_{n-j}} \frac{\mathrm{d}}{\mathrm{d}u}\Big(I_{-1}(u) - I_0(u)\log u\Big)\,\mathrm{d}u = \int_{r_{n-j}}^{r_n} \frac{I_1(u)\log u}{u}\,\mathrm{d}u.
\tag{2.20}
$$

We can bound $\log u \geqslant \log r_{n-j}$. The integral of $I_1(u)/u$ yields $I_0(r_n) - I_0(r_{n-j}) = j$ and we get the lower bound of the proposition. The upper bound is similar, using $\log u \leqslant \log r_n$. $\qquad\square$

## 3. Parameters with super-exponential growth

First we consider the regime when $\theta_n$ diverges fast enough. It is not too hard to show that only one cycle of length $n$ is present, meaning also that all finite cycles have disappeared.

**Theorem 3.1.** *Assume that $\theta_n > 0$ for all $n$, and that*

$$
\lim_{n\to\infty} \sum_{j=1}^{n-1} \frac{\theta_j \theta_{n-j}}{\theta_n} = 0.
$$

*Then*

$$
\lim_{n\to\infty} \mathbb{P}_n(L_1 = n) = 1.
$$

It immediately follows that $\mathbb{P}_n(K = 1) \to 1$ and $\mathbb{P}_n(R_j = 0) \to 1$ for all fixed $j$.

Let us check that the theorem applies to the parameters $\theta_n = \mathrm{e}^{n^\gamma}$ with $\gamma > 1$. We have

$$\frac{\theta_j \theta_{n-j}}{\theta_n} = \mathrm{e}^{-n^\gamma[1-(\frac{j}{n})^\gamma-(1-\frac{j}{n})^\gamma]}. \tag{3.1}$$

It is easy to check that $(1-s)^\gamma \leqslant 1 - cs$ for $0 \leqslant s \leqslant \frac{1}{2}$ with $c = 2(1 - 2^{-\gamma}) > 1$. Then for $j \leqslant \frac{n}{2}$,

$$\frac{\theta_j \theta_{n-j}}{\theta_n} \leqslant \mathrm{e}^{-n^\gamma[c\frac{j}{n}-(\frac{j}{n})^\gamma]} \leqslant \mathrm{e}^{-(c-1)n^{\gamma-1}j}. \tag{3.2}$$

It follows that

$$\sum_{j=1}^{n-1} \frac{\theta_j \theta_{n-j}}{\theta_n} \leqslant 2 \sum_{j=1}^{n/2} \frac{\theta_j \theta_{n-j}}{\theta_n} \leqslant 2 \sum_{j \geqslant 1} \mathrm{e}^{-(c-1)n^{\gamma-1}j}, \tag{3.3}$$

which clearly goes to 0 as $n \to \infty$.

*Proof of theorem 3.1.* By the assumption of the theorem, there exists $N$ such that $\theta_j \theta_{n-j} < \theta_n$ for all $n \geqslant N$. Let $C$ such that $h_n \leqslant C\theta_n$ for all $n \leqslant N$. We now prove by induction that this upper bound holds for all $n$. By (2.4) and the induction hypothesis,

$$h_{n+1} \leqslant \frac{C}{n+1} \sum_{j=1}^{n+1} \theta_j \theta_{n+1-j} \leqslant C\theta_{n+1}. \tag{3.4}$$

We have $\mathbb{P}_n(L_1 = n) = \frac{\theta_n}{nh_n}$ by Proposition 2.1 (a). Using (2.4), we get

$$\frac{nh_n}{\theta_n} = \sum_{j=1}^{n} \frac{\theta_j h_{n-j}}{\theta_n} = 1 + \sum_{j=1}^{n-1} \frac{\theta_j h_{n-j}}{\theta_n}. \tag{3.5}$$

Using $h_{n-j} \leqslant C\theta_{n-j}$ and the assumption of the theorem, we see that $\frac{nh_n}{\theta_n} \to 1$ as $n \to \infty$. $\qquad\square$

## 4. PARAMETERS WITH SUB-EXPONENTIAL GROWTH

Our goal here is to understand the regime of parameters that grow sub-exponentially, $\theta \approx \mathrm{e}^{n^\gamma}$ with $0 < \gamma < 1$. It turns out to be difficult to tackle this case directly and we appeal to an indirect approach, by focusing on the generating function rather than its parameters.

Let $A, a, b, c$ be positive parameters to be chosen later, and let

$$G_\theta(z) = A(1-z)^{-c} \mathrm{e}^{a(1-z)^{-b}}. \tag{4.1}$$

Then the parameters, $\theta_n$, are the coefficients of $I_0$ or, equivalently, the shifted coefficients of $G_\theta$:

$$\theta_n = [z^{n-1}]G_\theta(z). \tag{4.2}$$

It is not hard to check (by repeated differentiation) that $\theta_n > 0$ for all $n \geqslant 1$. Notice also that the radius of convergence of $G_\theta$ is 1.

**Proposition 4.1.**

$$\theta_{n+1} = \frac{A(ab)^{\frac{1}{b+1}(\frac{1}{2}-c)}}{\sqrt{2\pi(b+1)}} n^{\frac{1}{b+1}(c-\frac{b+2}{2})} \exp\left\{ \left[a(b+1)(ab)^{-\frac{b}{b+1}}\right]n^{\frac{b}{b+1}} + \left[\frac{1}{2}(ab)^{\frac{2}{b+1}}\right]n^{\frac{b-1}{b+1}} + o\left(n^{\frac{b-1}{b+1}\vee 0}\right) \right\}.$$

*Proof.* We use the saddle point method, which has long been used for this class of functions [20]. We have

$$\alpha(z) = z \frac{d}{dz} \log G_\theta(z) = z \left[ \frac{c}{1-z} + \frac{ab}{(1-z)^{b+1}} \right],$$ (4.3)

$$I_1(z) = I_0(z)(\alpha(z) + 1) = I_0(z) \left[ abz(1-z)^{-b-1} + cz(1-z)^{-1} + 1 \right].$$

Notice that, as $z \to 1$,

$$\alpha'(z) = ab(b+1)(1-z)^{-b-2}(1+o(1)).$$ (4.4)

Defining $\rho_n$ by $\alpha(\rho_n) = n$ we obtain

$$\rho_n = 1 - \left( \frac{ab}{n} \right)^{\frac{1}{b+1}} \left[ \rho_n + \frac{c}{ab}(1-\rho_n)^b - \frac{c}{ab}(1-\rho_n)^{b+1} \right]^{\frac{1}{b+1}}.$$ (4.5)

By the implicit function theorem, we get

$$\rho_n = 1 - \left( \frac{ab}{n} \right)^{\frac{1}{b+1}} + \frac{1}{b+1} \left( \frac{ab}{n} \right)^{\frac{2}{b+1}} - \frac{c}{b+1} \left( \frac{1}{n} \right) + o\left( n^{-(1 \wedge \frac{2}{b+1})} \right)$$

$$= \exp \left\{ -\left( \frac{ab}{n} \right)^{\frac{1}{b+1}} + \frac{1}{2} \frac{1-b}{1+b} \left( \frac{ab}{n} \right)^{\frac{2}{b+1}} - \frac{c}{b+1} \left( \frac{1}{n} \right) + o\left( n^{-(1 \wedge \frac{2}{b+1})} \right) \right\}.$$ (4.6)

We compute the terms that appear in the formula (2.15) for $\theta_n$. First,

$$G_\theta(\rho_n) = A \left( \frac{n}{ab} \right)^{\frac{c}{b+1}} \exp \left\{ a \left( \frac{n}{ab} \right)^{\frac{b}{b+1}} + \frac{ab}{b+1} \left( \frac{n}{ab} \right)^{\frac{b-1}{b+1}} - \frac{c}{b+1} + o\left( n^{\frac{b-1}{b+1} \vee 0} \right) \right\},$$

$$\rho_n^{-n} = \exp \left\{ (ab)^{\frac{1}{b+1}} n^{\frac{b}{b+1}} + \frac{1}{2} \frac{b-1}{b+1} (ab)^{\frac{2}{b+1}} n^{\frac{b-1}{b+1}} + \frac{c}{b+1} + o\left( n^{\frac{b-1}{b+1} \vee 0} \right) \right\},$$ (4.7)

$$\alpha'(\rho_n) = (b+1)(ab)^{-\frac{1}{b+1}} n^{\frac{b+2}{b+1}}(1+o(1)).$$

We get the proposition by inserting these values into (2.15). □

We choose the numbers $A, a, b, c$ so that $\theta_n \approx e^{n^\gamma}$. Precisely, let

$$b = \frac{\gamma}{1-\gamma},$$

$$a = (1-\gamma)\gamma^{\frac{\gamma}{1-\gamma}},$$

$$c = \frac{b}{2} + 1,$$ (4.8)

$$A = \sqrt{2\pi(b+1)}(ab)^{-\frac{1}{b+1}(\frac{1}{2}-c)}.$$

They imply the following relations, that are often useful when checking the details of the calculations:

$$\gamma = \frac{b}{b+1}, \qquad ab = \gamma^{\frac{1}{1-\gamma}}, \qquad \frac{1}{b+1} = 1 - \gamma.$$ (4.9)

With these numbers, the precise asymptotic expression of $\theta_n$ is

$$\theta_n = \exp \left\{ n^\gamma + \frac{1}{2} \gamma^2 n^{2\gamma-1} + o\left( n^{(2\gamma-1) \vee 0} \right) \right\}.$$ (4.10)

Notice that $\theta_n = e^{n^\gamma}(1+o(1))$ when $\gamma < \frac{1}{2}$. The case $\gamma = \frac{1}{2}$ can be handled by modifying the number $A$. For the case $\gamma > \frac{1}{2}$, the correction $n^{2\gamma-1}$ is present in the exponential and it cannot be removed easily.

It is time to state the main result of this section.

**Theorem 4.2.** *Consider the set of parameters $\theta_1, \theta_2, \ldots$ whose generating function is given by Eq. (4.1) with $A, a, b, c$ specialized as in Eq. (4.8). Then*

(a) $\dfrac{L_1}{(\log n)^{1/\gamma}} \Rightarrow (1-\gamma)^{-1/\gamma}.$

(b) $R_1, R_2, \ldots$ *converge weakly to independent Poisson random variables with respective means* $\theta_1, \frac{\theta_2}{2}, \ldots$.

*Proof.* Let $B = (1 - \gamma)^{-1/\gamma}$. We show that for any $\varepsilon > 0$,

$$\lim_{n \to \infty} \mathbb{P}_n \Big( \Big| \frac{L_1}{(\log n)^{1/\gamma}} - B \Big| > \varepsilon \Big) = 0. \tag{4.11}$$

By Proposition 2.1 (a), we have

$$\mathbb{P}_n \Big( \Big| \frac{L_1}{(\log n)^{1/\gamma}} - B \Big| > \varepsilon \Big) = \sum_{j: | \frac{j}{(\log n)^{1/\gamma}} - B | > \varepsilon} \frac{\theta_j h_{n-j}}{n h_n}. \tag{4.12}$$

We use the saddle point method for the generating function $G_h = \mathrm{e}^{I-1}$. The equation $I_0(r_n) = n$ implies that

$$r_n = 1 - \Big[ \frac{1}{a} \log \frac{n(1 - r_n)^c}{A r_n} \Big]^{-1/b}. \tag{4.13}$$

(We keep using $A, a, b, c$ rather than $\gamma$ for convenience.) By the implicit function theorem, we get

$$r_n = 1 - a^{\frac{1}{b}} (\log n)^{-\frac{1}{b}} + O \Big( \frac{\log \log n}{(\log n)^{b+1}} \Big)$$
$$= \exp \Big\{ -a^{\frac{1}{b}} (\log n)^{-\frac{1}{b}} + O \Big( \frac{\log \log n}{(\log n)^{b+1}} \vee (\log n)^{-\frac{2}{b}} \Big) \Big\}. \tag{4.14}$$

It follows that $I_1(r_n) = n (\frac{1}{a} \log n)^{1/\gamma} (1 + o(1))$. We use Proposition 2.3 to get

$$\frac{h_{n-j}}{h_n} \leqslant \sqrt{\frac{n}{n-j}} \Big( \frac{\log n}{\log(n-j)} \Big)^{1/2\gamma} \exp \Big\{ -j a^{\frac{1}{b}} (\log n)^{-\frac{1}{b}} \Big( 1 + O \Big( \frac{\log \log n}{\log n} \vee \frac{1}{(\log n)^{1/b}} \Big) \Big) \Big\}. \tag{4.15}$$

The cases $j = n$ and $j = n-1$ need actually to be handled separately. Using the expression (4.10) for $\theta_j$ and the bound above, it is easy to check that

$$\lim_{n \to \infty} \sum_{j=n/2}^{n} \frac{\theta_j h_{n-j}}{n h_n} = 0. \tag{4.16}$$

For $1 \leqslant j \leqslant n/2$, we have

$$\sum_{j: | \frac{j}{(\log n)^{1/\gamma}} - B | > \varepsilon} \frac{\theta_j h_{n-j}}{n h_n} \leqslant C \sum_{j: | \frac{j}{(\log n)^{1/\gamma}} - B | > \varepsilon} \exp \Big\{ j^\gamma - j a^{\frac{1}{b}} (\log n)^{-\frac{1}{b}} - \log n$$
$$+ O(j^{(2\gamma-1) \vee 0}) + O \Big( \frac{\log \log n}{\log n} \vee \frac{1}{(\log n)^{1/b}} \Big) \Big\}. \tag{4.17}$$

Let us make the change of variables $j = i (\log n)^{1/\gamma}$. Then

$$\sum_{j: | \frac{j}{(\log n)^{1/\gamma}} - B | > \varepsilon} \frac{\theta_j h_{n-j}}{n h_n} \leqslant C \sum_{\substack{i \in (\log n)^{-1/\gamma} \mathbb{N} \\ |i - B| > \varepsilon}} \mathrm{e}^{- \log n [a^{\frac{1}{b}} i - i^\gamma + 1 + o(1)]}. \tag{4.18}$$

It is easy to see that the function $f(x) = a^{\frac{1}{b}} x - x^\gamma + 1$ is convex with a minimum at $B = (1 - \gamma)^{-1/\gamma}$, where it takes value 0. For $|x - B| > \varepsilon$ we can bound $f(x) > \delta |x - B|$. We can estimate the sum by an integral, in order to get

$$\sum_{j: | \frac{j}{(\log n)^{1/\gamma}} - B | > \varepsilon} \frac{\theta_j h_{n-j}}{n h_n} \leqslant C (\log n)^{1/\gamma} \int_{|x - B| > \varepsilon} \mathrm{e}^{-(\log n) \delta |x - B|} \, \mathrm{d}x, \tag{4.19}$$

which clearly vanishes in the limit $n \to \infty$. This proves (a).

It is clear from Proposition 2.3 and Eq. (4.14) that $h_{n-1}/h_n \to 1$ as $n \to \infty$, so that (b) follows immediately from Corollary 2.2. $\qquad \square$

## 5. Parameters with algebraic growth

We again work with a generating function rather than parameters. Recall that $\gamma > 0$, and let

$$I_0(z) = \frac{\Gamma(\gamma + 1)}{(1 - z)^{\gamma + 1}} - \Gamma(\gamma + 1). \tag{5.1}$$

One easily checks that

$$\frac{d^n}{dz^n} I_0(z) = \frac{\Gamma(n + \gamma + 1)}{(1 - z)^{\gamma + n + 1}}. \tag{5.2}$$

One then gets the parameters:

$$\theta_n = [z^n] I_0(z) = \frac{1}{n!} \frac{d^n}{dz^n} I_0(0) = \frac{\Gamma(\gamma + n + 1)}{n!}. \tag{5.3}$$

By a straightforward application of Stirling's formula one sees that the parameters grow algebraically:

$$\theta_n = n^\gamma (1 + o(1)). \tag{5.4}$$

**Theorem 5.1.** *Choose $\theta_1, \theta_2, \ldots$ such that their generating function is given by Eq. (5.1). Then*

(a) $L_1/n^{\frac{1}{1+\gamma}}$ *converges weakly to the Gamma random variable with parameters $(\gamma + 1, a)$ with $a = \Gamma(\gamma + 1)^{\frac{1}{\gamma + 1}}$. In other words, we have*

$$\lim_{n \to \infty} \mathbb{P}_n \Big( \frac{L_1}{n^{1/(1+\gamma)}} < s \Big) = \int_0^s x^\gamma e^{-ax} \, dx.$$

(b) $\displaystyle \lim_{n \to \infty} n^{-\frac{\gamma}{\gamma + 1}} \mathbb{E}_n(K) = \big( \Gamma(\gamma)/\gamma^\gamma \big)^{\frac{1}{\gamma + 1}}.$

(c) *The distribution of number of finite cycles converges weakly to independent Poisson random variables with means $\theta_1, \frac{\theta_2}{2}, \ldots$.*

*Proof.* We use the saddle point method. Let $r_n$ be defined by $I_0(r_n) = n$. Then

$$r_n = 1 - \Big( \tfrac{n}{\Gamma(1+\gamma)} + 1 \Big)^{-\frac{1}{1+\gamma}}. \tag{5.5}$$

It is enough for our purpose to retain

$$r_n = 1 - an^{-\frac{1}{1+\gamma}} + O(n^{-\frac{2+\gamma}{1+\gamma}}) = \exp\Big\{ -an^{-\frac{1}{1+\gamma}} + O(n^{-\frac{2}{1+\gamma}}) \Big\}. \tag{5.6}$$

In order to use Proposition 2.3, we check that $r_{n-j}^j$ is close to $r_n^j$. We assume from now on that $j < Cn^{1/(1+\gamma)}$ for a constant $C$ independent of $n$. A few calculations yield

$$r_{n-j}^j = \exp\Big\{ -ajn^{-\frac{1}{1+\gamma}} + O(j^2 n^{-\frac{2+\gamma}{1+\gamma}} \vee jn^{-\frac{2}{1+\gamma}}) \Big\}. \tag{5.7}$$

Then

$$r_{n-j}^j = r_n^j (1 + o(1)). \tag{5.8}$$

We also have

$$I_1(z) = \Gamma(\gamma + 2) \, z \, (1 - z)^{-\gamma - 2}. \tag{5.9}$$

Then

$$\frac{I_1(r_n)}{I_1(r_{n-j})} = \frac{r_n}{r_{n-j}} \left(\frac{n}{n-j}\right)^{\frac{2+\gamma}{1+\gamma}} (1 + o(1))$$

$$= e^{\frac{aj}{1+\gamma} n^{-\frac{\gamma+2}{\gamma+1}}} (1 + o(1))$$

(5.10)

It follows that

$$\frac{h_{n-j}}{h_n} = r_n^j \sqrt{\frac{I_1(r_n)}{I_1(r_{n-j})}} (1 + o(1))$$

$$= e^{-ajn^{-1/(1+\gamma)}} (1 + o(1)).$$

(5.11)

We can now proceed to the calculation of the distribution of $L_1$. Using Proposition 2.1 (a), we have

$$\mathbb{P}_n \left(\frac{L_1}{n^{1/(1+\gamma)}} < s\right) = \sum_{j=1}^{s\, n^{1/(1+\gamma)}} \frac{j^\gamma}{n} e^{-ajn^{-1/(1+\gamma)}} (1 + o(1)).$$

(5.12)

(We can use the asymptotic value for $\theta_j$ because finite $j$ contribute a vanishing amount.) We rescale the variables in order to recognize a Riemann integral:

$$\mathbb{P}_n \left(\frac{L_1}{n^{1/(1+\gamma)}} < s\right) = \frac{1}{n^{\frac{1}{1+\gamma}}} \sum_{j=1}^{s\, n^{1/(1+\gamma)}} \left(\frac{j}{n^{1/(1+\gamma)}}\right)^\gamma e^{-a\frac{j}{n^{1/(1+\gamma)}}} (1 + o(1)).$$

(5.13)

As $n \to \infty$, this converges to the probability that the Gamma random variable with parameters $(\gamma + 1, a)$ be less than $s$.

For part (b) we use Proposition 2.1 (b). Using $\theta_j = j^\gamma(1 + o(1))$ and Eq. (5.11), we have

$$\mathbb{E}_n(K) = \sum_{j=1}^{n} j^{\gamma-1} e^{-ajn^{1/(\gamma+1)}} (1 + o(1)).$$

(5.14)

Notice that the contribution of finite $j$ vanishes, which justifies using the asymptotic expression for $\theta_j$. Introducing the appropriate scaling that leads to a Riemann integral, we rewrite the expression as

$$\frac{a^\gamma}{n^{\frac{\gamma}{\gamma+1}}} \mathbb{E}_n(K) = \frac{a}{n^{\frac{1}{\gamma+1}}} \sum_{j=1}^{n} \left(\frac{aj}{n^{\frac{1}{\gamma+1}}}\right)^{\gamma-1} e^{-ajn^{-\frac{1}{\gamma+1}}} (1 + o(1)).$$

(5.15)

The right side converges to $\int_0^\infty x^{\gamma-1} e^{-x} \, dx = \Gamma(\gamma)$ and we obtain the claim (b).

Part (c) follows from (5.11) and Corollary 2.2.                                    □

## 6. Asymptotic Ewens parameters

Past studies of the Ewens distribution have focused on the number of cycles. It was shown in particular that the number of cycles with length less than $n^s$ is approximately equal to $\theta s \log n$ for all $0 < s \leqslant 1$, and that it satisfies a central limit theorem [15] and a large deviation principle [13]. In this section we consider the case where $\theta_j \to \theta$ as $j \to \infty$. We look at the distribution of finite cycles and at the joint distribution of the largest cycles. Let $L^{(1)}, L^{(2)}, \dots$ denote the cycle lengths in nonincreasing order (for all $\sigma \in \mathcal{S}_n$ we have $\sum_j j R_j(\sigma) = \sum_i L^{(i)} = n$).

The large cycle lengths converge to the Poisson-Dirichlet distribution. In order to define it, first consider a sequence of i.i.d. beta random variables with parameters $(1, \theta)$, $(X_1, X_2, \dots )$. That is, $\mathbb{P}(X > s) = (1 - s)^\theta$ for $0 \leqslant s \leqslant 1$. Then form the sequence $(X_1, (1 - X_1)X_2, (1 - X_1)(1 - X_2)X_3, \dots )$. It is not hard to check that it is a random partition of $[0, 1]$, which is called the Griffiths-Engen-McCloskey distribution. Reorganizing these numbers in nonincreasing order gives another random partition of $[0, 1]$, and the corresponding distribution is called Poisson-Dirichlet.

**Theorem 6.1.** *Assume that $\theta_n \to \theta$. Then, as $n \to \infty$,*
  (a) *the random variables $R_1, R_2, R_3, \dots$ converge weakly to independent Poisson with respective means $\theta_1, \frac{\theta_2}{2}, \frac{\theta_3}{3}, \dots$;*
  (b) *the total number of cycles is logarithmic:* $\displaystyle \lim_{n \to \infty} \frac{\mathbb{E}_n(K)}{\log n} = \theta$;
  (c) *the joint distribution of $\frac{L^{(1)}}{n}, \frac{L^{(2)}}{n}, \dots$ converges weakly to Poisson-Dirichlet with parameter $\theta$.*

The last result involves only the limit $\theta$ and not the individual parameters $\theta_j$s. This is not surprising as the longest cycles become infinite as $n \to \infty$. The theorems of [15, 13] also concern cycles of diverging lengths and they should remain valid in the asymptotic Ewens case without modifications. On the other hand, the distribution of finite cycles depends explicitly on the $\theta_j$s.

The rest of this section is devoted to the proof of this theorem. It relies on estimates for the normalization $h_n$. Let us introduce the function $\Lambda(x)$, $x \geqslant 1$, by

$$\Lambda\left(\frac{1}{1-s}\right) = \exp \sum_{j \geqslant 1} \frac{\theta_j - \theta}{j} s^j, \tag{6.1}$$

where $0 \leqslant s < 1$.

**Lemma 6.2.** *The function $\Lambda$ is "slowly varying" in a strong sense. Namely, let $(x_n)$ and $(y_n)$ be any two diverging sequences such that there exists a constant $C > 1$ with*

$$\frac{1}{C} \leqslant \frac{x_n}{y_n} \leqslant C$$

*for all $n$. Then*

$$\lim_{n \to \infty} \frac{\Lambda(x_n)}{\Lambda(y_n)} = 1.$$

*Proof.* We need to show that

$$\sum_{j \geqslant 1} \frac{\theta_j - \theta}{j} \left[ \left(1 - \tfrac{1}{x_n}\right)^j - \left(1 - \tfrac{1}{y_n}\right)^j \right] \tag{6.2}$$

converges to 0 as $n \to \infty$. Given $\varepsilon > 0$, let $N_\varepsilon$ such that $|\theta_j - \theta| < \varepsilon$ for all $j > N_\varepsilon$. The sum over the first $N_\varepsilon$ terms of (6.2) goes to 0 as $n \to \infty$. The rest is less than

$$\varepsilon \sum_{j \geqslant 1} \frac{1}{j} \left| \left(1 - \tfrac{1}{x_n}\right)^j - \left(1 - \tfrac{1}{y_n}\right)^j \right| = \varepsilon \left| \log \frac{x_n}{y_n} \right| \leqslant \varepsilon \log C. \tag{6.3}$$

The expression (6.2) is then as small as we want when $n$ is large enough. $\qquad \square$

Using the definition (6.1) and recognizing the Taylor series of the logarithm, we have

$$G_h(s) = (1 - s)^{-\theta} \Lambda\left(\frac{1}{1-s}\right). \tag{6.4}$$

**Proposition 6.3.**
$$h_n = \frac{n^{\theta-1}}{\Gamma(\theta)} \Lambda(n) \left(1 + o(1)\right).$$

*Proof.* The generating function of $h_n$ being given by (6.4) with $\Lambda$ a slowly varying function, we can use the Tauberian theorem of Hardy-Littlewood-Karamata (see Theorem 9 of [6]) to obtain

$$\frac{1}{n} \sum_{j=0}^{n-1} h_j = \frac{n^{\theta-1}\Lambda(n)}{\Gamma(\theta+1)} \left(1 + o(1)\right). \tag{6.5}$$

We need to remove the Cesàro average in the left side. From (2.4), we have

$$h_n = \frac{\theta}{n} \sum_{j=0}^{n-1} h_j + \frac{1}{n} \sum_{j=0}^{n-1} (\theta_{n-j} - \theta) h_j. \tag{6.6}$$

The first term of the right side can be combined with (6.5) and it gives the right result. We need to check that the correction due to the second term is irrelevant, i.e., we need to check that

$$\lim_{n\to\infty} \frac{1}{n^\theta \Lambda(n)} \sum_{j=0}^{n-1} (\theta_{n-j} - \theta) h_j = 0. \tag{6.7}$$

Let $\varepsilon > 0$. Using (6.5), we first have

$$\frac{1}{n^\theta \Lambda(n)} \sum_{j=0}^{(1-\varepsilon)n} |\theta_{n-j} - \theta| h_j \leqslant \left(\sup_{j \geqslant \varepsilon n} |\theta_j - \theta|\right) \frac{((1-\varepsilon)n)^\theta \Lambda((1-\varepsilon)n)}{n^\theta \Lambda(n)\Gamma(\theta+1)} \left(1 + o(1)\right), \tag{6.8}$$

which clearly vanishes in the limit $n \to \infty$. Second, let $C = \sup_j |\theta_j - \theta|$, and observe that

$$\sum_{j=(1-\varepsilon)n}^{n-1} h_j = \sum_{j=0}^{n-1} h_j - \sum_{j=0}^{(1-\varepsilon)n} h_j = \frac{n^\theta \Lambda(n)}{\Gamma(\theta+1)} \left[1 + o(1) - (1-\varepsilon)^\theta \frac{\Lambda((1-\varepsilon)n)}{\Lambda(n)} \left(1 + o(1)\right)\right]. \tag{6.9}$$

Then

$$\limsup_{n\to\infty} \frac{1}{n^\theta \Lambda(n)} \sum_{j=(1-\varepsilon)n}^{n-1} |\theta_{n-j} - \theta| h_j \leqslant \frac{C}{\Gamma(\theta+1)} \left[1 - (1-\varepsilon)^\theta\right], \tag{6.10}$$

which is arbitrarily small since $\varepsilon$ is arbitrary. □

We can now prove the theorem.

*Proof of Theorem 6.1.* The claim (a) easily follows from Lemma 6.2, Proposition 6.3, and Corollary 2.2.

For the claim (b), we first observe that the number of cycles of length larger than $\frac{n}{\sqrt{\log n}}$ is less than $\sqrt{\log n}$, so we only need to consider smaller cycles. This means that we can sum up to $\frac{n}{\sqrt{\log n}}$ in the expression of Proposition 2.1 (b) for $\mathbb{E}_n(K)$. By Proposition 6.3, the ratio $\frac{h_{n-j}}{h_n}$ converges to 1 as $n \to \infty$, uniformly in $1 \leqslant j \leqslant n/\sqrt{\log n}$. It follows that

$$\lim_{n\to\infty} \frac{\mathbb{E}_n(K)}{\log n} = \lim_{n\to\infty} \frac{1}{\log n} \sum_{j=1}^{n/\sqrt{\log n}} \frac{\theta_j}{j} = \theta. \tag{6.11}$$

We turn to part (c). Let $\tilde{L}_1, \tilde{L}_2, \ldots$ denote the lengths of the cycles when they have been ordered e.g. according to their smallest element. That is, $\tilde{L}_1 = L_1$ is the length of

the cycle that contains the index 1; $\tilde{L}_2$ is the length of the cycle that contains the smallest index that is not in the first cycle; and so on... We show that for all $k$,

$$\left(\frac{\tilde{L}_1}{n}, \frac{\tilde{L}_2}{n - \tilde{L}_1}, \frac{\tilde{L}_3}{n - \tilde{L}_1 - \tilde{L}_2}, \ldots, \frac{\tilde{L}_k}{n - \tilde{L}_1 - \cdots - \tilde{L}_{k-1}}\right)$$

converges to i.i.d. beta random variables with parameters $(1, \theta)$. This implies that $(\frac{\tilde{L}_1}{n}, \frac{\tilde{L}_2}{n}, \ldots)$ converges weakly to $\mathrm{GEM}(\theta)$; and reordering the cycle lengths in nonincreasing order yields $\mathrm{PD}(\theta)$. It is enough to show that for any $k$ and any $a_1, \ldots, a_k \in (0, 1)$, we have

$$\lim_{n \to \infty} \mathbb{P}_n\left(\frac{\tilde{L}_1}{n} \leqslant a_1, \ldots, \frac{\tilde{L}_k}{n - \tilde{L}_1 - \cdots - \tilde{L}_{k-1}} \leqslant a_k\right) = \prod_{i=1}^{k} \left[1 - (1 - a_i)^\theta\right]. \qquad (6.12)$$

The right side is the beta measure of the product of intervals $\times_{i=1}^{k}(0, a_i)$.

We proceed by induction on $k$, starting with $k = 1$. By Proposition 2.1 (a), we have

$$\mathbb{P}_n\left(\frac{L_1}{n} \leqslant a_1\right) = \frac{1}{n} \sum_{j=0}^{a_1 n} \theta_j \frac{h_{n-j}}{h_n} = \frac{1}{n} \sum_{j=0}^{a_1 n} \theta_j \left(1 - \frac{j}{n}\right)^{\theta-1} \frac{\Lambda(n-j)}{\Lambda(n)} (1 + o(1)). \qquad (6.13)$$

We used Proposition 6.3 to get the second identity. By Lemma 6.2 the ratio $\frac{\Lambda(n-j)}{\Lambda(n)}$ converges to 1. We clearly have a Riemann sum, so that

$$\lim_{n \to \infty} \mathbb{P}_n\left(\frac{L_1}{n} \leqslant a_1\right) = \theta \int_0^{a_1} (1 - x)^{\theta-1} \mathrm{d}x = 1 - (1 - a_1)^\theta. \qquad (6.14)$$

Next, we assume that the claim has been proved for $k$ and we prove it for $k + 1$. Let

$$A = \left\{(\ell_1, \ldots, \ell_k) \in \{1, \ldots, n\}^k : \frac{\ell_1}{n} \leqslant a_1, \ldots, \frac{\ell_k}{n - \ell_1 - \cdots - \ell_{k-1}} \leqslant a_k\right\}. \qquad (6.15)$$

It is not hard to verify that, on $A$,

$$n - \ell_1 - \cdots - \ell_k \geqslant n \prod_{i=1}^{k} (1 - a_i). \qquad (6.16)$$

We have

$$\mathbb{P}_n\left((\tilde{L}_1, \ldots, \tilde{L}_k) \in A, \frac{\tilde{L}_{k+1}}{n - \tilde{L}_1 - \cdots - \tilde{L}_k} \leqslant a_{k+1}\right)$$
$$= \sum_{(\ell_1, \ldots, \ell_k) \in A} \mathbb{P}_n(\tilde{L}_1 = \ell_1, \ldots, \tilde{L}_k = \ell_k) \, \mathbb{P}_n\left(\frac{\tilde{L}_{k+1}}{n - \tilde{L}_1 - \cdots - \tilde{L}_k} \leqslant a_{k+1} \Big| \tilde{L}_1 = \ell_1, \ldots, \tilde{L}_k = \ell_k\right).$$
$$(6.17)$$

Now we use the self-similarity of weighted permutations: Having chosen the first $k$ cycles, the distribution of the $(k+1)$th cycle is identical but with less indices available. Precisely, we have

$$\mathbb{P}_n\left(\frac{\tilde{L}_{k+1}}{n - \tilde{L}_1 - \cdots - \tilde{L}_k} \leqslant a_{k+1} \Big| \tilde{L}_1 = \ell_1, \ldots, \tilde{L}_k = \ell_k\right) = \mathbb{P}_{n-\ell_1-\cdots-\ell_k}\left(\frac{\tilde{L}_1}{n - \ell_1 - \cdots - \ell_k} < a_{k+1}\right).$$
$$(6.18)$$

(Notice that the $(k+1)$th cycle in the left side has become the 1st cycle in the right side.) The right side of the equation converges to the beta measure of $(0, a_{k+1})$. Convergence is uniform in $(\ell_1, \ldots, \ell_k) \in A$ because of (6.16). The right side of (6.17) then converges to the beta measure of the product of intervals $\times_{i=1}^{k+1}(0, a_i)$ by the induction hypothesis. $\qquad \square$

## 7. Parameters with sub-exponential decay

The second regime with long cycles occurs for parameters that go slowly to 0, such as $\theta_n = n^{-\gamma}$ with $\gamma > 0$, or $\theta_n = e^{-n^\gamma}$ with $0 < \gamma < 1$. It is not hard to check that the assumptions of the theorem below are satisfied in both these cases. Notice that the results about the $R_j$s and about $K$ have already been proved in [4] in the case $\theta_n \sim n^{-\gamma}$.

**Theorem 7.1.** *Assume that* $0 < \frac{\theta_{n-j}\theta_j}{\theta_n} < c_j$ *for all* $n$ *and all* $1 \leqslant j \leqslant \frac{n}{2}$, *with constants* $c_j$ *that satisfy* $\sum_{j \geqslant 1} \frac{c_j}{j} < \infty$. *Assume also that* $\frac{\theta_{n+1}}{\theta_n} \to 1$ *as* $n \to \infty$. *Then* $\sum_j h_j < \infty$, *and*

$$\lim_{n \to \infty} \mathbb{P}_n(L_1 = n - m) = \frac{h_m}{\sum_{j \geqslant 0} h_j}.$$

*In addition,* $R_1, R_2, R_3, \dots$ *converge weakly to independent Poisson random variables with respective means* $\theta_1, \frac{\theta_2}{2}, \frac{\theta_3}{3}, \dots$, *and* $K - 1$ *converges to Poisson with mean* $\sum_j \frac{\theta_j}{j}$.

*Proof.* The claim about $L_1$ was proved in [9]. The claim about the $R_j$s follows from Corollary 2.2 and from the fact that

$$h_n = \frac{C\theta_n}{n}\big(1 + o(1)\big) \tag{7.1}$$

with $C = \sum h_j$. This was proved in [9], see Eq. (3.12) there.

In order to prove that $K - 1$ converges to a Poisson random variable, let $m, k$ be fixed. We consider the set of permutations

$$A = \{\sigma : R_1(\sigma) + \cdots + R_m(\sigma) = k - 1\}, \tag{7.2}$$

and $B$ the set of permutations where exactly one cycle has length larger than $m$. We have, for all $n > 2m$,

$$A \cap B \subset \{\sigma : K(\sigma) = k\}, \qquad \{\sigma : L_1(\sigma) \geqslant n - m\} \subset B. \tag{7.3}$$

Then

$$\mathbb{P}_n(K = k) \geqslant \mathbb{P}_n(A) - \mathbb{P}(\{L_1 \geqslant n - m\}^c) = \mathbb{P}_n(A) - 1 + \sum_{j=0}^{m} \mathbb{P}_n(L_1 = n - j). \tag{7.4}$$

We take the limit $n \to \infty$. Since $R_1 + \cdots + R_m$ converges to Poisson with mean $\sum_{j=1}^{m} \frac{\theta_j}{j}$, we get

$$\liminf_{n \to \infty} \mathbb{P}_n(K = k) \geqslant \frac{1}{(k-1)!}\Big(\sum_{j=1}^{m} \frac{\theta_j}{j}\Big)^{k-1} e^{-\sum_{j=1}^{m} \frac{\theta_j}{j}} - 1 + \frac{\sum_{j=0}^{m} h_j}{\sum_{j \geqslant 0} h_j}. \tag{7.5}$$

We now take the limit $m \to \infty$ and we get

$$\liminf_{n \to \infty} \mathbb{P}_n(K = k) \geqslant \frac{1}{(k-1)!}\Big(\sum_{j \geqslant 1} \frac{\theta_j}{j}\Big)^{k-1} e^{-\sum_{j \geqslant 1} \frac{\theta_j}{j}}. \tag{7.6}$$

Summing over $k \geqslant 1$, the left side is less or equal to 1 by Fatou's lemma; the right side yields 1. This shows that the inequality above is actually an identity, and $K - 1$ is indeed Poisson in the limit $n \to \infty$. $\qquad\square$

## 8. Parameters with super-exponential decay

We conclude our study of random permutations with cycle weights by discussing the case $\theta_n = \mathrm{e}^{-n^\gamma}$ with $\gamma > 1$. It was actually studied in [9], where the typical cycle length was proved to be a fractional power of $\log n$, namely

$$\frac{L_1}{((\gamma - 1)\log n)^{1/\gamma}} \Rightarrow 1. \tag{8.1}$$

We complement this result with a claim about the number of finite cycles. It is actually not very sharp, but it provides useful information nonetheless.

**Theorem 8.1.** *As $n \to \infty$, we have*

$$\mathbb{E}_n(R_j) = \exp\left\{j\gamma\left(\frac{\log n}{\gamma - 1}\right)^{\frac{\gamma-1}{\gamma}} + o\left((\log n)^{\frac{\gamma-1}{\gamma}}\right)\right\}.$$

*Proof.* The radius of convergence of $G_\theta$ and $G_h$ is now infinite. Let $r_n$ satisfy $I_0(r_n) = n$. It was shown in [9], see Eq. (4.32) there, that

$$r_n = \exp\left\{\gamma\left(\frac{\log n}{\gamma - 1}\right)^{\frac{\gamma-1}{\gamma}}(1 + o(1))\right\}. \tag{8.2}$$

It follows that

$$r_{n-j} = \exp\left\{\gamma\left(\frac{\log n}{\gamma - 1}\right)^{\frac{\gamma-1}{\gamma}}\left(1 + \frac{\log(1 - \frac{j}{n})}{\log n}\right)^{\frac{\gamma-1}{\gamma}}(1 + o(1))\right\}. \tag{8.3}$$

We can then express $r_{n-j}$ in term of $r_n$,

$$r_{n-j} = r_n\,\mathrm{e}^{o\left((\log n)^{\frac{\gamma-1}{\gamma}}\right)}, \tag{8.4}$$

where the precise meaning of $o(\cdot)$ is that for any $\varepsilon > 0$, there exists $N$ such that

$$\left|\frac{o\left((\log n)^{\frac{\gamma-1}{\gamma}}\right)}{(\log n)^{\frac{\gamma-1}{\gamma}}}\right| < \varepsilon \tag{8.5}$$

for all $j, n$ such that $n > N$ and $n - j > N$.

Next, we observe that the parameters satisfy

$$\mathrm{e}^{-(j-1)^\gamma} = \mathrm{e}^{-j^\gamma + \gamma j^{\gamma-1} + O(j^{\gamma-2})} \tag{8.6}$$

so that $j\,\mathrm{e}^{-j^\gamma} \leqslant \mathrm{e}^{-(j-1)^\gamma}$ for all $j$ large enough. It follows that if $r$ is large enough,

$$I_1(r) = \sum_{j \geqslant 1} j\,\mathrm{e}^{-j^\gamma}\,r^j \leqslant \sum_{j \geqslant 1} \mathrm{e}^{-(j-1)^\gamma}\,r^j = r(I_0(r) + 1). \tag{8.7}$$

Since $I_1(r)$ is increasing in $r$ and $r_n$ is increasing in $n$, we have for $n$ large enough,

$$1 \leqslant \frac{I_1(r_n)}{I_1(r_{n-j})} \leqslant r_n\frac{I_0(r_n) + 1}{I_0(r_{n-j})}. \tag{8.8}$$

Using Proposition 2.3 and Eq. (8.2), we have

$$\frac{h_{n-j}}{h_n} = \exp\left\{j\gamma\left(\frac{\log n}{\gamma - 1}\right)^{\frac{\gamma-1}{\gamma}} + o\left(j(\log n)^{\frac{\gamma-1}{\gamma}}\right)\right\}. \tag{8.9}$$

A special case of Proposition 2.1 (c) is $\mathbb{E}_n(R_j) = \frac{h_{n-j}}{h_n}\frac{\theta_j}{j}$. Combining this with the previous equation, and neglecting $\theta_j/j$ which is less than the error, we get the claim of the theorem for all $j$ finite. $\qquad\square$

## 9. Uniform saddle point estimates

Since the two cases of generating functions that we have considered are entirely similar, we initially restrict attention to the case of the generating function for the $h_n$ (as specified in (2.16)). $G_h(z) = \exp I_{-1}(z)$ is analytic in the unit disc and hence the normalization coefficients we want to study are naturally given by the Cauchy representation

$$h_n = \frac{1}{2\pi i} \oint_{\mathcal{C}_n} G_h(z) \frac{dz}{z^{n+1}} \tag{9.1}$$

where $\mathcal{C}_n$ is the circle centered at 0 of radius $r_n$. With respect to polar coordinates along $\mathcal{C}_n$ this becomes

$$h_n = \frac{1}{2\pi} \int_{-\pi}^{\pi} \exp\left(I_{-1}\left(r_n\, e^{i\phi}\right) - n \log\left(r\, e^{i\phi}\right)\right) d\phi. \tag{9.2}$$

We observe that the function

$$F_n(z) = I_{-1}(z) - n \log(z) \tag{9.3}$$

which is positive and continuous on $(0,1)$ approaches $\infty$ at both endpoints and therefore attains a minimum value at $r_n$. This point is unique and, as we have already observed, explicitly given as the critical point satisfying

$$0 = F_n'(z) = \frac{d}{dz} I_{-1}(z) - \frac{n}{z}, \tag{9.4}$$

or equivalently

$$r_n = I_0^{-1}(n). \tag{9.5}$$

We now consider a complex neighborhood (in $z$) of $r_n$. Since $F_n(z)$ is analytic in the unit disc minus the origin, one may assume that $F_n$ is analytic on the chosen neighborhood of $r_n$ and hence $r_n$ must be a saddle point of $F_n$ (by the maximum principle). The integral (9.2) is complex-valued, so it is natural to try to apply the method of steepest descent [17] here. One may describe this approach in terms of a dynamical system; viz., the Cauchy-Riemann equations for the analytic function $F_n(z)$ may be viewed as a gradient dynamical system with potential $\operatorname{Re} F_n(z)$. The critical point $z = r_n$ is a fixed point of this system and the locus $\operatorname{Im} F(n) = 0$ cuts out the stable and unstable manifolds of this fixed point. The real axis is the stable manifold in all the cases we consider. The steepest descent curves are the components of the unstable manifold; $F_n(z) = \operatorname{Re} F_n(z)$ decreases monotonically along these curves as one moves away from the fixed point $r_n$. The situation is illustrated in the left graphic of Figure 1 below which depicts these stable and unstable manifolds for a particular case of algebraically growing parameters. For background the reader is referred to section 2.3 of [17]. We begin by Taylor expanding $F_n(z)$, as given by (9.3), near $r_n$, and applying Taylor's form of the remainder theorem to derive the representation

$$F_n(z) = (I_{-1}(r_n) - n\log(r_n)) + \frac{I_1(r_n)}{2r_n^2}(z - r_n)^2 + \frac{I_2(\tilde{r_n}) - 3I_1(\tilde{r_n}) + 2I_0(\tilde{r_n}) - n}{6\tilde{r_n}^3}(z - r_n)^3$$

$$= F_n(r_n) + A_n(z - r_n)^2 + B_n(z - r_n)^3(1 + o(1)). \tag{9.6}$$

where

$$
\tilde{r_n} = r_n(1 + o(1))
$$
$$
A_n = \frac{I_1(r_n)}{2r_n^2}
$$
$$
B_n = \frac{I_2(r_n) - 3I_1(r_n) + n}{6r_n^3}.
$$
(9.7)

With $z = x + iy$, the local structure of the stable and unstable manifolds is given by the locus

$$
\begin{aligned}
\operatorname{Im} F_n(z) &= 2A_n(x - r_n)y + B_n\left(3(x - r_n)^2 y - y^3\right) \\
&= y\left(2A_n(x - r_n) + B_n\left(3(x - r_n)^2 - y^2\right)\right) \\
&= 0.
\end{aligned}
$$
(9.8)

Indeed, $y = 0$ locally describes the stable manifold which we have already seen to be the $x$-axis while the remaining factor, which to leading orders has the form

$$
y^2 = \frac{2A_n}{B_n}(x - r_n),
$$
(9.9)

locally describes a parabolic arc for the unstable manifold (steepest descent curves), consistent with the example shown in Figure 1.

One may similarly expand the real part of $F_n$ (first line below) and then restrict it to the unstable manifold (second line below),

$$
\begin{aligned}
\operatorname{Re} F_n(z) - F_n(r_n) &= A_n\left[(x - r_n)^2 - y^2\right] + B_n\left[(x - r_n)^3 - 3(x - r_n)y^2\right](1 + o(1)) \\
&= \left(A_n(x - r_n)^2 + B_n(x - r_n)^3\right)(1 + o(1))
\end{aligned}
$$
(9.10)

where in the second line we have used (9.9) and the fact, which will be seen below, that $B_n$ dominates $A_n$ as $n \to \infty$.

We next apply these observations to the contour integral (9.1). By Cauchy's Theorem the contour of integration, $\mathcal{C}_n$ may be deformed within a region of analyticity without affecting the value of the integral. We will deform to a contour of the form $\mathcal{C} = D_+ + C - D_-$ where $D_+$ is the sub-locus of the steepest descent curve in the upper half plane starting at $r_n$ and terminating at a point $z_0$, inside the unit disc, to be determined. $D_-$ is the conjugate reflection of $D_+$ in the lower half plane. $C$ is the circular arc of radius $|z_0|$ starting at $z_0$ and terminating at $\bar{z}_0$. (Note that although $\mathcal{C}_n$ is not equal to the steepest descent path, it is tangent to that path at $r_n$.)

We concentrate first on the integral over the steepest descent contours of $\mathcal{C}$. At the end of this section it will be shown that $z_0$ may be chosen, depending on $n$, so that that the quadratic term in (9.10) goes to infinity with $n$ while the cubic term goes to zero. $z_0$ itself will tend to 1 along with $r_n$ as $n \to \infty$. It is then natural to make the change of variables along $D_\pm$:

$$
\begin{aligned}
\frac{\sigma^2}{2} &= F(r_n) - F_n(z) \\
&= -\frac{I_1(r_n)}{2r_n^2}(x - r_n)^2 + o(1).
\end{aligned}
$$
(9.11)

With this we have

$$\frac{1}{2\pi\mathrm{i}} \int_{D_+ - D_-} \mathrm{e}^{F_n(z)} \frac{\mathrm{d}z}{z} = \frac{\mathrm{e}^{F_n(r_n)}}{2\pi\mathrm{i}} \int_{\bar{z_0}}^{z_0} \mathrm{e}^{F_n(z) - F_n(r_n)} \frac{\mathrm{d}z}{z}$$

$$= \frac{\mathrm{e}^{F_n(r_n)}}{\pi\mathrm{i}} \int_{x_0 - r_n}^{0} \mathrm{e}^{-\sigma^2/2} \frac{\mathrm{d}x}{r_n + x} \tag{9.12}$$

$$= \frac{\mathrm{e}^{F_n(r_n)}}{\pi\sqrt{I_1(r_n)}} \int_{0}^{\sigma_0} \mathrm{e}^{-\sigma^2/2} \, \mathrm{d}\sigma(1 + o(1))$$

where $\sigma_0 = \frac{\sqrt{I_1(r_n)}}{r_n}(x_0 - r_n)$. In the last line the change of variables (9.11) was implemented. As already mentioned, at the end of this section it will be shown that a choice of $z_0$ can be made consistent with all prior estimates and for which $\sigma_0 \to \infty$ as $n \to \infty$. It follows that

$$\frac{1}{2\pi\mathrm{i}} \int_{D_+ - D_-} \mathrm{e}^{F_n(z)} \frac{\mathrm{d}z}{z} = \frac{\mathrm{e}^{F_n(r_n)}}{\sqrt{2\pi I_1(r_n)}}(1 + o(1)). \tag{9.13}$$

To complete the verification of (2.17), as well as the similar argument for (2.10), one still needs to argue that the global error coming from the integral (9.2), restricted to $C$, is asymptotically negligible in comparison to (9.13). We illustrate the situation with two images from the case of algebraic growth (specifically, the instance of $G_h$ for (5.3) where $\gamma = 1$ with $n = 100$).
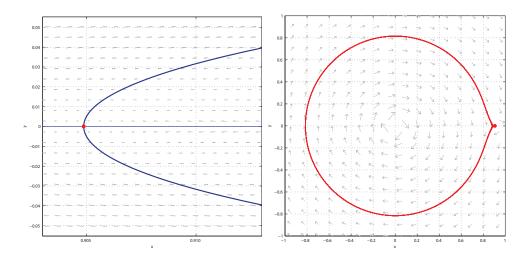


FIGURE 1.   saddle point and level curve at $r_n$

As has already been described, the graphic on the left in Figure 1 shows the saddle point $r_n$ with paths on the real axis ascending from the saddle and the other two curves descending from the saddle (*steepest descent curves*) into the upper and lower half planes respectively. This illustrates the fact, stated before, that the contour $\mathcal{C}_n$ is tangent to the steepest descent curves. The graphic on the right shows the level curve, passing through $r_n$, of the real part of $F_n(z)$. This level curve is also a locus where the magnitude of the integrand of (9.2) is constant. Note that this level curve is quite close to being circular away from a small neighborhood of $r_n$. This property is shared by other nearby

level curves (for level values different than $F_n(r_n)$). This suggests that the order of the magnitude of the global error is bounded by the order of the absolute value of the integrand of (9.2) evaluated at $r_n \, e^{i\phi_0}$. We will take a slightly different tack here which essentially accomplishes the same estimate but is easier to implement. Namely, we return to $C$ and observe that the value of $F(z)$ (which equals $\operatorname{Re} F(z)$ along the unstable manifolds) is decreasing along the unstable manifolds as one moves away from $r_n$. Hence, the value of the integrand in (9.1) at the respective endpoints $z_0, \bar{z}_0 = |z_0|e^{i\phi_0}$ of $C$ is exponentially smaller (in $n$) than its value at $r_n$. We further observe that

$$
\begin{aligned}
\left| \frac{1}{2\pi i} \int_C \frac{e^{nI_{-1}(z)}}{z^{n+1}} dz \right| &\leqslant \frac{1}{\pi} \frac{1}{|z_0|^n} \int_{\phi_0}^{\pi} e^{n\operatorname{Re} I_{-1}(|z_0|\,e^{i\phi})} d\phi \\
&= \frac{1}{\pi} \frac{1}{|z_0|^n} e^{n\operatorname{Re} I_{-1}(|z_0|\,e^{i\phi_0})} \int_{\phi_0}^{\pi} e^{n\operatorname{Re} \left[ I_{-1}(|z_0|\,e^{i\phi}) - I_{-1}|z_0|\,e^{i\phi_0}) \right]} d\phi \quad (9.14) \\
&\leqslant \frac{1}{\pi} \frac{1}{|z_0|^n} e^{n\operatorname{Re} I_{-1}(|z_0|\,e^{i\phi_0})} (\pi - \phi_0),
\end{aligned}
$$

where the last inequality follows from the fact that

$$
\operatorname{Re} \left[ I_{-1}(|z_0|\,e^{i\phi}) - I_{-1}(|z_0|\,e^{i\phi_0}) \right] = \sum_{j \geqslant 1} \frac{\theta_j}{j} |z_0|^j \left( \cos(j\phi) - \cos(j\phi_0) \right) \leqslant 0 \qquad (9.15)
$$

for $\phi \in (\phi_0, \pi)$. It follows that the integral over $C$ is exponentially negligible in comparison to (9.13).

Finally we return to the claim made just prior to (9.11) that $z_0$ may be chosen so that, in (9.10), the quadratic term grows to infinity with $n$ while the cubic term decreases. Given the analysis presented in sections 4 and 5 and in particular the estimates (4.8), (4.14) and (5.6, 5.9) it suffices to show that the order of $x_0 - r_n$ may be chosen so that

$$
\begin{aligned}
I_1(r_n)(x_0 - r_n)^2 &\to \infty \\
I_2(r_n)(x_0 - r_n)^3 &\to 0
\end{aligned}
$$

as $n \to \infty$. The following table summarizes the orders in $n$ of the relevant terms and presents a choice for the orders of $(x_0 - r_n)$ for each of the cases of generating functions that we consider in this paper. In each case the choice is given in terms of a weighted geometric mean of the growth rates for $I_1(r_n)$ and $I_2(r_n)$. (Note that in the case of $G_\theta$, $r_n$ should be replaced by $\rho_n$.)

| | $\mathcal{O}(I_1(r_n))$ | $\mathcal{O}(I_2(r_n))$ | $\mathcal{O}(x_0 - r_n)$ | $\mathcal{O}(I_1(r_n)(x_0-r_n)^2)$ | $\mathcal{O}(I_2(r_n)(x_0-r_n)^3)$ |
|---|---|---|---|---|---|
| $G_h(z)$, alg. | $n^{\frac{\gamma+2}{\gamma+1}}$ | $n^{\frac{\gamma+3}{\gamma+1}}$ | $n^{-\frac{1}{12}\frac{5\gamma+12}{\gamma+1}}$ | $n^{\frac{1}{6}\frac{\gamma}{\gamma+1}}$ | $n^{-\frac{1}{4}\frac{\gamma}{\gamma+1}}$ |
| $G_\theta(z)$, sub-exp. | $n^{2-\gamma}$ | $n^{3-2\gamma}$ | $n^{-1+\frac{7}{12}\gamma}$ | $n^{\frac{1}{6}\gamma}$ | $n^{-\frac{1}{4}\gamma}$ |
| $G_h(z)$, sub-exp. | $n(\log n)^{\frac{1}{\gamma}}$ | $n(\log n)^{\frac{2}{\gamma}}$ | $n^{-\frac{5}{12}}(\log n)^{-\frac{7}{12}\frac{1}{\gamma}}$ | $\left( \frac{n}{(\log n)^{1/\gamma}} \right)^{\frac{1}{6}}$ | $\left( \frac{(\log n)^{\frac{1}{\gamma}}}{n} \right)^{\frac{1}{4}}$ |

The arguments in this section follow the general strategy of *Hayman's method*. We refer the reader to Chapter VIII of [14] for a nice overview of this technique.

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