

DECORATED GEOMETRIC CRYSTALS, POLYHEDRAL AND MONOMIAL REALIZATIONS OF CRYSTAL BASES

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Dedicated to Professor Michio Jimbo on the occasion of his 60th birthday

ABSTRACT. We shall show that for type A_n the realization of crystal bases obtained from the decorated geometric crystals in [2] coincides with our polyhedral realizations of crystal bases. We also observe certain relations of decorations and monomial realizations of crystal bases.

1. INTRODUCTION

In [2], Berenstein and Kazhdan introduced the notion of decorated geometric crystals for reductive algebraic groups. Geometric crystals are geometric analogue to the Kashiwara's crystal bases ([1]). We, indeed, treated geometric crystals in the affine/Kac-Moody settings ([10, 11, 13]), but we do not need such general settings and then we shall consider the (semi-)simple settings below. Let I be a finite index set. Associated with a Cartan matrix $A = (a_{i,j})_{i,j \in I}$, define the decorated geometric crystal $\mathcal{X} = (\chi, f)$, which is a pair of geometric crystal $\chi = (X, \{e_i\}_i, \{\gamma_i\}_i, \{\varepsilon_i\}_i)$ and a certain special rational function f such that

$$f(e_i^c(x)) = f(x) + (c - 1)\varphi_i(x) + (c^{-1} - 1)\varepsilon_i(x),$$

for any $i \in I$, where e_i^c is the rational \mathbb{C}^\times action on X , and φ_i and $\varepsilon_i = \varepsilon_i \cdot \gamma_i$ are the rational functions on X .

If we apply the procedure called “ultra-discretization”(UD) to “positive geometric crystals” (see 3.3), then we would obtain certain free-crystals for the transposed Cartan matrix ([1, 13]). As for a positive decorated geometric crystal (χ, f, T', θ) applying UD to the function f and considering the convex polyhedral domain defined by the inequality $UD(f) \geq 0$, we get the crystal with the property “normal”([8]). Moreover, abstracting a connected component with the highest weight λ , we obtain the Langlands dual Kashiwara's crystal $B(\lambda)$ with the highest weight λ .

This result makes us recall the “polyhedral realization” of crystal bases ([14, 16]) since it has very similar way to get the crystal $B(\lambda)$ from certain free-crystals, defined by the system of linear inequalities. Thus, one of the main aims of this article is to show that the crystals obtained by UD from positive decorated geometric crystals and the polyhedral realizations of crystals coincide with each other for type A_n .

One more aim of this article is to describe the relations between the function f_B for certain decorated geometric crystal $(TB_{w_0}^-, f_B)$ and monomial realization of crystals ([9, 12]). We shall propose the conjecture of their relations and present the affirmative answer for type A_n . Let us mention the statement of the conjecture: for the function f_B and certain positive structure $T\Theta_1^-$ on $TB_{w_0}^-$, the

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function $f_B(t\Theta_1^-(c))$ is expressed as a sum of monomials in the crystal $\mathcal{Y}(p)$ with positive coefficients (for more details, see Conjecture 6.4 below.).

Observing this relation, we can deduce the refined polyhedral realization of crystals induced from the monomial realizations. Indeed, for the original polyhedral realizations we are forced the condition “ample”, which is some technical condition to guarantee the non-emptiness of the underlying crystal (see Theorem 2.5). But, if the relations among the polyhedral realizations, the UD of decorated geometric crystals and the monomial realizations are established, it would be possible to remove the condition “ample” and it would become easier to obtain polyhedral realizations of crystals than applying the present method.

The organization of the article is as follows: in Sect.2, we review the theory of crystals and their polyhedral realizations. In Sect.3, first we introduce the theory of decorated geometric crystals following [2]. Next, we define the decoration by using the elementary characters and certain special positive decorated geometric crystal on $\mathbb{B}_w = TB_w^-$. Finally, the ultra-discretization of TB_w^- is described explicitly. We calculate the function f_B exactly for type A_n in Sect.4. In Sect.5, for the type A_n the coincidence of the polyhedral realization $\Sigma_t[\lambda]$ and the ultra-discretization $B_{f_B, \Theta_{i_0}^-}(\lambda)$ will be clarified by using the result in Sect.4. In the last section, we review the monomial realization of crystals ([9, 12]) and the function f_B is expressed in terms of the monomials in the monomial realizations of crystals for type A_n . Finally, the conjecture is proposed and under the validity of the conjecture, we shall state the refined polyhedral realizations associated with the monomial realizations.

The results for other simple Lie algebras are mentioned in the forthcoming paper.

2. CRYSTAL AND ITS POLYHEDRAL REALIZATION

2.1. Notations. We list the notations used in this paper. Indeed, the settings below are originally Kac-Moody ones, but in the article we do not need them and then we restrict the settings to semi-simple ones. Let \mathfrak{g} be a semi-simple Lie algebra over \mathbb{Q} with a Cartan subalgebra \mathfrak{t} , a weight lattice $P \subset \mathfrak{t}^*$, the set of simple roots $\{\alpha_i : i \in I\} \subset \mathfrak{t}^*$, and the set of coroots $\{h_i : i \in I\} \subset \mathfrak{t}$, where I is a finite index set. Let $\langle h, \lambda \rangle = \lambda(h)$ be the pairing between \mathfrak{t} and \mathfrak{t}^* , and (α, β) be an inner product on \mathfrak{t}^* such that $(\alpha_i, \alpha_i) \in 2\mathbb{Z}_{\geq 0}$ and $\langle h_i, \lambda \rangle = \frac{2(\alpha_i, \lambda)}{(\alpha_i, \alpha_i)}$ for $\lambda \in \mathfrak{t}^*$ and $A := (\langle h_i, \alpha_j \rangle)_{i,j}$ is the associated Cartan matrix. Let $P^* = \{h \in \mathfrak{t} : \langle h, P \rangle \subset \mathbb{Z}\}$ and $P_+ := \{\lambda \in P : \langle h_i, \lambda \rangle \in \mathbb{Z}_{\geq 0}\}$. We call an element in P_+ a *dominant integral weight*. The quantum algebra $U_q(\mathfrak{g})$ is an associative $\mathbb{Q}(q)$ -algebra generated by the e_i, f_i ($i \in I$), and q^h ($h \in P^*$) satisfying the usual relations. The algebra $U_q^-(\mathfrak{g})$ is the subalgebra of $U_q(\mathfrak{g})$ generated by the f_i ($i \in I$).

For the irreducible highest weight module of $U_q(\mathfrak{g})$ with the highest weight $\lambda \in P_+$, we denote it by $V(\lambda)$ and its *crystal base* we denote $(L(\lambda), B(\lambda))$. Similarly, for the crystal base of the algebra $U_q^-(\mathfrak{g})$ we denote $(L(\infty), B(\infty))$ (see [6, 7]). Let $\pi_\lambda : U_q^-(\mathfrak{g}) \rightarrow V(\lambda) \cong U_q^-(\mathfrak{g})/\sum_i U_q^-(\mathfrak{g})f_i^{1+\langle h_i, \lambda \rangle}$ be the canonical projection and $\hat{\pi}_\lambda : L(\infty)/qL(\infty) \rightarrow L(\lambda)/qL(\lambda)$ be the induced map from π_λ . Here note that $\hat{\pi}_\lambda(B(\infty)) = B(\lambda) \sqcup \{0\}$.

By the terminology *crystal* we mean some combinatorial object obtained by abstracting the properties of crystal bases. Indeed, crystal constitutes a set B and the maps $wt : B \rightarrow P$, $\varepsilon_i, \varphi_i : B \rightarrow \mathbb{Z} \sqcup \{-\infty\}$ and $\tilde{e}_i, \tilde{f}_i : B \sqcup \{0\} \rightarrow B \sqcup \{0\}$ ($i \in I$) satisfying several axioms (see [8], [16], [14]). In fact, $B(\infty)$ and $B(\lambda)$ are the typical examples of crystals.

Let B_1 and B_2 be crystals. A *strict morphism* of crystals $\psi : B_1 \rightarrow B_2$ is a map $\psi : B_1 \sqcup \{0\} \rightarrow B_2 \sqcup \{0\}$ satisfying the following conditions: $\psi(0) = 0$, $wt(\psi(b)) = wt(b)$, $\varepsilon_i(\psi(b)) = \varepsilon_i(b)$, $\varphi_i(\psi(b)) = \varphi_i(b)$ if $b \in B_1$ and $\psi(b) \in B_2$, and the map $\psi : B_1 \sqcup \{0\} \rightarrow B_2 \sqcup \{0\}$ commutes with all \tilde{e}_i and \tilde{f}_i . An injective strict morphism is called an *embedding* of crystals.

It is well-known that $U_q(\mathfrak{g})$ has a Hopf algebra structure. Then the tensor product of $U_q(\mathfrak{g})$ -modules has a $U_q(\mathfrak{g})$ -module structure. The crystal bases have very nice properties for tensor operations.

Indeed, if (L_i, B_i) is a crystal base of $U_q(\mathfrak{g})$ -module M_i ($i = 1, 2$), $(L_1 \otimes_A L_2, B_1 \otimes B_2)$ is a crystal base of $M_1 \otimes_{\mathbb{Q}(q)} M_2$ ([7]). Consequently, we can consider the tensor product of crystals and then they constitute a tensor category.

2.2. Polyhedral Realization of $B(\infty)$. Let us recall the results in [16].

Consider the infinite \mathbb{Z} -lattice

$$(2.1) \quad \mathbb{Z}^\infty := \{(\cdots, x_k, \cdots, x_2, x_1) : x_k \in \mathbb{Z} \text{ and } x_k = 0 \text{ for } k \gg 0\};$$

we will denote by $\mathbb{Z}_{\geq 0}^\infty \subset \mathbb{Z}^\infty$ the subsemigroup of nonnegative sequences. To the rest of this section, we fix an infinite sequence of indices $\iota = \cdots, i_k, \cdots, i_2, i_1$ from I such that

$$(2.2) \quad i_k \neq i_{k+1} \text{ and } \#\{k : i_k = i\} = \infty \text{ for any } i \in I.$$

We can associate to ι a crystal structure on \mathbb{Z}^∞ and denote it by \mathbb{Z}_ι^∞ ([16, 2.4]).

Proposition 2.1 ([8], See also [16]). *There is a unique strict embedding of crystals (called Kashiwara embedding)*

$$(2.3) \quad \Psi_\iota : B(\infty) \hookrightarrow \mathbb{Z}_{\geq 0}^\infty \subset \mathbb{Z}_\iota^\infty,$$

such that $\Psi_\iota(u_\infty) = (\cdots, 0, \cdots, 0, 0)$, where $u_\infty \in B(\infty)$ is the vector corresponding to $1 \in U_q^-(\mathfrak{g})$.

Consider the infinite dimensional vector space

$$\mathbb{Q}^\infty := \{x = (\cdots, x_k, \cdots, x_2, x_1) : x_k \in \mathbb{Q} \text{ and } x_k = 0 \text{ for } k \gg 0\},$$

and its dual space $(\mathbb{Q}^\infty)^* := \text{Hom}(\mathbb{Q}^\infty, \mathbb{Q})$. We will write a linear form $\varphi \in (\mathbb{Q}^\infty)^*$ as $\varphi(x) = \sum_{k \geq 1} \varphi_k x_k$ ($\varphi_j \in \mathbb{Q}$) for $x \in \mathbb{Q}^\infty$.

For the fixed infinite sequence $\iota = (i_k)$ and $k \geq 1$ we set $k^{(+)} := \min\{l : l > k \text{ and } i_k = i_l\}$ and $k^{(-)} := \max\{l : l < k \text{ and } i_k = i_l\}$ if it exists, or $k^{(-)} = 0$ otherwise. We set for $x \in \mathbb{Q}^\infty$, $\beta_0(x) = 0$ and

$$(2.4) \quad \beta_k(x) := x_k + \sum_{k < j < k^{(+)}} \langle h_{i_k}, \alpha_{i_j} \rangle x_j + x_{k^{(+)}} \quad (k \geq 1).$$

We define the piecewise-linear operator $S_k = S_{k, \iota}$ on $(\mathbb{Q}^\infty)^*$ by

$$S_k(\varphi) := \begin{cases} \varphi - \varphi_k \beta_k & \text{if } \varphi_k > 0, \\ \varphi - \varphi_k \beta_{k^{(-)}} & \text{if } \varphi_k \leq 0. \end{cases}$$

Here we set

$$(2.5) \quad \Xi_\iota := \{S_{j_l} \cdots S_{j_2} S_{j_1} x_{j_0} \mid l \geq 0, j_0, j_1, \cdots, j_l \geq 1\},$$

$$(2.6) \quad \Sigma_\iota := \{x \in \mathbb{Z}^\infty \subset \mathbb{Q}^\infty \mid \varphi(x) \geq 0 \text{ for any } \varphi \in \Xi_\iota\}.$$

We impose on ι the following positivity assumption:

$$(2.7) \quad \text{if } k^{(-)} = 0 \text{ then } \varphi_k \geq 0 \text{ for any } \varphi(x) = \sum_k \varphi_k x_k \in \Xi_\iota.$$

Theorem 2.2 ([16]). *Let ι be a sequence of indices satisfying (2.2) and (2.7). Then we have $\text{Im}(\Psi_\iota) (\cong B(\infty)) = \Sigma_\iota$.*

2.3. Structure of $\mathbb{Z}_\iota^\infty[\lambda]$. Let $R_\lambda := \{r_\lambda\}$ be the crystal which consists of one element r_λ ([14]). Consider the crystal $\mathbb{Z}_\iota^\infty \otimes R_\lambda$ and denote it by $\mathbb{Z}_\iota^\infty[\lambda]$. Here note that since the crystal R_λ has only one element, as a set we can identify $\mathbb{Z}_\iota^\infty[\lambda]$ with \mathbb{Z}_ι^∞ but their crystal structures are different. So we review an explicit crystal structure of $\mathbb{Z}^\infty[\lambda]$ in [14]. Fix a sequence of indices $\iota := (i_k)_{k \geq 1}$ satisfying the condition (2.2) and a weight $\lambda \in P$ (Here we do not necessarily assume that λ is dominant.). $\mathbb{Z}^\infty[\lambda]$ can be regarded as a subset of \mathbb{Q}^∞ , and then we denote an element in $\mathbb{Z}^\infty[\lambda]$ by $x = (\cdots, x_k, \cdots, x_2, x_1)$. For $x = (\cdots, x_k, \cdots, x_2, x_1) \in \mathbb{Q}^\infty$ we define the linear functions

$$(2.8) \quad \sigma_k(x) := x_k + \sum_{j>k} \langle h_{i_k}, \alpha_{i_j} \rangle x_j, \quad (k \geq 1)$$

$$(2.9) \quad \sigma_0^{(i)}(x) := -\langle h_i, \lambda \rangle + \sum_{j \geq 1} \langle h_i, \alpha_{i_j} \rangle x_j, \quad (i \in I)$$

Here note that since $x_j = 0$ for $j \gg 0$ on \mathbb{Q}^∞ , the functions σ_k and $\sigma_0^{(i)}$ are well-defined. Let $\sigma^{(i)}(x) := \max_{k: i_k = i} \sigma_k(x)$, and $M^{(i)} := \{k : i_k = i, \sigma_k(x) = \sigma^{(i)}(x)\}$. Note that $\sigma^{(i)}(x) \geq 0$, and that $M^{(i)} = M^{(i)}(x)$ is a finite set if and only if $\sigma^{(i)}(x) > 0$. Now we define the maps $\tilde{e}_i : \mathbb{Z}^\infty[\lambda] \sqcup \{0\} \rightarrow \mathbb{Z}^\infty[\lambda] \sqcup \{0\}$ and $\tilde{f}_i : \mathbb{Z}^\infty[\lambda] \sqcup \{0\} \rightarrow \mathbb{Z}^\infty[\lambda] \sqcup \{0\}$ by setting $\tilde{e}_i(0) = \tilde{f}_i(0) = 0$, and

$$(2.10) \quad (\tilde{f}_i(x))_k = x_k + \delta_{k, \min M^{(i)}} \text{ if } \sigma^{(i)}(x) > \sigma_0^{(i)}(x); \text{ otherwise } \tilde{f}_i(x) = 0,$$

$$(2.11) \quad (\tilde{e}_i(x))_k = x_k - \delta_{k, \max M^{(i)}} \text{ if } \sigma^{(i)}(x) > 0 \text{ and } \sigma^{(i)}(x) \geq \sigma_0^{(i)}(x); \text{ otherwise } \tilde{e}_i(x) = 0,$$

where $\delta_{i,j}$ is the Kronecker's delta. We also define the functions wt , ε_i and φ_i on $\mathbb{Z}^\infty[\lambda]$ by

$$(2.12) \quad wt(x) := \lambda - \sum_{j=1}^{\infty} x_j \alpha_{i_j},$$

$$(2.13) \quad \varepsilon_i(x) := \max(\sigma^{(i)}(x), \sigma_0^{(i)}(x))$$

$$(2.14) \quad \varphi_i(x) := \langle h_i, wt(x) \rangle + \varepsilon_i(x).$$

Note that by (2.12) we have

$$(2.15) \quad \langle h_i, wt(x) \rangle = -\sigma_0^{(i)}(x).$$

2.4. Polyhedral Realization of $B(\lambda)$. In this subsection, we review the result in [14]. In the rest of this section, λ is supposed to be a dominant integral weight. Here we define the map

$$(2.16) \quad \Phi_\lambda : (B(\infty) \otimes R_\lambda) \sqcup \{0\} \rightarrow B(\lambda) \sqcup \{0\},$$

by $\Phi_\lambda(0) = 0$ and $\Phi_\lambda(b \otimes r_\lambda) = \hat{\pi}_\lambda(b)$ for $b \in B(\infty)$. We set

$$\tilde{B}(\lambda) := \{b \otimes r_\lambda \in B(\infty) \otimes R_\lambda \mid \Phi_\lambda(b \otimes r_\lambda) \neq 0\}.$$

Theorem 2.3 ([14]). (i) *The map Φ_λ becomes a surjective strict morphism of crystals $B(\infty) \otimes R_\lambda \rightarrow B(\lambda)$.*

(ii) *$\tilde{B}(\lambda)$ is a subcrystal of $B(\infty) \otimes R_\lambda$, and Φ_λ induces the isomorphism of crystals $\tilde{B}(\lambda) \xrightarrow{\sim} B(\lambda)$.*

By Theorem 2.3, we have the strict embedding of crystals $\Omega_\lambda : B(\lambda) (\cong \tilde{B}(\lambda)) \hookrightarrow B(\infty) \otimes R_\lambda$. Combining Ω_λ and the Kashiwara embedding Ψ_ι , we obtain the following:

Theorem 2.4 ([14]). *There exists the unique strict embedding of crystals*

$$(2.17) \quad \Psi_\iota^{(\lambda)} : B(\lambda) \xrightarrow{\Omega_\lambda} B(\infty) \otimes R_\lambda \xrightarrow{\Psi_\iota \otimes \text{id}} \mathbb{Z}_\iota^\infty[\lambda],$$

such that $\Psi_\iota^{(\lambda)}(u_\lambda) = (\cdots, 0, 0, 0) \otimes r_\lambda$.

We fix a sequence of indices ι satisfying (2.2) and take a dominant integral weight $\lambda \in P_+$. For $k \geq 1$ let $k^{(\pm)}$ be the ones in 2.2. Let $\beta_k^{(\pm)}(x)$ be linear functions given by

$$(2.18) \quad \beta_k^{(+)}(x) = \sigma_k(x) - \sigma_{k^{(+)}}(x) = x_k + \sum_{k < j < k^{(+)}} \langle h_{i_k}, \alpha_{i_j} \rangle x_j + x_{k^{(+)}} ,$$

$$(2.19) \quad \beta_k^{(-)}(x) = \begin{cases} \sigma_{k^{(-)}}(x) - \sigma_k(x) = x_{k^{(-)}} + \sum_{k^{(-)} < j < k} \langle h_{i_k}, \alpha_{i_j} \rangle x_j + x_k & \text{if } k^{(-)} > 0, \\ \sigma_0^{(i_k)}(x) - \sigma_k(x) = -\langle h_{i_k}, \lambda \rangle + \sum_{1 \leq j < k} \langle h_{i_k}, \alpha_{i_j} \rangle x_j + x_k & \text{if } k^{(-)} = 0, \end{cases}$$

Here note that $\beta_k^{(+)} = \beta_k$ and $\beta_k^{(-)} = \beta_{k^{(-)}}$ if $k^{(-)} > 0$.

Using this notation, for every $k \geq 1$, we define an operator $\widehat{S}_k = \widehat{S}_{k,\iota}$ for a linear function $\varphi(x) = c + \sum_{k \geq 1} \varphi_k x_k$ ($c, \varphi_k \in \mathbb{Q}$) on \mathbb{Q}^∞ by:

$$\widehat{S}_k(\varphi) := \begin{cases} \varphi - \varphi_k \beta_k^{(+)} & \text{if } \varphi_k > 0, \\ \varphi - \varphi_k \beta_k^{(-)} & \text{if } \varphi_k \leq 0. \end{cases}$$

For the fixed sequence $\iota = (i_k)$, in case $k^{(-)} = 0$ for $k \geq 1$, there exists unique $i \in I$ such that $i_k = i$. We denote such k by $\iota^{(i)}$, namely, $\iota^{(i)}$ is the first number k such that $i_k = i$. Here for $\lambda \in P_+$ and $i \in I$ we set

$$(2.20) \quad \lambda^{(i)}(x) := -\beta_{\iota^{(i)}}^{(-)}(x) = \langle h_i, \lambda \rangle - \sum_{1 \leq j < \iota^{(i)}} \langle h_i, \alpha_{i_j} \rangle x_j - x_{\iota^{(i)}}.$$

For ι and a dominant integral weight λ , let $\Xi_\iota[\lambda]$ be the set of all linear functions generated by $\widehat{S}_k = \widehat{S}_{k,\iota}$ from the functions x_j ($j \geq 1$) and $\lambda^{(i)}$ ($i \in I$), namely,

$$(2.21) \quad \begin{aligned} \Xi_\iota[\lambda] &:= \{ \widehat{S}_{j_l} \cdots \widehat{S}_{j_1} x_{j_0} : l \geq 0, j_0, \dots, j_l \geq 1 \} \\ &\cup \{ \widehat{S}_{j_k} \cdots \widehat{S}_{j_1} \lambda^{(i)}(x) : k \geq 0, i \in I, j_1, \dots, j_k \geq 1 \}. \end{aligned}$$

Now we set

$$(2.22) \quad \Sigma_\iota[\lambda] := \{ x \in \mathbb{Z}_\iota^\infty[\lambda] (\subset \mathbb{Q}^\infty) : \varphi(x) \geq 0 \text{ for any } \varphi \in \Xi_\iota[\lambda] \}.$$

For a sequence ι and a dominant integral weight λ , a pair (ι, λ) is called *ample* if $\Sigma_\iota[\lambda] \ni \vec{0} = (\dots, 0, 0)$.

Theorem 2.5 ([14]). *Suppose that (ι, λ) is ample. Then we have $\text{Im}(\Psi_\iota^{(\lambda)}) (\cong B(\lambda)) = \Sigma_\iota[\lambda]$, where the explicit form of ε_i on $\Sigma_\iota[\lambda]$ is as follows:*

$$(2.23) \quad \varepsilon_i(x) = \sigma^{(i)}(x).$$

The other formula for φ_i , \tilde{e}_i and \tilde{f}_i are same as above.

Proof. The formula (2.23) slightly differs from (2.13). Indeed, by (2.11) we know that for $x \in \Sigma_\iota[\lambda]$ unless $\sigma^{(i)}(x) > 0$ and $\sigma^{(i)}(x) \geq \sigma_0^{(i)}(x)$, we find $\tilde{e}_i(x) = 0$. Furthermore, for any $x = (\dots, x_2, x_1) \in \Sigma_\iota[\lambda]$ it follows from the definition of $\Sigma_\iota[\lambda]$ that $0 \leq \lambda^{(i)}(x) = \sigma_{\iota^{(i)}}(x) - \sigma_0^{(i)}(x)$ which implies that $\sigma^{(i)}(x) \geq \sigma_0^{(i)}(x)$ and then we can obtain (2.23). \square

2.5. A_n -case. We shall apply the results in the previous subsection to the case $\mathfrak{g} = A_n$. Let us identify the index set I with $[1, n] := \{1, 2, \dots, n\}$ in the standard way; thus, the Cartan matrix $(a_{i,j} = \langle h_i, \alpha_j \rangle)_{1 \leq i,j \leq n}$ is given by $a_{i,i} = 2$, $a_{i,j} = -1$ for $|i - j| = 1$, and $a_{i,j} = 0$ otherwise. As the infinite sequence ι let us take the following periodic sequence

$$\iota = \dots, \underbrace{n, \dots, 2, 1}, \dots, \underbrace{n, \dots, 2, 1}, \underbrace{n, \dots, 2, 1}.$$

Following [16, Sect.5], we shall change the indexing set for \mathbb{Z}^∞ from $\mathbb{Z}_{\geq 1}$ to $\mathbb{Z}_{\geq 1} \times [1, n]$, which is given by the bijection $\mathbb{Z}_{\geq 1} \times [1, n] \rightarrow \mathbb{Z}_{\geq 1} ((j; i) \mapsto (j-1)n + i)$. According to this, we will write an element $x \in \mathbb{Z}^\infty$ as a doubly-indexed family $(x_{j;i})_{j \geq 1, i \in [1, n]}$. We will adopt the convention that $x_{j;i} = 0$ unless $j \geq 1$ and $i \in [1, n]$; in particular, $x_{j;0} = x_{j;n+1} = 0$ for all j .

Theorem 2.6. *Let $\lambda = \sum_{1 \leq i \leq n} \lambda_i \Lambda_i$ ($\lambda_i \in \mathbb{Z}_{\geq 0}$) be a dominant integral weight. In the above notation, the image $\text{Im}(\Psi_\iota^{(\lambda)})$ is the set of all integer families $(x_{j;i})$ such that*

$$(2.24) \quad x_{1;i} \geq x_{2;i-1} \geq \cdots \geq x_{i;1} \geq 0 \text{ for } 1 \leq i \leq n$$

$$(2.25) \quad x_{j;i} = 0 \text{ for } i + j > n + 1,$$

$$(2.26) \quad \lambda_i \geq x_{j,i-j+1} - x_{j,i-j} \text{ for } 1 \leq j \leq i \leq n.$$

Observing (2.25), we can rewrite the theorem in the following form: Let ι_0 be one of the reduced longest words of type A_n :

$$(2.27) \quad \iota_0 = \underbrace{1}, \underbrace{2, 1}, \underbrace{3, 2, 1}, \cdots, \underbrace{n, n-1, \cdots, 2, 1}.$$

Corollary 2.7. *Associated with ι_0 , we define*

$$(2.28) \quad \mathbb{Z}_{\iota_0}[\lambda] := \{(x_{j;i} | 1 \leq i + j \leq n + 1) \in \mathbb{Z}^{\frac{n(n+1)}{2}} | (x_{j;i}) \text{ satisfies (2.24) and (2.26)}\}$$

There exists the crystal structure on $\mathbb{Z}_{\iota_0}[\lambda]$ induced from the one on $\mathbb{Z}_l[\lambda]$ and then the crystal $\mathbb{Z}_{\iota_0}[\lambda]$ is isomorphic to $B(\lambda)$.

3. DECORATED GEOMETRIC CRYSTALS

The basic reference for this section is [1, 2].

3.1. Definitions. Let $A = (a_{ij})_{i,j \in I}$ be an indecomposable Cartan matrix with a finite index set I (though we can consider more general Kac-Moody setting.). Let $(\mathfrak{t}, \{\alpha_i\}_{i \in I}, \{h_i\}_{i \in I})$ be the associated root data satisfying $\alpha_j(h_i) = a_{ij}$. Let $\mathfrak{g} = \mathfrak{g}(A) = \langle \mathfrak{t}, e_i, f_i (i \in I) \rangle$ be the simple Lie algebra associated with A over \mathbb{C} and $\Delta = \Delta_+ \sqcup \Delta_-$ be the root system associated with \mathfrak{g} , where Δ_\pm is the set of positive/negative roots.

Define the simple reflections $s_i \in \text{Aut}(\mathfrak{t})$ ($i \in I$) by $s_i(h) := h - \alpha_i(h)h_i$, which generate the Weyl group W . Let G be the simply connected simple algebraic group over \mathbb{C} whose Lie algebra is $\mathfrak{g} = \mathfrak{n}_+ \oplus \mathfrak{t} \oplus \mathfrak{n}_-$, which is the usual triangular decomposition. Let $U_\alpha := \exp \mathfrak{g}_\alpha$ ($\alpha \in \Delta$) be the one-parameter subgroup of G . The group G (resp. U^\pm) is generated by $\{U_\alpha | \alpha \in \Delta\}$ (resp. $\{U_\alpha | \alpha \in \Delta_\pm\}$). Here U^\pm is a unipotent radical of G and $\text{Lie}(U^\pm) = \mathfrak{n}_\pm$. For any $i \in I$, there exists a unique group homomorphism $\phi_i: SL_2(\mathbb{C}) \rightarrow G$ such that

$$\phi_i \left(\begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix} \right) = \exp(te_i), \quad \phi_i \left(\begin{pmatrix} 1 & 0 \\ t & 1 \end{pmatrix} \right) = \exp(tf_i) \quad (t \in \mathbb{C}).$$

Set $\alpha_i^\vee(c) := \phi_i \left(\begin{pmatrix} c & 0 \\ 0 & c^{-1} \end{pmatrix} \right)$, $x_i(t) := \exp(te_i)$, $y_i(t) := \exp(tf_i)$, $G_i := \phi_i(SL_2(\mathbb{C}))$, $T_i := \alpha_i^\vee(\mathbb{C}^\times)$ and $N_i := N_{G_i}(T_i)$. Let T be a maximal torus of G which has P as its weight lattice and $\text{Lie}(T) = \mathfrak{t}$. Let $B^\pm (\supset T)$ be the Borel subgroup of G . We have the isomorphism $\phi: W \xrightarrow{\sim} N/T$ defined by $\phi(s_i) = N_i T/T$. An element $\bar{s}_i := x_i(-1)y_i(1)x_i(-1)$ is in $N_G(T)$, which is a representative of $s_i \in W = N_G(T)/T$.

Definition 3.1. Let X be an affine algebraic variety over \mathbb{C} , $\gamma_i, \varepsilon_i, f$ ($i \in I$) rational functions on X , and $e_i: \mathbb{C}^\times \times X \rightarrow X$ a unital rational \mathbb{C}^\times -action. A 5-tuple $\chi = (X, \{e_i\}_{i \in I}, \{\gamma_i\}_{i \in I}, \{\varepsilon_i\}_{i \in I}, f)$ is a G (or \mathfrak{g})-decorated geometric crystal if

- (i) $(\{1\} \times X) \cap \text{dom}(e_i)$ is open dense in $\{1\} \times X$ for any $i \in I$, where $\text{dom}(e_i)$ is the domain of definition of $e_i: \mathbb{C}^\times \times X \rightarrow X$.

- (ii) The rational functions $\{\gamma_i\}_{i \in I}$ satisfy $\gamma_j(e_i^c(x)) = c^{a_{ij}}\gamma_j(x)$ for any $i, j \in I$.
- (iii) The function f satisfies

$$(3.1) \quad f(e_i^c(x)) = f(x) + (c-1)\varphi_i(x) + (c^{-1}-1)\varepsilon_i(x),$$

for any $i \in I$ and $x \in X$, where $\varphi_i := \varepsilon_i \cdot \gamma_i$.

- (iv) e_i and e_j satisfy the following relations:

$$\begin{aligned} e_i^{c_1} e_j^{c_2} &= e_j^{c_2} e_i^{c_1} && \text{if } a_{ij} = a_{ji} = 0, \\ e_i^{c_1} e_j^{c_1 c_2} e_i^{c_2} &= e_j^{c_2} e_i^{c_1 c_2} e_j^{c_1} && \text{if } a_{ij} = a_{ji} = -1, \\ e_i^{c_1} e_j^{c_1 c_2} e_i^{c_1 c_2} e_j^{c_2} &= e_j^{c_2} e_i^{c_1 c_2} e_j^{c_1 c_2} e_i^{c_1} && \text{if } a_{ij} = -2, a_{ji} = -1, \\ e_i^{c_1} e_j^{c_1 c_2} e_i^{c_1 c_2} e_j^{c_1 c_2} e_i^{c_2} e_j^{c_2} &= e_j^{c_2} e_i^{c_1 c_2} e_j^{c_1 c_2} e_i^{c_1 c_2} e_j^{c_1 c_2} e_i^{c_1} && \text{if } a_{ij} = -3, a_{ji} = -1. \end{aligned}$$

- (v) The rational functions $\{\varepsilon_i\}_{i \in I}$ satisfy $\varepsilon_i(e_i^c(x)) = c^{-1}\varepsilon_i(x)$ and $\varepsilon_i(e_j^c(x)) = \varepsilon_i(x)$ if $a_{i,j} = a_{j,i} = 0$.

We call the function f in (iii) the *decoration* of χ and the relations in (iv) are called *Verma relations*. If $\chi = (X, \{e_i\}, \{\gamma_i\}, \{\varepsilon_i\})$ satisfies the conditions (i), (ii), (iv) and (v), we call χ a *geometric crystal*. *Remark.* The definitions of ε_i and φ_i are different from the ones in e.g., [2] since we adopt the definitions following [10, 11]. Indeed, if we flip $\varepsilon_i \rightarrow \varepsilon_i^{-1}$ and $\varphi_i \rightarrow \varphi_i^{-1}$, they coincide with ours.

3.2. Characters. Let $\widehat{U} := \text{Hom}(U, \mathbb{C})$ be the set of additive characters of U . The *elementary character* $\chi_i \in \widehat{U}$ and the *standard regular character* $\chi^{\text{st}} \in \widehat{U}$ are defined by

$$\chi_i(x_j(c)) = \delta_{i,j} \cdot c \quad (c \in \mathbb{C}, i \in I), \quad \chi^{\text{st}} = \sum_{i \in I} \chi_i.$$

Let us define an anti-automorphism $\eta : G \rightarrow G$ by

$$\eta(x_i(c)) = x_i(c), \quad \eta(y_i(c)) = y_i(c), \quad \eta(t) = t^{-1} \quad (c \in \mathbb{C}, t \in T),$$

which is called the *positive inverse*.

The rational function f_B on G is defined by

$$(3.2) \quad f_B(g) = \chi^{\text{st}}(\pi^+(w_0^{-1}g)) + \chi^{\text{st}}(\pi^+(w_0^{-1}\eta(g))),$$

for $g \in B\overline{w}_0B$, where $\pi^+ : B^{-}U \rightarrow U$ is the projection by $\pi^+(bu) = u$.

For a split algebraic torus T over \mathbb{C} , let us denote its lattice of (multiplicative) characters (resp. co-characters) by $X^*(T)$ (resp. $X_*(T)$). By the usual way, we identify $X^*(T)$ (resp. $X_*(T)$) with the weight lattice P (resp. the dual weight lattice P^*).

3.3. Positive structure and ultra-discretization. In this subsection, we review the notion positive structure and the ultra-discretization, which is called the tropicalization in [1, 2].

Definition 3.2. Let T, T' be split algebraic tori over \mathbb{C} .

- (i) A regular function $f = \sum_{\mu \in X^*(T)} c_\mu \cdot \mu$ on T is *positive* if all coefficients c_μ are non-negative numbers. A rational function on T is said to be *positive* if there exist positive regular functions g, h such that $f = \frac{g}{h}$ ($h \neq 0$).
- (ii) Let $f : T \rightarrow T'$ be a rational map between T and T' . Then we say that f is *positive* if for any $\xi \in X^*(T')$ we have that $\xi \circ f$ is positive in the above sense.

Note that if f, g are positive rational functions on T , then $f \cdot g$, f/g and $f + g$ are all positive.

Definition 3.3. Let $\chi = (X, \{e_i\}_{i \in I}, \{\text{wt}_i\}_{i \in I}, \{\varepsilon_i\}_{i \in I}, f)$ be a decorated geometric crystal, T' an algebraic torus and $\theta : T' \rightarrow X$ a birational map. The birational map θ is called *positive structure* on χ if it satisfies:

- (i) For any $i \in I$ the rational functions $\gamma_i \circ \theta, \varepsilon_i \circ \theta, f \circ \theta : T' \rightarrow \mathbb{C}$ are all positive in the above sense.
- (ii) For any $i \in I$, the rational map $e_{i,\theta} : \mathbb{C}^\times \times T' \rightarrow T'$ defined by $e_{i,\theta}(c, t) := \theta^{-1} \circ e_i^c \circ \theta(t)$ is positive.

Let $v : \mathbb{C}(c) \setminus \{0\} \rightarrow \mathbb{Z}$ be a map defined by $v(f(c)) := \deg(f(c^{-1}))$, which is different from that in e.g., [10, 11, 13, 15]. Note that this definition of the map UD is called tropicalization in [1] and much simpler than the one in [2] since it is sufficient in this article. Here, we have the formula for positive rational functions f and g :

$$(3.3) \quad v(f \cdot g) = v(f) + v(g), \quad v(f/g) = v(f) - v(g), \quad v(f + g) = \min(v(f), v(g)).$$

Let $f : T \rightarrow T'$ be a positive rational mapping of algebraic tori T and T' . We define a map $\widehat{f} : X_*(T) \rightarrow X_*(T')$ by

$$\langle \chi, \widehat{f}(\xi) \rangle = v(\chi \circ f \circ \xi),$$

where $\chi \in X^*(T')$ and $\xi \in X_*(T)$.

Let \mathcal{T}_+ be the category whose objects are algebraic tori over \mathbb{C} and whose morphisms are positive rational maps. Then, we obtain the functor

$$\begin{array}{ccc} UD : & \mathcal{T}_+ & \longrightarrow \\ & T & \mapsto X_*(T) \\ (f : T \rightarrow T') & \mapsto & (\widehat{f} : X_*(T) \rightarrow X_*(T')). \end{array}$$

Let $\theta : T \rightarrow X$ be a positive structure on a decorated geometric crystal $\chi = (X, \{e_i\}_{i \in I}, \{\text{wt}_i\}_{i \in I}, \{\varepsilon_i\}_{i \in I, f})$. Applying the functor UD to positive rational morphisms $e_{i,\theta} : \mathbb{C}^\times \times T' \rightarrow T'$ and $\gamma \circ \theta : T' \rightarrow T$ (the notations are as above), we obtain

$$\begin{aligned} \tilde{e}_i &:= UD(e_{i,\theta}) : \mathbb{Z} \times X_*(T) \rightarrow X_*(T) \\ \text{wt}_i &:= UD(\gamma_i \circ \theta) : X_*(T') \rightarrow \mathbb{Z}, \\ \tilde{\varepsilon}_i &:= UD(\varepsilon_i \circ \theta) : X_*(T') \rightarrow \mathbb{Z}, \\ \tilde{f} &:= UD(f \circ \theta) : \end{aligned}$$

Now, for given positive structure $\theta : T' \rightarrow X$ on a geometric crystal $\chi = (X, \{e_i\}_{i \in I}, \{\text{wt}_i\}_{i \in I}, \{\varepsilon_i\}_{i \in I})$, we associate the quadruple $(X_*(T'), \{\tilde{e}_i\}_{i \in I}, \{\text{wt}_i\}_{i \in I}, \{\tilde{\varepsilon}_i\}_{i \in I})$ with a free pre-crystal structure (see [1, 2.2]) and denote it by $UD_{\theta, T'}(\chi)$. We have the following theorem:

Theorem 3.4 ([1, 2, 13]). *For any geometric crystal $\chi = (X, \{e_i\}_{i \in I}, \{\gamma_i\}_{i \in I}, \{\varepsilon_i\}_{i \in I})$ and positive structure $\theta : T' \rightarrow X$, the associated pre-crystal $UD_{\theta, T'}(\chi) = (X_*(T'), \{e_i\}_{i \in I}, \{\text{wt}_i\}_{i \in I}, \{\tilde{\varepsilon}_i\}_{i \in I})$ is a Kashiwara's crystal.*

Remark. The definition of $\tilde{\varepsilon}_i$ is different from the one in [2, 6.1.] since our definition of ε_i corresponds to ε_i^{-1} in [2].

Now, for a positive decorated geometric crystal $\mathcal{X} = ((X, \{e_i\}_{i \in I}, \{\gamma_i\}_{i \in I}, \{\varepsilon_i\}_{i \in I, f}), \theta, T')$, set

$$(3.4) \quad \tilde{B}_{\tilde{f}} := \{\tilde{x} \in X_*(T') \mid \tilde{f}(\tilde{x}) \geq 0\},$$

where $X_*(T')$ is identified with $\mathbb{Z}^{\dim(T')}$. Define

$$(3.5) \quad B_{f,\theta} := (\tilde{B}_{\tilde{f}}, \text{wt}_i|_{\tilde{B}_{\tilde{f}}}, \varepsilon_i|_{\tilde{B}_{\tilde{f}}}, e_i|_{\tilde{B}_{\tilde{f}}})_{i \in I}.$$

Proposition 3.5 ([2]). *For a positive decorated geometric crystal $\mathcal{X} = ((X, \{e_i\}_{i \in I}, \{\gamma_i\}_{i \in I}, \{\varepsilon_i\}_{i \in I, f}), \theta, T')$, the quadruple $B_{f,\theta}$ in (3.5) is a normal crystal.*

3.4. Decorated geometric crystal on \mathbb{B}_w . For a Weyl group element $w \in W$, define B_w^- by

$$(3.6) \quad B_w^- := B^- \cap U\overline{w}U.$$

Now, set $\mathbb{B}_w := TB_w^-$. Let $\gamma_i : \mathbb{B}_w \rightarrow \mathbb{C}$ be the rational function defined by

$$(3.7) \quad \gamma_i : \mathbb{B}_w \hookrightarrow B^- \xrightarrow{\sim} T \times U^- \xrightarrow{\text{proj}} T \xrightarrow{\alpha_i^\vee} \mathbb{C}.$$

For any $i \in I$, there exists the natural projection $pr_i : B^- \rightarrow B^- \cap \phi(SL_2)$. Hence, for any $x \in \mathbb{B}_w$ there exists unique $v = \begin{pmatrix} b_{11} & 0 \\ b_{21} & b_{22} \end{pmatrix} \in SL_2$ such that $pr_i(x) = \phi(v)$. Using this fact, we define the rational function ε_i on \mathbb{B}_w by

$$(3.8) \quad \varepsilon_i(x) = \frac{b_{22}}{b_{21}} \quad (x \in \mathbb{B}_w).$$

The rational \mathbb{C}^\times -action e_i on \mathbb{B}_w is defined by

$$(3.9) \quad e_i^c(x) := x_i((c-1)\varphi_i(x)) \cdot x \cdot x_i((c^{-1}-1)\varepsilon_i(x)) \quad (c \in \mathbb{C}^\times, x \in \mathbb{B}_w),$$

if $\varepsilon_i(x)$ is well-defined, that is, $b_{21} \neq 0$, and $e_i^c(x) = x$ if $b_{21} = 0$.

Remark. The definition (3.8) is different from the one in [2]. Indeed, if we take $\varepsilon_i(x) = b_{21}/b_{22}$, then it coincides with the one in [2].

Proposition 3.6 ([2]). *For any $w \in W$, the 5-tuple $\chi := (\mathbb{B}_w, \{e_i\}_i, \{\gamma_i\}_i, \{\varepsilon_i\}_i, f_B)$ is a decorated geometric crystal, where f_B is in (3.2), γ_i is in (3.7), ε_i is in (3.8) and e_i is in (3.9).*

For the longest Weyl group element $w_0 \in W$, let $\mathbf{i}_0 = i_1 \dots i_N$ be one of its reduced expressions and define the positive structure on $B_{w_0}^- \Theta_{\mathbf{i}_0}^- : (\mathbb{C}^\times)^N \rightarrow B_{w_0}^-$ by

$$\Theta_{\mathbf{i}_0}^-(c_1, \dots, c_N) := \mathbf{y}_{i_1}(c_1) \cdots \mathbf{y}_{i_N}(c_N),$$

where $\mathbf{y}_i(c) = y_i(c)\alpha^\vee(c^{-1})$, which is different from $Y_i(c)$ in [13, 14, 10, 11]. Indeed, $Y_i(c) = \mathbf{y}_i(c^{-1})$. We also define the positive structure on \mathbb{B}_{w_0} as $T\Theta_{\mathbf{i}_0}^- : T \times (\mathbb{C}^\times)^N \rightarrow \mathbb{B}_{w_0}$ by $T\Theta_{\mathbf{i}_0}^-(t, c_1, \dots, c_N) = t\Theta_{\mathbf{i}_0}^-(c_1, \dots, c_N)$.

Now, for this positive structure, we describe the geometric crystal structure on $\mathbb{B}_{w_0} = TB_{w_0}^-$ explicitly. In fact, it is quite similar to that of the Schubert variety associated with w_0 as in [13] and then we obtain the following formula by the similar method in [13].

Proposition 3.7. *The action e_i^c on $t\Theta_{\mathbf{i}_0}^-(c_1, \dots, c_N)$ is given by*

$$e_i^c(t\Theta_{\mathbf{i}_0}^-(c_1, \dots, c_N)) = t\Theta_{\mathbf{i}_0}^-(c'_1, \dots, c'_N)$$

where

$$(3.10) \quad c'_j := c_j \cdot \frac{\sum_{1 \leq m < j, i_m = i} c \cdot c_1^{a_{i_1, i}} \cdots c_{m-1}^{a_{i_{m-1}, i}} c_m + \sum_{j \leq m \leq N, i_m = i} c_1^{a_{i_1, i}} \cdots c_{m-1}^{a_{i_{m-1}, i}} c_m}{\sum_{1 \leq m \leq j, i_m = i} c \cdot c_1^{a_{i_1, i}} \cdots c_{m-1}^{a_{i_{m-1}, i}} c_m + \sum_{j < m \leq N, i_m = i} c_1^{a_{i_1, i}} \cdots c_{m-1}^{a_{i_{m-1}, i}} c_m}.$$

The explicit forms of rational functions ε_i and γ_i are:

$$(3.11) \quad \varepsilon_i(t\Theta_{\mathbf{i}_0}^-(c)) = \left(\sum_{1 \leq m \leq N, i_m = i} \frac{1}{c_m c_{m+1}^{a_{i_{m+1}, i}} \cdots c_N^{a_{i_N, i}}} \right)^{-1}, \quad \gamma_i(t\Theta_{\mathbf{i}_0}^-(c)) = \frac{\alpha_i(t)}{c_1^{a_{i_1, i}} \cdots c_N^{a_{i_N, i}}}.$$

Proof. If we rewrite $t\Theta_{\mathbf{i}_0}^-(c)$ in the form,

$$t \cdot \alpha_{i_1}^\vee(c_1^{-1} \cdots \alpha_{i_N}^\vee(c_N^{-1})y_{i_1}(d_1) \cdots y_{i_N}(d_N),$$

then we easily get $d_m = c_m c_{m+1}^{a_{i_{m+1}, i}} \cdots c_N^{a_{i_N, i}}$ for $m = 1, \dots, N$. Thus, we obtain the explicit form of ε_i as above. To find the explicit form of the action e_i^c , the following formula is crucial:

$$x_i(a)y_j(b) = \begin{cases} y_i(\frac{b}{1+ab})\alpha_i^\vee(1+ab)x_i(\frac{a}{1+ab}) & \text{if } i = j, \\ y_j(b)x_i(a) & \text{if } i \neq j. \end{cases}$$

Applying this formula to (3.9) repeatedly, we have the above explicit action of e_i^c . \square

3.5. Ultra-Discretization of $\mathbb{B}_w = TB_w^-$. Applying the ultra-discretization functor to \mathbb{B}_w , we obtain the free crystal $\mathcal{UD}(\mathbb{B}_{w_0}) = X_*(T) \times \mathbb{Z}^N$, where N is the length of the longest element w_0 . Then define the map $\tilde{h} : \mathcal{UD}(\mathbb{B}_{w_0}) = X_*(T) \times \mathbb{Z}^N \rightarrow X_*(T) (= P^*)$ as the projection to the left component and set

$$B_{w_0}(\lambda^\vee) := \tilde{h}^{-1}(\lambda^\vee), \quad B_{f_B, \Theta_{\mathbf{i}_0}^-}(\lambda^\vee) := B_{w_0}(\lambda^\vee) \cap B_{f_B, \Theta_{\mathbf{i}_0}^-},$$

for $\lambda^\vee \in X_*(T) = P^*$. Set $P_+^* := \{h \in P^* \mid \Lambda_i(h) \geq 0 \text{ for any } i \in I\}$ and for $\lambda^\vee = \sum_i \lambda_i h_i$, we define $\lambda = \sum_i \lambda_i \Lambda_i \in P_+$. Then, we have

Theorem 3.8 ([2]). *The set $B_{f_B, \Theta_{\mathbf{i}_0}^-}(\lambda^\vee)$ is non-empty if $\lambda^\vee \in P_+^*$ and in that case, $B_{f_B, \Theta_{\mathbf{i}_0}^-}(\lambda^\vee)$ is isomorphic to $B(\lambda)^L$, which is the Langlands dual crystal associated with \mathfrak{g}^L .*

It follows from (3.10) and (3.11) that we have

Theorem 3.9. *Let $\lambda^\vee \in P_+^*$. The explicit crystal structure of $B_{f_B, \Theta_{\mathbf{i}_0}^-}(\lambda^\vee)$ is as follows: For $x = (x_1, \dots, x_N) \in B_{f_B, \Theta_{\mathbf{i}_0}^-}(\lambda^\vee) \subset \mathbb{Z}^N$, we have*

$$(3.12) \quad \tilde{e}_i^n(x) = (x'_1, \dots, x'_N),$$

where

$$(3.13) \quad x'_j = x_j + \min \left(\min_{1 \leq m < j, i_m = i} (n + \sum_{k=1}^m a_{i_k, i} x_k), \min_{j \leq m \leq N, i_m = i} (\sum_{k=1}^m a_{i_k, i} x_k) \right) \\ - \min \left(\min_{1 \leq m \leq j, i_m = i} (n + \sum_{k=1}^m a_{i_k, i} x_k), \min_{j < m \leq N, i_m = i} (\sum_{k=1}^m a_{i_k, i} x_k) \right),$$

$$(3.14) \quad \text{wt}_i(x) = \lambda(h_i) - \sum_{k=1}^N a_{i_k, i} x_k,$$

$$(3.15) \quad \varepsilon_i(x) = \max_{1 \leq m \leq N, i_m = i} (x_m + \sum_{k=m+1}^N a_{i_k, i} x_k),$$

and $x = (x_1, \dots, x_N)$ belongs to $B_{f_B, \Theta_{\mathbf{i}_0}^-}(\lambda^\vee)$ if and only if $\mathcal{UD}(f_B)(x) \geq 0$.

It follows immediately from (3.13):

Lemma 3.10. *Set $X_m := \sum_{k=1}^m a_{i_k, i} x_k$, $\mathcal{X}^{(i)} := \min\{X_m \mid 1 \leq m \leq N, i_m = i\}$ ($i \in I$) and $M^{(i)} := \{l \mid 1 \leq l \leq N, i_l = i, X_l = \mathcal{X}^{(i)}\}$. Define $m_e := \max(M^{(i)})$ and $m_f := \min(M^{(i)})$: for $x \in B_{f_B, \Theta_{\mathbf{i}_0}^-}(\lambda^\vee)$,*

we get

$$(3.16) \quad \tilde{e}_i(x) = \begin{cases} (x_1, \dots, x_{m_e} - 1, \dots, x_N) & \text{if } \mathcal{UD}(f_B)(x_1, \dots, x_{m_e} - 1, \dots, x_N) \geq 0, \\ 0 & \text{otherwise,} \end{cases}$$

$$(3.17) \quad \tilde{f}_i(x) = \begin{cases} (x_1, \dots, x_{m_f} + 1, \dots, x_N) & \text{if } \mathcal{UD}(f_B)(x_1, \dots, x_{m_f} + 1, \dots, x_N) \geq 0, \\ 0 & \text{otherwise.} \end{cases}$$

Finally, due to the results in Sect.2 and in this section, we obtain the following theorem

Theorem 3.11. *If we have $B_{f_B, \Theta_{\mathbf{i}_0}^-}(\lambda^\vee) = \Sigma_{\mathbf{i}_0^{-1}}[\lambda]^L$ as a set. Then they are isomorphic each other as crystals, where L means the Langlands dual crystal, that is, it is defined by the transposed Cartan matrix and \mathbf{i}_0^{-1} means the opposite order of \mathbf{i}_0 .*

Proof. The coincidence of the actions \tilde{e}_i and \tilde{f}_i are shown by comparing (2.10) and (2.11) with (3.16) and (3.17) since the following are equivalent:

(a) X_k is the minimum.

(b) $\sigma^{(i)}(x) = \sigma_i^{(0)}(x) + \lambda_i - X_k$ is the maximum.

Similarly, comparing (2.12) with (3.14) and (2.23) with (3.15) respectively, we obtain the coincidence of ε_i and wt_i . \square

4. EXPLICIT FORM OF THE DECORATION f_B OF TYPE A_n

4.1. Generalized Minors and the function f_B . For this subsection, see [3, 4, 5]. Let G be a simply connected simple algebraic groups over \mathbb{C} and $T \subset G$ a maximal torus. Let $X^*(T) := \text{Hom}(T, \mathbb{C}^\times)$ and $X_*(T) := \text{Hom}(\mathbb{C}^\times, T)$ be the lattice of characters and co-characters respectively. We identify P (resp. P^*) with $X^*(T)$ (resp. $X_*(T)$) as above.

Definition 4.1. For $\mu \in P_+$, the *principal minor* $\Delta_\mu : G \rightarrow \mathbb{C}$ is defined as

$$\Delta_\mu(u^- t u^+) := \mu(t) \quad (u^\pm \in U^\pm, t \in T).$$

Let $\gamma, \delta \in P$ be extremal weights such that $\gamma = u\mu$ and $\delta = v\mu$ for some $u, v \in W$. Then the *generalized minor* $\Delta_{\gamma, \delta}$ is defined by

$$\Delta_{\gamma, \delta}(g) := \Delta_\mu(\bar{u}^{-1} g \bar{v}) \quad (g \in G),$$

which is a regular function on G .

Lemma 4.2 ([2]). *Suppose that G is simply connected.*

- (i) *For $u \in U$ and $i \in I$, we have $\Delta_{\mu, \mu}(u) = 1$ and $\chi_i(u) = \Delta_{\Lambda_i, s_i \Lambda_i}(u)$, where Λ_i be the i th fundamental weight.*
- (ii) *Define the map $\pi^+ : B^- \cdot U \rightarrow U$ by $\pi^+(bu) = u$ for $b \in B^-$ and $u \in U$. For any $g \in G$, we have*

$$(4.1) \quad \chi_i(\pi^+(g)) = \frac{\Delta_{\Lambda_i, s_i \Lambda_i}(g)}{\Delta_{\Lambda_i, \Lambda_i}(g)}.$$

Proposition 4.3 ([2]). *The function f_B in (3.2) is described as follows:*

$$(4.2) \quad f_B(g) = \sum_i \frac{\Delta_{w_0 \Lambda_i, s_i \Lambda_i}(g) + \Delta_{w_0 s_i \Lambda_i, \Lambda_i}(g)}{\Delta_{w_0 \Lambda_i, \Lambda_i}(g)}$$

Let $\mathbf{i} = i_1 \cdots i_N$ be a reduced word for the longest Weyl group element w_0 . For $t\Theta_{\mathbf{i}}^-(c) \in \mathbb{B}_{w_0} = T \cdot B_{w_0}^-$, we get the following formula.

$$(4.3) \quad f_B(t\Theta_{\mathbf{i}}^-(c)) = \sum_i \Delta_{w_0 \Lambda_i, s_i \Lambda_i}(\Theta_{\mathbf{i}}^-(c)) + \alpha_i(t) \Delta_{w_0 s_i \Lambda_i, \Lambda_i}(\Theta_{\mathbf{i}}^-(c)).$$

4.2. Bilinear Forms. Let $\omega : \mathfrak{g} \rightarrow \mathfrak{g}$ be the anti involution

$$\omega(e_i) = f_i, \quad \omega(f_i) = e_i, \quad \omega(h) = h,$$

and extend it to G by setting $\omega(x_i(c)) = y_i(c)$, $\omega(y_i(c)) = x_i(c)$ and $\omega(t) = t$ ($t \in T$).

There exists a \mathfrak{g} (or G)-invariant bilinear form on the finite-dimensional irreducible \mathfrak{g} -module $V(\lambda)$ such that

$$\langle au, v \rangle = \langle u, \omega(a)v \rangle, \quad (u, v \in V(\lambda), a \in \mathfrak{g}(\text{or } G)).$$

For $g \in G$, we have the following simple fact:

$$\Delta_{\Lambda_i}(g) = \langle gu_{\Lambda_i}, u_{\Lambda_i} \rangle.$$

Hence, for $w, w' \in W$ we have

$$(4.4) \quad \Delta_{w\Lambda_i, w'\Lambda_i}(g) = \Delta_{\Lambda_i}(\bar{w}^{-1}g\bar{w}') = \langle \bar{w}^{-1}g\bar{w}' \cdot u_{\Lambda_i}, u_{\Lambda_i} \rangle = \langle g\bar{w}' \cdot u_{\Lambda_i}, \bar{w} \cdot u_{\Lambda_i} \rangle,$$

where u_{Λ_i} is a properly normalized highest weight vector in $V(\Lambda_i)$ and note that $\omega(\bar{s}_i^\pm) = \bar{s}_i^\mp$.

4.3. Explicit form of $f_B(t\Theta_i^-(c))$ of type A_n . Now, we consider the type A_n , that is, $G = SL_{n+1}(\mathbb{C})$. We fix the reduced longest word $\mathbf{i}_0 = \underbrace{1, 2, \dots, n}, \underbrace{1, 2, \dots, n-1}, \dots, \underbrace{1, 2, 3, 1, 2, 1}$. This is

just the opposite order $\iota_0 = \mathbf{i}_0^{-1}$ as in Sect.2. To obtain the explicit form of $f_B(t\Theta_{\mathbf{i}_0}^-(c))$, by (4.3) it suffices to know $\Delta_{w_0\Lambda_j, s_j\Lambda_j}(\Theta_{\mathbf{j}_0}^-(c))$ and $\Delta_{w_0s_j\Lambda_j, \Lambda_j}(\Theta_{\mathbf{j}_0}^-(c))$ for

$$c = (c_{i,j} | i+j \leq n+1) = (c_{1,1}, c_{1,2}, \dots, c_{1,n}, c_{2,1}, c_{2,2}, \dots, c_{2,n-1}, \dots, c_{n-1,1}, c_{n-1,2}, c_{n,1}) \in (\mathbb{C}^\times)^N.$$

Theorem 4.4. For $c \in (\mathbb{C}^\times)^N$ as above, we have the following explicit forms:

$$(4.5) \quad \Delta_{w_0\Lambda_j, s_j\Lambda_j}(\Theta_{\mathbf{i}_0}^-(c)) = c_{n-j+1,1} + \frac{c_{n-j+1,2}}{c_{n-j+2,1}} + \frac{c_{n-j+1,3}}{c_{n-j+2,2}} + \dots + \frac{c_{n-j+1,j}}{c_{n-j+2,j-1}},$$

$$(4.6) \quad \Delta_{w_0s_j\Lambda_j, \Lambda_j}(\Theta_{\mathbf{i}_0}^-(c)) = \frac{1}{c_{j,1}} + \frac{c_{j-1,1}}{c_{j-1,2}} + \frac{c_{j-2,2}}{c_{j-2,3}} + \dots + \frac{c_{1,j-1}}{c_{1,j}}, \quad (j \in I).$$

The proof of this theorem will be given in the next subsection.

4.4. Proof of Theorem 4.4. Let $V_1 := V(\Lambda_1)$ be the vector representation of $\mathfrak{sl}_{n+1}(\mathbb{C})$ with the standard basis $\{v_1, \dots, v_{n+1}\}$, and $\{e_i, f_i, h_i\}_{i=1, \dots, n}$ the Chevalley generators of $\mathfrak{sl}_{n+1}(\mathbb{C})$. Their actions on the basis vectors are as follows:

$$(4.7) \quad e_i v_j = \begin{cases} v_i & \text{if } j = i+1, \\ 0 & \text{otherwise,} \end{cases} \quad f_i v_j = \begin{cases} v_{i+1} & \text{if } j = i, \\ 0 & \text{otherwise,} \end{cases} \quad h_i v_j = \begin{cases} v_i & \text{if } j = i, \\ -v_{i-1} & \text{if } j = i-1 \text{ and } i \neq 1, \\ 0 & \text{otherwise,} \end{cases}$$

By these explicit actions we know that $e_i^2 = f_i^2 = 0$ on V_1 . Thus, we can write $\mathbf{x}_i(c) := \alpha_i^\vee(c^{-1})x_i(c) = c^{-h_i}(1 + c \cdot e_i)$ and $\mathbf{y}_i(c) := y_i(c)\alpha_i^\vee(c^{-1}) = (1 + c \cdot f_i)c^{-h_i}$ on V_1 and then

$$(4.8) \quad \mathbf{x}_i(c)v_j = \begin{cases} cv_{i+1} + v_i & \text{if } j = i+1, \\ cv_i & \text{if } j = i, \\ v_j & \text{otherwise,} \end{cases} \quad \mathbf{y}_i(c)v_j = \begin{cases} c^{-1}v_i + v_{i+1} & \text{if } j = i, \\ cv_i & \text{if } j = i-1, \\ v_j & \text{otherwise.} \end{cases}$$

For $c = (c_1, \dots, c_i) \in (\mathbb{C}^\times)^i$ set $X^{(i)}(c) := \mathbf{x}_i(c_i) \cdots \mathbf{x}_1(c_1)$ and its action on the basis vector is as follows:

$$(4.9) \quad X^{(i)}(c)v_k = \begin{cases} c_k^{-1}c_{k-1}v_k + v_{k-1} & \text{if } k < i+1, \\ c_i v_{i+1} + v_i & \text{if } k = i+1, \\ v_k & \text{if } k > i+1. \end{cases}$$

Now, for $c = (c_{k,i})_{1 \leq i, k \leq n, i+k \leq n+1} \in (\mathbb{C}^\times)^{\frac{n(n+1)}{2}}$ set $c^{(k)} := (c_{n+1-k,k}, c_{n+1-k,k-1}, \dots, c_{n+1-k,2}, c_{n+1-k,1})$ and

$$X(c) := X^{(1)}(c^{(1)})X^{(2)}(c^{(2)}) \dots X^{(n-1)}(c^{(n-1)})X^{(n)}(c^{(n)})$$

Here, note that

$$(4.10) \quad \omega(\Theta_{\mathbf{i}_0}^-(c)) = X(c).$$

Writing

$$X(c)v_i = \sum_{k=1}^i {}^i\Xi_k v_k,$$

we shall get the explicit form of the coefficient ${}^i\Xi_k$ with the direct calculations: For $i = 1, \dots, n$ and $k = 1, \dots, i$, set

$${}^i\mathbf{m}_k := \{M | M \subset \{1, \dots, n-k+1\}, \#M = n-i+1\}.$$

For $M \in {}^i\mathbf{m}_k$, write $M = M_1 \sqcup \dots \sqcup M_{i-k+1}$ where each M_j ($j = 1, \dots, s := i-k+1$) is a consecutive subsequence of M satisfying $\min(M_l) - \max(M_j) = l-j+1$ for any $1 \leq j < l \leq s$ if both M_j and M_l are non-empty, which is called a segment of M . For $M = M_1 \sqcup \dots \sqcup M_s \in {}^i\mathbf{m}_k$, write each segment:

$$M_1 = \{1, 2, \dots, j_1-1\}, \quad M_2 = \{j_1+1, j_1+2, \dots, j_2-1\}, \quad M_s = \{j_{i-k}+1, j_{i-k}+2, \dots, n-k+1\},$$

where $1 \leq j_1 < j_2 < \dots < j_{i-k} \leq n-k+1$ and set

$$c^M := \frac{c_{1,i-1} \dots c_{j_1-1,i-1} \dots c_{j_{i-k}+1,k-1} \dots c_{n-k+1,k-1}}{c_{1,i} \dots c_{j_1-1,i} \dots c_{j_{i-k}+1,k} \dots c_{n-k+1,k}}.$$

Proposition 4.5. *We have the following explicit form of ${}^i\Xi_k$.*

$$(4.11) \quad {}^i\Xi_k = \sum_{M \in {}^i\mathbf{m}_k} c^M.$$

This formula is obtained by direct calculations.

For the module $V(\Lambda_j)$ ($j > 1$), let us denote its normalized highest (resp. lowest) weight vector by u_{Λ_j} (resp. v_{Λ_j}). Set

$$[i_1, \dots, i_j] := v_{i_1} \wedge v_{i_2} \wedge \dots \wedge v_{i_j} \in \bigwedge^j V_1$$

$$I_j := \{[i_1, i_2, \dots, i_j] \mid 1 \leq i_1 < i_2 < \dots < i_j \leq n+1\}.$$

I_j is a normal basis of $V(\Lambda_j)$ with the weight $\sum_{k=1}^j (\Lambda_{i_k} - \Lambda_{i_k-1})$. Indeed, $u_{\Lambda_j} = v_1 \wedge v_2 \wedge \dots \wedge v_j$ and $v_{\Lambda_j} = v_{n-k} \wedge v_{n-k+1} \wedge \dots \wedge v_{n+1}$. The actions of e_i and f_i on the vector $[i_1, \dots, i_j]$ are given by

$$(4.12) \quad e_i[i_1, \dots, i_j] = \begin{cases} [i_1, \dots, i_{k-1}, i, i_{k+1}, \dots, i_j] & \text{if } i_k = i+1, i_{k-1} < i \text{ for some } k, \\ 0 & \text{otherwise,} \end{cases}$$

$$(4.13) \quad f_i[i_1, \dots, i_j] = \begin{cases} [i_1, \dots, i_{k-1}, i+1, i_{k+1}, \dots, i_j] & \text{if } i_k = i, i_{k+1} > i+1 \text{ for some } k, \\ 0 & \text{otherwise.} \end{cases}$$

It follows from the formula (4.4) and (4.10) that

$$(4.14) \quad \Delta_{w_0\Lambda_j, s_i\Lambda_j}(\Theta_{\mathbf{i}_0}^-(c)) = \langle \Theta_{\mathbf{i}_0}^-(c) \bar{s}_j \cdot u_{\Lambda_j}, \bar{w}_0 \cdot u_{\Lambda_j} \rangle = \langle \bar{s}_j \cdot u_{\Lambda_j}, X(c) \bar{w}_0 \cdot u_{\Lambda_j} \rangle.$$

Since $\bar{s}_j \cdot u_{\Lambda_j} = [1, 2, \dots, j-1, j+1]$ and $\bar{w}_0 \cdot u_{\Lambda_j} = v_{\Lambda_j}$, to obtain $\Delta_{w_0\Lambda_j, s_i\Lambda_j}(\Theta_{\mathbf{i}_0}^-(c))$ it suffices to find the coefficient of $[1, 2, \dots, j-1, j+1]$ in $X(c)v_{\Lambda_j}$.

Lemma 4.6. *We have*

$$(4.15) \quad X^{(j+1)}(c^{(j+1)}) \cdots X^{(n)}(c^{(n)})v_{\Lambda_j} = [2, 3, \dots, j, j+1] + \sum_{1 \leq i_1 < \dots < i_j < j+1} c_{i_1, \dots, i_j} [i_1, \dots, i_j],$$

where $c_{i_1, \dots, i_j} \in \mathbb{C}$ is the coefficient.

Proof. First, let us see $W_n := X^{(n)}(c^{(n)})v_{\Lambda_j}$. It is easily to see that

$$(4.16) \quad W_n = \mathbf{x}_n(c_{1,n}) \cdots \mathbf{x}_1(c_{1,1})v_{\Lambda_j} \\ = [n+1-j, n+2-j, \dots, n-1, n] + \sum_{1 \leq i_1 < \dots < i_{j-1} < n+1} c_{i_1, \dots, i_{j-1}, n+1} [i_1, \dots, i_{j-1}, n+1],$$

since the term $c_{1,n}^{-h_n} c_{1,n} e_n \cdot c_{1,n-1}^{-h_{n-1}} c_{1,n-1} e_{n-1} \cdots c_{1,n+1-j}^{-h_{n+1-j}} c_{1,n+1-j} e_{n+1-j} \cdot c_{1,n-j}^{-h_{n-j}} \cdots c_{1,1}^{-h_1}$ in $X^{(n)}(c^{(n)})$ gives the leading term $[n+1-j, n+2-j, \dots, n-1, n]$ in (4.16). Indeed, the basis vectors appearing in $X^{(n-1)}(c^{(n-1)})[i_1, \dots, n+1]$ are in the form $[\dots, n+1]$, which means that we may see only the vector $X^{(n-1)}(c^{(n-1)})[n+1-j, n+2-j, \dots, n-1, n]$ in $X^{(n-1)}(c^{(n-1)})X^{(n)}(c^{(n)})v_{\Lambda_j}$. By considering similarly, we obtain

$$X^{(n-1)}(c^{(n-1)})X^{(n)}(c^{(n)})v_{\Lambda_j} = [n-j, n+1-j, \dots, n-2, n-1] + \sum_{1 \leq i_1 < \dots < i_j \geq n} c_{i_1, \dots, i_j} [i_1, \dots, i_j].$$

Repeating this process, we get the desired result. \square

We shall see the action of $\overline{X}_j := X^{(1)}(c^{(1)}) \cdots X^{(j)}(c^{(j)})$ on the vector $[2, 3, \dots, j, j+1]$. The following lemma is shown easily.

Lemma 4.7. *In the expansion of*

$$\overline{X}_j = c_{n,1}^{-h_n} (1 + c_{n,1} e_1) \cdots c_{j,1}^{-h_1} (1 + c_{j,1} e_1),$$

the only terms $E_m := A \cdot B_m \cdot C_m \cdot D_m$ ($m = 0, \dots, j-1$) produce the vector $[1, 2, \dots, j-1, j+1]$ in $X^{(1)}(c^{(1)}) \cdots X^{(j)}(c^{(j)})[2, 3, \dots, j, j+1]$, where

$$\begin{aligned} A &:= c_{n,1}^{-h_1} c_{n-1,2}^{-h_2} c_{n-1,1}^{-h_1} \cdots c_{n-j+3,j-2}^{-h_{j-2}} c_{n-j+3,j-3}^{-h_{j-3}} \cdots c_{n-j+3,1}^{-h_1}, \\ B_m &:= c_{n-j+2,j-1}^{-h_{j-1}} c_{n-j+2,j-1} e_{j-1} \cdots c_{n-j+2,m+2}^{-h_{m+2}} c_{n-j+2,m+1}^{-h_{m+1}} c_{n-j+2,m+1} e_{m+1}, \\ C_m &:= c_{n-j+2,m}^{-h_m} \cdots c_{n-j+2,2}^{-h_2} c_{n-j+2,1}^{-h_1} c_{n-j+1,j}^{-h_j} c_{n-j+1,j-1}^{-h_{j-1}} \cdots c_{n-j+1,m+1}^{-h_{m+1}}, \\ D_m &:= c_{n-j+1,m}^{-h_m} c_{n-j+1,m} e_m \cdots c_{n-j+1,2}^{-h_2} c_{n-j+1,2} e_2 \cdots c_{n-j+1,1}^{-h_1} c_{n-j+1,1} e_1, \end{aligned}$$

where we understand $D_0 = 1$.

For $m = 1, 2, \dots, j-1$ it is trivial that

$$C_m \cdot D_m [2, 3, \dots, j, j+1] = \frac{c_{n-j+1,m+1}}{c_{n-j+2,m}} [1, 2, \dots, m, m+2, m+3, \dots, j, j+1].$$

And then

$$A \cdot B_m \cdot C_m \cdot D_m [2, 3, \dots, j, j+1] = \frac{c_{n-j+1,m+1}}{c_{n-j+2,m}} [1, 2, 3, \dots, j-1, j+1].$$

For $m = 0$, we have

$$A \cdot B_0 \cdot C_0 \cdot D_0 [2, 3, \dots, j, j+1] = c_{n-j+1,1} [1, 2, \dots, j-1, j+1].$$

Finally, we obtain that the coefficient of $[1, 2, \dots, j-1, j+1]$ in $X(c)v_{\Lambda_j}$ is

$$(4.17) \quad c_{n-j+1,1} + \sum_{m=1}^{j-1} \frac{c_{n-j+1,m+1}}{c_{n-j+2,m}},$$

which is just $\Delta_{w_0\Lambda_j, s_i\Lambda_j}(\Theta_{\mathbf{i}_0}^-(c))$ and then we have shown (4.5) in Theorem 4.4. The formula (4.6) would be shown by the similar way to (4.5). \square

Note that for $j = 1$ and $k = 2$, we find ${}^1\Xi_2 = \Delta_{w_0\Lambda_1, s_1\Lambda_1}(\Theta_{\mathbf{i}_0}^-(c))$.

5. ULTRA-DISCRETIZATION AND POLYHEDRAL REALIZATIONS OF TYPE A_n

In this section, we shall only treat the type A_n . Then we identify P with P^* by $\lambda \leftrightarrow \lambda^\vee$. Let us describe the explicit form of $B_{f_B, \Theta_{\mathbf{i}_0}^-}(\lambda)$ for type A_n applying the result in Theorem 3.9 and show the coincidence of the crystals $B_{f_B, \Theta_{\mathbf{i}_0}^-}(\lambda)$ and $\Sigma_{\iota_0}[\lambda]$ in Sect.2 using Theorem 3.11.

For $\mathfrak{g} = \mathfrak{sl}_{n+1}(\mathbb{C})$, let \mathbf{i}_0 be as in 4.3. Then we have the following:

Lemma 5.1. *The crystal $B_{f_B, \Theta_{\mathbf{i}_0}^-}(\lambda)$ is defined by*

$$(5.1) \quad B_{f_B, \Theta_{\mathbf{i}_0}^-}(\lambda) := \left\{ (x_{k,l} | k+l \leq n+1) \in \mathbb{Z}^N \mid \begin{array}{l} x_{1,i} \geq x_{2,i-1} \geq \cdots \geq x_{i,1} \geq 0 \text{ for } 1 \leq i \leq n \\ \lambda_i \geq x_{j,i-j+1} - x_{j,i-j} \text{ for } 1 \leq j \leq i \leq n. \end{array} \right\},$$

where $N = \frac{n(n+1)}{2}$.

Proof. We shall see the explicit form of $\mathcal{UD}(f_B)(x)$. Indeed, by virtue of (4.3), it is sufficient to know the forms of $\Delta_{w_0\Lambda_j, s_j\Lambda_j}(\Theta_{\mathbf{i}_0}^-(c))$ and $\Delta_{w_0s_j\Lambda_j, \Lambda_j}(\Theta_{\mathbf{i}_0}^-(c))$, which are given in (4.5) and (4.6). Thus, we have

$$\mathcal{UD}(f_B)(t, x) = \min_{1 \leq j \leq n} (\mathcal{UD}(\Delta_{w_0\Lambda_j, s_j\Lambda_j}(\Theta_{\mathbf{i}_0}^-))(x), \mathcal{UD}(\alpha_j(t)) + \mathcal{UD}(\Delta_{w_0s_j\Lambda_j, \Lambda_j}(\Theta_{\mathbf{i}_0}^-))(x))$$

and

$$(5.2) \quad \mathcal{UD}(\Delta_{w_0\Lambda_j, s_j\Lambda_j}(\Theta_{\mathbf{i}_0}^-))(x) = \min_{k=1, \dots, j} (x_{n-j+1, k} - x_{n-j+2, k-1}),$$

$$(5.3) \quad \mathcal{UD}(\Delta_{w_0s_j\Lambda_j, \Lambda_j}(\Theta_{\mathbf{i}_0}^-))(x) = \min_{k=1, \dots, j} (x_{j-k+1, k-1} - x_{j-k+1, k}),$$

where $x_{j,k} = \mathcal{UD}(c_{j,k})$ and we understand $x_{m,0} = 0$. Hence, if we identify $\mathcal{UD}(\alpha_j(t))$ with λ_j , then the condition $\mathcal{UD}(f_B)(\lambda, x) \geq 0$ in Theorem 3.9 is equivalent to the condition in (5.1). \square

Theorem 5.2. *For any dominant integral weight λ , there exists the following isomorphism of crystals $B_{f_B, \Theta_{\mathbf{i}_0}^-}[\lambda] \cong \Sigma_{\iota_0}[\lambda]$ where $\Sigma_{\iota_0}[\lambda]$ is as in Corollary 2.7 and $\iota_0 = \mathbf{i}_0^{-1}$.*

Proof. By Theorem 3.11 it is necessarily for us to show that $B_{f_B, \Theta_{\mathbf{i}_0}^-}[\lambda] = \Sigma_{\iota_0}[\lambda]$ as a set, which is shown by Corollary 2.7 and Lemma 5.1. \square

6. ELEMENTARY CHARACTERS AND MONOMIAL REALIZATION OF CRYSTALS

We shall see the elementary characters as in Sect4 from the different point of view, that is, the monomial realization of crystals.

Let us introduce the monomial realization of crystals (See [9, 12]). For variables $\{Y_{m,i} | i \in I, m \in \mathbb{Z}\}$, define the set of monomials

$$\mathcal{Y} := \{Y = \prod_{m \in \mathbb{Z}, i \in I} Y_{m,i}^{l_{m,i}} | l_{m,i} \in \mathbb{Z} \setminus \{0\} \text{ except for finitely many } (m, i)\}.$$

Fix a set of integers $p = (p_{i,j})_{i,j \in I, i \neq j}$ such that $p_{i,j} + p_{j,i} = 1$. For this $p := (p_{i,j})_{i,j \in I, i \neq j}$ and a generalized Cartan matrix $(a_{i,j})_{i,j \in I}$, set

$$A_{m,i} = Y_{m,i} Y_{m+1,i} \prod_{j \neq i} Y_{m+p_{j,i}, j}^{a_{j,i}}.$$

Note that for any cyclic order $\iota = \cdots (i_1 i_2 \cdots i_n)(i_1 i_2 \cdots i_n) \cdots$ s.t. $\{i_1, \dots, i_n\} = I$, we can associate the following $(p_{i,j})$ by:

$$p_{i_a, i_b} = \begin{cases} 1 & a < b, \\ 0 & a > b. \end{cases}$$

For example, if we take $\iota = \cdots (213)(213) \cdots$, then we have $p_{2,1} = p_{1,3} = p_{2,3} = 1$ and $p_{1,2} = p_{3,1} = p_{3,2} = 0$. Thus, we can identify a cyclic order $\cdots (i_1 \cdots i_n)(i_1 \cdots i_n) \cdots$ with such $(p_{i,j})$.

For a monomial $Y = \prod_{m,i} Y_{m,i}^{l_{m,i}}$, set

$$\begin{aligned} wt(Y) &= \sum_{i,m} l_{m,i} \Lambda_i, \quad \varphi_i(Y) = \max_{k \in \mathbb{Z}} \left\{ \sum_{k \leq m} l_{m,i} \right\}, \quad \varepsilon_i(Y) = \varphi_i(Y) - wt(Y)(h_i), \\ \tilde{f}_i(Y) &= \begin{cases} A_{n_f, i}^{-1} \cdot Y & \text{if } \varphi_i(Y) > 0, \\ 0 & \text{if } \varphi_i(Y) = 0, \end{cases} \quad \tilde{e}_i(Y) = \begin{cases} A_{n_e, i} \cdot Y & \text{if } \varepsilon_i(Y) > 0, \\ 0 & \text{if } \varepsilon_i(Y) = 0, \end{cases} \\ n_f &= \min\{n | \varphi_i(Y) = \sum_{k \leq n} m_{k,i}\}, \quad n_e = \max\{n | \varphi_i(Y) = \sum_{k \leq n} m_{k,i}\}. \end{aligned}$$

Theorem 6.1 ([9, 12]). (i) In the above setting, \mathcal{Y} is a crystal, which is denoted by $\mathcal{Y}(p)$.
(ii) If $Y \in \mathcal{Y}(p)$ satisfies $\varepsilon_i(Y) = 0$ for any $i \in I$, then the connected component containing Y is isomorphic to $B(wt(Y))$.

In the above setting, for type A_n take $(p_{i,j})_{i,j \in I, i \neq j}$ such that $p_{i,j} = 1$ for $i < j$, $p_{i,j} = 0$ for $i > j$, which corresponds to the cyclic order $\mathbf{i} = (12 \cdots n)(12 \cdots n) \cdots$. Then we obtain

Proposition 6.2. The crystal containing the monomial $Y_{n-i+1,1}$ (resp. $Y_{i,1}^{-1}$) is isomorphic to $B(\Lambda_1)$ (resp. $B(\Lambda_n)$) and all basis vectors are given by

$$\begin{aligned} \tilde{f}_k \cdots \tilde{f}_2 \tilde{f}_1(Y_{n-i+1,1}) &= \frac{Y_{n-i+1,k+1}}{Y_{n-i+2,k}} \in B(\Lambda_1), \\ \tilde{e}_k \cdots \tilde{e}_2 \tilde{e}_1(Y_{i,1}^{-1}) &= \frac{Y_{i-k,k}}{Y_{i-k,k+1}} \in B(\Lambda_n) \quad (k = 1, \dots, n). \end{aligned}$$

Proof. The explicit form of $A_{m,i}$ ($m \in \mathbb{Z}, i \in I$) is as follows:

$$(6.1) \quad A_{m,i} = \begin{cases} Y_{m,1} Y_{m,2}^{-1} Y_{m+1,1} & \text{if } i = 1, \\ Y_{m,i} Y_{m,i+1}^{-1} Y_{m+1,i-1}^{-1} Y_{m+1,i} & \text{if } i \neq 1, n, \\ Y_{m,n} Y_{m+1,n-1}^{-1} Y_{m+1,n} & \text{if } i = n. \end{cases}$$

Then, applying \tilde{e}_i and \tilde{f}_i repeatedly, we obtain the results. For example,

$$\tilde{f}_1(Y_{n-i+1,1}) = Y_{n-i+1,1} \cdot A_{n-i+1,1}^{-1} = \frac{Y_{n-i+1,2}}{Y_{n-i+2,1}}.$$

Applying this results to Theorem 4.4 and changing the variable $Y_{m,l}$ to $c_{m,l}$, we find: □

Proposition 6.3. For $j = 1, \dots, n$ we have

$$\begin{aligned} \chi_j(\pi^+(w_0^{-1} t \Theta_{\mathbf{i}}^-(c))) &= \Delta_{w_0 \Lambda_j, s_j \Lambda_j}(\Theta_{\mathbf{i}}^-(c)) = \sum_{k=0}^{j-1} \tilde{f}_k \cdots \tilde{f}_2 \tilde{f}_1(c_{n-j+1,1}), \\ \chi_j(\pi^+(w_0^{-1} \eta(t \Theta_{\mathbf{i}}^-(c)))) &= \alpha_j(t) \Delta_{w_0 s_j \Lambda_j, \Lambda_j}(\Theta_{\mathbf{i}}^-(c)) = \alpha_j(t) \sum_{k=0}^{j-1} \tilde{e}_k \cdots \tilde{e}_2 \tilde{e}_1(c_{j,1}^{-1}). \end{aligned}$$

Note that $\{\tilde{f}_k \cdots \tilde{f}_2 \tilde{f}_1(c_{n-i+1,1}) | 0 \leq k < i\} = B(\Lambda_1)s_{k-1} \cdots s_2 s_1$ is the Demazure crystal associated with the Weyl group element $s_{k-1} \cdots s_2 s_1$ ([8]).

Observing Proposition 6.3, we present the following conjecture:

Conjecture 6.4. *There exists certain reduced longest word $\mathbf{i} = (i_1, \dots, i_N)$ and $p = (p_{i,j})_{i \neq j}$ such that for any $i \in I$, there exist Demazure crystal $B_w^-(i) \subset B(\Lambda_k)$, Demazure crystal $B_{w'}^+(i) \subset B(\Lambda_j)$ and positive integers $\{a_b, a_{b'} | b \in B_w^-, b' \in B_{w'}^+\}$ satisfying*

$$\begin{aligned} \chi_i(\pi^+(w_0^{-1}t\Theta_{\mathbf{i}}^-(c))) &= \Delta_{w_0\Lambda_i, s_i\Lambda_i}(\Theta_{\mathbf{i}}^-(c)) = \sum_{b \in B_w^-(i)} a_b m_b(c), \\ \chi_i(\pi^+(w_0^{-1}\eta(t\Theta_{\mathbf{i}}^-(c)))) &= \alpha_i(t)\Delta_{w_0s_i\Lambda_i, \Lambda_i}(\Theta_{\mathbf{i}}^-(c)) = \alpha_j(t) \sum_{b' \in B_{w'}^+(i)} a_{b'} m_{b'}(c), \end{aligned}$$

where $m_b(c) \in \mathcal{Y}(p)$ is the monomial corresponding to $b \in B(\Lambda_k)$ associated with $p = (p_{i,j})_{i \neq j}$.

We would see the answers to this conjecture for other type of Lie algebras in the forthcoming papers.

Suppose that this conjecture is right and then we can deduce the following:

Corollary 6.5. *In the setting of the above conjecture, define the linear function $\tilde{m}_b(x) := \mathcal{UD}(m_b)(x)$ ($x \in \mathbb{Z}^N$) and set*

$$\tilde{\Sigma}_{\mathbf{i}-1}[\lambda] := \{x = (x_N, \dots, x_1) \in \mathbb{Z}_{\mathbf{i}-1}^N[\lambda] \mid \tilde{m}_b \geq 0, \lambda_i + \tilde{m}_{b'} \geq 0 \text{ for any } b \in B_w^-(i), b' \in B_{w'}^+(i) (i \in I)\}.$$

Then this is equipped with the crystal structure and isomorphic to the crystal $B(\lambda)$.

Proof. Since $\mathcal{UD}(f_B)(\lambda, x) \geq 0$ is equivalent to the condition of the set $\tilde{\Sigma}_{\mathbf{i}-1}[\lambda]$, we know that the set $\tilde{\Sigma}_{\mathbf{i}-1}[\lambda]$ coincides with $B_{f_B, \Theta_{\mathbf{i}_0}^-}(\lambda^\vee)$. \square

We call $\tilde{\Sigma}_{\mathbf{i}-1}[\lambda]$ the *refined polyhedral realization* associated with the monomial realizations $\mathcal{Y}(p)$.

REFERENCES

- [1] Berenstein A. and Kazhdan D., Geometric crystals and Unipotent crystals, GAFA 2000(Tel Aviv,1999), Geom. Funct. Anal. 2000, Special Volume, Part I, 188–236.
- [2] Berenstein A. and Kazhdan D., Geometric and unipotent crystals. II. From unipotent bicrystals to crystal bases. Quantum groups, 13–88, Contemp. Math., 433, Amer. Math. Soc., Providence, RI, 2007.
- [3] Berenstein A., Fomin S. and Zelevinsky A., Parametrizations of canonical bases and totally positive matrices, Adv. Math. 122 (1996), 49–149.
- [4] Berenstein A. and Zelevinsky A., Total positivity in Schubert varieties, Comment. Math. Helv. 72 (1997), 128–166.
- [5] Berenstein A. and Zelevinsky A., Tensor product multiplicities, canonical bases and totally positive varieties, Invent. Math. 143 (2001), 77–128.
- [6] Kashiwara M. Crystallizing the q -analogue of universal enveloping algebras. Comm. Math. Phys. **1990**, 133, 249–260.
- [7] Kashiwara M. On crystal bases of the q -analogue of universal enveloping algebras, Duke Math. J. **1991**, 63 (2), 465–516.
- [8] Kashiwara M. Crystal base and Littelmann’s refined Demazure character formula. Duke Math. J. **1993**, 71 (3), 839–858.
- [9] Kashiwara M. Realizations of crystals. Combinatorial and geometric representation theory (Seoul, 2001), 133–139, Contemp. Math., **325**, 133–139, (2003).
- [10] Kashiwara M., Nakashima T. and Okado M., Affine geometric crystals and limit of perfect, Transactions in American Mathematical Society 360 (2008), no.7, crystals, math.QA/0512657 (to appear in Trans.Amer.Math.Soc.).
- [11] Kashiwara M., Nakashima T. and Okado M., Tropical R maps and Affine Geometric Crystals, Representation Theory **14** (2010), 446–509.
- [12] Nakajima H., t -analogs of q -characters of quantum affine algebras of type A_n and D_n . Contemp. Math. **325**, 141–160, (2003).
- [13] Nakashima T., Geometric crystals on Schubert varieties, Journal of Geometry and Physics, **53** (2), 197–225, (2005).

- [14] Nakashima T., Polyhedral realizations of crystal bases for integrable highest weight modules. J. Algebra **219**, no. 2, 571–597, (1999).
- [15] Nakashima T., Geometric crystals on unipotent groups and generalized Young tableaux, Journal of Algebra, **293**, No.1, 65–88, (2005).
- [16] Nakashima T., Zelevinsky A., Polyhedral realizations of crystal bases for quantized Kac-Moody algebras. Adv. Math. **131**, no. 1, 253–278, (1997).

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