Fibonacci numbers, consecutive patterns, and inverse peaks

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Abstract

We give multiple proofs of two formulas concerning the enumeration of permutations avoiding a monotone consecutive pattern with a certain value for the inverse peak number or inverse left peak number statistic. The enumeration in both cases is given by a sequence related to Fibonacci numbers. We also show that there is exactly one permutation whose inverse peak number is zero among all permutations with any fixed descent composition, and we give a few elementary consequences of this fact. Our proofs involve generating functions, symmetric functions, regular expressions, and monominodomino tilings.

Keywords: permutations, peaks, consecutive patterns, Fibonacci numbers

1. Introduction

Let \mathfrak{S}_n denote the symmetric group of permutations on the set $[n] := \{1, 2, ..., n\}$. We write permutations in one-line notation—that is, $\pi = \pi_1 \pi_2 \cdots \pi_n$ —and the π_i are called *letters* of π . The *length* of π is the number of letters in π , so that π has length n whenever $\pi \in \mathfrak{S}_n$.

For a sequence of n distinct integers w, the standardization of w—denoted std(w)—is defined to be the permutation in \mathfrak{S}_n obtained by replacing the smallest entry of w with 1, the second smallest with 2, and so on. As an example, we have std(83614) = 52413. Given permutations $\pi \in \mathfrak{S}_n$ and $\sigma \in \mathfrak{S}_m$, we say that π contains σ (as a consecutive pattern) if $\operatorname{std}(\pi_i\pi_{i+1}\cdots\pi_{i+m-1}) = \sigma$ for some $i \in [n-m+1]$. If π does not contain σ , then we say that π avoids σ (as a consecutive pattern). Let $\mathfrak{S}_n(\sigma)$ denote the subset of permutations in \mathfrak{S}_n which avoid σ .

This paper supplements the recent paper [12], in which the second author proves a lifting of the Goulden–Jackson cluster method for permutations—a standard tool in the study

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of consecutive patterns—to the Malvenuto–Reutenauer algebra. By applying various homomorphisms to this generalized cluster method, the second author obtains in [12] various specializations which allows one to count permutations avoiding prescribed consecutive patterns while keeping track of certain "inverse" permutation statistics. These statistics include the inverse descent number, the inverse peak number, and the inverse left peak number.

In this paper, we prove two observations which were left unproven in [12]; these are stated in Theorems 1 and 2 below. Both theorems concern the enumeration of permutations avoiding a monotone pattern with a certain value for the inverse peak number or inverse left peak number statistic, and the enumeration in both cases is given by a sequence related to Fibonacci numbers.

Let us establish a few more definitions. For a permutation π in \mathfrak{S}_n , we say that $i \in \{2, 3, ..., n-1\}$ is a peak of π if $\pi_{i-1} < \pi_i > \pi_{i+1}$, and we say that $i \in [n-1]$ is a left peak of π if either i is a peak of π or if i = 1 and $\pi_1 > \pi_2$. Let $\operatorname{ipk}(\pi)$ be the number of peaks of π^{-1} , and let $\operatorname{ilpk}(\pi)$ be the number of left peaks of π^{-1} . For example, if $\pi = 23568714$, then $\pi^{-1} = 71283465$ and we have $\operatorname{ipk}(\pi) = 2$ and $\operatorname{ilpk}(\pi) = 3$.

The Fibonacci sequence of order k (also called the k-generalized Fibonacci sequence), denoted $\{f_n^{(k)}\}_{n\geq 0}$, is defined by the recursion

$$f_n^{(k)} = f_{n-1}^{(k)} + f_{n-2}^{(k)} + \dots + f_{n-k}^{(k)}$$

with $f_0^{(k)} := 1$ (and where we treat $f_n^{(k)}$ as 0 for n < 0). Hence, the Fibonacci sequence of order two is the usual Fibonacci sequence [10, A000045] and the Fibonacci sequence of order three is the *tribonacci sequence* [10, A000073].¹ The number $f_n^{(k)}$ counts tilings of a $1 \times n$ rectangle with tiles of size $1 \times j$ where j ranges from 1 to k [1, Section 3.4]; equivalently, $f_n^{(k)}$ is the number of compositions of n with no parts greater than k. Several other combinatorial interpretations for $f_n^{(k)}$ are known; see the references in [10, A092921]. Theorem 1 gives another combinatorial interpretation for $f_n^{(k)}$.

Theorem 1 ([12, Claim 4.6]). Let $n \ge 1$ and $m \ge 3$. The number of permutations π in $\mathfrak{S}_n(12\cdots m)$ with $\mathrm{ipk}(\pi) = 0$ —that is, such that π^{-1} has no peaks—is equal to $f_n^{(m-1)}$.

The next theorem concerns OEIS sequence A080145 [10, A080145]. As far as we know, this is the first combinatorial interpretation related to permutation enumeration for the numbers in [10, A080145]. Here, $f_n := f_n^{(2)}$ denotes the *n*th Fibonacci number.

Theorem 2 ([12, Claim 4.9]). Let $n \ge 1$. The number of permutations π in $\mathfrak{S}_n(321)$ with $ilpk(\pi) = 1$ —that is, such that π^{-1} has exactly one left peak—is equal to

$$\sum_{i=1}^{n-1} \sum_{k=1}^{i} f_{k-1} f_k = f_{n-1} f_n - \left\lfloor \frac{n+1}{2} \right\rfloor.$$

This paper is organized as follows. In Section 2, we prove both theorems using generating functions derived in [12]. Sections 3 and 4 provide alternative proofs for these theorems which we believe are more enlightening. In Section 3, we show that there is exactly one permutation

¹Note that the OEIS uses a different indexing for these sequences.

 π with ipk(π) = 0 among all permutations with any fixed "descent composition" (Theorem 5), and Theorem 1 follows as an immediate corollary of this result. We give two proofs of Theorem 5. Our first proof recovers Theorem 5 from two classical results of H. O. Foulkes from [4], a foundational paper relating symmetric function theory and representation theory to permutation enumeration. Our second proof is a direct bijective proof, which is in a sense equivalent to the symmetric function proof but does not rely on the machinery of symmetric functions. We end Section 3 with a few other elementary consequences of Theorem 5. Finally, in Section 4, we give a bijective proof of Theorem 2 using regular expressions and tilings.

2. Generating function proofs for Theorems 1 and 2

Given a consecutive pattern σ , let $P_{\sigma,n}^{\text{ipk}}(t)$ be defined by

$$P_{\sigma,n}^{\mathrm{ipk}}(t) \coloneqq \sum_{\pi \in \mathfrak{S}_n(\sigma)} t^{\mathrm{ipk}(\pi) + 1}$$

for $n \geq 1$ and $P_{\sigma,0}^{\text{ipk}}(t) := 1$. (It may seem more natural to define these polynomials without an extra factor of t, but the definition given above leads to nicer formulas.) The theorem below, derived in [12], gives a generating function formula for the polynomials $P_{12\cdots m,n}^{\text{ipk}}(t)$.

Theorem 3 ([12, Theorem 4.5 (c)]). Let $m \geq 2$. We have

$$\frac{1}{1-t} + \frac{1}{2} \sum_{n=1}^{\infty} \left(\frac{1+t}{1-t}\right)^{n+1} P_{12\cdots m,n}^{\text{ipk}} \left(\frac{4t}{(1+t)^2}\right) x^n$$

$$= 1 + \sum_{k=1}^{\infty} \left[1 - 2kx + \sum_{j=1}^{\infty} (c_{m,j,k} x^{jm} - c'_{m,j,k} x^{jm+1})\right]^{-1} t^k$$

where

$$c_{m,j,k} = 2\sum_{l=1}^{k} {l+jm-1 \choose l-1} {jm-1 \choose k-l}$$
 and $c'_{m,j,k} = 2\sum_{l=1}^{k} {l+jm \choose l-1} {jm \choose k-l}$.

We shall use Theorem 3 to prove Theorem 1. Given two formal power series f and g in the variable t, let us write $f \sim g$ if they have the same linear coefficients, i.e., [t] f = [t] g.

Proof of Theorem 1. Taking Theorem 3, replacing x with (1-t)x/(1+t), and performing a few algebraic manipulations gives

$$\sum_{n=1}^{\infty} P_{12\cdots m,n}^{\text{ipk}} \left(\frac{4t}{(1+t)^2} \right) x^n = \frac{2}{1+t} \left[(1-t) \left[1 + \sum_{k=1}^{\infty} \left[1 - \frac{2kx(1-t)}{1+t} \right] + \sum_{j=1}^{\infty} \left(c_{m,j,k} \left(\frac{x(1-t)}{1+t} \right)^{jm} - c'_{m,j,k} \left(\frac{x(1-t)}{1+t} \right)^{jm+1} \right) \right]^{-1} t^k \right] - 1 \right].$$

Let us replace the variable t with v, and let $t=4v/(1+v)^2$. It can be readily verified that solving $t=4v/(1+v)^2$ yields $v=2t^{-1}(1-\sqrt{1-t})-1$. Therefore, we have

$$\sum_{n=1}^{\infty} P_{12\cdots m,n}^{\text{ipk}}(t) x^n = \frac{2}{1+v} \left[(1-v) \left[1 + \sum_{k=1}^{\infty} \left[1 - \frac{2kx(1-v)}{1+v} \right] + \sum_{j=1}^{\infty} \left(c_{m,j,k} \left(\frac{x(1-v)}{1+v} \right)^{jm} - c'_{m,j,k} \left(\frac{x(1-v)}{1+v} \right)^{jm+1} \right) \right]^{-1} v^k \right] - 1 \right].$$

where v is as above. In fact, v is a formal power series in t, and we have

$$v = \frac{1}{4}t + \frac{1}{8}t^2 + \frac{5}{64}t^3 + \cdots,$$

$$\frac{2}{1+v} = 2 - \frac{1}{2}t - \frac{1}{8}t^2 - \frac{1}{16}t^3 + \cdots,$$

$$\frac{1-v}{1+v} = 1 - \frac{1}{2}t - \frac{1}{8}t^2 - \frac{1}{16}t^3 + \cdots.$$

and

As we are only concerned with the linear coefficients, we may truncate these series to obtain

$$\sum_{n=1}^{\infty} P_{12\cdots m,n}^{\text{ipk}}(t)x^{n}$$

$$\sim \left(2 - \frac{t}{2}\right) \left[\left(1 - \frac{t}{4}\right) \left[1 + \sum_{k=1}^{\infty} \left[1 - 2kx + \sum_{j=1}^{\infty} \left(c_{m,j,k}x^{jm} - c'_{m,j,k}x^{jm+1}\right)\right]^{-1} \left(\frac{t}{4}\right)^{k}\right] - 1\right]$$

$$\sim -t + 2\left[1 - 2x + \sum_{j=1}^{\infty} \left(c_{m,j,1}x^{jm} - c'_{m,j,1}x^{jm+1}\right)\right]^{-1} \frac{t}{4} + \frac{t}{2}.$$

Some additional algebraic manipulations show that

$$\sum_{n=1}^{\infty} P_{12\cdots m,n}^{\text{ipk}}(t)x^{n} \sim -t + 2\left[1 - 2x + \sum_{j=1}^{\infty} \left(c_{m,j,1}x^{jm} - c'_{m,j,1}x^{jm+1}\right)\right]^{-1} \frac{t}{4} + \frac{t}{2}$$

$$= \frac{t}{2} \left[1 - 2x + \sum_{j=1}^{\infty} \left(2x^{jm} - 2x^{jm+1}\right)\right]^{-1} - \frac{t}{2}$$

$$= \frac{t}{2} \cdot \frac{1}{1 - 2x + 2(1 - x)\sum_{j=1}^{\infty} x^{jm}} - \frac{t}{2}$$

$$= \frac{t}{2} \cdot \frac{1}{1 - 2x + 2(1 - x)\left(\frac{1}{1 - x^{m}} - 1\right)} - \frac{t}{2}$$

$$= \frac{t(x - x^{m})}{1 - 2x + x^{m}}$$

$$= t \left(\frac{1-x}{1-2x+x^m} - 1 \right).$$

Because $(1-x)/(1-2x+x^m)$ is the ordinary generating function for the Fibonacci numbers of order m-1 [10, A048887], the result follows.

Let us now turn our attention to Theorem 2. Given a consecutive pattern σ , define

$$P_{\sigma,n}^{\mathrm{ilpk}}(t) \coloneqq \sum_{\pi \in \mathfrak{S}_n(\sigma)} t^{\mathrm{ilpk}(\pi)}$$

for all $n \geq 0$. Then the following is an analogue of Theorem 3 for the polynomials $P_{m\cdots 21,n}^{\text{ilpk}}(t)$.

Theorem 4 ([12, Theorem 4.8 (c)]). Let $m \ge 2$. We have

$$\sum_{n=0}^{\infty} \frac{(1+t)^n}{(1-t)^{n+1}} P_{m\cdots 21,n}^{\text{ilpk}} \left(\frac{4t}{(1+t)^2} \right) x^n$$

$$= \frac{1}{1-x} + \sum_{k=1}^{\infty} \left[1 - (2k+1)x + \sum_{j=1}^{\infty} (e_{m,j,k} x^{jm} - e'_{m,j,k} x^{jm+1}) \right]^{-1} t^k$$

where

$$e_{m,j,k} = 4\sum_{l=1}^{k} {l+jm-1 \choose l-1} {jm-2 \choose k-l}$$
 and $e'_{m,j,k} = 4\sum_{l=1}^{k} {l+jm \choose l-1} {jm-1 \choose k-l}$.

Proof of Theorem 2. Following the proof of Theorem 1 given above, we obtain from Theorem 4 the formula

$$\sum_{n=0}^{\infty} P_{m\cdots 21,n}^{\text{ilpk}}(t) x^n = (1-v) \left[\frac{1}{1 - \frac{x(1-v)}{1+v}} + \sum_{k=1}^{\infty} \left[1 - \frac{(2k+1)x(1-v)}{1+v} + \sum_{j=1}^{\infty} \left(e_{m,j,k} \left(\frac{x(1-v)}{1+v} \right)^{jm} - e'_{m,j,k} \left(\frac{x(1-v)}{1+v} \right)^{jm+1} \right) \right]^{-1} v^k \right]$$

where again $v = 2t^{-1}(1 - \sqrt{1-t}) - 1$. Because

$$v = \frac{1}{4}t + \frac{1}{8}t^2 + \frac{5}{64}t^3 + \cdots,$$
$$\frac{1-v}{1+v} = 1 - \frac{1}{2}t - \frac{1}{8}t^2 - \frac{1}{16}t^3 + \cdots,$$

and

$$\frac{1}{1 - \frac{x(1-v)}{1+x}} = \frac{1}{1-x} - \frac{x}{2(1-x)^2}t + \frac{x(3x-1)}{8(1-x)^3}t^2 - \frac{x(5x^2-4x+1)}{16(1-x)^4}t^3 + \cdots,$$

we have

$$\begin{split} \sum_{n=0}^{\infty} P_{m\cdots 21,n}^{\text{ilpk}}(t) x^n &\sim \left(1 - \frac{t}{4}\right) \left[\frac{1}{1-x} - \frac{xt}{2(1-x)^2} + \sum_{k=1}^{\infty} \left[1 - (2k+1)x\right] \right. \\ &\qquad \qquad + \sum_{j=1}^{\infty} (e_{m,j,k} x^{jm} - e'_{m,j,k} x^{jm+1}) \right]^{-1} \left(\frac{t}{4}\right)^k \right] \\ &\sim \left(1 - \frac{t}{4}\right) \left[\frac{1}{1-x} - \frac{xt}{2(1-x)^2} + \left[1 - 3x + \sum_{j=1}^{\infty} (e_{m,j,1} x^{jm} - e'_{m,j,1} x^{jm+1})\right]^{-1} \frac{t}{4}\right] \\ &\sim -\frac{t}{4(1-x)} - \frac{xt}{2(1-x)^2} + \left[1 - 3x + \sum_{j=1}^{\infty} (e_{m,j,1} x^{jm} - e'_{m,j,1} x^{jm+1})\right]^{-1} \frac{t}{4} \\ &= -\frac{t}{4(1-x)} - \frac{xt}{2(1-x)^2} + \left[1 - 3x + \sum_{j=1}^{\infty} (4x^{jm} - 4x^{jm+1})\right]^{-1} \frac{t}{4} \\ &= -\frac{t}{4(1-x)} - \frac{xt}{2(1-x)^2} + \frac{1}{1 - 3x + 4(1-x)\sum_{j=1}^{\infty} x^{jm}} \cdot \frac{t}{4} \\ &= -\frac{t}{4(1-x)} - \frac{xt}{2(1-x)^2} + \frac{1}{1 - 3x + 4(1-x)\left(\frac{1}{1-x^m} - 1\right)} \cdot \frac{t}{4} \\ &= \frac{x^2(x^{m-2} - 1)t}{(1-x)^2(x^{m+1} - 3x^m + 3x - 1)}. \end{split}$$

For m=3, this specializes to

$$\sum_{n=0}^{\infty} P_{321,n}^{\text{ilpk}}(t) x^n \sim \frac{x^2(x-1)t}{(1-x)^2(x^4-3x^3+3x-1)}$$
$$= \frac{x^2t}{(1-x)^2(1-2x-2x^2+x^3)},$$

and since $x^2/((1-x)^2(1-2x-2x^2+x^3))$ is the ordinary generating function for [10, A080145] (shifted by 1), the result follows.

3. Descent compositions and inverse peaks: alternative proofs of Theorem 1

Every permutation can be decomposed into a sequence of increasing runs—maximal increasing consecutive subsequences—and the descent composition of a permutation π is the integer composition whose parts are the lengths of the increasing runs of π in the order that they appear. For example, the increasing runs of $\pi = 85712643$ are 8, 57, 126, 4, and 3, and the descent composition of π is (1,2,3,1,1). It is easy to see that a permutation avoids the consecutive pattern $12 \cdots m$ if and only if its descent composition has every part less than m.

The following theorem is the main result of this section.

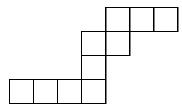


Figure 1: The ribbon shape corresponding to the composition (4, 1, 2, 3).

Theorem 5. For any composition L of $n \ge 1$, there exists exactly one permutation $\pi \in \mathfrak{S}_n$ with descent composition L such that $ipk(\pi) = 0$.

By restricting Theorem 5 to compositions with all parts less than m, we see that permutations π in $\mathfrak{S}_n(12\cdots m)$ with $\mathrm{ipk}(\pi)=0$ are in one-to-one correspondence with compositions of n with all parts less than m. Hence Theorem 1 is an immediate corollary of Theorem 5.

In this section we give two (related) proofs for Theorem 5, thus resulting in two additional proofs of Theorem 1. For the first proof, we will assume familiarity with some basic definitions from the theory of symmetric functions at the level of Stanley [11, Chapter 7], but let us briefly establish notation and review a few elementary facts which will be needed for the proof.

Given a partition λ , let s_{λ} denote the Schur function of shape λ , and if μ is a partition contained inside λ , then let $s_{\lambda/\mu}$ denote the skew Schur function of shape λ/μ . Recall that a connected skew shape with no 2×2 square is called a *ribbon shape*.² Given a composition $L = (L_1, L_2, \ldots, L_k)$, let r_L denote the skew Schur function of the ribbon shape with L_i squares in row k - i + 1 for each i. See Figure 1 for an example.

Let $\langle \cdot, \cdot \rangle$ denote the usual scalar product on symmetric functions, and let $c_{\mu,\nu}^{\lambda} = \langle s_{\lambda/\mu}, s_{\nu} \rangle$. Recall that the $c_{\mu,\nu}^{\lambda}$ are called *Littlewood–Richardson coefficients* and are the structure constants for products of Schur functions when expanded back in the Schur basis; that is,

$$s_{\mu}s_{\nu} = \sum_{\lambda} c_{\mu,\nu}^{\lambda} s_{\lambda}.$$

We next state a couple lemmas from which Theorem 5 follows. The first lemma is equivalent to Theorem 6.2 of Foulkes [4], although it was first stated in the form below by Gessel [5, Theorem 5]; see also [11, Corollary 7.23.8].

Lemma 6. Let L and M be compositions of $n \ge 1$. Then $\langle r_L, r_M \rangle$ is the number of permutations π with descent composition L such that π^{-1} has descent composition M.

The second lemma is Theorem 7.1 of Foulkes [4].

Lemma 7. Let λ , μ , and ν be partitions of $n \ge 1$ such that λ has p columns and $\nu = (n-r, 1^r)$ for some $0 \le r \le n$. Then

$$c_{\mu,\nu}^{\lambda} = \delta_{n-r,p} = \begin{cases} 1, & if \ n-r = p, \\ 0, & otherwise. \end{cases}$$

²Ribbon shapes are also commonly called "skew-hooks" (e.g., by Foulkes [4]) or "border strips" (e.g., by Stanley [11]).

First proof of Theorem 5. Let r_L be the ribbon Schur function corresponding to the composition L. Furthermore, let us write $r_L = s_{\lambda/\mu}$ and let p be the number of columns of λ . It is easy to see that a permutation in \mathfrak{S}_n has no peaks if and only if its descent composition is of the form $(1^r, n-r)$ for some $0 \le r \le n$. Also, the ribbon shape corresponding to the composition $(1^r, n-r)$ is precisely the Young diagram corresponding to the partition $(n-r, 1^r)$, so that $r_{(1^r,n-r)} = s_{(n-r,1^r)}$. Thus, in light of Lemma 6, the number of permutations $\pi \in \mathfrak{S}_n$ with descent composition L such that π^{-1} has no peaks is equal to

$$\sum_{r=0}^{n} \langle r_{L}, r_{(1^{r}, n-r)} \rangle = \sum_{r=0}^{n} \langle s_{\lambda/\mu}, s_{(n-r, 1^{r})} \rangle$$

$$= \sum_{r=0}^{n} c_{\mu, (n-r, 1^{r})}^{\lambda} \qquad (\text{since } c_{\mu, \nu}^{\lambda} = \langle s_{\lambda/\mu}, s_{\nu} \rangle)$$

$$= \sum_{r=0}^{n} \delta_{n-r, p} \qquad (\text{by Lemma 7})$$

$$= 1.$$

Hence the result follows.

Now, we provide a bijective proof of Theorem 5.

Second proof of Theorem 5. Let $L = (L_1, L_2, ..., L_k)$ be a composition of n. We construct a permutation π with descent composition L and whose inverse has no peaks by inserting the letters 1, 2, ..., n in a line with n positions in the following way:

- 1. Let 1 be the first letter of the kth increasing run, let 2 be the first letter of the (k-1)th increasing run, and so on. This step determines the positions of the letters $1, 2, \ldots, k$.
- 2. Insert the remaining letters $k+1, k+2, \ldots, n$ into the remaining positions in ascending order.

For example, suppose that L=(3,2,3,1). Then the resulting permutation would be $\pi=456372891$, which indeed has descent composition L and whose inverse $\pi^{-1}=964123578$ has no peaks. In general, by placing the smallest k letters in the initial positions of each increasing run and the remaining letters in ascending order, it is clear that the resulting permutation must have descent composition L. Furthermore, observe that π^{-1} consists of a decreasing run formed by the positions of the smallest k letters in π , followed by an increasing run formed by the positions of the remaining letters in π . Thus π^{-1} has no peaks.

We now argue that any permutation whose inverse has no peaks must have the form given above. Let π be such a permutation in \mathfrak{S}_n with $\pi^{-1}(k) = 1$. Since π^{-1} has no peaks, the first k letters of π^{-1} are descending and the remaining letters of π^{-1} are ascending. This means that π is a shuffle of the decreasing subsequence $k \cdots 21$ (with k in the first position of π) and the increasing subsequence $(k+1)(k+2)\cdots n$. Hence, π is of the form given above, which completes the proof.

There is a simple idea underlying the above proof: the permutations whose inverse has no peaks are precisely the permutations whose entries can be drawn on a "<" shape. This set of permutations is an example of a *monotone grid class*, a type of permutation class arising in the study of permutation patterns [2, 7, 8].

Also, we note that the bijection used in the above proof is closely related to a bijection between ribbon shapes and standard Young tableaux of hook shape which Foulkes uses to prove Lemma 7. Thus, our second proof of Theorem 5 is in a sense equivalent to our first proof, but described without using the language of symmetric functions.

Finally, we give a few other elementary consequences of Theorem 5. Recall that a permutation π in \mathfrak{S}_n is alternating if $\pi_1 < \pi_2 > \pi_3 < \cdots$, and is reverse-alternating if $\pi_1 > \pi_2 < \pi_3 > \cdots$.

Corollary 8. For any $n \geq 1$, there is exactly one alternating permutation π in \mathfrak{S}_n with $ipk(\pi) = 0$, and exactly one reverse-alternating permutation π in \mathfrak{S}_n with $ipk(\pi) = 0$.

Proof. It is easy to see that a permutation in \mathfrak{S}_n is alternating if and only if its descent composition is of the form (2, 2, ..., 2) or (2, 2, ..., 2, 1), depending on the parity of n. Similarly, a permutation in \mathfrak{S}_n is reverse-alternating if and only if its descent composition is of the form (1, 2, 2, ..., 2, 1) or (1, 2, 2, ..., 2). Therefore, the result follows from Theorem 5.

Given a permutation π in \mathfrak{S}_n , we say that $i \in [n-1]$ is a *descent* of π if $\pi_i > \pi_{i+1}$. Let $des(\pi)$ denote the number of descents of π , which is one less than the number of parts of the descent composition of π . Also, let $pk(\pi)$ be the number of peaks of π and $lpk(\pi)$ the number of left peaks of π .

Corollary 9. For any $n \ge 1$ and $k \ge 0$, the number of permutations π in \mathfrak{S}_n with $\operatorname{des}(\pi) = k$ and $\operatorname{ipk}(\pi) = 0$ is equal to $\binom{n-1}{k}$.

Proof. This is an immediate consequence of Theorem 5 and the well-known fact that, for any integers $n, k \ge 1$, there are $\binom{n-1}{k-1}$ compositions of n into k parts [10, A097805].

Corollary 10. For any $n \ge 1$ and $k \ge 0$, the number of permutations π in \mathfrak{S}_n with $pk(\pi) = k$ and $pk(\pi) = 0$ is equal to $\binom{n}{2k+1}$.

Proof. The number of peaks of a permutation π is equal to the number of increasing runs of π of length greater than 1, not including the final increasing run [6, Lemma 2.1 (a)]. Furthermore, it is known that there are $\binom{n}{2k}$ compositions of n with k parts greater than 1 [10, A034839]. Therefore, by conditioning on the number of letters of π prior to the final increasing run and using Theorem 5, we see that there are $\sum_{j=0}^{n-1} \binom{j}{2k}$ permutations π in \mathfrak{S}_n with $\mathrm{pk}(\pi) = k$ and $\mathrm{ipk}(\pi) = 0$. The result then follows from the identity $\sum_{j=0}^{n-1} \binom{j}{2k} = \binom{n}{2k+1}$ [10, A034867].

Corollary 11. For any $n \geq 1$ and $k \geq 0$, the number of permutations π in \mathfrak{S}_n with $\operatorname{lpk}(\pi) = k$ and $\operatorname{ipk}(\pi) = 0$ is equal to $\binom{n}{2k}$.

Proof. Let us call $i \in \{2, 3, ..., n\}$ a right peak of π if either i is a peak of π or if i = n and $\pi_n > \pi_{n-1}$. Let $\operatorname{rpk}(\pi)$ denote the number of right peaks of π . Define $C_{n,k}^{\text{lpk}}$ to be the set of compositions which are descent compositions of permutations in \mathfrak{S}_n with k left peaks,

and define $C_{n,k}^{\text{rpk}}$ in the analogous way but for right peaks. It is known that $\text{rpk}(\pi)$ is equal to the number of increasing runs of π of length greater than 1 [6, Lemma 2.1 (d)]. Again using the fact that there are $\binom{n}{2k}$ compositions of n with k parts greater than 1, we see that $|C_{n,k}^{\text{rpk}}| = \binom{n}{2k}$. In light of Theorem 5, it suffices to show that $|C_{n,k}^{\text{lpk}}| = \binom{n}{2k}$; we do this by giving a bijection between $C_{n,k}^{\text{lpk}}$ and $C_{n,k}^{\text{rpk}}$.

Given a permutation $\pi = \pi_1 \pi_2 \cdots \pi_n$ in \mathfrak{S}_n , define the *reverse* of π to be $\pi^r := \pi_n \cdots \pi_2 \pi_1$. Clearly, the reversal operation $\pi \mapsto \pi^r$ is an involution on \mathfrak{S}_n which toggles between left peaks and right peaks. Moreover, reversal induces an involution on compositions: if L is a composition, then define L^r to be the descent composition of π^r , where π is any permutation whose descent composition is L. (It is easy to see that L^r does not depend on the specific choice of π .) This involution on compositions restricts to a bijection between $C_{n,k}^{\text{lpk}}$ and $C_{n,k}^{\text{rpk}}$, and we are done.

4. A bijective proof of Theorem 2

In this final section, we give a bijective proof of Theorem 2. Here, we assume basic familiarity with regular expressions. For an introduction to regular expressions as they arise in the theory of computation, see [9, Sec. 1.3]; for an explication of the role of regular expressions in symbolic combinatorics, see [3, Secs. I.4 & A.7].

Our proof consists of multiple stages, and before we begin the proof, let us establish some new notation and outline the steps of our proof. Define the sets N_n and N'_n by

$$N_n := \{ \pi \in \mathfrak{S}_n : \operatorname{lpk}(\pi) = 1 \} \text{ and } N'_n := \{ \pi \in N_n : \pi^{-1} \in \mathfrak{S}_n(321) \}.$$

It is clear that N'_n is in bijection with permutations in $\mathfrak{S}_n(321)$ satisfying ilpk $(\pi) = 1$, which are precisely the permutations counted by Theorem 2.

Let A^n denote the set of words of length n on the alphabet $\{a,b,c\}$, and let W_n denote the set of words $w \in A^n$ of the form $w = a^i cuac^j$, where $i, j \geq 0$ and $u \in A^{n-i-j-2}$, such that w avoids the subwords bba, bbb, cba, and cbb. (We say that w avoids v if w does not contain v as a subword, i.e., we cannot write $u = \alpha v\beta$ for some—possibly empty—words α and β .) The first step of our proof will be to establish the following proposition.

Proposition 12. The sets N'_n and W_n are in bijection for every $n \geq 1$.

Upon proving Proposition 12, it remains to show $|W_n| = \sum_{i=1}^{n-1} \sum_{k=1}^i f_{k-1} f_k$. It is easily checked that $\sum_{i=1}^{n-1} \sum_{k=1}^i f_{k-1} f_k = \sum_{k=1}^{n-1} \sum_{j=0}^{n-k-1} f_{k-1} f_k$, so we will instead show that

$$|W_n| = \sum_{k=1}^{n-1} \sum_{j=0}^{n-k-1} f_{k-1} f_k. \tag{1}$$

To do so, we partition W_n into subsets $W_n^{j,k}$ corresponding to the summands in Equation (1), and then construct a bijection between each $W_n^{j,k}$ and a certain set of tilings whose cardinality is $f_{k-1}f_k$. To define the subsets $W_n^{j,k}$, we shall need to characterize the words in W_n as those matching a certain regular expression. Let θ^* denote the Kleene star of a regular expression θ , and let $\theta^+ := \theta^*\theta$.

Proposition 13. Let $n \ge 1$ and $w \in A^n$. Then we have $w \in W_n$ if and only if w matches the regular expression $a^*c (c \cup bc \cup a^+b \cup a^+c)^* a^+c^*$.

Note that $a^*c (c \cup bc \cup a^+b \cup a^+c)^* a^+c^*$ is an unambiguous regular expression, in the sense that every word that matches it does so in a unique way (see [3, Sec. A.7]). We could then "translate" this regular expression into a generating function for the words that match it. This generating function turns out to be $x^2/((1-x)^2(1-2x-2x^2+x^3))$, which appears in our earlier generating function proof of Theorem 2. Our present focus, however, is on bijective proof.

Let Z_k be the set of words in A^k that match the regular expression $a^*c (c \cup bc \cup a^+b \cup a^+c)^*$. If w matches the regular expression $a^*c (c \cup bc \cup a^+b \cup a^+c)^* a^+c^*$, then we can write w uniquely in the form $w = za^{n-j-k}c^j$, where $1 \le k \le n-1$, $0 \le j \le n-k-1$, and $z \in Z_k$. Thus we can define

$$W_n^{j,k} := \{ za^{n-j-k}c^j \in A^n : z \in Z_k \}.$$

Observe that each $W_n^{j,k}$ is a subset of W_n by Proposition 13. Hence, for each $n \geq 1$, the sets $W_n^{j,k}$ (ranging over $1 \leq k \leq n-1$ and $0 \leq j \leq n-k-1$) form a partition of W_n . Moreover, there is a bijection from $W_n^{j,k}$ to Z_k : namely, remove the $a^{n-j-k}c^j$ suffix. Therefore, we have

$$|W_n| = \sum_{k=1}^{n-1} \sum_{j=0}^{n-k-1} |W_n^{j,k}| = \sum_{k=1}^{n-1} \sum_{j=0}^{n-k-1} |Z_k|$$

and the last remaining step will be to prove the following proposition.

Proposition 14. We have $|Z_k| = f_{k-1}f_k$ for every $k \ge 1$.

Now that we have outlined our bijective proof of Theorem 2, we will now fill in the details by proving Propositions 12–14.

4.1. Proof of Proposition 12

Recall that N_n is the set of permutations in \mathfrak{S}_n with exactly one left peak. Given $\pi \in \mathfrak{S}_n$, observe that $\pi \in N_n$ if and only if (i) $\pi \neq 12 \cdots n$ and (ii) π can be written in the form $\pi = \alpha \beta \gamma$, where α , β , and γ are subwords of π (possibly of length 0) such that α is increasing, β is decreasing, and γ is increasing.

For example, if $\pi = 125101286437911 \in \mathfrak{S}_{12}$, then $\pi \in N_{12}$, and we can take $\alpha = 12510$ and $\beta = 128643$ and $\gamma = 7911$. See Figure 2; the reason we have named this set N_n is because the permutations are shaped like the letter N. Like the set of permutations whose inverse has no peaks, the set $\bigsqcup_{n\geq 0}(N_n\cup\{12\cdots n\})$ is an example of a monotone grid class (see the remark after the second proof of Theorem 5 in Section 3).

We can also characterize N_n using valleys. Given π in \mathfrak{S}_n , we say that $i \in \{2, 3, \dots, n-1\}$ is a valley of π if $\pi_{i-1} > \pi_i < \pi_{i+1}$, and we say that $i \in \{2, 3, \dots, n\}$ is a right valley of π if either i is a valley of π or if i = n and $\pi_{n-1} > \pi_n$. Then N_n is the set of permutations in \mathfrak{S}_n with exactly one right valley. Note that if $\pi \in N_n$, then the left peak of π must be less than the right valley of π .

The decomposition $\alpha\beta\gamma$ is not unique. Given $\pi \in N_n$, let i be the left peak of π and j the right valley of π . In the decomposition $\pi = \alpha\beta\gamma$, the letters $\pi_1, \pi_2, \dots \pi_{i-1}$ must be in α ;

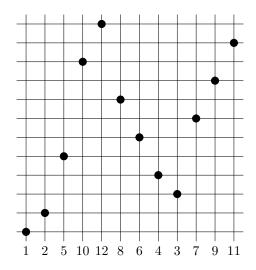


Figure 2: The permutation $\pi = 125101286437911 \in \mathfrak{S}_{12}$ can be decomposed into an increasing sequence $\alpha = 12510$ followed by a decreasing sequence $\beta = 128643$ followed by an increasing sequence $\gamma = 7911$. The permutation is shaped like an N.

the letters $\pi_{i+1}, \ldots, \pi_{j-1}$ must be in β ; and the letters π_{j+1}, \ldots, π_n must be in γ . The only choices are to place π_i in either α or β and to place π_j in either β or γ . Thus there are $2 \cdot 2 = 4$ possible decompositions $\pi = \alpha \beta \gamma$. Of these four, we define the *canonical decomposition* of π to be the one which places π_i in α and places π_j in γ .

Continuing the example above with $\pi = 125101286437911$, we have $\pi_i = 12$ and $\pi_j = 3$, and the canonical decomposition is $\alpha \mid \beta \mid \gamma = 1251012 \mid 864 \mid 37911$. (We shall often write $\alpha\beta\gamma$ as $\alpha \mid \beta \mid \gamma$ to clearly demarcate the blocks of the decomposition.) Observe that the canonical decomposition of a permutation is the one with the shortest possible β . From this point forward, we may refer to the canonical decomposition simply as "the decomposition".

Recall that A^n is the set of words of length n on the alphabet $\{a, b, c\}$. A permutation $\pi \in N_n$ determines a word $\phi(\pi) = w_1 w_2 \cdots w_n \in A^n$ in the following way. Let $\pi = \alpha \beta \gamma$ be the decomposition of π . Then, for each $i \in [n]$, set $w_i = a$ if the value i is in α , set $w_i = b$ if i is in β , and set $w_i = c$ if i is in γ . This map $\phi \colon N_n \to A^n$ is injective, because each w_i tells us which block i is placed in, which uniquely determines $\pi \in N_n$.

Continuing the example above with $\pi = \alpha \mid \beta \mid \gamma = 1251012 \mid 864 \mid 37911$, we have that $w_1 = a$ since 1 is found in α , that $w_2 = a$ since 2 is found in α , that $w_3 = c$ since 3 is found in γ , and so on. Therefore, $\phi(\pi) = aacbabcbcaca$.

Lemma 15. Let $w \in A^n$. We have $w \in \phi(N_n)$ if and only if w has the form $w = a^i cuac^j$, where $i, j \geq 0$ and $u \in A^{n-i-j-2}$.

Proof. Assume $w \in \phi(N_n)$, so $w = \phi(\pi)$ for some $\pi \in N_n$. Consider the decomposition $\pi = \alpha \beta \gamma$. If 1 is in α , then let i be the largest number such that $1, 2, \ldots, i$ are all in α ; otherwise set i = 0. This implies that $w_1 w_2 \cdots w_i = a^i$ and that w_{i+1} is either b or c. Since i + 1 is the smallest letter in β or γ , it must be the letter located at the right valley of π , which means that it is in γ ; thus, $w_{i+1} = c$. This proves that w begins with $a^i c$.

If n is in γ , then let j be the largest number such that the values $n-j+1,\ldots,n$ are all found in γ ; otherwise, let j=0. A similar argument as above shows that w ends with ac^j . Therefore, either $w=a^icuac^j$ for some word u, or $w=a^ic^j$. But if $w=a^ic^j$ then $\pi=12\cdots n$, contradicting the definition of N_n . Therefore, we have $w=a^icuac^j$ for some word u.

Conversely, assume that w is in the form $w = a^i cuac^j$, with i, j, and u as in the statement of the lemma. Define $\pi \in \mathfrak{S}_n$ as follows: For each $p \in [n]$, place p into α if $w_p = a$, place p into β if $w_p = b$, and place p into γ if $w_p = c$; now sort α in increasing order, sort β in decreasing order, sort γ in increasing order, and set $\pi = \alpha \beta \gamma$. We know that i + 1 is placed in γ and n - j is placed in α because $w_{i+1} = c$ and $w_{n-j} = a$; thus $\pi \neq 12 \cdots n$, and so $\pi \in N_n$.

We will now show that $\alpha\beta\gamma$ is the canonical decomposition of π ; from this, and from the way we defined π , it immediately follows that $\phi(\pi) = w$. Since $w_1w_2 \cdots w_{i+1} = a^ic$, the values $1, 2, \ldots, i$ are all in α ; thus i+1 is the lowest value in β or γ , so it must be the letter located at the right valley of π . Therefore, the letter located at the right valley of π is in γ . A similar argument with $w_{n-j+1}w_{n-j+2}\cdots w_n = ac^j$ shows that the letter located at the left peak of π is in α . Therefore, $\alpha\beta\gamma$ is the canonical decomposition of π .

Recall that N'_n is the subset of N_n containing those permutations whose inverse avoids the consecutive pattern 321. Equivalently, N'_n is the set of permutations $\pi \in N_n$ such that, for all $i \in [n-2]$, the letters i, i+1, and i+2 do not occur in decreasing order in π —that is, if i+1 occurs to the left of i, then i+2 must not occur to the left of i+1.

Lemma 16. Let $\pi \in \mathfrak{S}_n$. We have $\pi \in N'_n$ if and only if $\phi(\pi)$ avoids the subwords bba, bbb, cba, and cbb.

Proof. Set $w = w_1 w_2 \cdots w_n = \phi(\pi)$. Observe that i is an inverse descent of π —that is, a descent of π^{-1} —if and only if i+1 appears to the left of i in π . For a given $i \in [n-1]$, we characterize the words $w \in W_n$ such that i is an inverse descent of $\phi(w)$:

- If $w_i w_{i+1} = aa$, then both i and i+1 are in the increasing block α of π , so i+1 appears to the right of i. Thus i is not an inverse descent of π .
- If $w_i w_{i+1} = cc$, then both i and i+1 are in the increasing block γ , so i is not an inverse descent of π for the same reason as above.
- If $w_i w_{i+1} = bb$, then both i and i+1 are in the decreasing block β , so i+1 appears to the left of i. Thus i is an inverse descent of π .
- If $w_i w_{i+1} \in \{ab, ac, bc\}$, then the block $(\alpha, \beta, \text{ or } \gamma)$ that contains i is to the left of the block that contains i+1, so i+1 appears to the right of i. Thus i is not an inverse descent of π .
- If $w_i w_{i+1} \in \{ba, ca, cb\}$, then the same reasoning as the previous case shows that i is an inverse descent of π .

Therefore, i is an inverse descent of π if and only if $w_i w_{i+1} \in \{ba, bb, ca, cb\}$.

Finally, i is the starting position of an occurrence of 321 in π^{-1} if and only if i and i+1 are inverse descents of π , if and only if $w_i w_{i+1}$ and $w_{i+1} w_{i+2}$ are both in $\{ba, bb, ca, cb\}$, if and only if $w_i w_{i+1} w_{i+2} \in \{bba, bbb, cba, cbb\}$.

Recall that W_n was defined to be the set of words in A^n that satisfy the conditions from both Lemma 15 and Lemma 16. Hence, Lemmas 15 and 16 imply that ϕ restricted to N'_n is a bijection onto W_n . This completes the proof of Proposition 12.

4.2. Proof of Proposition 13

Our proof of Proposition 13 relies on the following lemma.

Lemma 17. Let v be a word that does not end with a. If cva avoids the subwords bba, bbb, cba, and cbb, then v matches the regular expression $(c \cup bc \cup a^+b \cup a^+c)^*$.

Proof. We induct on the length of v. The empty word matches the regular expression, so let us assume that v is not empty.

First, suppose v starts with c. Let us write v = cv' for a word v'. Since cva = ccv'a avoids the given subwords, so does cv'a, and since v does not end with a, neither does v'. Thus, by the induction hypothesis v' matches $(c \cup bc \cup a^+b \cup a^+c)^*$, and so v = cv' also does.

Next, suppose v starts with b. If v = b, then cva = cba would be a forbidden subword, so v has length at least 2. Since cv starts with cb, it must start with cbc—otherwise cv would start with a forbidden subword cbb or cba. Let us write v = bcv' for a word v'. Since cva = cbcv'a avoids the given subwords, so does cv'a, and since v does not end with a, neither does v'. Thus, by the induction hypothesis v' matches $(c \cup bc \cup a^+b \cup a^+c)^*$, and so v = bcv' also does.

Finally, suppose v starts with a. Let i be the largest number such that v starts with a^i . We know that i is not the length of v, because v does not end with a. So the (i+1)th letter of v is either b or c; without loss of generality, suppose it is b. Write $v = a^i b v'$ for a word v'. Since $cva = ca^i b v'a$ avoids the given subwords, so does bv'a, and since v does not end with a, neither does v'. Thus, by the induction hypothesis v' matches $(c \cup bc \cup a^+b \cup a^+c)^*$, and so $v = a^i b v'$ also does because $i \ge 1$.

We are now ready to prove Proposition 13.

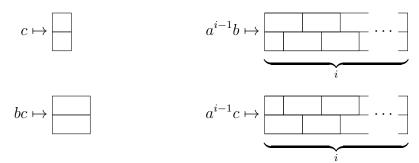
Proof of Proposition 13. Suppose w matches the regular expression $a^*c (c \cup bc \cup a^+b \cup a^+c)^* a^+c^*$. Clearly $w = a^icuac^j$ for some $i, j \ge 0$ and some word u, and a routine case-checking argument verifies that w avoids bba, bbb, cba, and cbb. Therefore, $w \in W_n$ (where n is the length of w).

Now, suppose $w \in W_n$, and write $w = a^i cuac^j$. Since the prefix $a^i c$ matches $a^* c$ and the suffix ac^j matches ac^* , it remains to prove that u matches $(c \cup bc \cup a^+b \cup a^+c)^* a^*$. Let l be the largest number such that u ends with a^l (so l = 0 if u does not end with a). We can write $u = va^l$, where v is a word that does not end with a. Since a^l matches a^* , and because Lemma 17 implies v matches $(c \cup bc \cup a^+b \cup a^+c)^*$, we conclude that $u = va^l$ matches $(c \cup bc \cup a^+b \cup a^+c)^* a^*$ as desired.

4.3. Proof of Proposition 14

A classical combinatorial model for Fibonacci numbers is given by monomino-domino tilings of rectangles, that is, f_k is the number of tilings of a $1 \times k$ rectangle with 1×1 blocks (monominoes) and 1×2 blocks (dominoes). Furthermore, f_{k-1} is the number of such tilings in which the leftmost block is a monomino. Thus, $f_{k-1}f_k$ is the number of ordered pairs of these tilings in which the first tiling begins with a monomino (and with no restrictions on the second tiling). By stacking the first tiling on top of the second, we get a monomino-domino tiling of a $2 \times k$ rectangle—where all dominoes are placed horizontally—with a monomino in the top-left corner. Let T_k be the set of these tilings; then $|T_k| = f_{k-1}f_k$.

To complete the proof of Proposition 14, we will construct a bijection between Z_k and T_k . Recall that Z_k is the set of words in A^k that match the regular expression $a^*c (c \cup bc \cup a^+b \cup a^+c)^*$, and observe that this regular expression is unambiguous: every word in Z_k has a unique decomposition into a subword matching a^*c followed by a sequence of subwords matching c, bc, a^+b , or a^+c . Equivalently, this is a decomposition into a sequence of subwords matching c, bc, a^+b , or a^+c such that the first subword matches c or a^+c . We now map a word c in its decomposition to indecomposable horizontal segments of the tiling as follows:



(Here, $i \geq 2$. Note that for the tilings corresponding to $a^{i-1}b$ and $a^{i-1}c$, the right end of the tiling depends on whether i is even or odd.) The condition that each word in Z_k starts with c or a^+c is equivalent to the condition that each tiling in T_k has a monomino in the top-left corner. It is clear that this mapping is bijective, and we have established Proposition 14.

For example, if $z = aacbcccaaabbcac \in Z_{15}$, then z decomposes as aac|bc|c|c|aaab|bc|ac, so z maps to



Our bijective proof of Theorem 2 involved several different bijections; let us now describe the composite bijection. Since we had to partition W_n into subsets $W_n^{j,k}$, our bijection must keep track of j and k as well as the resulting tiling. Thus, we are really mapping N'_n onto the set of ordered triples (j, k, τ) such that $1 \le k \le n - 1$, $0 \le j \le n - k - 1$, and $\tau \in T_k$. The number of such ordered triples is the right side of Equation (1). We illustrate the bijection using the example $\pi = 128910141617|12114|35671315181920 \in N'_{20}$. We have $\phi(\pi) = aacbcccaaabbcacaaccc \in W_{20}$, and in fact $aacbcccaaabbcac|aa|ccc \in W_{20}^{3,15}$ due to the prefix $aacbcccaaabbcac \in Z_{15}$ and the suffix a^2c^3 . Finally, aacbcccaaabbcac maps to the tiling τ that we saw in the previous example, so π maps to $(3, 15, \tau)$.

4.4. Remarks on avoiding $m \cdots 21$

Parts of the construction we have described can be generalized to permutations avoiding the consecutive pattern $m \cdots 21$, for any $m \geq 3$. We outline the more general construction here but omit the details.

Let $W_n^{(m)}$ denote the set of words w in A^n of the form $w = a^i cuac^j$, where $i, j \geq 0$ and $u \in A^{n-i-j-2}$, such that w avoids the subwords $b^{m-1}a$, b^m , $cb^{m-2}a$, and cb^{m-1} . Then the map $\phi: N_n \to A^n$ restricts to a bijection between $\{\pi \in N_n : \pi^{-1} \in \mathfrak{S}_n(m \cdots 21)\}$ and $W_n^{(m)}$; this is a generalization of Proposition 12.

Next, we can generalize Proposition 13 to the fact that, for $w \in A^n$, we have $w \in W_n^{(m)}$ if and only if w matches the regular expression

$$a^*c \left[b^{\leq m-3} (c \cup bc \cup a^+b \cup a^+c) \right]^* b^{\leq m-3} a^+c^*,$$
 (2)

where $b^{\leq t} := (\varepsilon \cup b \cup b^2 \cup \cdots \cup b^t)$. Setting m = 3 recovers the regular expression from Proposition 13.

It can be shown that $x^2(x^{m-2}-1)/((1-x)^2(x^{m+1}-3x^m+3x-1))$ is the generating function for words matching the regular expression in (2); this generating function appeared in our first proof of Theorem 2.

Unfortunately, the words matching the regular expression in (2) do not have a nice interpretation in terms of tilings that would generalize the mapping from Section 4.3. We have found a few ways of turning these words into tilings of $2 \times k$ rectangles, but none are as natural-seeming as the one for m = 3; more importantly, none can be broken down as a one-row tiling stacked on top of another one-row tiling, so these would not yield a simple product formula like $f_{k-1}f_k$ in the case of m = 3. The best we can do is to use (2) to derive a rational generating function, but there does not seem to be anything of great significance about this generating function or the associated sequence of numbers.

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