Liquidity Jump, Liquidity Diffusion, and Portfolio of Assets with Extreme Liquidity

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Abstract

We model a portfolio of crypto assets that does not respond well to multivariate autoregressive models because of discontinuity in conditional covariance matrix and posterior covariance matrix caused by extreme liquidity. We adjust asset-level return and volatility with liquidity to reduce such discontinuity, and restore the effectiveness of a set of liquidity-adjusted VECM-DCC/ADCC-BL models at extreme liquidity. We establish two distinctive yet complementary portfolio liquidity measures: portfolio liquidity jump that quantifies the effect of liquidity adjustment in forecasting the conditional covariance matrix, and portfolio liquidity diffusion that quantifies the effect of liquidity adjustment in estimating the posterior covariance matrix.

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1. Introduction

Deng and Zhou (2024a) show that the traditional ARMA-GARCH/EGARCH representations are not effective in modeling assets with extreme liquidity, as extreme liquidity manifests price jumps that do not respond well to the autoregressive models. They further demonstrate that, if the return and volatility (of assets with jumps) be adjusted with appropriate liquidity measures, the well-behaved autoregressive properties would be restored, and the AMRA-GARCH/EGARCH models would become effective again in modeling these assets with much improved predictability at extreme liquidity. They provide empirical evidence for such improved predictability by comparing the portfolio performance of a series of Liquidity-Adjusted Mean Variance (LAMV) and traditional Mean Variance (TMV) portfolios and observe clear advantage for the LAMV over the TMV.[1] Deng and Zhou (2024a) use a selected portfolio of crypto assets to represent assets with extreme liquidity, which may subject to manipulative trading activities called "wash trades" (Cong et al., 2023). In order to combat the potential negative effect of wash trading, they deploy a treatment designed to remove the wash trades.

Deng and Zhou (2024b) further divide the asset-level liquidity into two components: liquidity jump and liquidity diffusion, and find that the liquidity diffusion has a higher correlation with crypto washing trading than the liquidity jump. They demonstrate that while the treatment of washing trading significantly reduces the liquidity diffusion, it only marginally reduces the

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[1] Deng and Zhou (2024a) use the standard mean-variance (MV) method to optimize portfolios with forecasted returns produced by the liquidity-adjusted ARMA-GARCH/EGARCH models and name them the "LAMV" portfolios. They also use the standard MV method to optimize portfolios with forecasted returns produced by the traditional ARMA-GARCH/EGARCH models and name them the "TMV" portfolios. We adopt their terminology throughout this paper.

liquidity jump to a level to that is inadequate to restore the effectiveness of the autoregressive models. They recommend that treatment on wash trading is not needed in modeling established crypto assets that trade in mainstream exchanges as they are not heavily infested, and that such treatment may remove legitimate high-volume trades and therefore deteriorate the effectiveness of the autoregressive models. They further demonstrate that liquidity adjustment is adequate to restore the effectiveness of the autoregressive models. In this paper, we follow the recommendation of Deng and Zhou (2024b) not to include treatment on wash trading, as we indeed investigate 10 established crypto assets with high market cap and long trading history that trade in Binance, the largest crypto exchange by trading amount.[2]

While the univariate liquidity-adjusted ARMA-GARCH/EGARCH models of Deng and Zhou (2024a) offer improved predictability on the conditional return and volatility of individual assets, they are not multivariate models and can model the conditional covariance of a multivariate financial time series, i.e., a portfolio of assets. Ling and McAleer (2003) provide an asymptotic vectorized ARMA-ARCH (VARMA-ARCH) framework to link the VARMA-produced conditional residuals to a multivariate ARCH model. Engle and Granger (1987) introduce the concepts of co-integration and error correction for multivariate time series, which Phillips and Ouliaris (1990) extend to co-integrated multivariate time series with a VECM (Vectorized Error Correction Model) representation. Engle and Sheppard (2001) establish the Dynamic Conditional Correlation (DCC) model, and Chong and Miffre (2010) further identify that the DCC(1,1) specification, essentially a multivariate GARCH(1,1) extension, is more effective in analyzing

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time-varying conditional correlations and covariance in multivariate time series across asset classes. Cappiello et al. (2006) introduce an asymmetric DCC (ADCC) model to accommodate asymmetries among conditional covariance and structural break induced conditional correlation increases. Linking the VECM (Phillips and Ouliaris, 1990) to the DCC (Engle and Sheppard, 2001) and ADCC (Cappiello et al., 2006), Deng (2018) develops a generalized VECM-DCC/ADCC-based Black-Litterman (BL) framework, which we refer to as the VECM-DCC/ADCC-BL model for the rest of the paper, to predict the posterior return vector and covariance matrix of a multivariate financial time series.

This paper seeks to extend Deng and Zhou (2024a, 2024b) to modeling portfolios of assets with extreme liquidity, represented by a portfolio of established crypto assets trade on Binance. We hypothesize that, for a portfolio consisting of assets with extreme liquidity and therefore does not respond well to multivariate autoregressive models, if the return and volatility of individual assets are properly adjusted by the measures of their liquidity, the well-behaved autoregressive properties will be restored to the portfolio, which will respond well again to multivariate autoregressive models. Based on the above hypothesis and following Deng (2018), we develop a set of liquidity-adjusted multivariate VECM-DCC/ADCC-BL time series constructs to forecast the daily covariance matrix of a portfolio of selected crypto assets that represent assets with extreme liquidity.[3] We theoretically demonstrate that the liquidity adjustment on the asset-level by Deng and Zhou (2024a) "smoothens" the daily conditional covariance and produces less irrational

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[3] We start from the standard VECM-DCC/ADCC-BL to model the covariance of liquidity-adjusted portfolios, based on which we develop a set of VECM-DCC/ADCC-BL models with built-in liquidity adjustment, which we name the "liquidity-adjusted VECM-DCC/ADCC-BL" models. For the purpose of comparisons, we also use the standard VECM-DCC/ADCC-BL to model the covariance of regular (non-liquidity-adjusted) portfolios, and we call these models "traditional VECM-DCC/ADCC-BL" models. We use these terms throughout this paper.

increment in the daily covariance matrices under extreme liquidity, therefore the liquidity-adjusted VECM-DCC/ADCC-BL models offer much improved predictability. To support our hypothesis with empirical evidence, we apply the forecasted daily portfolio covariance by the liquidity-adjusted VECM-DCC/ADCC-BL models as inputs to a set of LAMV constructs, also proposed by Deng and Zhou (2024a), for portfolio optimization. For comparison, we duplicate the procedure with the forecasted daily regular (non-liquidity-adjusted) covariance being fed into a set of TMV constructs. We observe a clear advantage for the LAMV over the TMV in portfolio performance.

The contribution of this paper is three-fold. First, we derive and formulate portfolio-level liquidity-adjusted return and volatility, based on which we develop two complementary portfolio-level liquidity measures: the portfolio liquidity jump that measures the magnitude of portfolio liquidity; and the portfolio liquidity diffusion that measures the volatility of portfolio liquidity. Second, we adjust asset-level return and volatility with their respective liquidity measures, which greatly reduces the discontinuity of daily portfolio conditional covariance and posterior covariance matrices, and in turn restores the effectiveness of the VECM-DCC/ADCC-BL models with much improved predictability at extreme liquidity. Third, we establish that the portfolio liquidity jump quantifies the effect of liquidity adjustment in forecasting the portfolio conditional covariance, while the portfolio liquidity diffusion quantifies the effect of liquidity adjustment in estimating the daily covariance matrix. While we use crypto assets to demonstrate our models, they are effective on other assets with extreme liquidity.

The rest of the paper proceeds as follows. Section 2 reviews existing literature on portfolio liquidity. Section 3 introduces portfolio liquidity jump and diffusion, and portfolio-level liquidity-adjusted return and volatility. Section 4 provides descriptive statistics of the dataset and discussions on the distributions of portfolio-level liquidity measures. Section 5 presents the

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liquidity-adjusted VECM-DCC/ADCC-BL models and forecasts of liquidity-adjusted daily covariance matrix. Section 6 optimizes LAMV portfolios enhanced with forecasts from the liquidity-adjusted VECM-DCC/ADCC-BL models. Section 7 concludes the paper.

2. Literature Review on Portfolio Liquidity

We refer to Deng and Zhou (2024a, 2024b) for a thorough literature review on asset-level liquidity measures, liquidity costs, volatility of liquidity, components of liquidity, and models of assets with extreme liquidity. In this section, we review the recent development in portfolio liquidity modeling in the context of portfolio optimization.

In literature, liquidity has been used as a constraint in portfolio optimization. Among a vast volume of portfolio optimization literature (see, for example, Kolm, Tütüncü and Fabozzi, 2014), only a selected few explicitly include liquidity as a constraint in portfolio optimization. Lo, Petrov and Wierzbicki (2006) define portfolio liquidity as a weighted average liquidity of individual assets, named as Weighted Average Liquidity (WAL). Vieira and Filomena (2019) consider the monetary value of constructed portfolios, referred to as Financial Value Liquidation (FVL), and model liquidity with parameters used by portfolio managers. Vieira et al. (2023) utilize both WAL and FCL to understand the impact of liquidity constraints on index tracking and find that portfolios subjected to liquidity constraints are more liquid than portfolios free of liquidity constraints. On the methodology level, a considerable stream of literature seeks to solve the dynamic portfolio optimization problem with liquidity costs in closed-form, both transient (e.g., Çetin and Rogers, 2007; Ly Vath, Mnif and Pham, 2007; Ma, Song and Zhang, 2013) and permanent market impact (e.g., Gârleanu and Pedersen, 2013; Lim and Wimonkittiwat, 2014; Gaigi et al., 2016; Mei, DeMiguel and Nogales, 2016). In order to broaden the application range, the Least Squares Monte

Carlo (LSMC) algorithm is applied to dynamic portfolio optimization for semi closed-form and numerical solutions (e.g., Brandt et al., 2005; Cong and Oosterlee, 2016, 2017; Zhang et al., 2019).

Another stream of studies focuses on the impact of liquidity, also as a constraint, on portfolio value-at-risk (VaR) with a variety of modeling approaches, which is in general more relevant to the methodology developed in this paper. Al Janabi (2011) argues that regular VaR models assess the downward risk in mark-to-market portfolio value over a given time horizon but do not account for the actual trading risk of liquidation and introduces a multivariate Liquidity-Adjusted VaR (LVaR) subject to constraints on expected return, trading volume and liquidation horizon. Al Janabi (2013) extends Al Janabi (2011) with a so-called LVaR-GARCH-M(1,1) model, which essentially incorporates a GARCH-M technique to forecast conditional volatility and expected return in a multivariate LVaR construct. Hung et al. (2020) apply multivariate GARCH-t and GJR-GARCH-t models to seize the liquidity property embedded in the individual asset returns and evaluate their accuracy and efficiency in computing LVaR forecasts for portfolios. Weiß and Supper (2013) model the joint distribution of bid-ask spreads and log returns of a stock portfolio by using Autoregressive Conditional Double Poisson and GARCH processes and vine copulas. Al Janabi, Ferrer, and Shahzad (2019) develop a LVaR optimization technique based on vine copulas and apply it to multi-asset portfolios. In addition, Al Janabi et al. (2017) propose a LVaR model, in which the linear correlations between assets are replaced by a multivariate nonlinear Dynamic Conditional Correlation (DCC) *t*-copula modeling. Al Janabi (2021) provides a thorough review on LVaR-based multivariate portfolio optimization algorithms.

On the other hand, liquidity has seldom been used a direct parameter in portfolio optimization. At the theoretical level, the existing literature does not model portfolio-level (and for that matter, asset-level) return and volatility with explicit adjustment of liquidity. We aim to bridge this gap

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by providing models that explicitly adjust return and volatility with liquidity, which serves as our first motivation. At the methodological level, although a few studies provide certain autoregressive models that address portfolio-level liquidity risk (e.g., Al Janabi, 2013; Weiß and Supper, 2013; Al Janabi et al., 2017; Hunt et al., 2020), again these models do not incorporate explicit liquidity-adjustment on the variables they actually model (i.e., portfolio-level conditional return and covariance). In this paper, we seek to develop a new set of liquidity-adjusted multivariate autoregressive models that are applicable to the assets with extreme liquidity, with improved predictability. This serves as our second motivation.

3. Liquidity-adjusted Return and Volatility, and Liquidity Measures

3.1 Liquidity-adjusted Return and Volatility, and Liquidity Measures on Asset Level

In this subsection, we briefly review the asset-level liquidity-adjusted return and volatility, as well as liquidity jump and liquidity diffusion proposed by Deng and Zhou (2024a, 2024b). Using the tick-level trading data from the most dominant crypto asset exchange, Binance, Deng and Zhou (2024a) model the minute-level liquidity-adjusted volatility $\sigma^2{}_T^\ell$ and return r_t^ℓ at equilibrium for time-period T (a 24-hour/1440-minute trading day) as follows:

$$\sigma^2{}_T^\ell = \frac{1}{T}\sum_{t=1}^T \eta_T \frac{|r_t|/\overline{|r_t|}}{A_t/\overline{A_t}} (r_t - \overline{r_t})^2 = \frac{1}{T}\sum_{t=1}^T \left(r_t^\ell - \overline{r_t^\ell}\right)^2$$

$$r_t^\ell = \sqrt{\eta_T \frac{|r_t|/\overline{|r_t|}}{A_t/\overline{A_t}}}\, r_t$$

$$\text{subject to: } \sum_{t=1}^T \eta_T \frac{|r_t|/\overline{|r_t|}}{A_t/\overline{A_t}} = T \Rightarrow \eta_T = \frac{T}{\sum_{t=1}^T \frac{|r_t|/\overline{|r_t|}}{A_t/\overline{A_t}}}$$

where:
1. *r_t is the observed return at minute t, $\overline{r_t}$ is its arithmetic average in that day,*
2. *$|r_t|$ is the absolute value of r_t, $\overline{|r_t|}$ is its arithmetic average in that day,*
3. *r_t^ℓ is the liquidity-adjusted return at minute t, $\overline{r_t^\ell}$ is its arithmetic average in that day,*
4. *A_t is the dollar (USDT) amount traded at minute t, $\overline{A_t}$ is its arithmetic average in that day,*
5. *η_T is the daily normalization factor on day T and is a constant for day T,*
6. *$T = 1440$ as there are 1440 minutes (24 hours) in a crypto asset trading day.*

The daily (day-level) regular and liquidity-adjusted returns for time-period T is obtained by aggregating the intraday (minute-level) returns are given as:

$$r_{TT} = (1 + r_t)^T - 1$$
$$r_{TT}^{\ell} = \left(1 + r_t^{\ell}\right)^T - 1$$

The realized and unobservable daily (intraday on minute-level) variance for time-period T is:

$$\sigma^2{}_{TT}^{\ell} = T\sigma^2{}_T^{\ell}$$

The "daily liquidity premium Beta," $\beta_{r_{TT}}^{\ell}$, for time-period T is thus defined as follows:

$$\beta_{r_{TT}}^{\ell} = \left|r_{TT}/r_{TT}^{\ell}\right| \subset \begin{cases} > 1; high\ daily\ liquidity\ magnitude \\ = 1; equilibrium\ daily\ liquidity\ magnitude \\ < 1; low\ daily\ liquidity\ magnitude \end{cases} \qquad (1)$$

In addition, Deng and Zhou (2024b) introduce a "daily liquidity volatility Beta," $\beta_{\sigma_{TT}}^{\ell}$, which reflects the volatility of liquidity for time-period T, defined as follows:

$$\beta_{\sigma_{TT}}^{\ell} = \left|\sigma_{TT}/\sigma_{TT}^{\ell}\right| \subset \begin{cases} > 1; high\ daily\ liquidity\ volatility \\ = 1; equilibrium\ daily\ liquidity\ volatility \\ < 1; low\ daily\ liquidity\ volatility \end{cases} \qquad (2)$$

Deng and Zhou (2024b) name $\beta_{r_{TT}}^{\ell}$ and $\beta_{\sigma_{TT}}^{\ell}$ the asset-level "daily liquidity Beta" pair, which are also regarded as the asset-level daily "liquidity jump" and "liquidity diffusion," respectively, and the two constituent components of the asset-level daily liquidity. The liquidity jump $\beta_{r_{TT}}^{\ell}$ measures the magnitude of liquidity, and is an indicator of jumps (extreme liquidity) when $\beta_{r_{TT}}^{\ell}$ is much greater than 1 ($\beta_{r_{TT}}^{\ell} \gg 1$); the liquidity diffusion $\beta_{\sigma_{TT}}^{\ell}$ models the time variability of liquidity, and high liquidity volatility occurs when $\beta_{\sigma_{TT}}^{\ell}$ is much greater than 1 ($\beta_{\sigma_{TT}}^{\ell} \gg 1$).[4]

3.2 Liquidity-adjusted Return and Volatility, and Liquidity Measures on Portfolio Level

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[4] Practically, liquidity magnitude is "extreme" if $\beta_{r_{TT}}^{\ell} \geq 4$, and liquidity volatility is "extreme" when $\beta_{\sigma_{TT}}^{\ell} \geq 2.5$.

Derived from Deng and Zhou (2024a, 2024b), at the portfolio level (with N assets, $N > 1$), the daily regular and liquidity-adjusted return vectors, and how they are connected, are given as:[5]

$$Q_t = \begin{bmatrix} r_{t_1} \\ \dots \\ r_{t_i} \\ \dots \\ r_{t_N} \end{bmatrix}; \quad Q_t^\ell = \begin{bmatrix} r_{t\,1}^\ell \\ \dots \\ r_{t\,i}^\ell \\ \dots \\ r_{t\,N}^\ell \end{bmatrix}; \quad i \in [1, N] \tag{3}$$

$$Q_{TT} = \begin{bmatrix} r_{t_1} \\ \dots \\ r_{t_i} \\ \dots \\ r_{t_N} \end{bmatrix} = \begin{bmatrix} \beta_{r_{t_1}}^\ell r_{t\,1}^\ell \\ \dots \\ \beta_{r_{t_i}}^\ell r_{t\,i}^\ell \\ \dots \\ \beta_{r_{t_N}}^\ell r_{t\,N}^\ell \end{bmatrix} = \begin{bmatrix} \beta_{r_{t_1}}^\ell & \dots & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & \beta_{r_{t_i}}^\ell & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & 0 & \dots & \beta_{r_{t_N}}^\ell \end{bmatrix} \begin{bmatrix} r_{t\,1}^\ell \\ \dots \\ r_{t\,i}^\ell \\ \dots \\ r_{t\,N}^\ell \end{bmatrix} = B_{r_t}^{\mathcal{P}\ell} Q_t^\ell; \quad Q_t^\ell = B_{r_t}^{\mathcal{P}\ell^{-1}} Q_t \tag{4}$$

where:

$$B_{r_t}^{\mathcal{P}\ell} = \begin{bmatrix} \beta_{r_{t_1}}^\ell & \dots & 0 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & \beta_{r_{t_i}}^\ell & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & 0 & \dots & \beta_{r_{t_N}}^\ell \end{bmatrix}; \quad \left| B_{r_t}^{\mathcal{P}\ell} \right| = \prod_{i=1}^N \beta_{r_{t_i}}^\ell, \left| B_{r_t}^{\mathcal{P}\ell} \right| \text{ is the determinant of } B_{r_t}^{\mathcal{P}\ell} \tag{5}$$

In Equation 5, we introduce a "portfolio liquidity premium Beta matrix" $B_{r_t}^{\mathcal{P}\ell}$, which is a diagonal matrix with elements being asset-level $\beta_{r_{t_i}}^\ell$'s. The portfolio-level regular and liquidity-adjusted daily covariance matrices, and their connection, are given as follows:[6]

$$\Sigma_{r_t}^{TT} = T \begin{bmatrix} \sigma_{t\,11}^2 & \dots & \sigma_{t\,1i} & \dots & \sigma_{t\,1N} \\ \dots & \dots & \dots & \dots & \dots \\ \sigma_{t\,i1} & \dots & \sigma_{t\,ii}^2 & \dots & \sigma_{t\,iN} \\ \dots & \dots & \dots & \dots & \dots \\ \sigma_{t\,N1} & \dots & \sigma_{t\,Ni} & \dots & \sigma_{t\,NN}^2 \end{bmatrix}; \quad \Sigma_{r_t^\ell}^{TT} = T \begin{bmatrix} \sigma_{t\,11}^{2\,\ell} & \dots & \sigma_{t\,1i}^\ell & \dots & \sigma_{t\,1N}^\ell \\ \dots & \dots & \dots & \dots & \dots \\ \sigma_{t\,i1}^\ell & \dots & \sigma_{t\,ii}^{2\,\ell} & \dots & \sigma_{t\,iN}^\ell \\ \dots & \dots & \dots & \dots & \dots \\ \sigma_{t\,N1}^\ell & \dots & \sigma_{t\,Ni}^\ell & \dots & \sigma_{t\,NN}^{2\,\ell} \end{bmatrix} \tag{6}$$

$$\Sigma_{r_t}^{TT} = B_{\sigma_t}^{\mathcal{P}\ell} \Sigma_{r_t^\ell}^{TT} B_{\sigma_t}^{\mathcal{P}\ell H} \tag{7}$$

where: $\sigma_{t_{ij}}$ is the intraday (minute-level) covariance between assets i and j, $i, j \in [1, N], i \neq j$

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[5] From this point on (for all subsequent sections and appendices), we discuss only daily level data. In order to simplify data notation, we replace the subscript *TT* with *t*. The subscript *t* refers to a point in time with a daily interval (not minute interval), i.e., day *t*.

[6] From this point forward we use the phrase "daily covariance matrices" to refer to the daily (minute-level intraday) covariance matrices $\Sigma_{r_t}^{TT}$ and $\Sigma_{r_t^\ell}^{TT}$ to avoid being verbose.

In Equation 7 we introduce a "portfolio liquidity volatility Beta matrix" $B_{\sigma_t}^{\mathcal{P}\ell}$, which is a symmetric matrix and can be solved by the Conditional Singular Value Decomposition (Conditional SVD) proposed by Deng (2024) (see Appendix 1 for details). Collectively $B_{r_t}^{\mathcal{P}\ell}$ and $B_{\sigma_t}^{\mathcal{P}\ell}$ form the "portfolio liquidity Beta matrix pair." The determinant of $B_{r_t}^{\mathcal{P}\ell}$, $\left| B_{r_t}^{\mathcal{P}\ell} \right|$, is the "portfolio liquidity jump" that measures the magnitude of portfolio liquidity; the determinant of $B_{\sigma_t}^{\mathcal{P}\ell}$, $\left| B_{\sigma_t}^{\mathcal{P}\ell} \right|$, is the "portfolio liquidity diffusion" that measures the volatility of portfolio liquidity. Their significance becomes apparent in Sections 4 and 5.

4. Dataset and Descriptive Statistics

We follow Deng and Zhou (2024a) to select 10 largest crypto assets by their market values with at least 4.5 years of historical data (April 27, 2019 to February 8, 2024 with 1,749 trading days) for 3.5-year back-tests (April 27, 2020 to February 8, 2024, 1,383 trading days) with a 1-year (365 days) rolling window that trade on Binance. The selected crypto assets are ADA, BNB, BTC, ETC, ETH, LINK, LTC, MATIC, XMR and XRP. We first collect and aggregate the tick-level trading data of these assets to construct and calculate asset-level daily data, from which we construct the return vectors and portfolio covariance matrices, both regular (Q_t and $\Sigma_{r_t}^{TT}$) and liquidity-adjusted (Q_t^ℓ and $\Sigma_{r_t^\ell}^{TT}$), as well as the portfolio liquidity jump and diffusion matrices $B_{r_t}^{\mathcal{P}\ell}$ and $B_{r_t}^{\mathcal{P}\ell}$. We report the descriptive statistics of asset-level $\beta_{r_t}^\ell$ and portfolio-level $\left| B_{r_t}^{\mathcal{P}\ell} \right|$ in Panel A of Table 1, and that of asset-level $\beta_{\sigma_t}^\ell$ and portfolio-level $\left| B_{\sigma_t}^{\mathcal{P}\ell} \right|$ in Panel B of Table 1. We

provide the scatter plot of $\left|B_{r_t}^{\mathcal{P}\ell}\right|$-$\left|B_{\sigma_t}^{\mathcal{P}\ell}\right|$ in Plot A of Figure 1, the visualizations of distributions of portfolio-level $\left|B_{r_t}^{\mathcal{P}\ell}\right|$ in Plot B of Figure 1, and that of $\left|B_{\sigma_t}^{\mathcal{P}\ell}\right|$ in Plot C of Figure 1.[7]

Deng and Zhou (2024a) observe that extreme liquidity occurs at about the same time (time series) for all the assets (cross-sectional). Therefore, when extreme liquidity occurs for at least some of the assets, there is also extreme liquidity on the portfolio level. From Panel A of Table 1, we observe that the distribution of portfolio liquidity jump $\left|B_{r_t}^{\mathcal{P}\ell}\right|$ is even more asymmetric than that of $\beta_{r_t}^{\ell}$ for the individual assets. The mean of $\left|B_{r_t}^{\mathcal{P}\ell}\right|$ is at 3.84, much higher than that of the $\beta_{r_t}^{\ell}$ for individual assets (1.49 for BTC to 2.02 for ADA), and the number of days with extreme value (= 10) is high at 553 (31.62% out of 1,749), with a remarkably high weight of the overall $\left|B_{r_t}^{\mathcal{P}\ell}\right|$ load at 82.43%. On the other hand, the median of $\left|B_{r_t}^{\mathcal{P}\ell}\right|$ at 0.63 is significantly lower than that of the $\beta_{r_t}^{\ell}$'s (0.85 for XRP to 1.06 for ADA). A histogram of $\left|B_{r_t}^{\mathcal{P}\ell}\right|$ (Plot B of Figure 1) provides a visualization of such a highly asymmetric distribution with long right tail (and a "tall" bar at $\left|B_{r_t}^{\mathcal{P}\ell}\right| = 10$). The results show that, as $\left|B_{r_t}^{\mathcal{P}\ell}\right|$ represents portfolio liquidity jump, there is higher magnitude of liquidity on the portfolio level than on the asset level.

From Panel B of Table 1, we observe that the distribution of portfolio liquidity diffusion $\left|B_{\sigma_t}^{\mathcal{P}\ell}\right|$ is also more asymmetric than that of $\beta_{\sigma_t}^{\ell}$ for all the assets. The mean of $\left|B_{\sigma_t}^{\mathcal{P}\ell}\right|$ is at 1.83 and noticeably higher than that of the $\beta_{\sigma_t}^{\ell}$ (0.82 for BTC to 1.49 for XMR). The number of days with extreme value (=10) is only 185 (10.58% of 1,749) but with a substantially high weight of the

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overall $\left|B_{\sigma_t}^{\mathcal{P}\ell}\right|$ load at 57.91%. On the other hand, the median of $\left|B_{\sigma_t}^{\mathcal{P}\ell}\right|$ is 0.32 and much lower than that of the $\beta_{\sigma_t}^{\ell}$ (0.82 for BTC to 1.18 for ETC). A histogram of $\left|B_{\sigma_t}^{\mathcal{P}\ell}\right|$ (Plot C of Figure 1) provides a visualization of the distribution. The results show that, as $\left|B_{\sigma_t}^{\mathcal{P}\ell}\right|$ represents portfolio liquidity diffusion, there is higher volatility of liquidity on the portfolio level than on the asset level.

The scatter plot of $\left|B_{\sigma_t}^{\mathcal{P}\ell}\right|$-$\left|B_{r_t}^{\mathcal{P}\ell}\right|$ (Plot A of Figure 1) visualizes the 2D distribution of portfolio liquidity along the jump (y-axis) and diffusion (x-axis) dimensions. We observe that there are both extreme liquidity jump ($\left|B_{r_t}^{\mathcal{P}\ell}\right| \gg 1$) and extreme liquidity diffusion ($\left|B_{\sigma_t}^{\mathcal{P}\ell}\right| \gg 1$), and that there are a very small number days with both extreme liquidity jump and diffusion, presumably from wash trades, an direct support and extension to Deng and Zhou (2024b) that treatment on wash trading is not necessary for established crypto assets. We will report additional descriptive statistics with visualizations for the purposely constructed portfolios in Section 6.

5. Multivariate Autoregressive Models and Posterior Daily Covariance Matrix

5.1 Regular VECM-DCC/ADCC-BL Models

In this section we first present a multivariate autoregressive VECM-DCC/ADCC-BL time series construct to model portfolio covariance. The model can be further divided into three portions: the VECM/VAR portion that models the portfolio conditional return vector, the DCC/ADCC portion that models the portfolio conditional variance matrix, and the BL portion that estimates the of posterior portfolio covariance matrix.

Following Deng (2018), we first apply the VECM/VAR model to the daily return series over the sample period with 1,749 days (with 1,383 days being out-of-sample predictions and a rolling window of 365 days) for the 10 selected crypto assets. We conduct a set of Johansen tests on the

full sample and confirm that the r_t of ten assets are cointegrated, and so as the r_t^{ℓ} (Table 2).

Therefore, both return vectors can be modelled by an autoregressive VECM/VAR(p) construct.

We then establish a specific time-series VECM/VAR(p) model (see Appendix 2 Subsection A2.1

for details) given in Equation 8. For each rolling window, we identify the VAR order p based on

the AIC value ($p \leq 5$), fit the model, and use the residual (fitting error) vector as the (E_t) input to

the next-stage DCC/ADCC analysis. The VECM/VAR specification of Equation 8 produces a

one-period ahead ($t+1$) forecasted return vector $\hat{Q}_{t+1}$, of which the residual vector $\hat{E}_{t+1}$ is given

in Equation 9 (see Appendix 2 Subsection A2.1 for details).

Second, we apply a DCC(1,1) and an ADCC(1,1) specifications to estimate the time-varying

conditional covariance matrix in the residual error vector $\hat{E}_{t+1}$. For each rolling window, we fit

both DCC(1,1) and ADCC(1,1) of Equation 10 on $\hat{E}_{t+1}$, and choose either DCC(1,1) or

ADCC(1,1) with a higher log-likelihood to produce the conditional covariance matrix, $\hat{\Omega}_{t+1}$. (steps

of deriving the conditional variance matrix $\hat{\Omega}_{t+1}$ are given in Appendix 2, Subsection A2.1).

Finally, with the forecasted $\hat{\Omega}_{t+1}$ we estimate the posterior (forecasted) covariance matrix for

day $t+1$, $\hat{\Sigma}_{r_{t+1}}^{TT}$, which is analytically expressed in a BL-style Equation 11 (steps of deriving the $\hat{\Sigma}_{r_{t+1}}^{TT}$

are given in Appendix 2 Subsection A2.1).[8] We consolidate Equations 8-11 for the regular

VECM-DCC/ADCC-BL models in the following (see Appendix 2 Subsection A2.1 for details):

$$Q_t = \sum_{i=1}^{p} \Phi_i Q_{t-i} + E_t \qquad (8)$$
$$\hat{E}_{t+1} = \hat{Q}_{t+1} - Q_{t+1} \qquad (9)$$
$$\hat{E}_{t+1} | \Psi_t \sim N(0, \hat{\Omega}_{t+1}) \qquad (10)$$

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[8] We use $\Sigma_{r_t}^{TT}$ and $\Sigma_{r_t^{\ell}}^{TT}$ to refer to the regular and liquidity-adjusted daily covariance matrices, respectively, to avoid the confusion with Σ_{r_t} and $\Sigma_{r_t^{\ell}}$, which are the regular and liquidity-adjusted covariance matrices for a given period of time (the 365-day rolling window in the context of this paper), respectively.

$$\hat{\Sigma}_{r_{t+1}}^{TT} = \Sigma_{r_t}^{TT} + \left[\left(\tau\Sigma_{r_t}^{TT}\right)^{-1} + \hat{\Omega}_{t+1}^{-1}\right]^{-1} \tag{11}$$

Where:
1) Q_t is the portfolio return vector, E_t is the residual vector, and Φ_i is the coefficient matrix for VAR lag I,
2) $\hat{Q}_{t+1}$ is the forecasted portfolio return vector (out-of-sample) at time t+1,
3) Q_{t+1} is the actual observed return vector (out-of-sample) at time t+1,
4) $\hat{E}_{t+1}$ is the conditional residual vector (out-of-sample) at time t_1, from the VECM/VAR stage,
5) $\hat{\Omega}_{t+1}$ is the conditional covariance matrix of $\hat{E}_{t+1}$ in the rolling window.

5.2 Liquidity-Adjusted VECM-DCC/ADCC-BL Models

We repeat the procedure in Subsection 5.1 to derive the liquidity-adjusted VECM-DCC/ADCC-BL models in Equations 12-18 (see Appendix 2 Subsection A2.2 for details):

$$Q_t^{\ell} = \sum_{i=1}^{p} \Phi_i^{\ell} Q_{t-i}^{\ell} + E_t^{\ell} \tag{12}$$

$$\hat{E}_{t+1}^{\ell} = \hat{Q}_{t+1}^{\ell} - Q_{t+1}^{\ell} \tag{13}$$

$$\hat{E}_{t+1}^{\ell}|\Psi_t \sim N\left(0, \hat{\Omega}_{t+1}^{\ell}\right) \tag{14}$$

$$\hat{\Sigma}_{r_{t+1}^{\ell}}^{TT} = \Sigma_{r_t^{\ell}}^{TT} + \left[\left(\tau\Sigma_{r_t^{\ell}}^{TT}\right)^{-1} + \left(B_t^{\mathcal{P}\ell}\hat{\Omega}_{t+1}^{\ell}B_t^{\mathcal{P}\ell^H}\right)^{-1}\right]^{-1} \tag{15}$$

where:

$$E_t^{\ell} = B_{r_t}^{\mathcal{P}\ell^{-1}} E_t \tag{16}$$

$$\hat{\Omega}_{t+1}^{\ell} = B_{r_t}^{\mathcal{P}\ell^{-1}} \hat{\Omega}_{t+1} = B_{r_t}^{\mathcal{P}\ell^{-\frac{1}{2}}} \hat{\Omega}_{t+1} B_{r_t}^{\mathcal{P}\ell^{-\frac{1}{2}}} \tag{17}$$

$$B_t^{\mathcal{P}\ell} = B_{\sigma_t}^{\mathcal{P}\ell^{-1}} B_{r_t}^{\mathcal{P}\ell^{\frac{1}{2}}} \Rightarrow \left|B_t^{\mathcal{P}\ell}\right| = \left|B_{\sigma_t}^{\mathcal{P}\ell^{-1}}\right|\left|B_{r_t}^{\mathcal{P}\ell^{\frac{1}{2}}}\right| \tag{18}$$

In Equation 18, we establish a "portfolio liquidity Beta matrix" $B_t^{\mathcal{P}\ell}$, which is the matrix product of the inverse of "portfolio liquidity premium Beta matrix" $B_{r_t}^{\mathcal{P}\ell}$ and the matrix square root of "portfolio liquidity volatility Beta matrix" $B_{\sigma_t}^{\mathcal{P}\ell}$, and is a scaling factor matrix for the conditional covariance $\hat{\Omega}_{t+1}^{\ell}$ in constructing the expected covariance $\hat{\Sigma}_{r_{t+1}^{\ell}}^{TT}$. The determinant of $B_t^{\mathcal{P}\ell}$, $\left|B_t^{\mathcal{P}\ell}\right|$, is a combination of the portfolio liquidity jump $\left|B_{r_t}^{\mathcal{P}\ell}\right|$ and portfolio liquidity diffusion $\left|B_{\sigma_t}^{\mathcal{P}\ell}\right|$. It is thus apparent that both the portfolio liquidity jump and portfolio liquidity diffusion have direct role in estimating the posterior portfolio covariance. The descriptive statistics of $\left|B_t^{\mathcal{P}\ell}\right|$ is given in Table 3, column "liquidity combined," and its histogram is given in Plot D of Figure 1. The distribution of $\left|B_t^{\mathcal{P}\ell}\right|$ is kind of "in the middle" between that of $\left|B_{r_t}^{\mathcal{P}\ell}\right|$ and $\left|B_{\sigma_t}^{\mathcal{P}\ell}\right|$, reflecting that the

two liquidity measures work towards opposite directions in forming it. The liquidity-adjusted models of Equations 12-15 have the similar form as the regular models of Equations 8-11, and their error terms and conditional covariance matrices are connected through Equations 16 and 17.

5.3 Linkage of Regular and Liquidity-Adjusted VECM-DCC/ADCC-BL Models

From Equation 17, we observe that when portfolio liquidity jump is high ($\left|B_{r_t}^{\mathcal{P}\ell}\right| > 1$), the liquidity-adjusted conditional covariance is reduced from its regular counterpart ($\hat{\Omega}_{t+1}^{\ell} = B_{r_t}^{\mathcal{P}\ell-1}\hat{\Omega}_{t+1}$), while portfolio liquidity jump is low ($\left|B_{r_t}^{\mathcal{P}\ell}\right| < 1$) the opposite is true. Conceptually this means that, through liquidity adjustment, the effect of portfolio liquidity jump has been incorporated into the daily conditional covariance. This highlights the significance of the portfolio liquidity jump, which is that it quantifies the effect of liquidity adjustment in restoring the portfolio-level autoregressive property of continuity to the portfolio conditional covariance.

Furthermore, in the BL portion, the expected increments of $\hat{\Sigma}_{r_{t+1}}^{TT}$ and $\hat{\Sigma}_{r_{t+1}^{\ell}}^{TT}$ on day $t+1$ with 1st order approximation are connected as (see Appendix 2 Subsection A2.3 for details):

$$\Delta\hat{\Sigma}_{r_{t+1}}^{TT} \sim \hat{\Omega}_{t+1} \tag{19}$$

$$\Delta\hat{\Sigma}_{r_{t+1}^{\ell}}^{TT} \sim B_t^{\mathcal{P}\ell}\hat{\Omega}_{t+1}^{\ell}B_t^{\mathcal{P}\ell H} = B_{\sigma_t}^{\mathcal{P}\ell-1}\hat{\Omega}_{t+1}B_{\sigma_t}^{\mathcal{P}\ell-1H} \tag{20}$$

Equation 20 demonstrates that when portfolio liquidity diffusion is high ($\left|B_{\sigma_t}^{\mathcal{P}\ell}\right| > 1$), the expected increment of $\hat{\Sigma}_{r_{t+1}}^{TT}$ is lower than that of $\hat{\Sigma}_{r_{t+1}}^{TT}$, and when it is low ($\left|B_{\sigma_t}^{\mathcal{P}\ell}\right| < 1$) the opposite is true. Conceptually this means, through liquidity adjustment, the effect of portfolio liquidity diffusion has been incorporated into the expected increment of daily covariance matrix. This signifies the importance of portfolio liquidity diffusion, that it quantifies the effect of liquidity adjustment in estimating the liquidity-adjusted daily covariance matrix $\hat{\Sigma}_{r_{t+1}^{\ell}}^{TT}$ in conjunction with the conditional covariance matrix (both $\hat{\Omega}_{t+1}$ and $\hat{\Omega}_{t+1}^{\ell}$). We use the Frobenius distance as the forecast

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accuracy criteria (Table 3). The results in Table 2 show that $\hat{\Sigma}^{TT}_{r^\ell_{t+1}}$ has a higher Frobenius distance (hence lower forecasting accuracy) of 0.0589 than $\hat{\Sigma}^{TT}_{r_{t+1}}$ at 0.0149. The reason is that there are fewer number of days with a $\left|B^{\mathcal{P}\ell}_{\sigma_t}\right|$ being greater than 1 (456 days or 26.07% of 1,749 days). As such, the "smoothness" of the daily covariance matrix through liquidity adjustment is actually more important than the accuracy of it through the BL model prediction.

In Section 6 we will demonstrate that the LAMV portfolios have clearly superior performance over the TMV ones. This provides a direct empirical support to our hypothesis: if the return and volatility of individual assets are properly adjusted by liquidity, under extreme liquidity, the magnitude of liquidity is reduced (scaled down by portfolio liquidity jump) and the DCC/ADCC model produces "smoother" portfolio conditional covariance, and the volatility of liquidity is also reduced (scaled down by portfolio liquidity diffusion) and the BL model produces less irrational expected increment in the daily covariance matrix.

6. Empirical Tests on Liquidity-Adjusted Autoregressive Models

In this section we provide empirical evidence that the liquidity-adjusted models of Section 5 offer better predictability on portfolio covariance matrix than their traditional counterparts, through comparing the performance of a set of MV portfolios taking the forecasts of the daily covariance matrix from the liquidity-adjusted VECM-DCC/ADCC-BL as inputs against that of the traditional MV portfolios. To combine the findings of Deng and Zhou (2024a) with this paper, we construct a set of MV portfolios enhanced by the forecasted returns of individual assets from the liquidity-adjusted ARMA-GARCH/EGARCH models. Finally, in order to present the full merit of adjusting return and volatility on both asset and portfolio levels, we create an additional set of

MV portfolios enhanced by the forecasts from both the liquidity-adjusted ARMA-GARCH/EGARCH and VECM-DCC/ADCC-BL models.

6.1 Benchmark Portfolios

We duplicate the benchmark portfolios of Deng and Zhou (2024) as follows:

1. Portfolio 1: An equal-weight portfolio with each asset assigned an equal weight of 10%.

2. Portfolio 2: A market (equilibrium) portfolio with each asset assigned a weight proportional to its market weight (in amount).

3. Portfolio 3: A liquidity-weight portfolio with each asset assigned a weight proportional to its daily liquidity premium Beta $\left(\sim \bar{\beta}_{r_t}^{\ell}\right)$, for the short-term investors that take advantage of low transaction costs.

4. Portfolio 4: An inverse-liquidity-weight portfolio with each asset assigned a weight that is inversely proportional to its daily liquidity premium Beta $\left(\sim 1/\beta_{r_t}^{\ell}\right)$, for the long-term investors that seek a liquidity premium.

All the portfolios do not include the risk-free asset USDT (zero weight).

6.2 Standard MV Portfolios

We then construct two MV portfolios: traditional and liquidity-adjusted. The standard daily-optimized MV in a time-series construct can be analytically expressed as the following quadratic programming problem with constraints:

$$\max_{W_t} \left(\bar{\mu}_t W_t - \frac{\lambda_t}{2} W_t^H \bar{\Sigma}_t W_t \right); H \text{ is Transpose} \tag{21}$$

$$subject\ to:$$
$$\sum_i^N w_t^i = 1; \ i = USDT, ADA, BNB, BTC, ETC, ETH, LINK, LTC, MATIC, XMR, XRP; \ N = 11$$
$$w_t^i \geq 0; \ long\text{-}only$$
$$w_t^{USDT} \leq 1$$
$$w_t^i \leq 0.300 \ (3 \times equal\ weight); \ i \neq USDT$$

where:

$$\lambda_t = \frac{r_{t_{mkt}}^P - r_t^{rf}}{\sigma_{t\,mkt}^2} = \frac{r_{t_{mkt}}^P - r_t^{USDT}}{\sigma_{t\,mkt}^2} = \frac{r_{t_{mkt}}^P}{\sigma_{t\,mkt}^2}$$

$r_{t_{mkt}}^P$ is the return of the market or equilibrium portfolio on day t, $\sigma_{t\,mkt}^2$ is its variance of the rolling window;
r_t^{rf} and r_t^{USDT} are the returns of a risk-free asset and USDT, respectively. USDT is regarded as risk-free with 0 return

In the standard MV construct of Equation 21, $\bar{\mu}_t$ is the mean (row) return vector of the ten-asset portfolio over a 365-day rolling window ending on day t, and $\bar{\Sigma}_t$ is the covariance matrix of daily returns of the constituent assets in that rolling window (note: it is not the daily minute-level covariance matrix, Σ_t^{TT}, see the rest of this section). Both $\bar{\mu}_t$ and $\bar{\Sigma}_t$ are realized and derived from available information up to day t. In addition, W_t is the portfolio (column) weight vector to be optimized for day t. The daily MV portfolios are:

5. Portfolio 5: A TMV portfolio; $\bar{\mu}_t$ is the mean vector of r_t's over the rolling window ending on day t, or $\bar{\mu}_{r_t}$; $\bar{\Sigma}_t$ is the covariance matrix of r_t's for the rolling window, or $\bar{\Sigma}_{r_t}$.

6. Portfolio 6: A LAMV portfolio; $\bar{\mu}_t$ is the mean vector of r_t^ℓ's over the rolling window ending on day t, or $\bar{\mu}_{r_t^\ell}$; $\bar{\Sigma}_t$ is the covariance matrix of r_t^ℓ's for the rolling window, or $\bar{\Sigma}_{r_t^\ell}$.

6.3 VECM-DCC/ADCC-BL-enhanced MV Portfolios

To demonstrate the improved predictability of the liquidity-adjusted VECM-DCC/ADCC-BL models of Section 5, we construct two MV portfolios with forecasted daily covariance matrix (Portfolios 7 and 8). Built upon Portfolios 5 and 6, we further construct two portfolios with VECM-DCC/ADCC-BL enhancement by rewriting Equation 21 to retain $\bar{\mu}_t$ and to replace $\bar{\Sigma}_t$ by the VECM-DCC/ADCC-BL-forecasted posterior covariance matrix on day $t+1$, $\hat{\Sigma}_{t+1}^{TT}$. The portfolios are constructed as:

$$\max_{W_t} \left(\bar{\mu}_t W_t - \frac{\lambda_t}{2} W_t^H \hat{\Sigma}_{t+1}^{TT} W_t \right) \tag{22}$$

All the constraints for Equation 22 are the same as those for Equation 21. The VECM/VAR-DCC/ADCC-BL-enhanced MV portfolios are:

7. Portfolio 7: A VECM-DCC/ADCC-enhanced TMV portfolio; $\bar{\mu}_t$ is the mean vector of r_t's over the rolling window ending on day t, or $\bar{\mu}_{r_t}$; $\hat{\Sigma}_{t+1}^{TT}$ is the VECM-DCC/ADCC-BL-forecasted daily regular minute-level (intraday) covariance matrix for day $t+1$, $\hat{\Sigma}_{r_{t+1}}^{TT}$.

8. Portfolio 8: A VECM-DCC/ADCC-enhanced LAMV portfolio; $\bar{\mu}_t$ is the mean vector of r_t^ℓ's over the rolling window ending on day t, or $\bar{\mu}_{r_t^\ell}$; $\hat{\Sigma}_{t+1}^{TT}$ is the VECM-DCC/ADCC-BL-forecasted daily liquidity-adjusted minute-level (intraday) covariance matrix, $\hat{\Sigma}_{r_{t+1}^\ell}^{TT}$.

6.4 ARMA-GARCH/EGARCH-enhanced MV Portfolios

We follow Deng and Zhou (2024) to construct two additional ARMA-GARCH/EGARCH-enhanced MV portfolios with forecasted daily return vector. We rewrite Equation 21 by retaining $\bar{\Sigma}_t$ and replacing $\bar{\mu}_t$ with the ARMA-GARCH/EGARCH forecasted return vector, $\hat{\mu}_{t+1}^{arga}$. The portfolios are constructed as:

$$\max_{W_t} \left(\hat{\mu}_{t+1}^{arga} W_t - \frac{\lambda_t}{2} W_t^H \bar{\Sigma}_t W_t \right) \qquad (23)$$

Again, all the constraints for Equation 23 are the same as those for Equation 21. The proposed ARMA-GARCH/EGARCH-enhanced MV portfolios are:

9. Portfolio 9: An ARMA-GARCH/EGARCH-enhanced TMV portfolio; $\hat{\mu}_{t+1}^{arga}$ is the return vector of ARMA-GARCH/EGARCH forecasted r_t values for day $t+1$, $\hat{\mu}_{r_{t+1}}^{arga}$; $\bar{\Sigma}_t$ is the covariance matrix of r_t's for the rolling window, or $\bar{\Sigma}_{r_t}$.

10. Portfolio 10: An ARMA-GARCH/EGARCH-enhanced LAMV portfolio; $\hat{\mu}_{t+1}^{arga}$ is the return vector of ARMA-GARCH/EGARCH forecasted r_t^ℓ values for day $t+1$, $\hat{\mu}_{r_{t+1}^\ell}^{arga}$; $\bar{\Sigma}_t$ is the covariance matrix of r_t^ℓ's for the rolling window, or $\bar{\Sigma}_{r_t^\ell}$.

6.5 VECM-DCC/ADCC and ARMA-GARCH/EGARCH-BL-enhanced MV Portfolios

Finally, we rewrite Equation 22 by replacing $\bar{\mu}_t$ by the ARMA-GARCH/EGARCH-forecasted return vector, $\hat{\mu}_{t+1}^{arga}$ while retaining the VECM-DCC/ADCC-forecasted posterior covariance matrix on day $t+1$, $\hat{\Sigma}_{t+1}^{TT}$. The portfolios are constructed as:

$$\max_{W_t} \left(\hat{\mu}_{t+1}^{arga} W_t - \frac{\lambda_t}{2} W_t^H \hat{\Sigma}_{t+1}^{TT} W_t \right) \tag{24}$$

All the constraints for Equation 24 are the same as those for Equation 21. The final ARMA-GARCH/EGARCH and VECM/VAR-DCC/ADCC-enhanced MV portfolios are:

11. Portfolio 11: A VECM-DCC/ADCC-BL and ARMA-GARCH/EGARCH-enhanced TMV portfolio; $\hat{\mu}_{t+1}^{arga}$ is the return vector of ARMA-GARCH/EGARCH-forecasted r_t's for day $t+1$, $\hat{\mu}_{r_{t+1}}^{arga}$; $\hat{\Sigma}_{t+1}^{TT}$ is the VECM -DCC/ADCC-forecasted daily regular minute-level (intraday) covariance matrix for day $t+1$, $\hat{\Sigma}_{r_{t+1}}^{TT}$.

12. Portfolio 12: A VECM-DCC/ADCC-BL and ARMA-GARCH/EGARCH-enhanced LAMV portfolio; $\hat{\mu}_{t+1}^{arga}$ is the vector of ARMA-GARCH/EGARCH-forecasted r_t^{ℓ} values for day $t+1$, $\hat{\mu}_{r_{t+1}^{\ell}}^{arga}$; $\hat{\Sigma}_{t+1}^{TT}$ is the VECM-DCC/ADCC-forecasted daily liquidity-adjusted minute-level (intraday) covariance matrix for day $t+1$, $\hat{\Sigma}_{r_{t+1}^{\ell}}^{TT}$.

6.6 Descriptive Statistics of the Portfolios

In Table 4 we capture the descriptive statistics of daily portfolio return (Panel A) and volatility (Panel B) of the 12 portfolios. The benchmark portfolios (Portfolios 1 to 4) are on the top. The eight MV portfolios (Portfolios 5 to 12) are arranged as such: the TMV portfolios with incremental forecast enhancement are listed on the left (Portfolios 5, 7, 9, 11), while their corresponding LAMV portfolios are shown on the right (Portfolios 6, 8, 10, 12). That way, it is easier to observe the improvement after applying each type of enhancement methodology vertically within the TMV and LAMV, while at the same time conveniently compare the differences between the TMV and LAMV after applying each specific incremental enhancement methodology horizontally.

From Panel A of Table 4, the most important observations we make are, that among the TMV portfolios, the standard portfolio (Portfolio 5) has a very high maximum daily return at 134.96%, which drops to a lower level for the portfolios with forecast enhancements (Portfolios 7, 9 and 11, at 73.51%, 69.77% and 73.37%, respectively); while that among the LAMV portfolios (Portfolios, 6, 8, 10 and 12), the maximum daily return is much lower with no discontinuity from the standard portfolio (Portfolio 6 at 35.18%) to the forecast-enhanced portfolios (Portfolios 8, 10 and 12 at 42.83%, 50.84% and 43.30%, respectively). This indicates that jumps indeed exist on the portfolio level in the (unadjusted) return, which autoregressive models try to "smoothen" with "extra effort," and when such jumps are too severe (high liquidity jump) the correction is far from being adequate, making the autoregressive models ineffective. However, with appropriate liquidity adjustment, jumps are essentially removed, and the (liquidity-adjusted) return can thus be modeled by the autoregressive models efficiently. From the visualizations of portfolio return distribution for each of the 12 portfolios in Figure 2, we observe that, the TMV portfolios (all with "_rr" at the end of name) all have symmetric distribution but with long right tail, while the LAMV portfolios (all with "_rrlq" at the end of name) have symmetric distributions with no extreme values. Indeed, jumps (high liquidity jump) are essentially removed by liquidity adjustment.

It is interesting that the forecast-enhanced Portfolios 8, 10 and 12 actually have somewhat higher maximum daily return (42.83%, 50.84% and 43.30%, respectively) than the standard Portfolio 6 (35.18%). The reason is that, with the high liquidity magnitude ($\left|B_{r_t}^{\mathcal{P}\ell}\right| \gg 1$) having been removed, the autoregressive models are actually dominated by very low liquidity magnitude ($\left|B_{r_t}^{\mathcal{P}\ell}\right| \ll 1$), resulting in higher level of forecast error (E_t^{ℓ} in Equation 16) that may produce higher forecasted portfolio return (Q_t^{ℓ} in Equation 12). Also, the maximum daily return of ARMA-GARCH/EGARCH-enhanced Portfolio 10 (50.84%) is higher than that of the VECM-

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DCC/ADCC-BL-enhanced Portfolios 8 and 12 (42.83% and 43.30%, respectively), due to that the estimation of covariance matrix increment has an additional layer of reducing high portfolio liquidity volatility ($\left|B_{\sigma_t}^{\mathcal{P}\ell}\right| \gg 1$) ($\Delta\hat{\Sigma}_{r_{t+1}^\ell}^{TT}$ in Equation 20), which may further reduce jumps.

Panel B of Table 4 captures the descriptive statistics of daily portfolio volatility for all the 12 portfolios. Unlike the portfolio return, the portfolio volatility exhibits the similar pattern of change across the incrementally enhanced TMV and LAMV portfolios (vertically), and similar values between the TMV and LAMV portfolios with the same incremental enhancement (horizontally). The visualizations of volatility distribution of each of the 12 portfolios are given in Figure 3, which confirm that there is no obvious difference in portfolio volatility between the equivalent TMV and LAMV portfolios. Still, it is worth pointing out that for the VECM-DCC/ADCC-BL-enhanced portfolios, the LAMV portfolios actually have higher level of portfolio volatility than their corresponding TMV portfolios (38.97% of LAMV Portfolio 8 vs. 27.13% of TMV Portfolio 7; 39.91% of LAMV Portfolio 12 vs. 27.13% of TMV Portfolio 11). An explanation is that as the estimation of covariance matrix increment having reduced high portfolio liquidity volatility ($\left|B_{\sigma_t}^{\mathcal{P}\ell}\right| \gg 1$), the BL portion of the model is now dominated by low portfolio liquidity volatility ($\left|B_{\sigma_t}^{\mathcal{P}\ell}\right| \ll 1$) and produces higher $\Delta\hat{\Sigma}_{r_{t+1}^\ell}^{TT}$ (Equation 20), thus higher level of portfolio volatility.

To further understand the impact of liquidity-adjustment on portfolio return and volatility, we compare the return and volatility of the TMV portfolios and their corresponding LAMV counterparts for each specific incremental forecast enhancement (horizontally). The comparisons, expressed as the ratios of portfolio returns and volatilities, respectively, of a TMV portfolio to its equivalent LAMV portfolio (i.e., $r_{t+1}^{P-TMV}/r_{t+1}^{P-LAMV}$ and $\sigma_{t+1}^{P-TMV}/\sigma_{t+1}^{P-LAMV}$), are given in Table 5. In Panel A of Table 5, we observe that the ratio $r_{t+1}^{P-TMV}/r_{t+1}^{P-LAMV}$ for all four comparable pairs of

TMV-LAMV have a mean value greater than 1 and a maximum value capped at 10. This also provides direct evidence that the liquidity adjustment essentially removes the jumps. It is worth mentioning that the mean return ratio of the VECM-DCC/ADCC-BL portfolios (Ratios 2 and 4: 1.39 and 1.41) have somewhat higher values than that of other portfolios (Ratios 1 and 3: 1.18 and 1.20), which indicates higher level of reduction for jumps, consistent with the portfolio level descriptive statistics. The visualizations of $r_{t+1}^{P-TMV}/r_{t+1}^{P-LAMV}$ confirms that it has an asymmetric distribution with a long right tail (Figure 4). The ratio $\sigma_{t+1}^{P-TMV}/\sigma_{t+1}^{P-LAMV}$ from Panel B in Table 5, on the other hand, has a mean value less than 1 for all portfolio pairs (0.81 to 0.89), and a maximum value far less than 10 (1.61 to 2.95), indicating again that the LAMV portfolios actually have higher level of portfolio volatility than their corresponding TMV portfolios, which is consistent with the portfolio descriptive statistics. The visualization of the ratio $\sigma_{t+1}^{P-TMV}/\sigma_{t+1}^{P-LAMV}$ (Figure 5) confirms that it has a less asymmetric and relatively narrow distribution. Figure 6 provides the scatter plot of the 2D distribution of the two ratios ($r_{t+1}^{P-TMV}/r_{t+1}^{P-LAMV}$ vs. $\sigma_{t+1}^{P-TMV}/\sigma_{t+1}^{P-LAMV}$), which confirms that the liquidity adjustment essentially removes the portfolio jumps, and essentially all the portfolio jumps, while its impact on portfolio volatility is not obvious.

6.7 Performance Comparisons of Portfolios

We use the annualized Sharpe Ratio (SR_a) to compare the performance between portfolios:

$$SR_a = \frac{r_a^P - r_a^{rf}}{\sigma_a^P} = \frac{r_a^P - r_a^{USDT}}{\sigma_a^P} = \frac{r_a^P}{\sigma_a^P} \tag{25}$$

Where:
1. r_a^P, σ_a^P are the annualized realized regular daily portfolio return and standard deviation.
2. r_a^{rf} and r_a^{USDT} are the annualized realized daily returns for the risk-free asset and USDT, respectively.

Table 6 summarizes the performance of all 12 portfolios. The LAMV portfolios demonstrate a clear incremental improvement with each enhancement, while the TMV portfolios exhibit a performance deterioration in general. Among the TMV portfolios, the standard MV portfolio

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(Portfolio 5) without any forecast enhancement has a SR of 1.32. With $\hat{\Sigma}_{r_{t+1}}^{TT}$ replacing $\bar{\Sigma}_t$, Portfolio 7 has a marginally improved SR of 1.44 from Portfolio 5. This indicates that, the forecast produced by VECM-DCC/ADCC-BL models with no liquidity adjustment, $\hat{\Sigma}_{r_{t+1}}^{TT}$, does offer somewhat improved perspective of the portfolio covariance on day $t+1$ than the covariance matrix of the rolling window, $\bar{\Sigma}_t$, indicating that VECM-DCC/ADCC-BL models are robust but not fully effective without liquidity adjustment. In parallel, with the mean return vector $\bar{\mu}_{r_t}$ being replaced by $\hat{\mu}_{r_{t+1}}^{arga}$, Portfolio 9 has a drastic performance decline from Portfolio 5, with the SR dropping to 0.78. This result supports Deng and Zhou (2024a) that without the embedded information on liquidity, the forecast of ARMA-GARCH/EGARCH models on r_{t+1} is actually worse than just averaging r_t's with a rolling window. Enhanced by both forecasts ($\hat{\Sigma}_{r_{t+1}}^{TT}$ and $\hat{\mu}_{r_{t+1}}^{arga}$), Portfolio 11 has a comparable but slightly lower SR (0.75) than Portfolio 9, indicating that any improvement from the VECM-DCC/ADCC-BL models is overwhelmed by the deterioration brought about by the ARMA-GARCH/EARCH models. It is interesting that the VECM-DCC/ADCC-BL models are more robust than the ARMA-GARCH/EGARCH models, and the reason is that, again, the estimation of covariance matrix increment has an additional layer of reducing high portfolio liquidity volatility ($\left|B_{\sigma_t}^{\mathcal{P}\ell}\right| \gg 1$) ($\Delta\hat{\Sigma}_{r_{t+1}^\ell}^{TT}$ in Equation 20), which may further reduce jumps.

On the other hand, each incremental forecast enhancement to the LAMV portfolios improves performance. Portfolio 6 is the standard LAMV portfolio with a SR of 1.41. Portfolio 8 incorporates $\hat{\Sigma}_{r_{t+1}^\ell}^{TT}$, with a greatly improved the SR of 1.87, indicating that the liquidity-adjusted VECM-DCC/ADCC-BL is highly effective in modeling the liquidity-adjusted daily covariance with the additional information on liquidity. In parallel, Portfolio 10 incorporates $\hat{\mu}_{r_{t+1}^\ell}^{arga}$ and raises the SR to 1.81 from Portfolio 6. This result provides a clear empirical support that the liquidity-

adjusted ARMA-GARCH/EGARCH is more effective in modeling the liquidity-adjusted return r_t^ℓ, which is consistent with Deng and Zhou (2024a). It is worth noticing that Portfolio 8 has a higher SR (1.87) than Portfolio 10 (SR=1.81), for the reason, yet again, that the VECM-DCC/ADCC-BL models reduce both jump and volatility in portfolio covariance matrix, which the ARMA-GARCH/EGARCH models only reduce jump in portfolio return, and therefore the former are more robust. Finally, Portfolio 12 that incorporates both $\hat{\Sigma}_{r_{t+1}^\ell}^{TT}$ and $\hat{\mu}_{r_{t+1}^\ell}^{arga}$ has a SR of 1.85, which is comparable to that of Portfolio 8 (SR=1.87) but not higher. However, it is preferable as it incorporates enhancements from both the models, and thus more robust.

We now compare the TMV and LAMV portfolios with the same forecast enhancement. From Table 6, the standard LAMV Portfolio 6 has a marginally higher SR than the standard TMV Portfolio 5 (1.41 vs. 1.32). For the VECM/VAR-DCC/ADCC-BL-enhanced portfolios, the LAMV Portfolio 8 has a clear advantage over the TMV Portfolio 7 with much higher SR (1.87 vs. 1.44). For the ARMA-GARCH/EGARCH-enhanced portfolios, the LAMV Portfolio 10 clearly has superior performance over the TMV Portfolio 9 with a significantly higher SR (1.81 vs. 0.78). And for the portfolios with both enhancements, the LAMV Portfolio 12 also demonstrates vastly superior performance over the TMV Portfolio 11 through SR (1.85 vs. 0.75). As the only difference between the corresponding TMV and LAMV portfolios is that the latter is enhanced with liquidity adjustment, we conclude that it is the sole source of superior performance. In addition, the LAMV models enhanced by either autoregressive models have better performance than the standard LAMV model, as the forecasts by these models offer better perspectives on the variables they model than simply calculating the mean return or the covariance matrix in a rolling window. The visualizations of the portfolio performance are given in Figure 7.

In summary, based on the portfolio performance, the VECM/VAR-DCC/ADCC-BL models offer improved predictability for the liquidity-adjusted daily conditional covariance $\hat{\Omega}_{t+1}^\ell$, which in turn produces less irrational expected daily increment in the daily covariance matrix $\hat{\Sigma}_{r_{t+1}^\ell}^{TT}$, and ultimately improves the portfolio performance through the LAMV constructs.

7. Conclusions

In this paper, we thoroughly examine how the distribution of liquidity-adjusted return and volatility affects the predictability of assets through the portfolio-level liquidity-adjusted VECM-DCC/ADCC-BL models. To empirically test whether the liquidity-adjusted autoregressive models improve predictability, we construct a set of MV portfolios to compare their performance.

We use a selected portfolio of established crypto assets that are not overly exposed to wash trades. We use trading data of the crypto assets to construct and derive the portfolio-level daily return vector and covariance matrix $(Q_t^\ell, \Sigma_{r_t}^\ell)$, based on which we establish the portfolio liquidity Beta matrix pair $(B_{r_t}^{\mathcal{P}\ell}, B_{\sigma_t}^{\mathcal{P}\ell})$, as well as the portfolio liquidity jump and diffusion, $(\left|B_{r_t}^{\mathcal{P}\ell}\right|, \left|B_{\sigma_t}^{\mathcal{P}\ell}\right|)$. The portfolio liquidity jump $\left|B_{r_t}^{\mathcal{P}\ell}\right|$ measures the magnitude of portfolio liquidity, and the portfolio liquidity diffusion $\left|B_{\sigma_t}^{\mathcal{P}\ell}\right|$ measures the volatility of portfolio liquidity. We find that the distribution of portfolio liquidity jump $\left|B_{r_{TT}}^{\mathcal{P}\ell}\right|$ is highly asymmetric with long right tail than that of the asset-level liquidity jump $\beta_{r_t}^\ell$, and that the distribution of portfolio liquidity diffusion $\left|B_{\sigma_{TT}}^{\mathcal{P}\ell}\right|$ is more asymmetric than that of the asset-level liquidity diffusion $\beta_{\sigma_{TT}}^\ell$.

We develop a set of multivariate VECM-DCC/ADCC-BL models at extreme liquidity to predict the daily conditional covariate matrix $(\hat{\Omega}_{t+1}^\ell)$ and to estimate the posterior daily covariance matrix $(\hat{\Sigma}_{r_{t+1}^\ell}^{TT})$. In order to empirically prove that these models offer improved predictability at extreme

liquidity, we incorporate the forecasted portfolio covariance matrix $\hat{\Sigma}_{r_{t+1}^{TT}}^{TT}$ as input to a LAMV construct for portfolio optimization. Furthermore, we incorporate the forecasted return of individual assets from the ARMA-GARCH/EGARCH models of Deng and Zhou (2024a) $\hat{\mu}_{r_{t+1}^\ell}^{arga}$, as well as both the forecasted return and covariance, as inputs to a set of LAMV constructs, so that we can evaluate the effects of forecast enhancements on both the asset and portfolio levels. We repeat the procedure to produce TMV constructs with no liquidity adjustment so that we can evaluate the effectiveness of liquidity adjustment by comparing the LAMV constructs and their TMV equivalents. We find that the LAMV constructs have a clear advantage over their TMV counterparts in portfolio performance, which provides empirical support to that the liquidity adjustment greatly reduces the discontinuity in portfolio conditional covariance and posterior covariance matrices, and in turn restores the effectiveness of the liquidity-adjusted VECM-DCC/ADCC-BL models with much improved predictability at extreme liquidity.

We establish that the portfolio liquidity jump $\left|B_{r_t}^{\mathcal{P}\ell}\right|$ quantifies the effect of liquidity adjustment in forecasting the portfolio conditional covariance, while portfolio liquidity diffusion $\left|B_{\sigma_t}^{\mathcal{P}\ell}\right|$ quantifies the effect of liquidity adjustment in estimating the daily covariance matrix. We demonstrate that the VECM-DCC/ADCC-BL models are more robust than the ARMA-GARCH/EGARCH models, as the former reduces both the portfolio liquidity magnitude ($\left|B_{r_t}^{\mathcal{P}\ell}\right|$) and high portfolio liquidity volatility ($\left|B_{\sigma_t}^{\mathcal{P}\ell}\right|$) in the estimation of posterior covariance matrix increment, while the latter only reduces the portfolio liquidity magnitude in forecasting the portfolio conditional covariance matrix. Our models provide a viable and robust alternative for modeling portfolio jumps and can be utilized to model other asset classes with high liquidity risk.

Appendix 1 – Deriving the Portfolio Liquidity Volatility Beta Matrix $B_{\sigma_t}^{\mathcal{P}\ell}$

Deng (2024) proposes a Conditional Singular Value Decomposition (Conditional SVD) in the form of $A_{\{mn\}} = H_{\{mk\}} B_{\{kl\}} M_{\{ln\}}^*$ for given general matrices $A_{\{mn\}}$ and $B_{\{kl\}}$, and provides a special case, that when $m = n = k = l$, a reduced conditional SVD of the following exists:

$$A = HBH^*; \quad where: A, B \in \mathbb{C}^{n \times n} \tag{A1-1}$$

A and B have the SVD decompositions as:

$$A = U_A \Sigma_A U_A^*; B = U_B \Sigma_B U_B^* \tag{A1-2a, 2b}$$

> where:
> $U's$ are square complex unitary matrices; $\Sigma's$ are rectangular diagonal matrices with non-negative real numbers.

And there exists a decomposition between Σ_A and Σ_B as:

$$\Sigma_A = R\Sigma_B R^* \Rightarrow \Sigma_A = RR^* \Sigma_B = RR\Sigma_B \Rightarrow R = (\Sigma_A \Sigma_B^{-1})^{\frac{1}{2}} \tag{A1-3}$$

> where: R is a diagonal matrix with real numbers

By substituting Σ_A in Equation A1-2a with the RHS of Equation A1-3 we get:

$$A = U_A(R\Sigma_B R^*)U_A^* \Rightarrow A = (U_A R)\Sigma_B (U_A R)^* \tag{A1-4a}$$

Also, substitute B in Equation A1-1 with the RHS of Equation A1-2b:

$$A = H(U_B \Sigma_B U_B^*)H^* \Rightarrow A = (HU_B)\Sigma_B (HU_B)^* \tag{A1-4b}$$

By comparing Equation A1-4a and Equation A1-4b we get:

$$U_A R = HU_B \Rightarrow H = U_A R U_B^* \Rightarrow H = U_A (\Sigma_A \Sigma_B^{-1})^{\frac{1}{2}} U_B^* \tag{A1-5}$$

Equation A1-5 solves H in proposition $A = HBH^*$, with HH^* being a symmetric matrix.

Equation 7 in Subsection 5.2 is a special case of the special case in Deng (2024), in which A, B are symmetric square matrices with non-negative elements, and non-zero values on the diagonal:

Let $\Sigma_{r_t}^{TT} \equiv A$, $\Sigma_{r_t^\ell}^{TT} \equiv B$ and $B_{\sigma_t}^{\mathcal{P}\ell} \equiv H$, we get:

$$\Sigma_{r_t}^{TT} = B_{\sigma_t}^{\mathcal{P}\ell} \Sigma_{r_t^\ell}^{TT} B_{\sigma_t}^{\mathcal{P}\ell H}; \quad where: \Sigma_{r_t}^{TT} = U_A \Sigma_A U_A^H; \Sigma_{r_t^\ell}^{TT} = U_B \Sigma_B U_B^H; \Sigma_A = R\Sigma_B R^H \tag{A1-6a}$$

$$B_{\sigma_t}^{\mathcal{P}\ell} = U_A R U_B^{-1} \tag{A1-6b}$$

Appendix 2 –Liquidity-Adjusted VECM-DCC/ADCC-BL Models at Extreme Liquidity

A2.1 Regular VECM/VAR-DCC/ADCC-BL Models at Extreme Liquidity

The full expression of regular VECM/VAR(p) specification given by Equation 13 is:

$$Q_t = \sum_{i=1}^{p} \Phi_i Q_{t-i} + E_t \tag{A2-1}$$

$$\Delta Q_t = \Gamma Q_{t-1} + \sum_{i=1}^{p-1} \Phi_i^* \Delta Q_{t-i} + E_t$$

$$\Phi_i^* = -\sum_{j=i+1}^{p} \Phi_i, i = 1, \dots, p-1$$

$$\Gamma = -\left(I - \sum_{i=1}^{p} \Phi_i\right) = -\Phi(1)$$

Where Q_t is the portfolio return vector, E_t is the residual vector, and Φ_i is the coefficient matrix for VAR lag i.

The regular VECM/VAR(p) specification of Equation A1 produces a one-period ($t+1$) forecasted return vector, $\hat{Q}_{t+1}$, of which the residual vector, $\hat{E}_{t+1}$, is given by Equation 14:

$$\hat{E}_{t+1} = \hat{Q}_{t+1} - Q_{t+1} \tag{A2-2}$$

Where Q_{t+1} is the actual observed return vector (out-of-sample observation) at time t+1.

We then apply a DCC(1,1) specification to estimate the time-varying conditional covariance in the residual error vector $\hat{E}_{t+1}$. The DCC(1,1) specification is given as (Deng, 2018, Equation 5, with modifications on symbols):

$$\hat{E}_{t+1}|\Psi_t \sim N\left(0, \hat{\Omega}_{t+1} = \hat{H}_{t+1}\hat{P}_{t+1}\hat{H}_{t+1}\right) \tag{A2-3a}$$

$$\hat{H}_{t+1}^2 = H_0^2 + K E_t E_t^H + \Lambda H_t^2$$

$$\hat{P}_{t+1} = \hat{O}_{t+1}^* \hat{O}_{t+1} \hat{O}_{t+1}^*$$

$$\hat{O}_{t+1} = (1 - a - b)\bar{O} + a \Xi_t \Xi_t^H + b O_t$$

$$\Xi_t = H_t^{-1} E_t$$

$$a + b < 1$$

Where:
1) *$\hat{E}_{t+1}$ is the conditional residual vector from the VECM/VAR stage;*
2) *$\hat{\Omega}_{t+1}$ is the conditional covariance matrix of $\hat{E}_{t+1}$;*
3) *$\hat{H}_{t+1}$ is the normalization matrix for $\hat{P}_{t+1}$;*
4) *K and Λ are diagonal coefficient matrices for H_t;*
5) *$\hat{P}_{t+1}$ is the conditional correlation matrix of $\hat{E}_{t+1}$;*
6) *$\hat{O}_{t+1}$ and $\hat{O}_{t+1}^*$ are estimator matrices for $\hat{P}_{t+1}$;*
7) *$\bar{O}$ is the unconditional correlation matrix of E_t;*
8) *Ξ_t is the standardized residual vector of E_t.*

In order to accommodate asymmetries among conditional covariance and structural break induced conditional correlation increase, we also apply an ADCC(1,1) specification to $\hat{E}_{t+1}$. The ADCC(1,1) can be regarded as the multivariate variation of EGARCH(1,1) in that coefficient a captures the residual's magnitude effect and that g reflects its sign impact (Deng, 2018, Equation 6, with modifications on symbols):

$$\hat{E}_{t+1}|\Psi_t \sim N\left(0, \hat{\Omega}_{t+1} = \hat{H}_{t+1}\hat{P}_{t+1}\hat{H}_{t+1}\right) \qquad (A2\text{-}3b)$$

$$\hat{H}_{t+1}^2 = H_0^2 + KE_tE_t^H + \Lambda H_t^2$$

$$\hat{P}_{t+1} = \hat{O}_{t+1}^* \hat{O}_{t+1} \hat{O}_{t+1}^*$$

$$\hat{O}_{t+1} = (1 - a - b)\bar{O} - g\bar{N} + a\Xi_t\Xi_t^H + bO_t + gN_tN_t^H$$

$$\Xi_t = H_t^{-1}E_t$$

$$N_t = I[\xi_{i,t} < 0] \circ \Xi_t$$

$$a + b + g < 1$$

Where:
1) *N_t augments the asymmetric effect of the negative elements $\xi_{i,t} < 0$ in Ξ_t;*
2) *the matrix operator "∘" is the Hadamard product of two identically sized matrices/vectors, computed simply by element-wise multiplication;*
3) *all other parameters are defined the same way as in Equation A4.*

For each rolling window, we fit both DCC(1,1) and ADCC(1,1) on $\hat{E}_{t+1}$, and choose either DCC(1,1) or ADCC(1,1) with a higher log-likelihood to produce the conditional covariance matrix, $\hat{\Omega}_{t+1}$. With the forecasted $\hat{\Omega}_{t+1}$ we further estimate the posterior (forecasted) daily covariance matrix for day $t+1$, $\hat{\Sigma}_{r_{t+1}}^{TT}$, which is analytically expressed as (Deng, 2018, Equation 8, with modifications on symbols):

$$\hat{\Sigma}_{r_{t+1}}^{TT} = \Sigma_{r_t}^{TT} + \hat{M}_{t+1}^{-1} \qquad (A2\text{-}4a)$$

$$\hat{M}_{t+1}^{-1} = \left[\left(\tau\Sigma_{r_t}^{TT}\right)^{-1} + \widehat{Pm}_{t+1}^H \hat{\Omega}_{t+1}^{-1} \widehat{Pm}_{t+1}\right]^{-1} = \left[\left(\tau\Sigma_{r_t}^{TT}\right)^{-1} + I_{N\times N}\hat{\Omega}_{t+1}^{-1}I_{N\times N}\right]^{-1} = \left[\left(\tau\Sigma_{r_t}^{TT}\right)^{-1} + \hat{\Omega}_{t+1}^{-1}\right]^{-1} \quad (A2\text{-}4b)$$

where:
1) *$\hat{M}_{t+1}^{-1}$ is the adjustment to the covariance matrix at time t for the next time period t+1;*
2) *$\widehat{Pm}_{t+1}$ is the 1-period forward estimated weight matrix representing the investor's views and companion of $\hat{Q}_{t+1}$, thus it is a N × N matrix. Since $\hat{Q}_{t+1}$ is "absolute," as it is forecasted in an objective fashion, $\widehat{Pm}_{t+1}$ is an identity matrix of order N, $I_{N\times N}$, N is the number of assets in the portfolio;*
3) *τ is the "confidence" parameter for the forecasted values. It is typically between 0.01 and 10, and we choose a value of 1.0 through experimentation.*

We thus consolidate Equations A2-4a and A2-4b into Equation A2-4 and estimate the posterior regular daily regular covariance matrix for day $t+1$ as:

$$\hat{\Sigma}_{r_{t+1}}^{TT} = \Sigma_{r_t}^{TT} + \left[\left(\tau\Sigma_{r_t}^{TT}\right)^{-1} + \hat{\Omega}_{t+1}^{-1}\right]^{-1} \tag{A2-4}$$

Finally, we consolidate Equations A2-1, A2-2, A2-3a/3b and A2-4b of the regular VECM-DCC/ADCC model and the posterior regular daily covariance matrix at extreme liquidity as:

$$Q_t = \sum_{i=1}^{p} \Phi_i\, Q_{t-i} + E_t \tag{A2-1}$$

$$\hat{E}_{t+1} = \hat{Q}_{t+1} - Q_{t+1} \tag{A2-2}$$

$$\hat{E}_{t+1}|\Psi_t \sim N\left(0, \hat{\Omega}_{t+1}\right) \tag{A2-3}$$

$$\hat{\Sigma}_{r_{t+1}}^{TT} = \Sigma_{r_t}^{TT} + \left[\left(\tau\Sigma_{r_t}^{TT}\right)^{-1} + \hat{\Omega}_{t+1}^{-1}\right]^{-1} \tag{A2-4}$$

Where:
1) Q_t is the portfolio return vector, E_t is the residual vector, and Φ_i is the coefficient matrix for VAR lag I,
2) $\hat{Q}_{t+1}$ is the forecasted portfolio return vector (out-of-sample) at time $t+1$,
3) Q_{t+1} is the actual observed return vector (out-of-sample) at time $t+1$,
4) $\hat{E}_{t+1}$ is the conditional residual vector (out-of-sample) at time t_1, from the VECM/VAR stage,
5) $\hat{\Omega}_{t+1}$ is the conditional covariance matrix of $\hat{E}_{t+1}$ in the rolling window.

We then transfer Equations A2-1 to A2-4 back to Subsection 5.1 as Equations 12-15.

A2.2 Liquidity-Adjusted VECM/VAR-DCC/ADCC-BL Models at Extreme Liquidity

Equation 4 gives the connection between the daily regular and liquidity-adjusted return vectors, with the subscript substitution we have :

$$Q_t = B_{r_t}^{\mathcal{P}\ell} Q_t^\ell \tag{A2-5}$$

By substituting Q_t and Q_{t-i} in Equation A2-1 by the RHS of Equation A2-5 we get:

$$Q_t = \sum_{i=1}^{p} \Phi_i\, Q_{t-i} + E_t$$

$$\Rightarrow B_{r_t}^{\mathcal{P}\ell} Q_t^\ell = \sum_{i=1}^{p} \Phi_i\, B_{r_{t-i}}^{\mathcal{P}\ell} Q_{t-i}^\ell + E_t$$

$$\Rightarrow Q_t^\ell = \sum_{i=1}^{p} B_{r_t}^{\mathcal{P}\ell-1} \Phi_i\, B_{r_{t-i}}^{\mathcal{P}\ell} Q_{t-i}^\ell + B_{r_t}^{\mathcal{P}\ell-1} E_t$$

$$\Rightarrow Q_t^\ell = \sum_{i=1}^{p} \Phi_i^\ell\, Q_{t-i}^\ell + E_t^\ell \tag{A2-6a}$$

$$\text{where: } \Phi_i^\ell = B_{r_t}^{\mathcal{P}\ell-1} \Phi_i B_{r_{t-i}}^{\mathcal{P}\ell}; \ E_t^\ell = B_{r_t}^{\mathcal{P}\ell-1} E_t \tag{A2-6b}$$

The expected error vector $\hat{E}_{t+1}$ is given as:

$$\hat{E}_{t+1}^\ell = \hat{Q}_{t+1}^\ell - Q_{t+1}^\ell \tag{A2-7}$$

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The conditional covariance matrix $\hat{\Omega}_{t+1}$ of the expected error vector $\hat{E}_{t+1}$ is given by Equations A2-2 and A2-3 in DCC(1,1) and ADCC(1,1), respectively. Therefore, from Equation A2-6b we :

$$\hat{E}_{t+1}|\Psi_t \sim N(0, \hat{\Omega}_{t+1})$$
$$\Rightarrow B_{r_t}^{\mathcal{P}\ell}\hat{E}_{t+1}^{\ell}|\Psi_t \sim N(0, \hat{\Omega}_{t+1})$$
$$\Rightarrow \hat{E}_{t+1}^{\ell}|\Psi_t \sim N\left(0, B_{r_t}^{\mathcal{P}\ell-1}\hat{\Omega}_{t+1}\right)$$

We thus establish the following liquidity adjusted DCC/ADCC(1,1) under extreme liquidity:

$$\hat{E}_{t+1}^{\ell}|\Psi_t \sim N(0, \hat{\Omega}_{t+1}^{\ell}) \tag{A2-8a}$$
$$where: \hat{\Omega}_{t+1}^{\ell} = B_{r_t}^{\mathcal{P}\ell-1}\hat{\Omega}_{t+1} \Rightarrow \hat{\Omega}_{t+1} = B_{r_t}^{\mathcal{P}\ell}\hat{\Omega}_{t+1}^{\ell}$$

Taking advantage of that $B_{r_t}^{\mathcal{P}\ell}$ is a diagonal matrix, from the above equation we get:

$$\hat{\Omega}_{t+1} = B_{r_t}^{\mathcal{P}\ell\frac{1}{2}}B_{r_t}^{\mathcal{P}\ell\frac{1}{2}}\hat{\Omega}_{t+1}^{\ell}$$

$$where: B_{r_t}^{\mathcal{P}\ell\frac{1}{2}} \text{ is a diagonal matrix}$$

Furthermore, taking advantage of that $\hat{\Omega}_{t+1}^{\ell}$ and $\hat{\Omega}_{t+1}$ are conditional covariance matrices and therefore symmetric and that $B_{r_t}^{\mathcal{P}\ell\frac{1}{2}}$ is a diagonal matrix, and therefore $B_{r_t}^{\mathcal{P}\ell\frac{1}{2}}\hat{\Omega}_{t+1}^{\ell}$ is symmetric, we have the follows:

$$B_{r_t}^{\mathcal{P}\ell\frac{1}{2}}\hat{\Omega}_{t+1}^{\ell} = B_{r_t}^{\mathcal{P}\ell\frac{1}{2}^H}\hat{\Omega}_{t+1}^{\ell\,H} = \left(\hat{\Omega}_{t+1}^{\ell}B_{r_t}^{\mathcal{P}\ell\frac{1}{2}}\right)^H = \hat{\Omega}_{t+1}^{\ell}B_{r_t}^{\mathcal{P}\ell\frac{1}{2}}$$

$$\Rightarrow \hat{\Omega}_{t+1} = B_{r_t}^{\mathcal{P}\ell\frac{1}{2}}B_{r_t}^{\mathcal{P}\ell\frac{1}{2}}\hat{\Omega}_{t+1}^{\ell} = B_{r_t}^{\mathcal{P}\ell\frac{1}{2}}\left(B_{r_t}^{\mathcal{P}\ell\frac{1}{2}}\hat{\Omega}_{t+1}^{\ell}\right) = B_{r_t}^{\mathcal{P}\ell\frac{1}{2}}\hat{\Omega}_{t+1}^{\ell}B_{r_t}^{\mathcal{P}\ell\frac{1}{2}} \Rightarrow \hat{\Omega}_{t+1}^{\ell} = B_{r_t}^{\mathcal{P}\ell-\frac{1}{2}}\hat{\Omega}_{t+1}B_{r_t}^{\mathcal{P}\ell-\frac{1}{2}} \tag{A2-8b}$$

Also, Equation 7 gives the connection between the daily regular and liquidity-adjusted covariance matrices, and with substitution of scripts we have :

$$\Sigma_{r_t}^{TT} = B_{\sigma_t}^{\mathcal{P}\ell}\Sigma_{r_t^{\ell}}^{TT}B_{\sigma_t}^{\mathcal{P}\ell H} \Rightarrow \Sigma_{r_t^{\ell}}^{TT} = B_{\sigma_t}^{\mathcal{P}\ell-1}\Sigma_{r_t}^{TT}B_{\sigma_t}^{\mathcal{P}\ell H-1} \tag{A2-9}$$

In Equation A2-4 we estimate the regular posterior daily covariance matrix. Again we take similar approximation under extreme liquidity that when $\left|B_{\sigma_t}^{\mathcal{P}\ell}\right| \gg 1$ or $\left|B_{\sigma_t}^{\mathcal{P}\ell}\right| \ll 1$, $B_{\sigma_{t+1}}^{\mathcal{P}\ell} \approx B_{\sigma_t}^{\mathcal{P}\ell}$.

By substituting $\hat{\Sigma}^{TT}_{r_{t+1}}$, $\Sigma^{TT}_{r_t}$ with the RHS of Equation A2-9 in to Equation A2-4 we get the following modified regular daily covariance matrix, but expressed with liquidity-adjusted variable:

$$\hat{\Sigma}^{TT}_{r_{t+1}} = \Sigma^{TT}_{r_t} + \left[\left(\tau\Sigma^{TT}_{r_t}\right)^{-1} + \hat{\Omega}^{-1}_{t+1}\right]^{-1}$$

$$\Rightarrow B^{\mathcal{P}\ell^{-1}}_{\sigma_t}\hat{\Sigma}^{TT}_{r_{t+1}}B^{\mathcal{P}\ell^{H-1}}_{\sigma_t} = B^{\mathcal{P}\ell^{-1}}_{\sigma_t}\Sigma^{TT}_{r_t}B^{\mathcal{P}\ell^{H-1}}_{\sigma_t} + B^{\mathcal{P}\ell^{-1}}_{\sigma_t}\left[\left(\tau B^{\mathcal{P}\ell}_{\sigma_t}\Sigma^{TT}_{r^\ell_t}B^{\mathcal{P}\ell^{H}}_{\sigma_t}\right)^{-1} + \hat{\Omega}^{-1}_{t+1}\right]^{-1}B^{\mathcal{P}\ell^{H-1}}_{\sigma_t}$$

$$\Rightarrow \hat{\Sigma}^{TT}_{r^\ell_{t+1}} = \Sigma^{TT}_{r^\ell_t} + \left[B^{\mathcal{P}\ell^{H}}_{\sigma_t}\left(\left(\tau B^{\mathcal{P}\ell}_{\sigma_t}\Sigma^{TT}_{r^\ell_t}B^{\mathcal{P}\ell^{H}}_{\sigma_t}\right)^{-1} + \hat{\Omega}^{-1}_{t+1}\right)B^{\mathcal{P}\ell}_{\sigma_t}\right]^{-1}$$

$$\Rightarrow \hat{\Sigma}^{TT}_{r^\ell_{t+1}} = \Sigma^{TT}_{r^\ell_t} + \left[B^{\mathcal{P}\ell^{H}}_{\sigma_t}\left(\tau^{-1}B^{\mathcal{P}\ell^{H-1}}_{\sigma_t}\Sigma^{TT^{-1}}_{r^\ell_t}B^{\mathcal{P}\ell^{-1}}_{\sigma_t} + \hat{\Omega}^{-1}_{t+1}\right)B^{\mathcal{P}\ell}_{\sigma_t}\right]^{-1}$$

$$\Rightarrow \hat{\Sigma}^{TT}_{r^\ell_{t+1}} = \Sigma^{TT}_{r^\ell_t} + \left[\left(\tau\Sigma^{TT}_{r^\ell_t}\right)^{-1} + B^{\mathcal{P}\ell^{H}}_{\sigma_t}\hat{\Omega}^{-1}_{t+1}B^{\mathcal{P}\ell}_{\sigma_t}\right]^{-1}$$

$$\Rightarrow \hat{\Sigma}^{TT}_{r^\ell_{t+1}} = \Sigma^{TT}_{r^\ell_t} + \left[\left(\tau\Sigma^{TT}_{r^\ell_t}\right)^{-1} + \left(B^{\mathcal{P}\ell^{-1}}_{\sigma_t}\hat{\Omega}_{t+1}B^{\mathcal{P}\ell^{H-1}}_{\sigma_t}\right)^{-1}\right]^{-1}$$

$$\Rightarrow \hat{\Sigma}^{TT}_{r^\ell_{t+1}} = \Sigma^{TT}_{r^\ell_t} + \left[\left(\tau\Sigma^{TT}_{r^\ell_t}\right)^{-1} + \left(B^{\mathcal{P}\ell^{-1}}_{\sigma_t}\hat{\Omega}_{t+1}B^{\mathcal{P}\ell^{-1}^{H}}_{\sigma_t}\right)^{-1}\right]^{-1}$$

The above equation utilizes $B^{\mathcal{P}\ell^{H-1}}_{\sigma_t} = B^{\mathcal{P}\ell^{-1}^{H}}_{\sigma_t}$. We further substitute $\hat{\Omega}_{t+1}$ in the above equation with the RHS Equation A2-8b and get:

$$\Rightarrow \hat{\Sigma}^{TT}_{r^\ell_{t+1}} = \Sigma^{TT}_{r^\ell_t} + \left[\left(\tau\Sigma^{TT}_{r^\ell_t}\right)^{-1} + \left(B^{\mathcal{P}\ell^{-1}}_{\sigma_t}B^{\mathcal{P}\ell^{\frac{1}{2}}}_{r_t}\hat{\Omega}^\ell_{t+1}B^{\mathcal{P}\ell^{\frac{1}{2}}}_{r_t}B^{\mathcal{P}\ell^{-1}^{H}}_{\sigma_t}\right)^{-1}\right]^{-1}$$

$$\Rightarrow \hat{\Sigma}^{TT}_{r^\ell_{t+1}} = \Sigma^{TT}_{r^\ell_t} + \left[\left(\tau\Sigma^{TT}_{r^\ell_t}\right)^{-1} + \left(\left(B^{\mathcal{P}\ell^{-1}}_{\sigma_t}B^{\mathcal{P}\ell^{\frac{1}{2}}}_{r_t}\right)\hat{\Omega}^\ell_{t+1}\left(B^{\mathcal{P}\ell^{-1}}_{\sigma_t}B^{\mathcal{P}\ell^{\frac{1}{2}}}_{r_t}\right)^{H}\right)^{-1}\right]^{-1}$$

$$\Rightarrow \hat{\Sigma}^{TT}_{r^\ell_{t+1}} = \Sigma^{TT}_{r^\ell_t} + \left[\left(\tau\Sigma^{TT}_{r^\ell_t}\right)^{-1} + \left(B^{\mathcal{P}\ell}_t\hat{\Omega}^\ell_{t+1}B^{\mathcal{P}\ell^{H}}_t\right)^{-1}\right]^{-1} \tag{A2-10}$$

$$\text{where: } B^{\mathcal{P}\ell}_t = B^{\mathcal{P}\ell^{-1}}_{\sigma_t}B^{\mathcal{P}\ell^{\frac{1}{2}}}_{r_t} \tag{A2-11}$$

In Equation A2-11, we established a "portfolio liquidity Beta matrix" $B^{\mathcal{P}\ell}_t$, which is the matrix product of the inverse of "portfolio liquidity premium Beta matrix" $B^{\mathcal{P}\ell}_{r_t}$ and the matrix square root of "portfolio liquidity volatility Beta matrix" $B^{\mathcal{P}\ell}_{\sigma_t}$. Equation A2-10 is the liquidity-adjusted posterior daily covariance matrix.

Consolidating the above equations, we get the liquidity-adjusted VECM-DCC/ADCC model and posterior daily covariance matrix at extreme liquidity as:

$$Q_t^\ell = \sum_{i=1}^p \Phi_i \, Q_{t-i}^\ell + E_t^\ell \tag{A2-6a}$$

$$\hat{E}_{t+1}^\ell = \hat{Q}_{t+1}^\ell - Q_{t+1}^\ell \tag{A2-7}$$

$$\hat{E}_{t+1}^\ell | \Psi_t \sim N\left(0, \hat{\Omega}_{t+1}^\ell\right) \tag{A2-8a}$$

$$\hat{\Sigma}_{r_{t+1}^\ell}^{TT} = \Sigma_{r_t^\ell}^{TT} + \left[\left(\tau\Sigma_{r_t^\ell}^{TT}\right)^{-1} + \left(B_t^{\mathcal{P}\ell}\hat{\Omega}_{t+1}^\ell B_t^{\mathcal{P}\ell H}\right)^{-1}\right]^{-1} \tag{A2-10}$$

where:

$$E_t^\ell = B_{r_t}^{\mathcal{P}\ell-1} E_t \tag{A2-6b}$$

$$\hat{\Omega}_{t+1}^\ell = B_{r_t}^{\mathcal{P}\ell-\frac{1}{2}} \hat{\Omega}_{t+1} B_{r_t}^{\mathcal{P}\ell-\frac{1}{2}} \tag{A2-8b}$$

$$B_t^{\mathcal{P}\ell} = B_{\sigma_t}^{\mathcal{P}\ell-1} B_{r_t}^{\mathcal{P}\ell\frac{1}{2}} \tag{A2-11}$$

We then transfer the above equation block back to Subsection 5.1 as Equations 12-18.

A2.3 Linkage and Comparisons of Posterior Daily Covariance Matrices

In Equations A2-4 and A2-10 we estimate the regular and liquidity-adjusted posterior daily covariance matrices, from which we estimate the expected increment of these matrices as follows:

$$\hat{\Sigma}_{r_{t+1}}^{TT} = \Sigma_{r_t}^{TT} + \left[\left(\tau\Sigma_{r_t}^{TT}\right)^{-1} + \hat{\Omega}_{t+1}^{-1}\right]^{-1} \Rightarrow \Delta\hat{\Sigma}_{r_{t+1}}^{TT} = \left[\left(\tau\Sigma_{r_t}^{TT}\right)^{-1} + \hat{\Omega}_{t+1}^{-1}\right]^{-1}$$

$$\hat{\Sigma}_{r_{t+1}^\ell}^{TT} = \Sigma_{r_t^\ell}^{TT} + \left[\left(\tau\Sigma_{r_t^\ell}^{TT}\right)^{-1} + \left(B_t^{\mathcal{P}\ell}\hat{\Omega}_{t+1}^\ell B_t^{\mathcal{P}\ell H}\right)^{-1}\right]^{-1} \Rightarrow \Delta\hat{\Sigma}_{r_{t+1}^\ell}^{TT} = \left[\left(\tau\Sigma_{r_t^\ell}^{TT}\right)^{-1} + \left(B_t^{\mathcal{P}\ell}\hat{\Omega}_{t+1}^\ell B_t^{\mathcal{P}\ell H}\right)^{-1}\right]^{-1}$$

In general, $\hat{\Omega}_{t+1}$ and $\hat{\Omega}_{t+1}^\ell$ are structed to be orders or magnitude smaller than $\Sigma_{r_t}^{TT}$ and $\Sigma_{r_t^\ell}^{TT}$, respectively. As such, with 1st order approximation, the above can be further simplified as:

$$\Delta\hat{\Sigma}_{r_{t+1}}^{TT} \sim \hat{\Omega}_{t+1} \tag{A2-10a}$$

$$\Delta\hat{\Sigma}_{r_{t+1}^\ell}^{TT} \sim B_t^{\mathcal{P}\ell}\hat{\Omega}_{t+1}^\ell B_t^{\mathcal{P}\ell H} = \left(B_{\sigma_t}^{\mathcal{P}\ell-1} B_{r_t}^{\mathcal{P}\ell\frac{1}{2}}\right)\left(B_{r_t}^{\mathcal{P}\ell-\frac{1}{2}}\hat{\Omega}_{t+1} B_{r_t}^{\mathcal{P}\ell-\frac{1}{2}}\right)\left(B_{\sigma_t}^{\mathcal{P}\ell-1} B_{r_t}^{\mathcal{P}\ell\frac{1}{2}}\right)^H = B_{\sigma_t}^{\mathcal{P}\ell-1}\hat{\Omega}_{t+1} B_{\sigma_t}^{\mathcal{P}\ell-1 H} \tag{A2-10b}$$

We then transfer Equations A2-10a/b back to Subsection 5.3 as Equations 21 and 22.

References

Al Janabi, M.A., 2011. Dynamic equity asset allocation with liquidity-adjusted market risk criterion: Appraisal of efficient and coherent portfolios. *J. of Asset Management*, *12*, pp.378-394.

Al Janabi, M.A., 2013. Optimal and coherent economic-capital structures: evidence from long and short-sales trading positions under illiquid market perspectives. *Annals of Operations Research*, *205*(1), pp.109-139.

Al Janabi, M.A., 2021. Multivariate portfolio optimization under illiquid market prospects: a review of theoretical algorithms and practical techniques for liquidity risk management. *Journal of Modelling in Management*, *16*(1), pp.288-309.

Al Janabi, M.A., Ferrer, R. and Shahzad, S.J.H., 2019. Liquidity-adjusted value-at-risk optimization of a multi-asset portfolio using a vine copula approach. *Physica A: Statistical Mechanics and its Applications*, *536*, p.122579.

Al Janabi, M.A., Hernandez, J.A., Berger, T. and Nguyen, D.K., 2017. Multivariate dependence and portfolio optimization algorithms under illiquid market scenarios. *European Journal of Operational Research*, *259*(3), pp.1121-1131.

Brandt, M.W., Goyal, A., Santa-Clara, P. and Stroud, J.R., 2005. A simulation approach to dynamic portfolio choice with an application to learning about return predictability. *The Review of Financial Studies*, *18*(3), pp.831-873.

Cappiello, L., Engle, R.F. and Sheppard, K., 2006. Asymmetric dynamics in the correlations of global equity and bond returns. *Journal of Financial Econometrics 4*(4), pp.537-572.

Çetin, U. and Rogers, L.C.G., 2007. Modeling liquidity effects in discrete time. *Mathematical Finance*, *17*(1), pp.15-29.

Chong, J. and Miffre, J., 2010. Conditional correlation and volatility in commodity futures and traditional asset markets. *Journal of Alternative Investments*, *12*(3), p.61.

Cong, F. and Oosterlee, C.W., 2016. Multi-period mean–variance portfolio optimization based on Monte-Carlo simulation. *Journal of Economic Dynamics and Control*, *64*, pp.23-38.

Cong, F. and Oosterlee, C.W., 2017. Accurate and robust numerical methods for the dynamic portfolio management problem. *Computational Economics*, *49*, pp.433-458.

Cong, L. W., Li, X., Tang, K., and Yang, Y., 2023. Crypto wash trading. *Management Science 69*, pp.6427-6454.

Deng, Q., 2018. A generalized VECM/VAR-DCC/ADCC framework and its application in the Black-Litterman model: Illustrated with a China portfolio. *China Finance Review International*, *8*(4), pp.453-467.

Deng, Q., 2024. A conditional singular value decomposition. Available at arXiv: https://doi.org/10.48550/arXiv.2403.09696.

Deng, Q. and Zhou, Z.G., 2024a. Liquidity Premium, Liquidity-Adjusted Return and Volatility, and Extreme Liquidity. Available at arXiv: https://doi.org/10.48550/arXiv.2306.15807.

Deng, Q. and Zhou, Z.G., 2024b. Liquidity Jump, Liquidity Diffusion, and Treatment on Wash Trading of Crypto Assets. Available at Research Gate: http://dx.doi.org/10.13140/RG.2.2.22833.60002.

Engle, R.F. and Granger, C.W., 1987. Co-integration and error correction: representation, estimation, and testing. *Econometrica: journal of the Econometric Society*, pp.251-276.

Engle, R.F. and Sheppard, K., 2001. Theoretical and empirical properties of dynamic conditional correlation multivariate GARCH. NBER Working Paper 8554.

Gaigi, M.H., Ly Vath, V., Mnif, M. and Toumi, S., 2016. Numerical approximation for a portfolio optimization problem under liquidity risk and costs. *Applied Math & Op*, *74*, pp.163-195.

Gârleanu, N. and Pedersen, L.H., 2013. Dynamic trading with predictable returns and transaction costs. *The Journal of Finance*, *68*(6), pp.2309-2340.

Hung, J.C., Su, J.B., Chang, M.C. and Wang, Y.H., 2020. The impact of liquidity on portfolio value-at-risk forecasts. *Applied economics*, *52*(3), pp.242-259.

Kolm, P. N., Tütüncü, R., and Fabozzi, F. J., 2014. 60 years of portfolio optimization: Practical challenges and current trends. *European Journal of Operational Research, 234(2)*, pp.356-371.

Lim, A.E. and Wimonkittiwat, P., 2014. Dynamic portfolio selection with market impact costs. *Operations Research Letters*, *42*(5), pp.299-306.

Ling, S. and McAleer, M., 2003. Asymptotic theory for a vector ARMA-GARCH model. *Econometric theory*, *19*(2), pp.280-310.

Lo, A.W., Petrov, C. and Wierzbicki, M., 2006. It's 11 Pm—Do you know where your liquidity is? The mean–variance–liquidity frontier. In *The World of Risk Management*, pp.47-92.

Ly Vath, V., Mnif, M. and Pham, H., 2007. A model of optimal portfolio selection under liquidity risk and price impact. *Finance and Stochastics*, *11*, pp.51-90.

Ma, J., Song, Q., Xu, J. and Zhang, J., 2013. Optimal portfolio selection under concave price impact. *Applied Mathematics & Optimization*, *67*(3), pp.353-390.

Mei, X., DeMiguel, V. and Nogales, F.J., 2016. Multiperiod portfolio optimization with multiple risky assets and general transaction costs. *Journal of Banking & Finance*, *69*, pp.108-120.

Phillips, P.C. and Ouliaris, S., 1990. Asymptotic properties of residual based tests for cointegration. *Econometrica: journal of the Econometric Society*, pp.165-193.

Vieira, E.B.F. and Filomena, T.P., 2020. Liquidity constraints for portfolio selection based on financial volume. *Computational Economics*, *56*(4), pp.1055-1077.

Vieira, E.B.F., Filomena, T.P., Sant'anna, L.R. and Lejeune, M.A., 2023. Liquidity-constrained index tracking optimization models. *Annals of Operations Research*, *330*(1), pp.73-118.

Weiß, G.N. and Supper, H., 2013. Forecasting liquidity-adjusted intraday value-at-risk with vine copulas. *Journal of Banking & Finance*, *37*(9), pp.3334-3350.

Zhang, R., Langrené, N., Tian, Y., Zhu, Z., Klebaner, F. and Hamza, K., 2019. Dynamic portfolio optimization with liquidity cost and market impact: a simulation-and-regression approach. *Quantitative Finance*, *19*(3), pp.519-532.

Table 1 – Descriptive Statistics of Liquidity Jump and Liquidity Diffusion

Panel A reports descriptive statistics of daily liquidity jump ($\beta_{r_t}^{\ell}$) for each crypto asset and $\left|B_{r_t}^{\mathcal{P}\ell}\right|$ on the portfolio level, Panel B reports descriptive statistics of daily liquidity diffusion ($\beta_{\sigma_t}^{\ell}$) for each crypto asset and $\left|B_{\sigma_t}^{\mathcal{P}\ell}\right|$ on the portfolio level, over the entire sample period. The maximum daily liquidity Beta is capped at 10. All ten crypto assets and the portfolio are measured with their trading pairs with Tether or USDT, a "stable coin" pegged to the US dollar, which is regarded as the "risk-free" asset in portfolios with a 0% interest rate in terms of their market values.

Panel A	liquidity jump ($\beta_{r_t}^{\ell}$ and $\left\|B_{r_t}^{\mathcal{P}\ell}\right\|$)										
ticker	ADA	BNB	BTC	ETC	ETH	LINK	LTC	MATIC	XMR	XRP	port
count	1749	1749	1749	1749	1749	1749	1749	1749	1749	1749	1749
mean	2.02	1.78	1.49	1.99	1.60	1.97	1.99	1.88	1.98	1.59	3.84
std	2.51	2.30	1.95	2.49	2.00	2.50	2.53	2.46	2.66	2.14	4.51
min	0.00	0.00	0.00	0.00	0.01	0.00	0.00	0.00	0.00	0.00	0.00
25%	0.58	0.52	0.56	0.52	0.59	0.56	0.49	0.48	0.41	0.48	0.02
50% (median)	1.06	1.00	0.91	1.03	0.98	1.00	1.01	1.00	0.90	0.85	0.63
75%	2.15	1.85	1.43	2.22	1.64	2.05	2.16	1.92	2.11	1.61	10.00
max	10.00	10.00	10.00	10.00	10.00	10.00	10.00	10.00	10.00	10.00	10.00
highest days (= max)	91	77	44	90	52	92	78	89	110	53	553
% of total days	5.20%	4.40%	2.52%	5.15%	2.97%	5.26%	4.46%	5.09%	6.29%	3.03%	31.62%
weight in beta	25.78%	24.77%	16.85%	25.90%	18.61%	26.67%	22.42%	27.12%	31.81%	19.10%	82.43%
highest days (>= mean)	463	459	412	472	451	458	479	448	453	443	660
% of total days	26.47%	26.24%	23.56%	26.99%	25.79%	26.19%	27.39%	25.61%	25.90%	25.33%	37.74%
highest days (>= 1)	921	874	772	906	854	876	880	870	817	731	820
% of total days	52.66%	49.97%	44.14%	51.80%	48.83%	50.09%	50.31%	49.74%	46.71%	41.80%	46.88%
lowest days (<= 0.10)	67	89	58	65	62	72	79	108	112	73	651
% of total days	3.83%	5.09%	3.32%	3.72%	3.54%	4.12%	4.52%	6.17%	6.40%	4.17%	37.22%

Panel B	liquidity diffusion ($\beta_{\sigma_t}^{\ell}$ and $\left\|B_{\sigma_t}^{\mathcal{P}\ell}\right\|$)										
ticker	ADA	BNB	BTC	ETC	ETH	LINK	LTC	MATIC	XMR	XRP	port
count	1749	1749	1749	1749	1749	1749	1749	1749	1749	1749	1749
mean	1.33	0.90	0.82	1.40	0.87	1.38	1.11	1.14	1.49	0.92	1.83
std	1.29	0.15	0.07	0.67	0.08	1.01	0.68	0.85	1.34	0.33	3.18
min	0.55	0.58	0.43	0.55	0.52	0.56	0.65	0.48	0.60	0.57	0.00
25%	0.87	0.81	0.78	0.98	0.82	0.88	0.88	0.86	0.98	0.82	0.19
50% (median)	0.98	0.86	0.82	1.18	0.86	1.01	0.96	0.93	1.12	0.87	0.32
75%	1.27	0.93	0.86	1.59	0.91	1.41	1.11	1.11	1.39	0.96	1.13
max	10.00	2.53	1.15	8.89	1.45	10.00	10.00	10.00	10.00	10.00	10.00
highest days (= max)	18	0	0	0	0	3	2	7	19	1	185
% of total days	1.03%	0.00%	0.00%	0.00%	0.00%	0.17%	0.11%	0.40%	1.09%	0.06%	10.58%
weight in beta	7.74%	0.00%	0.00%	0.00%	0.00%	1.24%	1.03%	3.52%	7.29%	0.62%	57.91%
highest days (>= mean)	362	597	840	589	759	451	452	397	353	564	363
% of total days	20.70%	34.13%	48.03%	33.68%	43.40%	25.79%	25.84%	22.70%	20.18%	32.25%	20.75%
highest days (>= 1)	821	261	18	1248	108	888	692	645	1262	320	456
% of total days	46.94%	14.92%	1.03%	71.36%	6.17%	50.77%	39.57%	36.88%	72.16%	18.30%	26.07%
lowest days (<= 0.10)	0	0	0	0	0	0	0	0	0	0	78
% of total days	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	0.00%	4.46%

Table 2 – Descriptive Statistics of Liquidity Jump and Liquidity Diffusion

This tables reports descriptive statistics portfolio liquidity jump $\left|B_{r_t}^{\mathcal{P}\ell}\right|$, portfolio liquidity diffusion $\left|B_{\sigma_t}^{\mathcal{P}\ell}\right|$, and portfolio liquidity combined $\left|B_t^{\mathcal{P}\ell}\right|$ over the entire sample period. The maximum daily portfolio liquidity Beta is capped at 10.

measures	daily portfolio liquidity		
	liquidity jump $B_{r_t}^{\mathcal{P}\ell}$	liquidity diffusion $\left|B_{\sigma_t}^{\mathcal{P}\ell}\right|$	liquidity combined $\left|B_t^{\mathcal{P}\ell}\right|$
count	1749	1749	1749
mean	3.84	1.83	3.30
std	4.51	3.18	3.85
min	0.00	0.00	0.00
25%	0.02	0.19	0.20
50% (median)	0.63	0.32	1.03
75%	10.00	1.13	6.72
max	10.00	10.00	10.00
highest days (= max)	553	185	302
% of total days	31.62%	10.58%	17.27%
weight in beta	82.43%	57.91%	52.35%
highest days (>= mean)	660	363	619
% of total days	37.74%	20.75%	35.39%
highest days (>= 1)	820	456	882
% of total days	46.88%	26.07%	50.43%
lowest days (<= 0.10)	651	78	336
% of total days	37.22%	4.46%	19.21%

Table 3 – Johansen Tests on Daily Regular Return (r_t) and Daily Liquidity-Adjusted Return (r_t^ℓ) with VECM-DCC/ADCC Forecast Accuracy (Frobenius Distance)

This table reports the Johansen tests on daily regular returns (r_t) and daily liquidity-adjusted returns (r_t^ℓ). Panel A reports the full-sample test results on daily regular returns (r_t), and the out-of-sample VECM-DCC/ADCC forecast accuracy (Frobenius distance) on daily regular minute-level (intraday) covariance $\hat{\Sigma}_{r_{t+1}}^{TT}$; Panel B reports the full-sample test results on daily liquidity-adjusted returns (r_t^ℓ), and the out-of-sample VECM-DCC/ADCC forecast accuracy (Frobenius distance) on daily liquidity-adjusted minute-level (intraday) covariance $\hat{\Sigma}_{r_{t+1}^\ell}^{TT}$. ***, **, and * stand for the 1%, 5%, and 10% significant levels, respectively.

Panel A	daily regular return (r_{TT}) portfolio		
tested VECM/VAR order (AIC)			5
	test statistic	critical values of test	
		10pct	5pct	1pct
r <=9	213.24***	10.49	12.25	16.26
r <=8	221.94***	16.85	18.96	23.65
r <=7	245.59***	23.11	25.54	30.34
r <=6	260.74***	29.12	31.46	36.65
r <=5	290.40***	34.75	37.52	42.36
r <=4	314.37***	40.91	43.97	49.51
r <=3	369.47***	46.32	49.42	54.71
r <=2	385.56***	52.16	55.50	62.46
r <=1	428.40***	57.87	61.29	67.88
r <=0	469.13***	63.18	66.23	73.73
tested DCC covariance matrix Frobenius Distance (1,383 days out of sample)			0.0149

Panel B	daily liquidity-adjusted return (r_{TT}^ℓ) portfolio		
tested VECM/VAR order (AIC)			5
	test statistic	critical values of test	
		10pct	5pct	1pct
r <=9	54.45***	10.49	12.25	16.26
r <=8	187.24***	16.85	18.96	23.65
r <=7	208.93***	23.11	25.54	30.34
r <=6	226.26***	29.12	31.46	36.65
r <=5	245.66***	34.75	37.52	42.36
r <=4	265.26***	40.91	43.97	49.51
r <=3	290.32***	46.32	49.42	54.71
r <=2	326.10***	52.16	55.50	62.46
r <=1	369.31***	57.87	61.29	67.88
r <=0	412.94***	63.18	66.23	73.73
tested DCC covariance matrix Frobenius Distance (1,383 days out of sample)			0.0589

Table 4 – Descriptive Statistics of Portfolio Return (r_{t+1}^P) and Volatility (σ_{t+1}^P)

This table reports the descriptive statistics of portfolio return (r_{t+1}^P) and volatility (σ_{t+1}^P) for all 12 portfolios.

Panel A	Portfolio Return r_{t+1}^P			
Portfolio Number	1	2	3	4
Portfolio Description	equ	mkt	blq	blq_inv
mean	0.33%	0.29%	0.31%	0.30%
median	0.39%	0.21%	0.31%	0.30%
max	**77.78%**	**86.25%**	**68.51%**	**70.45%**
Portfolio Number	5		6	
Portfolio Description	MV_rr		MV_rrlq	
mean	0.37%		0.36%	
median	0.18%		0.36%	
max	**134.96%**		**35.18%**	
Portfolio Number	7		8	
Portfolio Description	MV_dcc_best_rr		MV_dcc_best_rrlq	
mean	0.25%		0.40%	
median	0.25%		0.35%	
max	**73.51%**		**42.83%**	
Portfolio Number	9		10	
Portfolio Description	MV_arga_rr		MV_arga_rrlq	
mean	0.33%		0.40%	
median	0.25%		0.38%	
max	**69.77%**		**50.84%**	
Portfolio Number	11		12	
Portfolio Description	MV_intraday_dcc_best_rr		MV_intraday_dcc_best_rrlq	
mean	0.24%		0.39%	
median	0.27%		0.35%	
max	**73.37%**		**43.30%**	

Panel B	Portfolio Volatility σ_{t+1}^P			
Portfolio Number	1	2	3	4
Portfolio Description	equ	mkt	blq	blq_inv
mean	3.75%	3.52%	3.93%	3.92%
median	3.21%	3.00%	3.40%	3.36%
max	**31.83%**	**31.51%**	**34.48%**	**29.89%**
Portfolio Number	5		6	
Portfolio Description	MV_rr		MV_rrlq	
mean	3.56%		4.29%	
median	2.88%		3.64%	
max	**41.09%**		**40.73%**	
Portfolio Number	7		8	
Portfolio Description	MV_dcc_best_rr		MV_dcc_best_rrlq	
mean	3.74%		4.32%	
median	3.64%		3.69%	
max	**27.13%**		**38.79%**	
Portfolio Number	9		10	
Portfolio Description	MV_arga_rr		MV_arga_rrlq	
mean	3.53%		4.19%	
median	2.99%		3.60%	
max	**41.09%**		**41.09%**	
Portfolio Number	11		12	
Portfolio Description	MV_intraday_dcc_best_rr		MV_intraday_dcc_best_rrlq	
mean	3.72%		4.25%	
median	3.25%		3.65%	
max	**27.13%**		**39.91%**	

Table 5 – Comparisons of TMV vs. LAMV

This table reports the descriptive statistics of the ratio between the TMV and LAMV portfolio returns ($r_{t+1}^{P-TMV}/r_{t+1}^{P-LAMV}$), and the ratio between the TMV and LAMV portfolio volatilities ($\sigma_{t+1}^{P-TMV}/\sigma_{t+1}^{P-LAMV}$), for the TMV and LAMV portfolios with the same incremental forecast enhancement.

Panel A	$r_{t+1}^{P-TMV}/r_{t+1}^{P-LAMV}$			
ratio number	1	2	3	4
portfolios - TMV	5: MV_rr	7: MV_dcc _best_rr	9: MV_arga_rr	11. MV_intraday _dcc_best_rr
portfolios - LAMV	6: MV_rrlq	8: MV_dcc _best_rrlq	10. MV_arga_rrlq	12. MV_intraday _dcc_best_rrlq
count	1383	1383	1383	1383
mean	1.18	1.39	1.20	1.41
std	1.63	1.79	1.50	1.79
min	0.00	0.00	0.00	0.00
25%	0.41	0.61	0.51	0.63
50% (median)	0.86	0.93	0.93	0.95
75%	1.20	1.34	1.20	1.35
max	10	10	10	10
highest days (= max)	29	32	23	29
% of total days	2.10%	2.31%	1.66%	2.10%
weight in beta	17.84%	16.69%	13.90%	14.90%
highest days (>= mean)	357	319	350	320
% of total days	25.81%	23.07%	25.31%	23.14%
highest days (>= 1)	524	596	579	627
% of total days	37.89%	43.09%	41.87%	45.34%
lowest days (<= 0.10)	77	42	57	38
% of total days	5.57%	3.04%	4.12%	2.75%

Panel B	$\sigma_{t+1}^{P-TMV}/\sigma_{t+1}^{P-LAMV}$			
ratio number	1	2	3	4
portfolios - TMV	5: MV_rr	7: MV_dcc _best_rr	9: MV_arga_rr	11. MV_intraday _dcc_best_rr
portfolios - LAMV	6: MV_rrlq	8: MV_dcc _best_rrlq	10. MV_arga_rrlq	12. MV_intraday _dcc_best_rrlq
count	1383	1383	1383	1383
mean	0.81	0.87	0.84	0.89
std	0.30	0.26	0.29	0.27
min	0.00	0.00	0.00	0.00
25%	0.60	0.76	0.67	0.78
50% (median)	0.93	0.93	0.96	0.94
75%	1.02	1.02	1.03	1.03
max	1.61	2.95	1.67	2.91
highest days (= max)	0	0	0	0
% of total days	0.00%	0.00%	0.00%	0.00%
weight in beta	0.00%	0.00%	0.00%	0.00%
highest days (>= mean)	892	855	915	853
% of total days	64.50%	61.82%	66.16%	61.68%
highest days (>= 1)	427	419	518	452
% of total days	30.87%	30.30%	37.45%	32.68%
lowest days (<= 0.10)	19	21	16	21
% of total days	1.37%	1.52%	1.16%	1.52%

Table 6 – Performance Comparisons of TMV vs. LAMV

This table compares the performance of 12 portfolios, including four benchmark portfolios (1-4), four TMV portfolios (5, 7, 9, 11) and four LAMV portfolios (6, 8, 10, 12). The risk-free rate (R_f) is assumed to be zero.

	Portfolio Performance			
Portfolio Number	1	2	3	4
Portfolio Description	equ	mkt	blq	blq_inv
Annualized Return	130.81%	105.64%	104.04%	100.85%
Annualized Standard Deviation	90.28%	86.37%	92.34%	92.38%
Annualized Sharpe Ratio (Rf=0%)	**1.45**	**1.22**	**1.13**	**1.09**
Portfolio Number	5		6	
Portfolio Description	MV_rr		MV_rrlq	
Annualized Return	140.93%		135.16%	
Annualized Standard Deviation	106.91%		96.08%	
Annualized Sharpe Ratio (Rf=0%)	**1.32**		**1.41**	
Portfolio Number	7		8	
Portfolio Description	MV_dcc_best_rr		MV_dcc_best_rrlq	
Annualized Return	126.70%		177.31%	
Annualized Standard Deviation	88.04%		94.78%	
Annualized Sharpe Ratio (Rf=0%)	**1.44**		**1.87**	
Portfolio Number	9		10	
Portfolio Description	MV_arga_rr		MV_arga_rrlq	
Annualized Return	71.20%		173.35%	
Annualized Standard Deviation	91.70%		95.90%	
Annualized Sharpe Ratio (Rf=0%)	**0.78**		**1.81**	
Portfolio Number	11		12	
Portfolio Description	MV_intraday_dcc_best_rr		MV_intraday_dcc_best_rrlq	
Annualized Return	67.88%		172.47%	
Annualized Standard Deviation	91.04%		93.48%	
Annualized Sharpe Ratio (Rf=0%)	**0.75**		**1.85**	

Figure 1 – Distributions of Portfolio-Level $\left|B_{r_t}^{\mathcal{P}\ell}\right|$ and $\left|B_{\sigma_t}^{\mathcal{P}\ell}\right|$

The scatter plot on the top left (Plot A) provides the distribution of portfolio liquidity jump vs. portfolio liquidity diffusion $\left|B_{r_t}^{\mathcal{P}\ell}\right|$ (y-axis) vs. $\left|B_{\sigma_t}^{\mathcal{P}\ell}\right|$ (x-axis); the histogram on the top right (Plot B) illustrates the distribution of the portfolio liquidity jump $\left|B_{r_t}^{\mathcal{P}\ell}\right|$; the histogram on the bottom left (Plot C) illustrates the distribution of the portfolio liquidity diffusion $\left|B_{\sigma_t}^{\mathcal{P}\ell}\right|$; the histogram on the bottom right (Plot D) illustrates the distribution of the portfolio liquidity combined $\left|B_{t}^{\mathcal{P}\ell}\right|$, over the entire sample period. The maximum values of $\left|B_{t}^{\mathcal{P}\ell}\right|$, $\left|B_{r_t}^{\mathcal{P}\ell}\right|$ and $\left|B_{t}^{\mathcal{P}\ell}\right|$ are all capped at 10.
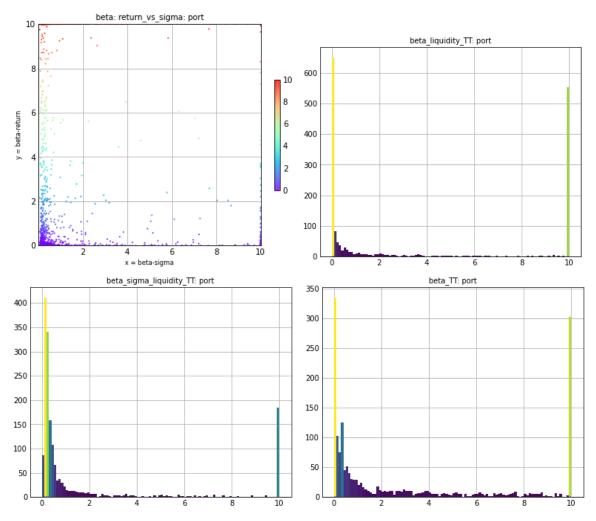
Figure 2 – Distribution of Portfolio Return r_{t+1}^P

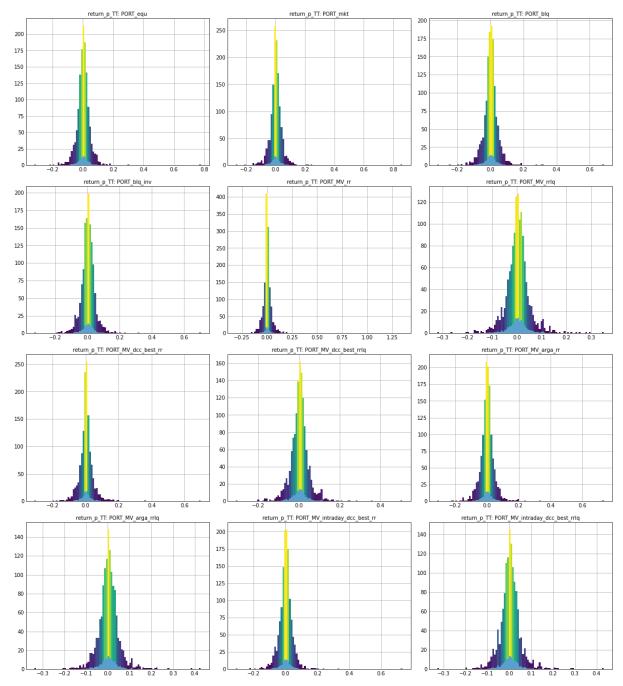This figure provides the histogram of daily portfolio return (r_{t+1}^P) for each of the 12 portfolios.

Figure 3 – Distribution of Portfolio Volatility σ_{t+1}^P

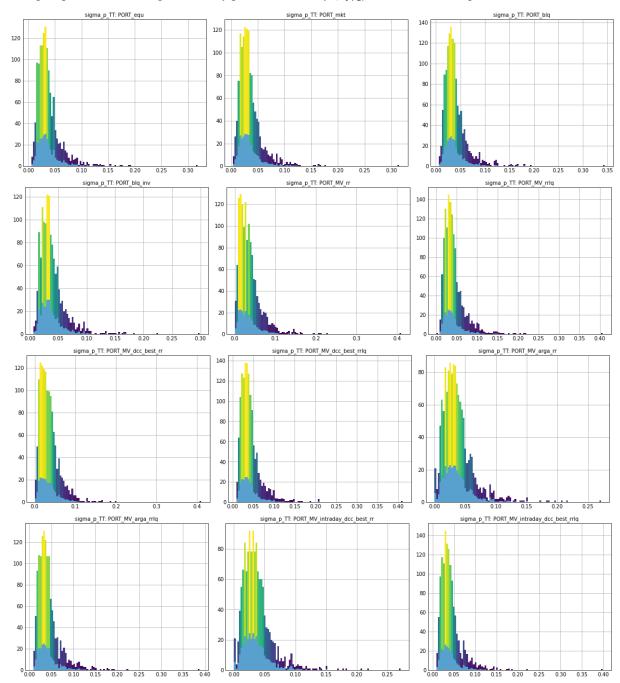This figure provides the histogram of daily portfolio volatility (σ_{t+1}^P) for each of the 12 portfolios.

Figure 4 – Distribution of Ratio $r_{t+1}^{P-TMV}/r_{t+1}^{P-LAMV}$

This figure provides visualizations of the ratio between the TMV and LAMV portfolio return ($r_{t+1}^{P-TMV}/r_{t+1}^{P-LAMV}$), for the TMV and LAMV portfolios with the same incremental forecast enhancement. Column 1 presents the histograms, Column 2 the scatter plots of (r_{t+1}^{P-TMV}-r_{t+1}^{P-LAMV}) overlapping a straight line with coefficient of 1.0, and Column 3 the 3D scatter plots of (r_{t+1}^{P-TMV}-r_{t+1}^{P-LAMV}-ratio)) with its 2D projection (r_{t+1}^{P-TMV}-r_{t+1}^{P-LAMV}).
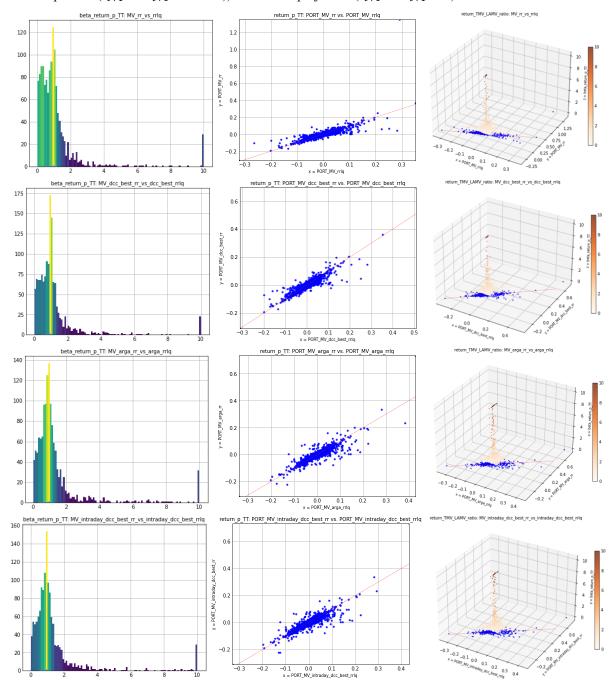
Figure 5 – Distribution of Ratio $\sigma_{t+1}^{P-TMV}/\sigma_{t+1}^{P-LAMV}$

This figure provides visualizations of the ratio between the TMV and LAMV portfolio volatility ($\sigma_{t+1}^{P-TMV}/\sigma_{t+1}^{P-LAMV}$), for the TMV and LAMV portfolios with the same incremental forecast enhancement. Column 1 presents the histograms, Column 2 the scatter plots of ($\sigma_{t+1}^{P-TMV}-\sigma_{t+1}^{P-LAMV}$) overlapping a straight line with coefficient of 1.0, and Column 3 the 3D scatter plots of ($\sigma_{t+1}^{P-TMV}-\sigma_{t+1}^{P-LAMV}$-ratio)) with its 2D projection ($\sigma_{t+1}^{P-TMV}-\sigma_{t+1}^{P-LAMV}$).
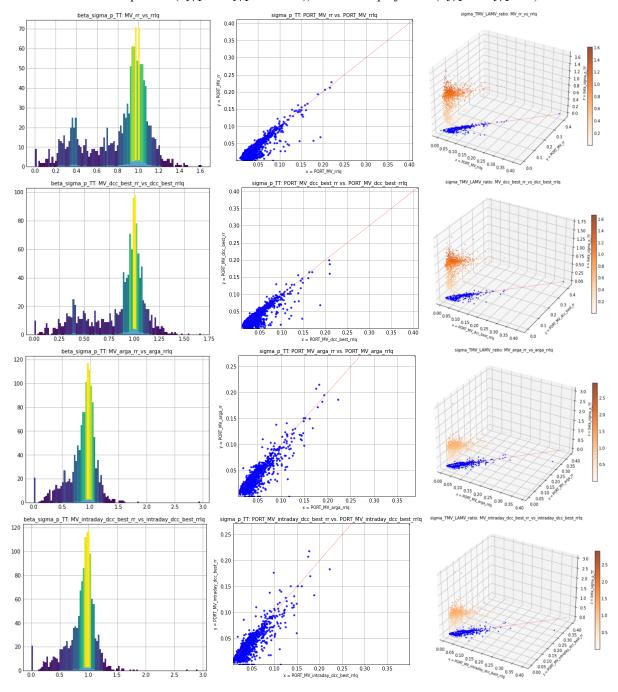
Figure 6 – Distribution of Portfolio Ratios
$$(r_{t+1}^{P-TMV}/r_{t+1}^{P-LAMV} \text{ vs. } \sigma_{t+1}^{P-TMV}/\sigma_{t+1}^{P-LAMV})$$

This figure provides the distributions of the ratios between the TMV and LAMV portfolio volatility TMV and LAMV portfolios, $r_{t+1}^{P-TMV}/r_{t+1}^{P-LAMV}$ (y-axis) vs. $\sigma_{t+1}^{P-TMV}/\sigma_{t+1}^{P-LAMV}$ (x-axis).
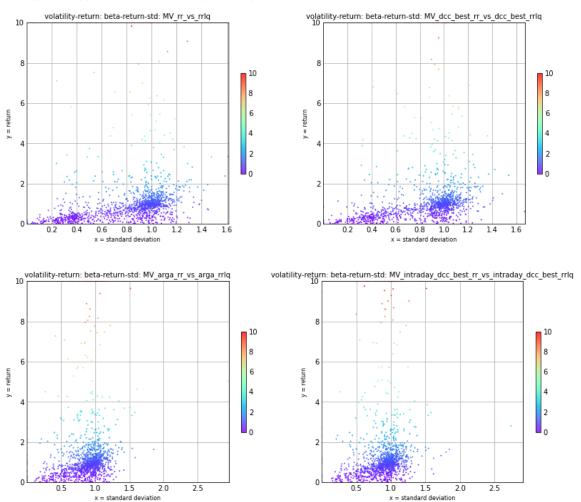
Figure 7 – Performance of Portfolios

Plot A provides visual comparison of the performance of four benchmark portfolios; Plot B provides visual comparison of the performance of four TMV portfolios; Plot C provides a visual comparison of the performance of four LAMV portfolios.