PART-1

Linear Differential Equations of n^{th} Order with Constant Coefficients.

CONCEPT OUTLINE

Differential Equation : An equation involving derivatives of one or more dependent variables with respect to one or more independent variables is called a differential equation.

For example, $\log\left(\dfrac{dy}{dx}\right) = ax + by$

$$(1 - x^2)(1 - y)dx = xy(1 + y)dy$$

$$\frac{dy}{dx} = \sec(x + y)$$

Order of a Differential Equation : The order of a differential equation is the order of the highest derivative involved in a differential equation.

For example, $\dfrac{d^4x}{dt^4} + \dfrac{d^2x}{dt^2} + \left(\dfrac{dx}{dt}\right)^5 = e^5$ is of 4^{th} order.

Degree of a Differential Equation : The degree of a differential equation is the power of the highest derivative which occurs in it, after the differential equation has been made free from radicals and fractions as far as the derivatives are concerned.

For example, $\dfrac{d^4x}{dt^4} + \dfrac{d^2x}{dt^2} + \left(\dfrac{dx}{dt}\right)^5 = e^t$, is of first degree.

Linear Differential Equation : A linear differential equation is an equation in which the dependent variable and its derivatives appear only in the first degree.

For example, $\dfrac{d^2y}{dx^2} + \dfrac{dy}{dx} - 9y = 4x^2 - 7$

The above equation is called a LDE (linear differential equation) with constant coefficients.

Questions-Answers

Long Answer Type and Medium Answer Type Questions

Que 1.1. **Write the procedure to find complementary function.**

Answer

Following are the steps to find complementary function :

Step I : Put the RHS of the given equation equals to zero. *i.e.*, $f(D)\, y = 0$

Step II : Replace $\frac{d}{dx} \approx D, \frac{d^2}{dx^2} \approx D^2$ and so on *i.e.*, convert the given equation in symbolic form.

Step III : Make an auxiliary equation replacing D by m.

e.g., $(D^2 + 4D + 7) = 0$ then its auxiliary equation is

$$m^2 + 4m + 7 = 0$$

Step IV : Find the roots of auxiliary equation (AE), CF will depend upon the type of root.

Case I : If all roots of the AE are real and distinct say $m_1, m_2, ..., m_n$

Then, $$CF = C_1 e^{m_1 x} + C_2 e^{m_2 x} + ... + C_n e^{m_n x}$$

Where $C_1, C_2, ..., C_n$ are constants.

Case II : If roots of AE are real and equal say

$$m_1 = m_2 .... = m_n = m \text{ (say).}$$

Then, $CF = (C_1 + C_2 x + C_3 x^2 + ... + C_n x^n)\, e^{mx}$

If some roots are equal, others are distinct say

$$m_1 = m_2 = m_3 = m$$

and $m_4, m_5, ..., m_n$

Then, $CF = (C_1 + C_2 x + C_3 x^2)\, e^{mx} + C_4 e^{m_4 x} + C_5 e^{m_5 x} + ... + C_n e^{m_n x}$

Case III : If the roots of AE are complex say

$$m = \alpha \pm i\beta, \text{ then}$$

$$CF = C_1 e^{(\alpha + i\beta)x} + C_2 e^{(\alpha - i\beta)x}$$

or $$CF = C_1 e^{\alpha x} e^{i\beta x} + C_2 e^{\alpha x} e^{-i\beta x}$$

$$CF = C_1 e^{\alpha x} (\cos \beta x + i \sin \beta x) + C_2 e^{\alpha x} (\cos \beta x - i \sin \beta x)$$

$$CF = e^{\alpha x} (C_1 + C_2) \cos \beta x + i e^{\alpha x} (C_1 - C_2) \sin \beta x$$

$$CF = e^{\alpha x} [A \cos \beta x + iB \sin \beta x]$$

or changing the constants

$$CF = e^{\alpha x} (C_1 \cos \beta x + C_2 \sin \beta x)$$

This expression may be written as

$$CF = C_1 e^{\alpha x} (\cos \beta x + C_2)$$

or $$CF = C_1 e^{\alpha x} (\sin \beta x + C_2)$$

Case IV : If the AE has irrational roots say

$$= \alpha \pm \sqrt{\beta}, \text{ where } \beta \text{ is positive}$$

Then, $$CF = e^{\alpha x} (C_1 \cosh \sqrt{\beta}\, x + C_2 \sinh \sqrt{\beta}\, x)$$

Que 1.2. **Explain the method to find out the particular integral when the function in RHS is $e^{ax}, f(a) \neq 0$ and $e^{ax}, f(a) = 0$.**

Answer

A. Case I :

When RHS function is $e^{ax}, f(a) \neq 0$,

Then, $$PI = \frac{1}{f(D)} e^{ax}$$

Now replace D by a so PI will be,

$$= \frac{e^{ax}}{f(a)}$$

If $f(a) = 0$, it will be a case of failure.

B. Case II :

When RHS of function is $e^{ax}, f(a) = 0$,

Then, $$PI = \frac{e^{ax}}{f(D)}$$

Now, $$PI = \frac{x e^{ax}}{f'(D)}$$

Multiply with x and differentiate denominator once.

Again if, $f'(a) = 0$ then, continue to multiply with x and differentiate denominator,

$$PI = x^n \frac{e^{ax}}{f^n(a)}$$

Que 1.3. **What is the procedure to find particular integral when the RHS function is either sin ax, cos ax while $f(-a^2) \neq 0$, or sin ax, cos ax while $f(-a^2) = 0$?**

Answer

Case I : When function is $\sin ax$ or $\cos ax$ and $f(-a^2) \neq 0$,

$$PI = \frac{\sin ax}{f(D^2)}$$

or $$\text{PI} = \frac{\cos ax}{f(D^2)}$$

In both cases replace D^2 by $-a^2$ but $f(-a)^2 \neq 0$. If after replacing D^2 by $-a^2$ any term of D exist in denominator then, multiply the operator by its conjugate, again D^2 by $-a^2$. Terms of D in numerator stands for differentiation of function.

Case II : When function is $\sin ax$ or $\cos ax$ and $f(-a^2) = 0$,

$$\text{PI} = \frac{\sin ax}{f(D^2)} = x\frac{\sin ax}{f'(-a^2)}$$

Repeat this step again if $f'(-a^2) = 0$.

Que 1.4. **Solve $\frac{d^2y}{dx^2} + 4y = \sin^2 2x$ with conditions $y(0) = 0$, $y'(0) = 0$.**

AKTU 2012-13, Marks 05

Answer

$$\frac{d^2y}{dx^2} + 4y = \sin^2 2x$$

$$\frac{d^2y}{dx^2} + 4y = \frac{1}{2} - \frac{\cos 4x}{2} \qquad \left[\because \cos 4x = 1 - 2\sin^2 2x,\ \sin^2 2x = \frac{1 - \cos 4x}{2}\right]$$

The auxiliary equation is

$$m^2 + 4 = 0$$
$$m = \pm 2i$$
$$\therefore \quad \text{CF} = C_1 \cos 2x + C_2 \sin 2x$$

$$\text{PI} = \frac{1}{D^2+4}\sin^2 2x = \frac{1}{2}\left[\frac{1}{D^2+4}(1 - \cos 4x)\right]$$

$$= \frac{1}{2}\left[\frac{1}{D^2+4}(e^{0x}) - \frac{1}{D^2+4}\cos 4x\right]$$

$$= \frac{1}{2}\left[\frac{1}{0+4} - \frac{1}{-16+4}\cos 4x\right]$$

$$= \frac{1}{2}\left[\frac{1}{4} + \frac{1}{12}\cos 4x\right]$$

$$= \frac{1}{8}\left[1 + \frac{1}{3}\cos 4x\right]$$

Now the complete solution is,

$$y = C_1 \cos 2x + C_2 \sin 2x + \frac{1}{8}\left[1 + \frac{1}{3}\cos 4x\right] \quad \text{...(1.4.1)}$$

$$y' = -2C_1 \sin 2x + 2C_2 \cos 2x - \frac{1}{6}\sin 4x \quad \text{...(1.4.2)}$$

Using boundary conditions,

$$y(0) = 0 \text{ and } y'(0) = 0$$

From eq. (1.4.1), we have

$$0 = C_1 + \frac{1}{6} \Rightarrow C_1 = -\frac{1}{6}$$

From eq. (1.4.2), we have

$$0 = 2C_2 \Rightarrow C_2 = 0$$

Putting the value of C_1 and C_2 in eq. (1.4.1), we get

$$y = -\frac{1}{6}\cos 2x + \frac{1}{8}\left[1 + \frac{1}{3}\cos 4x\right]$$

Que 1.5. **A function $n(x)$ satisfies the differential equation $\frac{d^2n(x)}{dx^2} - \frac{n(x)}{L^2} = 0$, where L is a constant. The boundary conditions are $n(0) = x$ and $n(\infty) = 0$. Find the solution to this equation.**

AKTU 2016-17, Marks 07

Answer

$$\frac{d^2n(x)}{dx^2} - \frac{n(x)}{L^2} = 0$$

The auxiliary equation is

$$m^2 - \frac{1}{L^2} = 0$$

$$m = \pm \frac{1}{L}$$

$$\text{CF} = C_1 e^{-\frac{1}{L}x} + C_2 e^{\frac{1}{L}x}$$

Complete solution,

$$n(x) = \text{CF} + \text{PI}$$

$$n(x) = C_1 e^{-\frac{x}{L}} + C_2 e^{\frac{x}{L}} \qquad (\because \text{ PI} = 0)$$

Boundary conditions are wrong. So we can't solve it further.

Que 1.6. **Solve $\dfrac{d^2x}{dt^2} + 9x = \cos 3t$.** **AKTU 2013-14, Marks 05**

Answer

$$\frac{d^2x}{dt^2} + 9x = \cos 3t$$

$$(D^2 + 9)\, x = \cos 3t$$

Auxiliary equation : $m^2 + 9 = 0$

$$m^2 = -9 \Rightarrow m = \pm 3i$$

$$\text{CF} = (C_1 \cos 3t + C_2 \sin 3t)$$

$$\text{PI} = \frac{1}{D^2 + 9} \cos 3t$$

$$\text{PI} = t\, \frac{1}{2D} \cos 3t = \frac{t}{2}\left(\frac{\sin 3t}{3}\right) = \frac{t \sin 3t}{6}$$

Complete solution, $x = \text{CF} + \text{PI} = C_1 \cos 3t + C_2 \sin 3t + \dfrac{t}{6} \sin 3t$

Que 1.7. **Find the particular solution of the differential equation $\dfrac{d^2y}{dx^2} + a^2y = \sec ax$** **AKTU 2016-17, Marks 07**

Answer

Auxiliary equation is,

$$m^2 + a^2 = 0$$

$$m = \pm ai$$

$$\text{CF} = C_1 \cos ax + C_2 \sin ax$$

$$\text{PI} = \frac{1}{D^2 + a^2} \sec ax$$

$$= \frac{1}{(D - ia)(D + ia)} \sec ax$$

$$= \frac{1}{2ia}\left[\frac{1}{D - ia} - \frac{1}{D + ia}\right] \sec ax$$

$$= \frac{1}{2ia}\left[\frac{1}{(D - ia)} \sec ax - \frac{1}{(D + ia)} \sec ax\right]$$

$$= \frac{1}{2ia}[P_1 - P_2]$$

Where,

$$P_1 = \frac{1}{D - ia} \sec ax$$

$$= e^{iax} \int e^{-iax} \sec ax\, dx$$

$$= e^{iax} \int (\cos ax - i \sin ax) \sec ax\, dx$$

$$= e^{iax} \int (1 - i \tan ax)\, dx$$

$$= e^{iax} \left\{x + i\left(\frac{\log \cos ax}{a}\right)\right\}$$

Similarly,

$$P_2 = \frac{1}{D + ia}(\sec ax) = e^{-iax}\left\{x - i\left(\frac{\log \cos ax}{a}\right)\right\}$$

Replacing i by $-i$)

$$\therefore \quad \text{PI} = \frac{1}{2ia}\left[e^{iax}\left\{x + i\left(\frac{\log \cos ax}{a}\right)\right\} - e^{-iax}\left\{x - i\left(\frac{\log \cos ax}{a}\right)\right\}\right]$$

$$= \frac{1}{2ia}\left[x(e^{iax} - e^{-iax}) + i\left(\frac{\log \cos ax}{a}\right)(e^{iax} + e^{-iax})\right]$$

$$= \frac{1}{2ia}\left[2\, ixa \sin ax + \frac{i}{a} \log \cos ax\; 2 \cos ax\right]$$

$$= \frac{1}{a}\left[x \sin ax + \frac{1}{a} \cos ax \log \cos ax\right]$$

Que 1.8. **Solve $(D^2 - 2D + 1)\, y = e^x \sin x$** **AKTU 2016-17, Marks 7.5**

Answer

$$(D^2 - 2D + 1)y = e^x \sin x$$

Auxiliary equation,

$$m^2 - 2m + 1 = 0$$

$$m^2 - m - m + 1 = 0$$
$$m(m-1) - 1(m-1) = 0$$
$$(m-1)^2 = 0$$
$$m = 1, 1$$
$$\text{CF} = (C_1 + C_2x)e^x$$
$$\text{PI} = \frac{1}{(D^2 - 2D + 1)} e^x \sin x$$
$$= e^x \frac{1}{(D+1)^2 - 2(D+1) + 1} \sin x$$
$$= e^x \frac{1}{(D^2 + 2D + 1 - 2D - 2 + 1)} \sin x = e^x \frac{\sin x}{D^2}$$

Replace D^2 by -1

$$= -e^x \sin x$$

$\therefore$ Complete solution $= \text{CF} + \text{PI}$

$$y = (C_1 + C_2x)e^x - e^x \sin x$$

Que 1.9. Solve : $(D^2 - 3D + 2)\,y = x^2 + 2x + 1.$

AKTU 2014-15, Marks 05

Answer

$$(D^2 - 3D + 2)\,y = x^2 + 2x + 1$$

Auxiliary equation,

$$m^2 - 3m + 2 = 0$$
$$(m-1)(m-2) = 0$$
$$m = 1, 2$$
$$\text{CF} = C_1 e^x + C_2 e^{2x}$$
$$\text{PI} = \frac{1}{(D^2 - 3D + 2)}(x^2 + 2x + 1)$$
$$= \frac{1}{2}\left[1 + \frac{D^2 - 3D}{2}\right]^{-1}(x^2 + 2x + 1)$$
$$= \frac{1}{2}\left[1 - \frac{D^2}{2} + \frac{3D}{2} + \left(\frac{D^2 - 3D}{2}\right)^2 \ldots\right](x^2 + 2x + 1)$$
$$= \frac{1}{2}\left[1 - \frac{D^2}{2} + \frac{3D}{2} + \frac{9D^2}{4}\right](x^2 + 2x + 1)$$

(Neglecting higher terms)

$$= \frac{1}{2}\left[x^2 + 2x + 1 - \frac{2}{2} + \frac{3}{2}(2x + 2) + \frac{9}{4} \times 2\right]$$
$$= \frac{1}{2}\left[x^2 + 2x + 3x + 3 + \frac{9}{2}\right]$$

$$= \frac{1}{2}\left[x^2 + 5x + \frac{15}{2}\right]$$
$$y = \text{CF} + \text{PI}$$
$$= C_1 e^x + C_2 e^{2x} + \frac{1}{2}\left[x^2 + 5x + \frac{15}{2}\right]$$

Que 1.10. Solve the differential equation $\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = xe^x \cos x$.

AKTU 2013-14, Marks 05

Answer

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = xe^x \cos x$$

Auxiliary equation,

$$(m^2 - 2m + 1) = 0$$
$$(m-1)^2 = 0$$
$$m = 1, 1$$
$$\text{CF} = (C_1 + C_2x)e^x$$
$$\text{PI} = \frac{1}{(D-1)^2} xe^x \cos x = \frac{1}{(D-1)^2} e^x (x \cos x)$$
$$= e^x \frac{1}{(D+1-1)^2} x \cos x$$
$$= e^x \frac{1}{D^2} x \cos x = e^x \frac{1}{D}[x \sin x + \cos x]$$
$$= e^x [-x \cos x + \sin x + \sin x]$$
$$\text{PI} = e^x [-x \cos x + 2 \sin x]$$

Complete solution is given by

$$y = \text{CF} + \text{PI}$$
$$y = (C_1 + C_2x)\,e^x + e^x(-x \cos x + 2 \sin x)$$

Que 1.11. Solve the following differential equation

$$\frac{d^2y}{dx^2} + 2\frac{dy}{dx} + y = x^2 e^{-x} \cos x.$$

AKTU 2011-12, Marks 05

Answer

Same as Q. 1.10, Page 1–10F, Unit-1.

(**Answer :** $y = (C_1 + C_2 x)\,e^{-x} + e^{-x}(-x^2 \cos x + 4x \sin x + 6 \cos x)$)

Que 1.12. Solve $(D^2 - 2D + 4)\,y = e^x \cos x + \sin x \cos 3x$.

AKTU 2017-18, Marks 07

Answer

Given equation, $(D^2 - 2D + 4)\,y = e^x \cos x + \sin x \cos 3x$

Auxiliary equation,

$$m^2 - 2m + 4 = 0$$

$$m = \frac{+2 \pm \sqrt{4-16}}{2}$$

$$m = \frac{2 \pm \sqrt{-12}}{2}$$

$$m = 1 \pm i\sqrt{3}$$

Complementary function is

$$CF = e^x (C_1 \cos \sqrt{3}x + C_2 \sin \sqrt{3}x)$$

Particular integral, PI = $P_1 + P_2$

$$P_1 = e^x \cos x$$

$$= \frac{1}{D^2 - 2D + 4} e^x \cos x$$

$$= e^x \frac{1}{(D+1)^2 - 2(D+1) + 4} \cos x$$

$$= e^x \frac{1}{D^2 + 3} \cos x$$

$$= e^x \frac{1}{-1+3} \cos x$$

$$= e^x \frac{\cos x}{2}$$

$$P_2 = \frac{1}{D^2 - 2D + 4} \sin x \cos 3x$$

$$= \frac{1}{2} \frac{1}{(D^2 - 2D + 4)} 2 \sin x \cos 3x$$

$$= \frac{1}{2} \frac{1}{D^2 - 2D + 4} \{\sin (x + 3x) + \sin (x - 3x)\}$$

$$= \frac{1}{2} \frac{1}{D^2 - 2D + 4} (\sin 4x - \sin 2x)$$

$$= \frac{1}{2}\left[\frac{1}{D^2 - 2D + 4} \sin 4x - \frac{1}{D^2 - 2D + 4} \sin 2x\right]$$

$$= \frac{1}{2}\left[\frac{1}{-(4)^2 - 2D + 4} \sin 4x - \frac{1}{-(2)^2 - 2D + 4} \sin 2x\right]$$

$$= \frac{1}{2}\left[\frac{1}{-12 - 2D} \sin 4x - \frac{1}{-2D} \sin 2x\right]$$

$$= \frac{1}{4}\left[\frac{-1}{D+6} \sin 4x + \frac{1}{D} \sin 2x\right]$$

$$= \frac{1}{4}\left[\frac{-(D-6)}{D^2 - 36} \sin 4x - \frac{\cos 2x}{2}\right]$$

$$= \frac{1}{4}\left[\frac{-(D-6)}{-52} \sin 4x - \frac{\cos 2x}{2}\right]$$

$$= \frac{1}{4}\left[\frac{4\cos 4x - 6 \sin 4x}{52} - \frac{\cos 2x}{2}\right]$$

$$= \frac{1}{4}\left[\frac{4\cos 4x - 6 \sin 4x}{52}\right] - \frac{\cos 2x}{8}$$

Complete solution,

$$y = CF + PI$$

$$= CF + P_1 + P_2$$

$$y = e^x \left(C_1 \cos \sqrt{3}x + C_2 \sin \sqrt{3}x\right) + e^x \frac{\cos x}{2} + \frac{1}{4}\left[\frac{4\cos 4x - 6\sin 4x}{52}\right] - \frac{\cos 2x}{8}$$

PART-2

Simultaneous Linear Differential Equations.

CONCEPT OUTLINE

Simultaneous Differential Equation : If two or more dependent variables are functions of a single independent variable, the equations which consist of the derivatives of such variables are called simultaneous differential equations.

Questions-Answers

Long Answer Type and Medium Answer Type Questions

Que 1.13. Solve the simultaneous equation $\frac{dx}{dt} + 5x - 2y = t$, $\frac{dy}{dt} + x + y = 0$ being given $x = 0, y = 0$ when $t = 0$.

AKTU 2014-15, Marks 10

Answer

$$(D + 5)\, x - 2y = t \quad \text{...(1.13.1)}$$
$$x + (D + 1)\, y = 0 \quad \text{...(1.13.2)}$$

On multiplying eq. (1.13.2) by $(D + 5)$ and subtracting from eq. (1.13.1), we get

$$(D + 1)(D + 5)\, y + 2y = -t$$
$$(D^2 + 6D + 5 + 2)\, y = -t$$

Auxiliary equation, $m^2 + 6m + 7 = 0$

$$m = \frac{-6 \pm \sqrt{36 - 28}}{2} \Rightarrow m = -3 \pm \sqrt{2}$$

$$\text{CF} = e^{-3t}(C_1 \cosh \sqrt{2}\, t + C_2 \sinh \sqrt{2}\, t)$$

$$\text{PI} = \frac{1}{D^2 + 6D + 7}(-t)$$

$$= \frac{-1}{7}\left(1 + \frac{D^2 + 6D}{7}\right)^{-1}(t) = -\frac{1}{7}\left(1 - \frac{6D}{7}\right)t$$

$$\text{PI} = -\frac{1}{7}\left(t - \frac{6}{7}\right)$$

$$y = e^{-3t}(C_1 \cosh \sqrt{2}\, t + C_2 \sinh \sqrt{2}\, t) - \frac{1}{7}\left(t - \frac{6}{7}\right) \quad \text{...(1.13.3)}$$

$$\frac{dy}{dt} = e^{-3t}(-C_1\sqrt{2} \sinh \sqrt{2}\, t + \sqrt{2}\, C_2 \cosh \sqrt{2}\, t)$$
$$- 3e^{-3t}(C_1 \cosh \sqrt{2}\, t + C_2 \sinh \sqrt{2}\, t) - \frac{1}{7}$$

From eq. (1.13.2),

$$x = -\frac{dy}{dt} - y$$

$$x = -e^{-3t}(-C_1\sqrt{2} \sinh \sqrt{2}\, t + \sqrt{2}\, C_2 \cosh \sqrt{2}\, t) + \frac{1}{7}$$
$$+ 3e^{-3t}(C_1 \cosh \sqrt{2}\, t + C_2 \sinh \sqrt{2}\, t)$$
$$- e^{-3t}(C_1 \cosh \sqrt{2}\, t + C_2 \sinh \sqrt{2}\, t) + \frac{1}{7}\left(t - \frac{6}{7}\right)$$

$$x = -e^{-3t}(-C_1\sqrt{2} \sinh \sqrt{2}\, t + \sqrt{2}\, C_2 \cosh \sqrt{2}\, t)$$
$$+ 2e^{-3t}(C_1 \cosh \sqrt{2}\, t + C_2 \sinh \sqrt{2}\, t) + \frac{t}{7} + \frac{1}{49} \quad \text{...(1.13.4)}$$

Boundary conditions

$$x(0) = 0, y(0) = 0$$

From eq. (1.13.3) and eq. (1.13.4), we have

$$0 = C_1 + \frac{6}{7}$$

$$C_1 = -\frac{6}{7}$$

and $$0 = -\sqrt{2}\, C_2 + 2C_1 + \frac{1}{49}$$

$$\sqrt{2}\, C_2 = -\frac{12}{7} + \frac{1}{49}$$

$$\sqrt{2}\, C_2 = -\frac{83}{49}$$

$$C_2 = -\frac{83}{49\sqrt{2}}$$

Now, $$y = e^{-3t}\left[-\frac{6}{7} \cosh \sqrt{2t} - \frac{83}{49\sqrt{2}} \sinh \sqrt{2t}\right] - \frac{1}{7}\left(t - \frac{6}{7}\right)$$

$$x = -e^{-3t}\left(-\frac{6}{7}\sqrt{2} \sinh \sqrt{2t} - \frac{83}{49} \cosh \sqrt{2t}\right)$$
$$+ 2e^{-3t}\left(t - \frac{6}{7}\sqrt{2t} - \frac{83}{49\sqrt{2}} \sinh \sqrt{2t}\right) + \frac{t}{7} + \frac{1}{49}$$

Que 1.14. Solve the following simultaneous equations.

$$\frac{d^2x}{dt^2} + y = \sin t$$

$$\frac{d^2y}{dt^2} + x = \cos t$$

AKTU 2015-16, Marks 10

Answer

Let $\frac{d}{dt} \equiv D$ then the given system of equations become

$$D^2x + y = \sin t \quad \text{...(1.14.1)}$$
$$x + D^2y = \cos t \quad \text{...(1.14.2)}$$

Multiplying eq. (1.14.1) by D^2, we get

$$D^4x + D^2y = -\sin t \quad \text{...(1.14.3)}$$

Subtracting eq. (1.14.2) from eq. (1.14.3), we get

$$(D^4 - 1)\,x = -\sin t - \cos t$$

Auxiliary equation is

$$m^4 - 1 = 0$$

$$(m^2 - 1)(m^2 + 1) = 0$$

$$\Rightarrow \quad m = 1, -1, \pm i$$

$$\therefore \quad \text{CF} = C_1e^t + C_2e^{-t} + C_3 \cos t + C_4 \sin t$$

$$\text{PI} = \frac{1}{D^4 - 1}(-\sin t - \cos t)$$

$$= -t\frac{1}{4D^3}(\sin t + \cos t) = \frac{t}{4}(-\cos t + \sin t)$$

$$\therefore \quad x = C_1e^t + C_2e^{-t} + C_3\cos t + C_4\sin t + \frac{t}{4}(\sin t - \cos t) \quad ...(1.14.4)$$

$$Dx = C_1e^t + C_2e^{-t} - C_3\sin t + C_4\cos t + \frac{t}{4}(\cos t + \sin t) + \frac{1}{4}(\sin t - \cos t)$$

$$D^2x = C_1e^t + C_2e^{-t} - C_3\cos t + C_4\sin t + \frac{t}{4}(-\sin t + \cos t) + \frac{1}{4}(\cos t + \sin t) + \frac{1}{4}(\cos t + \sin t)$$

From eq. (1.14.1), $y = \sin t - \dfrac{d^2x}{dt^2}$

$$y = -C_1e^t - C_2e^{-t} + C_3\cos t + C_4\sin t + \frac{t}{4}(\sin t - \cos t) + \frac{1}{2}(\sin t - \cos t)$$

Eq. (1.14.4) and eq. (1.14.5), when taken together, give the complete solution of the given system of equations.

Que 1.15. Solve the following :

$$\frac{dx}{dt} = 3x + 8y$$

$$\frac{dy}{dt} = -x - 3y \text{ with } x(0) = 6 \text{ and } y(0) = -2$$

AKTU 2013-14, Marks 05

Answer

$$\frac{dx}{dt} = 3x + 8y$$

$$\frac{dy}{dt} = -x - 3y$$

Let $\dfrac{d}{dt} = D$, so the given equation reduces to

$$(D - 3)\,x - 8y = 0 \quad ...(1.15.1)$$

$$x + (D + 3)\,y = 0 \quad ...(1.15.2)$$

Multiply by $(D + 3)$ in eq. (1.15.1) and multiply by 8 in eq. (1.15.2), then add both equations

$$(D^2 - 9 + 8)\,x = 0$$

$$(D^2 - 1)\,x = 0$$

Auxiliary equation is, $m^2 - 1 = 0$

$$m = \pm 1$$

$$\text{CF} = C_1\,e^{-t} + C_2\,e^t \quad ...(1.15.3)$$

$$\text{PI} = 0$$

$$\therefore \quad x = C_1\,e^{-t} + C_2\,e^t$$

From eq. (1.15.1),

$$8y(t) = \frac{dx(t)}{dt} - 3x(t)$$

or

$$8y = \frac{dx}{dt} - 3x$$

$$8y = C_1(-1)e^{-t} + C_2\,e^t - 3\,[C_1\,e^{-t} + C_2\,e^t]$$

$$8y = -4C_1e^{-t} - 2\,C_2e^t$$

$$y = -0.5\,C_1\,e^{-t} - 0.25\,C_2\,e^t \quad ...(1.15.4)$$

Apply boundary conduction,

$$x(0) = 6$$

From eq. (1.15.3), $6 = C_1 + C_2$ ...(1.15.5)

From eq. (1.15.4), $y(0) = -2 = -0.5\,C_1 - 0.25\,C_2$ ...(1.15.6)

By solving eq. (1.15.5) and eq. (1.15.6), we get

$$C_1 = 2$$

$$C_2 = 4$$

$$\therefore \quad x = 2\,e^{-t} + 4\,e^t$$

$$y = -e^{-t} - e^t$$

Que 1.16. Solve $\dfrac{dx}{dt} + 2x + 4y = 1 + 4t;\ \dfrac{dy}{dt} + x - y = \dfrac{3}{2}t^2.$

AKTU 2012-13, Marks 05

Answer

$$\frac{dx}{dt} + 2x + 4y = 1 + 4t,\quad \frac{dy}{dt} + x - y = \frac{3}{2}t^2$$

Writing D for $\dfrac{d}{dt}$, the given equation becomes

$$(D + 2)\,x + 4y = 1 + 4t \quad ...(1.16.1)$$

$$x + (D - 1)y = \text{[illegible]} \quad ...(1.16.2)$$

To eliminate y, multiplying eq. (1.16.1) by $(D-1)$ and multiplying eq. (1.16.2) by 4, then subtracting, we get

$$[(D+2)(D-1)-4]x = (D-1)(1) + 4(D-1)t - 6t^2$$

$$(D^2+2D-D-2-4)x = -1+4-4t-6t^2$$

$$(D^2+D-6)x = 3-4t-6t^2$$

Auxiliary equation is

$$m^2+m-6=0$$

$$m^2+3m-2m-6=0$$

$$m(m+3)-2(m+3)=0$$

$$(m+3)(m-2)=0 \Rightarrow m=2,-3$$

$\therefore$ $$\text{CF} = C_1e^{2t}+C_2e^{-3t}$$

$$\text{PI} = \frac{1}{(D^2+D-6)}(3-4t-6t^2)$$

$$= \frac{3}{(D^2+D-6)}e^{0t} - \frac{4t}{(D^2+D-6)} - \frac{6}{(D^2+D-6)}t^2$$

$$= -\frac{3}{6}+\frac{4}{6}\frac{1}{\left[1+\left(-\frac{D^2}{6}-\frac{D}{6}\right)\right]}t + \frac{6}{6}\frac{1}{\left[1+\left(-\frac{D^2}{6}-\frac{D}{6}\right)\right]}t^2$$

$$= -\frac{3}{6}+\frac{4}{6}\left[1+\left(-\frac{D}{6}-\frac{D^2}{6}\right)\right]^{-1}t + \left[1+\left(-\frac{D}{6}-\frac{D^2}{6}\right)\right]^{-1}t^2$$

$$= -\frac{3}{6}+\frac{4}{6}\left[1+\frac{D}{6}+\frac{D^2}{6}\right]t + \left[1-\left(-\frac{D}{6}-\frac{D^2}{6}\right)+\left(-\frac{D}{6}-\frac{D^2}{6}\right)^2\right]t^2$$

$$= -\frac{3}{6}+\frac{4t}{6}+\frac{4}{36}+t^2+\frac{2t}{6}+\frac{2}{6}+\frac{2}{36} = t^2+\frac{6t}{6}+\frac{(-18+4+12+2)}{36}$$

$$\text{PI} = t^2+t$$

So, $$x = C_1e^{2t}+C_2e^{-3t}+t^2+t$$

Now $$\frac{dx}{dt} = 2C_1e^{2t}-3C_2e^{-3t}+2t+1$$

Substituting the values of x and $\frac{dx}{dt}$ in eq. (1.16.1), we get

$$4y = -2C_1e^{2t}+3C_2e^{-3t}-2t-1-2C_1e^{2t}-2C_2e^{-3t} - 2t^2-2t+1+4t$$

$$y = -C_1e^{2t}+\frac{1}{4}C_2e^{-3t}-\frac{1}{2}t^2$$

Que 1.17. **Solve the simultaneous differential equations**

$$\frac{d^2x}{dt^2}-4\frac{dx}{dt}+4x = y \text{ and } \frac{d^2y}{dt^2}+4\frac{dy}{dt}+4y = 25x+16e^t.$$

AKTU 2017-18, Marks 07

Answer

$$(D^2-4D+4)x-y=0 \quad \text{...(1.17.1)}$$

$$-25x+(D^2+4D+4)y = 16e^t \quad \text{...(1.17.2)}$$

Multiplying eq. (1.17.1) by D^2+4D+4 and adding to eq. (1.17.2), we get

$$(D^2-4D+4)(D^2+4D+4)x-25y = 16e^t$$

$$(D^4-8D-9)x = 16e^t$$

Auxiliary equation is,

$$m^4-8m^2-9=0$$

$$\Rightarrow \quad (m^2-9)(m^2+1)=0 \Rightarrow m=\pm i, \pm 3$$

$\therefore$ $$\text{CF} = C_1e^{-3t}+C_2e^{-3t}+C_3\cos t+C_4\sin t$$

$$\text{PI} = \frac{1}{D^4-8D^2-9}(16e^t) = -e^t$$

$\therefore$ $$x = C_1e^{3t}+C_2e^{-3t}+C_3\cos t+C_4\sin t-e^t \quad \text{...(1.17.3)}$$

$$\frac{dx}{dt} = 3C_1e^{3t}-3C_2e^{-3t}+C_3(-\sin t)+C_4\cos t-e^t$$

$$\frac{d^2x}{dt^2} = 9C_1e^{3t}+9C_2e^{-3t}-C_3\cos t-C_4\sin t-e^t$$

From eq. (1.17.1), $$y = \frac{d^2x}{dt^2}-4\frac{dx}{dt}+4x$$

$$= 9C_1e^{3t}+9C_2e^{-3t}-C_3\cos t-C_4\sin t-e^t$$

$$-4(3C_1e^{3t}-3C_2e^{-3t}-C_3\sin t+C_4\cos t-e^t)$$

$$+4(C_1e^{3t}+C_2e^{-3t}+C_3\cos t+C_4\sin t-e^t)$$

$$\Rightarrow y = C_1e^{3t}+25C_2e^{-3t}+(3C_3-4C_4)\cos t+(4C_3+3C_4)\sin t-e^t \quad \text{...(1.17.4)}$$

Eq. (1.17.3) and eq. (1.17.4) when taken together give the complete solution.

PART-3

Second Order Linear Differential Equations with Variable Coefficients, Solution by Changing Independent Variable, Reduction of Order.

CONCEPT OUTLINE

Second Order Linear Differential Equation : A differential equation of the form $\frac{d^2y}{dx^2}+P\frac{dy}{dx}+Qy=R$ is known linear differential equation of second order, where P, Q and R are functions of x alone.

Method of Reduction of Order to Solve Second Order Linear Differential Equation :

Let $y = u$ be a part of the complementary function of the given differential equation

$$\frac{d^2y}{dx^2}+P\frac{dy}{dx}+Qy=R \quad ...(1)$$

Where u is a function of x, then, we have

$$\frac{d^2u}{dx^2}+P\frac{du}{dx}+Qu=R \quad ...(2)$$

Let $y = uv$, be the complete solution of eq. (1), where v is a function of x.

Differentiating y w.r.t x,

$$\frac{dy}{dx} = u\frac{dv}{dx}+\frac{du}{dx}v$$

Again
$$\frac{d^2y}{dx^2} = u\frac{d^2v}{dx^2}+2\frac{du}{dx}\frac{dv}{dx}+v\frac{d^2u}{dx^2}$$

Substituting the values of y, $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$ in eq. (1), we get

$$u\frac{d^2v}{dx^2}+2\frac{du}{dx}\frac{dv}{dx}+v\frac{d^2u}{dx^2}+P\left(u\frac{dv}{dx}+v\frac{du}{dx}\right)+Q(uv)=R$$

$$u\frac{d^2v}{dx^2}+\left(2\frac{du}{dx}+Pu\right)\frac{dv}{dx}+\left(\frac{d^2u}{dx^2}+P\frac{du}{dx}+Qu\right)v=R$$

$$u\frac{d^2v}{dx^2}+\left(2\frac{du}{dx}+Pu\right)\frac{dv}{dx}=R$$

$$\frac{d^2v}{dx^2}+\left(\frac{2}{u}\frac{du}{dx}+P\right)\frac{dv}{dx}=\frac{R}{u} \quad ...(3)$$

Put $\frac{dv}{dx}=p$ then, $\frac{d^2v}{dx^2}=\frac{dp}{dx}$

Now eq. (3) becomes, $\frac{dp}{dx}+\left(\frac{2}{u}\frac{du}{dx}+P\right)P=\frac{R}{u}$ ...(4)

Eq. (4), is a linear differential equation of first order in p and x.

$$\text{IF} = e^{\int\left(\frac{2}{u}\frac{du}{dx}+P\right)dx} = e^{\left(\int\frac{2}{u}du+\int P\,dx\right)} = u^2 e^{\int P\,dx}$$

Solution of eq. (4) is given by

$$pu^2 e^{\int P\,dx} = \int\frac{R}{u}u^2 e^{\int P\,dx}\,dx + C_1$$

Where C_1 is an arbitrary constant of integration.

$$\Rightarrow \quad p = \frac{1}{u^2}e^{-\int P\,dx}\left[\int Ru\,e^{\int P\,dx}\,dx + C_1\right]$$

$$\therefore \quad \frac{dv}{dx} = \frac{1}{u^2}e^{-\int P\,dx}\left[\int Ru\,e^{\int P\,dx}\,dx + C_1\right]$$

Integration yields, $v = \int\frac{1}{u^2}e^{-\int P\,dx}\left[\int Ru\,e^{\int P\,dx}\,dx + C_1\right]dx + C_2$

where C_2 is an arbitrary constant of integration.

Hence the complete solution of eq. (1) is given by,

$$y = uv$$

$$\Rightarrow \quad y = u\int\frac{1}{u^2}e^{-\int P\,dx}\left[\int Ru\,e^{\int P\,dx}\,dx + C_1\right]dx + C_2u$$

Questions-Answers

Long Answer Type and Medium Answer Type Questions

Que 1.18. Solve $(3x+2)^2\frac{d^2y}{dx^2}-(3x+2)\frac{dy}{dx}-12y=6x$.

Answer

$$(3x+2)^2\frac{d^2y}{dx^2}-(3x+2)\frac{dy}{dx}-12y=6x$$

Using $3x+2=e^t$, $(3x+2)^2\frac{d^2y}{dx^2}=9D(D-1)y$ and $(3x+2)\frac{dy}{dx}=3Dy$,

we get

$$9D(D-1)y-3Dy-12y=2(e^t-2)$$
$$(9D^2-9D-3D-12)y=2(e^t-2)$$

The auxiliary equation is

$$9m^2-12m-12=0$$
$$(m-2)\left(m+\frac{2}{3}\right)=0$$
$$m=2,-\frac{2}{3}$$

Therefore, the complementary function is

$$\text{CF} = C_1e^{2t}+C_2e^{\frac{-2t}{3}}$$

and
$$\text{PI} = \frac{1}{9D^2-12D+12}2(e^t-2)$$
$$= 2\left\{\frac{1}{9D^2-12D-12}e^t-2\frac{e^0}{9D^2-12D-12}\right\}$$

$$= 2\frac{1}{9-12-12}e^z - 4\frac{1}{0-0-12} = \frac{2e^z}{-15} + \frac{1}{3}$$

The solution is $y = CF + PI$

$$y = C_1 e^{2z} + C_2 e^{\frac{-2z}{3}} + \frac{1}{3} - \frac{2}{15}e^z$$

Using, $z = \log(3x + 2)$, we get

$$y = C_1 e^{2\log(3x+2)} + C_2 e^{\frac{-2\log(3x+2)}{3}} + \frac{1}{3} - \frac{2}{15}e^{\log(3x+2)}$$

$$= C_1(3x+2)^2 + C_2(3x+2)^{-2/3} + \frac{1}{3} - \frac{2}{15}(3x+2)$$

C_1 and C_2 are arbitrary constants of integration.

Que 1.19. Solve $\dfrac{d^2y}{dx^2} + \dfrac{1}{x}\dfrac{dy}{dx} = \dfrac{12\log x}{x^2}$

Answer

Given equation may be written as

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} = 12\log x \qquad \text{(Let, } z = \log x\text{)}$$

or $\{D(D-1) + D\}\, y = 12z$

$$D^2 y = 12z$$

Auxiliary equation is, $m^2 = 0$

$$m = 0, 0$$

$$CF = (C_1 + C_2 z)\, e^{0z} = C_1 + C_2 z$$

$$PI = \frac{1}{D^2} 12z = 12\frac{1}{D^2} z = 12\frac{z^3}{6} = 2z^3$$

Complete solution, $y = CF + PI$

$$y = C_1 + C_2 z + 2z^3$$

$$y = C_1 + C_2 \log x + 2(\log x)^3$$

Que 1.20. **Write the procedure for solving the linear differential equation by changing the independent variable.**

Answer

Let the given differential equation is

$$\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R \qquad \text{...(1.20.1)}$$

Let the independent variable be changed from x to z and $z = f(x)$

$$\frac{dy}{dx} = \frac{dy}{dz}\frac{dz}{dx}$$

and
$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{dy}{dz}\frac{dz}{dx}\right)$$

$$= \frac{d^2y}{dz^2}\left(\frac{dz}{dx}\right)^2 + \frac{dy}{dz}\frac{d^2z}{dx^2}$$

Substituting the values of dy/dx and d^2y/dx^2 in eq. (1.20.1), we have

$$\left(\frac{dz}{dx}\right)^2\frac{d^2y}{dz^2} + \left(\frac{d^2z}{dx^2} + P\frac{dz}{dx}\right)\frac{dy}{dz} + Qy = R$$

$$\frac{d^2y}{dz^2} + P_1\frac{dy}{dz} + Q_1 y = R_1 \qquad \text{...(1.20.2)}$$

Where,
$$P_1 = \frac{\dfrac{d^2z}{dx^2} + P\dfrac{dz}{dx}}{(dz/dx)^2},$$

$$Q_1 = \frac{Q}{(dz/dx)^2} \quad R_1 = \frac{R}{(dz/dx)^2}$$

P_1, Q_1, and R_1 are functions of x but may be expressed as functions of z by the given relation between z and x.
Here, we choose z to make the coefficient of dy/dx zero, *i.e.*,

$$P_1 = 0$$

and
$$\frac{d^2z}{dx^2} + P\frac{dz}{dx} = 0$$

or
$$\frac{d^2z/dx^2}{dz/dx} = -P$$

Integrating, we get

$$\ln\frac{dz}{dx} = -\int P\,dx$$

$$\frac{dz}{dx} = e^{-\int P dx}$$

Integrating again, we get

$$z = \int e^{-\int P dx} dx$$

Now, eq. (1.20.2) reduces to

$$\frac{d^2y}{dz^2} + Q_1 y = R_1$$

Which can be solved easily provided Q_1 comes out to be a constant or a constant multiplied by $1/z^2$. Again if we choose z such that,

$$Q_1 = \frac{Q}{(dz/dx)^2} = a^2 \quad \text{(Constant)}$$

$$a^2\left(\frac{dz}{dx}\right)^2 = Q$$

$$a\frac{dz}{dx} = \sqrt{Q}$$

$$az = \int \sqrt{Q}\, dx$$

Then eq. (1.20.2) reduces to

$$x\frac{d^2y}{dx^2} + P_1\frac{dy}{dz} + a^2y = R_1$$

Which can be solved easily provided P_1 comes out to be a constant.

Que 1.21. **Solve by changing the independent variable :**

$$\frac{d^2y}{dx^2} + (3\sin x - \cot x)\frac{dy}{dx} + 2y\sin^2 x = e^{-\cos x}\sin^2 x$$

AKTU 2014-15, Marks 05

Answer

$y'' + (3\sin x - \cot x)y' + 2y\sin^2 x = e^{-\cos x}\sin^2 x$

Changing independent variable

$$z = f(x)$$

$$\frac{dy}{dx} = \frac{dy}{dz}\frac{dz}{dx},\ \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dz}\frac{dz}{dx}\right) = \frac{d}{dx}\left(\frac{dy}{dz}\right)\frac{dz}{dx} + \frac{d^2z}{dx^2}$$

$$= \frac{d}{dz}\left(\frac{dy}{dz}\right)\left(\frac{dz}{dx}\right)\left(\frac{dz}{dx}\right) + \frac{d^2z}{dx^2}$$

$$= \frac{d^2y}{dz^2}\left(\frac{dz}{dx}\right)^2 + \frac{dy}{dz}\frac{d^2z}{dx^2}$$

Now from given equation,

$$\frac{d^2y}{dz^2}\left(\frac{dz}{dx}\right)^2 + \frac{d^2z}{dx^2} + (3\sin x - \cot x)\frac{dy}{dz}\frac{dz}{dx} + 2y\sin^2 x = e^{-\cos x}\sin^2 x$$

$$\frac{d^2y}{dz^2} + \frac{d^2z}{dx^2} + \frac{(3\sin x - \cot x)\frac{dy}{dz}\frac{dz}{dx}}{\left(\frac{dz}{dx}\right)^2} + \frac{2\sin^2 x}{\left(\frac{dz}{dx}\right)^2}y = e^{-\cos x}\sin^2 x$$

This can be written as

$$\frac{d^2y}{dz^2} + P_1\frac{dy}{dz} + Q_1y = R_1$$

Where,

$$P_1 = \frac{\frac{d^2y}{dz^2} + (3\sin x - \cot x)\frac{dz}{dx}}{\left(\frac{dz}{dx}\right)^2}$$

$$Q_1 = \frac{2\sin^2 x}{(dz/dx)^2},\ R_1 = \frac{e^{-\cos x}\sin^2 x}{(dz/dx)^2}$$

Choose $Q_1 = 2$, *i.e.*, $2 = \frac{2\sin^2 x}{\left(\frac{dz}{dx}\right)^2} \Rightarrow \left(\frac{dz}{dx}\right)^2 = \sin^2 x \Rightarrow \frac{dz}{dx} = \sin x$

$$z = -\cos x$$

$$\frac{d^2z}{dx^2} = \cos x$$

Now,

$$P_1 = \frac{\cos x + (3\sin x - \cot x)\sin x}{\sin^2 x}$$

$$= \frac{\cos x + 3\sin^2 x - \frac{\cos x}{\sin x}\sin x}{\sin^2 x} = 3$$

$$R_1 = \frac{e^{-\cos x}\sin^2 x}{\sin^2 x} = e^{-\cos x}$$

$$\frac{d^2y}{dz^2} + 3\frac{dy}{dz} + 2y = e^{-\cos x}$$

$$\frac{d^2y}{dz^2} + 3\frac{dy}{dz} + 2y = e^{-z}$$

Auxiliary equation is $m^2 + 3m + 2 = 0$

$$m = -1, -2$$

$$\text{CF} = C_1e^{-z} + C_2e^{-2z}$$

$$\text{PI} = \frac{1}{(D+2)(D+1)}e^{z} = \frac{1}{D^2 + 3D + 2}e^{z}$$

Put, $D = -1$

$$= \frac{1}{1+3+2}e^{z}\ \frac{e^{z}}{6}$$

$\therefore$ Complete solution $= \text{CF} + \text{PI} = C_1\frac{e^{z}}{6} + C_2e^{-z} + e^{-z} = C_1e^{-\cos x} + C_2e^{-\cos x} + e^{-\cos x/6}$

PART-4

Normal Form.

Questions-Answers

Long Answer Type and Medium Answer Type Questions

Que 1.22. How can we solve differential equation by removing the first derivative or converting in normal form ?

Answer

A part of the complementary function is needed to find the complete solution, it is not always possible to find an integral belonging to CF in such cases, we reduce the given equation to the form in which the term containing the first derivative is absent. For this, we shall change the dependent variable in the equation.

$$\frac{d^2y}{dx^2}+P\frac{dy}{dx}+Qy=R \quad \text{....(1.22.1)}$$

By putting $y = uv$, where u is some function of x, so that

$$\frac{dy}{dx} = u\frac{dv}{dx}+v\frac{du}{dx}$$

and
$$\frac{d^2y}{dx^2} = u\frac{d^2v}{dx^2}+2\frac{du}{dx}\frac{dv}{dx}+v\frac{d^2u}{dx^2}$$

On substituting dy/dx and d^2y/dx^2 in terms of u and v in eq. (1.22.1), we get

$$u\frac{d^2v}{dx^2}+\left(Pu+2\frac{du}{dx}\right)\frac{dv}{dx}+\left(\frac{d^2u}{dx^2}+P\frac{du}{dx}+Qu\right)v = R$$

$$\frac{d^2v}{dx^2}+\left(P+\frac{2}{u}\frac{du}{dx}\right)\frac{dv}{dx}+\left(\frac{1}{u}\frac{d^2u}{dx^2}+\frac{P}{u}\frac{du}{dx}+Q\right)v = R/u \quad \text{...(1.22.2)}$$

Let us choose u such that,

$$P+\frac{2}{u}\frac{du}{dx}=0$$

$$\frac{du}{dx} = -\frac{P}{2}u$$

$$\frac{du}{u} = -\frac{P}{2}dx$$

$$\therefore \quad u = e^{-1/2\int P dx}$$

Now, from eq. (1.22.2), we have

$$\frac{d^2v}{dx^2}+\left[\frac{1}{u}\left(-\frac{u}{2}\frac{dP}{dx}-\frac{P}{2}\frac{du}{dx}\right)+\frac{P}{u}\frac{du}{dx}+Q\right]v = R\,e^{1/2\int P dx}$$

$$\frac{d^2v}{dx^2}+\left[-\frac{1}{2}\frac{dP}{dx}-\frac{P}{2u}\left(-\frac{P}{2}u\right)+\frac{P}{2}\left(\frac{-P}{2}u\right)+Q\right]v = R\,e^{1/2\int P dx}$$

$$\frac{d^2v}{dx^2}+\left[Q-\frac{1}{2}\frac{dP}{dx}-\frac{1}{4}P^2\right]v = R\,e^{1/2\int P dx}$$

$$\left.\begin{aligned} \text{or} \quad & \frac{d^2v}{dx^2}+Xv=Y \\ \text{Where} \quad & X=Q-\frac{1}{2}\frac{dP}{dx}-\frac{1}{4}P^2 \\ \text{and} \quad & Y=R\,e^{1/2\int P dx} \end{aligned}\right\} \quad \text{...(1.22.3)}$$

Eq. (1.22.3) may easily be integrated and is known as normal form of eq. (1.22.1).

Que 1.23. **Solve the following equation by reducing into normal form.**

$$\frac{d^2y}{dx^2}+2x\frac{dy}{dx}+(x^2-8)y = x^2e^{-x^2/2}.$$

AKTU 2011-12, Marks 05

OR

Solve the following differential equation by reducing into normal form :

$$y''+2xy'+(x^2-8)y = x^2e^{-\frac{1}{2}x^2}.$$

AKTU 2012-13, Marks 05

Answer

$$\frac{d^2y}{dx^2}+2x\frac{dy}{dx}+(x^2-8)y = x^2e^{-x^2/2}$$

On comparison with, $\frac{d^2y}{dx^2}+P\frac{dy}{dx}+Qy=R$, we have

$$P=2x,\ Q=x^2-8,\ R=x^2e^{-x^2/2}$$

$$v = e^{-\frac{1}{2}\int P dx} = e^{-\frac{1}{2}\int 2x dx} = e^{-\frac{x^2}{2}}$$

We know that, u is given by

$$\frac{d^2u}{dx^2}+Q_1u=R_1 \quad \text{...(1.23.1)}$$

Where,
$$Q_1 = Q-\frac{1}{2}\frac{dP}{dx}-\frac{P^2}{4} = x^2-8-\frac{1}{2}(2)-\frac{4x^2}{4}$$

$$Q_1 = -9$$

$$R_1 = \frac{R}{v} = \frac{x^2e^{-x^2/2}}{e^{-x^2/2}} = x^2$$

On putting the value of Q_1 and R_1 in eq. (1.23.1), we get

$$\frac{d^2u}{dx^2}-9u = x^2$$

$$(D^2-9)u = x^2$$

Auxiliary equation, $m^2-9=0$

$$m = \pm 3$$

$$CF = C_1 e^{3x} + C_2 e^{-3x}$$

$$PI = \frac{1}{D^2 - 9} x^2 = -\frac{1}{9}\left(1 - \frac{D^2}{9}\right)^{-1} x^2 = -\frac{1}{9}\left(1 + \frac{D^2}{9}\right) x^2$$

$$PI = -\frac{1}{9}\left(x^2 + \frac{2}{9}\right)$$

Complete solution, $u = CF + PI = C_1 e^{3x} + C_2 e^{-3x} - \frac{1}{9}\left(x^2 + \frac{2}{9}\right)$

Thus $y = uv = \left[C_1 e^{3x} + C_2 e^{-3x} - \frac{1}{9}\left(x^2 + \frac{2}{9}\right)\right] e^{-\frac{x^2}{2}}$

Que 1.24. **Using normal form, solve :**

$$\frac{d^2y}{dx^2} - 4x\frac{dy}{dx} + (4x^2 - 1) y = -3e^{x^2} \sin 2x$$

AKTU 2013-14, Marks 05

Answer

Here, $P = -4x, Q = 4x^2 - 1, R = -3e^{x^2} \sin 2x$

Let $y = uv$ be the complete solution.

Now, $u = e^{-\frac{1}{2}\int(-4x)dx} = e^{x^2}$

$$Q_1 = Q - \frac{1}{2}\frac{dP}{dx} - \frac{P^2}{4}$$

$$= 4x^2 - 1 - \frac{1}{2}(-4) - \frac{1}{4}(16x^2) = 1$$

Also, $R_1 = \frac{R}{u} = \frac{-3e^{x^2} \sin 2x}{e^{x^2}} = -3\sin 2x$

Hence normal form is, $\frac{d^2v}{dx^2} + v = -3 \sin 2x$

Auxiliary equation, $m^2 + 1 = 0 \Rightarrow m = \pm i$

$$CF = C_1 \cos x + C_2 \sin x$$

$$PI = \frac{1}{D^2 + 1}(-3 \sin 2x) = \frac{-3}{(-4+1)} \sin 2x$$

$$PI = \sin 2x$$

Complete solution, $v = CF + PI = C_1 \cos x + C_2 \sin x + \sin 2x$

Hence the complete solution of given differential equation is

$$y = uv = e^{x^2}(C_1 \cos x + C_2 \sin x + \sin 2x)$$

PART-5

Method of Variation of Parameters.

CONCEPT OUTLINE

Method of Variation of Parameters : By this method the general solution is obtained by varying the arbitrary constants of the complementary function that is why the method is known as method of variation of parameters.

Procedure : First find the complementary function of the given differential equation.

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = X$$

Let it is be $CF = Ay_1 + By_2$ ...(1)

So that y_1 and y_2 satisfy given differential equation let us assume

$PI = u y_1 + v y_2$ ...(2)

Where u and v are given by

$$u = \int \frac{-X y_2}{y_1 y_2' - y_2 y_1'} dx$$

and

$$v = \int \frac{X y_1}{y_1 y_2' - y_2 y_1'} dx$$

Putting u and v in eq. (2), we can find PI and then complete solution

$$y = CF + PI$$

Questions-Answers

Long Answer Type and Medium Answer Type Questions

Que 1.25. **Apply method of variation of parameters to solve**

$$x^2\frac{d^2y}{dx^2} + 4x\frac{dy}{dx} + 2y = e^x$$

AKTU 2011-12, Marks 10

Answer

$$x^2\frac{d^2y}{dx^2} + 4x\frac{dy}{dx} + 2y = e^x$$

$$[D(D-1) + 4D + 2]y = e^{e^t} \quad [\because x = e^t]$$

$$(D^2 + 3D + 2)y = e^{e^t}$$

Auxiliary equation, $m^2 + 3m + 2 = 0 \Rightarrow m = -1, -2$

$$CF = C_1 e^{-t} + C_2 e^{-2t}$$

$$\text{PI} = \frac{1}{D^2+3D+2} e^{e^z}$$

(Using General method to find PI)

$$= \frac{1}{(D+1)(D+2)} e^{e^z} = \left(\frac{1}{D+1} - \frac{1}{D+2}\right) e^{e^z}$$

$$= \frac{1}{D+1} e^{e^z} - \frac{1}{D+2} e^{e^z}$$

$$= e^{-z}\int e^z e^{e^z}\, dz - e^{-2z}\int e^{2z} e^{e^z}\, dz$$

Let $e^z = t \Rightarrow e^z\, dz = dt$

$$= e^{-z}\int e^t\, dt - e^{-2z}\int t e^t\, dt = e^{-z} e^t - e^{-2z}(te^t - e^t)$$

$$= e^{-z} e^{e^z} - e^{-2z}(e^z e^{e^z} - e^{e^z}) = e^{-2z} e^{e^z}$$

Complete solution, $y = \text{CF} + \text{PI}$

$$y = C_1 e^{-z} + C_2 e^{-2z} + e^{-2z} e^{e^z}$$

$$y = C_1\left(\frac{1}{x}\right) + C_2\left(\frac{1}{x^2}\right) + \left(\frac{1}{x^2}\right) e^x$$

Que 1.26. **Using variation of parameters method, solve**

$$x^2\frac{d^2y}{dx^2} + 2x\frac{dy}{dx} - 12y = 0$$

AKTU 2015-16, Marks 10

Answer

Same as Q. 1.25, Page 1–28F, Unit-1.

(**Answer :** $y = C_1 x_3 + C_2 / x_4$)

Que 1.27. **Apply method of variation of parameters to find the general solution of**

$$\frac{d^2x}{dt^2} - 4\frac{dx}{dt} + 3x = \frac{e^t}{1+e^t}$$

AKTU 2012-13, Marks 10

Answer

$$\frac{d^2x}{dt^2} - 4\frac{dx}{dt} + 3x = \frac{e^t}{1+e^t}$$

$$(D^2 - 4D + 3)x = \frac{e^t}{1+e^t}$$

Auxiliary equation, $m^2 - 4m + 3 = 0$

$$m = 1, 3$$

$$\text{CF} = C_1 e^t + C_2 e^{3t}$$

Here, part of CF are $u = e^t$, $v = e^{3t}$. Also, $R = \dfrac{e^t}{1+e^t}$

Let $x = Ae^t + Be^{3t}$ be the complete solution of the given equation where A and B are suitable function of t.

To determine A and B, we have

$$A = \int \frac{-Rv}{uv_1 - u_1v}\, dt + C_1 = -\int \frac{e^t e^{3t}}{(1+e^t)(3e^{4t} - e^{4t})}\, dt + C_1$$

$$= -\int \frac{e^{4t}}{2(1+e^t)e^{4t}}\, dt + C_1 = -\int \frac{e^{-t}}{2(e^{-t}+1)}\, dt + C_1$$

$$= \frac{1}{2}\ln(e^{-t}+1) + C_1$$

$$B = \int \frac{Ru}{uv_1 - u_1v}\, dt + C_2$$

$$= \int \frac{e^t e^t}{(1+e^t)(3e^{4t} - e^{4t})}\, dt + C_2 = \int \frac{e^{2t}}{2(1+e^t)e^{4t}}\, dt + C_2$$

$$= \frac{1}{2}\int \frac{e^{-2t}}{(1+e^t)}\, dt + C_2 = \frac{1}{2}\int \frac{e^{-3t}}{(e^{-t}+1)}\, dt + C_2$$

$$= -\frac{1}{4}(e^{-t}+1)^2 - \frac{1}{2}\ln(e^{-t}+1) + C_2$$

Hence the complete solution is

$$x = \left[\frac{1}{2}\ln(e^{-t}+1) + C_1\right]e^t + \left[-\frac{1}{4}(e^{-t}+1)^2 \ln(e^{-t}+1) + (e^{-t}+1) + C_2\right]e^{3t}$$

Que 1.28. **Solve by method of variation of parameters for the differential equation :**

$$\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 9y = \left(\frac{e^{3x}}{x^2}\right)$$

AKTU 2016-17, Marks 07

Answer

$$\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 9y = \left(\frac{e^{3x}}{x^2}\right)$$

Auxiliary equation,

$$m^2 - 6m + 9 = 0$$

$$(m-3)^2 = 0$$

$$m = 3, 3$$

So, $\text{CF} = (C_1 + C_2x)e^{3x}$

Here $u = e^{3x}$ and $v = x\,e^{3x}$ are two parts of CF

Also, $R = \dfrac{e^{3x}}{x^2}$

Let the complete solution be

$$y = A e^{3x} + Bx e^{3x}$$

To determine the values of A and B, we have

$$A = \int -\frac{Rv}{uv_1 - u_1v} dx + C_1$$

$$A = \int -\frac{\left(\frac{e^{3x}}{x^2}\right) xe^{3x}}{e^{3x}(e^{3x} + 3x e^{3x}) - xe^{3x} 3e^{3x}} dx + C_1$$

$$A = -\int \frac{e^{6x}/x}{e^{6x}} dx + C_1$$

$$A = -\int \frac{1}{x} dx + C_1$$

$$A = -\log x + C_1$$

$$B = \int \frac{Ru}{uv_1 - u_1v} dx + C_2$$

$$B = \int \frac{\frac{e^{3x}}{x^2} e^{3x}}{e^{3x}(e^{3x} + 3x e^{3x}) - 3e^{3x} xe^{3x}} dx + C_2$$

$$B = \int \frac{1}{x^2} dx + C_2$$

$$B = -\frac{1}{x} + C_2$$

Hence the complete solution is

$$y = (-\log x + C_1) e^{3x} + \left(-\frac{1}{x} + C_2\right) x e^{3x}$$

Que 1.29. **Use variation of parameters method to solve the differential equation $x^2 y'' + xy' - y = x^2 e^x$.**

AKTU 2017-18, Marks 07

Answer

$$x^2 y'' + xy' - y = x^2 e^x. \quad \text{...(1.29.1)}$$

$$y'' + \frac{y'}{x} - \frac{y}{x^2} = e^x \quad \text{...(1.29.2)}$$

Here, $R = e^x$

Consider the equation $y'' + \frac{y'}{x} - \frac{y}{x^2} = 0$ for finding parts of CF

Put $x = e^z$ so that $z = \log x$

So, $[D(D-1) + D - 1] y = 0$

$$(D^2 - 1) y = 0 \quad \text{...(1.29.3)}$$

Auxiliary equation, $m^2 - 1 = 0 \Rightarrow m = \pm 1$

$\therefore$ $\text{CF} = C_1e^z + C_2e^{-z} = C_1x + C_2\frac{1}{x}$

Hence parts of CF are x and $\frac{1}{x}$

Let $u = x$ and $v = \frac{1}{x}$

Let $y = Ax + \frac{B}{x}$ be the complete solution, where A and B are some suitable functions of x. A and B are determined as follows :

$$A = -\int \frac{Rv}{uv_1 - u_1v} dx + C_1$$

$$= -\int \frac{e^x \frac{1}{x}}{x\left(\frac{-1}{x^2}\right) - 1\left(\frac{1}{x}\right)} dx + C_1$$

$$= -\int \frac{e^x \frac{1}{x}}{\left(\frac{-2}{x}\right)} dx + C_1 = \frac{1}{2}e^x + C_1$$

and $$B = \int \frac{Ru}{uv_1 - u_1v} dx + C_2 = \int \frac{e^x x}{x\left(\frac{-1}{x^2}\right) - \left(\frac{1}{x}\right)} dx + C_2$$

$$= \int \frac{e^x x}{\left(\frac{-2}{x}\right)} dx + C_2 = -\frac{1}{2}\int x^2 e^x \, dx + C_2$$

$$= -\frac{1}{2}\left[x^2 e^x - \int 2x e^x \, dx\right] + C_2 = -\frac{1}{2}[x^2 - 2(x-1) e^x] + C_2$$

$$= -\frac{1}{2}x^2e^x + (x-1) e^x \; C_2$$

Hence the complete solution is given by

$$y = Ax + \frac{B}{x} = \left(\frac{1}{2}e^x + C_1\right) x + \left[-\frac{1}{2}x^2 e^x + (x-1) e^x + C_2\right]\frac{1}{x}$$

$$y = C_1x + \frac{C_2}{x} + \left(1 - \frac{1}{x}\right)e^x$$

PART-6

Cauchy Euler Equation.

CONCEPT OUTLINE

Cauchy-Euler Equation : An equation of the form

$$x^n \frac{d^n y}{dx^n} + a_1 x^{n-1} \frac{d^{n-1}y}{dx^{n-1}} + a_2 x^{n-2} \frac{d^{n-2}y}{dx^{n-2}} + \ldots\ldots + a_{n-1} x \frac{dy}{dx} + a_n y = Q$$

Where a_i's are constants and Q is a function of x, called Cauchy's homogeneous linear equation. Such equations can be reduced to linear differential equations with constant coefficients by the substitution

$$x = e^t \quad \text{or} \quad z = \log x$$

Questions-Answers

Long Answer Type and Medium Answer Type Questions

Que 1.30. Solve : $x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = (\log x) \sin (\log x)$.

UPTU 2014-15, Marks 05

Answer

$$x^2 y'' + xy' + y = (\log x) \sin (\log x)$$

This is the Cauchy Euler equation.

Put $x = e^t$, $t = \log x$, $x^2 y'' = D(D-1)y$, and we get $xy' = Dy$

$$[D(D-1) + D + 1]y = t \sin t$$

$$[D^2 - D + D + 1]y = t \sin t$$

$$(D^2 + 1)y = t \sin t$$

Auxiliary equation, $m^2 + 1 = 0$, $m = \pm i$

$$\text{CF} = C_1 \cos t + C_2 \sin t$$

$$\text{PI} = \frac{1}{D^2+1} t \sin t$$

$$= \text{Imaginary part of } \frac{1}{D^2+1} e^{it} \sin t$$

Put $D = D + i$,

$$= \text{Imaginary part of } e^{it} \frac{1}{(D+i)^2+1} \sin t$$

$$= \text{Imaginary part of } e^{it} \frac{1}{D^2 - 1 + 2Di + 1} \sin t$$

$$= \text{Imaginary part of } e^{it} \frac{1}{D^2 + 2Di} \sin t$$

Put $D^2 = -1$,

$$= \text{Imaginary part of } e^{it} \frac{1}{2Di - 1} \sin t$$

$$= \text{Imaginary part of } e^{it} \frac{2Di+1}{(2Di+1)(2Di-1)} \sin t$$

$$= \text{Imaginary part of } e^{it} \frac{(2Di+1)}{-4D^2-1} \sin t$$

$$= \text{Imaginary part of } e^{it} \frac{(1+2Di)}{3} \sin t$$

$$= \text{Imaginary part of } \frac{1}{3} (\cos t + i \sin t)(\sin t - 2i \cos t)$$

$$= \frac{1}{3} (\sin^2 t - 2\cos^2 t)$$

$$\text{PI} = \frac{1}{3} (\sin^2 t - 2\cos^2 t)$$

Complete solution, $y = \text{CF} + \text{PI} = C_1 \cos t + C_2 \sin t + \frac{1}{3} (\sin^2 t - 2\cos^2 t)$

Where, $t = \log x$

$$y = C_1 \cos (\log x) + C_2 \sin (\log x) + \frac{1}{3} [\sin^2 (\log x) - 2\cos^2 (\log x)]$$

PART-7

Series Solution (Frobenius Method).

CONCEPT OUTLINE

Frobenius Method : Following are the steps of solving differential equation with the help of frobenius method :

1. Assume $y = a_0 x_m + a_1 x^{m+1} + a_2 x^{m+2} + \ldots$ ...(1)
2. Substitute from eq. (1) for y, $\frac{dy}{dx}, \frac{d^2y}{dx^2}$ in given equation
3. Equate to zero the coefficient of lowest power of x. This gives a quadratic equation in m which is known as the Indicial equation.
4. Equate to zero, the coefficients of other powers of x to find $a_1, a_2, a_3, \ldots$ in terms of a_0.
5. Substitute the values of $a_1, a_2, a_3, \ldots$ in eq. (1) to get the series solution of the given equation having a_0 as arbitrary constant. Obviously, this is not the complete solution of given equation since the complete solution must have two independent arbitrary constants.

Questions-Answers

Long Answer Type and Medium Answer Type Questions

Que 1.31. Find the series solution of the following differential equation.

$$2x(1-x)\frac{d^2y}{dx^2}+(1-x)\frac{dy}{dx}+3y=0$$

AKTU 2015-16, Marks 10

Answer

$$2x(1-x)y''+(1-x)y'+3y=0 \quad \text{...(1.31.1)}$$

Dividing eq. (1.31.1) by $2x(1-x)$, we get

$$y''+\frac{1}{2x}y'+\frac{3}{2x(1-x)}y=0 \quad \text{...(1.31.2)}$$

Comparing eq. (1.31.2) with $y''+P(x)y'+Q(x)y=0$, we get

$$P(x)=\frac{1}{2x} \text{ and } Q(x)=\frac{3}{2x(1-x)}$$

Here $P(x)$ and $Q(x)$ both are non-analytic at $x=0$. But $xP(x)=\frac{1}{2}$ and

$x^2Q(x)=\frac{3x}{(1-x)}$ are analytic therefore $x=0$ is a regular singular point.

Let the solution of the given differential equation is

$$y=\sum_{k=0}^{\infty}a_kx^{m+k}$$

$$y'=\sum_{k=0}^{\infty}a_k(m+k)x^{m+k-1}$$

$$y''=\sum_{k=0}^{\infty}a_k(m+k)(m+k-1)x^{m+k-2}$$

Putting all these values in given differential equation and collecting the like terms, we get

$$\sum_{k=0}^{\infty}a_k(m+k+1)(-2m-2k+3)x^{m+k}+\sum_{k=0}^{\infty}a_k(m+k)(2m+2k-1)x^{m+k-2}=0 \quad \text{...(1.31.3)}$$

Equating the coefficient of lowest degree term x^{m-2} to zero.

$$a_0m(2m-1)=0$$

$\because$ $a_0\neq 0$

$\therefore$ $m=0,\frac{1}{2}$

Roots are different and not differing by an integer. The general term is obtained by replacing k by $k+1$ in second summation of eq. (1.31.3).

$a_k(m+k+1)(-2m-2k+3)+a_{k+1}(m+k+1)(2m+2k+1)=0$

$\therefore$ $$a_{k+1}=\frac{-(m+k+1)(-2m-2k+3)}{(m+k+1)(2m+2k+1)}a_k$$

Thus, $$a_{k+1}=\frac{2m+2k-3}{2m+2k+1}a_k$$

Putting $k=0,1,2.........$

$$a_1=\frac{2m-3}{2m+1}a_0$$

$$a_2=\frac{(2m-1)}{(2m+3)}a_1$$

$$a_3=\frac{(2m+1)}{(2m+5)}a_2$$

$$a_4=\frac{(2m+3)}{(2m+7)}a_3$$

$$a_5=\frac{(2m+5)}{(2m+9)}a_4$$

At $m=0$, $a_1=-3a_0, a_2=a_0, a_3=\frac{1}{5}a_0, a_4=\frac{3}{35}a_0, a_5=\frac{1}{21}a_0$

$$y_1=y_{m=0}=x^m(a_0+a_1x+a_2x^2+a_3x^3+a_4x^4+a_5x^5+....)$$

$$=x^0a_0\left(1-3x+x^2+\frac{1}{5}x^3+\frac{3}{35}x^4+\frac{1}{21}x^5+....\right)$$

$$y_1=a_0\left(1-3x+\frac{3x^2}{1.3}+\frac{3}{3.5}x^3+\frac{3}{5.7}x^4+\frac{3}{7.9}x^5+......\right)$$

At $m=1/2$, $a_1=-a_0, a_2=0, a_3=0, a_4=a_5=a_6=............=0$

$$y_2=(y)_{m=1/2}=x^{1/2}a_0(1-x+0......)$$

$$y_2=\sqrt{x}\,a_0(1-x)$$

General solution is $y=Ay_1+By_2$

$$y=A\left(1-3x+\frac{3}{1.3}x^2+\frac{3}{3.5}x^3+\frac{3}{5.7}x^4+\frac{3}{7.9}x^5+.....\right)+B\sqrt{x}(1-x)$$

Que 1.32. Solve in series : $2x^2y''+x(2x+1)y'-y=0$.

AKTU 2014-15, Marks 10

Answer

$$2x^2y''+x(2x+1)y'-y=0 \quad \text{...(1.32.1)}$$

$x=0$ is a regular singular point.

Let,
$$y = \sum_{k=0}^{\infty} a_k x^{m+k}$$
$$y' = \sum_{k=0}^{\infty} a_k (m+k) x^{m+k-1}$$
$$y'' = \sum_{k=0}^{\infty} a_k (m+k) x^{m+k-2} (m+k-1)$$

Putting the value of y, y' and y'' in eq. (1.32.1), we get
$$2\sum_{k=0}^{\infty} a_k (m+k)(m+k-1) x^{m+k} + 2\sum_{k=0}^{\infty} a_k (m+k) x^{m+k+1}$$
$$+ \sum_{k=0}^{\infty} a_k (m+k) x^{(m+k)} - \sum_{k=0}^{\infty} a_k x^{m+k} = 0$$
$$2\sum_{k=0}^{\infty} a_k (m+k) x^{m+k+1} + \sum_{k=0}^{\infty} a_k [(m+k)(2m+2k-2+1)-1] x^{m+k} = 0$$
$$2\sum_{k=0}^{\infty} a_k (m+k) x^{m+k+1} + \sum_{k=0}^{\infty} a_k (m+k-1)(2m+2k+1) x^{m+k} = 0$$

Equating the lowest degree term to zero by putting $k = 0$ in second summation,
$$a_0 (m-1)(2m+1) = 0$$
$$a_0 \neq 0$$
$$m = 1, -\frac{1}{2}$$

Roots are different and their difference is not an integer.

Thus, $$y = C_1(y)_{m=1} + C_2(y)_{m=\frac{-1}{2}}$$

Equating the general terms,
$$2a_k (m+k) + a_{k+1}(m+k)(2m+2k+3) = 0$$
$$a_{k+1} = \frac{-2a_k}{(2m+2k+3)}$$

Putting $k = 0, 1, 2, \ldots$
$$a_1 = \frac{-2a_0}{2m+3}$$
$$a_2 = \frac{-2a_1}{(2m+5)}$$
$$a_3 = \frac{-2a_2}{(2m+7)} \text{ and so on}$$

At $m = 1$,
$$a_1 = \frac{-2a_0}{5}$$
$$a_2 = \frac{-2}{7}\left(\frac{-2a_0}{5}\right) = \frac{4a_0}{35}$$
$$a_3 = \frac{-2}{9}\left(\frac{4a_0}{35}\right) = \frac{-8a_0}{5.7.9}$$
$$a_4 = \frac{16a_0}{5.7.9.11}$$

At $m = -\frac{1}{2}$,
$$a_1 = \frac{-2a_0}{2} = -a_0$$
$$a_2 = \frac{-2}{4}(-a_0) = \frac{a_0}{2}$$
$$a_3 = \frac{-2}{6}\left(\frac{a_0}{2}\right) = \frac{-a_0}{6}$$
$$a_4 = \frac{-2}{8}\left(\frac{-a_0}{6}\right) = \frac{a_0}{24}$$

Thus, $$y = C_1 x\, a_0\left[1 - \frac{2}{5}x + \frac{4}{35}x^2 - \frac{8}{5.7.9}x^3 + \frac{16}{5.7.9.11}x^4 \ldots\right]$$
$$+ C_2 x^{-1/2} a_0\left[1 - x + \frac{1}{2}x^2 - \frac{1}{6}x^3 + \frac{1}{24}x^4 \ldots\right]$$

Que 1.33. **Use Frobenius series method to find the series solution of $(1-x^2)y'' - xy' + 4y = 0$**

AKTU 2011-12, Marks 10

Answer

$$(1-x^2)y'' - xy' + 4y = 0$$

Let $$x + 1 = t$$
$$t(2-t)y'' - (t-1)y' + 4y = 0 \quad \ldots(1.33.1)$$

Dividing eq. (1.33.1) by $t(2-t)$, we get
$$y'' - \frac{(t-1)}{t(2-t)}y' + \frac{4}{t(2-t)}y = 0 \quad \ldots(1.33.2)$$

Comparing eq. (1.33.2) with $y'' + P(t)y' + Q(t)y = 0$
$$P(t) = \frac{-(t-1)}{t(2-t)} \text{ and } Q(t) = \frac{4}{t(2-t)}$$

$t = 0$ is a singular point for the given differential equation.

Let, $$y = \sum_{k=0}^{\infty} a_k t^{m+k} \text{ is a solution}$$
$$y' = \sum a_k (m+k) t^{m+k-1}$$
$$y'' = \sum a_k (m+k)(m+k-1) t^{m+k-2}$$

From eq. (1.33.1),
$$t(2-t)\sum a_k (m+k)(m+k-1)t^{m+k-2} - (t-1)\sum a_k (m+k) t^{m+k-1} + 4\sum a_k t^{m+k} = 0$$
$$2\sum a_k (m+k)(m+k-1)t^{m+k-1} - \sum a_k (m+k)(m+k-1)t^{m+k}$$

$$-\sum a_k(m+k)\,t^{m+k} + \sum a_k(m+k)\,t^{m+k-1} + 4\sum a_k\,t^{m+k} = 0$$

$$\sum a_k(m+k)(2m+2k-2+1)\,t^{m+k-1} - \sum a_k[(m+k)(m+k)-4]t^{m+k} = 0$$

$$\sum a_k(m+k)(2m+2k-1)\,t^{m+k-1} - \sum a_k(m+k+2)(m+k-2)t^{m+k} = 0 \quad ...(1.33.3)$$

Putting $k = 0$ in lowest degree term, t^{m-1}

$$a_0 m(2m-1) = 0$$

$\because$ $a_0 \neq 0$

$\therefore$ $m = 0,\ 1/2$

Putting $k = k + 1$ in first summation and $k = k$ in second summation of eq. (1.33.3)

$$a_{k+1}(m+k+1)(2m+2k+1) - a_k(m+k+2)(m+k-2) = 0$$

$$a_{k+1} = \frac{(m+k+2)(m+k-2)}{(m+k+1)(2m+2k+1)}a_k$$

Putting $k = 0, 1, 2, 3, ...$

$$a_1 = \frac{(m+2)(m-2)}{(m+1)(2m+1)}a_0,\ a_2 = \frac{(m+3)(m-1)}{(m+2)(2m+3)}a_1,\ a_3 = \frac{(m+4)m}{(m+3)(2m+5)}a_2$$

At $m = 0$,

$$a_1 = \frac{-4}{1}a_0 = -4a_0$$

$$a_2 = \frac{-3}{6}a_1 = 2a_0$$

$$a_3 = 0$$

At $m = 1/2$,

$$a_1 = -\frac{5}{4}a_0$$

$$a_2 = \frac{7}{32}a_0$$

$$a_3 = \frac{3}{128}a_0$$

Thus,

$$y = C_1(y)_{m=0} + C_2(y)_{m=1/2}$$

$$y = C_1\left[\sum_{k=0}^{\infty} a_k t^{m+k}\right]_{m=0} + C_2\left[\sum_{k=0}^{\infty} a_k t^{m+k}\right]_{m=\frac{1}{2}}$$

$$= C_1[a_0 + a_1 t + a_2 t^2 + a_3 t^3 + ...] + C_2\left[a_0(t)^{\frac{1}{2}} + a_1(t)^{\frac{3}{2}} + a_2(t)^{\frac{5}{2}} + a_3(t)^{\frac{7}{2}} + ...\right]$$

$$= C_1[a_0 + (-4a_0)(1+x) + 2a_0(1+x)^2 + 0] +$$

$$C_2\left[a_0(1+x)^{1/2} + a_0\left(\frac{-5}{4}\right)(1+x)^{3/2} + \frac{7}{32}a_0(1+x)^{5/2} + \frac{3}{128}a_0(1+x)^{7/2} + ...\right]$$

$$= C_1 a_0[1 - 4 - 4x + 2 + 2x^2 + 4x] + C_2 a_0(1+x)^{1/2}\left[1 - \frac{5}{4}(1+x) + \frac{7}{32}(1+x)^2 + \frac{3}{128}(1+x)^3 + ...\right]$$

$$= C_1 a_0[1 + 2x^2] + C_2 a_0(1+x)^{1/2}\left[1 - \frac{5}{4}(1+x) + \frac{7}{32}(1+x)^2 + \frac{3}{128}(1+x)^3 + ...\right]$$

Que 1.34. **Find the Frobenius series solution of the following differential equation about $x = 0$.**

$$2x^2y'' + 7x(x+1)y' - 3y = 0.$$

AKTU 2012-13, Marks 10

Answer

$$2x^2y'' + 7x(x+1)y' - 3y = 0 \quad ...(1.34.1)$$

$x = 0$ is a regular singular point.

Let,

$$y = \sum_{k=0}^{\infty} a_k x^{m+k}$$

$$y' = \sum_{k=0}^{\infty} a_k(m+k)x^{m+k-1}$$

$$y'' = \sum_{k=0}^{\infty} a_k(m+k)x^{m+k-2}(m+k-1)$$

Putting the value of y, y' and y'' in eq. (1.34.1), we have

$$2x^2\sum_{k=0}^{\infty} a_k(m+k)(m+k-1)x^{m+k-2} + 7x^2\sum_{k=0}^{\infty} a_k(m+k)x^{m+k-1} + 7x\sum_{k=0}^{\infty} a_k(m+k)x^{m+k-1} - 3\sum_{k=0}^{\infty} a_k x^{m+k} = 0$$

$$\sum_{k=0}^{\infty} a_k(m+k)(2m+2k-2+7)x^{m+k} - 3\sum_{k=0}^{\infty} a_k x^{m+k} + 7\sum_{k=0}^{\infty} a_k(m+k)x^{m+k+1} = 0$$

$$\sum_{k=0}^{\infty} a_k(m+k)(2m+2k+5)x^{m+k} - 3\sum_{k=0}^{\infty} a_k x^{m+k} + 7\sum_{k=0}^{\infty} a_k(m+k)x^{m+k+1} = 0$$

Equating the lowest degree term to zero by putting $k = 0$ in first summation,

$$a_0 m(2m+5) - 3 = 0$$

$\because$ $a_0 \neq 0$

$\therefore$ $2m^2 + 5m - 3 = 0$

$$m = -3,\ \frac{1}{2}$$

Roots are different and their difference is not an integer,

Thus, $y = C_1(y)_{m=-3} + C_2(y)_{m=1/2}$

Equating the general terms,

$$a_{k+1}[(m+k+1)(2m+2k+2+5)-3] + 7a_k(m+k) = 0$$

$$a_{k+1} = \frac{-7a_k(m+k)}{[(m+k+1)(2m+2k+7)-3]}$$

Putting $k = 0, 1, 2, ...$

$$a_1 = \frac{-7a_0 m}{[(m+1)(2m+7)-3]}$$

$$a_2 = \frac{-7a_0(m+1)}{[(m+2)(2m+9)-3]}$$

$$= \frac{49\,a_0\,m(m+1)}{[(m+1)(2m+7)-3][(m+2)(2m+9)-3]}$$

$$a_3 = \frac{-7a_2(m+2)}{[(m+3)(2m+11)-3]}$$

$$= \frac{-343\,a_0\,m(m+1)(m+2)}{[(m+1)(2m+7)-3][(m+2)(2m+9)-3][(m+3)(2m+11)-3]}$$

At $m = \frac{1}{2}$,

$$a_1 = \frac{-7a_0}{18}$$

$$a_2 = \frac{-7\,a_1 \times (3/2)}{[(5/2)\times 10 - 3]} = \frac{49\,a_0}{264}$$

$$a_3 = \frac{-7\,a_2 \times (5/2)}{[(7/2)\times 12 - 3} = -\frac{1215\,a_0}{20592}$$

At $m = -3$,

$$a_1 = \frac{-21a_0}{5}$$

$$a_2 = \frac{-7\,a_1 \times (-2)}{[(-1)\times 3 - 3]} = \frac{49\,a_0}{5}$$

$$a_3 = 0$$

Thus,

$$y = C_1(y)_{m=-3} + C_2(y)_{m=1/2}$$

$$y = C_1\,[a_0 x^{-3} x^0 + a_1 x^{-3} x^1 + a_2 x^{-3} x^2 + a_3 x^{-3} x^3 \ldots]$$
$$+ C_2\,[a_0 x^{1/2} x^0 + a_1 x^{1/2} x^1 + a_2 x^{1/2} x^2 + a_3 x^{1/2} x^3 \ldots]$$

$$y = C_1\,a_0\,x^{-3}\left[1 - \frac{21}{5}x + \frac{49}{5}x^2 + \ldots\right]$$
$$+ C_2\,a_0\,x^{1/2}\left[1 - \frac{7}{18}x + \frac{49}{264}x^2 - \frac{1215}{20592}x^3 \ldots\right]$$

Que 1.35. **Find the series solution by Forbenius method for the differential equation $(1 - x^2)y'' - 2xy' + 20y = 0$**

AKTU 2016-17, Marks 07

Answer

Same as Q. 1.33, Page 1–38F, Unit-1.

Answer : $y = [A + B\log(x+1)]\left(1 - 10\,t + \frac{45}{2}(x+1)^2\,t^2 + \left(\frac{-35}{2}(x+1)^3 + \ldots\right)\right)$

☺☺☺

PART-1

Improper Integrals, Beta and Gama Functions and their Properties.

CONCEPT OUTLINE

Improper Integrals : By definition of a regular (or proper) definite integral $\int_a^b f(x)dx$, it is assumed that the limits of integration are finite and that the integrand $f(x)$ is continuous for every value of x in the interval $a \le x \le b$. If at least one of these conditions is violated, then the integral is known as an improper integral (or singular or generalized or infinite integral).

Beta Function : The definite integral $\int_0^1 x^{m-1}(1-x)^{n-1}dx$ is called the Beta function, where m and n are positive. Beta function is denoted by $\beta(m, n)$. Thus

$$\beta(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}dx$$

Property 1 : $\beta(m, n) = \beta(n, m)$

Property 2 : Transformation of Beta function is

$$\beta(m, n) = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}dx$$

Gamma Function : Gamma function for a positive number n is denoted by $\overline{|n}$ and is given by

$$\overline{|n} = \int_0^\infty e^{-x}x^{n-1}dx, n > 0$$

Questions-Answers

Long Answer Type and Medium Answer Type Questions

Que 2.1. Evaluate $\int_0^\infty \frac{x^4(1+x^5)}{(1+x)^{15}}dx$.

Answer

$$\int_0^\infty \frac{x^4(1+x^5)}{(1+x)^{15}}dx = \int_0^\infty \frac{x^4dx}{(1+x)^{15}} + \int_0^\infty \frac{x^9}{(1+x)^{15}}dx$$

$$= \int_0^\infty \frac{x^{5-1}}{(1+x)^{5+10}}dx + \int_0^\infty \frac{x^{10-1}}{(1+x)^{10+5}}dx$$

$$= \beta(5, 10) + \beta(10, 5)$$

$$= 2\,\beta(5, 10) \qquad [\because \beta(m, n) = \beta(n, m)]$$

Que 2.2. To prove $\overline{|n+1} = n\overline{|n}$.

Answer

We know that $\overline{|n} = \int_0^\infty e^{-x}x^{n-1}dx$

$$\overline{|n+1} = \int_0^\infty e^{-x}x^{n+1-1}dx = \int_0^\infty e^{-x}x^n dx$$

Integrating by parts,

$$\overline{|n+1} = \left[-x^n e^{-x}\right]_0^\infty - \int_0^\infty nx^{n-1}(-e^{-x})dx$$

$$= 0 + n\int_0^\infty e^{-x}x^{n-1}dx$$

$$\overline{|n+1} = n\overline{|n}$$

Que 2.3. Prove that $\beta(m, n) = \beta(m+1, n) + \beta(m, n+1)$.

Answer

$$\text{RHS} = \beta(m+1, n) + \beta(m, n+1)$$

$$= \int_0^1 x^{m+1-1}(1-x)^{n-1}dx + \int_0^1 x^{m-1}(1-x)^{n+1-1}dx$$

$$= \int_0^1 x^m(1-x)^{n-1}dx + \int_0^1 x^{m-1}(1-x)^n dx$$

$$= \int_0^1 x^{m-1}(1-x)^{n-1}(x+1-x)\,dx = \beta(m, n)$$

Que 2.4. Find the value of $\overline{|\tfrac{1}{2}}$.

Answer

$$\overline{|n} = \int_0^\infty e^{-x}x^{n-1}dx$$

$$\Gamma\left(\frac{1}{2}\right) = \int_0^\infty e^{-x}\, x^{-1/2}\, dx$$

Let, $x = y^2$

$$dx = 2y\, dy = \int_0^\infty e^{-y^2}\,\frac{1}{y}\,2y\, dy$$

$$\Gamma\left(\frac{1}{2}\right) = 2\int_0^\infty e^{-y^2}\, dy \qquad \text{...(2.4.1)}$$

Similarly,

$$\Gamma\left(\frac{1}{2}\right) = 2\int_0^\infty e^{-x^2}\, dy \qquad \text{...(2.4.2)}$$

Multiplying eq. (2.4.1) and eq. (2.4.2), we get

$$\left(\Gamma\left(\frac{1}{2}\right)\right)^2 = 4\int_0^\infty e^{-(x^2+y^2)}\, dx\, dy$$

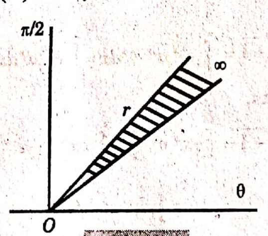


Fig. 2.4.1.

Changing this integral to polar coordinate by putting $x = r\cos\theta$, $y = r\sin\theta$ and $dx\, dy = r\, dr\, d\theta$.

Region of integration is the complete positive quadrant r will vary from 0 to ∞ and θ from 0 to $\pi/2$.

$$\left(\Gamma\left(\frac{1}{2}\right)\right)^2 = 4\int_0^{\pi/2}\int_0^\infty e^{-r^2}\, r\, dr\, d\theta$$

$$= 4\int_0^{\pi/2}\left[-\frac{1}{2}e^{-r^2}\right]_0^\infty d\theta = 2\int_0^{\pi/2} d\theta = \pi$$

$$\left(\Gamma\left(\frac{1}{2}\right)\right)^2 = \pi$$

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$

Que 2.5. **To prove that** $\beta(m, n) = \int_0^1 \dfrac{x^{m-1} + x^{n-1}}{(1+x)^{m+n}}\, dx$.

Answer

Let,

$$x = \frac{1}{1+y}$$

$$dx = \frac{-1}{(1+y)^2}\, dy$$

$$\beta(m, n) = \int_\infty^0 \left(\frac{1}{1+y}\right)^{m-1}\left(\frac{y}{1+y}\right)^{n-1}\left(\frac{-1}{(1+y)^2}\right) dy$$

$$= \int_0^\infty \frac{y^{n-1}}{(1+y)^{m+n}}\, dy$$

$$= \int_0^1 \frac{y^{n-1}}{(1+y)^{m+n}}\, dy + \int_1^\infty \frac{y^{n-1}}{(1+y)^{m+n}}\, dy \qquad \text{...(2.5.1)}$$

Now in the second integral,

Let,

$$y = \frac{1}{t}$$

$$dy = -\frac{1}{t^2}\, dt$$

$$\int_0^\infty \frac{y^{n-1}}{(1+y)^{m+n}}\, dy = \int_1^0 \frac{\left(\frac{1}{t}\right)^{n-1}}{\left(1+\frac{1}{t}\right)^{m+n}}\left(-\frac{1}{t^2}\right) dt$$

$$= \int_0^1 \frac{t^{m-1}}{(1+t)^{m+n}}\, dt = \int_0^1 \frac{y^{m-1}}{(1+y)^{m+n}}\, dy$$

From eq. (2.5.1),

$$\beta(m, n) = \int_0^1 \frac{y^{n-1}}{(1+y)^{m+n}}\, dy + \int_0^1 \frac{y^{m-1}}{(1+y)^{m+n}}\, dy = \int_0^1 \frac{x^{m-1} + x^{n-1}}{(1+x)^{m+n}}\, dx$$

Que 2.6. **Prove that :** $\beta(m, n) = \dfrac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$, $m > 0, n > 0$.

AKTU 2017-18, Marks 07

Answer

We know that, $\Gamma(n) = k^n \int_0^\infty e^{-kx}\, x^{n-1}\, dx$

Replacing k by z, $\Gamma(n) = z^n \int_0^\infty e^{-zx}\, x^{n-1}\, dx$

Multiplying both sides by $e^{-z} z^{m-1}$,

$$\Gamma(n)\, e^{-z} z^{n-1} = \int_0^\infty z^{n+m-1} e^{-z(1+x)} x^{n-1}\, dx$$

Integrating both sides w.r.t z from 0 to ∞,

$$\Gamma(n)\int_0^\infty e^{-z} z^{m-1} dz = \int_0^\infty x^{n-1}\left\{\int_0^\infty e^{-z(1+x)} z^{m+n-1}\, dz\right\} dx$$

Let, $z(1+x) = y$

$$dz = \frac{dy}{1+x}$$

$$\Gamma(n)\Gamma(m) = \int_0^\infty x^{n-1}\int_0^\infty e^{-y}\frac{y^{m+n-1}}{(1+x)^{m+n}}\, dy\, dx$$

$$\Gamma(n)\Gamma(m) = \int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}}\left\{\int_0^\infty e^{-y} y^{m+n-1} dy\right\} dx$$

$$= \Gamma(m+n)\int_0^\infty \frac{x^{n-1}}{(1+x)^{m+n}} dx$$

$$\Gamma(n)\Gamma(m) = \Gamma(m+n)\ \beta(m, n)$$

Thus, $$\beta(m, n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$$

Que 2.7. **Evaluate : $\int_0^\infty \cos x^2\, dx$**

Answer

We know that

$$\int_0^\infty e^{-ax} x^{n-1} \cos bx\, dx = \frac{\Gamma(n)\cos n\theta}{(a^2+b^2)^{n/2}}, \text{ where } \theta = \tan^{-1}\left(\frac{b}{a}\right)$$

Put $$a = 0,\ \int_0^\infty x^{n-1}\cos bx\, dx = \frac{\Gamma(n)}{b^n}\cos\frac{n\pi}{2}$$

Also putting $x^n = z$ so that $x^{n-1}dx = \frac{dz}{n}$ and $x = z^{1/n}$

$\therefore$ $$\int_0^\infty \cos bz^{1/n}\, dz = \frac{n\Gamma(n)}{b^n}\cos\frac{n\pi}{2}$$

or $$\int_0^\infty \cos(bx^{1/n})\, dx = \frac{\Gamma(n+1)}{b^n}\cos\frac{n\pi}{2}$$

Here $b = 1, n = 1/2$

$\therefore$ $$\int_0^\infty \cos x^2\, dx = \Gamma(3/2)\cos\frac{\pi}{4} = \frac{\sqrt{\pi}}{2}\frac{1}{\sqrt{2}} = \frac{1}{2}\sqrt{\frac{\pi}{2}}$$

Que 2.8. **Prove that $\int_0^{\pi/2} \sin^p\theta \cos^q\theta\, d\theta = \dfrac{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{q+1}{2}\right)}{2\Gamma\left(\frac{p+q+2}{2}\right)}$.**

Answer

We know that $\beta(m, n) = \int_0^1 x^{m-1}(1-x)^{n-1}\, dx$

Putting $x = \sin^2\theta$

$dx = 2\sin\theta\cos\theta\, d\theta$

$$= \int_0^{\pi/2} 2\sin^{2(m-1)}\theta\cos^{2(n-1)}\theta\sin\theta\cos\theta\, d\theta$$

$$\beta(m, n) = 2\int_0^{\pi/2} \sin^{2m-1}\theta\cos^{2n-1}\theta\, d\theta$$

$$\frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)} = 2\int_0^{\pi/2} \sin^{2m-1}\theta\cos^{2n-1}\theta\, d\theta \qquad \left[\because \beta(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}\right]$$

Let, $2m-1 = p$ and $2n-1 = q$, $m = p+1/2$, $n = q+1/2$

$\therefore$ $$\frac{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{q+1}{2}\right)}{\Gamma\left(\frac{p+q+2}{2}\right)} = 2\int_0^{\pi/2}\sin^p\theta\cos^q\theta\, d\theta$$

$$\int_0^{\pi/2}\sin^p\theta\cos^q\theta\, d\theta = \frac{\Gamma\left(\frac{p+1}{2}\right)\Gamma\left(\frac{q+1}{2}\right)}{2\Gamma\left(\frac{p+q+2}{2}\right)}$$

Que 2.9. **State and prove the duplication formula.**

Answer

A. Duplication Formula :

$$\Gamma(m)\Gamma\left(m+\frac{1}{2}\right) = \frac{\sqrt{\pi}}{2^{2m-1}}\Gamma(2m), \text{ where } m \text{ is positive.}$$

B. Proof : We know that

$$\beta(m, n) = 2\int_0^{\pi/2}\sin^{2m-1}\theta\cos^{2n-1}\theta\, d\theta$$

$$\frac{\Gamma(m)\Gamma(n)}{2\Gamma(m+n)} = \int_0^{\pi/2}\sin^{2m-1}\theta\cos^{2n-1}\theta\, d\theta \qquad ...(2.9.1)$$

Let, $2n-1 = 0$

$n = 1/2$

Now from eq. (2.9.1)

$$\int_0^{\pi/2}\sin^{2m-1}\theta\, d\theta = \frac{\Gamma(m)\Gamma\left(\frac{1}{2}\right)}{2\Gamma\left(m+\frac{1}{2}\right)} \qquad ...(2.9.2)$$

Again in eq. (2.9.1), let $n = m$

$$\frac{\Gamma m \Gamma m}{2\Gamma 2m} = \int_0^{\pi/2} (\sin\theta\cos\theta)^{2m-1} d\theta = \frac{1}{2^{2m-1}}\int_0^{\pi/2} (\sin 2\theta)^{2m-1} d\theta$$

Let, $2\theta = \phi$

$2d\theta = d\phi$

$$= \frac{1}{2^{2m-1}}\int_0^{\pi} (\sin\phi)^{2m-1}\frac{d\phi}{2} = \frac{1}{2^{2m}}\int_0^{\pi} \sin^{2m-1}\phi \, d\phi$$

$$\frac{(\Gamma m)^2}{2\Gamma 2m} = \frac{2}{2^{2m}}\int_0^{\pi/2} \sin^{2m-1}\theta \, d\theta$$

[Using property of definite integral]

$$\frac{(\Gamma m)^2 2^{2m-1}}{2\Gamma 2m} = \int_0^{\pi/2} \sin^{2m-1}\theta \, d\theta \quad ...(2.9.3)$$

From eq. (2.9.2) and eq. (2.9.3),

$$\frac{\Gamma m \Gamma \frac{1}{2}}{2\Gamma\left(m+\frac{1}{2}\right)} = \frac{(\Gamma m)^2 2^{2m-1}}{2\Gamma 2m}$$

$$\Gamma m \,\Gamma\left(m+\frac{1}{2}\right) = \frac{\sqrt{\pi}}{2^{2m-1}}\Gamma 2m$$

Que 2.10. Prove that $\int_0^{\pi/2}\sqrt{\tan\theta}\, d\theta = \frac{\pi}{\sqrt{2}}$.

Answer

$$\int_0^{\pi/2}\sqrt{\tan\theta}\, d\theta = \int_0^{\pi/2}\sqrt{\tan\left(\frac{\pi}{2}-\theta\right)}\, d\theta$$

$$= \int_0^{\pi/2}\sqrt{\cot\theta}\, d\theta \qquad \left(\because \int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx\right)$$

$$= \int_0^{\pi/2}\cos^{1/2}\theta\sin^{-1/2}\theta\, d\theta = \frac{\Gamma\left(\frac{\frac{1}{2}+1}{2}\right)\Gamma\left(\frac{-\frac{1}{2}+1}{2}\right)}{2\Gamma\left(\frac{\frac{1}{2}-\frac{1}{2}+2}{2}\right)}$$

$$= \frac{\Gamma\frac{3}{4}\Gamma\frac{1}{4}}{2\Gamma 1} = \frac{1}{2}\Gamma\frac{1}{4}\Gamma\left(1-\frac{1}{4}\right) = \frac{1}{2}\frac{\pi}{\sin\frac{\pi}{4}} = \frac{\pi}{\sqrt{2}}$$

$$\left(\because \Gamma n\,\Gamma(1-n) = \frac{\pi}{\sin n\pi}\right)$$

Que 2.11. Using Beta and Gamma functions, evaluate $\int_0^{\infty}\frac{dx}{1+x^4}$.

AKTU 2011-12, Marks 05

Answer

$$I = \int_0^{\infty}\frac{dx}{1+x^4}$$

Let, $x^2 = \tan\theta$

$2x\,dx = \sec^2\theta\, d\theta$

$$dx = \frac{\sec^2\theta\, d\theta}{2\sqrt{\tan\theta}} = \frac{1}{2\sqrt{\sin\theta}\cos^{3/2}\theta}d\theta$$

$$dx = \frac{1}{2}\sin^{-1/2}\theta\cos^{-3/2}\theta\, d\theta$$

$$I = \int_0^{\pi/2}\frac{1}{2}\frac{\sin^{-1/2}\theta\cos^{-3/2}\theta}{\sec^2\theta}d\theta$$

$$I = \frac{1}{2}\int_0^{\pi/2}\sin^{-1/2}\theta\cos^{1/2}\theta\, d\theta$$

$$\left[\because \int_0^{\pi/2}\sin^p\theta\cos^q\theta\, d\theta = \frac{\Gamma\frac{p+1}{2}\Gamma\frac{q+1}{2}}{2\Gamma\frac{p+q+2}{2}}\right]$$

$$I = \frac{1}{2}\frac{\Gamma\frac{1}{4}\Gamma\frac{3}{4}}{2\Gamma 1} = \frac{1}{4}\Gamma\frac{1}{4}\Gamma\frac{3}{4}$$

$$= \frac{1}{4}\frac{\pi}{\sin\frac{3\pi}{4}} = \frac{\pi}{4}\sqrt{2} \Rightarrow I = \frac{\pi}{2\sqrt{2}} \quad \left[\because \frac{\pi}{\sin n\pi} = \Gamma n\,\Gamma(1-n)\right]$$

Que 2.12. Using Beta and Gamma function, evaluate $\int_0^1\left(\frac{x^3}{1-x^3}\right)^{\frac{1}{2}}dx$.

AKTU 2014-15, Marks 3.5

Answer

Same as Q. 2.11, Page 2–9F, Unit-2.

$$\left(\textbf{Answer: } I = \frac{1}{3}\frac{\sqrt{\pi}\,\Gamma 5/6}{\Gamma 4/3}\right)$$

Que 2.13. For the Gamma function, show that

$$\frac{\Gamma\left(\frac{1}{3}\right)\Gamma\left(\frac{5}{6}\right)}{\Gamma\left(\frac{2}{3}\right)} = (2)^{1/3}\sqrt{\pi}.$$

AKTU 2016-17, Marks 07

Answer

$$\text{LHS} = \frac{\Gamma\frac{1}{3}\,\Gamma\frac{5}{6}}{\Gamma\frac{2}{3}} = \frac{\Gamma\frac{1}{3}\,\Gamma\left(\frac{1}{3}+\frac{1}{2}\right)}{\Gamma\frac{2}{3}} = \frac{\sqrt{\pi}}{(2)^{2\cdot\frac{1}{3}-1}}\,\frac{\Gamma\left(2\times\frac{1}{3}\right)}{\Gamma\frac{2}{3}}$$

$$\left[\because \text{By Duplication formula, } \Gamma m\,\Gamma\left(m+\frac{1}{2}\right) = \frac{\sqrt{\pi}}{2^{2m-1}}\,\Gamma 2m\right]$$

$$= \frac{\sqrt{\pi}}{(2)^{-\frac{1}{3}}} = (2)^{1/3}\sqrt{\pi} = \text{RHS}$$

PART-2

Dirichlet's Integral and its Applications.

CONCEPT OUTLINE

Dirichlet's Integral : Dirichlet's integral is given as,

$$\iint_D x^{l-1}y^{m-1}\,dx\,dy = \frac{\Gamma l\,\Gamma m}{\Gamma(l+m+1)}\,a^{l+m}$$

Where D is the domain $x \ge 0, y \ge 0$ and $x + y \le a$.

Dirichlet's Integral for Three Variables :

$$\iiint_D x^{l-1}y^{m-1}z^{n-1}\,dx\,dy\,dz = \frac{\Gamma l\,\Gamma m\,\Gamma n}{\Gamma(l+m+n+1)}$$

Where D is the domain $x \ge 0, y \ge 0, z \ge 0$ and $x + y + z \le 1$.

Questions-Answers

Long Answer Type and Medium Answer Type Questions

Que 2.14. State and prove Dirichlet's integral for two variables.

Answer

A. **Dirichlet's Integral for Two Variables :** The Dirichlet's integral for two variables is given by,

$$\iint_D x^{l-1}y^{m-1}\,dx\,dy = \frac{\Gamma l\,\Gamma m}{\Gamma(l+m+1)}\,a^{l+m}$$

Where D is the domain $x \ge 0, y \ge 0$ and $x + y \le a$

B. **Proof :** Let, $x = aX$

$y = aY$

Therefore, given integral becomes $\iint_{D'} (aX)^{l-1}(aY)^{m-1}a^2\,dX\,dY$

Where D' is the domain and $X \ge 0, Y \ge 0$ and $X + Y \le 1$

$$= a^{l+m}\iint_{D'} X^{l-1}Y^{m-1}\,dX\,dY$$

$$= a^{l+m}\int_0^1\int_0^{1-X} X^{l-1}Y^{m-1}\,dX\,dY = a^{l+m}\int_0^1 X^{l-1}\left[\frac{Y^m}{m}\right]_0^{1-X} dX$$

$$= \frac{a^{l+m}}{m}\int_0^1 X^{l-1}(1-X)^m\,dX$$

$$= \frac{a^{l+m}}{m}\int_0^1 X^{l-1}(1-X)^{m+1-1}\,dX = \frac{a^{l+m}}{m}\beta(l, m+1)$$

$$= \frac{a^{l+m}}{m}\,\frac{\Gamma l\,\Gamma(m+1)}{\Gamma(l+m+1)} = a^{l+m}\,\frac{\Gamma l\,\Gamma m}{\Gamma(l+m+1)}$$

Que 2.15. State and prove Dirichlet's integral for three variables.

Answer

A. **Dirichlet's Integral for Three Variables :**

$$\iiint_D x^{l-1}y^{m-1}z^{n-1}\,dx\,dy\,dz = \frac{\Gamma l\,\Gamma m\,\Gamma n}{\Gamma(l+m+n+1)}$$

Where D is the domain $x \ge 0, y \ge 0, z \ge 0$ and $x + y + z \le 1$

B. **Proof :** $x + y + z \le 1$

$y + z \le 1 - x = a$ (let)

Therefore given integral becomes $\int_0^1\int_0^{1-x}\int_0^{1-x-y} x^{l-1}y^{m-1}z^{n-1}\,dz\,dy\,dx$

$$= \int_0^1 x^{l-1}\left[\int_0^a\int_0^{a-y} y^{m-1}z^{n-1}\,dz\,dy\right]dx$$

$$= \int_0^1 x^{l-1} \frac{\Gamma m\,\Gamma n}{\Gamma(m+n+1)} a^{m+n}\, dx$$

$$= \frac{\Gamma m\,\Gamma n}{\Gamma(m+n+1)} \int_0^1 x^{l-1}(1-x)^{m+n+1-1}\, dx$$

$$= \frac{\Gamma m\,\Gamma n}{\Gamma(m+n+1)} \beta(l, m+n+1)$$

$$= \frac{\Gamma m\,\Gamma n}{\Gamma(m+n+1)} \frac{\Gamma l\,\Gamma(m+n+1)}{\Gamma(l+m+n+1)} = \frac{\Gamma l\,\Gamma m\,\Gamma n}{\Gamma(l+m+n+1)}$$

Que 2.16. Evaluate $\iiint_V (ax^2 + by^2 + cz^2)\, dx\, dy\, dz$ where V is the region bounded by $x^2 + y^2 + z^2 \le 1$. **AKTU 2012-13, Marks 10**

Answer

$\iiint_V (ax^2 + by^2 + cz^2)\, dx\, dy\, dz$, Where $V = x^2 + y^2 + z^2 \le 1$

Let, $x^2 = u, y^2 = v, z^2 = w$

$$\therefore \qquad x = \sqrt{u}, \quad y = \sqrt{v}, \quad z = \sqrt{w}$$

And,
$$dx = \frac{1}{2\sqrt{u}}, \quad dy = \frac{1}{2\sqrt{v}}, \quad dz = \frac{1}{2\sqrt{w}}$$

$$= \iiint_{V'} (au + bv + cw) \frac{1}{8\sqrt{uvw}}\, du\, dv\, dw, \text{ where } V' = u + v + w \le 1$$

$$= \iiint_{V'} \frac{a}{8} u^{1/2} v^{-1/2} w^{-1/2}\, du\, dv\, dw + \frac{b}{8}\iiint u^{-1/2} v^{1/2} w^{-1/2}\, du\, dv\, dw + \frac{c}{8}\iiint u^{-1/2} v^{-1/2} w^{1/2}\, du\, dv\, dw$$

$$= \frac{a}{8}\iiint_{V'} u^{\frac{3}{2}-1} v^{\frac{1}{2}-1} w^{\frac{1}{2}-1}\, du\, dv\, dw + \frac{b}{8}\iiint_{V'} u^{\frac{1}{2}-1} v^{\frac{3}{2}-1} w^{\frac{1}{2}-1}\, du\, dv\, dw + \frac{c}{8}\iiint_{V'} u^{\frac{1}{2}-1} v^{\frac{1}{2}-1} w^{\frac{3}{2}-1}\, du\, dv\, dw$$

$$= \frac{a}{8} \frac{\Gamma\frac{3}{2}\Gamma\frac{1}{2}\Gamma\frac{1}{2}}{\Gamma\left(\frac{3}{2}+\frac{1}{2}+\frac{1}{2}+1\right)} + \frac{b}{8} \frac{\Gamma\frac{3}{2}\Gamma\frac{1}{2}\Gamma\frac{1}{2}}{\Gamma\frac{7}{2}} + \frac{c}{8} \frac{\Gamma\frac{3}{2}\Gamma\frac{1}{2}\Gamma\frac{1}{2}}{\Gamma\frac{7}{2}} = \frac{\Gamma\frac{3}{2}\Gamma\frac{1}{2}\Gamma\frac{1}{2}}{8\Gamma\frac{7}{2}} (a+b+c)$$

$$= \frac{\frac{1}{2}\pi\sqrt{\pi}}{8 \cdot \frac{5}{2} \cdot \frac{3}{2} \cdot \frac{1}{2}\sqrt{\pi}} (a+b+c) = \frac{\pi}{30}(a+b+c)$$

Que 2.17. Prove that : $\sqrt{\pi}\,\Gamma(2n) = 2^{2n-1}\,\Gamma(n)\,\Gamma\left(n+\frac{1}{2}\right)$, where n is not a negative integer or zero. **AKTU 2012-13, Marks 10**

Answer

We know that
$$\frac{\Gamma\frac{p+1}{2}\Gamma\frac{q+1}{2}}{2\Gamma\frac{p+q+2}{2}} = \int_0^{\pi/2} \sin^p\theta \cos^q\theta\, d\theta$$

Let, $q = p$

$$\therefore \qquad \frac{\Gamma\frac{p+1}{2}\Gamma\frac{p+1}{2}}{2\Gamma(p+1)} = \int_0^{\pi/2} (\sin\theta\cos\theta)^p\, d\theta$$

$$= \frac{1}{2^p}\int_0^{\pi/2} (\sin 2\theta)^p\, d\theta$$

Let, $2\theta = t$

$$= \frac{1}{2^{p+1}}\int_0^{\pi} \sin^p t\, dt = \frac{1}{2^p}\int_0^{\pi/2} \sin^p t\, dt = \frac{1}{2^p} \frac{\Gamma\frac{p+1}{2}\Gamma\frac{0+1}{2}}{2\Gamma\frac{p+2}{2}}$$

$$\therefore \qquad \frac{\Gamma\frac{p+1}{2}\Gamma\frac{p+1}{2}}{2\Gamma(p+1)} = \frac{1}{2^p} \frac{\Gamma\frac{p+1}{2}\Gamma\frac{1}{2}}{2\Gamma\frac{p+2}{2}}$$

$$\frac{\Gamma\frac{p+1}{2}}{\Gamma(p+1)} = \frac{1}{2^p} \frac{\sqrt{\pi}}{\Gamma\frac{p+2}{2}}$$

Let, $\frac{p+1}{2} = n$ or $p = 2n - 1$

$$\frac{\Gamma n}{\Gamma 2n} = \frac{1}{2^{2n-1}} \frac{\sqrt{\pi}}{\Gamma\frac{2n+1}{2}}$$

or
$$\sqrt{\pi}\,\Gamma 2n = 2^{2n-1}\,\Gamma n\,\Gamma\left(n+\frac{1}{2}\right)$$

Que 2.18. Find the volume and the mass contained in the solid region in the first octant of the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$, if the density at any point $\rho(x, y, z) = kxyz$. **AKTU 2014-15, Marks 10**

Answer

Volume of the solid bounded by the ellipsoid $= 8\iiint\limits_D dx\,dy\,dz$

Let, $\dfrac{x^2}{a^2} = u, \dfrac{y^2}{b^2} = v, \dfrac{z^2}{c^2} = w$

$2x\,dx = a^2\,du$

$$dx = \frac{a\,du}{2\sqrt{u}}$$

Similarly, $dy = \dfrac{b\,dv}{2\sqrt{v}}$

$$dz = \frac{c\,dw}{2\sqrt{w}}$$

Required volume,

$$V = 8\iiint\limits_{D'} \frac{abc}{8\sqrt{u\,v\,w}}\,du\,dv\,dw$$

Where D' is the region when $u \ge 0, v \ge 0, w \ge 0$ and $u + v + w = 1$

$$= 8\frac{abc}{8}\iiint\limits_{D'} u^{-1/2}\,v^{-1/2}\,w^{-1/2}\,du\,dv\,dw$$

$$= 8\frac{abc}{8}\iiint\limits_{D'} u^{\frac{1}{2}-1}\,v^{\frac{1}{2}-1}\,w^{\frac{1}{2}-1}\,du\,dv\,dw$$

Using Dirichlet's integral,

$$= 8\frac{abc}{8}\frac{\Gamma\frac{1}{2}\Gamma\frac{1}{2}\Gamma\frac{1}{2}}{\Gamma\left(\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+1\right)} = 8\frac{abc}{8}\frac{(\sqrt{\pi})^3}{\frac{3}{2}\cdot\frac{1}{2}\sqrt{\pi}}$$

$$= \frac{4}{3}\pi\,abc \text{ cubic unit}$$

Mass = Volume × Density $= \iiint\limits_D kxyz\,dx\,dy\,dz$

Let, $\dfrac{x^2}{a^2} = u, \dfrac{y^2}{b^2} = v$ and $\dfrac{z^2}{c^2} = w$

$x = a\sqrt{u}$ and $dx = \dfrac{a}{2\sqrt{u}}\,du$

Similarly, $y = a\sqrt{v}$ and $dy = \dfrac{b}{2\sqrt{v}}\,dv$

$z = c\sqrt{w}$ and $dz = \dfrac{c}{2\sqrt{w}}\,dw$

$$\text{Mass} = \iiint\limits_{D'} \frac{abc}{8}\,u^{-1/2}\,v^{-1/2}\,w^{-1/2}\,k\,a\sqrt{u}\,b\sqrt{v}\,c\sqrt{w}\,du\,dv\,dw$$

$$= \frac{k\,a^2b^2c^2}{8}\iiint\limits_{D'} u^0\,v^0\,w^0\,du\,dv\,dw$$

Where D' is the domain,

$u \ge 0, v \ge 0, w \ge 0,\ u + v + w = 1$

$$= \frac{ka^2b^2c^2}{8}\iiint\limits_{D'} u^{1-1}\,v^{1-1}\,w^{1-1}\,du\,dv\,dw$$

$$= \frac{ka^2b^2c^2}{8}\,\frac{\Gamma 1\,\Gamma 1\,\Gamma 1}{\Gamma(1+1+1+1)} = \frac{ka^2b^2c^2}{48}$$

Que 2.19. Find the mass of a solid $\left(\dfrac{x}{ab}\right)^p + \left(\dfrac{y}{b}\right)^q + \left(\dfrac{z}{c}\right)^r = 1$, the density at any point being $\rho = kx^{l-1}y^{m-1}z^{n-1}$, where x, y, z are all positive.

AKTU 2015-16, Marks 10

Answer

Let us take

$\left(\dfrac{x}{ab}\right)^p = u$ or $\dfrac{x}{ab} = u^{1/p}$ or $x = ab\,u^{1/p}$

$\left(\dfrac{y}{b}\right)^q = v$ or $\dfrac{y}{b} = v^{1/q}$ or $y = bv^{1/q}$

$\left(\dfrac{z}{c}\right)^r = w$ or $\dfrac{z}{c} = w^{1/r}$ or $z = cw^{1/r}$

Now $dx = \dfrac{ab}{p}u^{\left(\frac{1}{p}-1\right)}du$

$dy = \dfrac{b}{q}v^{\left(\frac{1}{q}-1\right)}dw$

$dz = \dfrac{c}{r}w^{\left(\frac{1}{r}-1\right)}dw$

$\therefore$ Volume $= \iiint dxdydz = \iiint \dfrac{ab}{p}u^{\left(\frac{1}{p}-1\right)}du\,\dfrac{b}{q}v^{\left(\frac{1}{q}-1\right)}\dfrac{c}{r}w^{\left(\frac{1}{r}-1\right)}dw\,dv$

$$= \iiint \frac{ab^2c}{pqr}u^{\left(\frac{1}{p}-1\right)}v^{\left(\frac{1}{q}-1\right)}w^{\left(\frac{1}{r}-1\right)}du\,dv\,dw$$

Mass = Volume × Density

$$= \iiint \frac{ab^2c}{pqr} u^{\left(\frac{1}{p}-1\right)} v^{\left(\frac{1}{q}-1\right)} w^{\left(\frac{1}{r}-1\right)} kx^{(l-1)} y^{(m-1)} z^{(n-1)}\, du\, dw\, dv$$

$$= \iiint \frac{ab^2c}{pqr} u^{\left(\frac{1}{p}-1\right)} v^{\left(\frac{1}{q}-1\right)} w^{\left(\frac{1}{r}-1\right)} k(ab)^{(l-1)}$$

$$u^{\left(\frac{l-1}{p}\right)} b^{(m-1)} v^{\left(\frac{m-1}{q}\right)} c^{(n-1)} w^{\left(\frac{n-1}{r}\right)}\, du\, dv\, dw$$

$$= \iiint \frac{ab^2c}{pqr} u^{\left(\frac{1}{p}-1\right)} v^{\left(\frac{1}{q}-1\right)} w^{\left(\frac{1}{r}-1\right)} k(ab)^{(l-1)}$$

$$b^{(m-1)} c^{(n-1)} u^{(l/p-1/p)} v^{(m/q-1/q)} w^{(n/r-1/r)}\, du\, dv\, dw$$

$$= \iiint \frac{ka^l b^{(l+m-1)} c^n}{pqr} u^{\left(\frac{1}{p}-1\right)} v^{\left(\frac{1}{q}-1\right)} w^{\left(\frac{1}{r}-1\right)}\, du\, dv\, dw$$

$$= \frac{ka^l b^{m+l} c^n}{pqr} \frac{\lceil l/p \lceil m/q \lceil n/r}{\lceil l/p+m/q+n/r+1} \quad \text{(By using Dirichlet's integral unit)}$$

Que 2.20. **Find the volume of the solid bounded by the co-ordinate planes and the surface $\sqrt{\frac{x}{a}} + \sqrt{\frac{y}{b}} + \sqrt{\frac{z}{c}} = 1$.**

Answer

Put $\sqrt{\frac{x}{a}} = u, \sqrt{\frac{y}{b}} = v, \sqrt{\frac{z}{c}} = w$ then $u \ge 0, v \ge 0, w \ge 0$ and $u + v + w = 1$

Also, $dx = 2au\, du$, $dy = 2bv\, dv$, $dz = 2cw\, dw$

Required volume $= \iiint_D dx\, dy\, dz$

$$= \iiint_D 8\,abc\, uvw\, du\, dv\, dw, \text{ where } u + v + w = 1$$

$$= 8\,abc \iiint_D u^{2-1} v^{2-1} w^{2-1}\, du\, dv\, dw$$

$$= 8\,abc \frac{\lceil 2 \lceil 2 \lceil 2}{\lceil (2+2+2+1)} = 8\,abc \cdot \frac{1.1.1}{\lceil 7} = \frac{abc}{90}$$

Que 2.21. **Find the mass of a plate which is formed by the coordinate planes and the plane $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$, the density is given by $\rho = kxyz$.**

AKTU 2017-18, Marks 3.5

AKTU 2011-12, Marks 05

Answer

Same as Q. 2.18, Page 2-13F, Unit-2.

(**Answer :** $M = \frac{ka^2b^2c^2}{720}$)

PART-3

Applications of Definite Integrals to Evaluate Surface Areas and Volume of Revolutions.

CONCEPT OUTLINE

Surface of the Solid of Revolution : The curved surface of the solid generated by the revolution, about the x-axis, of the area bounded by the curve $y = f(x)$, the x-axis and the ordinates $x = a$, $x = b$ is

$$\int_{x=a}^{x=b} 2\pi\, y\, ds$$

Where s is the length of the arc of the curve measured from a fixed point on it to any point (x, y).

Three Practical Forms of Surface Formula :

i. **Surface Formula for Cartesian Equation :** The curved surface of the solid generated by the revolution about the x-axis, of the area bounded by the curve $y = f(x)$, the x-axis and the ordinates $x = a$, $x = b$ is

$$\int_{x=a}^{x=b} 2\pi\, y \frac{ds}{dx} dx, \text{ where } \frac{ds}{dx} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}$$

ii. **Surface Formula for Parametric Equation :** The curved surface of the solid generated by the revolution about the x-axis, of the area bounded by the curve $x = f(t)$, $y = \phi(t)$, the x-axis and the ordinates at the point, where $t = a$, $t = b$ is

$$\int_{x=a}^{x=b} 2\pi\, y \frac{ds}{dx} dt, \text{ where } \frac{ds}{dx} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dx}\right)^2}$$

iii. **Surface Formula for Polar Equation :** The curved surface of the solid generated by the revolution, about the initial line, of the area bounded by the curve $r = f(\theta)$ and the radii vectors $\theta = \alpha$, $\theta = \beta$ is

$$\int_{\theta=\alpha}^{\theta=\beta} 2\pi\, y \frac{ds}{dx} d\theta, \text{ where } \frac{ds}{d\theta} = \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}$$

and $\qquad y = r \sin\theta.$

Revolution about y-axis : The curved surface of the solid generated by the revolution about the y-axis of the area bounded by the curve $x = f(y)$, the y-axis and the abscissa $y = a$, $y = b$ is

$$\int_{y=a}^{y=b} 2\pi x \, ds$$

Volume between Two Solids : The volume of the solid generated by the revolution about the x-axis, of the arc bounded by the curves $y = f(x)$, $y = \phi(x)$, and the ordinates $x = a$, $x = b$ is

$$\int_b^a \pi (y_1^2 - y_2^2)\, dx$$

Where y_1 is the 'y' of the upper curve and y_2 that of the lower curve.

Volume Formula for Parametric Equations :

i. The volume of the solid generated by the revolution about the x-axis, of the area bounded by the curve $x = f(t)$, $y = \phi(t)$, the x-axis and the ordinates, where $t = a$, $t = b$ is

$$\int_a^b \pi y^2 \frac{dx}{dt} dt$$

ii. The volume of the solid generated by the revolution about the y-axis, of the area bounded by the curves $x = f(t)$, $y = \phi(t)$, the y-axis and the abscissa at the points, where $t = a$, $t = b$ is

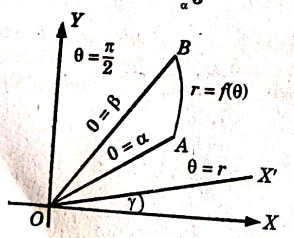
$$\int_a^b \pi x^2 \frac{dy}{dt} dt$$

Volume Formulae for Polar Curves : The volume of the solid generated by the revolution of the area bounded by the curves $r = f(\theta)$, and the radii vectors $\theta = \alpha$, $\theta = \beta$

i. About the initial line OX $(\theta = 0)$ is $\int_\alpha^\beta \frac{2}{3} \pi r^3 \sin\theta \, d\theta$

ii. About the line OY $\left(\theta = \frac{\pi}{2}\right)$ is $\int_\alpha^\beta \frac{2}{3} \pi r^3 \cos\theta \, d\theta$

iii. About any line OX' $(\theta = \gamma)$ is $\int_\alpha^\beta \frac{2}{3} \pi r^3 \sin(\theta - \gamma) \, d\theta$



Questions-Answers

Long Answer Type and Medium Answer Type Questions

Que 2.22. **Find the area of the surface formed by the revolution of the parabola $y^2 = 4ax$ about the x-axis by the arc from the vertex to one end of the latus rectum.**

Answer

The equation of the parabola is $y^2 = 4ax$ ...(2.22.1)

Differentiating wrt x, we get

$$2y \frac{dy}{dx} = 4a \text{ or } \frac{dy}{dx} = \frac{2a}{y}$$

$$\therefore \quad \frac{ds}{dx} = \sqrt{1 + \left(\frac{dy}{dx}\right)^2} = \sqrt{1 + \frac{4a^2}{y^2}} = \sqrt{1 + \frac{4a^2}{4ax}} = \sqrt{\frac{x+a}{x}}$$

For the arc from the vertex O to L, the end of the latus rectum, x varies from 0 to a.

$$\therefore \text{ Required surface} = \int_{x=0}^{a} 2\pi y \frac{ds}{dx} dx$$

$$= \int_{x=0}^{a} 2\pi \sqrt{4ax} \sqrt{\frac{x+a}{x}} \, dx$$

[$\because$ From eq. (2.22.1) $y = \sqrt{4ax}$]

$$= 4\pi \sqrt{a} \int_{x=0}^{a} (x+a)^{1/2} dx$$

$$= 4\pi \sqrt{a} \, \frac{2}{3} \left[(x+a)^{3/2}\right]_0^a = \frac{8\pi a^2}{3} (2\sqrt{2} - 1)$$

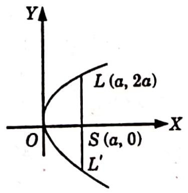


Fig. 2.22.1.

Que 2.23. **The curve $r = a(1 + \cos\theta)$ revolves about the initial line. Find the surface of the figure so formed.**

Answer

The equation of the cardioid is $r = a(1 + \cos\theta)$ ...(2.23.1)

The cardioid is symmetrical about the initial line and for the upper half of the curve, θ varies from 0 to π.

Now from eq. (2.23.1), $\frac{dr}{d\theta} = -a\sin\theta$

$$\therefore \quad \frac{ds}{d\theta} = \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}$$

$$= \sqrt{a^2(1+\cos\theta)^2 + a^2\sin^2\theta}$$

$$= a\sqrt{2(1+\cos\theta)} = 2a\cos\frac{\theta}{2}$$

$$\therefore \quad \text{Required surface} = \int 2\pi y \frac{ds}{d\theta}\, d\theta, \text{ where } y = r\sin\theta$$

$$= 2\pi\int_0^{\pi} a\sin\theta(1+\cos\theta)\, 2a\cos\frac{\theta}{2}\, d\theta \quad (\because r = a(1+\cos\theta))$$

$$= 2\pi\int_0^{\pi} a2\sin\frac{\theta}{2}\cos\frac{\theta}{2}\, 2\cos^2\frac{\theta}{2}\, 2a\cos\frac{\theta}{2}\, d\theta$$

$$= 16\pi a^2\int_0^{\pi}\cos^4\frac{\theta}{2}\sin\frac{\theta}{2}\, d\theta$$

$$= 16\pi a^2\left[\frac{-\cos^5\theta/2}{5\times\frac{1}{2}}\right]_0^{\pi}$$

$$= -\frac{32}{5}\pi a^2(0-1) = \frac{32}{5}\pi a^2$$

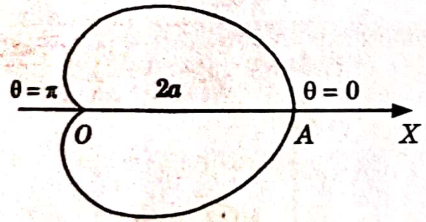


Fig. 2.23.1.

Que 2.24. **The arc of the cardioid $r = a(1+\cos\theta)$ included between $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$ is rotated about the line $\theta = \frac{\pi}{2}$. Find the area of surface generated.**

Answer

The cardioid is $r = a(1+\cos\theta)$ ...(2.24.1)

The arc CAB $\left(\text{from } \theta = -\frac{\pi}{2} \text{ to } \theta = \frac{\pi}{2}\right)$ revolves about the line

$\theta = \frac{\pi}{2}$, *i.e.*, the y-axis.

Also the curve is symmetrical about the initial line or x-axis.

From eq. (2.24.1), $\frac{dr}{d\theta} = -a\sin\theta$

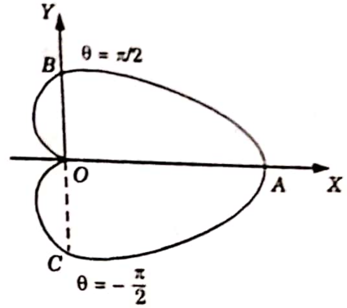


Fig. 2.24.1.

$$\therefore \quad \frac{ds}{d\theta} = \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} = \sqrt{a^2(1+\cos\theta)^2 + a^2\sin^2\theta}$$

$$= a\sqrt{2(1+\cos\theta)} = 2a\cos\frac{\theta}{2}$$

$\therefore$ Required surface area

= 2 × Surface generated by the revolution of arc AB

$$= 2\int_0^{\pi/2} 2\pi x\frac{ds}{d\theta}\, d\theta \quad [\because \text{ For the arc } AB, \theta \text{ varies from 0 to } \pi/2]$$

$$= 4\pi\int_0^{\pi/2} r\cos\theta\, 2a\cos\frac{\theta}{2}\, d\theta \quad (\because x = r\cos\theta)$$

$$= 8\pi a\int_0^{\pi/2} a(1+\cos\theta)\cos\theta\cos\frac{\theta}{2}\, d\theta$$

$$= 8\pi a^2\int_0^{\pi/2}\left(2 - 2\sin^2\frac{\theta}{2}\right)\left(1 - 2\sin^2\frac{\theta}{2}\right)\cos\frac{\theta}{2}\, d\theta$$

Put $\sin\frac{\theta}{2} = T \quad \therefore \quad \frac{1}{2}\cos\frac{\theta}{2}\, d\theta = dt$

Now the limits are given as follows,

When $\theta = 0$, $t = 0$ and when $\theta = \pi/2$, $t = 1/\sqrt{2}$.

$$\text{Now, surface area} = 16\pi a^2\int_0^{1/\sqrt{2}}(1 - 3t^2 + 2t^4)\, 2dt$$

$$= 32\pi a^2\left[t - t^3 + \frac{2t^5}{5}\right]_0^{1/\sqrt{2}} = \frac{96}{5\sqrt{2}}\pi a^2$$

Que 2.25. **Find the volume of the solid generated by the revolution of $r = 2a \cos\theta$ about the initial line.**

Answer

The equation of the curve is

$$r = 2a\cos\theta \qquad ...(2.25.1)$$

Eq. (2.25.1) is clearly a circle passing through the pole. The curve is symmetrical about the initial line and for the upper half of the circle θ varies from 0 to $\frac{\pi}{2}$.

$\therefore$ Required volume

$$= \int_0^{\pi/2}\frac{2}{3}\pi r^3 \sin\theta\, d\theta = \frac{2}{3}\pi\int_0^{\pi/2}(2a\cos\theta)^3 \sin\theta\, d\theta$$

$$= \frac{16}{3}\pi a^3\int_0^{\pi/2}\cos^3\theta \sin\theta\, d\theta$$

$$= -\frac{16}{3}\pi a^3\left[\frac{\cos^4\theta}{4}\right]_0^{\pi/2} = -\frac{4}{3}\pi a^3(0-1) = \frac{4}{3}\pi a^3$$

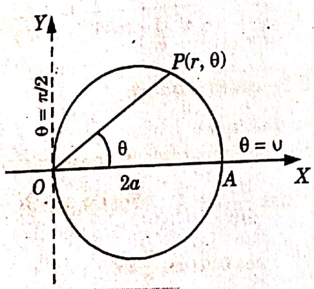


Fig. 2.25.1.

Que 2.26. **Show that the volume of the solid formed by the revolution of the curve $r = a + b\cos\theta$ ($a > b$) about the initial line is $\frac{4}{3}\pi a(a^2 + b^2)$.**

Answer

The equation of the curve is

$$r = a + b\cos\theta \; (a > b) \qquad ...(2.26.1)$$

The curve is symmetrical about the initial line and for the upper half of the curve θ varies from 0 to π.

$\therefore$ Required volume

$$= \int_0^{\pi}\frac{2}{3}\pi r^3 \sin\theta\, d\theta$$

$$= \frac{2}{3}\pi\int_0^{\pi}(a + b\cos\theta)^3 \sin\theta\, d\theta$$

$$= -\frac{2}{3}\frac{\pi}{b}\int_0^{\pi}(a + b\cos\theta)^3(-b\sin\theta\, d\theta)$$

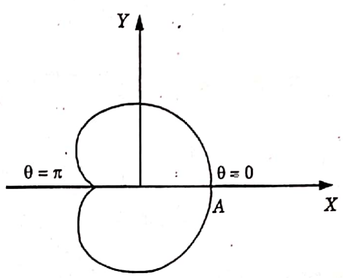


Fig. 2.26.1.

$$= -\frac{2}{3}\frac{\pi}{b}\left[\frac{(a + b\cos\theta)^4}{4}\right]_0^{\pi}$$

$$= -\frac{2\pi}{3b}\left[\frac{(a-b)^4}{4} - \frac{(a+b)^4}{4}\right]$$

$$= \frac{\pi}{6b}[(a+b)^4 - (a-b)^4] = \frac{4}{3}\pi a(a^2 + b^2)$$

Que 2.27. **Find the volume of the solid formed by the revolution of the cissoid $y^2(2a - x) = x^3$ about its asymptote.**

Answer

The equation of the curve is $y^2(2a-x) = x^3$ or $y^2 = \dfrac{x^3}{2a-x}$ ...(2.27.1)

The curve is symmetrical about the x-axis and the asymptote is the line $2a - x = 0$ or $x = 2a$.

If $P(x, y)$ be any point on the curve and $PM \perp$ on the asymptote (the axis of revolution), and $PN \perp OX$.

Then $PM = NA = OA - ON = 2a - x$ and $AM = NP = y$,

where A is the point of intersection of the asymptote and the x-axis.

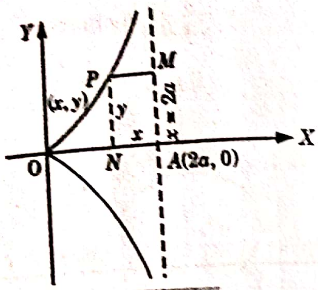


Fig. 2.27.1.

$\therefore$ Required volume $= 2\int \pi (PM)^2 \, d(AM)$ ...(2.27.2)

Now, $AM = y = \dfrac{x^{3/2}}{\sqrt{2a-x}}$ [From eq. (2.27.1)]

$\therefore$ $d(AM) = dy$

$$= \frac{(2a-x)^{1/2}\frac{3}{2}x^{1/2} - x^{3/2}\frac{1}{2}(2a-x)^{-1/2}(-1)}{2a-x}\,dx$$

$$= \frac{3x^{1/2}(2a-x)+x^{3/2}}{2(2a-x)^{3/2}}\,dx = \frac{\sqrt{x}\,(3a-x)}{(2a-x)^{3/2}}\,dx$$

From eq. (2.27.2), we get

$\therefore$ Required volume

$$= 2\pi\int_0^{2a}(2a-x)^2\,\frac{\sqrt{x}(3a-x)}{(2a-x)^{3/2}}\,dx$$

$$= 2\pi\int_0^{2a}(3a-x)^2\,\sqrt{x}\,\sqrt{2a-x}\,dx$$

Put $x = 2a\sin^2\theta$ $\therefore$ $dx = 4a\sin\theta\cos\theta\,d\theta$

Now the limits of the integral are given as follows,

When $x = 0$, $\theta = 0$, and when $x = 2a$, $\theta = \dfrac{\pi}{2}$

Now, required volume

$$= 2\pi\int_0^{\pi/2}(3a - 2a\sin^2\theta)\sqrt{2a\sin^2\theta}\,\sqrt{2a(1-\sin^2\theta)}\;4a\sin\theta\cos\theta\,d\theta$$

$$= 16\pi a^3\int_0^{\pi/2}(3 - 2\sin^2\theta)\sin^2\theta\cos^2\theta\,d\theta$$

$$= 16\pi a^3\int_0^{\pi/2}(3\sin^2\theta\cos^2\theta - 2\sin^4\theta\cos^2\theta)\,d\theta$$

$$= 16\pi a^3\left[3\cdot\frac{1.1}{4.2}\cdot\frac{\pi}{2} - 2\cdot\frac{3.1.1}{6.4.2}\cdot\frac{\pi}{2}\right] = 16\pi a^3\left[\frac{3\pi}{16} - \frac{\pi}{16}\right] = 2\pi^2a^3$$

☺☺☺

3 UNIT

Sequence and Series

CONTENTS

PART-1

Definition of Sequence and Series with Examples, Convergence of Sequence and Series.

CONCEPT OUTLINE

Sequence : An ordered set of real number $a_1, a_2, a_3, ..., a_n$ is called a sequence and is denoted by (a_n). If the number of terms is unlimited, then the sequence is said to be an infinite sequence and a_n is its general term.

Series : If $u_1, u_2, u_3, ..., u_n, ...$ be an infinite sequence of real numbers, then

$$u_1 + u_2 + u_3 + ... + u_n + ... \infty$$

is called an infinite series. An infinite series is denoted by $\Sigma\, u_n$ and the sum of its first n terms is denoted by s_n.

Convergence, Divergence and Oscillation of a Sequence :

If $\lim\limits_{n\to\infty}(a_n) = l$ is finite and unique, the sequence is said to be convergent.

If $\lim\limits_{n\to\infty}(a_n)$ is infinite ($\pm\infty$), the sequence is said to be divergent.

If $\lim\limits_{n\to\infty}(a_n)$ is not unique, the sequence is said to be oscillatory.

Convergence, Divergence and Oscillation of a Series : Consider the infinite series $\Sigma u_n = u_1 + u_2 + u_3 + ... + u_n + ... \infty$ and let the sum of the first n terms be $s_n = u_1 + u_2 + u_3 + ... + u_n$

Clearly, s_n is a function of n and as n increases indefinitely three possibilities arises :

i. If s_n tends to a finite limit as $s_n \to \infty$, the series Σu_n is said to be convergent.

ii. If s_n tends to $\pm\infty$ as $n \to \infty$, the series Σu_n is said to be divergent.

iii. If s_n does not tend to a unique limit as $n \to \infty$, then the series Σu_n is said to be oscillatory or non-convergent.

Questions-Answers

Long Answer Type and Medium Answer Type Questions

Que 3.1. **Examine the following sequence for convergence :**

i. $a_n = \dfrac{n^2 - 2n}{3n^2 + n}$, ii. $a_n = 2^n$ iii. $a_n = 3 + (-1)^n$.

Answer

i. $\lim_{n\to\infty}\left(\frac{n^2-2n}{3n^2+n}\right)=\lim_{n\to\infty}\frac{1-2/n}{3+1/n}$ = 1/3 which is finite and unique. Hence the sequence (a_n) is convergent.

ii. $\lim_{n\to\infty}(2^n)=\infty$. Hence the sequence (a_n) is divergent.

iii. $\lim_{n\to\infty}[3+(-1)^n] = 3 + 1 = 4$, when n is even
$= 3 - 1 = 2$, when n is odd
i.e., this sequence doesn't have a unique limit. Hence it oscillates.

Que 3.2. **Examine the following series for convergence :**

i. $1+2+3+....+n+....\infty$
ii. $5-4-1+5-4-1+5-4-1+....\infty$

Answer

i. Here, $s_n = 1+2+3+.....+n=\frac{n(n+1)}{2}$

$\therefore \lim_{n\to\infty} s_n = \frac{1}{2}\lim_{n\to\infty} n(n+1)\to\infty$ Hence this series is divergent.

ii. Here, $s_n = 5-4-1+5-4-1+5-4-1+....n$ terms
$= 0, 5$ or 1
Clearly in this case, s_n does not tend to a unique limit. Hence the series is oscillatory.

Que 3.3. **Test the following series for convergence :**

i. $\frac{1}{1.2.3}+\frac{3}{2.3.4}+\frac{5}{3.4.5}+.....\infty$

ii. $1+\frac{1}{2^2}+\frac{2^2}{3^3}+\frac{3^3}{4^4}+....\infty$

Answer

i. We have $u_n = \frac{2n-1}{n(n+1)(n+2)}=\frac{1}{n^2}\frac{2-1/n}{(1+1/n)(1+2/n)}$

Taking $v_n = 1/n^2$, we have

$\therefore \lim_{n\to\infty}\frac{u_n}{v_n} = \lim_{n\to\infty}\frac{2-1/n}{(1+1/n)(1+2/n)}=\frac{2-0}{(1+0)(1+0)}$

= 2, which is finite and non zero.

Hence, both Σu_n and Σv_n converge or diverge together but $\Sigma v_n = \Sigma 1/n^2$ is known to be convergent. Hence Σu_n is also convergent.

ii. Here $u_n = \frac{n^n}{(n+1)n+1}=\frac{1}{n+1}\cdot\left(\frac{n}{n+1}\right)^2$, ignoring the first term.

Taking $v_n = 1/n$, we have

$\lim_{n\to\infty}\left(\frac{u_n}{v_n}\right) = \lim_{n\to\infty}\frac{n}{n+1}.\lim_{n\to\infty}\left(\frac{n}{n+1}\right)^n$

$= \lim_{n\to\infty}\left(\frac{1}{1+1/n}\right).\lim_{n\to\infty}\frac{1}{(1+1/n)^n}=1.\frac{1}{e}\neq 0$

Now since Σv_n is divergent, therefore Σv_n is also divergent.

Que 3.4. **Determine the nature of the series :**

i. $\frac{\sqrt{2}-1}{3^3-1}+\frac{\sqrt{3}-1}{4^3-1}+\frac{\sqrt{4}-1}{5^3-1}+....\infty$ **ii.** $\sum\frac{1}{n}\sin\frac{1}{n}$

Answer

i. We have $u_n = \frac{\sqrt{(n+1)}-1}{(n+2)^3-1}=\frac{\sqrt{n}\,[(1+1/n)-1/\sqrt{n}]}{n^3\,[(1+2/n)^3-1/n^3]}$

Taking $v_n = 1/n^{5/2}$, we have

$\lim_{n\to\infty}\frac{u_n}{v_n} = \lim_{n\to\infty}\frac{\sqrt{[(1+1/n)}-1/\sqrt{n}]}{[(1+2/n)^3-1/n^3]}=1\neq 0$

Since Σv_n is convergent, therefore Σu_n is also convergent.

ii. Here $u_n = \frac{1}{n}\sin\frac{1}{n}=\frac{1}{n}\left[\frac{1}{n}-\frac{1}{3!\,n^3}+\frac{1}{5!\,n^5}-.....\right]$

$= \frac{1}{n^2}\left[1-\frac{1}{3!\,n^2}+\frac{1}{5!\,n^4}-....\right]$

Taking $v_n = 1/n^2$, we have

$\lim_{n\to\infty}\frac{u_n}{v_n} = \lim_{n\to\infty}\left[1-\frac{1}{3!\,n^2}+\frac{1}{5!\,n^4}-....\right]=1\neq 0$

Since Σv_n is convergent, therefore Σu_n is also convergent.

PART-2

Tests for Convergence of Series (Ratio Test, D'Alembert's Test, Raabe's Test).

Questions-Answers

Long Answer Type and Medium Answer Type Questions

Que 3.5. **Discuss in detail about D' Alembert's test or ratio test. Also give its limitations.**

Answer

A. D'Alembert's Test or Ratio Test :

In a positive term series Σu_n, if

$$\lim_{n\to\infty} \frac{u_{n+1}}{u_n} = \lambda,$$ then the series converges for $\lambda < 1$ and diverges for $\lambda > 1$.

Case I : When, $\lim\limits_{n\to\infty} \dfrac{u_{n+1}}{u_n} = \lambda < 1$

By definition of a limit, we can find a positive number r (< 1) such that

$\dfrac{u_{n+1}}{u_n} < r$ for all $n > m$

Leaving out the first m terms, let the series be $u_1 + u_2 + u_3 + ...$

So that $\dfrac{u_2}{u_1} < r$, $\dfrac{u_3}{u_2} < r$, $\dfrac{u_4}{u_3} < r$,.... and so on. Then $u_1 + u_2 + u_3 + ... \infty$

$$= u_1\left(1 + \frac{u_2}{u_1} + \frac{u_3}{u_2}\frac{u_2}{u_1} + \frac{u_4}{u_3}\frac{u_3}{u_2}\frac{u_2}{u_1} + ... \infty\right)$$

$$< u_1 (1 + r + r^2 + r^3 + ... \infty)$$

$$= \frac{u_1}{1 - r},$$ which is finite quantity. Hence Σu_n is convergent. [$\because r < 1$]

Case II : When, $\lim\limits_{n\to\infty} \dfrac{u_{n+1}}{u_n} = \lambda > 1$

By definition of limit, we can find m, such that $\dfrac{u_{n+1}}{u_n} \geq 1$ for all $n \geq m$.

Leaving out the first m terms, let the series be

$u_1 + u_2 + u_3 + ...$ so that $\dfrac{u_2}{u_1} \geq 1$, $\dfrac{u_3}{u_2} \geq r$, $\dfrac{u_4}{u_3} \geq 1$,.... and so on.

$$\therefore \quad u_1 + u_2 + u_3 + u_4 + ... + u_n = u_1\left(1 + \frac{u_2}{u_1} + \frac{u_3}{u_2}\frac{u_2}{u_1} + ...\right)$$

$$\geq u_1 (1 + 1 + 1 + ... \text{ to } n \text{ terms}) = nu_1$$

$\therefore \lim\limits_{n\to\infty} (u_1 + u_2 + ... + u_n) \geq \lim\limits_{n\to\infty} (nu_1)$, which tends to infinity. Hence Σu_n is divergent

B. Limitations of D'Alembert's Test :

1. Ratio test fails when $\lambda = 1$.
2. This test makes no reference to the magnitude of u_{n+1}/u_n but concerns only with the limit of this ratio.

Que 3.6. **Test for convergence of the following series :**

i. $\dfrac{1}{2\sqrt{1}} + \dfrac{x^2}{3\sqrt{2}} + \dfrac{x^4}{4\sqrt{3}} + \dfrac{x^6}{5\sqrt{4}} + ... \infty$

ii. $1 + \dfrac{2}{5}x + \dfrac{6}{9}x^2 + \dfrac{14}{17}x^3 + ... + \dfrac{2^n - 2}{2^n + 1}x^{n-1} + ...(x > 0)$

Answer

i. We have, $u_n = \dfrac{x^{2n-2}}{(n+1)\sqrt{n}}$ and $u_{n+1} = \dfrac{x^{2n}}{(n+2)\sqrt{(n+1)}}$

$$\therefore \quad \lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \lim_{n\to\infty} \frac{x^{2n-2}}{(n+1)\sqrt{n}} \frac{(n+2)\sqrt{(n+1)}}{x^{2n}}$$

$$= \lim_{n\to\infty} \left[\frac{n+2}{n+1}\left(\frac{n+1}{n}\right)^{½}\right] x^{-2}$$

$$= \lim_{n\to\infty} \left[\frac{1+2/n}{1+1/n}\sqrt{(1+1/n)}\right] x^{-2} = x^{-2}$$

Hence Σu_n converges if $x^{-2} > 1$ *i.e.*, for $x^2 < 1$ and diverges for $x^2 > 1$.

If $x^2 = 1$, then, $u_n = \dfrac{1}{(n+1)\sqrt{n}} = \dfrac{1}{n^{3/2}} \dfrac{1}{1+1/n}$

Taking $v_n = \dfrac{1}{n^{3/2}}$, we get $\lim\limits_{n\to\infty} \dfrac{u_n}{v_n} = \lim\limits_{n\to\infty} \dfrac{1}{1+1/n} = 1$, a finite quantity.

$\therefore$ Both Σu_n and Σv_n converge or diverge together. But $\Sigma v_n = \sum \dfrac{1}{n^{3/2}}$ is a convergent series.

$\therefore$ Σu_n is also convergent. Hence the given series converges if $x^2 \leq 1$ and diverges if $x^2 > 1$.

ii. Here, $\dfrac{u_n}{u_{n+1}} = \dfrac{2^n - 2}{2^n + 1} x^{n-1} \dfrac{2^{n+1} + 1}{2^{n+1} - 2} \dfrac{1}{x^n} = \dfrac{1 - \dfrac{2}{2^n}}{1 + \dfrac{1}{2^n}} \dfrac{2 + \dfrac{1}{2^n}}{2 - \dfrac{2}{2^n}} \dfrac{1}{x}$

$$\therefore \lim_{n\to\infty} \frac{u_n}{u_{n+1}} = \frac{1-0}{1+0}\,\frac{2+0}{2-0}\,\frac{1}{x} = \frac{1}{x}$$

Thus by ratio test, Σu_n converges for $x^{-1} > 1$ *i.e.*, for $x < 1$ and diverges for $x > 1$. But it fails for $x = 1$.

When $x = 1$, $\lim_{n\to\infty} u_n = \lim_{n\to\infty} \dfrac{2^n - 2}{2^n + 1} = \lim_{n\to\infty} \dfrac{1 - \frac{1}{2^n}}{1 + \frac{1}{2^n}} = 1 \neq 0$

$\therefore$ Σu_n diverges for $x = 1$. Hence the given series converges for $x < 1$ and diverges for $x \geq 1$.

Que 3.7. **Discuss the convergence of the series.**

$$1 + \frac{2!}{2^2} + \frac{3!}{3^3} + \frac{4!}{4^4} + ... \infty$$

Answer

Given series is

$$\Sigma u_n = \sum_{n=1}^{\infty} \frac{n!}{n^n}$$

Here, $\dfrac{u_n}{u_{n+1}} = \dfrac{n!}{n^n} \dfrac{(n+1)^{n+1}}{(n+1)!} = \dfrac{(n+1)^n}{n^n} = \left(1 + \dfrac{1}{n}\right)^n$

$\therefore$ $\lim_{n\to\infty} \dfrac{u_n}{u_{n+1}} = \lim_{n\to\infty} \left(1 + \dfrac{1}{n}\right)^n = e$, which is > 1. Hence the given series is convergent.

Que 3.8. **Examine the convergence of the series :**

$$\frac{x}{1+x} + \frac{x^2}{1+x^2} + \frac{x^3}{1+x^3} + ... \infty$$

Answer

Here, $u_n = \dfrac{x^n}{1+x^n}$ and $u_{n+1} = \dfrac{x^{n+1}}{1+x^{n+1}}$

$\therefore$ $\lim_{n\to\infty} \dfrac{u_n}{u_{n+1}} = \lim_{n\to\infty} \left(\dfrac{x^n}{x^{n+1}} \dfrac{1+x^{n+1}}{1+x^n}\right) = \lim_{n\to\infty} \left(\dfrac{1+x^{n+1}}{x+x^{n+1}}\right)$

$= \dfrac{1}{x}$, if $x < 1$ $\qquad [\because x^{n+1} \to 0$ and $n \to \infty]$

Also, $\lim_{n\to\infty} \dfrac{u_n}{u_{n+1}} = \lim_{n\to\infty} \left(\dfrac{1 + 1/x^{n+1}}{1 + x/x^{n+1}}\right) = 1$, if $x > 1$.

By ratio test, Σu_n converges for $x < 1$ and fails for $x \geq 1$.

When $x = 1$, $\Sigma u_n = \dfrac{1}{2} + \dfrac{1}{2} + \dfrac{1}{2} + ... \infty$, which is divergent.

Hence the given series converges for $x < 1$ and diverges for $x \geq 1$.

Que 3.9. **Explain Raabe's test in brief.**

Answer

In the positive term series Σu_n, if $\lim_{n\to\infty} n\left(\dfrac{u_n}{u_{n+1}} - 1\right) = k$, then the series converges for $k > 1$ and diverges for $k < 1$, but the test fails for $k = 1$.

When $k > 1$, choose a number p such that $k > p > 1$, and compare Σu_n with the series $\sum \dfrac{1}{n^p}$ which is convergent since $p > 1$.

$\therefore$ Σu_n will converge, if from and after some term,

$$\frac{u_n}{u_{n+1}} > \frac{(n+1)^p}{n^p} \text{ or } \left(1 + \frac{1}{n}\right)^p$$

or if, $\dfrac{u_n}{u_{n+1}} > 1 + \dfrac{p}{n} + \dfrac{p(p-1)}{2n^2} + ...$

or if, $n\left(\dfrac{u_n}{u_{n+1}} - 1\right) > p + \dfrac{p(p-1)}{2n} + ...$

or if, $\lim_{n\to\infty} n\left(\dfrac{u_n}{u_{n+1}} - 1\right) > \lim_{n\to\infty} \left[p + \dfrac{p(p-1)}{2n} + ...\right]$

i.e., if $k > p$, which is true. Hence, Σu_n is convergent.
The other case when $k < 1$ can be proved similarly.

Que 3.10. **Test for convergence of the following the series :**

i. $\sum \dfrac{4 \cdot 7 ... (3n+1)}{1 \cdot 2 ... n} x^{2n}$ $\qquad$ **ii.** $\sum \dfrac{(n!)^2}{(2n)!} x^{2n}$

Answer

i. Here, $\dfrac{u_n}{u_{n+1}} = \dfrac{4 \cdot 7 ...(3n+1)}{1 \cdot 2....n} x^n \div \dfrac{4 \cdot 7.....(3n+4)}{1 \cdot 2.....(n+1)} x^{n+1} = \dfrac{n+1}{3n+4} \dfrac{1}{x}$

$= \left[\dfrac{1 + 1/n}{3 + 4/n}\right] \dfrac{1}{x}$

$\therefore$ $\lim_{n\to\infty} \dfrac{u_n}{u_{n+1}} = \dfrac{1}{3x}$

Thus by ratio test, the series converges for $\dfrac{1}{3x} > 1$, *i.e.* for $x < \dfrac{1}{3}$ and diverges for $x > \dfrac{1}{3}$. But it fails for $x = \dfrac{1}{3}$.

$\therefore$ Let us try the Raabe's test

Now, $\dfrac{u_n}{u_{n+1}} = \left(1+\dfrac{1}{n}\right)\left(1+\dfrac{4}{3n}\right)^{-1}$ [Expand by binomial theorem]

$$= \left(1+\frac{1}{n}\right)\left(1-\frac{4}{3n}+\frac{16}{9n^2}-....\right) = 1-\frac{1}{3n}+\frac{4}{9n^2}+...$$

$$\therefore n\left(\frac{u_n}{u_{n+1}}-1\right) = -\frac{1}{3}+\frac{4}{9n}+....$$

$$\therefore \lim_{n\to\infty} n\left(\frac{u_n}{u_n+1}-1\right) = -\frac{1}{3} \text{ which is } < 1$$

Thus by Raabe's test, the series diverges.

Hence the given series converges for $x < (1/3)$ and diverges for $x \geq (1/3)$.

ii. Here, $\dfrac{u_n}{u_{n+1}} = \left(\dfrac{n!}{(n+1)!}\right)^2 \dfrac{[2(n+1)]!}{(2n)!} \dfrac{x^{2n}}{x^{2(n+1)}}$

$$= \frac{(2n+1)(2n+2)}{(n+1)^2}\frac{1}{x^2} = \frac{2(2n+1)}{n+1}\frac{1}{x^2}$$

$$\therefore \lim_{n\to\infty}\frac{u_n}{u_{n+1}} = \lim_{n\to\infty}\frac{2(2+1/n)}{1+1/n}\frac{1}{x^2} = \frac{4}{x^2}$$

Thus by ratio test, the series converges for $x^2 < 4$ diverges for $x^2 < 4$ and diverges for $x^2 > 4$. But fails for $x^2 = 4$.

When $x^2 = 4$, $n\left(\dfrac{u_n}{u_{n+1}}-1\right) = n\left(\dfrac{2n+1}{2n+2}-1\right) = -\dfrac{n}{2n+2}$

$$\therefore \lim_{n\to\infty} n\left(\frac{u_n}{u_{n+1}}-1\right) = -\frac{1}{2} < 1$$

Thus by Raabe's test, the series diverges.

Hence the given series converges for $x^2 < 4$ and diverges for $x^2 \geq 4$.

PART-3

Fourier Series.

CONCEPT OUTLINE

Fourier Series in the Interval $C < x < C + 2\pi$: The Fourier series for the function $f(x)$ in the interval $C < x < C + 2\pi$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

Where a_0, a_n and b_n are called Fourier coefficients, and given as

$$a_0 = \frac{1}{\pi}\int_C^{C+2\pi} f(x)dx$$

$$a_n = \frac{1}{\pi}\int_C^{C+2\pi} f(x)\cos nx\, dx$$

$$b_n = \frac{1}{\pi}\int_C^{C+2\pi} f(x)\sin nx\, dx$$

Fourier Series when Interval is Changed : Fourier series in the interval $C < x < C + 2L$ is given as

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L} + \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}$$

Where, $a_0 = \dfrac{1}{L}\displaystyle\int_C^{C+2L} f(x)dx$

$$a_n = \frac{1}{L}\int_C^{C+2L} f(x)\cos\frac{n\pi x}{L}dx$$

and $b_n = \dfrac{1}{L}\displaystyle\int_C^{C+2L} f(x)\sin\frac{n\pi x}{L}dx$

Note :

i. If $C = -L$, then interval is $-L < x < L$ and

$$a_0 = \frac{1}{L}\int_{-L}^{L} f(x)dx$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}dx$$

ii. If $f(x)$ is an odd function then,

$a_n = a_0 = 0$.

$$b_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}dx$$

iii. If $f(x)$ is an even function then,

$b_n = 0$ and $a_0 = \dfrac{2}{L}\displaystyle\int_0^L f(x)dx$,

$$a_n = \frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)dx,$$

Questions-Answers

Long Answer Type and Medium Answer Type Questions

Que 3.11. Find the Fourier series expansion of the function

$$f(x) = \begin{cases} -1, & \text{for } -\pi < x < -\pi/2 \\ 0, & \text{for } -\pi/2 < x < \pi/2 \\ 1, & \text{for } \pi/2 < x < \pi \end{cases}$$

Hence deduce that $\dfrac{\pi}{4} = 1 - \dfrac{1}{3} + \dfrac{1}{5} - \dfrac{1}{7} + \ldots$

AKTU 2011-12, Marks 10

Answer

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{1}{\pi}\left[\int_{-\pi}^{-\pi/2}(-1)dx + \int_{\pi/2}^{\pi}(1)dx\right]$$

$$= \frac{1}{\pi}\left[-\left(-\frac{\pi}{2}+\pi\right)+\left(\pi-\frac{\pi}{2}\right)\right]$$

$$a_0 = 0$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\, dx$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{-\pi/2} -\cos nx\, dx + \int_{\pi/2}^{\pi} \cos nx\, dx\right]$$

$$= \frac{1}{\pi}\left[-\left\{\frac{\sin nx}{n}\right\}_{-\pi}^{-\pi/2} + \left\{\frac{\sin nx}{n}\right\}_{\pi/2}^{\pi}\right]$$

$$= \frac{1}{\pi}\left[\frac{\sin n\pi/2}{n} - \frac{\sin n\pi}{n} + \frac{\sin n\pi}{n} - \frac{\sin n\pi/2}{n}\right]$$

$$a_n = 0$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\, dx$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{-\pi/2} -\sin nx\, dx + \int_{\pi/2}^{\pi} \sin nx\, dx\right]$$

$$= \frac{1}{\pi}\left[\left\{\frac{\cos nx}{n}\right\}_{-\pi}^{-\pi/2} - \left\{\frac{\cos nx}{n}\right\}_{\frac{\pi}{2}}^{\pi}\right]$$

$$= \frac{1}{\pi}\left[\frac{\cos n\pi/2}{n} - \frac{\cos n\pi}{n} - \frac{\cos n\pi}{n} + \frac{\cos n\pi/2}{n}\right]$$

$$b_n = \frac{2}{\pi n}\left[\cos\frac{n\pi}{2} - \cos n\pi\right]$$

Hence required series is,

$$f(x) = \sum_{n=1}^{\infty} \frac{2}{n\pi}\left(\cos\frac{n\pi}{2} - \cos n\pi\right)\sin nx$$

Putting $x = \pi/2$ in the above series,

$$[f(x)]_{x=\pi/2} = \sum \frac{2}{\pi n}\left(\cos\frac{n\pi}{2} - \cos n\pi\right)\sin\frac{n\pi}{2}$$

$$\frac{0+1}{2} = \frac{2}{\pi}\sum\frac{1}{n}\left(\cos\frac{n\pi}{2} - \cos n\pi\right)\sin\frac{n\pi}{2}$$

Putting $n = 1, 2, 3, 4, \ldots\ldots$

$$\frac{\pi}{4} = \frac{1}{1}\left(\cos\frac{\pi}{2} - \cos\pi\right)\sin\frac{\pi}{2} + 0$$

$$+\frac{1}{3}\left(\cos\frac{3\pi}{2} - \cos 3\pi\right)\sin\frac{3\pi}{2} + 0$$

$$+\frac{1}{5}\left(\cos\frac{5\pi}{2} - \cos 5\pi\right)\sin\frac{5\pi}{2} + 0 + \ldots$$

$$\frac{\pi}{4} = 1 + \frac{1}{3}(-1) + \frac{1}{5}(1) + \frac{1}{7}(-1) + \ldots$$

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots$$

Que 3.12. Find the Fourier series to represent the function $f(x)$ given by

$$f(x) = \begin{cases} \pi x & ; \ 0 \le x \le 1 \\ \pi(2-x) & ; \ 1 \le x \le 2 \end{cases}$$

AKTU 2013-14, Marks 10

Answer

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\pi x + \sum_{n=1}^{\infty} b_n \sin n\pi x$$

Then $\quad a_0 = \int_0^2 f(x)dx = \int_0^1 \pi x\, dx + \int_1^2 \pi(2-x)\, dx$

$$a_0 = \pi\left[\frac{x^2}{2}\right]_0^1 + \pi\left[2x - \frac{x^2}{2}\right]_1^2 = \pi\left(\frac{1}{2}\right) + \pi\left[(4-2) - \left(2 - \frac{1}{2}\right)\right]$$

$$a_0 = \pi$$

$$a_n = \int_0^2 f(x)\cos n\pi x\, dx$$

$$= \int_0^1 \pi x \cos n\pi x\, dx + \int_1^2 \pi(2-x)\cos n\pi x\, dx$$

$$a_n = \left[\pi x \frac{\sin n\pi x}{n\pi} - \pi\left(-\frac{\cos n\pi x}{n^2\pi^2}\right)\right]_0^1 + \left[\pi(2-x)\frac{\sin n\pi x}{n\pi} - (-\pi)\left(-\frac{\cos n\pi x}{n^2\pi^2}\right)\right]_1^2$$

$$= \left[\frac{\cos n\pi}{n^2\pi} \frac{-1}{n^2\pi}\right] + \left[\frac{-\cos 2n\pi}{n^2\pi} + \frac{\cos n\pi}{n^2\pi}\right]$$

$$= \frac{2}{n^2\pi}[\cos n\pi - 1] = \frac{2}{n^2\pi}[(-1)^n - 1]$$

$$= \begin{cases} 0, & \text{if } n \text{ is even} \\ -\dfrac{4}{\pi n^2}, & \text{if } n \text{ is odd} \end{cases}$$

$$b_n = \int_0^2 f(x) \sin n\pi x \, dx$$

$$= \int_0^1 \pi x \sin n\pi x \, dx + \int_1^2 \pi(2-x) \sin n\pi x \, dx$$

$$= \left[\pi x\left(-\frac{\cos n\pi x}{n\pi}\right) - \pi\left(\frac{-\sin n\pi x}{n^2\pi^2}\right)\right]_0^1 + \left[\pi(2-x)\left(\frac{-\cos n\pi x}{n\pi}\right) - (-\pi)\left(\frac{-\sin n\pi x}{n^2\pi^2}\right)\right]_1^2$$

$$= \left[-\frac{\cos n\pi}{n}\right] + \left[\frac{\cos n\pi}{n}\right] = 0$$

$$f(x) = \frac{\pi}{2} - \frac{4}{\pi}\left(\frac{\cos \pi x}{1^2} + \frac{\cos 3\pi x}{3^2} + \frac{\cos 5\pi x}{5^2} + \dots\dots\dots\right)$$

Que 3.13. **Express $f(x) = |x|$; $-\pi < x < \pi$ as Fourier series.**

AKTU 2013-14, Marks 10

Answer

Since $f(-x) = |-x| = |x| = f(x)$

$\therefore$ $f(x)$ is an even function and hence $b_n = 0$

Let $f(x) = |x| = \dfrac{a_0}{2} + \displaystyle\sum_{n=1}^{\infty} a_n \cos nx$

Where $a_0 = \dfrac{2}{\pi}\displaystyle\int_0^{\pi} f(x)\,dx = \dfrac{2}{\pi}\int_0^{\pi}|x|\,dx = \dfrac{2}{\pi}\int_0^{\pi} x\,dx = \dfrac{2}{\pi}\left[\dfrac{x^2}{2}\right]_0^{\pi} = \pi$

$$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx \, dx = \frac{2}{\pi}\int_0^{\pi}|x|\cos nx \, dx$$

$$a_n = \frac{2}{\pi}\int_0^{\pi} x \cos nx \, dx$$

$$= \frac{2}{\pi}\left[x\left(\frac{\sin nx}{n}\right) - 1\left(\frac{-\cos nx}{n^2}\right)\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\frac{\cos n\pi}{n^2} - \frac{1}{n^2}\right] = \frac{2}{\pi n^2}[(-1)^n - 1]$$

$$a_n = \begin{cases} 0, & \text{if } n \text{ is even} \\ \dfrac{-4}{\pi n^2}, & \text{if } n \text{ is odd} \end{cases}$$

$$|x| = \frac{\pi}{2} - \frac{4}{\pi}\left(\cos x + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \dots\dots\dots\right)$$

Que 3.14. **Expand $f(x) = x \sin x$ as a Fourier series in $0 < x < 2\pi$.**

AKTU 2014-15, Marks 10

Answer

$$f(x) = x \sin x \quad ; \quad 0 < x < 2\pi$$

$$a_0 = \frac{1}{\pi}\int_0^{2\pi} x \sin x \, dx = \frac{1}{\pi}[x(-\cos x) + \sin x]_0^{2\pi} = \frac{1}{\pi}[-2\pi]$$

$$a_0 = -2$$

$$a_n = \frac{1}{\pi}\int_0^{2\pi} x \sin x \cos nx \, dx$$

$$= \frac{1}{2\pi}\int_0^{2\pi} x[\sin(1+n)x + \sin(1-n)x]\,dx$$

$$= \frac{1}{2\pi}\left[-x\frac{\cos(n+1)x}{n+1} + \frac{\sin(n+1)x}{(n+1)^2} + \frac{x\cos(n-1)x}{n-1} - \frac{\sin(n-1)x}{(n-1)^2}\right]_0^{2\pi}$$

$$= \frac{1}{2\pi}\left[\frac{-2\pi}{n+1} + \frac{2\pi}{n-1}\right] = \frac{1}{n-1} - \frac{1}{n+1}$$

$$a_n = \frac{2}{n^2-1}, n \neq 1$$

When $n = 1$, we have

$$a_1 = \frac{1}{\pi}\int_0^{2\pi} x \sin x \cos x \, dx = \frac{1}{2\pi}\int_0^{2\pi} x \sin 2x \, dx$$

$$= \frac{1}{2\pi}\left[x\left\{\frac{-\cos 2x}{2}\right\} + \frac{\sin 2x}{4}\right]_0^{2\pi} = \frac{1}{2\pi}\left[\frac{-2\pi}{2}\right]$$

$$a_1 = -\frac{1}{2}$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} x \sin x \sin nx \, dx$$

$$= \frac{1}{2\pi}\int_0^{2\pi} x[\cos(1-n)x - \cos(1+n)x]dx$$

$$= \frac{1}{2\pi}\left[x\frac{\sin(n-1)x}{n-1} + \frac{\cos(n-1)x}{(n-1)^2} - \frac{x\sin(n+1)x}{(n+1)} - \frac{\cos(n+1)x}{(n+1)^2}\right]_0^{2\pi}$$

$$= \frac{1}{2\pi}\left[\frac{1}{(n-1)^2} - \frac{1}{(n+1)^2} - \frac{1}{(n-1)^2} + \frac{1}{(n+1)^2}\right]$$

$$b_n = 0$$

When $n = 1$, we have

$$b_1 = \frac{1}{\pi}\int_0^{2\pi} x \sin x \sin x \, dx = \frac{1}{2\pi}\int_0^{2\pi} x(1-\cos 2x)\,dx$$

$$= \frac{1}{2\pi}\left[x\left(x - \frac{\sin 2x}{2}\right) - 1.\left(\frac{x^2}{2} + \frac{\cos 2x}{4}\right)\right]_0^{2\pi}$$

$$= \frac{1}{2\pi}\left[2\pi(2\pi) - \frac{4\pi^2}{2} - \frac{1}{4} + \frac{1}{4}\right] = \frac{1}{2\pi}(2\pi^2) = \pi$$

$$\therefore \quad f(x) = \frac{a_0}{2} + a_1\cos x + b_1 \sin x + \sum_{n=2}^{\infty} a_n \cos nx + \sum_{n=2}^{\infty} b_n \sin nx$$

$$= -1 - \frac{1}{2}\cos x + \pi \sin x + \sum_{n=2}^{\infty}\frac{2}{n^2-1}\cos nx$$

Que 3.15. **Find the Fourier series to represent the function $f(x)$ given by**

$$f(x) = \begin{cases} -K & \text{for } -\pi < x < 0 \\ K & \text{for } \ 0 < x < \pi \end{cases}$$

Hence show that $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots = \frac{\pi}{4}$. **AKTU 2015-16, Marks 10**

Answer

$$f(x) = \begin{cases} -K & \text{for } -\pi < x < 0 \\ K & \text{for } \ 0 < x < \pi \end{cases}$$

$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{1}{\pi}\int_{-\pi}^{0}(-K)dx + \frac{1}{\pi}\int_0^{\pi} K\,dx$$

$$a_0 = 0$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{0} -K\cos nx\,dx + \frac{1}{\pi}\int_0^{\pi} K\cos nx\,dx$$

$$a_n = -\frac{K}{\pi}\left[\frac{\sin nx}{n}\right]_{-\pi}^{0} + \frac{K}{\pi}\left[\frac{\sin nx}{n}\right]_0^{\pi}$$

$$a_n = 0$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$$

$$= \frac{1}{\pi}\int_{-\pi}^{0}(-K\sin nx)dx + \frac{1}{\pi}\int_0^{\pi} K\sin nx\,dx$$

$$= \frac{1}{\pi}\left[-K\left(\frac{-\cos nx}{n}\right)\right]_{-\pi}^{0} + \frac{1}{\pi}\left[-K\left(\frac{-\cos nx}{n}\right)\right]_{-\pi}^{0}$$

$$= \frac{K}{\pi}\left[\frac{1}{n} - \frac{(-1)^n}{n} - \frac{(-1)^n}{n} + \frac{1}{n}\right]$$

$$b_n = \frac{K}{\pi}\left[\frac{2}{n} - \frac{2(-1)^n}{n}\right]$$

$$= \begin{cases} 0, & \text{if } n \text{ is even} \\ \dfrac{4K}{n\pi}, & \text{if } n \text{ is odd} \end{cases}$$

$$f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

$$f(x) = \sum_{n=1}^{\infty} b_n \sin nx$$

$$= b_1 \sin x + b_2 \sin 2x + b_3 \sin 3x + \ldots.$$

$$f(x) = \frac{4K}{\pi}\left[\sin x + \frac{1}{3}\sin 3x + \frac{1}{5}\sin 5x + \ldots.\right]$$

Now putting $x = \frac{\pi}{2}$

$$f\left(\frac{\pi}{2}\right) = K = \frac{4K}{\pi}\left[1 + \frac{1}{3}(-1) + \frac{1}{5}(1) + \frac{1}{7}(-1) + \ldots..\right]$$

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots..$$

Que 3.16. Find the Fourier series expansion of the following function of period 2π, defined as

$$f(x) = \begin{cases} -1, & -\pi < x < 0 \\ 1, & 0 \le x < \pi \end{cases}$$

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots$$

AKTU 2012-13, Marks 10

Answer

Same as Q. 3.15, Page 3–15F, Unit-3, (Putting $K = 1$).

Que 3.17. Find the Fourier series of

$$f(x) = x^3 \text{ in } (-\pi, \pi)$$

AKTU 2015-16, Marks 05

Answer

$f(x) = x^3$ is an odd function.

$\therefore$ $a_0 = 0$ and $a_n = 0$

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x)\sin nx\,dx = \frac{2}{\pi}\int_0^{\pi} x^3 \sin nx\,dx$$

$$\left[\int uv = uv_1 - u'v_2 + u''v_3 - u'''v_4 + \ldots\right]$$

$$= \frac{2}{\pi}\left[x^3\left(\frac{-\cos nx}{n}\right) - 3x^2\left(-\frac{\sin nx}{n^2}\right) + 6x\left(\frac{\cos nx}{n^3}\right) - 6\left(\frac{\sin nx}{n^4}\right)\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[-\frac{\pi^3\cos n\pi}{n} + \frac{6\pi\cos n\pi}{n^3}\right] = 2(-1)^n\left[-\frac{\pi^2}{n} + \frac{6}{n^3}\right]$$

$$\therefore \quad f(x) = x^3 = 2\left[-\left(-\frac{\pi^2}{1} + \frac{6}{1^3}\right)\sin x + \left(-\frac{\pi^2}{2} + \frac{6}{2^3}\right)\sin 2x - \left(-\frac{\pi^2}{3} + \frac{6}{3^3}\right)\sin 3x \ldots\right]$$

Que 3.18. Obtain Fourier series for the function

$$f(x) = \begin{cases} x, & -\pi < x < 0 \\ -x, & 0 < x < \pi \end{cases}$$ and hence show that $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots$

AKTU 2017-18, Marks

Answer

Let the Fourier series be

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx \quad \ldots(3.18.1)$$

Now, $a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{1}{\pi}\left\{\int_{-\pi}^{0} x\,dx + \int_0^{\pi} -x\,dx\right\}$

$$= \frac{1}{\pi}\left\{\left[\frac{x^2}{2}\right]_{-\pi}^{0} - \left[\frac{x^2}{2}\right]_0^{\pi}\right\} = \frac{1}{\pi}\left\{0 - \frac{\pi^2}{2} - \frac{\pi^2}{2}\right\} = -\pi$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\,dx$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} x\cos nx\,dx + \int_0^{\pi} -x\cos nx\,dx\right]$$

$$= \frac{1}{\pi}\left\{\left[\frac{x\sin nx}{n}\right]_{-\pi}^{0} - \int_{-\pi}^{0} 1\frac{\sin nx}{n}dx + \left[\frac{-x\sin nx}{n}\right]_0^{\pi} - \int_0^{\pi}(-1)\frac{\sin nx}{n}dx\right\}$$

$$= \frac{1}{\pi}\left\{\frac{1}{n^2}(\cos nx)_{-\pi}^{0} - \frac{1}{n^2}(\cos nx)_0^{\pi}\right\}$$

$$= \frac{1}{\pi}\left[\left\{\frac{1-(-1)^n}{n^2}\right\} - \left\{\frac{(-1)^n - 1}{n^2}\right\}\right] = \frac{1}{\pi}\left[\frac{2(1-(-1)^n)}{n^2}\right]$$

$$= \frac{2}{\pi n^2}\{1-(-1)^n\}$$

$$= \begin{cases} 0, & \text{if } n \text{ is even.} \\ \dfrac{4}{\pi n^2}, & \text{if } n \text{ is odd.} \end{cases}$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin nx\,dx$$

$$= \frac{1}{\pi}\left[\int_{-\pi}^{0} x\sin nx\,dx + \int_0^{\pi}(-x)\sin nx\,dx\right]$$

$$= \frac{1}{\pi}\left[\left\{x\left(\frac{-\cos nx}{n}\right)\right\}_{-\pi}^{0} - \int_{-\pi}^{0} 1\left(\frac{-\cos nx}{n}\right)dx + \left\{(-x)\left(\frac{-\cos nx}{n}\right)\right\}_0^{\pi} - \int_0^{\pi}(-1)\left(\frac{-\cos nx}{n}\right)dx\right]$$

$$= \frac{1}{\pi}\left[\left\{0 - \frac{\pi}{n}\cos n\pi\right\} + \frac{1}{n}\left[\frac{\sin nx}{n}\right]_{-\pi}^{0} + \left\{\frac{\pi(-1)^n}{n} - 0\right\} - \frac{1}{n}\left[\frac{\sin nx}{n}\right]_0^{\pi}\right]$$

$$= \frac{1}{\pi}\left[-\frac{\pi}{n}(-1)^n + \frac{1}{n}\pi(-1)^n\right]$$

= 0, whatever be the value of n.

Therefore, the Fourier series is

$$f(x) = \frac{-\pi}{2} + \frac{4}{\pi}\left[\frac{\cos x}{1^2} + \frac{\cos 3x}{3^2} + \frac{\cos 5x}{5^2} + \ldots\right] \quad \ldots(3.18.2)$$

Since the function $f(x)$ is discontinuous at $x = 0$, by Dirichlet's condition

$$f(0) = \frac{1}{2}[\text{LHL} + \text{RHL}] = (1/2)[f(0-0) + f(0+0)] = 0$$

Put $x = 0$ in eq. (3.18.2), we get

$$0 = \frac{-\pi}{2} + \frac{4}{\pi}\left[\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots\right]$$

$$\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \ldots\ldots = \frac{\pi^2}{8}$$

PART-4

Half Range Fourier Sine and Cosine Series.

CONCEPT OUTLINE

Half Range Series : Half series is found when a periodic function is expanded in half range of its period *i.e.*, to expand $f(x)$ in range $(0, L)$ having a period of $2L$.

A function $f(x)$ defined in the interval $(0, L)$ has two half range series that are called Fourier cosine and Fourier sine series.

Half Range Cosine Series : The half range cosine series is given as

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L}$$

Where
$$a_0 = \frac{2}{L}\int_0^L f(x)\,dx$$

$$a_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}\,dx$$

Half Range Sine Series : The half range sine series is given as

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}$$

Where,
$$b_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L}\,dx$$

Questions-Answers

Long Answer Type and Medium Answer Type Questions

Que 3.19. **Expand $f(x) = x$ as a half range**

i. **Sine series in $0 < x < 2$**

ii. **Cosine series in $0 < x < 2$.**

Answer

i. Let
$$x = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{2} \quad \ldots(3.19.1)$$

Where,
$$b_n = \int_0^2 x\sin\frac{n\pi x}{2}\,dx$$

$$= \left\{x\left(\frac{-\cos\frac{n\pi x}{2}}{\frac{n\pi}{2}}\right)\right\}_0^2 - \int_0^2\left(\frac{-\cos\frac{n\pi x}{2}}{\frac{n\pi}{2}}\right)dx$$

$$= -\frac{4}{n\pi}\cos n\pi = -\frac{4}{n\pi}(-1)^n$$

Hence from eq. (3.19.1),

$$x = -\frac{4}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\sin\frac{n\pi x}{2}$$

ii. Let
$$x = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos\frac{n\pi x}{2} \quad \ldots(3.19.2)$$

Where,
$$a_0 = \int_0^2 x\,dx = \left(\frac{x^2}{2}\right)_0^2 = 2$$

and
$$a_n = \int_0^2 x\cos\frac{n\pi x}{2}\,dx = \left(x\,\frac{\sin\frac{n\pi x}{2}}{\frac{n\pi}{2}}\right)_0^2 - \int_0^2 \frac{\sin\frac{n\pi x}{2}}{\frac{n\pi}{2}}\,dx$$

$$= -\frac{2}{n\pi}\left(\frac{-\cos\frac{n\pi x}{2}}{\frac{n\pi}{2}}\right) = \frac{4}{n^2\pi^2}(\cos n\pi - 1)$$

Hence,
$$x = 1 + \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{(\cos n\pi - 1)}{n^2}\cos\frac{n\pi x}{2}$$

Que 3.20. Find the half range cosine series expansion of

$$f(x) = x - x^2, \quad 0 < x < 1$$

AKTU 2011-12, 2012-13; Marks 05

Answer

$$f(x) = x - x^2, \quad 0 < x < 1$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{1}$$

$$a_0 = 2\int_0^1 f(x)dx = 2\int_0^1 (x - x^2)dx$$

$$= 2\left[\frac{x^2}{2} - \frac{x^3}{3}\right]_0^1 = 2\left[\frac{1}{2} - \frac{1}{3}\right] = \frac{2}{6} = \frac{1}{3}$$

$$a_n = \frac{2}{1}\int_0^1 f(x)\cos\frac{n\pi x}{1} dx = 2\int_0^1 (x - x^2)\cos n\pi x \, dx$$

$$= 2\left[(x - x^2)\left(\frac{\sin n\pi x}{n\pi}\right) - (1 - 2x)\left(\frac{-\cos n\pi x}{n^2\pi^2}\right) + (-2)\left(-\frac{\sin n\pi x}{n^3\pi^3}\right)\right]_0^1$$

$$= 2\left[(-1)\frac{\cos n\pi}{n^2\pi^2} - \frac{1}{n^2\pi^2}\right] = 2\left[\frac{(-1)^{n+1}}{n^2\pi^2} - \frac{1}{n^2\pi^2}\right]$$

$$f(x) = \frac{1}{6} + \sum_{n=1}^{\infty} \frac{2}{n^2\pi^2}[(-1)^{n+1} - 1]\cos n\pi x$$

Que 3.21. Find the Fourier half range sine series for

$$f(x) = (x + 1) \text{ for } 0 < x < \pi.$$

AKTU 2013-14, Marks 05

Answer

$$f(x) = x + 1$$

$$x + 1 = \sum_{n=1}^{\infty} b_n \sin nx$$

Where,

$$b_n = \frac{2}{\pi}\int_0^{\pi} (x + 1)\sin nx \, dx$$

$$= \frac{2}{\pi}\left[\left[(x + 1)\left(\frac{-\cos nx}{n}\right)\right]_0^{\pi} - \int_0^{\pi} (1)\left(-\frac{\cos nx}{n}\right) dx\right]$$

$$= \frac{2}{\pi}\left[(\pi + 1)\left(\frac{-\cos n\pi}{n}\right) + \frac{\cos 0^\circ}{n}\right] + \left[\frac{\sin nx}{n^2}\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\left[\frac{(\pi + 1)\left(-(-1)^n\right) + 1}{n}\right] + \left[\frac{\sin n\pi - \sin 0^\circ}{n^2}\right]\right]$$

$$= \frac{2}{\pi n}\,[1 - (1 + \pi)(-1)^n]$$

$$= \begin{cases} -\dfrac{2}{n}; \text{ If } n \text{ is even} \\ \dfrac{2}{\pi n}(2 + \pi); \text{ If } n \text{ is odd} \end{cases}$$

Hence Fourier sine series is

$$\therefore \quad f(x) = x + 1 = \frac{2(2 + \pi)}{\pi}\left[\sin x + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \ldots\ldots\right]$$

$$- 2\left[\frac{\sin 2x}{2} + \frac{\sin 4x}{4} + \frac{\sin 6x}{6} + \ldots\right]$$

Que 3.22. Find the half range sine expansion of

$$f(t) = \begin{cases} t & ; \ 0 < t < 2 \\ 4 - t & ; \ 2 < t < 4 \end{cases}$$

AKTU 2014-15, Marks 05

Answer

$$b_n = \frac{1}{2}\int_0^4 f(t)\sin\frac{n\pi t}{4} dt$$

$$= \frac{1}{2}\left[\int_0^2 t\sin\left(\frac{n\pi t}{4}\right)dt + \int_2^4 (4 - t)\sin\left(\frac{n\pi t}{4}\right)dt\right]$$

$$= \frac{1}{2}\left[\left\{t\left(-\frac{4}{n\pi}\cos\frac{n\pi t}{4}\right) + \frac{16}{n^2\pi^2}\sin\left(\frac{n\pi t}{4}\right)\right\}_0^2\right.$$

$$\left. + \left\{(4 - t)\left(-\frac{4}{n\pi}\cos\frac{n\pi t}{4}\right) - \frac{16}{n^2\pi^2}\sin\frac{n\pi t}{4}\right\}_2^4\right]$$

$$= \frac{1}{2}\left[-\frac{8}{n\pi}\cos\frac{n\pi}{2} + \frac{16}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right) + \frac{8}{n\pi}\cos\left(\frac{n\pi}{2}\right) + \frac{16}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right)\right]$$

$$= \frac{16}{n^2\pi^2}\sin\left(\frac{n\pi}{2}\right)$$

Hence the Fourier series is,

$$f(x) = \sum_{n=0}^{\infty} \frac{16}{n^2\pi^2} \sin\left(\frac{n\pi}{2}\right) \sin\left(\frac{n\pi t}{4}\right)$$

Que 3.23. **Obtain the Fourier expansion of $f(x) = x \sin x$ as cosine series in $(0, \pi)$ and hence show that**

$$\frac{1}{1\times 3} - \frac{1}{3\times 5} + \frac{1}{5\times 7} - \text{..........} = \left(\frac{\pi - 2}{4}\right)$$

AKTU 2016-17, Marks 07

Answer

Let the Fourier series be

$$f(x) = x\sin x = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

Now,
$$a_0 = \frac{1}{\pi}\int_{-\pi}^{\pi} x\sin x\, dx = \frac{2}{\pi}\int_0^{\pi} x\sin x\, dx$$

[$\because$ $x \sin x$ is an even function]

Using $\int uv\,dx = uv_1 - u'v_2 + ...$, we have

$$= \frac{2}{\pi}[x(-\cos x) + (\sin x)]_0^{\pi}$$

$$= \frac{2}{\pi}(-\pi\cos\pi) = 2$$

And,
$$a_n = \frac{2}{\pi}\int_0^{\pi} x\sin x \cos nx\, dx$$

$$= \frac{1}{\pi}\int_0^{\pi} x(2\cos nx \sin x)\, dx$$

$$= \frac{1}{\pi}\int_0^{\pi} x[\sin(n+1)x - \sin(n-1)x]dx$$

[$\because$ $2\cos A \sin B = \sin(A+B) - \sin(A-B)$]

$$= \frac{1}{\pi}\left[x\left\{\frac{-\cos(n+1)x}{n+1} + \frac{\cos(n-1)x}{n-1}\right\} - \left\{\frac{-\sin(n+1)x}{(n+1)^2} + \frac{\sin(n-1)x}{(n-1)^2}\right\}\right]_0^{\pi}$$

$$= \frac{1}{\pi}\left[\pi\left\{\frac{-\cos(n+1)\pi}{n+1} + \frac{\cos(n-1)\pi}{n-1}\right\} - \left\{\frac{-\sin(n+1)\pi}{(n+1)^2} + \frac{\sin(n-1)\pi}{(n-1)^2}\right\} - 0\right]$$

$$= \frac{1}{\pi}\left[\pi\left\{\frac{-\cos(n+1)\pi}{n+1} + \frac{\cos(n-1)\pi}{n-1}\right\} - \{0 - 0\}\right]$$

$$= \frac{1}{\pi}\left[\pi\left\{\frac{-\cos(n+1)\pi}{n+1} + \frac{\cos(n-1)\pi}{n-1}\right\}\right], n \neq 1$$

When n is odd, $n \neq 1$, $(n-1)$ and $(n+1)$ are even.

$$\therefore \quad a_n = \frac{1}{\pi}\left[\pi\left\{-\frac{1}{n+1} + \frac{1}{n-1}\right\}\right] = \frac{1}{n-1} - \frac{1}{n+1} = \frac{2}{n^2 - 1}$$

When n is even, $(n-1)$ and $(n+1)$ are odd, therefore $\cos(n+1)\pi$ and $\cos(n-1)\pi$ are -1.

$$\therefore \quad a_n = -\frac{1}{n-1} + \frac{1}{n+1} = \frac{-2}{n^2-1}$$

When $n = 1$,
$$a_1 = \frac{2}{\pi}\int_0^{\pi} x\sin x \cos x\, dx = \frac{1}{\pi}\int_0^{\pi} x\sin 2x\, dx$$

$$= \frac{1}{\pi}\left[x\left(\frac{-\cos 2x}{2}\right) - 1.\left(\frac{-\sin 2x}{4}\right)\right]_0^{\pi}$$

$$= \frac{1}{\pi}\left\{\frac{-\pi\cos 2\pi}{2}\right\} = -\frac{1}{2}$$

Now the Fourier series is,

$$f(x) = x\sin x = 1 - \frac{1}{2}\cos x - 2\left\{\frac{\cos 2x}{2^2-1} - \frac{\cos 3x}{3^2-1} + \frac{\cos 4x}{4^2-1} - \frac{\cos 5x}{5^2-1}.....\right\} \quad ...(3.23.1)$$

Putting $x = \frac{\pi}{2}$ in eq. (3.23.1), we get

$$\frac{\pi}{2}\sin\frac{\pi}{2} = 1 - 2\left(\frac{-1}{2^2-1} + \frac{1}{4^2-1} - \frac{1}{6^2-1} + ...\right)$$

$$\frac{\pi}{2} - 1 = 2\left(\frac{1}{3} - \frac{1}{15} + \frac{1}{35} - ...\right)$$

$$\frac{\pi - 2}{4} = \frac{1}{1.3} - \frac{1}{3.5} + \frac{1}{5.7}.....$$

Que 3.24. **Obtain half range cosine series for e^x the function**

$$f(t) = \begin{cases} 2t & , 0 < t < 1 \\ 2(2-t) & , 1 < t < 2 \end{cases}$$

AKTU 2017-18, Marks 07

Answer

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi t}{l}$$

$$a_0 = \frac{2}{l}\int_0^l f(t)\, dt = \frac{2}{2}\left[\int_0^1 2t\, dt + \int_1^2 2(2-t)dt\right]$$

$$= \left[\left(\frac{2t^2}{2}\right)_0^1 + \left(4t - t^2\right)_1^2\right]$$

$$a_0 = [1 + 1] = 2$$

$$a_n = \frac{2}{2}\int_0^l f(t) \cos \frac{n\pi t}{l}\, dt$$

$$= \frac{2}{2}\left[\int_0^1 2t \cos \frac{n\pi t}{2}\, dt + \int_1^2 2(2-t) \cos \frac{n\pi t}{2}\, dt\right]$$

Using integration by parts

$$= \frac{2}{2}\left[\left(2t \frac{2}{n\pi} \sin \frac{n\pi t}{2} + 2\frac{2^2}{n^2\pi^2} \cos \frac{n\pi t}{2}\right)_0^1 + \left(2(2-t)\frac{2}{n\pi} \sin \frac{n\pi t}{2} - 2\frac{2^2}{n^2\pi^2} \cos \frac{n\pi t}{2}\right)_1^2\right]$$

$$= \left[\left(\frac{4}{n\pi} \sin \frac{n\pi}{2} + \frac{8}{n^2\pi^2}\left(\cos \frac{n\pi}{2} - 1\right)\right) + \left(-\frac{8}{n^2\pi^2} \cos n\pi - \frac{4}{n\pi} \sin \frac{n\pi}{2} + \frac{8}{n^2\pi^2} \cos \frac{n\pi}{2}\right)\right]$$

$$= \frac{16}{n^2\pi^2} \cos \frac{n\pi}{2} - \frac{8}{n^2\pi^2} - \frac{8}{n^2\pi^2} \cos n\pi$$

$$= \frac{8}{n^2\pi^2}\left[2 \cos \frac{n\pi}{2} - 1 - \cos n\pi\right]$$

When n is odd, $\cos \frac{n\pi}{2} = 0$ and $\cos n\pi = -1$

$\therefore$ $a_n = 0 \Rightarrow a_1 = a_3 = a_5 \text{.....} = 0$

When n is even, $a_2 = \frac{8}{2^2\pi^2}\left[2 \cos \frac{2\pi}{2} - 1 - \cos 2\pi\right] = -\frac{8}{\pi^2}$

$$a_4 = \frac{8}{4^2\pi^2}\left[2 \cos \frac{4\pi}{2} - 1 - \cos 4\pi\right] = 0$$

$$a_6 = \frac{8}{6^2\pi^2}\left[2 \cos \frac{6\pi}{2} - 1 - \cos 6\pi\right] = -\frac{8}{9\pi^2}$$

$$f(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \cos \frac{n\pi t}{l}$$

$$= 1 + \left(-\frac{8}{\pi^2} + 0 - \frac{8}{9\pi^2}\right) = 1 - \frac{8}{\pi^2}\left(1 + \frac{1}{9} + \text{.....}\right)$$

☺☺☺

4

UNIT

Complex Variable Differentiation

CONTENTS

PART-1

Limit, Continuity and Differentiability.

CONCEPT OUTLINE

Limit : The function $f(x, y)$ tends to the limit l as $x \to a$ and $y \to b$ if and only if the limit l is independent of the path followed by the point (x, y) as $x \to a$ and $y \to b$. Then

$$\lim_{\substack{x \to a \\ y \to b}} f(x, y) = l$$

The function $f(x, y)$ in region R tends to the limit l as $x \to a$ and $y \to b$ if and only if corresponding to a positive number $\in (a, b)$, there exists another positive number δ such that

$$|f(x, y) - l| < \in \text{ for } 0 < (x-a)^2 + (y-b)^2 < \delta^2$$

for every point (x, y) in R.

Continuity : A function $f(x, y)$ is said to be continuous at the point (a, b) if $\lim\limits_{\substack{x \to a \\ y \to b}} f(x, y) = f(a, b)$ irrespective of the path along with $x \to a$, $y \to b$.

Questions-Answers

Long Answer Type and Medium Answer Type Questions

Que 4.1. Evaluate $\lim\limits_{\substack{x \to 1 \\ y \to 2}} \dfrac{3x^2 y}{x^2 + y^2 + 5}$.

Answer

$$\lim_{\substack{x \to 1 \\ y \to 2}} \frac{3x^2 y}{x^2 + y^2 + 5} = \lim_{x \to 1} \left[\lim_{y \to 2} \frac{3x^2 y}{x^2 + y^2 + 5} \right] = \lim_{x \to 1} \frac{3x^2 (2)}{x^2 + (2)^2 + 5}$$

$$= \lim_{x \to 1} \frac{6x^2}{x^2 + 9} = \frac{6}{1+9} = \frac{3}{5}$$

Que 4.2. Evaluate $\lim\limits_{\substack{x \to \infty \\ y \to 2}} \dfrac{xy + 4}{x^2 + 2y^2}$.

Answer

$$\lim_{\substack{x \to \infty \\ y \to 2}} \frac{xy + 4}{x^2 + 2y^2} = \lim_{x \to \infty} \left[\lim_{y \to 2} \frac{xy + 4}{x^2 + 2y^2} \right] = \lim_{x \to \infty} \left[\frac{x(2) + 4}{x^2 + 2(2)^2} \right] = \lim_{x \to \infty} \frac{2x + 4}{x^2 + 8}$$

$$= \lim_{x\to\infty} \frac{2+\frac{4}{x}}{x+\frac{8}{x}} = \frac{2+0}{\infty+0} = 0$$

Que 4.3. **Show that the function $f(x, y) = x - y$ is continuous for all $(x, y) \in R^2$.**

Answer

Let $(a, b) \in R^2$ then $f(a, b) = a - b$

$$\therefore \quad |f(x,y) - f(a,b)| = |(x-y)-(a-b)|$$
$$= |(x-a)+(b-y)|$$
$$\le |x-a| + |y-b| \qquad [\because |x| = |-x|] \quad ...(4.3.1)$$

Let $\varepsilon > 0$. Choose $\delta = \frac{\varepsilon}{2}$ then for $|x-a| < \delta$ and $|y-b| < \delta$, we have from eq. (4.3.1)

$$|f(x,y) - f(a,b)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

Hence, the function $f(x, y) = x - y$ is continuous for all $(a, b) \in R^2$. But (a, b) is an arbitrary element of R^2, so $f(x, y) = x - y$ is continuous for all $(x, y) \in R^2$.

Que 4.4. **If $f(x,y) = \dfrac{x^3 - y^3}{x^2 + y^2}$ when $x \ne 0, y \ne 0$ and $f(x, y) = 0$ when $x = 0, y = 0$, find out whether the function $f(x, y)$ is continuous at origin.**

Answer

First calculate the limit of the function :

I. $$\lim_{\substack{x\to 0\\ y\to 0}} \frac{x^3-y^3}{x^2+y^2} = \lim_{y\to 0}\left(\frac{-y^3}{y^2}\right) = \lim_{y\to 0}(-y) = 0$$

II. $$\lim_{\substack{y\to 0\\ x\to 0}} \frac{x^3-y^3}{x^2+y^2} = \lim_{x\to 0}\left(\frac{x^3}{x^2}\right) = \lim_{x\to 0}(x) = 0$$

III. $$\lim_{\substack{y\to mx\\ x\to 0}} \frac{x^3-y^3}{x^2+y^2} = \lim_{x\to 0} \frac{x^3 - m^3x^3}{x^2+m^2x^2} = \lim_{x\to 0} \frac{(1-m^3)}{(1+m^2)} x = 0$$

IV. $$\lim_{\substack{y\to mx^2\\ x\to 0}} \frac{x^3-y^3}{x^2+y^2} = \lim_{x\to 0} \frac{x^3 - m^3 - m^3x^6}{x^2+m^2x^4} = \lim_{x\to 0} \frac{x^3(1-m^3x^3)}{x^2(1+m^2x^2)} \lim_{x\to 0} \frac{(1-m^3x^2)}{(1+m^2x^2)} x = 0$$

Since the limit along any path is same, the limit exists and equal to zero which is the value of the function $f(x, y)$ at the origin. Hence, the function f is continuous at the origin.

PART-2

Functions of Complex Variable, Analytic Functions, Cauchy Riemann Equations (Cartesian and Polar Form).

CONCEPT OUTLINE

Cauchy-Riemann or C-R Equation :

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$

and $$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Questions-Answers

Long Answer Type and Medium Answer Type Questions

Que 4.5. **Define analytic function and state the necessary and sufficient condition for function to be analytic.**

Answer

A. **Analytic Function :** A function $f(z)$ is said to be analytic at a point z_0 if it is one valued and differentiable not only at z_0 but at every point of some neighbourhood of z_0.

B. **Necessary and Sufficient Conditions for $f(z)$ to be Analytic :** The necessary and sufficient conditions for the function

$$w = f(z) = u(x, y) + iv(x, y)$$

to be analytic in a region R, are

i. $\dfrac{\partial u}{\partial x}, \dfrac{\partial u}{\partial y}, \dfrac{\partial v}{\partial x}, \dfrac{\partial v}{\partial y}$ are continuous functions of x and y in the region R.

ii. $\dfrac{\partial u}{\partial x} = \dfrac{\partial v}{\partial y}, \dfrac{\partial u}{\partial y} = -\dfrac{\partial v}{\partial x}$

The conditions in (ii) are known as Cauchy-Riemann equations or briefl[y] C-R equations.

Que 4.6. **Define analytic function. Discuss the analyticity of $f(z) = \text{Re}(z^3)$ in the complex plane.** **AKTU 2013-14 (III), Marks 05**

Answer

A. **Analytic Function :** Refer Q. 4.5, Page 4–4F, Unit-4.

B. **Numerical :**

$$z = (x + iy)$$

$$z^3 = (x + iy)^3 = x^3 - iy^3 + 3xiy\,(x + iy)$$

$$= (x^3 - 3xy^2) + (3x^2y - y^3)i$$

$$u = x^3 - 3xy^2$$

$$\frac{\partial u}{\partial x} = 3x^2 - 3y^2, \quad \frac{\partial u}{\partial y} = -6xy$$

$$v = (3x^2y - y^3)$$

$$\frac{\partial v}{\partial x} = 6xy, \quad \frac{\partial v}{\partial y} = 3x^2 - 3y^2$$

$$\because \quad \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

$\therefore$ $f(z) = \text{Re}\,(z^3)$ is analytic function.

Que 4.7. **Show that $f(z) = \log z$ is analytic everywhere in the complex plane except at the origin.** **AKTU 2013-14 (IV), Marks 05**

Answer

Here $f(z) = u + iv = \log z = \log(x + iy)$ $\quad (\because \; z = x + iy)$

Let $x = r\cos\theta$ and $y = r\sin\theta$ so that

$x + iy = r(\cos\theta + i\sin\theta) = re^{i\theta}$

$$\log(x + iy) = \log(r\,e^{i\theta}) = \log r + i\theta = \frac{1}{2}\log(x^2 + y^2) + i\tan^{-1}\left(\frac{y}{x}\right)$$

Separating real and imaginary parts, we get

$$u = \frac{1}{2}\log(x^2 + y^2) \text{ and } v = \tan^{-1}\left(\frac{y}{x}\right)$$

Now, $$\frac{\partial u}{\partial x} = \frac{x}{x^2 + y^2}, \quad \frac{\partial u}{\partial y} = \frac{y}{x^2 + y^2}$$

And $$\frac{\partial v}{\partial x} = \frac{-y}{x^2 + y^2}, \quad \frac{\partial v}{\partial y} = \frac{x}{x^2 + y^2}$$

We observe that the Cauchy-Riemann equations

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

are satisfied except when $x^2 + y^2 = 0$ *i.e.*, when $x = 0, y = 0$

Hence, the function $f(z) = \log z$ is analytic everywhere in the complex plane except at the origin.

Que 4.8. **Find the values of c_1 and c_2 such that the function $f(z) = x^2 + c_1 y^2 - 2xy + i\,(c_2 x^2 - y^2 + 2xy)$ is analytic. Also find $f'(z)$.** **AKTU 2016-17 (III), Marks 05**

Answer

$$f(z) = x^2 + c_1 y^2 - 2xy + i\,(c_2 x^2 - y^2 + 2xy)$$

$$u + iv = x^2 + c_1 y^2 - 2xy + i\,(c_2 x^2 - y^2 + 2xy)$$

Comparing real and imaginary parts, we get

$$u = x^2 + c_1 y^2 - 2xy$$

And $$v = c_2 x^2 - y^2 + 2xy$$

$$\frac{\partial u}{\partial x} = 2x - 2y \text{ and } \frac{\partial v}{\partial x} = 2c_2 x + 2y$$

$$\frac{\partial u}{\partial y} = 2c_1 y - 2x \text{ and } \frac{\partial v}{\partial y} = -2y + 2x$$

C-R equations are

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$$

$$2x - 2y = -2y + 2x \qquad ...(4.8.1)$$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

$$2c_1 y - 2x = -2c_2 x - 2y \qquad ...(4.8.2)$$

From eq. (4.8.1) and eq. (4.8.2), equating the coefficient of x and y, we get

$$2c_1 = -2 \Rightarrow c_1 = -1$$

$$-2 = -2c_2 \Rightarrow c_2 = 1$$

Now, $$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = (2x - 2y) + i\,(2x + 2y)$$

$$= (2x - 2y) + i\,(2x + 2y)$$

$$= 2[x + ix + (-y + iy) = 2[(1 + i)\,x + i\,(1 + i)y]$$

$$= 2(1 + i)\,(x + iy) = 2(1 + i)z$$

Que 4.9. **Find p such that the function $f(z)$ expressed in polar coordinates as $f(z) = r^2\cos 2\theta + ir^2\sin p\theta$ in analytic.**

Answer

Let $f(z) = u + iv$, then $u = r^2\cos 2\theta$, $v = r^2\sin p\theta$

$$\frac{\partial u}{\partial r} = 2r\cos 2\theta, \quad \frac{\partial v}{\partial r} = 2r\sin p\theta$$

$$\frac{\partial u}{\partial \theta} = -2r^2\sin 2\theta, \quad \frac{\partial v}{\partial \theta} = pr^2\cos p\theta$$

For $f(z)$ to be analytic, $\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta}$ and $\frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}$

$\therefore$ $2r\cos 2\theta = pr\cos p\theta$ and $2r\sin p\theta = 2r\sin 2\theta$

Both these equations are satisfied if $p = 2$.

Que 4.10. **Show that the function defined by $f(z) = \sqrt{|xy|}$ is not regular at the origin, although Cauchy-Riemann equations are satisfied.**

AKTU 2016-17 (IV), Marks 05

Answer

$f(z) = u(x, y) + iv(x, y) = \sqrt{|xy|}$ then $u(x, y) = \sqrt{|xy|}$, $v(x, y) = 0$

At the origin (0, 0), we have

$$\frac{\partial u}{\partial x} = \lim_{x \to 0}\frac{u(x,0)-u(0,0)}{x} = \lim_{x\to 0}\frac{0-0}{x} = 0$$

$$\frac{\partial u}{\partial y} = \lim_{y \to 0}\frac{u(0,y)-u(0,0)}{y} = \lim_{y\to 0}\frac{0-0}{y} = 0$$

$$\frac{\partial v}{\partial x} = \lim_{x \to 0}\frac{v(x,0)-v(0,0)}{x} = \lim_{x\to 0}\frac{0-0}{x} = 0$$

$$\frac{\partial v}{\partial y} = \lim_{y \to 0}\frac{v(0,y)-v(0,0)}{y} = \lim_{y\to 0}\frac{0-0}{y} = 0$$

Clearly, $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

Hence, C-R equations are satisfied at the origin.

Now $$f'(0) = \lim_{z\to 0}\frac{f(z)-f(0)}{z} = \lim_{z\to 0}\frac{\sqrt{|xy|}-0}{x+iy}$$

If $z \to 0$ along the line $y = mx$, we get

$$f'(0) = \lim_{x\to 0}\frac{\sqrt{|mx^2|}}{x(1+im)} = \lim_{x\to 0}\frac{\sqrt{|m|}}{1+im}$$

Now this limit is not unique since it depends on m. Therefore, $f'(0)$ does not exist.

Hence, the function $f(z)$ is not regular at the origin.

Que 4.11. **Prove that the function sinh z is analytic and find its derivation.**

Answer

Here $f(z) = u + iv = \sinh z = \sinh(x + iy) = \sinh x\cos y + i\cosh x\sin y$

$$\frac{\partial u}{\partial x} = \cosh x\cos y, \quad \frac{\partial u}{\partial y} = -\sinh x\sin y$$

$$\frac{\partial v}{\partial x} = \sinh x\sin y, \quad \frac{\partial v}{\partial y} = \cosh x\cos y$$

$\therefore$ $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

Thus C-R equations are satisfied.

Since $\sinh x$, $\cosh x$, $\sin y$ and $\cos y$ are continuous function, $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}$ and $\frac{\partial v}{\partial y}$ are also continuous functions satisfying C-R equations.

Hence $f(z)$ is analytic everywhere.

Now, $$f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \cosh x\cos y + i\sinh x\sin y = \cosh(x+iy) = \cosh z.$$

Que 4.12. **Using C – R equations show that $f(z) = |z|^2$ is not analytical at any point.**

AKTU 2014-15 (IV), Marks 05

Answer

Let $w = f(z) = u + iv = |z|^2$

$u + iv = x^2 + y^2$

Comparing both sides,

$$u = x^2 + y^2, \quad \frac{\partial u}{\partial x} = 2x, \quad \frac{\partial u}{\partial y} = 2y$$

$$v = 0, \quad \frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} = 0$$

Using C-R equations,

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \Rightarrow 2x = 0 \Rightarrow x = 0$$

And $$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \Rightarrow 2y = 0 \Rightarrow y = 0$$

At (0, 0) C-R equations are satisfied and the function is differentiable. Hence, the function is not analytic anywhere except at origin.

Que 4.13. If
$$f(z) = \frac{x^3 y(y - ix)}{x^6 + y^2} \quad \text{when } z \neq 0$$
$$= 0 \quad \text{when } z = 0$$

Prove that $\frac{f(z) - f(0)}{z} \to 0$ as $z \to 0$ along any radius vector but not as $z \to 0$ in any manner.

AKTU 2012-13 (III), Marks 05

Answer

$$f(z) = u + iv = \frac{x^3 y(y - ix)}{x^6 + y^2}, z \neq 0$$

$$u = \frac{x^3 y^2}{x^6 + y^2}, v = \frac{-x^4 y}{x^6 + y^2}$$

$$\frac{\partial u}{\partial x} = \lim_{h \to 0} \frac{u(0 + h, 0) - u(0, 0)}{h} = \lim_{h \to 0} \frac{0 / h^6}{h} = 0$$

$$\frac{\partial u}{\partial y} = \lim_{k \to 0} \frac{u(0, 0 + k) - u(0, 0)}{k} = \lim_{k \to 0} \frac{0 / k^2}{k} = 0$$

Similarly, $\frac{\partial v}{\partial x} = 0$ and $\frac{\partial v}{\partial y} = 0$

Thus $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

Hence, C-R equations are satisfied at origin.

Now, $$\frac{f(z) - f(0)}{z} = \left[\frac{x^3 y(y - ix)}{x^6 + y^2} - 0\right] \frac{1}{x + iy}$$

$$= \frac{x^3 y(y - ix)}{x^6 + y^2} \frac{1}{(x + iy)} = \frac{-ix^3 y}{x^6 + y^2}$$

Let $z \to 0$ along radius vector $y = mx$, then

$$\lim_{z \to 0} \frac{f(z) - f(0)}{z} = \lim_{x \to 0} \frac{-ix^3(mx)}{x^6 + m^2 x^2} = \lim_{x \to 0} \frac{-imx^2}{x^4 + m^2} = 0$$

Hence $\frac{f(z) - f(0)}{z} \to 0$ as $z \to 0$ along any radius vector

Let $z \to 0$ along $y = x^3$ then,

$$\lim_{z \to 0} \frac{f(z) - f(0)}{z} = \lim_{x \to 0} \frac{-ix^3 x^3}{x^6 + x^6} = \frac{-i}{2}$$

Thus $f'(0)$ does not exist, hence $f(z)$ is not analytic at $z = 0$.

Que 4.14. **Examine the nature of the function**

$$f(z) = \frac{x^2 y^5 (x + iy)}{x^4 + y^{10}}; z \neq 0, f(0) = 0$$

In the region including the origin. **AKTU 2015-16 (III), Marks 10**

Answer

Same as Q. 4.13, Page 4–9F, Unit-4.

(**Answer :** $f'(0)$ does not exist. Hence, $f(z)$ is not analytic at origin).

Que 4.15. **Prove that the function $f(z)$ defined by**

$$f(z) = \begin{cases} \frac{x^3(1+i) - y^3(1-i)}{x^2 + y^2} & z \neq 0 \\ 0 & z = 0 \end{cases}$$

is continuous and the Cauchy-Riemann equations are satisfied at the origin, yet $f'(0)$ does not exist. **AKTU 2016-17 (IV), Marks 10**

Answer

$$f(z) = \frac{x^3(1+i) - y^3(1-i)}{x^2 + y^2} = u + iv$$

where, $$u = \frac{x^3 - y^3}{x^2 + y^2}, v = \frac{x^3 + y^3}{x^2 + y^2}$$

$\because$ The value of $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ at (0, 0) we get $\frac{0}{0}$, so we apply first principle method.

At the origin,

$$\frac{\partial u}{\partial x} = \lim_{h \to 0} \frac{u(0 + h, 0) - u(0,0)}{h} = \lim_{h \to 0} \left(\frac{h^3}{h^2}\right) / h = 1$$

$$\frac{\partial u}{\partial y} = \lim_{k \to 0} \frac{u(0, 0 + k) - u(0,0)}{k} = \lim_{k \to 0} \left(\frac{-k^3}{k^2}\right) / k = -1$$

$$\frac{\partial v}{\partial x} = \lim_{h \to 0} \frac{v(0 + h, 0) - v(0,0)}{h} = \lim_{h \to 0} \left(\frac{h^3}{h^2}\right) / h = 1$$

$$\frac{\partial v}{\partial y} = \lim_{k \to 0} \frac{v(0, 0 + k) - v(0, 0)}{k} = \lim_{k \to 0} \left(\frac{k^3}{k^2}\right) / k = 1$$

$$\therefore \quad \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Hence, C-R equations are satisfied at origin.

Now $f'(0) = \lim_{z \to 0} \frac{f(z)-f(0)}{z} = \lim_{z \to 0}\left[\frac{x^3 - y^3 + i(x^3 + y^3)}{x^2 + y^2} \cdot \frac{1}{x+iy}\right]$

Now let $z \to 0$ along $y = mx$, then

$$f'(0) = \lim_{x \to 0}\left[\frac{x^3 - m^3x^3 + i(x^3 + m^3x^3)}{x^2 + m^2x^2} \cdot \frac{1}{x+imx}\right]$$

$$= \lim_{x \to 0}\left[\frac{1-m^3 + i(1+m^3)}{(1+m^2)(1+im)}\right] = \frac{m^3(-1+i)+(1+i)}{(1+m^2)(1+im)}$$

$\because$ The value of $f'(0)$ depends on m, therefore $f'(0)$ is not unique.
Hence, the function is not analytic at $z = 0$.

PART-3

Harmonic Function, Method to Find Analytic Functions.

CONCEPT OUTLINE

Harmonic Function : A function of (x, y) which possesses continuous partial derivatives of the first and second orders and satisfies Laplace equation is called a harmonic function.

Questions-Answers

Long Answer Type and Medium Answer Type Questions

Que 4.16. **If $f(z)$ is a harmonic function of z, show that**

$$\left\{\frac{\partial}{\partial x}|f(z)|\right\}^2 + \left\{\frac{\partial}{\partial y}|f(z)|\right\}^2 = |f'(z)|^2$$

Answer

We have, $f(z) = u + iv$ ...(4.16.1)

$\therefore$ $|f(z)| = \sqrt{u^2 + v^2}$ ...(4.16.2)

Partially differentiating eq. (4.16.2) w.r.t x and y, we get

$$\frac{\partial}{\partial x}|f(z)| = \frac{1}{2}(u^2 + v^2)^{-1/2}\left(2u\frac{\partial u}{\partial x} + 2v\frac{\partial v}{dx}\right)$$

$$= \frac{u\frac{\partial u}{\partial x} + v\frac{\partial v}{\partial x}}{|f(z)|} \quad ...(4.16.3)$$

Similarly, $\frac{\partial}{\partial y}|f(z)| = \frac{u\frac{\partial u}{\partial y} + v\frac{\partial v}{\partial y}}{|f(z)|}$ ...(4.16.4)

Squaring and adding eq. (4.16.3) and eq. (4.16.4), we get

$$\left\{\frac{\partial}{\partial x}|f(z)|\right\}^2 + \left\{\frac{\partial}{\partial y}|f(z)|\right\}^2 = \frac{\left(u\frac{\partial u}{\partial x} + v\frac{\partial v}{\partial x}\right)^2 + \left(u\frac{\partial u}{\partial y} + v\frac{\partial v}{\partial y}\right)^2}{|f(z)|^2}$$

$$= \frac{\left(u\frac{\partial u}{\partial x} + v\frac{\partial v}{\partial x}\right)^2 + \left(-u\frac{\partial u}{\partial x} + v\frac{\partial v}{\partial x}\right)^2}{|f(z)|^2}$$

(Using C-R equation)

$$= \frac{(u^2 + v^2)\left[\left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2\right]}{|f(z)|^2}$$

$$= \left(\frac{\partial u}{\partial x}\right)^2 + \left(\frac{\partial v}{\partial x}\right)^2 \qquad (\because |f(z)|^2 = u^2 + v^2)$$

$$= |f'(z)|^2 \qquad \left(\because f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)$$

Que 4.17. **Verify that the function on $u_1(x, y) = xy$ is harmonic and find its conjugate harmonic function. Express $u + iv$ as an analytic function $f(z)$.**

$u = x^2 - y^2 - y$

AKTU 2015-16 (III), Marks 05

Answer

$$u(x, y) = xy$$

$$\frac{\partial u}{\partial x} = y \qquad \therefore \frac{\partial^2 u}{\partial x^2} = 0$$

$$\frac{\partial u}{\partial y} = x \qquad \therefore \frac{\partial^2 u}{\partial y^2} = 0$$

For a function to be harmonic, it must satisfy Laplace equation.

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

Hence, function $u(x, y)$ is harmonic.
Using Cauchy-Riemann equation,

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Total differentiation of v is given as,

$$dv = \frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy = -xdx + ydy$$

$$\therefore \quad v = \frac{-x^2}{2} + \frac{y^2}{2} + c$$

u and v are said complex conjugate.

Again, $u = x^2 - y^2 - y$

$$\frac{\partial u}{\partial x} = 2x, \quad \frac{\partial u}{\partial y} = -2y - 1$$

Using Cauchy-Riemann equation,

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \text{ and, } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

$$\because \quad dv = \frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy = (2y+1)\,dx + 2xdy = d(2xy + x)$$

$$\therefore \quad v = 2xy + x + c$$

Then, $f(x, y) = u + iv = (x^2 - y^2 - y) + i(2xy + x + c)$

Que 4.18. **Show that $v(x, y) = e^{-x}(x \cos y + y \sin y)$ is harmonic. Find its harmonic conjugate.**

AKTU 2013-14 (III), Marks 05

Answer

$$v(x, y) = e^{-x}(x \cos y + y \sin y)$$

$$\frac{\partial v}{\partial x} = -e^{-x}(x \cos y + y \sin y) + e^{-x}(\cos y)$$

$$\frac{\partial v}{\partial y} = e^{-x}(-x \sin y + y \cos y + \sin y)$$

$$\frac{\partial^2 v}{\partial x^2} = -[-e^{-x}(x \cos y + y \sin y) + e^{-x}(\cos y)] - e^{-x}(\cos y)$$

$$\frac{\partial^2 v}{\partial y^2} = e^{-x}[-x \cos y + (\cos y - y \sin y) + \cos y]$$

$$= e^{-x}[2 \cos y - y \sin y - x \cos y]$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = e^{-x}[x \cos y + y \sin y - \cos y - \cos y] + e^{-x}[2 \cos y - y \sin y + x \cos y]$$

$$= e^{-x}[x \cos y + y \sin y - 2 \cos y + 2 \cos y - y \sin y - x \cos y]$$

$$= 0$$

Since, v satisfies the Laplace equation hence v is harmonic function.

$$du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy = \left(\frac{\partial v}{\partial y}\right)dx + \left(\frac{-\partial v}{\partial x}\right)dy$$

$$du = [e^{-x}(-x \sin y + y \cos y + \sin y)]\,dx + e^{-x}[x \cos y + y \sin y - \cos y]\,dy$$

$$u = \int_{y=\text{cont}} e^{-x}(-x\sin y + y \cos y + \sin y)dx + \int_{x=\text{cont}} e^{-x}(x\cos y + y \sin y - \cos y)dy$$

$$u = -\int e^{-x} x \sin y\, dx + y \cos y \int e^{-x}dx + \sin y \int e^{-x}\, dx + xe^{-x}\int \cos y\, dy + e^{-x}\int y \sin y\, dy - e^{-x}\int \cos y\, dy$$

$$u = -(-2x\, e^{-x}) \sin y - e^{-x} y \cos y - e^{-x} \sin y + x\, e^{-x} \sin y + e^{-x}(-y \cos y - y \sin y) - e^{-x} \sin y$$

$$u = 2x\, e^{-x} \sin y - e^{-x} y \cos y - e^{-x} \sin y + x\, e^{-x} \sin y - e^{-x} y \cos y - e^{-x} y \sin y - e^{-x} \sin y$$

$$u = 3x\, e^{-x} \sin y - 2e^{-x} y \cos y - e^{-x} y \sin y - 2e^{-x} \sin y$$

Here u is the harmonic conjugate of v.

Que 4.19. **Find an analytic function whose imaginary part is $e^{-x}(x \cos y + y \sin y)$.**

AKTU 2013-14 (IV), Marks 05

Answer

Let $f(z) = u + iv$ be the required analytic function.

Here, $v = e^{-x}(x \cos y + y \sin y)$

$$\frac{\partial v}{\partial y} = e^{-x}(-x \sin y + y \cos y + \sin y) = \psi_1(x, y)$$

$$\frac{\partial v}{\partial x} = e^{-x} \cos y - e^{-x}(x \cos y + y \sin y) = \psi_2(x, y)$$

$$\therefore \quad \psi_1(z, 0) = 0, \; \psi_2(z, 0) = e^{-z} - e^{-z} z = (1 - z)\, e^{-z}$$

By Milne's Thomson method,

$$f(z) = \int [\psi_1(z,0) + i\, \psi_2(z,0)]\, dz + c = i\int (1 - z)\, e^{-z}\, dz + c$$

$$= i\left[(1 - z)(-e^{-z}) - \int (-1)(-e^{-z})dz\right] + c$$

$$= i\,[(z - 1)\, e^{-z} + e^{-z}] + c$$

$$f(z) = ize^{-z} + c$$

Que 4.20. **Find the analytic function whose real part is $e^{2x}(x \cos 2y - y \sin 2y)$.**

AKTU 2014-15 (IV), Marks 05

Answer

Let, $u = e^{2x}(x \cos 2y - y \sin 2y)$

$$\frac{\partial u}{\partial x} = e^{2x}(\cos 2y) + 2e^{2x}(x \cos 2y - y \sin 2y) = \phi_1(x, y)$$

$$\frac{\partial u}{\partial y} = e^{2x}[-2x \sin 2y - 2y \cos 2y - \sin 2y] = \phi_2(x, y)$$

On replacing x by z and y by 0,

$$f(z) = \int \phi_1(z, 0)\, dz - i \int \phi_2(z, 0)\, dz + c$$

$$= \int \left[e^{2z} \cos 0 + 2e^{2z}(z)\right] dz - \int 0\, dz + c$$

$$= \int (e^{2z} + 2ze^{2z})\, dz + c = \frac{1}{2}e^{2z} + 2\left[z\frac{e^{2z}}{2} - \frac{e^{2z}}{4}\right] + c$$

$$f(z) = z\,e^{2z} + c$$

Que 4.21. **If $u = 3x^2y - y^3$ find the analytic function $f(z) = u + iv$.**

AKTU 2012-13 (III), Marks 05

Answer

$$u = 3x^2y - y^3$$

$$\frac{\partial u}{\partial x} = 6xy, \quad \frac{\partial u}{\partial y} = 3x^2 - 3y^2$$

Now,
$$dv = \frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy = \left(-\frac{\partial u}{\partial y}\right)dx + \left(\frac{\partial u}{\partial x}\right)dy$$

$$dv = (-(3x^2 - 3y^2))dx + 6xy\, dy$$

On integrating,
$$v = \underset{(y \text{ as constant})}{\int M\, dx} + \underset{(\text{Ignoring terms of } x)}{\int N\, dy}$$

$\therefore$
$$v = \int (3y^2 - 3x^2)dx + 0 = 3xy^2 - x^3 + c$$

Now,
$$u + iv = 3x^2y - y^3 + i(3xy^2 - x^3 + c)$$
$$= [3x^2y - y^3 + i(3xy^2 - x^3)] + ic$$
$$= -i(x^3 - iy^3 - 3xy^2 + 3ix^2y) + ic$$
$$= -i(x + iy)^3 + ic = -iz^3 + ic$$
$$u + iv = -i(z^3 - c)$$

Que 4.22. **Show that $e^x \cos y$ is harmonic function, find the analytic function of which it is real part.**

Answer

Let,
$$u = e^x \cos y$$

$\therefore$
$$\frac{\partial u}{\partial x} = e^x \cos y \Rightarrow \frac{\partial^2 u}{\partial x^2} = e^x \cos y$$

$$\frac{\partial u}{\partial y} = -e^x \sin y \Rightarrow \frac{\partial^2 u}{\partial y^2} = -e^x \cos y$$

Since $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$, therefore u is a harmonic function.

Let
$$d_v = \frac{\partial u}{\partial x}dx + \frac{\partial v}{\partial y}dy$$
$$= \left(-\frac{\partial u}{\partial x}\right)dx + \left(\frac{\partial v}{\partial y}\right)dy \quad \text{(By C-R equation)}$$
$$= e^x \sin y\, dx + e^x \cos y\, dy$$
$$= d(e^x \sin y)$$

Integration yields,
$$v = e^x \sin y + c$$

Hence
$$f(z) = u + iv = e^x \cos y + i(e^x \sin y + c)$$
$$= e^x(\cos y + i \sin y) + c_1 \quad (\text{where } c_1 = ic)$$
$$= e^{x+iy} + c_1 = e^z + c_1.$$

Que 4.23. **Show that the function $u = \frac{1}{2}\log(x^2 + y^2)$ is harmonic. Find the harmonic conjugate of u.**

AKTU 2014-15 (III), Marks 05

Answer

$$u = \frac{1}{2}\log(x^2 + y^2)$$

$$\frac{\partial u}{\partial x} = \frac{x}{x^2 + y^2}, \quad \frac{\partial u}{\partial y} = \frac{y}{x^2 + y^2}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{x^2 + y^2 - 2x^2}{(x^2 + y^2)^2} = \frac{y^2 - x^2}{(x^2 + y^2)^2}$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{x^2 + y^2 - 2y^2}{(x^2 + y^2)^2} = \frac{x^2 - y^2}{(x^2 + y^2)^2}$$

$\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$, hence u is harmonic.

Now,
$$dv = \frac{\partial v}{\partial x}dx + \frac{\partial v}{\partial y}dy = -\frac{\partial u}{\partial y}dx + \frac{\partial u}{\partial x}dy$$

$$dv = \frac{-y}{x^2 + y^2}dx + \frac{x}{x^2 + y^2}dy$$

$$dv = \frac{x\,dy - y\,dx}{x^2 + y^2} = d\left[\tan^{-1}\left(\frac{y}{x}\right)\right]$$

Integration yields, $v = \tan^{-1}\left(\dfrac{y}{x}\right) + c$

This is the required harmonic conjugate function of u.

Que 4.24. **If $f(z) = u + iv$ is analytic function and $u - v = e^x (\cos y - \sin y)$, find $f(z)$ in terms of z.**

AKTU 2015-16 (III), Marks 05

Answer

$$u + iv = f(z)$$
$$i(u + iv) = if(z) \quad ...(4.24.1)$$
$$iu - v = if(z) \quad ...(4.24.2)$$

On adding eq. (4.24.1) and eq. (4.24.2),

$$u - v + i(u + v) = (1 + i) f(z)$$
$$U + iV = F(z)$$

Where, $U = u - v = e^x (\cos y - \sin y)$

$V = u + v$

$$(1 + i) f(z) = F(z)$$

Now using Milne's Thomson method,

$$\frac{\partial U}{\partial x} = \phi_1 = e^x (\cos y - \sin y)$$

So, $\phi_1(z, 0) = e^z (\cos 0 - \sin 0)$

$$\phi_1(z, 0) = e^z$$

Now $\dfrac{\partial U}{\partial y} = \phi_2 = e^x (-\sin y - \cos y)$

$$\phi_2(z, 0) = e^z (-\sin 0 - \cos 0)$$
$$\phi_2(z, 0) = -e^z$$

According to Milne's Thomson method,

$$F(z) = \int \{\phi_1(z, 0) - i\,\phi_2(z, 0)\}\, dz + c$$
$$= \int (e^z + ie^z)\, dz + C = \int e^z (1 + i)\, dz + c$$
$$F(z) = (1 + i) e^z + c$$

or $(1 + i) f(z) = (1 + i) e^z + c$

$$f(z) = e^z + \frac{c}{1 + i}$$

Que 4.25. **Determine an analytic function $f(z)$ in term of z if**

$$u + v = 2\frac{\sin 2x}{e^{2y}} + e^{2y} - 2\cos 2x.$$

AKTU 2017-18 (IV), Marks 07

Answer

Let $f(z) = u + iv$

$\therefore$ $if(z) = iu - v$

$$(1 + i) f(z) = (u - v) + i(u + v)$$
$$F(z) = U + iV$$

Where, $U = (u - v)$ and $V = u + v$

Hence, $V = u + v = \dfrac{2\sin 2x}{e^{2y}} + e^{2y} - 2\cos 2x$

Now, $\dfrac{\partial V}{\partial x} = \dfrac{4\cos 2x}{e^{2y}} + 4\sin 2x = \psi_2(x, y)$

and $\dfrac{\partial V}{\partial y} = \dfrac{-4\sin 2x}{e^{2y}} + 2e^{2y} = \psi_1(x, y)$

$\therefore$ $\psi_1(z, 0) = -4\sin 2z + 2$

$$\psi_2(z, 0) = 4\cos 2z + 4\sin 2z$$

By Milne's Thomson method,

$$F(z) = \int \{\psi_1(z, 0) + i\,\psi_2(z, 0)\}\, dz + c$$
$$= \int \{(-4\sin 2z + 2) + i(4\cos 2z + 4\sin 2z)\}\, dz + c$$
$$= \left(\frac{4\cos 2z}{2} + 2z\right) + i\left(\frac{4\sin 2z}{2} - \frac{4\cos 2z}{2}\right) + c$$
$$= (2\cos 2z + 2z) + i(2\sin 2z - 2\cos 2z) + c$$

or $(1 + i) f(z) = (2\cos 2z + 2z) + i(2\sin 2z - 2\cos 2z) + c$

or $f(z) = \dfrac{2(\cos 2z + z)}{(1 + i)} + \dfrac{2i(\sin 2z - \cos 2z)}{(1 + i)} + \dfrac{c}{(1 + i)}$

Multiply and divide by $(1 - i)$ on RHS, we get

$$f(z) = \frac{2(\cos 2z + z)}{(1 + i)}\left(\frac{1 - i}{1 - i}\right) + \frac{2i(\sin 2z - \cos 2z)}{(1 + i)}\left(\frac{1 - i}{1 - i}\right) + \frac{c}{(1 + i)}\left(\frac{1 - i}{1 - i}\right)$$

$$= \frac{2(1 - i)(z + \cos 2z)}{1^2 - i^2} + \frac{2i(1 - i)(\sin 2z - \cos 2z)}{1^2 - i^2} + c_1$$

{Where, c_1 = Constant}

$$= \frac{2(1 - i)(z + \cos 2z)}{2} + \frac{2i(1 - i)(\sin 2z - \cos 2z)}{2} + c_1 \quad (\because i^2 = -1)$$

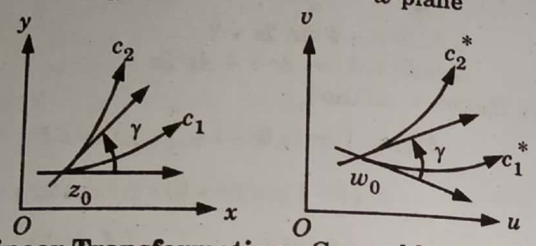
$$= (z + \cos 2z) - i(z + \cos 2z) + (i + 1)(\sin 2z - \cos 2z)$$
$$= (z + \cos 2z + \sin 2z - \cos 2z) + i(-z - \cos 2z + \sin 2z - \cos 2z)$$
$$f(z) = (z + \sin 2z) + i(\sin 2z - 2\cos 2z - z)$$

PART-4

Conformal Mapping.

CONCEPT OUTLINE

Conformal Mapping : A mapping $w = f(z)$ is said to be conformal if the angle between any two smooth curves c_1, c_2 in the z-plane intersecting at the point z_0 is equal in magnitude and sense to the angle between their images c_1^*, c_2^* in the w-plane at the point $w_0 = f(z_0)$.



General Linear Transformation : General linear transformation or simply linear transformation defined by the function

$$w = f(z) = az + b \quad \text{...(1)}$$

($a \neq 0$, and b are arbitrary complex constants) maps conformally the extended complex z-plane onto the extended w-plane, since this function is analytic and $f'(z) = a \neq 0$ for any z. If $a = 0$, eq. (1) reduces to a constant function.

Special Cases of Linear Transformation :

i. **Identity Transformation :** In this, $w = z$ for $a = 1$, $b = 0$, which maps a point z onto itself.
ii. **Translation :** In this, $w = z + b$ for $a = 1$, which translates (shifts) z through a distance $|b|$ in the direction of b.
iii. **Rotation :** In this, $w = e^{i\theta_0} + z$ for $a = e^{i\theta_0}$, $b = 0$ which rotates (the radius vector of point) z through a scalar angle θ_0 (counterclockwise if $\theta_0 > 0$, while clockwise of $\theta_0 < 0$).
iv. **Stretching (Scaling) :** In this, $w = az$ for 'a' real stretches if $a > 1$ (contracts if $0 < a < 1$) the radius sector by a factor 'a'.

Questions-Answers

Long Answer Type and Medium Answer Type Questions

Que 4.26. **State and prove condition for conformality.**

Answer

Statement : A mapping $w = f(z)$ is conformal at each point z_0 where $f(z)$ is analytic and $f'(z_0) \neq 0$.

Proof : Since f is analytic, f' exists and since $f' \neq 0$, we have at a point z_0

$$R_0 e^{i\theta_0} = f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z)}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \frac{\Delta w}{\Delta z}$$

$$= \lim_{\Delta z \to 0} \left(\left| \frac{\Delta w}{\Delta z} \right| + i \arg \frac{\Delta w}{\Delta z} \right)$$

So $$\theta_0 = \lim_{\Delta z \to 0} \left(\arg \frac{\Delta w}{\Delta z} \right)$$

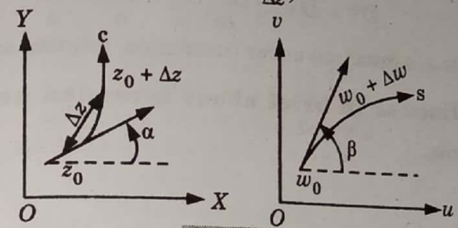


Fig. 4.26.1.

Since $$\Delta w = \frac{\Delta w}{\Delta z} \Delta z$$

$$\arg \Delta w = \arg \frac{\Delta w}{\Delta z} + \arg \Delta z$$

As $\Delta z \to 0$, $\quad \beta = \theta_0 + \alpha$

Thus the directed tangent to curve c at z_0 is rotated through an angle $\theta_0 = \arg f'(z_0)$, which is same for all curves through z_0. Let α_1, α_2 be angles of inclination of two curves c_1 and c_2 and β_1 and β_2 be the corresponding angles for their images S_1 and S_2.

Then $\quad \beta_1 = \alpha_1 + \theta_0$ and $\beta_2 = \alpha_2 + \theta_0$

Thus $\quad \beta_2 - \beta_1 = \alpha_2 - \alpha_1 = \gamma$

Hence, the angle γ between the curves c_1 and c_2 and their images S_1 and S_2 is same both in magnitude and sense.

Que 4.27. **Show that circles are invariant under translation, rotation and stretching.**

Answer

Linear transformation preserves circles *i.e.*, a circle in the z-plane under linear transformation maps to a circle in the w-plane.

Consider any circle in the z-plane

$$A(x^2 + y^2) + Bx + Cy + D = 0 \quad \text{...(4.27.1)}$$

Let $\quad w = f(z) = az + b$

From above $\quad u + iv = w = az + b = a(x + iy) + (b_1 + ib_2)$

or $\quad u = ax + b_1, v = ay + b_2$

or $$x = \frac{u - b_1}{a},\ y = \frac{v - b_2}{a},\ a \neq 0 \quad ...(4.27.2)$$

Substituting the value of x and y from eq. (4.27.2) in eq. (4.27.1), we get

$$A^*(u^2 + v^2) + B^* u + C^* v + D^* = 0 \quad ...(4.27.3)$$

Which is circle in the w-plane.

Where, $$A^* = \frac{A}{a^2},\ B^* = \frac{B - 2Ab_1}{a},\ C^* = \frac{C - 2Ab_2}{a}$$

and $$D^* = D + A\left(\frac{b_1^2 + b_2^2}{a^2}\right) - \frac{Bb_1}{a} - \frac{Cb_2}{a}$$

Thus circles are invariant under translation, rotation and stretching.

Que 4.28. **Discuss in brief about inversion and reflection transformation.**

Answer

Consider, $$w = \frac{1}{z} \quad \text{for } z \neq 0 \quad ...(4.28.1)$$

In polar coordinates,

$$Re^{i\phi} = \frac{1}{re^{i\theta}} = \frac{1}{r} e^{-i\theta}$$

So $R = \frac{1}{r}$, $\phi = -\theta$. Thus this transformation consists of an inversion in the unit circle ($Rr = 1$) followed by a mirror reflection about the real axis.

Also $|w| = \frac{1}{|z|}$. So the unit circle $|z| = 1$ maps onto the unit circle $|w| = \frac{1}{1} = 1$. Further the interior of the unit circle $|z| = 1$ (point lying within $|z| = 1$) are transformed to the exterior of the unit circle $|w| = 1$ (points lying outside $|w| = 1$) or vice versa (Fig. 4.28.1).

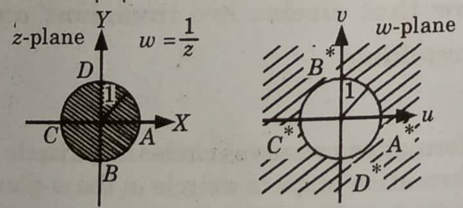


Fig. 4.28.1.

Que 4.29. **Find and plot the image of triangular region with vertices at (0, 0), (1, 0), (0, 1) under the transformation $w = (1 - i)z + 3$ (Fig. 4.29.1).**

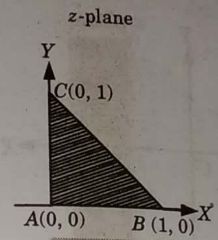


Fig. 4.29.1.

Answer

Here, $$u + iv = w = (1 - i)(x + iy) + 3$$
$$= x + iy - ix + y + 3$$

So $$u(x, y) = x + y + 3,\ v(x, y) = y - x$$

At AB, $y = 0$, $u = x + 3$, $v = -x$

or $$u = -v + 3$$

$\therefore$ $$v = 3 - u \text{ gives } A^* B^*$$

At AC, $x = 0$, $u = y + 3$, $v = y$,

or $$u = v + 3$$

$\therefore$ $$v = u - 3 \text{ gives } A^*C^*$$

At BC, $x + y = 1$, or substituting $u = (x + y) + 3$
$$= 1 + 3 = 4,$$
$$u = 4 \text{ gives } B^*C^*$$

So the image is the triangular region with vertices at $A^*(3, 0)$, $B^*(4, -1)$, $C^*(4, 1)$. Let $D\left(\frac{1}{4}, \frac{1}{4}\right)$ be any interior point of ABC. Its image is $D^*(3, 5, 0)$ which is also an interior point of $A^*B^*C^*$.

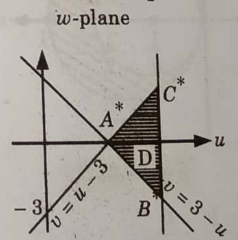


Fig. 4.29.2.

Que 4.30. **Find the graph for the strip $1 < x < 2$ under the mapping $w = \frac{1}{z}$ (Fig. 4.30.1).**

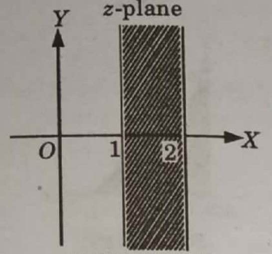


Fig. 4.30.1.

Answer

Here, $u + iv = w = \dfrac{1}{z} = \dfrac{x - iy}{x^2 + y^2}$

So $x = \dfrac{u}{u^2 + v^2}, \quad y = \dfrac{-v}{u^2 + v^2}$

Since $1 < x < 2$ so $1 < \dfrac{u}{u^2 + v^2} < 2$

or $u^2 + v^2 - u < 0$ and $2(u^2 + v^2) - u > 0$

Rewriting $\left(u - \dfrac{1}{2}\right)^2 + v^2 < \dfrac{1}{4}$ and $\left(u - \dfrac{1}{4}\right)^2 + v^2 > \dfrac{1}{16}$

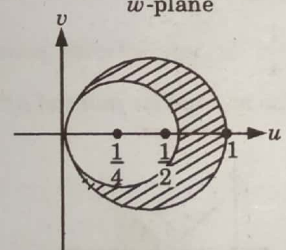


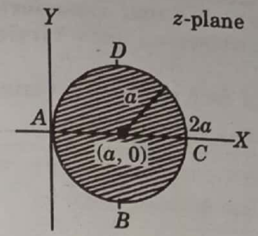
Fig. 4.30.2.

or $\left|w - \dfrac{1}{2}\right| < \dfrac{1}{2}$ and $\left|w - \dfrac{1}{4}\right| > \dfrac{1}{4}$

i.e., interior of the circle with centre at $\left(\dfrac{1}{2}, 0\right)$ and radius $\dfrac{1}{2}$ and exterior of the circle with centre at $\left(\dfrac{1}{4}, 0\right)$ and radius $\dfrac{1}{4}$.

Thus the infinite strip maps to the region shaded in the w-plane.

Que 4.31. **Determine and graph the image of $|z - a| = a$ under $w = z^2$ (Fig. 4.31.1).**



$|z - a| = a$

Fig. 4.31.1.

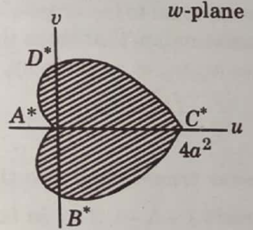
Answer

The given region is a circle in the z-plane with centre at $(a, 0)$ and radius a, *i.e.*,

$$z - a = ae^{i\theta} \quad \text{or} \quad z = a + ae^{i\theta} = a(1 + e^{i\theta})$$

So
$$\begin{aligned} w &= z^2 = a^2(1 + e^{i\theta})^2 = a^2(1 + \cos\theta + i\sin\theta)^2 \\ &= 2a^2(\cos^2\theta + \cos\theta + i\sin\theta\cos\theta + i\sin\theta) \\ Re^{i\phi} = w &= 2a^2(1 + \cos\theta)(\cos\theta + i\sin\theta) \\ &= 2a^2(1 + \cos\theta)e^{i\theta} \end{aligned}$$

Thus
$$\begin{aligned} R &= 2a^2(1 + \cos\theta) \\ &= 2a^2(1 + \cos\phi) \quad (\because \phi = \theta) \end{aligned}$$



$R = 2a^2(1 + \cos\phi)$ cardioid

Fig. 4.31.2.

PART-5

Mobius Transformation and their Properties.

CONCEPT OUTLINE

Mobius Transformation : It is also known as bilinear transformation. Bilinear transformation is the function w of a complex variable z of the form

$$w = f(z) = \frac{az + b}{cz + d}$$

Where a, b, c, d are complex or real constants subject to $ad - bc \neq 0$.

Properties of Mobius or Bilinear Transformation :

1. Circles are transformed into circles under bilinear transformation.
2. The cross-ratio of four points is invariant under a bilinear transformation.

Questions-Answers

Long Answer Type and Medium Answer Type Questions

Que 4.32. **How could you determine the bilinear transformation ?**

Answer

1. A bilinear transformation can be uniquely determined by three given conditions. To find the unique bilinear transformation which maps three given distinct points z_1, z_2, z_3 onto three distinct images w_1, w_2, w_3, consider w which is the image of a general point z under this transformation.
2. Now by theorem 2 which states that the cross-ratio of four points is invariant under a bilinear transformation, the cross-ratio of the four point w_1, w_2, w_3, w must be equal to the cross-ratio of z_1, z_2, z_3, z. Hence the unique bilinear transformation that maps three given point z_1, z_2, z_3 on to three given images w_1, w_2, w_3 is given by,

$$\frac{(w_1 - w_2)(w_3 - w)}{(w_1 - w)(w_3 - w_2)} = \frac{(z_1 - z_2)(z_3 - z)}{(z_1 - z)(z_3 - z_2)}$$

Que 4.33. **Find the bilinear transformation that maps the point 0, 1, i in z-plane onto the points $1 + i, -i, 2 - i$ in the w-plane.**

Answer

The required bilinear transformation is

$$\frac{(w_1 - w_2)(w_3 - w)}{(w_1 - w)(w_3 - w_2)} = \frac{(z_1 - z_2)(z_3 - z)}{(z_1 - z)(z_3 - z_2)}$$

$$\frac{(1+i+i)(2-i-w)}{(1+i-w)(2-i+i)} = \frac{(0-1)(i-z)}{(0-z)(i-z_1)}$$

$$\frac{(1+2i)}{2}\,\frac{(2-i-w)}{(1+i-w)} = (i-1)\left(\frac{i-z}{z}\right)$$

$$\frac{2-i-w}{1+i-w} = \frac{2(3i+1)}{5}\left(\frac{i-z}{z}\right)$$

Solving for w,

$$5z(2-i-w) = 2(3i+1)(1+i-w)(i-z)$$

$$w = \frac{(6i+2)(1+i)(i-z) - (2-i)5z}{-5z + (6i+2)(i-z)}$$

or

$$w = \frac{z(6+3i)+(8+4i)}{z(7+6i)+(6-2i)}$$

Que 4.34. **Determine the Mobius transformation having 1 and i as fixed (invariant) points and maps 0 to – 1.**

Answer

The Mobius transformation having α and β as fixed points is given by

$$w = \frac{\gamma z - \alpha\beta}{z - \alpha - \beta + \gamma}$$

For $\alpha = 1$, $\beta = i$, we have

$$w = \frac{\gamma z - i}{z - 1 - i + \gamma}$$

Since $z = 0$ is mapped to $w = -1$,

$$-1 = \frac{0 - i}{0 - 1 - i + \gamma}$$

or $\gamma = 2i + 1$

Thus the required transformation is

$$w = \frac{(2i+1)z - i}{z + i}$$

Que 4.35. **Find a bilinear transformation which maps the upper half of the z-plane into the interior of a unit circle in the w-plane. Verify the transformation (Fig. 4.35.1).**

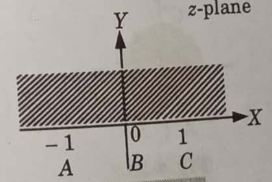


Fig. 4.35.1.

Answer

Suppose any three points in the upper half of z-plane say $A : -1$, $B : 0$, $C = 1$ gets mapped to any three points in the interior of the circle $|w| = 1$ in the w-plane, say $A^* : -i$, $B^* : 1$, $C^* : i$. Thus the required bilinear transformation is the one which maps $-1, 0, 1$ from z-plane to $-i, 1, i$ in the w-plane.

Now according to cross-ratio property,

$$\frac{(z_1 - z_2)(z_3 - z)}{(z_1 - z)(z_3 - z_2)} = \frac{(w_1 - w_2)(w_3 - w)}{(w_1 - w)(w_3 - w_2)}$$

$$\frac{(-1-0)(1-z)}{(-1-z)(1-0)} = \frac{(-i-1)(i-w)}{(-i-w)(i-1)}$$

or $$\frac{1-z}{1+z} = \frac{1+iw}{i+w}$$

On solving, $$w = \frac{i-z}{i+z}$$

Verification : $$|w| = \left|\frac{i-z}{i+z}\right| \le 1$$

or $$|i - z| \le |i + z|$$

$$\sqrt{x^2 + (1-y)^2} \le \sqrt{x^2 + (1+y)^2}$$

$$4y \ge 0$$

Thus the bilinear transformation $w = \dfrac{i-z}{i+z}$ transforms interior of unit circle in w-plane onto the upper half plane in z-plane.

Also, $$|w| = \left|\frac{i-z}{i+z}\right| = \sqrt{\frac{x^2 + (1-y)^2}{x^2 + (1+y)^2}}$$

For $y = 0$, $|w| = \sqrt{\dfrac{x^2+1}{x^2+1}} = 1$. Thus the real axis ($y = 0$) gets mapped to the unit circle $|w| = 1$.

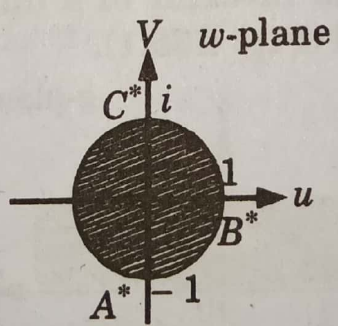


Fig. 4.35.2.

☺☺☺

5
UNIT

Complex Variable Integration

CONTENTS

PART-1

Complex Integrals, Contour Integrals, Cauchy-Goursat Theorem, Cauchy Integral Formula.

CONCEPT OUTLINE

Contour Integral : If the initial point and final point coincide so that C is a closed curve then this integral is called contour integral and is denoted by $\oint_c f(z)\,dz$.

If $\quad f(z) = u(x,y) + iv(x,y)$

since $\quad dz = dx + idy$

$$\int_c f(z)dz = \int_c (u+iv)(dx+idy) = \int_c (u\,dx - v\,dy) + i\int_c (v\,dx + u\,dy)$$

which shows that the evaluation of line integral of a complex function can be reduced to the evaluation of two line integrals of real functions.

Cauchy's Integral Theorem : If $f(z)$ is an analytic function and $f'(z)$ is continuous at each point within a simple closed curve C, then

$$\oint_C f(z)dz = 0$$

For multiple connected regions,

$$\oint_{C_1} f(z)dz = \oint_{C_2} f(z)dz$$

when integral along each curve is taken in anticlockwise direction.

Cauchy's-Goursat Theorem : Cauchy's theorem without the assumption that $f'(z)$ is continuous is known as Cauchy's-Goursat theorem.

Cauchy's Integral Formula : If $f(z)$ is analytic within and on a closed curve C and if a is any point within C, then

$$f(a) = \frac{1}{2\pi i}\oint_z \frac{f(z)}{(z-a)}dz$$

Cauchy's Integral Formula for Derivative of an Analytic Function : If a function $f(z)$ is analytic in a domain D, then at any point $z = a$ of D, $f(z)$ has derivatives of all orders, all of which are again analytic functions in D and are given by

$$f'(a) = \frac{n!}{2\pi i}\int_C \frac{f(z)}{(z-a)^{n+1}}dz$$

Where C is any closed curve in D surrounding the point $z = a$.

Questions-Answers

Long Answer Type and Medium Answer Type Questions

Que 5.1. **State Cauchy's Integral theorem and derive it.**

Answer

A. Statement : If $f(z)$ is an analytic function and $f'(z)$ is continuous at each point within and on a simple closed curve C, then

$$\oint_C f(z)dz = 0$$

B. Proof : Let R be the region bounded by the curve C.

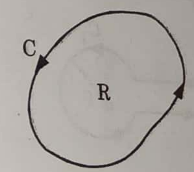


Fig. 5.1.1.

Let, $\quad f(z) = u(x, y) + iv(x, y)$, then

$$\oint_C f(z)dz = \oint_C (u+iv)(dx+idy)$$
$$= \oint_C (udx - vdy) + i\oint_C (vdx + udy) \quad \text{...(5.1.1)}$$

Since $f'(z)$ is continuous, the partial derivatives $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}, \frac{\partial v}{\partial y}$ are also continuous in R. Hence by Green's theorem, we have

$$\oint_C f(z)dz = \iint_R \left(-\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)dxdy + i\iint_R \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right)dxdy \quad \text{...(5.1.2)}$$

Now $f(z)$ being analytic at each point of the region R, by Cauchy-Riemann equations, we have

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

Thus, the two double integrals in eq. (5.1.2) vanish.

Hence $\quad \oint_C f(z)dz = 0$

Que 5.2. **State and prove Cauchy's integral formula.**

AKTU 2017-18 (III), Marks 10

Answer

A. Statement : If $f(z)$ is analytic within and on a closed curve C and a is any point within C, then

$$f(a) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z-a}\, dz$$

B. Proof : Consider the function $\frac{f(z)}{z-a}$, which is analytic at every point within C except at $z = a$. Draw a circle C_1 with a as centre and radius ρ such that C_1 lies entirely inside C. Thus $\frac{f(z)}{z-a}$ is analytic in the region between C and C_1.

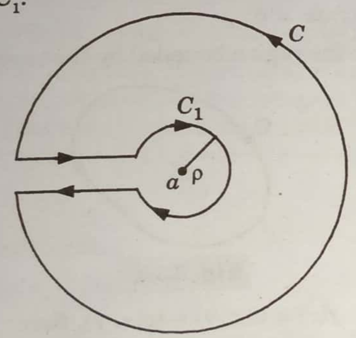


Fig. 5.2.1.

$\therefore$ By Cauchy's theorem, we have

$$\oint_C \frac{f(z)}{z-a}\, dz = \oint_{C_1} \frac{f(z)}{z-a}\, dz \quad \text{...(5.2.1)}$$

Now, the equation of circle C_1 is $|z-a| = \rho$ or $z - a = \rho e^{i\theta}$

So that $\quad dz = i\rho e^{i\theta}\, d\theta$

$$\therefore \oint_{C_1} \frac{f(z)}{z-a}\, dz = \int_0^{2\pi} \frac{f(a+\rho e^{i\theta})}{\rho e^{i\theta}}\, i\rho e^{i\theta}\, d\theta = i\int_0^{2\pi} f(a+\rho e^{i\theta})d\theta$$

Hence by eq. (5.2.1), we have

$$\oint_C \frac{f(z)}{z-a}\, dz = i\int_0^{2\pi} f(a+\rho e^{i\theta})d\theta \quad \text{...(5.2.2)}$$

In the limiting form, as the circle C_1 shrinks to the point a, *i.e.*, $\rho \to 0$, then from eq. (5.2.2),

$$\oint_C \frac{f(z)}{z-a}\, dz = i\int_0^{2\pi} f(a)d\theta = if(a)\int_0^{2\pi} d\theta = 2\pi i f(a)$$

Hence $\quad f(a) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z-a}\, dz$

Que 5.3. **State Cauchy integral theorem for an analytic function. Verify this theorem by integrating the function $z^3 + iz$ along the boundary of the rectangle with vertices $1, -1, i, -i$.**

AKTU 2014-15 (III), Marks 05

Answer

A. Cauchy's Integral Theorem : Refer Q. 5.1, Page 5–3F, Unit-5.

B. Numerical :

$$\int_C f(z)\,dz = \int_{AB} f(z)\,dz + \int_{BC} f(z)\,dz + \int_{CD} f(z)\,dz + \int_{DA} f(z)\,dz$$

$$\int_{AB} f(z)\,dz = \int (x+iy)^3 + i(x+iy)\{dx + i\,dy\} = 0 \quad \text{...(5.3.1)}$$

$$\int_{BC} f(z)\,dz = \int ((x+(x-1))^3 + i(2x-1)\{2\,dx\}$$

$$= 2\int_1^0 \{(2x-1)^3 + i(2x-1)\}\,dx = -i \quad \text{...(5.3.2)}$$

$$\int_{CD} f(z)\,dz = \int\left[\{x + i(+ix+i)\}^3 + i(-ix+i)\right](0) = 0 \quad \text{...(5.3.3)}$$

$$\int_{DA} f(z)\,dz = 2\int\left[\{x - i(ix+i)\}^3 + i(2x+1)\right]dx$$

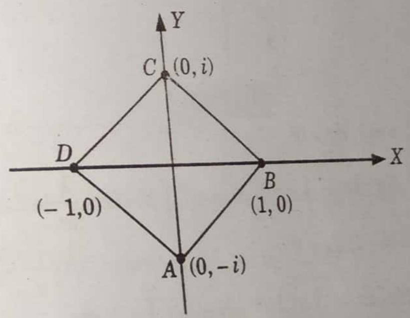


Fig. 5.3.1.

$$= 2\int [(2x+1)^3 + i(2x+1)]\,dx$$

$$= 2\left[\frac{(2x+1)^4}{8} + i\frac{(2x+1)}{2}\right]_{-1}^{0} = i \quad \text{...(5.3.4)}$$

From eq. (5.3.1), eq. (5.3.2), eq. (5.3.3) and eq. (5.3.4), we have

$$\int_C f(z)\,dz = -i + 0 + 0 + i = 0 \text{ (Hence proved)}$$

Que 5.4. **Verify Cauchy's theorem by integrating e^{iz} along the boundary of the triangle with the vertices at the points $1+i, -1+i$ and $-1-i$.**

AKTU 2017-18 (II), Marks 10

Answer

$$\oint_C f(z)\,dz = \int_{AB} f(z)\,dz + \int_{BC} f(z)\,dz + \int_{CA} f(z)\,dz \quad \text{...(5.4.1)}$$

Along AB, $y = x,\ dy = dx$

$f(z) = e^{iz} = e^{i(x+iy)}$

$f(x) = e^{i(1+i)x}$

$$\int_{AB} f(z)\,dz = \int_{-1}^{1} e^{i(1+i)x}\,(dx + i dx)$$

$$= (1+i)\left[\frac{e^{i(1+i)x}}{i(1+i)}\right]_{-1}^{1} = \frac{(i+1)}{(i-1)}\,[e^{i-1} - e^{-i+1}]$$

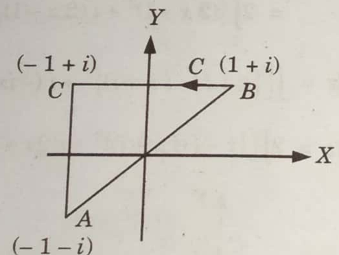


Fig. 5.4.1.

Along BC, $y = 1,\ dy = 0$

$$\int_{BC} f(z)\,dz = \int_{1}^{-1} e^{i(x+i)}\,dx = e^{-1}\int_{1}^{-1} e^{ix}\,dx = \frac{1}{ie}(e^{-i} - e^{i})$$

Along CA, $x = -1,\ dx = 0$

$$\int_{CA} f(z)\,dz = \int_{1}^{-1} e^{i(-1+iy)}\,i dy = ie^{-i}\int_{+1}^{-1} e^{-y}\,dy$$

$$= -ie^{-i}(e^{+1} - e^{-1}) = -ie^{-i}(e - e^{-1})$$

From eq. (5.4.1)

$$\oint_C f(z)\,dz = \frac{(i+1)^2}{-2}\left[\frac{e^{i}}{e} - ee^{-i}\right] + \frac{e^{-i}}{ie} - \frac{e^{i}}{ie} - ie^{-i}e + \frac{ie^{-i}}{e}$$

$$= -\frac{ie^{i}}{e} + iee^{-i} + \frac{e^{-i}}{ie} - \frac{e^{i}}{ie} - ie^{-i}e + \frac{ie^{-i}}{e}$$

$$= -ie^{i-1} + ie^{-i+1} - ie^{-i-1} + ie^{i-1} - ie^{-i+1} + ie^{-i-1}$$

$\therefore$ $\oint_C f(z)\,dz = 0$ (Hence proved)

Que 5.5. **State Cauchy's integral formula. Hence, Evaluate $\oint_C \frac{dz}{z^2(z^2-4)e^z}$, where C is $|z| = 1$**

AKTU 2012-13 (IV), Marks 05

Answer

A. Cauchy's Integral Formula : Refer Q. 5.2, Page 5–3F, Unit-5.

B. Numerical :

Let, $$I = \oint_C \frac{dz}{z^2(z^2-4)e^z},\ C = |z| = 1$$

$$I = \oint_C \frac{e^{-z}dz}{z^2(z^2-4)}$$

Poles are $z = 0$ (of order 2), $z = \pm 2$

$z = 0$ is the only pole which lie inside C.

$\therefore$ $$I = \oint_C \frac{e^{-z}/(z^2-4)}{z^2}\,dz = 2\pi i\left[\frac{d}{dz}\left(\frac{e^{-z}}{z^2-4}\right)\right]_{z=0}$$

$$I = 2\pi i\left[\frac{-(z^2-4)e^{-z} - 2ze^{-z}}{(z^2-4)^2}\right]_{z=0}$$

$$I = -2\pi i\left[\frac{-4+0}{16}\right]$$

$$I = \frac{\pi i}{2}$$

Thus $$\oint_C \frac{1}{z^2(z^2-4)e^z}\,dz = \frac{\pi i}{2}$$

Que 5.6. **State Cauchy's integral formula. Hence evaluate :**

$$\int_C \frac{2z+1}{z^2+z}\,dz, \text{ where C is } |z| = \frac{1}{2}.$$

AKTU 2014-15 (IV), Marks 05

Answer

A. Cauchy Integral Formula : Refer Q. 5.2, Page 5–3F, Unit-5.

B. Numerical :

Poles are given by $z^2 + z = 0,\ z = 0, -1$

$|z| = \frac{1}{2}$ is a circle with centre at origin and radius $\frac{1}{2}$. Pole $z = 0$ enclosed in $|z| = \frac{1}{2}$.

$$\int_C \frac{2z+1}{z(z+1)}dz = \int_C \frac{\frac{2z+1}{z+1}}{z} = 2\pi i\left[\frac{2z+1}{z+1}\right]_{z=0}$$

$$\int_C \frac{2z+1}{z(z+1)}dz = 2\pi i$$

Que 5.7. **Use Cauchy's integral formula to show that**

$$\int_C \frac{e^{zt}}{z^2+1}dz = 2\pi i \sin t \text{ if } t > 0 \text{ and } C \text{ is the circle } |z| = 3.$$

AKTU 2013-14 (IV), Marks 05

Answer

Poles of the integrand are given by

$$z^2 + 1 = 0,\ z = \pm i (\text{order } 1)$$

The circle $|z| = 3$ has centre at $z = 0$ and radius 3. It encloses both the singularities $z = i$ and $z = -i$.

$$\text{Now}, \int_C \frac{e^{zt}}{z^2+1}dz = \int_C \frac{e^{zt}}{(z+i)(z-i)}dz$$

$$= \int_{C_1} \frac{\left(\frac{e^{zt}}{z-i}\right)}{z+i}dz + \int_{C_2} \frac{\left(\frac{e^{zt}}{z+i}\right)}{z-i}dz = 2\pi i\left(\frac{e^{zt}}{z+i}\right)\Bigg|_{z=i} + 2\pi i\left(\frac{e^{zt}}{z-i}\right)\Bigg|_{z=-i}$$

$$= \pi(e^{it} - e^{-it}) = 2\pi i \sin t$$

Que 5.8. **Evaluate by Cauchy's integral formula**

$$\oint_C \frac{z^2-2z}{(z+1)^2(z^2+4)}dz, \text{ where } C \text{ is the circle } |z| = 3.$$

AKTU 2015-16 (III), Marks 05

Answer

Here, we have $\int_C \frac{z^2-2z}{(z+1)^2(z^2+4)}dz$

The poles are determined by putting the denominator equal to zero.

$$(z+1)^2(z^2+4) = 0$$

$$z = -1, -1 \text{ and } z = \pm 2i$$

The circle $|z| = 3$ with centre at origin and radius = 3 encloses a pole at $z = -1$ of second order and simple poles $z = \pm 2i$.

Let the given integral $= I_1 + I_2 + I_3$ ...(5.8.1)

$$I_1 = \int_{C_1} \frac{z^2-2z}{(z+1)^2(z^2+4)}dz = \int_{C_1} \frac{\frac{z^2-2z}{z^2+4}}{(z+1)^2}dz$$

$$= 2\pi i\left[\frac{d}{dz}\frac{z^2-2z}{z^2+4}\right]_{z=-1}$$

$$= 2\pi i\left[\frac{(z^2+4)(2z-2)-(z^2-2z)2z}{(z^2+4)^2}\right]_{z=-1}$$

$$= 2\pi i\left[\frac{(1+4)(-2-2)-(1+2)2(-1)}{(1+4)^2}\right]$$

$$= 2\pi i\left(-\frac{14}{25}\right) = \frac{-28\pi i}{25}$$

$$I_2 = \int_{C_2} \frac{\frac{z^2-2z}{(z+1)^2(z+2i)}}{(z-2i)}dz = 2\pi i\left[\frac{z^2-2z}{(z+1)^2(z+2i)}\right]_{z=2i}$$

$$= 2\pi i\left[\frac{-4-4i}{(2i+1)^2(2i+2i)}\right] = 2\pi i\frac{(1+i)}{4+3i}$$

$$I_3 = \int_{C_3} \frac{\frac{z^2-2z}{(z+1)^2(z-2i)}}{(z+2i)}dz = 2\pi i\left[\frac{z^2-2z}{(z+1)^2(z-2i)}\right]_{z=-2i}$$

$$= 2\pi i\left[\frac{-4+4i}{(-2i+1)^2(-2i-2i)}\right] = 2\pi i\frac{(i-1)}{(3i-4)}$$

Now putting the value of I_1, I_2 and I_3 in eq. (5.8.1), we get

$$\int_C \frac{z^2-2z}{(z+1)^2(z^2+4)}dz = \frac{-28\pi i}{25} + 2\pi i\left(\frac{1+i}{4+3i}\right) + 2\pi i\left(\frac{i-1}{3i-4}\right)$$

$$= 2\pi i\left[\frac{-14}{25} + \frac{1+i}{(4+3i)} + \frac{(i-1)}{(3i-4)}\right]$$

$$= 2\pi i\left[\frac{-14}{25} + \frac{(1+i)(3i-4)+(i-1)(4+3i)}{(-9-16)}\right]$$

$$= \frac{2\pi i}{-25}[14 + (3i-4-3-4i) + (4i-3-4-3i)]$$

$$= 0$$

Que 5.9. Evaluate the integral $\int \frac{e^{2z}}{(z+1)^5}\,dz$, around the boundary of the circle $|z| = 2$.

AKTU 2014-15 (III), Marks 05

Answer

Poles are $z = -1$ of order 5 will lie in $|z| = 2$

Using cauchy integral formula, we get

$$\int \frac{e^{2z}}{(z+1)^5}\,dz = \frac{2\pi i}{4!}\left[\frac{d^4}{dz^4}(e^{2z})\right]_{z=-1}$$

$$= \frac{2\pi i}{4!}(16e^{2z})_{z=-1} = \frac{32\pi i}{24} \times e^{-2} = \frac{4\pi i}{3e^2}$$

Que 5.10. Using Cauchy's integral formula evaluate $\int_C \frac{e^{2z}}{(z+1)^4}\,dz$, where C is the circle $|z| = 3$.

AKTU 2016-17 (IV), Marks 10

Answer

Same as Q. 5.9, Page 5–10F, Unit-5. $\left(\textbf{Answer}: \frac{8\pi i}{3e^2}\right)$.

Que 5.11. Evaluate $\int_C \frac{(1+z)\sin z}{(2z-3)^2}\,dz$, where C is the circle $|z - i| = 2$ counter clockwise.

AKTU 2013-14 (III), Marks 05

Answer

The given integral is $\int \frac{(1+z)\sin z}{(2z-3)^2}\,dz$

Poles of integrand,

$$(2z-3)^2 = 0$$

$$z = \frac{3}{2}, \frac{3}{2}$$

$\therefore$ Pole lie inside the circle of radius 2.

By Cauchy's integral formula,

$$\int_c \frac{(1+z)\sin z}{(2z-3)^2}\,dz = 2\pi i\left[\frac{d}{dz}(1+z)\sin z\right]_{z=3/2}$$

$$= 2\pi i\,[(1+z)\cos z + \sin z]_{z=3/2}$$

$$= 2\pi i\left(\frac{5}{2}\cos\frac{3}{2} + \sin\frac{3}{2}\right)$$

Que 5.12. Evaluate $\int_0^{3+i} (\bar{z})^2\,dz$, along the real axis from $z = 0$ to $z = 3$ and then along a line parallel to imaginary axis from $z = 3$ to $z = 3 + i$.

AKTU 2012-13 (IV), Marks 05

Answer

$$\int_0^{3+i} (\bar{z})^2\,dz = \int_0^{3+i} (x - iy)^2\,(dx + idy) = \int_{OA} x^2\,dx + \int_{AB} (3 - iy)^2\,idy$$

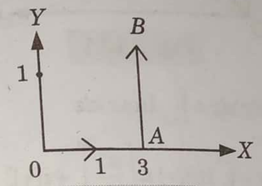


Fig. 5.12.1.

Along OA, $y = 0$, $dy = 0$, x varies 0 to 3

Along AB, $x = 3$, $dx = 0$, and y varies 0 to 1

$$\therefore \int_0^{3+i} (\bar{z})^2\,dz = \int_0^3 x^2\,dx + \int_0^1 (3 - iy)^2\,idy = \left[\frac{x^3}{3}\right]_0^3 + i\int_0^1 (9 - 6iy - y^2)dy$$

$$= \left[\frac{x^3}{3}\right]_0^3 + i\left[9y - 3iy^2 - \frac{y^3}{3}\right]_0^1 = \frac{27}{3} + i\left[\frac{26}{3} - 3i\right] = 12 + \frac{26i}{3}$$

Que 5.13. Integrate $f(z) = \text{Re}(z)$ from $z = 0$ to $z = 1 + 2i$, (i) along straight line joining $z = 0$ to $z = 1 + 2i$, (ii) along the real axis from $z = 0$ to $z = 1$ and then along a line parallel to imaginary axis from $z = 1$ to $z = 1 + 2i$.

AKTU 2013-14 (III), Marks 05

Answer

i. $$\int_0^{1+2i} f(z)\,dz = \int_0^{1+2i} \text{Re}(z)\,dz$$

Equation of OB is,

$$y - 0 = \frac{2-0}{1-0}(x - 0)$$

$$y = 2x$$

$$dy = 2dx$$

$$z = x + iy$$

$$dz = dx + idy = dx + i2dx$$

$$\int_0^{1+2i} \text{Re}(z)dz = \int_0^1 x(dx + idy)$$

$$= \int_0^1 x(dx + 2idx) = (1+2i)\int_0^1 x dx = (1+2i)\left[\frac{x^2}{2}\right]_0^1 = \frac{1+2i}{2}$$

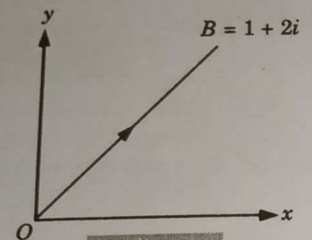


Fig. 5.13.1.

ii. $$\int f(z)dz = \int_{OA} \text{Re}(z)dz + \int_{AB} \text{Im}(z)dz$$

$$= \int_0^1 x\,dx + \int_0^2 1(idy) = \left[\frac{x^2}{2}\right]_0^1 + i[y]_0^2 = \frac{1}{2} + 2i = \frac{1+4i}{2}$$

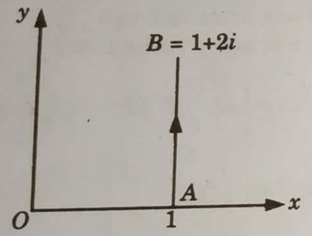


Fig. 5.13.2.

Que 5.14. **Evaluate : $\int_0^\infty \frac{\sin mx}{x} dx$, $m > 0$.**

AKTU 2017-18 (IV), Marks 10

Answer

Consider the integral $\int_C \frac{e^{miz}}{z} dz = \int_C f(z)\,dz$ where C consists of

i. The real axis from r to R.
ii. The upper half of the circle C_R : $|z| = R$,
iii. The real axis $-R$ to $-r$,
iv. The upper half of the circle C_r : $|z| = r$ (Fig. 5.14.1)
Since $f(z)$ has no singularity inside C (its only singular point being a simple pole at $z = 0$ which has been deleted by drawing C_r), we have by Cauchy's theorem :

$$\int_r^R f(x)dx + \int_{C_R} f(z)dz + \int_{-R}^{-r} f(x)dx + \int_{C_r} f(z)dz = 0 \quad ...(5.14.1)$$

Now $$\int_{C_R} f(z)\,dz = \int_0^\pi \frac{e^{imR(\cos\theta + i\sin\theta)}}{Re^{i\theta}} Rie^{\theta}\,d\theta \qquad [\because z = Re^{i\theta}]$$

$$= i\int_0^\pi e^{imR(\cos\theta + i\sin\theta)}\,d\theta$$

Since $|e^{imR(\cos\theta + i\sin\theta)}| = |e^{-mR\sin\theta + imR\cos\theta}| = e^{-mR}\sin\theta$

$$\therefore \left|\int_{C_R} f(z)\,dz\right| \le \int_0^\pi e^{-mR\sin\theta}\,d\theta = 2\int_0^{\pi/2} e^{-mR\sin\theta}\,d\theta$$

$$= 2\int_0^{\pi/2} e^{-2mR\theta/\pi}\,d\theta \qquad [\because \text{for } 0 \le \theta \le \pi/2, \sin\theta/\theta \ge 2/\pi]$$

$$= \frac{\pi}{mR}(1 - e^{-mR}) \text{ which} \to 0 \text{ as } R \to \infty,$$

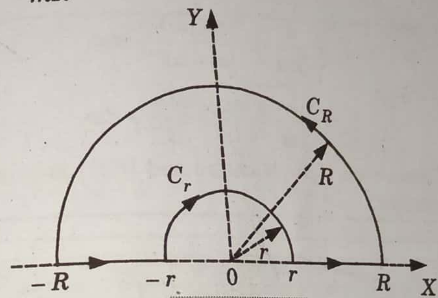


Fig. 5.14.1.

Also $\int_{C_r} f(z)\,dz = i\int_\pi^0 e^{imr(\cos\theta + i\sin\theta)}\,d\theta \to i\int_\pi^0 d\theta$ i.e. $-i\pi$ as $r \to 0$

Hence as $r \to 0$ and $R \to \infty$, we get from eq. (5.14.1).

$$\int_{-\infty}^{\infty} f(x)dx + 0 + \int_{-\infty}^{0} f(x)dx - i\pi = 0$$

or $$\int_{-\infty}^{\infty} f(x)dx = i\pi \text{ i.e. } \int_{-\infty}^{\infty} \frac{e^{imx}}{x}dx = i\pi \quad ...(5.14.2)$$

Equating imaginary parts from both sides,

$$\int_{-\infty}^{\infty} \frac{\sin mx}{x} dx = x$$

Hence $$\int_0^\infty \frac{\sin mx}{x} dx = \frac{\pi}{2}$$

PART-2

Taylor's Series, Laurent's Series, Liouville's Theorem.

CONCEPT OUTLINE

Taylor's Series : A function $f(z)$ which is analytic at all points within a circle C with centre at a can be represented uniquely as a convergent power series known as Taylor's series.

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n$$

Where, $$a_n = \frac{f^n(a)}{n!}.$$

Laurent's Series : If $f(z)$ is analytic inside and on the boundary of the annular (ring shaped) region R bounded by two concentric circles C_1 and C_2 of radii r_1 and r_2 ($r_1 > r_2$) respectively having centre at a, then for all z in R

$$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n + \sum_{n=1}^{\infty} b_n (z-a)^{-n}$$

Where, $$a_n = \frac{1}{2\pi i} \oint_{C_1} \frac{f(w)}{(w-a)^{n+1}} dw$$

and $$b_n = \frac{1}{2\pi i} \oint_{C_2} \frac{f(w)}{(w-a)^{-n+1}} dw$$

Liouville's Theorem : If $f(z)$ is entire and $|f(z)|$ is bounded for all z, then $f(z)$ is constant.

Questions-Answers

Long Answer Type and Medium Answer Type Questions

Que 5.15. Expand $\dfrac{1}{z^2 - 3z + 2}$ in the region $1 < |z| < 2$.

AKTU 2014-15 (IV), Marks 05

Answer

$$f(z) = \frac{1}{z^2 - 3z + 2} = \frac{1}{(z-2)} - \frac{1}{(z-1)} = -\frac{1}{2}\left(1 - \frac{z}{2}\right)^{-1} - \frac{1}{z}\left(1 - \frac{1}{z}\right)^{-1}$$

$$= -\frac{1}{2}\left[1 + \frac{z}{2} + \frac{z^2}{4} + \frac{z^3}{8} \ldots\right] - \frac{1}{z}\left[1 + \frac{1}{z} + \frac{1}{z^2} + \ldots\right]$$

After rearranging, we get,

$$f(z) = \ldots\ldots - z^{-3} - z^{-2} - z^{-1} - \frac{1}{2} - \frac{1}{4}z - \frac{1}{8}z^2 - \frac{1}{16}z^3 \ldots\ldots$$

Que 5.16. Obtain the Taylor's series expansion of $f(z) = \dfrac{1}{z^2 - 4z + 3}$ about the point $z = 4$. Find its region of convergence.

AKTU 2013-14 (IV), Marks 05

Answer

If the centre of the circle is at $z = 4$, then the distances of the singularities $z = 1$ and $z = 3$ from centre are 3 and 1.

Hence if a circle is drawn with centre at $z = 4$ and radius 1 then within circle $|z - 4| = 1$, the given function $f(z)$ is analytic hence it can be expanded in Taylor's series within the circle $|z - 4| = 1$ which is therefore the region of convergence.

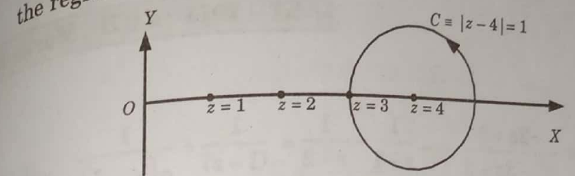


Fig. 5.16.1.

$$f(z) = \frac{1}{(z-1)(z-3)} = \frac{1}{2}\left[\frac{1}{z-3} - \frac{1}{z-1}\right] = \frac{1}{2}\left[\frac{1}{z-4+1} - \frac{1}{z-4+3}\right]$$

$$= \frac{1}{2}\left[\{1 + (z-4)\}^{-1} - \frac{1}{3}\left\{1 + \left(\frac{z-4}{3}\right)\right\}^{-1}\right]$$

$$f(z) = \frac{1}{2}\left[\sum_{n=0}^{\infty} (-1)^n (z-4)^n - \frac{1}{3}\sum_{n=0}^{\infty} (-1)^n \left(\frac{z-4}{3}\right)^n\right]$$

Que 5.17. Find the Taylor series expansion of the function $\tan^{-1} z$ about the point $z = \pi/4$.

AKTU 2014-15 (III), Marks 05

Answer

$$f(z) = \tan^{-1} z$$

$$f'(z) = \frac{1}{1+z^2}$$

$$f''(z) = \frac{-2z}{(1+z^2)^2}$$

$$f'''(z) = -2\left[\frac{(1+z^2)^2 - 4z^2(1+z^2)}{(1+z^2)^4}\right] = -2\left[\frac{1+z^2-4z^2}{(1+z^2)^3}\right] = \frac{2(3z^2-1)}{(1+z^2)^3}$$

$$f'\left(\frac{\pi}{4}\right) = \tan^{-1}\left(\frac{\pi}{4}\right) = 0.6658, f'\left(\frac{\pi}{4}\right) = 0.6185$$

$$f''\left(\frac{\pi}{4}\right) = \frac{-2(0.785)}{2.6142} = -0.60087$$

Thus,

$$\tan^{-1} z = 0.6658 + \left(z - \frac{\pi}{4}\right)(0.6185) + \frac{\left(z-\frac{\pi}{4}\right)^2}{2!}(-0.60087) + ...$$

Que 5.18. **Find all Taylor and Laurent series expansion of the following function about $z = 0$**

$$f(z) = \frac{-2z+3}{z^2-3z+2}$$

AKTU 2013-14 (III), Marks 05

Answer

$$f(z) = \frac{-2z+3}{z^2-3z+2} = -\frac{1}{z-1} - \frac{1}{z-2} = \frac{1}{(1-z)} + \frac{1}{2\left(1-\frac{z}{2}\right)} \quad ...(5.18.1)$$

$$= (1-z)^{-1} + \frac{1}{2}\left(1-\frac{z}{2}\right)^{-1}$$

Now expanding by binomial expansion

$$f(z) = (1 + z + z^2 + z^3 + ...) + \frac{1}{2}\left[1 + \frac{z}{2} + \left(\frac{z}{2}\right)^2 + \left(\frac{z}{2}\right)^3 + ...\right]$$

or
$$f(z) = \sum_{n=0}^{n} (1)^n z^n + \frac{1}{2}\sum_{n=0}^{n}(1)^n \left(\frac{z}{2}\right)^n$$

This is the Taylor's series expansion of given function.

Eq. (5.18.1) can also be written as,

$$f(z) = -\frac{1}{z}\left(1-\frac{1}{z}\right)^{-1} - \frac{1}{z}\left(1-\frac{2}{z}\right)^{-1}$$

Now expanding by binomial expansion we get

$$f(z) = -\frac{1}{z}\left[1 + \frac{1}{z} + \left(\frac{1}{z}\right)^2 + \left(\frac{1}{z}\right)^3 + ...\right] - \frac{1}{z}\left[1 + \frac{2}{z} + \left(\frac{2}{z}\right)^2 + \left(\frac{2}{z}\right)^3 + ...\right]$$

$$f(z) = -\frac{1}{z}\sum_{n=0}^{n}(1)^n \frac{1}{z^n} - \frac{1}{z}\sum_{n=0}^{n}(1)^n\left(\frac{2}{z}\right)^n$$

This is the Laurent's series expansion of given function.

Que 5.19. **Find the Laurent series for the function**

$$f(z) = \frac{7z^2+9z-18}{z^3-9z}$$, **z is complex variable valid for the regions**

i. $0 < |z| < 3$ **ii.** $|z| > 3$

AKTU 2015-16 (IV), Marks 10

AKTU 2012-13 (IV), Marks 05

Answer

$$f(z) = \frac{7z^2+9z-18}{z^3-9z}$$

Using partial fraction,

$$\frac{7z^2+9z-18}{z^3-9z} = \frac{A}{z} + \frac{B}{z-3} + \frac{C}{z+3}$$

$$A = \left.\frac{7z^2+9z-18}{(z-3)(z+3)}\right|_{z=0} = \frac{-18}{-3\times 3} = 2$$

$$B = \left.\frac{7z^2+9z-18}{z(z+3)}\right|_{z=3} = 4$$

$$C = \left.\frac{7z^2+9z-18}{z(z-3)}\right|_{z=-3} = 1$$

i. $0 < |z| < 3$

Rearrangement of function $f(z)$,

$$f(z) = \frac{2}{z} - \frac{4}{3\left(1-\frac{z}{3}\right)} + \frac{1}{3}\left(1+\frac{z}{3}\right)^{-1}$$

$$f(z) = \frac{2}{z} - \frac{4}{3}\left(1-\frac{z}{3}\right)^{-1} + \frac{1}{3}\left(1+\frac{z}{3}\right)^{-1}$$

$$\therefore \quad f(z) = \frac{2}{z} - \frac{4}{3}\sum_{n=0}^{\infty}\left(\frac{z}{3}\right)^n + \frac{1}{3}\sum_{n=0}^{\infty}(-1)^n\left(\frac{z}{3}\right)^n$$

ii. $|z| > 3$

$$f(z) = \frac{2}{z} + \frac{4}{z\left(1-\frac{3}{z}\right)} + \frac{1}{z\left(1+\frac{3}{z}\right)} = \frac{2}{z} + \frac{4}{z}\left(1-\frac{3}{z}\right)^{-1} + \frac{1}{z}\left(1+\frac{3}{z}\right)^{-1}$$

$$\therefore \quad f(z) = \frac{2}{z} + \frac{4}{z}\sum_{n=0}^{\infty}\left(\frac{3}{z}\right)^n + \frac{1}{z}\sum_{n=0}^{\infty}(-1)^n\left(\frac{3}{z}\right)^n$$

Que 5.20. **Expand $f(z) = \frac{z}{(z-1)(2-z)}$ in Laurent series valid for**

i. $|z-1| > 1$ **and** **ii.** $0 < |z-2| < 1$.

AKTU 2012-13 (III), Marks 05

Answer

$$f(z) = \frac{z}{(z-1)(2-z)}$$

$$f(z) = \frac{1}{z-1} - \frac{2}{z-2}$$

i. $|z-1| > 1$

$$f(z) = \frac{1}{z-1} - \frac{2}{(z-1)-1} = \frac{1}{z-1} - \frac{2}{(z-1)}\left[1-\frac{1}{z-1}\right]^{-1}$$

$$= \frac{1}{z-1} - \frac{2}{(z-1)}\left[1+\frac{1}{z-1}+\frac{1}{(z-1)^2}+\ldots\right]$$

$$f(z) = \frac{1}{z-1} - 2\sum_{n=0}^{\infty}\frac{1}{(z-1)^{n+1}}$$

ii. $0 < |z-2| < 1$

$$f(z) = \frac{1}{(z-2)+1} - \frac{2}{z-2} = [1+(z-2)]^{-1} - \frac{2}{z-2}$$

$$f(z) = [1-(z-2)+(z-2)^2-(z-2)^3+\ldots] - \frac{2}{z-2}$$

$$f(z) = -\left(\frac{2}{z-2}\right) + \sum_{n=0}^{\infty}(-1)^n (z-2)^n$$

Que 5.21. **Find the Laurent series expansion of**

$f(z) = \dfrac{7z-2}{z(z+1)(z+2)}$ **in the region** $1 < |z+1| < 3$.

AKTU 2016-17 (III), Marks 05

Answer

Same as Q. 5.20, Page 5–17F, Unit-5.

$$\left(\textbf{Answer : } f(z) = -\sum_{n=0}^{\infty}\frac{1}{(z+1)^{n+1}} + \frac{9}{z+1} - 8\sum_{n=0}^{\infty}\frac{(-1)^n}{(z+1)^{n+1}}\right)$$

PART-3

Singularities, Classification of Singularities, Zeros of Analytic Function.

CONCEPT OUTLINE

Singularity : A singularity of a function $f(z)$ is a point at which the function ceases to be analytic.

Types of Singularities :

i. **Isolated Singularity :** If $z = a$ is a singularity of $f(z)$ such that $f(z)$ is analytic at each point in its neighbourhood (*i.e.*, there exists a circle with centre a which has no other singularity), then $z = a$ is called an isolated singularity.

In such a case, $f(z)$ can be expanded in a Laurent's series around $z = a$, giving

$f(z) = a_0 + a_1(z-a) + a_2(z-a)^2 + \ldots + a_{-1}(z-a)^{-1} + a_{-2}(z-a)^{-2} + \ldots$ ....(1)

For example, $f(z) = \cot(\pi/z)$ is not analytic where as $\tan(\pi/z) = 0$ *i.e.*, at the points $\pi/z = 4\pi$ or $z = 1/n$ $(n = 1, 2, 3, \ldots)$.

Thus $z = 1, 1/2, 1/3, \ldots$ are all isolated singularities as there is no other singularity in their neighbourhood.

But when n is large, $z = 0$ is such a singularity that there are infinite number of other singularities in its neighbourhood.

Thus $z = 0$ is the non-isolated singularity of $f(z)$.

ii. **Removable singularity :** If all the negative powers of $(z-a)$ in eq. (1) are zero, then $f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n$. Here the singularity can be removed by defining $f(z)$ at $z = a$ in such a way that it becomes analytic at $z = a$. Such a singularity is called removable singularity.

Thus if $\lim_{z \to 0}$ exists finitely, then $z = a$ is a removable singularity.

iii. **Poles :** If all the negative powers of $(z-a)$ in eq. (1) after the n^{th} are missing, then the singularity at $z = a$ is called a pole of order n. A pole of first order is called a simple pole.

iv. **Essential Singularity :** If the number of negative powers of $(z-a)$ in eq. (1) is infinite, then $z = a$ is called an essential singularity. In this case, $\lim_{z \to a} f(z)$ does not exist.

Zeros of an Analytic Function : A zero of an analytic function $f(z)$ is that value for z for which $f(z) = 0$

Questions-Answers

Long Answer Type and Medium Answer Type Questions

Que 5.22. **Find the nature and location of singularities of the following functions :**

i. $\dfrac{z - \sin z}{z^2}$

ii. $(z+1)\sin\dfrac{1}{z-2}$

iii. $\dfrac{1}{\cos z - \sin z}$

Answer

i. Here, $z = 0$ is a singularity.

Also, $\dfrac{z-\sin z}{z^2} = \dfrac{1}{z^2}\left\{z-\left(z-\dfrac{z^3}{3!}+\dfrac{z^5}{5!}-\dfrac{z^7}{7!}+.....\right)\right\} = \dfrac{z}{3!}-\dfrac{z^3}{5!}+\dfrac{z^5}{7!}.....$

Since there are no negative powers of z in the expansion, $z = 0$ is a removable singularity.

ii. $$(z+1)\sin\frac{1}{z-2} = (t+2+1)\sin\frac{1}{t} \qquad \text{Where, } t = z-2$$

$$= (t+3)\left\{\frac{1}{t}-\frac{1}{3!\,t^3}+\frac{1}{5!\,t^5}-...\right\}$$

$$= \left(1-\frac{1}{3!\,t^2}+\frac{1}{5!\,t^4}-...\right)+\left(\frac{3}{t}-\frac{1}{2t^3}+\frac{3}{5!\,t^5}-...\right)$$

$$= 1+\frac{3}{t}-\frac{1}{6t^2}-\frac{1}{2t^3}+\frac{1}{120t^4}-.....$$

$$= 1+\frac{3}{z-2}-\frac{1}{6(z-2)^2}-\frac{1}{2(z-2)^3}+....$$

Since there are infinite number of terms in the negative powers of $(z-2)$, $z = 2$ is an essential singularity.

iii. Poles of $f(z) = \dfrac{1}{\cos z - \sin z}$ are given by equating the denominator to zero, *i.e.*, $\cos z - \sin z = 0$ or $\tan z = 1$ or $z = \pi/4$. Clearly $z = \dfrac{\pi}{4}$ is a simple pole of $f(z)$.

PART-4

Residues, Methods of Finding Residues, Cauchy Residue Theorem.

CONCEPT OUTLINE

Residues : The coefficient of $(z-a)^{-1}$ in the expansion of $f(z)$ around an isolated singularity is called the residue of $f(z)$ at that point. Thus in the Laurent's series expansion of $f(z)$ around $z = a$ *i.e.*, $f(z) = a_0 + a_1(z-a) + a_2(z-a)^2 + ... + a_{-1}(z-a)^{-1} + a_{-2}(z-a)^{-2} + ...$, the residue of $f(z)$ at $z = a$ is a_{-1}.

$$\therefore \qquad \text{Res } f(a) = \frac{1}{2\pi i}\oint_C f(z)dz$$

i.e., $$\oint_C f(z)dz = 2\pi i \text{ Res } f(a)$$

Cauchy's Residue Theorem or Theorem of Residues :

If a function $f(z)$ is analytic, except at a finite number of poles within a closed contour C and continuous on the boundary C, then

$$\oint_C f(z)dz = 2\pi i\,\Sigma R$$

$$= 2\pi i\begin{Bmatrix}\text{Sum of residues of } f(z) \text{ at its}\\ \text{poles within } C\end{Bmatrix}$$

Questions-Answers

Long Answer Type and Medium Answer Type Questions

Que 5.23. **Find the residues of $f(z) = \dfrac{z-3}{z^2+2z+5}$ at its poles. Hence or otherwise evaluate $\oint_C \dfrac{z-3}{z^2+2z+5}$, where C is the circle $|z+1-i| = 2$.**

AKTU 2012-13 (IV), Marks 05

Answer

The poles of $f(z) = \dfrac{z-3}{z^2+2z+5}$ are given by

$$z^2 + 2z + 5 = 0 \Rightarrow z = -1 \pm 2i$$

Only the pole $z = -1 + 2i$ lies inside the circle $|z+1-i| = 2$

Residue of $f(z)$ at $z = -1 + 2i$ is

$$= \lim_{z\to-1+2i}(z+1-2i)f(z)$$

$$= \lim_{z\to-1+2i}\frac{(z-\alpha)(z-3)}{z^2+2z+5}, \text{ where } \alpha = -1+2i\left(\text{Form}\frac{0}{0}\right)$$

$$= \lim_{z\to\alpha}\frac{(z-\alpha)+(z-3)}{2z+2} \qquad \text{(By L' Hospital's Rule)}$$

$$= \frac{\alpha-3}{2\alpha+2} = \frac{-1+2i-3}{-2+4i+2} = \frac{i-2}{2i}$$

$\therefore$ By Cauchy' residue theorem,

$$\oint_C \frac{z-3}{z^2+2z+5}dz = 2\pi i\left(\frac{i-2}{2i}\right) = \pi(i-2)$$

Que 5.24. **Determine the poles and residues at each pole for $f(z) = \dfrac{z-1}{(z+1)^2(z-2)}$ and hence evaluate $\oint_C f(z)dz$ where C is the circle $|z-i| = 2$.**

AKTU 2013-14 (IV), Marks 05

Answer

Poles of $f(z)$ are given by

$(z+1)^2(z-2) = 0$, $z = -1$ (Order 2), 2 (Simple pole)

Residue of $f(z)$ at $z = -1$ is,

$$R_1 = \frac{1}{(2-1)!}\left[\frac{d}{dz}\left\{(z+1)^2\frac{z-1}{(z+1)^2(z-2)}\right\}\right]_{z=-1}$$

$$= \left[\frac{d}{dz}\left(\frac{z-1}{z-2}\right)\right]_{z=-1} = \left[\frac{-1}{(z-2)^2}\right]_{z=-1} = \frac{-1}{9}$$

Residue of $f(z)$ at $z = 2$ is,

$$R_2 = \lim_{z\to 2}(z-2)\frac{z-1}{(z+1)^2(z-2)} = \lim_{z\to 2}\frac{z-1}{(z+1)^2} = \frac{1}{9}$$

The given curve $C = |z-i| = 2$ is a circle whose centre is at $z = i$ [*i.e.*, at (0, 1)] and radius is 2. Clearly, only the pole $z = -1$ lies inside the curve C.

Hence, by Cauchy's residue theorem

$$\oint_C f(z)\,dz = 2\pi i\,(R_1) = 2\pi i\left(\frac{-1}{9}\right) = -\frac{2\pi i}{9}$$

Que 5.25. **Determine the poles of the following function and residue at each pole :**

$$f(z) = \frac{z^2}{(z-1)^2(z+2)}$$ **and hence evaluate**

$\int_C f(z)\,dz$, **where** $C : |z| = 3$.

AKTU 2014-15 (IV), Marks 05

Answer

Same as Q. 5.24, Page 5–21F, Unit-5. (**Answer :** $2\pi i$)

Que 5.26. **Find the poles (with its order) and residue at each poles of the following function :**

$$f(z) = \frac{1-2z}{z(z-1)(z-2)^2}$$

AKTU 2016-17 (III), Marks 05

Answer

Same as Q. 5.24, Page 5–21F Unit-5. $\left(\textbf{Answer :} \text{ Residues are } -\frac{1}{4}, -1, \frac{5}{4}\right)$

Que 5.27. **Evaluate** $\int_C \frac{e^z}{\cos \pi z}\,dz$, **where C is the unit circle** $|z| = 1$.

Answer

Here, $$f(z) = \frac{e^z}{\cos \pi z} = \frac{e^z}{\left[1 - \frac{(\pi z)^2}{2!} + \frac{(\pi z)^4}{4!} - \dots\right]}$$

It has simple poles at $z = \pm\frac{1}{2}, \pm\frac{3}{2}, \pm\frac{5}{2}, \dots$, of which only $z = \pm\frac{1}{2}$ lie inside the circle $|z| = 1$.

Residue of $f(z)$ at $z = \frac{1}{2}$ is

$$\lim_{z\to\frac{1}{2}}\left(z-\frac{1}{2}\right)f(z) = \lim_{z\to\frac{1}{2}}\frac{\left(z-\frac{1}{2}\right)e^z}{\cos \pi z} \qquad \left[\text{Form } \frac{0}{0}\right]$$

$$= \lim_{z\to\frac{1}{2}}\frac{\left(z-\frac{1}{2}\right)e^z}{-\pi \sin \pi z} \qquad \text{[By L' Hospital's Rule]}$$

$$= \frac{e^{1/2}}{-\pi}$$

Similarly, residue of $f(z)$ at $z = \frac{1}{-2}$ is $\frac{e^{-1/2}}{\pi}$

$\therefore$ By residue theorem,

$$\oint_C \frac{e^z}{\cos \pi z}\,dz = 2\pi i \quad \text{(Sum of residues)}$$

$$= 2\pi i\left(-\frac{e^{1/2}}{\pi} + \frac{e^{-1/2}}{\pi}\right) = -4i\left(\frac{e^{1/2} - e^{-1/2}}{2}\right) = -4i \sinh\frac{1}{2}$$

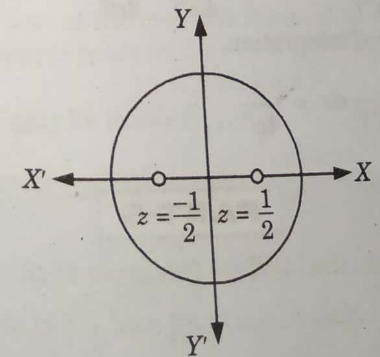


Fig. 5.27.1.

Que 5.28. Using calculus of residue, evaluate the following integral

$$\int_0^\infty \frac{dx}{(a^2+x^2)^2}.$$

AKTU 2015-16 (IV), Marks 10

Answer

Let, $$I = \int_0^\infty \frac{dx}{(x^2+a^2)^2}$$

$$f(x) = \frac{1}{(x^2+a^2)^2}$$

Poles, $(x^2+a^2)^2 = 0$

$\Rightarrow$ $x^2+a^2 = 0$

$\therefore$ $x = \pm ai$

Only one pole but repeated nature.

Residue at $x = ai$

$$= \frac{1}{(2-1)!}\left[\frac{d}{dx}\left\{(x-ai)^2 \times \frac{1}{(x-ai)^2(x+ai)^2}\right\}\right]_{x=ai}$$

$$= \frac{1}{(2-1)!}\left[\frac{d}{dx}\left(\frac{1}{x+ai}\right)^2\right]_{x=ai} = \left[\frac{-2}{(x+ai)^3}\right]_{x=ai}$$

$$= \frac{-1}{-4a^3i} = \frac{1}{4a^3i}$$

Using Cauchy's Residue theorem,

$$\int_{-\infty}^{\infty} f(x)dx = 2\pi i \text{ [Sum of residue]}$$

$$\int_{-\infty}^{\infty} \frac{1}{(x^2+a^2)^2}dx = 2\pi i \times \frac{1}{4a^3i} = \frac{\pi}{2a^3}$$

Using property of integration,

$$\int_0^\infty \frac{1}{(x^2+a^2)^2}dx = \frac{\pi}{4a^3}$$

PART-5

Evaluation of Real Integrals of the Type $\int_{-\infty}^{\infty} f(\cos\theta, \sin\theta)\, d\theta$ *and* $\int_{-\infty}^{\infty} f(x)\, dx$.

CONCEPT OUTLINE

Evaluation of Real Integral of Type $\int_0^{2\pi} f(\cos\theta, \sin\theta)d\theta$, $\int_{-\infty}^{+\infty} f(x)dx$ **:**

Integrals of the type $\int_0^{2\pi} f(\cos\theta, \sin\theta)d\theta$, where $f(\cos\theta, \sin\theta)$ is a rational function of $\cos\theta$ and $\sin\theta$.

$$\int_0^{2\pi} f(\cos\theta, \sin\theta)d\theta = \oint_C f\left(\frac{z+z^{-1}}{2}, \frac{z-z^{-1}}{2i}\right)\frac{dz}{iz}$$

where C is a unit circle of $|z| = 1$.

Questions-Answers

Long Answer Type and Medium Answer Type Questions

Que 5.29. Use calculus of residue to evaluate the following integral

$$\int_{-\infty}^{\infty} \frac{\cos x}{(x^2+a^2)(x^2+b^2)}dx$$

AKTU 2016-17 (III), Marks 10

Answer

We consider $\int_{-\infty}^{\infty} \frac{\cos x}{(x^2+a^2)(x^2+b^2)}dx = \int_C f(z)dz$

Where C is the contour consisting of the semi-circle C_R of radius R together with the part of the real axis from $-R$ to $+R$.

The integral has simple poles at

$$z = \pm ai, z = \pm bi$$

of which $z = ai, bi$ only lie inside C.

The residue (at $z = ai$) $= \lim_{z \to ai}(z-ai)\frac{\cos z\, dz}{(z^2+a^2)(z^2+b^2)}$

$$= \lim_{z\to ai}(z-ai)\frac{\cos z\, dz}{(z-ai)(z+ai)(z^2+b^2)} = \lim_{z\to ai}\frac{\cos z\, dz}{(z+ai)(z^2+b^2)}$$

$$= \left[\frac{\cos ai}{(ai+ai)((ai)^2+b^2)}\right] = \frac{\cos ai}{2ai(b^2-a^2)}$$

The residue (at $z = bi$) $= \lim_{z\to bi}(z-bi)\dfrac{\cos z\,dz}{(z^2+a^2)(z-bi)(z+bi)}$

$$= \lim_{z\to bi}\frac{\cos z\,dz}{(z^2+a^2)(z+bi)} = \left[\frac{\cos bi}{((bi)^2+a^2)(bi+bi)}\right] = \frac{\cos bi}{(a^2-b^2)2bi}$$

Sum of Residues $(R) = \dfrac{\cos ai}{2ai(b^2-a^2)} + \dfrac{\cos bi}{(a^2-b^2)2bi}$

$$= \frac{1}{2i}\left[\frac{\cos ai}{a(b^2-a^2)} + \frac{\cos bi}{b(a^2-b^2)}\right] = \frac{1}{2i}\left[-\frac{\cos ai}{a(a^2-b^2)} + \frac{\cos bi}{b(a^2-b^2)}\right]$$

$$= \frac{1}{2i}\left[\frac{\cos bi}{b(a^2-b^2)} - \frac{\cos ai}{a(a^2-b^2)}\right] = \frac{1}{2i(a^2-b^2)}\left[\frac{\cos bi}{b} - \frac{\cos ai}{a}\right]$$

$\therefore$ Using Cauchy's Residue theorem,

$$\int_{-\infty}^{\infty}\frac{\cos x\,dx}{(x^2+a^2)(x^2+b^2)} = 2\pi i\frac{1}{2i(a^2-b^2)}\left[\frac{\cos bi}{b} - \frac{\cos ai}{a}\right]$$

$$= \text{Re}\left[\frac{\pi}{(a^2-b^2)}\left(\frac{\cos bi}{b} - \frac{\cos ai}{a}\right)\right]$$

Que 5.30. **Using complex integration method, evaluate** $\displaystyle\int_0^{\pi}\frac{1}{3+\sin^2\theta}\,d\theta$.

AKTU 2012-13 (IV), Marks 05

Answer

$$I = \int_0^{\pi}\frac{1}{3+\sin^2\theta}\,d\theta = \int_0^{\pi}\frac{1}{3+\frac{1}{2}(1-\cos 2\theta)}\,d\theta$$

$$\left[\because \sin^2\theta = \frac{1-\cos 2\theta}{2}\right]$$

$$= 2\int_0^{\pi}\frac{1}{7-\cos 2\theta}\,d\theta$$

Put $2\theta = \phi$, $d\theta = \dfrac{d\phi}{2}$

$$= \int_0^{2\pi}\frac{1}{7-\cos\phi}\,d\phi = \int_0^{2\pi}\frac{1}{7-\dfrac{(e^{i\phi}+e^{-i\phi})}{2}}\,d\phi \quad \left[\because \cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2}\right]$$

$$= \int_0^{2\pi}\frac{2}{14-(e^{i\phi}+e^{-i\phi})}\,d\phi \qquad ...(5.30.1)$$

But $z = e^{i\phi}$ so that $d\phi = \dfrac{dz}{iz}$ then eq. (5.30.1) reduces to,

$$I = 2\int_C\frac{1}{14-\left(z+\frac{1}{z}\right)}\frac{dz}{iz} = \frac{2}{i}\int_C\frac{dz}{14z-z^2-1}$$

$$= 2i\int_C\frac{dz}{z^2-14z+1} = \int_C\frac{2i}{(z-\alpha)(z-\beta)}\,dz \qquad [\because\ z^2-14z+1=0]$$

Where, $\alpha = 7+4\sqrt{3}$ and $\beta = 7-4\sqrt{3}$

Here $\beta < 1$, so only β lies inside C.

Residue at $(z=\beta) = \lim_{z\to\beta}(z-\beta)\times\dfrac{2i}{(z-\alpha)(z-\beta)}$

$$= \frac{2i}{\beta-\alpha} = \frac{2i}{7-4\sqrt{3}-7-4\sqrt{3}} = -\frac{i}{4\sqrt{3}}$$

By Cauchy Residue theorem,

$$\int_0^{\pi}\frac{1}{3+\sin^2\theta}\,d\theta = 2\pi i\left(\frac{-i}{4\sqrt{3}}\right) = \frac{2\pi}{4\sqrt{3}} = \frac{\pi}{2\sqrt{3}}$$

Que 5.31. **Use contour integral to evaluate** $\displaystyle\int_0^{2\pi}\frac{d\theta}{3-2\cos\theta+\sin\theta}$.

AKTU 2012-13 (III), 2013-14 (III); Marks 05

Answer

Here; $\displaystyle I = \int_0^{2\pi}\frac{d\theta}{3-2\cos\theta+\sin\theta}$

$$= \oint_C f(z)\,dz, \text{ where } C \text{ is the unit circle } |z| = 1.$$

We know that, $\quad z = e^{i\theta}$ and $d\theta = \dfrac{dz}{iz}$,

$$\therefore\ I = \oint_C\frac{1}{3-\left(\dfrac{z^2+1}{z}\right)+\left(\dfrac{z^2-1}{2iz}\right)}\frac{dz}{iz} \quad \left[\begin{aligned}&\because \cos\theta = \frac{e^{\theta i}+e^{-\theta i}}{2} = \frac{z^2+1}{2z}\\ &\text{and } \sin\theta = \frac{e^{\theta i}-e^{-\theta i}}{2i} = \frac{z^2-1}{2iz}\end{aligned}\right]$$

$$I = \oint_C\frac{1}{3-z-\dfrac{1}{z}-\dfrac{zi}{2}+\dfrac{i}{2z}}\frac{dz}{iz} = \frac{2}{i}\oint_C\frac{dz}{6z-2z^2-2-iz^2+i}$$

$$= -\frac{2}{i}\oint_C\frac{dz}{(i+2)z^2-6z-(i-2)} = -\frac{2}{i}\oint_C\frac{dz}{(i+2)z^2-5z-z-(i-2)}$$

$$= -\frac{2}{i}\oint_C\frac{dz}{z[(i+2)z-5]-1[z+(i-2)]}$$

$$= -\frac{2}{i}\oint_C\frac{dz}{z[(i+2)z-5]-\dfrac{1}{(i+2)}[(i+2)z+(i-2)(i+2)]}$$

$$= -\frac{2}{i}\oint_C \frac{dz}{[z(i+2)-5]\left[z-\frac{1}{i+2}\right]} = -\frac{2}{i}\oint_C \frac{dz}{[z(i+2)-5]\left[z+\frac{i-2}{5}\right]}$$

Poles are $(2-i)$ and $\left(\frac{2-i}{5}\right)$. The only pole which lie inside C is

$z = \frac{2-i}{5}$.

$$\text{Residue at } z = \frac{2-i}{5} = \lim_{z\to\frac{2-i}{5}} \left(z+\frac{i-2}{5}\right) f(z)$$

$$= \lim_{z\to\left(\frac{2-i}{5}\right)} \left(-\frac{2}{i}\,\frac{1}{z(i+2)-5}\right) = \frac{1}{2i}$$

By Cauchy's residue theorem,

$$\oint_C f(z)\,dz = 2\pi i \text{ (Sum of all residues)}$$

$$\int_0^{2\pi} \frac{d\theta}{3-2\cos\theta+\sin\theta} = 2\pi i\left(\frac{1}{2i}\right) = \pi$$

Que 5.32. **Evaluate $\int_0^{2\pi} \frac{\cos 3\theta}{5+4\cos\theta}\,d\theta$.**

AKTU 2013-14 (IV), Marks 05

Answer

Let $I = \int_0^{2\pi} \frac{\cos 3\theta}{5+4\cos\theta}\,d\theta$ = Real part of $\int_0^{2\pi} \frac{e^{3i\theta}}{5+2(e^{i\theta}+e^{-i\theta})}\,d\theta$

$$= \text{Real part of } \oint_C \left|\frac{z^3}{5+2\left(z+\frac{1}{z}\right)}\right| \frac{dz}{iz} \qquad \left(\text{Writing } e^{i\theta}=z,\ d\theta=\frac{dz}{iz}\right)$$

$$= \text{Real part of } \frac{1}{i}\int_C \frac{z^3}{(2z+1)(z+2)}\,dz$$

Singularities of $f(z)$ are given by, $(2z+1)(z+2) = 0$

$$z = -\frac{1}{2},\ -2$$

Only, $z = -\frac{1}{2}$ lies within the unit circle $|z| = 1$.

$$\therefore \text{ Residue of } f(z) \left(\text{at } z=-\frac{1}{2}\right) = \lim_{z\to-\frac{1}{2}} \left(z+\frac{1}{2}\right) \times \frac{z^3}{i(2z+1)(z+2)}$$

$$= \lim_{z\to-\frac{1}{2}} \frac{z^3}{2i(z+2)} = \frac{1}{2i}\left(\frac{-1}{8}\right)\times\left(\frac{2}{3}\right) = \frac{-1}{24i}$$

Hence by Cauchy's Residue theorem

$$I = \oint_C f(z)\,dz = 2\pi i\left(\frac{-1}{24i}\right) = -\frac{\pi}{12}$$

Que 5.33. **Evaluate the integral $\int_0^{\pi} \frac{\cos^2 3\theta}{5-4\cos 2\theta}\,d\theta$.**

AKTU 2014-15 (III), Marks 05

Answer

Same as Q. 5.32, Page 5–28F, Unit-5. $\left(\textbf{Answer}: \frac{3\pi}{32}\right)$

Que 5.34. **Evaluate : $\int_0^{2\pi} \frac{d\theta}{a+b\sin\theta}$ if $a > |b|$**

AKTU 2016-17 (IV), Marks 05

Answer

Consider the integration round a unit circle C = $|z| = 1$

So that $\qquad z = e^{i\theta} \quad \therefore \quad d\theta = \frac{dz}{iz}$

Also, $\qquad \sin\theta = \frac{1}{2i}(e^{i\theta}-e^{-i\theta}) = \frac{1}{2i}\left(z-\frac{1}{z}\right)$

Then the given integral reduces to

$$I = \oint_C \frac{1}{\left[a+\frac{b}{2i}\left(z-\frac{1}{z}\right)\right]}\left(\frac{dz}{iz}\right) = \oint_C \frac{2iz}{bz^2+2iaz-b}\left(\frac{dz}{iz}\right)$$

$$= \frac{2}{b}\oint_C \frac{dz}{z^2+\frac{2ia}{b}z-1}$$

Poles are given by, $z^2 + \frac{2ia}{b}z - 1 = 0$

$$z = \frac{-\frac{2ia}{b} \pm \sqrt{-\frac{4a^2}{b^2}+4}}{2} = \frac{-ia}{b} \pm \frac{\sqrt{b^2-a^2}}{b}$$

$$= \frac{-ia}{b} \pm \frac{i\sqrt{a^2-b^2}}{b} = \alpha, \beta \text{ (simple poles)}$$

where, $\alpha = \frac{-ia}{b} + \frac{i\sqrt{a^2-b^2}}{b}$ and $\beta = \frac{-ia}{b} - \frac{i\sqrt{a^2-b^2}}{b}$

Clearly, $|\beta| > 1$

But $\alpha\beta = -1$

$\therefore$ $|\alpha\beta| = 1$

$|\alpha|\,|\beta| = 1$

$|\alpha| < 1$

Hence $z = \alpha$ is the only pole which lies inside circle $C = |z| = 1$.

Residue of $f(z)$ at $(z = \alpha)$ is

$$R = \lim_{z\to\alpha}(z-\alpha) \times \frac{2}{b(z-\alpha)(z-\beta)} = \frac{2}{b(\alpha-\beta)}$$

$$= \frac{2}{b\left(\frac{2i\sqrt{a^2-b^2}}{b}\right)} = \frac{1}{i\sqrt{a^2-b^2}}$$

$\therefore$ By Cauchy's Residue theorem,

$$I = 2\pi i(R) = 2\pi i\left(\frac{1}{i\sqrt{a^2-b^2}}\right) = \frac{2\pi}{\sqrt{a^2-b^2}}$$

$$\therefore \quad \int_0^{2\pi} \frac{d\theta}{a + b\sin\theta} = \frac{2\pi}{\sqrt{a^2-b^2}}$$

Que 5.35. **Evaluate the integral :** $\int_0^{2\pi} \frac{d\theta}{5 - 3\cos\theta}$

AKTU 2014-15 (IV), Marks 05

Answer

Same as Q. 5.34, Page 5–29F, Unit-5. $\left(\textbf{Answer : } \frac{\pi}{2}\right)$

Que 5.36. **Using complex variable techniques evaluate the real integral** $\int_0^{2\pi} \frac{\sin 2\theta}{5 - 4\cos\theta}\, d\theta$

AKTU 2017-18 (III), Marks 10

Answer

The given integral, $I = \int_0^{2\pi} \frac{\sin 2\theta}{5 - 4\cos\theta}\, d\theta$ ...(5.36.1)

$$\sin 2\theta = \frac{1}{2i}(e^{2i\theta} - e^{-2i\theta})$$

$$\cos\theta = \frac{1}{2}(e^{i\theta} + e^{-i\theta})$$

Putting $z = e^{i\theta}$, $d\theta = \frac{dz}{iz}$ in eq. (5.36.1), we get

$$I = \oint_C \frac{\frac{1}{2i}\left(z^2 - \frac{1}{z^2}\right)}{5 - 4 \times \frac{1}{2}\left(z + \frac{1}{z}\right)} \frac{dz}{iz}$$

$$= \frac{1}{2i} \oint_C \frac{z^4 - 1}{z^2\left(5 - 2\left(\frac{z^2+1}{z}\right)\right)} \frac{dz}{iz}$$

$$= \frac{1}{2i^2} \oint_C \frac{z^4 - 1}{z^2 \frac{(5z - 2z^2 - 2)}{z}} \frac{dz}{z}$$

$$= \frac{1}{2} \oint_C \frac{z^4 - 1}{z^2 (2z^2 - 5z + 2)} dz$$

$$= \frac{1}{2} \oint_C \frac{z^4 - 1}{z^2 (2z-1)(z-2)} dz$$

$$= \frac{1}{2} \oint_C f(z)dz, \text{ where } C \text{ is the unit circle } |z| = 1.$$

Now $f(z)$ has a pole of order 2 at $z = 0$ and simple poles at $z = 1/2$ and $z = 2$, of these only $z = 0$ and $z = 1/2$ lie within the circle.

$$\therefore \quad \text{Res } f\left(\frac{1}{2}\right) = \lim_{z\to 1/2}\left(z - \frac{1}{2}\right) \frac{(z^4 - 1)}{z^2(2z-1)(z-2)}$$

$$= \lim_{z\to 1/2}\left[\frac{z^4 - 1}{2z^2(z-2)}\right]$$

$$= \frac{\frac{1}{16} - 1}{2 \times \frac{1}{4}\left(\frac{1}{2} - 2\right)} = \frac{\frac{-15}{16}}{\frac{1}{2} \times \left(\frac{-3}{2}\right)} = \frac{5}{4}$$

$$\text{Res } f(0) = \frac{1}{(n-1)!}\left\{\frac{d^{n-1}}{dz^{n-1}}[(z-0)^n f(z)]\right\}_{z=0}$$

$$= \frac{1}{(2-1)!} \frac{d^{2-1}}{dz^{2-1}}\left[(z-0)^2 \times \frac{z^4 - 1}{z^2(2z-1)(z-2)}\right]_{z=0}$$

$(\because\ n = 2)$

$$= \left[\frac{d}{dz} \times z^2 \frac{(z^4 - 1)}{z^2(2z-1)(z-2)}\right]_{z=0}$$

$$= \left[\frac{d}{dz}\frac{z^4 - 1}{(2z-1)(z-2)}\right]_{z=0}$$

$$= \left\{\frac{(2z-1)(z-2)(4z^3) - (z^4 - 1)[(2z-1) + (z-2)2]}{[(2z-1)(z-2)]^2}\right\}_{z=0}$$

$$= \frac{0 - (-1)(-1-4)}{[-1(-2)]^2} = \frac{-5}{4}$$

Hence $I = \frac{1}{2}\{2\pi i\,[\text{Res } f(1/2) + \text{Res } f(0)]\} = 2i\left(\frac{5}{4} - \frac{5}{4}\right) = 0$

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