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Engineering Mathematics - I

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Quantum
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RAS 103 : ENGINEERING MATHEMATICS - I

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UNIT

Differential Calculus-I

PART-1

Successive Differentiation

CONCEPT OUTLINE : PART-1

Successive Differentiation :

If $y = f(x)$ is a differentiable function of x, then dy/dx is called the first differential coefficient of y w.r.t. x.

$\dfrac{d}{dx}\left(\dfrac{dy}{dx}\right) = \dfrac{d^2y}{dx^2}$ is called the second differential coefficient and in a similar way $\dfrac{d^ny}{dx^n} = \dfrac{d}{dx}\left(\dfrac{d^{n-1}y}{dx^{n-1}}\right)$ is the n^{th} differential coefficient of y w.r.t. x. This type of differentiation is called successive differentiation. n^{th} differential coefficient can be denoted by various ways as

$\dfrac{d^ny}{dx^n}$, D^ny, y_n, $f^n(x)$ etc.

To Find the n^{th} Derivative of :

1.
$$y = (ax+b)^m$$
$$y_1 = m(ax+b)^{m-1}a$$
$$y_2 = m(m-1)(ax+b)^{m-2}a^2$$
$$y_3 = m(m-1)(m-2)(ax+b)^{m-3}a^3$$
$$\vdots$$
$$y_n = m(m-1)(m-2)\ldots(m-n+1)a^n(ax+b)^{m-n}$$

If
$$m = n$$
$$y_n = n!\,a^n(ax+b)^0$$
$$y_n = n!\,a^n$$
$$y_n = n!\,a^n$$

2. $D^n(\log x)$
$$y = \log x$$
$$y_1 = \frac{1}{x}$$
$$y_2 = (-1)\frac{1}{x^2}$$
$$y_3 = (-1)(-2)\frac{1}{x^3}$$
$$y_4 = (-1)(-2)(-3)\frac{1}{x^4}$$
$$y_n = (-1)^{n-1}(n-1)!\frac{1}{x^n}$$

3. $D^n\sin(ax+b)$ $\quad y = \sin(ax+b)$

1
UNIT

Differential Calculus-I

PART-1

Successive Differentiation

CONCEPT OUTLINE : PART-1

Successive Differentiation :
If $y = f(x)$ is a differentiable function of x, then dy/dx is called the first differential coefficient of y w.r.t. x.

$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2}$ is called the second differential coefficient and in a

similar way $\frac{d^ny}{dx^n} = \frac{d}{dx}\left(\frac{d^{n-1}y}{dx^{n-1}}\right)$ is the n^{th} differential coefficient of y

w.r.t. x. This type of differentiation is called successive differentiation.
n^{th} differential coefficient can be denoted by various ways as :

$\frac{d^ny}{dx^n}$, D^ny, y_n, $f^n(x)$ etc.

To Find the n^{th} Derivative of :

1.
$$y = (ax + b)^m$$
$$y_1 = m\,(ax + b)^{(m-1)}\,a$$
$$y_2 = m\,(m - 1)\,(ax + b)^{m-2}\,a^2$$
$$y_3 = m\,(m - 1)\,(m - 2)\,(ax + b)^{m-3}\,a^3$$
$$\vdots \quad \vdots \quad \vdots \quad \vdots$$
$$y_n = m(m - 1)\,(m - 2).....(m - n + 1)a^n(ax + b)^{m-n}$$

If
$$m = n$$
$$y_m = m!\,a^m\,(ax + b)^0$$
$$y_m = m!\,a^m$$
$$y_n = n!\,a^n$$

2. $D^n(\log x)$
$$y = \log x$$
$$y_1 = \frac{1}{x}$$
$$y_2 = (-1)\,\frac{1}{x^2}$$
$$y_3 = (-1)\,(-2)\,\frac{1}{x^3}$$
$$y_4 = (-1)\,(-2)\,(-3)\,\frac{1}{x^4}$$

$$y_1 = a\cos(ax+b) = a\sin\left(\frac{\pi}{2}+ax+b\right)$$

$$y_2 = a^2\cos\left(\frac{\pi}{2}+ax+b\right) = a^2\sin\left(2\frac{\pi}{2}+ax+b\right)$$

$$y_n = a^n\sin\left(n\frac{\pi}{2}+ax+b\right)$$

Some Standard Results for n^{th} Derivative are :

1. $D^n(ax+b)^m = m(m-1)(m-2)......(m-n+1)\,a^n(ax+b)^{m-n}$
2. $D^n(ax+b)^{-1} = (-1)^n\, n!\, a^n(ax+b)^{-n-1}$
3. $D^n e^{ax+b} = a^n e^{ax+b}$
4. $D^n a^x = a^x(\log a)^n$
5. $D^n \log(ax+b) = \dfrac{(-1)^{n-1}(n-1)!\,a^n}{(ax+b)^n}$
6. $D^n \log x = \dfrac{(-1)^{n-1}(n-1)!}{x^n}$
7. $D^n \sin(ax+b) = a^n\sin\left(n\frac{\pi}{2}+ax+b\right)$
8. $D^n \cos(ax+b) = a^n\cos\left(n\frac{\pi}{2}+ax+b\right)$
9. $D^n e^{ax}\sin(bx+c) = r^n e^{ax}\sin(bx+c+n\phi)$

   where $r = \sqrt{a^2+b^2}$

   $\phi = \tan^{-1}(b/a)$
10. $D^n e^{ax}\cos(bx+c) = r^n e^{ax}\cos(bx+c+n\phi)$

Questions-Answers

Long Answer Type and Medium Answer Type Questions

Que 1.1. **Find the n^{th} differential coefficient of**

a. $\cos^4 x$

b. $\sin ax \cos bx$

c. $\sin x \cos 3x$

Answer

a. Let $y = \cos^4 x = \left[\frac{1}{2}(1+\cos 2x)\right]^2$

$$= \frac{1}{4}(1 + 2\cos 2x + \cos^2 2x)$$

$$= \frac{1}{4}\left[1 + 2\cos 2x + \left(\frac{1+\cos 4x}{2}\right)\right]$$

$$y = \frac{3}{8} + \frac{1}{2}\cos 2x + \frac{1}{8}\cos 4x$$

$\because$ $$D^n \cos(ax+b) = a^n \cos\left(ax + b + \frac{n\pi}{2}\right)$$

$$y_n = 0 + \frac{1}{2} \times 2^n \cos\left(2x + \frac{n\pi}{2}\right) + \frac{1}{8} \times 4^n \cos\left(4x + \frac{n\pi}{2}\right)$$

$$y_n = 2^{n-1}\cos\left(2x + \frac{n\pi}{2}\right) + 2^{2n-3}\cos\left(4x + \frac{n\pi}{2}\right)$$

b. $$y = \cos bx \sin ax$$

$$y = \frac{1}{2}(2\sin ax \cos bx)$$

$$= \frac{1}{2}[\sin(a+b)x + \sin(a-b)x]$$

$$y_n = \frac{1}{2}\left[(a+b)^n \sin\left\{(a+b)x + \frac{n\pi}{2}\right\} + (a-b)^n \sin\left\{(a-b)x + \frac{n\pi}{2}\right\}\right]$$

c. $$y = \sin x \cos 3x$$

$$y = \frac{1}{2}(\sin 4x - \sin 2x)$$

$$y_n = \frac{1}{2}\left[4^n \sin\left(4x + \frac{n\pi}{2}\right) - 2^n \sin\left(2x + \frac{n\pi}{2}\right)\right]$$

Que 1.2. **Find the n^{th} derivative of : $\dfrac{1}{(x-1)^3(x-2)}$**

Answer

Let, $$y = \frac{1}{(x-1)^3(x-2)}$$

$$= -\frac{1}{(x-1)^3} - \frac{1}{(x-1)^2} - \frac{1}{(x-1)} + \frac{1}{(x-2)}$$

$$= -(x-1)^{-3} - (x-1)^{-2} - (x-1)^{-1} + (x-2)^{-1}$$

$$D^n (ax+b)^{-p} = (-p)(-p-1)(-p-2) \ldots\ldots \{-p-(n-1)\}\, a^n (ax+b)^{-p-n}$$

$$= (-1)^n \frac{(p+n-1)!}{(p-1)!}\, a^n (ax+b)^{-p-n}$$

$\therefore$ $$y_n = \frac{(-1)^n (3+n-1)!}{(3-1)!}(x-1)^{-3-n}$$

$$\frac{(-1)^n(2+n-1)!}{(2-1)!}(x-1)^{-2-n} - \frac{(-1)^n(1+n-1)!}{(1-1)!}(x-1)^{-1-n} + (-1)^n\, n!\,(x-2)^{-n-1}$$

$$y_n = \frac{-(-1)^n(2+n)!}{2!}(x-1)^{-3-n} - \frac{-(-1)^n(1+n)!}{1!}(x-1)^{-2-n} - (-1)^n\, n!\,(x-1)^{-1-n} + (-1)^n\, n!\,(x-2)^{-n-1}$$

$$= (-1)^{n+1} n!\left[\frac{(n+2)(n+1)}{2(x-1)^{n+3}} + \frac{(n+1)}{(x-1)^{n+2}} + \frac{1}{(x-1)^{n+1}} - \frac{1}{(x-2)^{n+1}}\right]$$

Que 1.3. **Find the n^{th} derivative of $\tan^{-1}\left\{\dfrac{2x}{1-x^2}\right\}$.**

Answer

Let,
$$y = \tan^{-1}\left\{\frac{2x}{1-x^2}\right\}$$
$$y = 2\tan^{-1} x$$
$$y_1 = \frac{2}{1+x^2} = \frac{1}{i}\left[\frac{1}{x-i} - \frac{1}{x+i}\right]$$

Differentiating $(n-1)$ times,
$$y_n = \frac{1}{i}\,[(-1)^{n-1}(n-1)!\,\{(x-i)^{-n} - (x+i)^{-n}\}]$$

Let,
$$x = r\cos\theta,\ 1 = r\sin\theta$$
$$y_n = \frac{(-1)^{n-1}(n-1)!}{i}[r^{-n}(\cos\theta - i\sin\theta)^{-n} - r^{-n}(\cos\theta + i\sin\theta)^{-n}]$$

By De Moivre's Theorem,
$$y_n = \frac{(-1)^{n-1}(n-1)!}{i}r^{-n}\,[(\cos n\theta + i\sin n\theta) - (\cos n\theta - i\sin n\theta)]$$
$$= 2(-1)^{n-1}(n-1)!\,(r^{-n}\sin n\theta)$$
$$= 2(-1)^{n-1}(n-1)!\left(\frac{1}{\sin\theta}\right)^{-n}\sin n\theta$$
$$= 2(-1)^{n-1}(n-1)!\,\sin^n\theta\,\sin n\theta$$

Where
$$\theta = \tan^{-1}\left(\frac{1}{x}\right)$$

Que 1.4. **If $x = a(t - \sin t)$, $y = a(1 + \cos t)$, prove that**

$$\frac{d^2y}{dx^2} = \frac{1}{4a}\operatorname{cosec}^4\left(\frac{t}{2}\right)$$

Answer

$$x = a(t - \sin t)$$

$$\frac{dx}{dt} = a(1 - \cos t)$$

$$\frac{dy}{dt} = a(-\sin t)$$

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{-a \sin t}{a(1 - \cos t)}$$

$$= \frac{-\sin t}{1 - \cos t} = \frac{-2 \sin \frac{t}{2} \cos \frac{t}{2}}{1 - \left(1 - 2 \sin^2 \frac{t}{2}\right)}$$

$$\frac{dy}{dx} = -\cot \frac{t}{2}$$

$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dt}\left(\frac{dy}{dx}\right)\frac{dt}{dx}$$

$$\frac{d^2y}{dx^2} = +\frac{1}{2} \operatorname{cosec}^2 \frac{t}{2} \frac{1}{a(1 - \cos t)}$$

$$= +\frac{1}{2a} \operatorname{cosec}^2 \frac{t}{2} \frac{1}{(2\sin^2 t/2)}$$

$$\frac{d^2y}{dx^2} = +\frac{1}{4a} \operatorname{cosec}^4 \frac{t}{2}$$

PART-2

Leibnitz Theorem

CONCEPT OUTLINE : PART-2

Leibnitz Theorem : If u and v are any two functions of x such that all their desired differential coefficients exist, then the n^{th} differential coefficient of their product is given by

$$D^n(uv) = {}^nC_0 D^n u.v + {}^nC_1 D^{n-1} u.Dv + {}^nC_2 D^{n-2} u D^2v + \ldots\ldots + {}^nC_r D^{n-r} u D^r v + \ldots\ldots + u.D^n v$$

Determination of the value of the n^{th} derivative of a function for $x = 0$.

Step 1 : Suppose the given function equals to y.

Step 2 : Find $y_1 = \frac{dy}{dx}$

(*a*) Simplify the expression by taking LCM if possible.
(*b*) Square both sides to avoid square root.

(c) Convert y_1 in terms of y (if possible).

Step 3 : Find $y_2 = \dfrac{d^2y}{dx^2}$

Step 4 : Differentiate n times by Leibnitz theorem.
Step 5 : Put $x = 0$ in Steps (1), (2), (3) and (4).
Step 6 : Put $n = 1, 2, 3, 4$ in last equation of Step 5.
Step 7 : Discuss the cases for n *i.e.*, even or odd.

Questions-Answers

Long Answer Type and Medium Answer Type Questions

Que 1.5. **If $y = e^{m\cos^{-1}x}$, then show that $(1-x^2)\,y_{n+2} - (2n+1)xy_{n+1} - (n^2+m^2)y_n = 0$ and hence calculate y_n, when $x = 0$.**

UPTU 2013-14, Marks 10

OR

If $\cos^{-1}x = \log(y)^{1/m}$, then show $(1-x^2)y_{n+2} - (2n+1)xy_{n+1} - (n^2+m^2)y_n = 0$ and hence Calculate y_n when $x = 0$.

UPTU 2015-16, Marks 10

Answer

$$y = e^{m\cos^{-1}x} \quad ...(1.5.1)$$

$$y_1 = e^{m\cos^{-1}x}\left(\frac{-m}{\sqrt{1-x^2}}\right) = -\frac{me^{m\cos^{-1}x}}{\sqrt{1-x^2}} \quad ...(1.5.2)$$

$$\sqrt{1-x^2}\;y_1 = -my$$

Squaring on both sides

$\Rightarrow$ $$(1-x^2)\,y_1^2 = m^2y^2$$

Differentiating again, we get

$$(1-x^2)\,2y_1y_2 - 2x\,y_1^2 = m^2\,2y\,y_1$$

On dividing by $2y_1$ on both sides

$$\Rightarrow (1-x^2)\,y_2 - xy_1 - m^2y = 0 \quad ...(1.5.3)$$

Differentiating n times by Leibnitz's theorem, we get

$$(1-x^2)\,y_{n+2} + n(-2x)\,y_{n+1} + \frac{n(n-1)}{2}(-2)\,y_n - xy_{n+1} - ny_n - m^2y_n = 0$$

$$\Rightarrow (1-x^2)\,y_{n+2} - (2n+1)\,xy_{n+1} - (n^2+m^2)\,y_n = 0 \quad ...(1.5.4)$$

From equation (1.5.1) $y(0) = e^{m\cos^{-1}0} = e^{\frac{m\pi}{2}}$

From equation (1.5.2) $y_1(0) = \dfrac{-me^{m\cos^{-1}0}}{\sqrt{1-0}} = -me^{\frac{m\pi}{2}}$

From equation (1.5.3), $y_2(0) = m^2\, y(0) = m^2\, e^{\frac{m\pi}{2}}$

Put $n = 1, 2, 3, 4, \ldots\ldots$ in equation (1.5.4), we get

$$y_3(0) = (1^2 + m^2)\, y_1(0) = (1^2 + m^2)\left(-me^{\frac{m\pi}{2}}\right) = -m(1^2 + m^2)\, e^{\frac{m\pi}{2}}$$

$$y_4(0) = (2^2 + m^2)\, y_2(0) = (2^2 + m^2)\left(m^2 e^{\frac{m\pi}{2}}\right)$$

$$= m^2\,(2^2 + m^2)\, e^{\frac{m\pi}{2}}$$

$$y_5(0) = (3^2 + m^2)\, y_3(0) = (3^2 + m^2)\left(-m(1^2 + m^2)e^{\frac{m\pi}{2}}\right)$$

$$y_5(0) = -m\,(1^2 + m^2)\,(3^2 + m^2)\; e^{\frac{m\pi}{2}}$$

$$y_6(0) = (4^2 + m^2)\, y_4(0) = m^2\,(2^2 + m^2)\,(4^2 + m^2)\; e^{\frac{m\pi}{2}}$$

$$\therefore \quad y_n(0) = \begin{cases} -m(1^2 + m^2)(3^2 + m^2)\ldots\ldots[(n-2)^2 + m^2]\, e^{\frac{m\pi}{2}}, & \text{for odd } n \\ m^2(2^2 + m^2)(4^2 + m^2)\ldots\ldots[(n-2)^2 + m^2]\, e^{\frac{m\pi}{2}}, & \text{for even } n \end{cases}$$

Que 1.6. **If $y = a\cos(\log x) + b\sin(\log x)$. Find $(y_n)_0$.**

UPTU 2011-12, Marks 10

Answer

$$y = a\cos(\log x) + b\sin(\log x)$$

$$y_1 = -\frac{a}{x}\sin(\log x) + \frac{b}{x}\cos(\log x)$$

$$xy_1 = -a\sin(\log x) + b\cos(\log x)$$

$$xy_2 + y_1 = -\frac{a}{x}\cos(\log x) + \left(\frac{-b}{x}\right)\sin(\log x)$$

$$x^2y_2 + xy_1 = -y$$

$$x^2y_2 + xy_1 + y = 0$$

Differentiating n times using Leibnitz theorem,

$$x^2y_{n+2} + n(2x)\,y_{n+1} + n(n-1)\,y_n + \underline{xy_{n+1} + ny_n + y_n} = 0$$

$$x^2y_{n+2} + (2n + 1)\,xy_{n+1} + (n^2 + 1)\,y_n = 0$$

Put $x = 0$,

$$0 + 0 + (n^2 + 1)\,y_n(0) = 0$$

$$y_n(0) = 0$$

Que 1.7. **If $y = \tan^{-1}\left(\dfrac{a+x}{a-x}\right)$, prove that :**

$(a^2 + x^2)\, y_{n+1} + 2(n+1)x\, y_{n+1} + n(n+1)y_n = 0.$

UPTU 2011-12, Marks 05

Answer

$$y = \tan^{-1}\left(\frac{a+x}{a-x}\right)$$

$$y_1 = \frac{1}{1+\left(\dfrac{a+x}{a-x}\right)^2}\left[\frac{(a-x)+(a+x)}{(a-x)^2}\right]$$

$$y_1 = \frac{2a}{\dfrac{(a-x)^2+(a+x)^2}{(a-x)^2}\times(a-x)^2}$$

$$y_1 = \frac{2a}{2(a^2+x^2)}$$

$$y_1 = \frac{a}{a^2+x^2}$$

$$(a^2+x^2)\, y_1 = a$$

Differentiating $(n+1)$ times by Leibnitz theorem,

$(a^2+x^2)y_{n+2} + 2x(n+1)\, y_{n+1} + n(n+1)\, y_n = 0$

Hence proved.

Que 1.8. **If $y^{1/m} + y^{-1/m} = 2x$, prove that**

$(x^2-1)\, y_{n+2} + (2n+1)\, y_{n+1}x + (n^2-m^2)\, y_n = 0.$

Answer

$$y^{1/m} + y^{-1/m} = 2x$$

Let $z = y^{1/m}$

$$z + z^{-1} = 2x$$

$$z^2 + 1 - 2xz = 0$$

$$z = [x \pm \sqrt{x^2-1}]$$

$$y = [x \pm \sqrt{x^2-1}]^m$$

$$y_1 = m[x \pm \sqrt{x^2-1}]^{m-1}\left[1 \pm \frac{2x}{2\sqrt{x^2-1}}\right]$$

$$= m\,[x \pm \sqrt{x^2-1}]^m\ \frac{1}{\sqrt{x^2-1}}$$

$\Rightarrow \sqrt{x^2-1}\, y_1 = my$

Squaring on both sides

$(x^2 - 1)\, y_1^{\,2} = m^2 y^2$

Differentiate again,

$(x^2 - 1)\, 2y_1 y_2 + 2xy_1^{\,2} = 2m^2 yy_1$

On dividing $2y_1$,

$(x^2 - 1)\, y_2 + xy_1 - m^2 y = 0$

Differentiating n times by Leibnitz theorem,

$$(x^2-1)\, y_{n+2} + n.2x\, y_{n+1} + \frac{n(n-1)}{2!}\, 2.y_n + xy_{n+1} + ny_n - m^2 y_n = 0$$

$$\Rightarrow \quad (x^2-1)\, y_{n+2} + (2n+1)\, x\, y_{n+1} + (n^2 - m^2)\, y_n = 0$$

Que 1.9. **If $y^{\frac{1}{m}} + y^{\frac{-1}{m}} = 2x$ prove that $(x^2 - 1)y_{n-2} + (2n + 1)xy_{n-1} + (n^2 - m^2)y_n = 0$.**

UPTU 2014-15, Marks 05

Answer

Same as Q. 1.8, Page 12A, Unit-1.

Use $n - 2$ in place of $n + 2$ and $n - 1$ in place of $n + 1$.

Que 1.10. **If $y = (x + \sqrt{1+x^2})^m$, then find the n^{th} derivative of y at $x = 0$.**

UPTU 2012-13, Marks 10

Answer

$$y = (x + \sqrt{1+x^2}\,)^m \quad \text{...(1.10.1)}$$

$$y_1 = m(x + \sqrt{1+x^2}\,)^{m-1}\left(1 + \frac{x}{\sqrt{1+x^2}}\right)$$

$$\sqrt{1+x^2}\; y_1 = my \quad \text{...(1.10.2)}$$

Squaring both sides,

$(1 + x^2)y_1^{\,2} = m^2 y^2$

Differentiate again,

$(1 + x^2)2y_1 y_2 + 2xy_1^{\,2} = 2m^2 yy_1$

On dividing by $2y_1$

$$(1 + x^2)y_2 + xy_1 - m^2 y = 0 \quad \text{...(1.10.3)}$$

Differentiating 'n' times using Leibnitz theorem,

$(1 + x^2)y_{n+2} + n(2x)y_{n+1} + n(n-1)y_n + xy_{n+1} + ny_n - m^2 y_n = 0$

or $$(1 + x^2)y_{n+2} + (2n + 1)xy_{n+1} + (n^2 - m^2)y_n = 0 \quad \text{...(1.10.4)}$$

Put $x = 0$ in equation (1.10.1), equation (1.10.2), equation (1.10.3) and equation (1.10.4) respectively,

$$y(0) = 1$$
$$y_1(0) = my(0)$$
$$y_1(0) = m$$
$$y_2(0) = m^2$$
$$y_{n+2}(0) = (m^2 - n^2)y_n(0) \quad ...(1.10.5)$$

Put $n = 1, 2, 3, 4, 5, \ldots\ldots$ in equation (1.10.5),

$$y_3(0) = (m^2 - 1^2)\, y_1(0) = (m^2 - 1^2)m$$
$$y_4(0) = (m^2 - 2^2)\, y_2(0) = (m^2 - 2^2)m^2$$
$$y_5(0) = (m^2 - 3^2)\, y_3(0) = (m^2 - 3^2)\,(m^2 - 1^2)m$$
$$y_6(0) = (m^2 - 4^2)\,(m^2 - 2^2)m^2$$

and so on.

Thus using above results, we have

$$y_n(0) = \begin{cases} m(m^2 - 1^2)\,(m^2 - 3^2) \ldots\ldots \{m^2 - (n-2)^2\}, \text{ if } n \text{ is odd.} \\ m^2(m^2 - 2^2)\,(m^2 - 4^2) \ldots\ldots \{m^2 - (n-2)^2\}, \text{ if } n \text{ is even.} \end{cases}$$

Que 1.11. **If $y = \sin^{-1}x$, find $(y_n)_0$.**

Answer

$$y = \sin^{-1} x \quad ...(1.11.1)$$
$$y_1 = \frac{1}{\sqrt{1-x^2}} \quad ...(1.11.2)$$
$$(1 - x^2)\, y_1^2 = 1$$
$$(1 - x^2)\, y_1^2 - 1 = 0$$

Differentiate again w.r.t. x,

$$(1 - x^2)\, 2y_1\, y_2 - 2xy_1^2 = 0$$
$$(1 - x^2)\, y_2 - xy_1 = 0 \quad ...(1.11.3)$$

Differentiate n times by Leibnitz theorem,

$$y_{n+2}\,(1 - x^2) + n\, y_{n+1}\,(-2x) + \frac{n\,(n-1)}{2!}\, y_n\,(-2) - xy_{n+1} - ny_n\,(1) = 0$$
$$(1 - x^2)\, y_{n+2} - (2n + 1)\, xy_{n+1} - n^2y_n = 0 \quad ...(1.11.4)$$

Put $x = 0$ in equation (1.11.1), equation (1.11.2) and equation (1.11.3),

$$(y_0)_0 = 0,\ (y_1)_0 = 1,\ (y_2)_0 = 0$$

Put $x = 0$ in equation (1.11.4),

$$(y_{n+2})_0 = n^2\,(y_n)_0 \quad ...(1.11.5)$$

Put $n = n - 2$ in equation (1.11.5),

$$(y_n)_0 = (n-2)^2\,(y_{n-2})_0 = (n-2)^2\,(n-4)^2\,(y_{n-4})_0$$

Case (1) : If n is odd, put $n = 1, 3, 5, \ldots\ldots\ldots$ in equation (1.11.5).

$$(y_3)_0 = 1^2\,(y_1)_0 = 1^2.1$$
$$(y_5)_0 = 3^2\,(y_3)_0 = 3^2.1^2.1$$
$$(y_7)_0 = 5^2\,.\,3^2\,.1^2\,.1 \text{ and so on.}$$

Thus, if n is odd, $(y_n)_0 = (n-2)^2\,(n-4)^2 \ldots .5^2\,.3^2\,.1^2\,.1$

Case (2) : If n is even, put $n = 2, 4, 6, \ldots\ldots\ldots$ in equation (1.11.5)

$$(y_4)_0 = 2^2\,(y_2)_0 = 0$$
$$(y_6)_0 = 0$$

Thus, if n is even $(y_n)_0 = 0$.

Que 1.12. **If $y = \tan^{-1}x$, find $(y_n)_0$.**

Answer

We have, $y = \tan^{-1} x$ ...(1.12.1)

$$y_1 = \frac{1}{1+x^2} \quad ...(1.12.2)$$

$$(1 + x^2)\, y_1 = 1 \quad ...(1.12.3)$$

$$(1 + x^2)\, y_2 + 2xy_1 = 0 \quad ...(1.12.4)$$

Differentiate 'n' times by Leibnitz theorem,

$$(1+x^2)\, y_{n+2} + n.\, 2x\, y_{n+1} + \frac{2n\,(n-1)}{2!}\, y_n + 2x\, y_{n+1} + 2n\, y_n = 0$$

$$(1 + x^2)\, y_{n+2} + 2\,(n + 1)\, x\, y_{n+1} + n\,(n + 1)\, y_n = 0 \quad ...(1.12.5)$$

Put $x = 0$ in equation (1.12.1), equation (1.12.2) and equation (1.12.4),

$$(y)_0 = 0,\ (y_1)_0 = 1,\ (y_2)_0 = 0$$

Put $x = 0$ in equation (1.12.5),

$$(y_{n+2})_0 = -\{n\,(n + 1)\}\,(y_n)_0 \quad ...(1.12.6)$$

$$(y_n)_0 = -\,(n - 2)\,(n - 1)\,(y_{n-2})_0$$

$$= [-\,(n - 1)\,(n - 2)]\ [-\,(n - 3)\,(n - 4)]\,(y_{n-4})_0$$

From equation (1.12.6), $(y_{n-2})_0 = -\,(n - 3)\,(n - 4)\,(y_{n-4})_0$

If n is even

$(y_n)_0 = [-\,(n - 1)\,(n - 2)]\ [-\,(n - 3)\,(n - 4)] .... [-\,(3)\,(2)]\,(y_2)_0 = 0$

When n is odd,

$(y_n)_0 = [-\,(n - 1)\,(n - 2)]\ [-\,(n - 3)\,(n - 4)] ...... [-\,(4)\,(3)]\ [-\,(2)\,(1)]\,(y_1)_0$

$$= (-\,1)^{(n-1)/2}\,(n - 1)!$$

PART-3

Limit and Continuity

CONCEPT OUTLINE : PART-3

Limit : The function $f(x, y)$ is said to tend to the limit l as $x \to a$ and $y \to b$ if and only if the limit l is independent of the path followed by the point (x, y) as $x \to a$ and $y \to b$. Then

$$\lim_{\substack{x \to a \\ y \to b}} f(x, y) = l$$

The function $f(x, y)$ in region R is said to tend to the limit l as $x \to a$ and $y \to b$ if and only if corresponding to a positive number $\in (a, b)$, there exists another positive number δ such that

$|f(x, y) - l| < \in$ for $0 < (x - a)^2 + (y - b)^2 < \delta^2$

for every point (x, y) in R.

Continuity : A function $f(x, y)$ is said to be continuous at the point (a, b) if

$\lim_{\substack{x \to a \\ y \to b}} f(x, y)$ exists and $= f(a, b)$

Note : Generally $\lim_{x \to a}\left[\lim_{y \to b} f(x, y)\right] = \lim_{y \to b}\left[\lim_{x \to a} f(x, y)\right]$

But it is not always true.

Questions-Answers

Long Answer Type and Medium Answer Type Questions

Que 1.13. Evaluate $\lim_{\substack{x \to 1 \\ y \to 2}} \dfrac{3x^2 y}{x^2 + y^2 + 5}$.

Answer

$$\lim_{\substack{x \to 1 \\ y \to 2}} \frac{3x^2 y}{x^2 + y^2 + 5} = \lim_{x \to 1}\left[\lim_{y \to 2} \frac{3x^2 y}{x^2 + y^2 + 5}\right] = \lim_{x \to 1} \frac{3x^2 (2)}{x^2 + (2)^2 + 5}$$

$$= \lim_{x \to 1} \frac{6x^2}{x^2 + 9} = \frac{6}{1 + 9} = \frac{3}{5}$$

Que 1.14. Evaluate $\lim_{\substack{x \to 0 \\ y \to 0}} \dfrac{2xy}{3x^2 + y^2}$.

Answer

$$\lim_{\substack{x \to 0 \\ y \to 0}} \frac{2xy}{3x^2 + y^2} = \lim_{x \to 0}\left[\lim_{y \to 0} \frac{2xy}{3x^2 + y^2}\right] = \lim_{x \to 0} \frac{0}{3x^2} = \frac{0}{0}$$

The limit does not exist.

Que 1.15. Evaluate $\lim_{\substack{x \to \infty \\ y \to 2}} \dfrac{xy + 4}{x^2 + 2y^2}$.

Answer

$$\lim_{\substack{x \to \infty \\ y \to 2}} \frac{xy + 4}{x^2 + 2y^2} = \lim_{x \to \infty}\left[\lim_{y \to 2} \frac{xy + 4}{x^2 + 2y^2}\right] = \lim_{x \to \infty} \frac{2x + 4}{x^2 + 8}$$

$$= \lim_{x \to \infty} \frac{2 + \frac{4}{x}}{x + \frac{8}{x}} = \frac{2 + 0}{\infty + 0} = 0$$

Que 1.16. If $f(x, y) = \dfrac{x + y}{2x - y}$, show that

$$\lim_{x \to 0}\left[\lim_{y \to 0} f(x, y)\right] \neq \lim_{y \to 0}\left[\lim_{x \to 0} f(x, y)\right]$$

Answer

$$\lim_{x \to 0}\left[\lim_{y \to 0} \frac{x + y}{2x - y}\right] = \lim_{x \to 0}\left(\frac{x}{2x}\right) = \frac{1}{2}$$

$$\lim_{y \to 0}\left[\lim_{x \to 0} \frac{x + y}{2x - y}\right] = \lim_{y \to 0}\left(\frac{y}{-y}\right) = -1$$

Hence, $\lim_{x \to 0}\left[\lim_{y \to 0} f(x, y)\right] \neq \lim_{y \to 0}\left[\lim_{x \to 0} f(x, y)\right]$

PART-4

Partial Differentiation

CONCEPT OUTLINE : PART-4

Partial Differentiation : If a derivative of function of several independent variables be found with respect to any one of them, keeping the others as constants, it is said to be a partial derivative. The operation of finding the partial derivatives of a function of more than one independent variable is called partial differentiation.

$\frac{\partial}{\partial x}, \frac{\partial}{\partial y}$ etc., are the symbols used for partial derivatives. $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}$ can also be denoted as u_x, u_y. Second order partial derivatives are denoted by $\frac{\partial^2 u}{\partial x^2}, \frac{\partial^2 u}{\partial x \partial y}, \frac{\partial^2 u}{\partial y^2}$ etc.

Note : If $u = f(x, y)$ and its partial derivatives are continuous, then $\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}$

Questions-Answers

Long Answer Type and Medium Answer Type Questions

Que 1.17. If, $z = x^y + y^x$ verify $\frac{\partial^2 z}{\partial x \partial y} = \frac{\partial^2 z}{\partial y \partial x}$.

Answer

$$\frac{\partial z}{\partial x} = yx^{y-1} + y^x \log y$$

$$\frac{\partial^2 z}{\partial y \partial x} = x^{y-1} + yx^{y-1}\log x + xy^{x-1}\log y + y^x \frac{1}{y}$$

$$= x^{y-1}(1 + y\log x) + y^{x-1}(1 + x\log y)$$

$$\frac{\partial z}{\partial y} = x^y \log x + xy^{x-1}$$

$$\frac{\partial^2 z}{\partial x \partial y} = yx^{y-1}\log x + x^y \frac{1}{x} + 1.y^{x-1} + xy^{x-1}\log y$$

$$= x^{y-1}(1 + y\log x) + y^{x-1}(1 + x\log y)$$

Thus,

$$\frac{\partial^2 z}{\partial y \partial x} = \frac{\partial^2 z}{\partial x \partial y}$$

Que 1.18. **If $z = f(y/x)$ show that $x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 0$.**

Answer

$$z = f(y/x)$$

$$\frac{\partial z}{\partial x} = f'\left(\frac{y}{x}\right)\left(\frac{-y}{x^2}\right)$$

$$x\frac{\partial z}{\partial x} = -\frac{y}{x} f'\left(\frac{y}{x}\right) \quad ...(1.18.1)$$

$$\frac{\partial z}{\partial y} = f'\left(\frac{y}{x}\right).\frac{1}{x}$$

$$y\frac{\partial z}{\partial y} = \frac{y}{x} f'\left(\frac{y}{x}\right) \quad ...(1.18.2)$$

Adding equation (1.18.1) and equation (1.18.2),

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = 0$$

Que 1.19. **If $u = f(r)$ where $r^2 = x^2 + y^2$, show that**

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = f''(r) + \frac{1}{r} f'(r)$$

UPTU 2015-16, Marks 10

Answer

$$r^2 = x^2 + y^2 \text{ so that } \frac{\partial r}{\partial x} = \frac{x}{r}$$

$$\frac{\partial u}{\partial x} = \frac{df}{dr} \cdot \frac{\partial r}{\partial x}$$

$$\frac{\partial u}{\partial x} = \frac{df}{dr} \cdot \frac{x}{r}$$

$\Rightarrow$ Differentiating again w.r.t. x, we get

$$\frac{\partial^2 u}{\partial x^2} = \left(\frac{d^2 f}{dr^2}\frac{\partial r}{\partial x}\right)\cdot\frac{x}{r} + \frac{df}{dr}\cdot\left[\frac{r.1 - x\frac{\partial r}{\partial x}}{r^2}\right]$$

$$= \left(\frac{d^2 f}{dr^2}\frac{x}{r}\right)\cdot\frac{x}{r} + \frac{df}{dr}\cdot\left[\frac{r.1 - x\cdot\frac{x}{r}}{r^2}\right]$$

$$= \frac{d^2 f}{dr^2}\frac{x^2}{r^2} + \frac{df}{dr}\cdot\frac{r^2 - x^2}{r^3}$$

$$= \frac{d^2 f}{dr^2}\frac{x^2}{r^2} + \frac{df}{dr}\frac{y^2}{r^3} \quad \text{...(1.19.1)}$$

Similarly, $$\frac{\partial^2 u}{\partial y^2} = \frac{d^2 f}{dr^2}\frac{y^2}{r^2} + \frac{df}{dr}\frac{x^2}{r^3} \quad \text{...(1.19.2)}$$

On adding equation (1.19.1) and equation (1.19.2), we get

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{d^2 f}{dr^2}\frac{x^2 + y^2}{r^2} + \frac{df}{dr}\frac{x^2 + y^2}{r^3}$$

$$= \frac{d^2 f}{dr^2} + \frac{df}{dr}\frac{1}{r} = f''(r) + \frac{1}{r}f'(r)$$

Que 1.20. If $z = \phi(x + ct) + f(x - ct)$ show that $\dfrac{\partial^2 z}{\partial t^2} = c^2\dfrac{\partial^2 z}{\partial x^2}$.

UPTU 2010-11, Marks 05

Answer

$$z = \phi(x + ct) + f(x - ct)$$

$$\frac{\partial z}{\partial x} = \phi'(x + ct) + f'(x - ct)$$

$$\frac{\partial^2 z}{\partial x^2} = \phi''(x + ct) + f''(x - ct) \quad \text{...(1.20.1)}$$

$$\frac{\partial z}{\partial t} = c\phi'(x + ct) - cf'(x - ct)$$

$$\frac{\partial^2 z}{\partial t^2} = c^2\phi''(x + ct) + c^2 f''(x - ct)$$

$$\frac{\partial^2 z}{\partial t^2} = c^2[\phi''(x + ct) + f''(x - ct)] \quad \text{...(1.20.2)}$$

From equation (1.20.1) and equation (1.20.2),

$$\frac{\partial^2 z}{\partial t^2} = c^2\frac{\partial^2 z}{\partial x^2}$$

Que 1.21. **If $u(x, y, z) = \log(\tan x + \tan y + \tan z)$, prove that $\sin 2x \dfrac{\partial u}{\partial x} + \sin 2y \dfrac{\partial u}{\partial y} + \sin 2z \dfrac{\partial u}{\partial z} = 2$.**

UPTU 2011-12, Marks

Answer

$$u = \log(\tan x + \tan y + \tan z)$$

$$\frac{\partial u}{\partial x} = \frac{1}{(\tan x + \tan y + \tan z)}(\sec^2 x)$$

$$\frac{\partial u}{\partial y} = \frac{\sec^2 y}{\tan x + \tan y + \tan z}$$

$$\frac{\partial u}{\partial z} = \frac{\sec^2 z}{\tan x + \tan y + \tan z}$$

Now,
$$\sin 2x \frac{\partial u}{\partial x} = \frac{\sin 2x \sec^2 x}{\tan x + \tan y + \tan z}$$

$$\sin 2x \frac{\partial u}{\partial x} = \frac{2 \sin x}{\cos x(\tan x + \tan y + \tan z)}$$

$$= \frac{2\tan x}{\tan x + \tan y + \tan z} \quad ...(1.2$$

Similarly,
$$\sin 2y \frac{\partial u}{\partial y} = \frac{2\tan y}{\tan x + \tan y + \tan z} \quad ...(1.2$$

and
$$\sin 2z \frac{\partial u}{\partial z} = \frac{2\tan z}{\tan x + \tan y + \tan z} \quad ...(1.2$$

On adding equation (1.21.1), equation (1.21.2) and equation (1.21.3

$$\sin 2x \frac{\partial u}{\partial x} + \sin 2y \frac{\partial u}{\partial y} + \sin 2z \frac{\partial u}{\partial z} = 2$$

Que 1.22. **If $z = f(x, y)$ where $x = e^u \cos v$, $y = e^u \sin v$, prove that**

$$\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2 = e^{-2u}\left[\left(\frac{\partial f}{\partial u}\right)^2 + \left(\frac{\partial f}{\partial v}\right)^2\right]$$

UPTU 2012-13, Marks

Answer

$$\frac{\partial f}{\partial u} = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial u} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial u}$$

$$\frac{\partial f}{\partial u} = \frac{\partial f}{\partial x}(e^u \cos v) + \frac{\partial f}{\partial y}(e^u \sin v) \quad ...(1.22.1)$$

Similarly, $$\frac{\partial f}{\partial v} = \frac{\partial f}{\partial x}(-e^u \sin v) + \frac{\partial f}{\partial y}(e^u \cos v) \quad ...(1.22.2)$$

Squaring and adding equation (1.22.1) and equation (1.22.2), we get

$$\left(\frac{\partial f}{\partial u}\right)^2 + \left(\frac{\partial f}{\partial v}\right)^2 = e^{2u}\left(\frac{\partial f}{\partial x}\right)^2 + e^{2u}\left(\frac{\partial f}{\partial y}\right)^2$$

$$e^{-2u}\left[\left(\frac{\partial f}{\partial u}\right)^2 + \left(\frac{\partial f}{\partial v}\right)^2\right] = \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2$$

Hence proved.

Que 1.23. **If $u = \log(x^3 + y^3 + z^3 - 3xyz)$ show that**

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = \frac{3}{x+y+z} \text{ and } \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)^2 u = \frac{-9}{(x+y+z)^2}.$$

UPTU 2003-04, Marks 05

Answer

$$u = \log(x^3 + y^3 + z^3 - 3xyz)$$

$$\frac{\partial u}{\partial x} = \frac{3x^2 - 3yz}{x^3 + y^3 + z^3 - 3xyz} \quad ...(1.23.1)$$

$$\frac{\partial u}{\partial y} = \frac{3y^2 - 3xz}{x^3 + y^3 + z^3 - 3xyz} \quad ...(1.23.2)$$

$$\frac{\partial u}{\partial z} = \frac{3z^2 - 3xy}{x^3 + y^3 + z^3 - 3xyz} \quad ...(1.23.3)$$

Adding equation (1.23.1), equation (1.23.2), and equation (1.23.3), we get

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = \frac{3(x^2 + y^2 + z^2) - 3(xy + yz + zx)}{x^3 + y^3 + z^3 - 3xyz}$$

$$= \frac{3(x^2 + y^2 + z^2 - xy - yz - xz)}{x^3 + y^3 + z^3 - 3xyz}$$

$$= \frac{3(x^2 + y^2 + z^2 - xy - yz - zx)}{(x+y+z)(x^2 + y^2 + z^2 - xy - yz - zx)}$$

$$= \frac{3}{x+y+z} \text{ Proved}$$

Again $$\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)^2 u = \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)\left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}\right)$$

$$= \left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y} + \frac{\partial}{\partial z}\right)\frac{3}{x+y+z}$$

$$= \frac{-3}{(x+y+z)^2} - \frac{3}{(x+y+z)^2} - \frac{3}{(x+y+z)^2}$$

$$= \frac{-9}{(x+y+z)^2} \text{ Proved}$$

Que 1.24. If $x = r\cos\theta$, $y = r\sin\theta$, prove that $\dfrac{\partial^2\theta}{\partial x^2} + \dfrac{\partial^2\theta}{\partial y^2} = 0$ except when $x = 0$, $y = 0$.

Answer

$x = r\cos\theta$, $y = r\sin\theta$, $r^2 = x^2 + y^2$, $\theta = \tan^{-1}\left(\dfrac{y}{x}\right)$

$$\frac{\partial\theta}{\partial x} = \frac{\partial}{\partial x}\left(\tan^{-1}\frac{y}{x}\right) = \frac{1}{1+\left(\frac{y}{x}\right)^2}\left(\frac{-y}{x^2}\right) = \frac{-y}{x^2+y^2}$$

$$\frac{\partial^2\theta}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial\theta}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{-y}{x^2+y^2}\right) = -y\,\frac{-2x}{(x^2+y^2)^2}$$

$$\frac{\partial^2\theta}{\partial x^2} = \frac{2xy}{(x^2+y^2)^2} \qquad \text{...(1.24.1)}$$

$$\frac{\partial\theta}{\partial y} = \frac{1}{1+\left(\frac{y}{x}\right)^2}\cdot\left(\frac{1}{x}\right) = \frac{x}{x^2+y^2}$$

$$\frac{\partial^2\theta}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{x}{x^2+y^2}\right) = \frac{-2xy}{(x^2+y^2)^2} \qquad \text{...(1.24.2)}$$

Adding equation (1.24.1) and equation (1.24.2), we get,

$$\frac{\partial^2\theta}{\partial x^2} + \frac{\partial^2\theta}{\partial y^2} = 0$$

But at $x = 0$, $y = 0$

both $\dfrac{\partial^2\theta}{\partial x^2}$ and $\dfrac{\partial^2\theta}{\partial y^2}$ are of the indeterminate form $\dfrac{0}{0}$.

Que 1.25. For what value of n, $u = r^n(3\cos^2\theta - 1)$ satisfies the equation

$$\frac{\partial}{\partial r}\left(r^2\frac{\partial u}{\partial r}\right) + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial u}{\partial\theta}\right) = 0\ ?$$

UPTU 2013-14, Marks 10

Answer

$$u = r^n (3\cos^2\theta - 1)$$

$$\frac{\partial}{\partial r}\left(r^2 \frac{\partial u}{\partial r}\right) + \frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial u}{\partial\theta}\right) = 0 \quad \text{...(1.25.1)}$$

$$\frac{\partial u}{\partial r} = \frac{\partial}{\partial r}(r^n (3\cos^2\theta - 1)) = nr^{n-1}(3\cos^2\theta - 1)$$

$$r^2 \frac{\partial u}{\partial r} = nr^{n+1}(3\cos^2\theta - 1)$$

$$\frac{\partial}{\partial r}\left(r^2 \frac{\partial u}{\partial r}\right) = \frac{\partial}{\partial r}(nr^{n+1}(3\cos^2\theta - 1))$$

$$= n(n+1) r^n (3\cos^2\theta - 1) \quad \text{...(1.25.2)}$$

$$u = r^n (3\cos^2\theta - 1) = r^n \left(\frac{3}{2}(1 + \cos 2\theta) - 1\right)$$

$$\frac{\partial u}{\partial\theta} = r^n \left(\frac{3}{2}(-2\sin 2\theta)\right) = -3r^n \sin 2\theta$$

$$\sin\theta \frac{\partial u}{\partial\theta} = -6r^n \sin^2\theta\cos\theta$$

$$\frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial u}{\partial\theta}\right) = -6r^n (2\sin\theta\cos^2\theta + \sin^2\theta(-\sin\theta))$$

$$\frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial u}{\partial\theta}\right) = -6r^n (2\sin\theta\cos^2\theta - \sin^3\theta)$$

$$\frac{1}{\sin\theta}\left(\frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial u}{\partial\theta}\right)\right) = -6r^n (2\cos^2\theta - \sin^2\theta)$$

$$\frac{1}{\sin\theta}\left(\frac{\partial}{\partial\theta}\left(\sin\theta \frac{\partial u}{\partial\theta}\right)\right) = -6r^n (3\cos^2\theta - 1) \quad \text{...(1.25.3)}$$

Now put all these values in equation (1.25.1), we get

$$n(n+1) r^n (3\cos^2\theta - 1) - 6r^n (3\cos^2\theta - 1) = 0$$

$$r^n (3\cos^2\theta - 1)[n^2 + n - 6] = 0$$

Since $r^n (3\cos^2\theta - 1) \neq 0$

$$\therefore \quad n^2 + n - 6 = 0$$

$$(n - 2)(n + 3) = 0$$

$$n = 2, -3$$

PART-5

Homogeneous Function, Euler's Theorem, Total Derivatives.

CONCEPT OUTLINE : PART-5

Homogeneous Function : An expression in which every term is of the same degree is called a homogeneous function. An expression of the type $x^n f\left(\frac{y}{x}\right)$ or $y^n f\left(\frac{x}{y}\right)$ is a homogeneous function of $\left(\frac{y}{x}\right)$ or $\left(\frac{x}{y}\right)$.

Euler's Theorem : If u is a homogeneous function of x and y of degree n then $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu$.

Prop. 1 : If u is a homogeneous function of degree n in x and y, then

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial y \partial x} + y^2\frac{\partial^2 u}{\partial y^2} = n(n-1)u$$

Prop. 2 : If $F(u) = V(x, y, z)$, where V is a homogeneous function in x, y, z of degree n, then

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = n\frac{F(u)}{F'(u)}$$

Total Derivatives :

If $u = f(x, y)$, where $x = f_1(t)$, $y = f_2(t)$

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial u}{\partial y}\cdot\frac{dy}{dt}$$

$\frac{du}{dt}$ is called the total differential coefficient of u w.r.t. 't'.

Prop. 1 : If u is a function of x and y and y is a function of x, then

$$\frac{du}{dx} = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\cdot\frac{dy}{dx}$$

Prop. 2 : If $u = f(x, y)$, $x = f_1(t_1, t_2)$, $y = f_2(t_1, t_2)$

Then $$\frac{\partial u}{\partial t_1} = \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial t_1} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial t_1}$$

and $$\frac{\partial u}{\partial t_2} = \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial t_2} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial t_2}$$

Questions-Answers

Long Answer Type and Medium Answer Type Questions

Que 1.26. **State and prove Euler's theorem for homogeneous functions.**

UPTU 2012-13, Marks 05

Answer

Euler's theorem : If u is a homogeneous function of x and y of degree n, then

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu$$

Proof : Let, $$u = x^n f\left(\frac{y}{x}\right)$$

$$\frac{\partial u}{\partial x} = nx^{n-1} f\left(\frac{y}{x}\right) - \left[x^n f'\left(\frac{y}{x}\right)\right]\frac{y}{x^2}$$

$$x\frac{\partial u}{\partial x} = nx^n f\left(\frac{y}{x}\right) - x^{n-1} y f'\left(\frac{y}{x}\right) \quad ...(1.26.1)$$

Similarly, $$\frac{\partial u}{\partial y} = x^n \cdot \frac{1}{x} f'\left(\frac{y}{x}\right)$$

$$y\frac{\partial u}{\partial y} = yx^{n-1} f'\left(\frac{y}{x}\right) \quad ...(1.26.2)$$

Adding equation (1.26.1) and equation (1.26.2), we get

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nx^n f\left(\frac{y}{x}\right)$$

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = nu$$

Hence proved.

Que 1.27. **Verify Euler's theorem for** $u = \sin^{-1}\frac{x}{y} + \tan^{-1}\frac{y}{x}$.

Answer

$$u = \sin^{-1}\frac{x}{y} + \tan^{-1}\frac{y}{x}$$

$$u = \text{cosec}^{-1}\frac{y}{x} + \tan^{-1}\frac{y}{x}$$

$$f(tx, ty) = t^0 f(x, y)$$

Thus u is a homogeneous function of degree 0 in x and y.

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0 \times u = 0$$

$$\frac{\partial u}{\partial x} = \frac{1}{\sqrt{1-\frac{x^2}{y^2}}} \cdot \frac{1}{y} + \frac{1}{1+\frac{y^2}{x^2}} \cdot \left(-\frac{y}{x^2}\right)$$

$$= \frac{1}{\sqrt{y^2 - x^2}} - \frac{y}{\sqrt{x^2 + y^2}}$$

$$x\frac{\partial u}{\partial x} = \frac{x}{\sqrt{y^2 - x^2}} - \frac{xy}{x^2 + y^2} \quad \text{...(1.27.1)}$$

$$\frac{\partial u}{\partial y} = \frac{1}{\sqrt{1-\frac{x^2}{y^2}}}\left(-\frac{x}{y^2}\right) + \frac{x}{x^2 + y^2}$$

$$y\frac{\partial u}{\partial y} = \frac{-x}{\sqrt{y^2 - x^2}} + \frac{xy}{x^2 + y^2} \quad \text{...(1.27.2)}$$

On adding equation (1.27.1) and equation (1.27.2), we get

$x\dfrac{\partial u}{\partial x} + y\dfrac{\partial u}{\partial y} = 0$, which verifies the Euler's theorem.

Que 1.28. **Prove that $xu_x + yu_y = \frac{5}{2}\tan u$ if $u = \sin^{-1}\left(\frac{x^3 + y^3}{\sqrt{x} + \sqrt{y}}\right)$.**

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Answer

$$u = \sin^{-1}\left(\frac{x^3 + y^3}{\sqrt{x} + \sqrt{y}}\right)$$

$\sin u$ is a homogeneous function of degree 5/2, so that by using **Euler's** theorem, we get

$$x\frac{\partial}{\partial x}(\sin u) + y\frac{\partial}{\partial y}(\sin u) = n\,(\sin u)$$

$$x\cos u\frac{\partial u}{\partial x} + y\cos u\frac{\partial u}{\partial y} = \frac{5}{2}\sin u$$

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{5}{2}\tan u$$

Que 1.29. **If $u = x\,\phi\left(\frac{y}{x}\right) + \psi\left(\frac{y}{x}\right)$ prove that**

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = 0$$

Answer

$$u = z_1 + z_2$$

z_1 is a homogeneous function of degree 1 and z_2 is of 0.

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = x\frac{\partial}{\partial x}(z_1 + z_2) + y\frac{\partial}{\partial y}(z_1 + z_2)$$

$$= x\frac{\partial z_1}{\partial x} + y\frac{\partial z_1}{\partial y} + x\frac{\partial z_2}{\partial x} + y\frac{\partial z_2}{\partial y}$$

$$= 1.z_1 + 0.z_2 = z_1 \quad \text{...(1.29.1)}$$

Differentiate equation (1.29.1) partially w.r.t. x and y respectively, we get

$$x\frac{\partial^2 u}{\partial x^2} + \frac{\partial u}{\partial x} + y\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial z_1}{\partial x} \quad \text{...(1.29.2)}$$

$$x\frac{\partial^2 u}{\partial x \partial y} + \frac{\partial u}{\partial y} + y\frac{\partial^2 u}{\partial y^2} = \frac{\partial z_1}{\partial y} \quad \text{...(1.29.3)}$$

Multiplying equation (1.29.2) by x and equation (1.29.3) by y and on adding, we get

$$x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} + x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = x\frac{\partial z_1}{\partial x} + y\frac{\partial z_1}{\partial y}$$

$$\Rightarrow \quad x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} + z_1 = 1.z_1$$

$$\Rightarrow \quad x^2\frac{\partial^2 u}{\partial x^2} + 2xy\frac{\partial^2 u}{\partial x \partial y} + y^2\frac{\partial^2 u}{\partial y^2} = 0$$

Que 1.30. If $u = \cos^{-1}\left(\frac{x+y}{\sqrt{x}+\sqrt{y}}\right)$, show that $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + \frac{1}{2}\cot u = 0$

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Answer

$$u = \cos^{-1}\left(\frac{x+y}{\sqrt{x}+\sqrt{y}}\right)$$

$$\cos u = \frac{x+y}{\sqrt{x}+\sqrt{y}} = f(x, y)$$

Let, $$v = \cos u = \frac{x+y}{\sqrt{x}+\sqrt{y}}$$

v is a homogeneous function of degree (1/2), using Euler's theorem,

$$x\frac{\partial v}{\partial x} + y\frac{\partial v}{\partial y} = nv$$

$$x\frac{\partial}{\partial x}(\cos u) + y\frac{\partial}{\partial y}(\cos u) = \frac{1}{2}\cos u$$

$$x(-\sin u)\frac{\partial u}{\partial x} + y(-\sin u)\frac{\partial u}{\partial y} = \frac{1}{2}\cos u$$

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = \frac{1}{2}\frac{\cos u}{-\sin u}$$

$$x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + \frac{1}{2}\cot u = 0$$

Que 1.31. Verify Euler's theorem for $z = \dfrac{x^{1/3} + y^{1/3}}{x^{1/2} + y^{1/2}}$.

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Answer

$$z = \frac{x^{1/3} + y^{1/3}}{x^{1/2} + y^{1/2}} = f(x, y)$$

$$z = \frac{t^{1/3}(x^{1/3} + y^{1/3})}{t^{1/2}(x^{1/2} + y^{1/2})}$$

$$z = t^{-1/6} f(x, y)$$

z is a homogeneous function of degree $(-1/6)$. Thus using Euler's theorem

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = -\frac{1}{6}z$$

Now,

$$\frac{\partial z}{\partial x} = \frac{(x^{1/2} + y^{1/2})\frac{1}{3}x^{-2/3} - (x^{1/3} + y^{1/3})\frac{1}{2}x^{-1/2}}{(x^{1/2} + y^{1/2})^2}$$

$$x\frac{\partial z}{\partial x} = \frac{\frac{1}{3}x^{1/3}(x^{1/2} + y^{1/2}) - \frac{1}{2}x^{1/2}(x^{1/3} + y^{1/3})}{(x^{1/2} + y^{1/2})^2}$$

Similarly,

$$y\frac{\partial z}{\partial y} = \frac{\frac{1}{3}y^{1/3}(x^{1/2} + y^{1/2}) - \frac{1}{2}y^{1/2}(x^{1/3} + y^{1/3})}{(x^{1/2} + y^{1/2})^2}$$

Now,

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = \frac{1}{(x^{1/2} + y^{1/2})^2}\left[\frac{1}{3}(x^{1/3} + y^{1/3})(x^{1/2} + y^{1/2}) - \frac{1}{2}(x^{1/2} + y^{1/2})(x^{1/3} + y^{1/3})\right]$$

$$= -\frac{1}{6}\frac{(x^{1/3} + y^{1/3})}{(x^{1/2} + y^{1/2})} = -\frac{1}{6}z$$

Thus

$$x\frac{\partial z}{\partial x} + y\frac{\partial z}{\partial y} = -\frac{1}{6}z$$

Que 1.32. **If z is a function of x and y, where $x = e^u + e^{-v}$ and $y = e^{-u} - e^v$, show that**

$$\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} = x\frac{\partial z}{\partial x} - y\frac{\partial z}{\partial y}.$$

Answer

z is a composite function of u and v.

$$\therefore \quad \frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial u} = \frac{\partial z}{\partial x}e^u + \frac{\partial z}{\partial y}(-e^{-u})$$

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial v}$$

$$= \frac{\partial z}{\partial x}(-e^{-v}) + \frac{\partial z}{\partial y}(-e^{v})$$

$$\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} = (e^u + e^{-v})\frac{\partial z}{\partial x} - (e^{-u} - e^v)\frac{\partial z}{\partial y}$$

$$= x\frac{\partial z}{\partial x} - y\frac{\partial z}{\partial y}$$

Que 1.33. **Find $\frac{du}{dt}$ as a total derivative and verify the result by direct substitution if $u = x^2 + y^2 + z^2$ and $x = e^{2t}$, $y = e^{2t}\cos 3t$, $z = e^{2t}\sin 3t$.**

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Answer

$$\frac{du}{dt} = \frac{\partial u}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial u}{\partial y}\cdot\frac{dy}{dt} + \frac{\partial u}{\partial z}\cdot\frac{dz}{dt} \quad \text{...(1.33.1)}$$

$$\frac{\partial u}{\partial x} = 2x, \ \frac{\partial u}{\partial y} = 2y, \ \frac{\partial u}{\partial z} = 2z$$

$$\frac{dx}{dt} = 2e^{2t}, \ \frac{dy}{dt} = 2e^{2t}\cos 3t - 3e^{2t}\sin 3t$$

$$\frac{dz}{dt} = 2e^{2t}\sin 3t + 3e^{2t}\cos 3t$$

Putting all values in equation (1.33.1), we get

$$\frac{du}{dt} = 2x(2e^{2t}) + 2y[2e^{2t}\cos 3t - 3e^{2t}\sin 3t] + 2z[2e^{2t}\sin 3t + 3e^{2t}\cos 3t]$$

$$= 4xe^{2t} + 4ye^{2t}\cos 3t - 6ye^{2t}\sin 3t + 4ze^{2t}\sin 3t + 6ze^{2t}\cos 3t$$

$$= 4e^{2t}[x + y\cos 3t + z\sin 3t] - 6e^{2t}[y\sin 3t - z\cos 3t]$$

$$[\because y = e^{2t}\cos 3t \text{ and } z = e^{2t}\sin 3t]$$

$$= 4e^{2t}x + 4e^{2t}[e^{2t}\cos^2 3t + e^{2t}\sin^2 3t] - 6e^{2t}[e^{2t}\cos 3t\sin 3t - e^{2t}\cos 3t\sin 3t]$$

$= 4e^{2t}x + 4e^{2t}[e^{2t}] = 4e^{2t}.e^{2t} + 4e^{2t+2t}$
$= 4e^{4t} + 4e^{4t}$

$$\frac{du}{dt} = 8e^{4t}$$

Now from direct substitution

$$u = (e^{2t})^2 + (e^{2t}\cos 3t)^2 + (e^{2t}\sin 3t)^2$$
$$u = e^{4t} + e^{4t}[\cos^2 3t + \sin^2 3t]$$
$$u = e^{4t} + e^{4t}$$
$$u = 2e^{4t}$$

Now differentiating it

$$\frac{du}{dt} = 8e^{4t}$$

So, the result is verified.

Que 1.34. **If $V = f(2x - 3y, 3y - 4z, 4z - 2x)$ prove that $6V_x + 4V_y + 3V_z = 0$.**

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Answer

$$V = f(P, Q, R)$$

where

$$P = 2x - 3y$$
$$Q = 3y - 4z$$
$$R = 4z - 2x$$

$$V_x = \frac{\partial V}{\partial P}\frac{\partial P}{\partial x} + \frac{\partial V}{\partial Q}\frac{\partial Q}{\partial x} + \frac{\partial V}{\partial R}\frac{\partial R}{\partial x}$$

$$V_x = 2\frac{\partial V}{\partial P} + 0 - 2\frac{\partial V}{\partial R}$$

$$6V_x = 12\frac{\partial V}{\partial P} - 12\frac{\partial V}{\partial R} \quad ...(1.34.1)$$

Similarly,

$$V_y = \frac{\partial V}{\partial P}(-3) + \frac{\partial V}{\partial Q}(3) + \frac{\partial V}{\partial R}(0)$$

$$4V_y = -12\frac{\partial V}{\partial P} + 12\frac{\partial V}{\partial Q} \quad ...(1.34.2)$$

$$V_z = \frac{\partial V}{\partial P}(0) + \frac{\partial V}{\partial Q}(-4) + \frac{\partial V}{\partial R}(4)$$

$$3V_z = -12\frac{\partial V}{\partial Q} + 12\frac{\partial V}{\partial R} \quad ...(1.34.3)$$

On adding equation (1.34.1), equation (1.34.2) and equation (1.34.3), we get

$$6V_x + 4V_y + 3V_z = 0$$

PART-6

Change of Independent Variables.

CONCEPT OUTLINE : PART-6

Change of Independent Variables : If there be an expression involving two variables x and y and containing the differential coefficients $\frac{dy}{dx}, \frac{d^2y}{dx^2}$, etc., it is sometimes desirable to change the independent variable into the dependent variable or to change the independent variable from x to some third variable z of which x is a known function.

To change the Independent Variable into the Dependent variable : If we make y independent variable instead of x, we have

$$\frac{dy}{dx} = \frac{1}{dx/dy} = \left(\frac{dx}{dy}\right)^{-1}$$

and
$$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{dx}\left(\frac{dx}{dy}\right)^{-1}$$

$$= -1\left(\frac{dx}{dy}\right)^{-2}\frac{d^2x}{dy^2}\cdot\frac{dy}{dx} = -\left(\frac{dx}{dy}\right)^{-3}\frac{d^2x}{dy^2}$$

Similarly,
$$\frac{d^3y}{dx^3} = 3\left(\frac{dx}{dy}\right)^{-5}\left(\frac{d^2x}{dy^2}\right)^2 - \left(\frac{dx}{dy}\right)^{-4}\frac{d^3x}{dy^3}$$

To change the independent variable x into another variable z, where $x = f(z)$.

$$\frac{dy}{dx} = \frac{dy}{dz}\cdot\frac{dz}{dx} = \left(\frac{dx}{dz}\right)^{-1}\frac{d}{dz}(y)$$

The operator $\frac{d}{dx}$ is thus equivalent to the operator $\left(\frac{dx}{dz}\right)^{-1}\frac{d}{dz}$ and we write $\frac{d}{dx} \equiv \left(\frac{dx}{dz}\right)^{-1}\frac{d}{dz}$.

$$\frac{d^2y}{dx^2} = \left(\frac{dx}{dz}\right)^{-3}\left[\frac{dx}{dz}\cdot\frac{d^2y}{dz^2} - \frac{d^2x}{dz^2}\frac{dy}{dz}\right]$$

Similarly,

$$\frac{d^3y}{dz^3} = \left(\frac{dx}{dz}\right)^{-5}\left[\left(\frac{dx}{dz}\right)\left\{\frac{dx}{dz}\frac{d^3y}{dz^3} - \frac{d^3x}{dz^3}\frac{dy}{dz}\right\} - 3\frac{d^2x}{dz^3}\left\{\frac{dx}{dz}\frac{d^2y}{dz^2} - \frac{d^2x}{dz^2}\frac{dy}{dz}\right\}\right]$$

Transformation in the case of two Independent Variables :

a. If $u = f(x, y)$, then $du = \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy$.

b. If $u = f(x, y)$, where $x = \phi(t)$ and $y = \psi(t)$,

Then $$\frac{du}{dt} = \frac{\partial u}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial u}{\partial y}\cdot\frac{dy}{dt}$$

c. If $u = f(x, y)$, where $x = \phi(t_1, t_2)$ and $y = \psi(t_1, t_2)$,

Then $$\frac{\partial u}{\partial t_1} = \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial t_1} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial t_1}$$

$$\frac{\partial u}{\partial t_2} = \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial t_2} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial t_2}$$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial t_1}\cdot\frac{\partial t_1}{\partial x} + \frac{\partial u}{\partial t_2}\cdot\frac{\partial t_2}{\partial x}$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial t_1}\cdot\frac{\partial t_1}{\partial y} + \frac{\partial u}{\partial t_2}\cdot\frac{\partial t_2}{\partial y}$$

Questions-Answers

Long Answer Type and Medium Answer Type Questions

Que 1.35. **Show that the equation $x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + y = 0$ becomes $\frac{d^2y}{dz^2} + y = 0$ by substituting e^z for x.**

Answer

We have, $$\frac{dy}{dx} = \frac{dy}{dz}\cdot\frac{dz}{dx}$$

But $$e^z = x$$

Therefore $$e^z\frac{dz}{dx} = 1$$

or $$\frac{dz}{dx} = \frac{1}{e^z} = \frac{1}{x}$$

$$\frac{dy}{dx} = \frac{1}{x}\frac{dy}{dz}$$

$$x\frac{dy}{dx} = \frac{dy}{dz}$$

$\therefore$ $$x\frac{d}{dx} = \frac{d}{dz}$$

or $$x\frac{d}{dx}\left(x\frac{dy}{dx}\right) = \frac{d}{dz}\left(\frac{dy}{dz}\right)$$

$\therefore$ $$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} = \frac{d^2y}{dz^2}$$

$\therefore$ Hence the given equation becomes

$\frac{d^2y}{dz^2} + y = 0$ by substituting e^z for x.

Que 1.36. Transform $\frac{d^2y}{dx^2}$ **to the new variables u and v, taking u as independent variable, given**

$$x = \frac{1}{v}, y = uv.$$

Answer

We have, $$x = \frac{1}{v} \Rightarrow \frac{dx}{du} = -\frac{1}{v^2}\frac{dv}{du}$$

Also $$y = uv \Rightarrow \frac{dy}{du} = v + u\frac{dv}{du}$$

Now, $$\frac{dy}{dx} = \frac{dy/du}{dx/du} = \frac{v + u\,(dv/du)}{-\frac{1}{v^2}\frac{dv}{du}}$$

$$= \frac{-v^3 - uv^2\frac{dv}{du}}{\frac{dv}{du}}$$

Again $$\frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d}{du}\left(\frac{dy}{dx}\right)\cdot\frac{du}{dx}$$

$$= \frac{d}{du}\left[\frac{-v^3 - uv^2\frac{dv}{du}}{\frac{dv}{du}}\right]\left(-v^2\frac{du}{dv}\right)$$

$$\left(\frac{dv}{du}\right)\left\{3v^2\left(\frac{dv}{du}\right)+2uv\left(\frac{dv}{du}\right)^2+v^2\left(\frac{d^2v}{du^2}\right)\right\}$$

$$= \left(v^2\frac{du}{dv}\right)\cdot\frac{-\frac{d^2v}{du^2}\left\{v^2+uv^2\frac{dv}{du}\right\}}{\left(\frac{dv}{du}\right)^2}$$

$$= v^2\left(\frac{dv}{du}\right)^{-3}\left[4v^2\left(\frac{dv}{du}\right)^2+2uv\left(\frac{dv}{du}\right)^3-u^3\frac{d^2v}{du^2}\right]$$

$$= 4v^4\left(\frac{dv}{du}\right)^{-1}+2uv^3-v^5\left(\frac{dv}{du}\right)^{-3}\frac{d^2v}{du^2}$$

Que 1.37. **If $x = e^{\theta}$, $y = e^{\phi}$, prove that**

$$e^{2\theta}\frac{\partial^2 v}{\partial x^2}+e^{2\phi}\frac{\partial^2 v}{\partial y^2}+e^{\theta}\frac{\partial v}{\partial x}+e^{\phi}\frac{\partial v}{\partial y}=\frac{\partial^2 v}{\partial \theta^2}+\frac{\partial^2 v}{\partial \phi^2}.$$

Answer

We have,
$$\frac{\partial v}{\partial \theta} = \frac{\partial v}{\partial x}\cdot\frac{\partial x}{\partial \theta}+\frac{\partial v}{\partial y}\cdot\frac{\partial y}{\partial \theta}$$

$$= e^{\theta}\frac{\partial v}{\partial x}+0.\frac{\partial v}{\partial y} = x\frac{\partial v}{\partial x}$$

$\therefore$
$$\frac{\partial}{\partial \theta} = x\frac{\partial}{\partial x}$$

$$\frac{\partial}{\partial \theta}\left(\frac{\partial v}{\partial \theta}\right) = x\frac{\partial}{\partial x}\left(x\frac{\partial v}{\partial x}\right)$$

or
$$\frac{\partial^2 v}{\partial \theta^2} = x^2\frac{\partial^2 v}{\partial x^2}+x\frac{\partial v}{\partial x} \quad ...(1.37.1)$$

Again
$$\frac{\partial v}{\partial \phi} = \frac{\partial v}{\partial x}\cdot\frac{\partial x}{\partial \phi}+\frac{\partial v}{\partial y}\cdot\frac{\partial y}{\partial \phi}$$

$$= 0\frac{\partial v}{\partial x}+e^{\phi}\frac{\partial v}{\partial y} = y\frac{\partial v}{\partial y}$$

$$\frac{\partial}{\partial \phi} = y\frac{\partial}{\partial y}$$

$\therefore$
$$\frac{\partial}{\partial \phi}\left(\frac{\partial v}{\partial \phi}\right) = y\frac{\partial}{\partial y}\left(y\frac{\partial v}{\partial y}\right)$$

$$\frac{\partial^2 v}{\partial \phi^2} = y^2\frac{\partial^2 v}{\partial y^2}+y\frac{\partial v}{\partial y} \quad ...(1.37.2)$$

Adding equation (1.37.1) and equation (1.37.2), we get

$$\frac{\partial^2 v}{\partial \theta^2}+\frac{\partial^2 v}{\partial \phi^2} = x^2\frac{\partial^2 v}{\partial x^2}+x\frac{\partial v}{\partial x}+y^2\frac{\partial^2 v}{\partial y^2}+y\frac{\partial v}{\partial y}$$

$$= e^{2\theta}\frac{\partial^2 v}{\partial x^2}+e^{2\phi}\frac{\partial^2 v}{\partial y^2}+e^{\theta}\frac{\partial v}{\partial x}+e^{\phi}\frac{\partial v}{\partial y}$$

PART-7

Curve Tracing

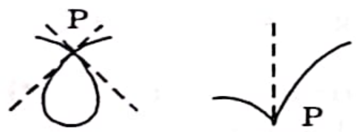
CONCEPT OUTLINE : PART-7

Curve Tracing : To find approximate shape of curves from their cartesian, polar or parametric equation without plotting a large number of data points.

A point through which two branches of a curve pass is called a double point. At such a point P, the curve has two tangents, one for each branch.

If the tangents are real and distinct, the double point is called a node Fig. (*a*)

If the tangents are real and coincident, the double point is called a cusp Fig. (*b*)



(*a*) (*b*)

If the tangents are imaginary, the double point is called a conjugate point (or an isolated point). Such a point cannot be shown in the Fig.

Procedure for Tracing Cartesian Curves :

1. **Symmetry :** See if the curve is symmetrical about any line.
   a. A curve is symmetrical about the X-axis, if only even powers of y occur in its equation. (e.g. $y^2 = 4ax$ is symmetrical about X-axis).

   OR

   If the equation remains same by replacing y by $-y$.
   b. A curve is symmetrical about the Y-axis, if only even powers of x occur in its equation. (e.g. $x^2 = 4ay$ is symmetrical about Y-axis).
   c. A curve is symmetrical about the line $y = x$, if on interchanging x and y its equation remains unchanged, (*e.g.* $x^3 + y^3 = 3axy$ is symmetrical about the line $y = x$).
2. **Origin :**
   a. See if the curve passes through the origin.

(A curve passes through the origin if there is no constant term in its equation).

b. If it does, find the equation of the tangents at that place by equating to zero the lowest degree terms.

c. If the origin is a double point, find whether the origin is a node, cusp or conjugate point

3. Asymptotes :

a. See if the curve has any asymptote parallel to the axes.

b. Then find the inclined asymptotes, if need be.

4. Points :

a. Find the points where the curve crosses the axes and the asymptotes.

b. Find the points where the tangent is parallel or perpendicular to the x-axis (*i.e.* the points where $dy/dx = 0$ or ∞).

c. Find the region (or regions) in which no portion of the curve exists.

Procedure for tracing polar curves :

1. Symmetry : See if the curve is symmetrical about any line.

a. A curve is symmetrical about the initial line OX, if only $\cos \theta$ (or $\sec \theta$) occur in its equation. (*i.e.* it remains unchanged when θ is changed to $-\theta$) e.g., $r = a (1 + \cos \theta)$ is symmetrical about the initial line.

b. A curve is symmetrical about the line through the pole $\perp$ to the initial line (*i.e.* OY), if only $\sin \theta$ (or $\operatorname{cosec} \theta$) occur in its equation. (*i.e.* it remains unchanged when θ is changed to $\pi - \theta$) e.g., $r = a \sin 3\theta$ is symmetrical about OY.

c. A curve is symmetrical about the pole, if only even powers of r occur in the equation (*i.e.*, it remains unchanged when r is changed to $-r$) e.g., $r^2 = a^2 \cos 2\theta$ is symmetrical about the pole.

2. Limits : See if r and θ are confined between certain limits.

a. Determine the numerically greatest value of r, so as to notice whether the curve lies within a circle or not e.g., $r = a \sin 3\theta$ lies wholly within the circle $r = a$.

b. Determine the region in which no portion of the curve lies by finding those values of θ for which r is imaginary. e.g., $r^2 = a^2 \cos 2\theta$ does not lie between the lines $\theta = \pi/4$ and $\theta = 3\pi/4$.

3. Asymptotes : If the curve possesses an infinite branch, find the asymptotes.

4. Points :

a. Giving successive values to θ, find the corresponding values of r.

b. Determine the points where the tangent coincides with the radius vector or is perpendicular to it (*i.e.*, the points where $\tan \phi = r\, d\theta/dr = 0$ or ∞).

Questions-Answers

Long Answer Type and Medium Answer Type Questions

Que 1.38. **Trace the curve $y^2(a - x) = x^3$.**

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Answer

Given curve is $y^2(a - x) = x^3$

Symmetry : All powers of y are even, the curve is symmetrical about X-axis.

Origin : Equation does not contain any constant term, therefore, it passes through origin.

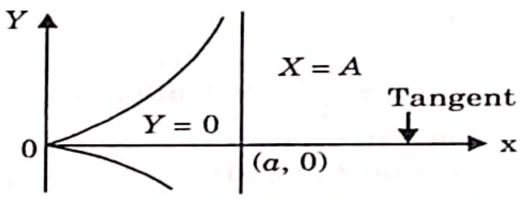

Fig. 1.39.1.

Region of absence of curve :

Tangent at origin : By equating the lowest degree term to zero

$$ay^2 = 0$$
$$y = 0, 0 \text{ (double point)}$$

Cusp : Two tangents are coincident, therefore, origin is a cusp.

Asymptote : Parallel to Y-axis by equating the coefficient of highest degree of y to zero

$$a - x = 0$$
$$x = a$$

Equation of Asymptote

$$y = \sqrt{\frac{x^3}{a - x}}$$

y^2 is negative for $x < 0$ and $x > a$.

Thus curve does not exist for $a < x < 0$.

Required curve is shown in the given Fig. 1.38.1.

Que 1.39. **Trace the curve $y^2(2a - x) = x^3$.**

UPTU 2014-15, 2015-16; Marks 05

Answer

Same as Q. 1.38, Page 37A, Unit-1.
Replace a by $2a$.

Que 1.40. **Trace the curve : $r^2 = a^2 \cos 2\theta$.**

UPTU 2011-12, 2014-15; Marks 05

Answer

Symmetry : The given curve is symmetrical about the initial line, about pole.

Tangent : Tangent at pole is given by putting $r = 0$

$$\cos 2\theta = 0$$

$$\cos 2\theta = \cos\left(\pm \frac{\pi}{2}\right)$$

$$2\theta = \pm\frac{\pi}{2}$$

$$\theta = \pm\frac{\pi}{4}, \text{ which are real.}$$

Also when $\theta = 0$, $r = \pm a$

The curve meets the line at $(\pm a, 0)$.

Greatest and least value of r :

When $\theta = 0$, $r = a$

$$\theta = \frac{\pi}{4}, r = 0$$

When $\frac{\pi}{4} < \theta < \frac{3\pi}{4}$, r is negative.

Thus curve does not lie within $\frac{\pi}{4} < \theta < \frac{3\pi}{4}$.

When $\theta = \frac{3\pi}{4}$, $r = 0$

When $\theta = \pi$, $r = a$

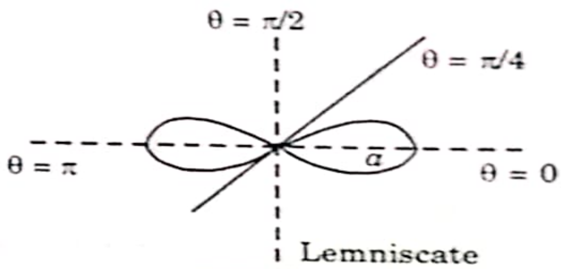

Fig. 1.40.1.

Que 1.41. **Trace the curve : $4ay^2 = x(x-2a)^2$.**

UPTU 2013-14, Marks 05

Answer

$$4ay^2 = x(x-2a)^2$$

1. **Origin :** The equation of the curve does not contain any constant term. Therefore it passes through origin.
2. **Symmetric :** Equation contains only even powers of y, therefore, it is symmetrical about X-axis.
3. **Point of intersection with X-axis :** On putting $y = 0$, we get

$$x(x-2a)^2 = 0$$
$$x = 0, 2a$$

On putting $x = 0$, we get

$$y = 0$$

So, the point of intersection is $(2a, 0)$.

4. **Region :** $4ay^2 = x(x-2a)^2$

$$y = \frac{\sqrt{x}}{\sqrt{4a}}(x-2a)$$

y becomes negative if $x < 0$.

$\therefore$ Curve does not lie on left of Y-axis.

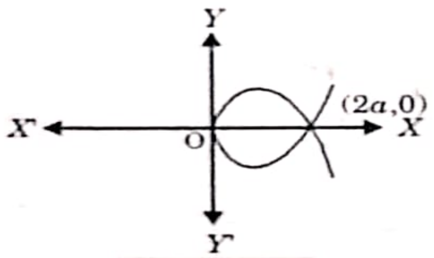

Fig. 1.41.1.

5. **Asymptotes :** The curve has no asymptote.
6. **Tangent :** Equating the lowest degree term to zero, we get $x = 0$.

$\therefore$ Tangent at origin is Y-axis.

Que 1.42. **Trace the curve $r = a \cos 3\theta$.**

Answer

a. The curve is symmetrical about the initial line.

b. We have $r = 0$, when $\cos 3\theta = 0$

i.e., $3\theta = \pm\frac{\pi}{2}, \pm\frac{3}{2}\pi, \pm\frac{5}{2}\pi$ etc. or $\theta = \pm\frac{\pi}{6}, \pm\frac{\pi}{2}, \pm\frac{5}{6}\pi$ etc.

Thus the lines $\theta = \pm\frac{\pi}{6}, \pm\frac{\pi}{2}$ etc., are the tangents to the curve at the pole.

c. Differentiate the equation of the curve, we get $\frac{dr}{d\theta} = -3a\sin 3\theta$

Therefore $\cot\phi = \frac{1}{r}\frac{dr}{d\theta} = \frac{1}{a\cos 3\theta}(-3a\sin 3\theta) = (-3\tan 3\theta)$

Now, $\phi = 90°$

When $\tan 3\theta = 0$

i.e., 3θ	:	0	π	2π	3π
θ	:	0	$\frac{\pi}{3}$	$\frac{2\pi}{3}$	π

At all these points the tangent to the curve is perpendicular to the radius vector. Also at each of these points the numerical value of r is a, which is the greatest value of the radius vector for this curve.

d.

θ	:	0	$\frac{\pi}{6}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{5\pi}{6}$	π
r	:	a	0	$-a$	0	a	0	$-a$

The curve is :

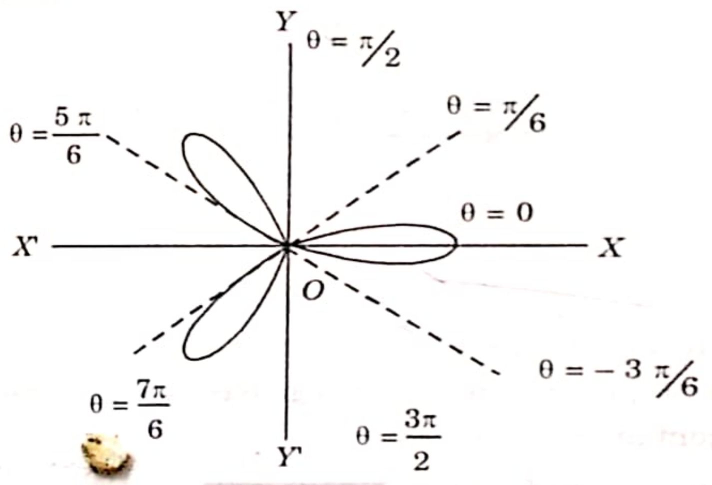

Fig. 1.43.1.

Que 1.43. **Trace the curve $r = a(1 - \cos\theta)$.**

Answer

a. The curve is symmetrical about the initial line.

b. We have $r = 0$ when $1 - \cos\theta = 0$ or $\cos\theta = 1$ or $\theta = 0$. Therefore the line $\theta = 0$ is tangent to the curve at the pole.

c. $$\frac{dr}{d\theta} = a \sin \theta$$

Therefore $$\cot \phi = \frac{1}{r}\frac{dr}{d\theta} = \frac{a \sin \theta}{a(1-\cos\theta)} = \cot \frac{\theta}{2}$$

Now, $\phi = \dfrac{\pi}{2}$ when $\dfrac{\theta}{2} = \dfrac{\pi}{2}$ *i.e.*, $\theta = \pi$.

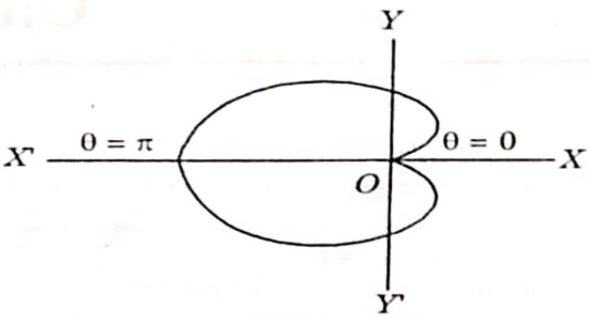

Fig. 1.43.1.

Thus at the point $\theta = \pi$ the tangent to the curve is perpendicular to the radius vector.

d. The table is :

θ	=	0	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	π
r	=	0	$\frac{a}{2}$	a	$\frac{3a}{2}$	$2a$

Thus as θ increases from 0 to π, r also increases from 0 to $2a$.
The curve is shown in Fig. 1.43.1.

☺☺☺

2 UNIT

Differential Calculus-II

PART-1

Taylor's and Maclaurin's Theorem, Expansion of function of several variables

CONCEPT OUTLINE : PART-1

Expansion of Function of one Variable :

Taylor's Series : Let $f(x)$ possesses continuous derivatives of all orders in the interval $[a, a+h]$, then for every positive integer value of n,

$$f(a+h) = f(a) + h f'(a) + \frac{h^2}{\lfloor 2} f''(a) + \ldots\ldots\ldots + \frac{h^n f^n(a)}{\lfloor n} \quad \ldots(1)$$

The series (1) is the Taylor's series for the expansion of $f(a+h)$ in powers of h.

Note :

a. Put $a + h = b$ or $h = b - a$, in equation (1)

$$f(b) = f(a) + (b-a) f'(a) + \frac{(b-a)^2}{\lfloor 2} f''(a) + \ldots\ldots + \frac{(b-a)^n}{\lfloor n} f^n(a)$$

b. Put $h = x - a$ in equation (1), we get

$$f(x) = f(a) + (x-a) f'(a) + \frac{(x-a)^2}{\lfloor 2} f''(a) + \ldots\ldots\ldots + \frac{(x-a)^n}{\lfloor n} f^n(a)$$

c. Put $a = 0$ and $h = x$ in equation (1), we get

$$f(x) = f(0) + x f'(0) + \frac{x^2}{\lfloor 2} f''(0) + \ldots\ldots\ldots + \frac{x^n}{\lfloor n} f^n(0)$$

This is the Maclaurin's series.

Maclaurin's Series : Suppose $f(x)$ possesses continuous derivatives of all orders in the interval $[0, x]$. Then for every positive integral value of n,

$$f(x) = f(0) + x f'(0) + \frac{x^2}{\lfloor 2} f''(0) + \ldots\ldots\ldots + \frac{x^n}{\lfloor n} f^n(0) \quad \ldots(2)$$

The series (2) is known as Maclaurin's infinite series for the expansion of $f(x)$ in powers of x.

Expansion of a Function of Two Variables : Taylor's theorem for a function of two variables,

$$f(x+h, y+k) = f(x, y) + \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right) f + \frac{1}{2!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^2 f + \frac{1}{3!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^3 f + \ldots\ldots\ldots$$

Corollary 1 : Put $x = a, y = b$

$$f(a+h, b+k) = f(a,b) + [h\, f_x(a,b) + k\, f_y(a,b)] + \frac{1}{2!}[h^2 f_{xx}(a,b) + 2hk\, f_{xy}(a,b) + k^2 f_{yy}(a,b)] + \ldots\ldots$$

Corollary 2 : In above expression put $a + h = x$, $b + k = y$

$$f(x, y) = f(a,b) + [(x-a)f_x(a,b) + (y-b)f_y(a,b)] + \frac{1}{2!}[(x-a)^2 f_{xx}(a,b) + 2(x-a)(y-b)\, f_{xy}(a,b) + (y-b)^2 f_{yy}(a,b)] + \ldots.$$

Corollary 3 : Put $a = 0$, $b = 0$ in above expression

$$f(x, y) = f(0, 0) + [xf_x(0,0) + yf_y(0,0)] + \frac{1}{2!}[x^2 f_{xx}(0, 0) + 2xy\, f_{xy}(0, 0) + y^2 f_{yy}(0, 0)] + \ldots$$

Questions-Answers

Long Answer Type and Medium Answer Type Questions

Que 2.1. Find the Taylor's series expansion of $f(xy) = x^3 + xy^2$ about point (2, 1). **UPTU 2011-12, Marks 02**

Answer

$$f(x, y) = x^3 + xy^2 \qquad f(2, 1) = 10$$

$$\frac{\partial f}{\partial x} = 3x^2 + y^2 \qquad f_x(2, 1) = 13$$

$$\frac{\partial^2 f}{\partial x^2} = 6x \qquad f_{xx}(2, 1) = 12$$

$$\frac{\partial f}{\partial y} = 2xy \qquad f_y(2, 1) = 4$$

$$f_{yy}(x, y) = 2x \qquad f_{yy}(2, 1) = 4$$

$$f_{xy}(x, y) = 2y \qquad f_{xy}(2, 1) = 2$$

Using Taylor's theorem,

$$f(x, y) = f(2, 1) + [(x-2)f_x(2, 1) + (y-1)\, f_y(2, 1)]$$

$$+ \frac{1}{2!}[(x-2)^2 f_{xx}(2, 1) + 2(x-2)(y-1)\, f_{xy}(2, 1) + (y-1)^2 f_{yy}(2, 1)] + \ldots$$

$$= 10 + 13(x-2) + 4(y-1) + \frac{1}{2!}[(x-2)^2\, 12 + 4(x-2)(y-1) + 4(y-1)^2] + \ldots$$

Que 2.2. **Expand $\sin x$ in ascending powers of $\left(x-\frac{\pi}{2}\right)$.**

Answer

$$f(x) = \sin x$$

$$f(x) = \sin\left(\frac{\pi}{2}+x-\frac{\pi}{2}\right) = f(a+h)$$

$$a = \frac{\pi}{2},\ h = x-\frac{\pi}{2}$$

$$\sin x = f\left(\frac{\pi}{2}\right)+\left(x-\frac{\pi}{2}\right)f'\left(\frac{\pi}{2}\right)+\frac{\left(x-\frac{\pi}{2}\right)^2}{2!}f''\left(\frac{\pi}{2}\right)$$
$$+\frac{\left(x-\frac{\pi}{2}\right)^3}{3!}f'''\left(\frac{\pi}{2}\right)+\ldots.. \qquad \ldots(2.2.1)$$

$$f(x) = \sin x \qquad f\left(\frac{\pi}{2}\right) = 1$$

$$f'(x) = \cos x \qquad f'\left(\frac{\pi}{2}\right) = 0$$

$$f''(x) = -\sin x \qquad f''\left(\frac{\pi}{2}\right) = -1$$

$$f'''(x) = -\cos x \qquad f'''\left(\frac{\pi}{2}\right) = 0$$

$$f''''(x) = \sin x \qquad f''''\left(\frac{\pi}{2}\right) = 1$$

Put these values in equation (2.2.1),

$$\sin x = 1+\left(x-\frac{\pi}{2}\right)(0)+\frac{\left(x-\frac{\pi}{2}\right)^2}{2!}(-1)+\frac{\left(x-\frac{\pi}{2}\right)^4}{4!}(1)+\ldots\ldots.$$

$$\sin x = 1-\frac{\left(x-\frac{\pi}{2}\right)^2}{2!}+\frac{\left(x-\frac{\pi}{2}\right)^4}{4!}+\ldots.$$

Que 2.3. **Expand $\log x$ in powers of $(x-1)$ by Taylor's theorem and hence find $\log(1.1)$.**

Answer

$$f(x) = \log x = \log(1+x-1)$$
$$= \log(a+h),\ a=1,\ h=x-1$$

$$f(x) = \log x \qquad f(1) = 0$$

$$f'(x) = \frac{1}{x} \qquad f'(1) = 1$$

$$f''(x) = -\frac{1}{x^2} \qquad f''(1) = -1$$

$$f'''(x) = \frac{2}{x^3} \qquad f'''(1) = 2$$

$$f''''(x) = \frac{-6}{x^4} \qquad f''''(1) = -6$$

Put in Taylor's series,

$$f(x) = \log x = 0 + (x-1)\,1 + \frac{(x-1)^2}{2!}(-1) + \frac{(x-1)^3}{3!}(2) + \frac{(x-1)^4}{4!}(-6) + \ldots\ldots$$

$$\log x = (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 - \frac{1}{4}(x-1)^4 + \ldots\ldots$$

Put $x = 1.1$

$$\log(1.1) = (1.1-1) - \frac{1}{2}(1.1-1)^2 + \frac{1}{3}(1.1-1)^3 - \frac{1}{4}(1.1-1)^4 + \ldots$$

$$\log(1.1) = 0.095305 \text{ (Approximate)}$$

Que 2.4. **Expand $e^{ax} \sin by$ in the powers of x and y as far as terms of third degree.**

UPTU 2010-11, Marks 05

Answer

$$\begin{aligned}
f(x, y) &= e^{ax} \sin by & f(0, 0) &= 0 \\
f_x(x, y) &= ae^{ax} \sin by & f_x(0, 0) &= 0 \\
f_{xx}(x, y) &= a^2e^{ax} \sin by & f_{xx}(0, 0) &= 0 \\
f_{xxx}(x, y) &= a^3e^{ax} \sin by & f_{xxx}(0, 0) &= 0 \\
f_y(x, y) &= be^{ax} \cos by & f_y(0, 0) &= b \\
f_{yy}(x, y) &= -b^2e^{ax} \sin by & f_{yy}(0, 0) &= 0 \\
f_{yyy}(x, y) &= -b^3e^{ax} \cos by & f_{yyy}(0, 0) &= -b^3 \\
f_{xy}(x, y) &= ab\, e^{ax} \cos by & f_{xy}(0, 0) &= ab \\
f_{xxy}(x, y) &= a^2 b\, e^{ax} \cos by & f_{xxy}(0, 0) &= a^2b \\
f_{xyy}(x, y) &= -ab^2e^{ax} \sin by & f_{xyy}(0, 0) &= 0
\end{aligned}$$

We know that,

$$f(x, y) = f(a, b) + [(x-a) f_x(a, b) + (y-b) f_y(a, b)]$$

$$+ \frac{1}{2!}[(x-a)^2 f_{xx}(a, b) + 2(x-a)(y-b) f_{xy}(a, b)$$

$$+ (y-b)^2 f_{yy}(a, b)] + \frac{1}{3!}[(x-a)^3 f_{xxx}(a, b) + 3(x-a)^2(y-b) f_{xxy}(a, b)$$

$$+ 3(x-a)(y-b)^2 f_{xyy}(a, b) + (y-b)^3 f_{yyy}(a, b)] + \ldots$$

At point (0, 0),

$$f(x, y) = 0 + x(0) + y(b) + \frac{1}{2!}[x^2(0) + 2xy(ab) + y^2(0)]$$
$$+ \frac{1}{3!}[(x)^3(0) + 3x^2y(a^2 b) + 3xy^2(0) + y^3(-b^3)] + \ldots$$

$$f(x, y) = by + abxy + \frac{a^2b\, x^2y}{2} - \frac{b^3\, y^3}{6} + \ldots$$

Que 2.5. Expand $f(x, y) = e^x \tan^{-1} y$ in powers of $(x - 1)$ and $(y - 1)$ up to two terms of degree 2.

UPTU 2012-13, Marks 05

Answer

$$f(x, y) = e^x \tan^{-1} y \Rightarrow f(1, 1) = e^1 \tan^{-1} 1 = \pi e/4$$
$$f_x(x, y) = e^x \tan^{-1} y \Rightarrow f_x(1, 1) = \pi e/4$$
$$f_{xx}(x, y) = e^x \tan^{-1} y \Rightarrow f_{xx}(1, 1) = e^1 \tan^{-1} 1 = \pi e/4$$
$$f_y(x, y) = e^x/(1 + y^2) \Rightarrow f_y(1, 1) = e/2$$
$$f_{yy}(x, y) = -2ye^x/(1 + y^2)^2 \Rightarrow f_{yy}(1, 1) = -2e/4 = -e/2$$
$$f_{xy}(x, y) = \frac{e^x}{1 + y^2} \Rightarrow f_{xy}(1, 1) = e/2$$

Using Taylor's expansion,

$$f(x, y) = f(1, 1) + [(x - 1) f_x(1, 1) + (y - 1) f_y(1, 1)]$$
$$+ \frac{1}{2!}[(x - 1)^2 f_{xx}(1, 1) + 2(x - 1)(y - 1) f_{xy}(1, 1) + (y - 1)^2 f_{yy}(1, 1)] + \ldots$$

$$= \frac{\pi e}{4} + (x - 1)\frac{\pi e}{4} + (y - 1)\frac{e}{2} + \frac{1}{2!}\left[(x - 1)^2 \frac{\pi e}{4} + 2(x - 1)(y - 1)\frac{e}{2} + (y - 1)^2\left(-\frac{e}{2}\right)\right] + \ldots$$

$$f(x, y) = \frac{\pi e}{4} + (x - 1)\frac{\pi e}{4} + (y - 1)\frac{e}{2} + \frac{1}{2!}\left[(x - 1)^2 \frac{\pi e}{4} + e(x - 1)(y - 1) - \frac{e}{2}(y - 1)^2\right] + \ldots$$

Que 2.6. Obtain Taylor's expansion of $\tan^{-1}\left(\frac{y}{x}\right)$, about (1, 1) up to and including the second degree terms.

Answer

Let $$f(x, y) = \tan^{-1}\left(\frac{y}{x}\right), \quad f(1, 1) = \frac{\pi}{4}$$

$$\frac{\partial f}{\partial x} = \frac{1}{1+\frac{y^2}{x^2}}\left(-\frac{y}{x^2}\right) = \frac{-y}{x^2+y^2}, f_x(1, 1) = -\frac{1}{2}$$

$$\frac{\partial f}{\partial y} = \frac{1}{1+\frac{y^2}{x^2}}\left(\frac{1}{x}\right) = \frac{x}{x^2+y^2}, f_y(1, 1) = \frac{1}{2}$$

$$\frac{\partial^2 f}{\partial x^2} = \frac{2xy}{(x^2+y^2)^2}, \quad f_{xx}(1, 1) = \frac{1}{2}$$

$$\frac{\partial^2 f}{\partial y^2} = \frac{-2xy}{(x^2+y^2)^2}, \quad f_{yy}(1, 1) = -\frac{1}{2}$$

$$\frac{\partial^2 f}{\partial x \partial y} = \frac{x^2+y^2-2x^2}{(x^2+y^2)^2} = \frac{y^2-x^2}{(x^2+y^2)^2}, f_{xy}(1, 1) = 0$$

By Taylor's theorem,

$$f(x, y) = f(a, b) + \left[(x-a)\frac{\partial f}{\partial x} + (y-b)\frac{\partial f}{\partial y}\right] +$$

$$\frac{1}{2!}\left[(x-a)^2\frac{\partial^2 f}{\partial x^2} + 2(x-a)(y-b)\frac{\partial^2 f}{\partial x \partial y} + (y-b)^2\frac{\partial^2 f}{\partial y^2}\right] + \ldots$$

$$\tan^{-1}\left(\frac{y}{x}\right) = \frac{\pi}{4} + (x-1)\left(-\frac{1}{2}\right) + (y-1)\frac{1}{2} + \frac{1}{2!}\left[(x-1)^2\left(\frac{1}{2}\right) + 0 + (y-1)^2\left(-\frac{1}{2}\right)\right]$$

$$= \frac{\pi}{4} - \frac{1}{2}(x-1) + \frac{1}{2}(y-1) + \frac{1}{4}(x-1)^2 - \frac{1}{4}(y-1)^2$$

Que 2.7. **Find the first six terms of the expansion of $e^x \log(1+y)$.**

UPTU 2014-15, Marks 10

Answer

$$f_x = e^x \log(1+y), \; f_x(0,0) = e^0 \log 1 = 0$$

$$f_{xx} = e^x \log(1+y), \; f_{xx}(0,0) = 0$$

$$f_{xxx} = e^x \log(1+y), \; f_{xxx}(0,0) = 0$$

$$f_{yxx} = \frac{e^x}{1+y}, \; f_{yxx}(0,0) = \frac{e^0}{1+0} = 1$$

$$f_y = \frac{e^x}{1+y}, \; f_y(0,0) = 1$$

$$f_{yy} = \frac{-e^x}{(1+y)^2}, \; f_{yy}(0,0) = -1$$

$$f_{xyy} = \frac{-e^x}{(1+y)^2}, \; f_{xyy}(0,0) = -1$$

$$f_{yyy} = \frac{2e^x}{(1+y)^3}, \; f_{yyy}(0,0) = 2$$

$$f_{xy} = \frac{e^x}{1+y}, \; f_{xy}(0,0) = 1$$

Put all these values in Maclaurin's series,

$$e^x \log(1+y) = y + xy - \frac{1}{2}y^2 + \frac{1}{2}x^2y - \frac{1}{2}xy^2 + \frac{1}{3}y^2$$

PART-2

Jacobian

CONCEPT OUTLINE : PART-2

Jacobian : If $u_1, u_2, \ldots\ldots\ldots u_n$ are functions of n independent variables $x_1, x_2, \ldots\ldots x_n$, then the determinant

$$\begin{vmatrix} \frac{\partial u_1}{\partial x_1} & \frac{\partial u_1}{\partial x_2} & \cdots\cdots & \frac{\partial u_1}{\partial x_n} \\ \frac{\partial u_2}{\partial x_1} & \frac{\partial u_2}{\partial x_2} & \cdots\cdots & \frac{\partial u_2}{\partial x_n} \\ \vdots & \vdots & & \vdots \\ \vdots & \vdots & & \vdots \\ \frac{\partial u_n}{\partial x_1} & \frac{\partial u_n}{\partial x_2} & \cdots\cdots & \frac{\partial u_n}{\partial x_n} \end{vmatrix}$$

is called the Jacobian of $u_1, u_2, \ldots\ldots\ldots u_n$ with respect to $x_1, x_2, \ldots\ldots\ldots x_n$ and is denoted either by

$\dfrac{\partial(u_1, u_2 \ldots\ldots u_n)}{\partial(x_1, x_2 \ldots\ldots x_n)}$ or by $J(u_1, u_2, \ldots\ldots\ldots, u_n)$.

Thus if u and v are functions of two independent variables x and y, then

$$\frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} = J(u, v)$$

Questions-Answers

Long Answer Type and Medium Answer Type Questions

Que 2.8. **If J is the Jacobian of u, v with respect to x, y and J' is the Jacobian of x, y with respect to u, v, then**

$$JJ' = 1 \text{ or } \frac{\partial(u,v)}{\partial(x,y)} \cdot \frac{\partial(x,y)}{\partial(u,v)} = 1$$

Answer

Let, $u = f_1(x, y)$, $v = f_2(x, y)$ ...(2.8.1)

Obviously x and y can also be expressed as functions of u and v. Differentiating equation (2.8.1) partially with respect to u and v, we get

$$1 = \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial u}, \quad 0 = \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial v} \quad \text{...(2.8.2)}$$

$$0 = \frac{\partial v}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial v}{\partial y}\cdot\frac{\partial y}{\partial u}, \quad 1 = \frac{\partial v}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial v}{\partial y}\cdot\frac{\partial y}{\partial v} \quad \text{...(2.8.3)}$$

Now,

$$\frac{\partial(u,v)}{\partial(x,y)} \cdot \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} \cdot \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

$$= \begin{vmatrix} \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial u} & \frac{\partial u}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial u}{\partial y}\cdot\frac{\partial y}{\partial v} \\ \frac{\partial v}{\partial x}\cdot\frac{\partial x}{\partial u} + \frac{\partial v}{\partial y}\cdot\frac{\partial y}{\partial u} & \frac{\partial v}{\partial x}\cdot\frac{\partial x}{\partial v} + \frac{\partial v}{\partial y}\cdot\frac{\partial y}{\partial v} \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1$$

Que 2.9. **If $u^3 + v^3 = x + y$ and $u^2 + v^2 = x^3 + y^3$ then show that**

$$\frac{\partial(u,v)}{\partial(x,y)} = \frac{1}{2}\frac{y^2 - x^2}{uv(u-v)}.$$

Answer

The given relation can be written as :

$$F_1 = u^3 + v^3 - x - y = 0$$

$$F_2 = u^2 + v^2 - x^3 - y^3 = 0$$

Now, $$\frac{\partial(u,v)}{\partial(x,y)} = (-1)^2 \frac{\partial(F_1,F_2)}{\partial(x,y)} \Big/ \frac{\partial(F_1,F_2)}{\partial(u,v)} \quad \text{...(2.9.1)}$$

We have, $$\frac{\partial(F_1,F_2)}{\partial(x,y)} = \begin{vmatrix} \frac{\partial F_1}{\partial x} & \frac{\partial F_1}{\partial y} \\ \frac{\partial F_2}{\partial x} & \frac{\partial F_2}{\partial y} \end{vmatrix} = \begin{vmatrix} -1 & -1 \\ -3x^2 & -3y^2 \end{vmatrix}$$

$$= 3y^2 - 3x^2 = 3(y^2 - x^2)$$

and $$\frac{\partial(F_1, F_2)}{\partial(u, v)} = \begin{vmatrix} \frac{\partial F_1}{\partial u} & \frac{\partial F_1}{\partial v} \\ \frac{\partial F_2}{\partial u} & \frac{\partial F_2}{\partial v} \end{vmatrix} = \begin{vmatrix} 3u^2 & 3v^2 \\ 2u & 2v \end{vmatrix}$$

$$= 6u^2v - 6uv^2 = 6uv\,(u - v)$$

From equation (2.9.1),

$$\frac{\partial(u, v)}{\partial(x, y)} = \frac{3(y^2 - x^2)}{6uv\,(u - v)} = \frac{1}{2}\,\frac{y^2 - x^2}{uv\,(u - v)}$$

Que 2.10. **If $x + y + z = u$, $y + z = uv$, $z = uvw$ show that $\frac{\partial(x, y, z)}{\partial(u, v, w)} = u^2v$.**

OR

If $x + y + z = u$, $y + z = uv$, $z = uvw$, then find $\frac{\partial(x, y, z)}{\partial(u, v, w)}$.

UPTU 2012-13, 2015-2016; Marks 05

Answer

The given relation can be written as :

$$F_1 = x + y + z - u = 0$$
$$F_2 = y + z - uv = 0$$
$$F_3 = z - uvw = 0$$

Now, $$\frac{\partial(x, y, z)}{\partial(u, v, w)} = (-1)^3\,\frac{\partial(F_1, F_2, F_3)}{\partial(u, v, w)} \Big/ \frac{\partial(F_1, F_2, F_3)}{\partial(x, y, z)} \qquad \text{...(2.10.1)}$$

We have, $$\frac{\partial(F_1, F_2, F_3)}{\partial(u, v, w)} = \begin{vmatrix} -1 & 0 & 0 \\ -v & -u & 0 \\ -vw & - & \end{vmatrix}$$

$$(-1)(-u)(-uv) = -u^2v$$

and $$\frac{\partial(F_1, F_2, F_3)}{\partial(x, y, z)} = \begin{vmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{vmatrix} = 1$$

$\therefore$ From equation (2.10.1),

$$\frac{\partial(x, y, z)}{\partial(u, v, w)} = \frac{-(-u^2v)}{1} = u^2v$$

Que 2.11. **The roots of the equation in λ, $(\lambda - x)^3 + (\lambda - y)^3 + (\lambda - z)^3 = 0$ are u, v, w.**

Prove that : $$\frac{\partial(u, v, w)}{\partial(x, y, z)} = -2\,\frac{(y - z)(z - x)(x - y)}{(v - w)(w - u)(u - v)}$$

OR

If u, v, w are the roots of the equation $(\lambda - x)^3 + (\lambda - y)^3 + (\lambda - z)^3 = 0$ in λ find $\dfrac{\partial(u,v,w)}{\partial(x,y,z)}$

UPTU 2015-16, Marks 10

Answer

The given equation in λ can be written as :

$$3\lambda^3 - 3\lambda^2 (x+y+z) + 3\lambda (x^2+y^2+z^2) - (x^3+y^3+z^3) = 0$$

Since u, v, w are the roots of this equation, therefore,

$$u + v + w = x + y + z$$
$$uv + vw + wu = x^2 + y^2 + z^2$$
$$uvw = \frac{1}{3}(x^3 + y^3 + z^3)$$

The above relation can be written as

$$F_1 = u + v + w - x - y - z = 0$$
$$F_2 = uv + vw + wu - x^2 - y^2 - z^2 = 0$$
$$F_3 = uvw - \frac{1}{3}(x^3 + y^3 + z^3) = 0$$

Now,
$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = (-1)^3 \frac{\partial(F_1,F_2,F_3)}{\partial(x,y,z)} \Big/ \frac{\partial(F_1,F_2,F_3)}{\partial(u,v,w)} \quad ...(2.11.1)$$

We have,

$$\frac{\partial(F_1,F_2,F_3)}{\partial(x,y,z)} = \begin{vmatrix} -1 & -1 & -1 \\ -2x & -2y & -2z \\ -x^2 & -y^2 & -z^2 \end{vmatrix} = -2\begin{vmatrix} 1 & 1 & 1 \\ x & y & z \\ x^2 & y^2 & z^2 \end{vmatrix}$$
$$= -2(y-z)(z-x)(x-y)$$

Also
$$\frac{\partial(F_1,F_2,F_3)}{\partial(u,v,w)} = \begin{vmatrix} 1 & 1 & 1 \\ v+w & u+w & u+v \\ vw & uw & uv \end{vmatrix}$$
$$= -(v-w)(w-u)(u-v)$$

Hence from equation (2.11.1),

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = (-1)^3 \frac{-2(y-z)(z-x)(x-y)}{-(v-w)(w-u)(u-v)}$$
$$= -2\frac{(y-z)(z-x)(x-y)}{(v-w)(w-u)(u-v)}$$

Que 2.12. If $u = x + 2y + z$, $v = x - 2y + 3z$ and $w = 2xy - xz + 4yz - 2z^2$ show that they are not independent. Find the relation between them.

Answer

We have,
$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} 1 & 2 & 1 \\ 1 & -2 & 3 \\ 2y-z & 2x+4z & -x+4y-4z \end{vmatrix}$$
$$= \begin{vmatrix} 1 & 0 & 0 \\ 1 & -4 & 2 \\ 2y-z & 2x-4y+6z & -x+2y-3z \end{vmatrix}$$
by $C_2 - 2C_1$ and $C_3 - C_1$
$$= -2\begin{vmatrix} 1 & 0 & 0 \\ 1 & 2 & 2 \\ 2y-z & -x+2y-3z & -x+2y-3z \end{vmatrix} = 0$$

Since Jacobian is zero, therefore these functions are not independent and so there must exist a relation between them.

We have,
$$u^2 - v^2 = (x+2y+z)^2 - (x-2y+3z)^2$$
$$= (x+2y+z+x-2y+3z)(x+2y+z-x+2y-3z)$$
$$= (2x+4z)(4y-2z)$$
$$= 4(x+2z)(2y-z) = 4(2xy - xz + 4yz - 2z^2)$$

Therefore $u^2 - v^2 = 4w$ is the required relation between u, v and w.

Que 2.13. Find $\dfrac{\partial(x,y,z)}{\partial(r,\theta,\phi)}$ if $x = \sqrt{vw}$, $y = \sqrt{uw}$, $z = \sqrt{uv}$ and $u = r\sin\theta\cos\phi$, $v = r\sin\theta\sin\phi$, $w = r\cos\theta$.

UPTU 2014-15, Marks 10

Answer

$x = \sqrt{vw}$, $y = \sqrt{uw}$, $z = \sqrt{uv}$

$$\frac{\partial(x,y,z)}{\partial(u,v,w)} = \begin{vmatrix} 0 & \frac{1}{2}\sqrt{\frac{w}{v}} & \frac{1}{2}\sqrt{\frac{v}{w}} \\ \frac{1}{2}\sqrt{\frac{w}{u}} & 0 & \frac{1}{2}\sqrt{\frac{u}{w}} \\ \frac{1}{2}\sqrt{\frac{v}{u}} & \frac{1}{2}\sqrt{\frac{u}{v}} & 0 \end{vmatrix}$$

Multiplying by $\sqrt{vw}$, $\sqrt{uw}$, $\sqrt{vu}$, we get

$$= \frac{1}{8}\frac{1}{\sqrt{vw}}\frac{1}{\sqrt{uw}}\frac{1}{\sqrt{vu}}\begin{vmatrix} 0 & w & v \\ w & 0 & u \\ v & u & 0 \end{vmatrix}$$

$$= \frac{1}{8}\frac{1}{uvw}[-w(-vu)+v(wu)]$$

$$= \frac{1}{8}\times\frac{1}{uvw}\times 2uvw = \frac{1}{4}$$

Again,

$$u = r\sin\theta\cos\phi,\ v = r\sin\theta\sin\phi,\ w = r\cos\theta$$

$$\frac{\partial(u,v,w)}{\partial(r,\theta,\phi)} = \begin{vmatrix} \sin\theta\cos\phi & r\cos\theta\cos\phi & -r\sin\theta\sin\phi \\ \sin\theta\sin\phi & r\cos\theta\sin\phi & r\sin\theta\cos\phi \\ \cos\theta & -r\sin\theta & 0 \end{vmatrix}$$

$$= r^2\sin\theta\begin{vmatrix} \sin\theta\cos\phi & \cos\theta\cos\phi & -\sin\phi \\ \sin\theta\sin\phi & \cos\theta\sin\phi & \cos\phi \\ \cos\theta & -\sin\theta & 0 \end{vmatrix}$$

$$= r^2\sin\theta\,[\sin\theta\cos\phi\,(0+\sin\theta\cos\phi) - \cos\theta\cos\phi\,(0-\cos\theta\cos\phi) - \sin\phi\,(-\sin^2\theta\sin\phi-\cos^2\theta\sin\phi)]$$

$$= r^2\sin\theta\,[\sin^2\theta\cos^2\phi+\cos^2\theta\cos^2\phi+\sin^2\theta\sin^2\phi + \cos^2\theta\sin^2\phi]$$

$$= r^2\sin\theta\,[\cos^2\phi+\sin^2\phi]$$

$$\frac{\partial(u,v,w)}{\partial(r,\theta,\phi)} = r^2\sin\theta$$

$$\frac{\partial(x,y,z)}{\partial(r,\theta,\phi)} = \frac{\partial(x,y,z)}{\partial(u,v,w)}\times\frac{\partial(u,v,w)}{\partial(r,\theta,\phi)} = \frac{1}{4}\times r^2\sin\theta$$

$$\frac{\partial(x,y,z)}{\partial(r,\theta,\phi)} = \frac{1}{4}r^2\sin\theta$$

Que 2.14. If $u_1 = \frac{x_2x_3}{x_1}$, $u_2 = \frac{x_3x_1}{x_2}$ and $u_3 = \frac{x_1x_2}{x_3}$ find the value of $\frac{\partial(u_1,u_2,u_3)}{\partial(x_1,x_2,x_3)}$.

UPTU 2010-11, Marks 05

OR

$y_1 = \frac{x_2x_3}{x_1}$, $y_2 = \frac{x_3x_1}{x_2}$ and $y_3 = \frac{x_1x_2}{x_3}$, find the value of $\frac{\partial(y_1,y_2,y_3)}{\partial(x_1,x_2,x_3)}$.

UPTU 2011-12, Marks 05

Answer

$$u_1 = \frac{x_2x_3}{x_1},\ u_2 = \frac{x_3x_1}{x_2},\ u_3 = \frac{x_1x_2}{x_3}$$

$$\frac{\partial u_1}{\partial x_1} = \frac{-x_2x_3}{x_1^2},\ \frac{\partial u_2}{\partial x_1} = \frac{x_3}{x_2},\ \frac{\partial u_3}{\partial x_1} = \frac{x_2}{x_3}$$

$$\frac{\partial u_1}{\partial x_2} = \frac{x_3}{x_1}, \frac{\partial u_2}{\partial x_2} = \frac{-x_3 x_1}{x_2^2}, \frac{\partial u_3}{\partial x_2} = \frac{x_1}{x_3}$$

$$\frac{\partial u_1}{\partial x_3} = \frac{x_2}{x_1}, \frac{\partial u_2}{\partial x_3} = \frac{x_1}{x_2}, \frac{\partial u_3}{\partial x_3} = -\frac{x_1 x_2}{x_3^2}$$

$$\frac{\partial(u_1, u_2, u_3)}{\partial(x_1, x_2, x_3)} = \begin{vmatrix} \frac{-x_2 x_3}{x_1^2} & \frac{x_3}{x_1} & \frac{x_2}{x_1} \\ \frac{x_3}{x_2} & \frac{-x_3 x_1}{x_2^2} & \frac{x_1}{x_2} \\ \frac{x_2}{x_3} & \frac{x_1}{x_3} & \frac{-x_1 x_2}{x_3^2} \end{vmatrix}$$

$$= \frac{1}{x_1^2 x_2^2 x_3^2} \begin{vmatrix} -x_2 x_3 & x_3 x_1 & x_2 x_1 \\ x_3 x_2 & -x_3 x_1 & x_1 x_2 \\ x_2 x_3 & x_1 x_3 & -x_1 x_2 \end{vmatrix}$$

$$= \frac{x_1^2 x_2^2 x_3^2}{x_1^2 x_2^2 x_3^2} \begin{vmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{vmatrix}$$

$$= -1(0) - 1(-1-1) + 1(1+1)$$

$$\frac{\partial(u_1, u_2, u_3)}{\partial(x_1, x_2, x_3)} = 4$$

Que 2.15. **Show that $u = y + z$, $v = x + 2z^2$, $w = x - 4yz - 2y^2$ are not independent. Find the relation between them.**

UPTU 2012-13, Marks 05

Answer

$$u = y + z$$
$$v = x + 2z^2$$
$$w = x - 4yz - 2y^2$$

$$\frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix} = \begin{vmatrix} 0 & 1 & 1 \\ 1 & 0 & 4z \\ 1 & -4z-4y & -4y \end{vmatrix}$$

$$= -1[-4y - 4z] + 1[-4z - 4y] = 0$$

Hence u, v, w are not independent.

$$w = x - 2y^2 - 4yz = v - 2z^2 - 2y^2 - 4yz \qquad [\because \quad x + 2z^2 = v]$$
$$= v - 2(z^2 + y^2) - 4yz$$

$$= v - 2(z^2 + y^2 + 2yz - 2yz) - 4yz = v - 2(z+y)^2 + 4yz - 4yz$$

$$w = v - 2u^2$$

or $$2u^2 = v - w$$

Que 2.16. **Show that the functions :**

$$u = x + y + z,$$
$$v = x^2 + y^2 + z^2 - 2xy - 2yz - 2zx, \text{ and}$$
$$w = x^3 + y^3 + z^3 - 3xyz$$

are functionally related. Find the relation between them.

UPTU 2013-14, Marks 10

Answer

Same as Q. 2.15, Page 55A, Unit-2.

Answer : $4w = 3\,uv + u^3$

PART-3

Approximation of Errors

CONCEPT OUTLINE : PART-3

Approximation of Errors : Let $f(x, y)$ be a continuous function of x and y. If δx and δy be the increments in x and y respectively, then new value of $f(x, y)$ will be $f(x + \delta x, y + \delta y)$ *i.e.*,

$$\delta f = f(x + \delta x, y + \delta y) - f(x, y)$$

Expanding using Taylor's series

$$\delta f = f(x, y) + \delta x \frac{\partial f}{\partial x} + \delta y \frac{\partial f}{\partial y} + \ldots\ldots\ldots - f(x, y)$$

Neglecting higher powers, $\delta f = \delta x \dfrac{\partial f}{\partial x} + \delta y \dfrac{\partial f}{\partial y}$. δx, δy and δf are small changes in x, y and f.

Relative Error : If δx is the error (small change) in x then relative

$$\text{error} = \frac{\partial x}{x}$$

$$\text{Percentage error} = \frac{\partial x}{x} \times 100\ \%$$

Que 2.17. **Find the possible percentage error in computing the resistance r from the formula $\frac{1}{r} = \frac{1}{r_1} + \frac{1}{r_2}$ if r_1, r_2 are both in error by 2 %.**

Answer

We have, $$\frac{1}{r} = \frac{1}{r_1} + \frac{1}{r_2}$$

Differentiating,

$$-\frac{1}{r^2}dr = -\frac{1}{r_1^2}dr_1 - \frac{1}{r_2^2}dr_2$$

$$\frac{1}{r}\left(100\frac{dr}{r}\right) = \frac{1}{r_1}\left(100\frac{dr_1}{r_1}\right) + \frac{1}{r_2}\left(100\frac{dr_2}{r_2}\right)$$

$$= \frac{1}{r_1}(2) + \frac{1}{r_2}(2) = 2\left(\frac{1}{r_1} + \frac{1}{r_2}\right) = 2\left(\frac{1}{r}\right)$$

$$100\frac{dr}{r} = 2$$

Percentage error in r is 2 %.

Que 2.18. **In estimating the number of bricks in a pile which is measured to be (5 m × 10 m × 5 m) count of bricks is taken as 100 bricks/m³. Find the error in the cost when the tape is stretched 2 % beyond its standard length. The cost of bricks is Rs. 2000 per thousand bricks.**

Answer

Volume $(V) = x\,y\,z$

Taking log on both sides,

$\log V = \log x + \log y + \log z$

Differentiating,

$$\frac{\delta V}{V} = \frac{\delta x}{x} + \frac{\delta y}{y} + \frac{\delta z}{z}$$

$$\frac{\delta V}{V} \times 100 = \frac{\delta x}{x} \times 100 + \frac{\delta y}{y} \times 100 + \frac{\delta z}{z} \times 100 = 2 + 2 + 2 = 6$$

$$\delta V = \frac{6V}{100} = \frac{6 \times 5 \times 10 \times 5}{100}$$

$$\delta V = 15 \text{ m}^3$$

Number of bricks in,

$$\delta V = 15 \times 100 = 1500$$

$$\text{Error in cost} = \frac{1500 \times 2000}{1000} = \text{Rs. } 3000$$

Que 2.19. **The period T of a simple pendulum is $T = 2\pi\sqrt{\dfrac{l}{g}}$. Find the maximum error in T due to possible error up to 1 % in l and 2.5 % in g.**

Answer

$$T = 2\pi\sqrt{\frac{l}{g}}$$

Taking log on both sides,

$$\log T = \log 2\pi + \frac{1}{2}[\log l - \log g]$$

Differentiating,

$$\frac{\delta T}{T} = \frac{1}{2}\frac{\delta l}{l} - \frac{1}{2}\frac{\delta g}{g}$$

$$\frac{\delta T}{T} \times 100 = \frac{1}{2}\frac{\delta l}{l} \times 100 - \frac{1}{2}\frac{\delta g}{g} \times 100$$

For maximum possible error,

$$\left|\frac{\delta T}{T} \times 100\right| = \frac{1}{2}\left|\frac{\delta l}{l} \times 100\right| + \frac{1}{2}\left|\frac{\delta g}{g} \times 100\right| = \frac{1}{2}(1) + \frac{1}{2}(2.5)$$

$$\frac{\delta T}{T} \times 100 = 1.75\ \%$$

Maximum error in T is 1.75 %.

Que 2.20. **A balloon in the form of right circular cylinder of radius 1.5 m and length 4 m is surmounted by hemispherical ends. If the radius is increased by 0.01 m and the length by 0.05 m, find the percentage change in the volume of the balloon.**

Answer

Given : Radius $(r) = 1.5$ m
Length $(h) = 4$ m

$$\text{Volume, } V = \pi r^2 h + \frac{2}{3}\pi r^3 + \frac{2}{3}\pi r^3 = \pi r^2 h + \frac{4}{3}\pi r^3$$

$$\delta V = 2\pi r h \delta r + \pi r^2 \delta h + \frac{4}{3}\pi \times 3r^2 \delta r$$

$$\frac{\delta V}{V} = \frac{2\pi r h \delta r + \pi r^2 \delta h + 4\pi r^2 \delta r}{\pi r^2 h + \frac{4}{3}\pi r^3}$$

$$= \frac{\pi r\,(2h\delta r + r\delta h + 4r\delta r)}{\pi r\left(rh + \frac{4}{3}\pi r^2\right)}$$

$$= \frac{2h\delta r + r\delta h + 4r\delta r}{rh + \frac{4}{3}r^2}$$

$$= \frac{2 \times 4 \times 0.01 + 1.5 \times 0.05 + 4 \times 1.5 \times 0.01}{1.5 \times 4 + \frac{4}{3} \times (1.5)^2} = \frac{0.08 + 0.075 + 0.06}{6 + 3}$$

$$\frac{\delta V}{V} \times 100 = \frac{0.215}{9} \times 100 = 2.389\ \%$$

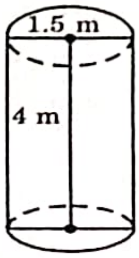

Fig. 2.20.1.

Que 2.21. **Find the percentage of error in calculating the area of an ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, when error of + 1 % is made in measuring the major and minor axes.** **UPTU 2010-11, Marks 05**

Answer

Area of the ellipse,

$$A = \pi ab$$

Taking log on both sides,

$$\log A = \log \pi + \log a + \log b$$

$$\frac{\delta A}{A} = 0 + \frac{\delta a}{a} + \frac{\delta b}{b}$$

$$\frac{\delta A}{A} \times 100 = \frac{\delta a}{a} \times 100 + \frac{\delta b}{b} \times 100$$

But, $\frac{\delta a}{a} \times 100 = 1, \ \frac{\delta b}{b} \times 100 = 1$

Thus,

Percentage error in area is

$$\frac{\delta A}{A} \times 100 = 1 + 1 = 2\ \%$$

Que 2.22. **The angles of a triangle are calculated from the sides a, b, c. If small changes δa, δb and δc are made in the sides, find δA, δB and δC where Δ is the area of the triangle and A, B, C are angles opposite to sides, a, b, c respectively. Also show that $\delta A + \delta B + \delta C = 0$.**

UPTU 2011-12, Marks 10

OR

Show that the approximate change in the angle of a triangle ABC due to small changes δa, δb, δc in the sides a, b and c is given by

$$\delta A = \frac{a}{2\Delta}(\delta a - \delta b \cos C - \delta c \cos B)$$

Where Δ is the area of the triangle and verify that $\delta A + \delta B + \delta C = 0$.

Answer

We know that,

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

$$b^2 + c^2 - a^2 = 2bc \cos A$$

$$a^2 = b^2 + c^2 - 2bc \cos A$$

$$2a\,\delta a = 2b\delta b + 2c\,\delta c - 2[c \cos A\,\delta b + b \cos A\,\delta c - bc \sin A\,\delta A]$$

$$bc \sin A\,\delta A = a\delta a - (b - c \cos A)\,\delta b - (c - b \cos A)\,\delta c$$

$$2\Delta\,\delta A = a\delta a - (a \cos C + c \cos A - c \cos A)\,\delta b - (a \cos B + b \cos A - b \cos A)\,\delta c$$

$$2\Delta\,\delta A = a\delta a - a\delta b \cos C - a\delta c \cos B$$

$$\delta A = \frac{a}{2\Delta}[\delta a - \delta b \cos C - \delta c \cos B] \quad ...(2.22.1)$$

Similarly,

$$\delta B = \frac{b}{2\Delta}[\delta b - \delta c \cos A - \delta a \cos C] \quad ...(2.22.2)$$

$$\delta C = \frac{c}{2\Delta}[\delta c - \delta a \cos B - \delta b \cos A] \quad ...(2.22.3)$$

On adding equation (2.22.1), equation (2.22.2) and equation (2.22.3), we get

$$\delta A + \delta B + \delta C = \frac{1}{2\Delta}[(a - b \cos C - c \cos B)\delta a + (b - a \cos C - c \cos A)\,\delta b + (c - a \cos B - b \cos A)\,\delta c]$$

$$= \frac{1}{2\Delta}[(a - a)\,\delta a + (b - b)\,\delta b + (c - c)\,\delta c] = 0$$

Que 2.23. **If the base radius and height of a cone are measured as 4 cm and 8 cm with a possible error of 0.04 and 0.08 inches respectively, calculate the percentage (%) error in calculating volume of the cone.**

UPTU 2011-12, Marks 05

Answer

Given :

$$r = 4 \text{ cm}, h = 8 \text{ cm}$$
$$\delta r = 0.04, h = 0.08$$
$$V = \frac{1}{3}\pi r^2 h \quad \text{...(2.23.1)}$$

Taking log on both sides of equation (2.23.1),

$$\log V = \log\left(\frac{\pi}{3}\right) + 2\log r + \log h$$

Differentiate both sides,

$$\frac{\delta V}{V} \times 100 = 0 + \frac{2\delta r}{r} \times 100 + \frac{\delta h}{h} \times 100$$
$$\frac{\delta V}{V} \times 100 = 2\left(\frac{0.04}{4} \times 100\right) + \frac{0.08}{8} \times 100 = 2 + 1$$
$$\frac{\delta V}{V} \times 100 = 3\ \%$$

Que 2.24. **The two sides of a triangle are measured as 50 cm and 70 cm, and the angle between them is 30°. If there are possible errors of 0.5 % in the measurement of the sides and 0.5 degree in that of the angle, find the maximum approximate percentage error in measuring the area of the triangle.** **UPTU 2012-13, Marks 05**

Answer

Given :

$$b = 50 \text{ cm}, c = 70 \text{ cm}$$
$$A = 30°, \frac{\delta b}{b} \times 100 = 0.5$$
$$\frac{\delta c}{c} \times 100 = 0.5$$
$$\delta A = 0.5$$

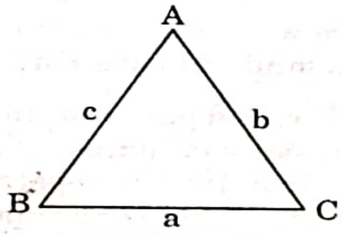

Fig. 2.24.1.

Area of triangle ABC is,

$$\Delta = \frac{1}{2} bc \sin A$$

Taking log on both sides,

$$\log \Delta = \log \frac{1}{2} + \log b + \log c + \log \sin A$$

Differentiate both sides,

$$\frac{\delta\Delta}{\Delta} \times 100 = \frac{\delta b}{b} \times 100 + \frac{\delta c}{c} \times 100 + \frac{\cos A}{\sin A}\, \delta A \times 100$$

$$\frac{\delta\Delta}{\Delta} \times 100 = 0.5 + 0.5 + \cot 30^\circ\,(0.5) \times 100 = 1 + 50(\sqrt{3}) = 87.60\ \%$$

PART-4

Extrema of Functions of Several Variables, Lagrangs's Method of Multipliers

CONCEPT OUTLINE : PART-4

Maxima and Minima :

(1) Functions of Single Independent Variable :

Definition : A function $f(x)$ is said to be maximum at $x = a$, if there exists a positive number δ such that

$f(a + h) < f(a)$ for all values of h, other than zero, in the interval, $(-\delta, \delta)$.

A function $f(x)$ is said to be minimum at $x = a$, if there exists a positive number δ such that

$f(a + h) > f(a)$ for all values of h, other than zero, in the interval $(-\delta, \delta)$.

Working Rule for Maxima and Minima of $f(x)$:

(*a*) Find $f'(x)$ and equate it to zero.

(*b*) Solve the resulting equation for x. Let its roots be a_1, a_2.......then $f(x)$ is stationary at $x = a_1, a_2$,..... Thus $x = a_1, a_2$, ... are the only points at which $f(x)$ can be maximum or minimum.

(*c*) Find $f''(x)$ and substitute in it by turns $x = a_1, a_2$, ...

(*d*) If $f''(a_1)$ is negative we have a maximum at $x = a_1$. If $f''(a_1)$ is positive, we have a minimum at $x = a_1$.

(*e*) If $f''(a_1) = 0$, find $f'''(x)$ and put $x = a_1$ in it. If $f'''(a_1) \neq 0$, there is neither a maximum nor a minimum at $x = a_1$. If $f'''(a_1) = 0$, find $f''''(x)$ and put $x = a_1$ in it. If f'''' is negative, we have a maximum at $x = a_1$, if it is positive, there is minimum at $x = a_1$. If $f''''(a_1)$ is zero, we must find $f'''''(x)$, and so on.

Repeat the above process for each root of the equation $f'(x) = 0$.

(2) Functions of Two Independent Variables :

Definition : Let $f(x, y)$ be any function of two independent variables x and y supposed to be continuous for all values of these variables in the neighbourhood of their values a and b respectively. Then $f(a, b)$ is said to be a maximum or a minimum values of $f(x, y)$ according to $f(a + h, b + k)$ is less or greater than $f(a, b)$ for all sufficiently small independent values of h and k, positive or negative, provided both of them are not equal to zero.

Working Rule for Maxima and Minima : Suppose $f(x, y)$ is a given function of x and y. Find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ and solve the simultaneous equations $\frac{\partial f}{\partial x} = 0$ and $\frac{\partial f}{\partial y} = 0$. In order to solve these equations we may either eliminate one of the variables, to factorise the equations. In the latter case each factor of the first equation must be solved in conjunction with each factor of the second equation. Suppose on solving these equations we get the pairs of values of x and y as (a_1, b_1), (a_2, b_2) etc. Then all these pairs of roots will give stationary values of $f(x, y)$.

To discuss the maximum or minimum at $x = a_1, y = b_1$ we should find

$$r = \left(\frac{\partial^2 f}{\partial x^2}\right)_{\substack{x=a_1\\y=b_1}}, \quad s = \left(\frac{\partial^2 f}{\partial x \partial y}\right)_{\substack{x=a_1\\y=b_1}}, \quad t = \left(\frac{\partial^2 f}{\partial y^2}\right)_{\substack{x=a_1\\y=b_1}}$$

Then calculate $rt - s^2$.

If $rt - s^2 > 0$ and r is negative, $f(x, y)$ is maximum at $x = a_1, y = b_1$.

If $rt - s^2 > 0$ and r is positive, $f(x, y)$ is minimum at $x = a_1, y = b_1$.

If $rt - s^2 < 0$, $f(x, y)$ is neither maximum nor minimum at $x = a_1, y = b_1$.

It $rt - s^2 = 0$, the case is doubtful and further investigation will be required to decide it.

Lagrange's Method of Undetermined Multipliers :

Let $f(x, y, z)$ be a function of three variables x, y and z and the variables are connected by the relation,

$$f(x, y, z) = 0. \qquad ...(1)$$

Then the necessary condition for $f(x, y, z)$ to be maximum or minimum is

$$\frac{\partial f}{\partial x} = 0, \frac{\partial f}{\partial y} = 0, \frac{\partial f}{\partial z} = 0$$

Thus, $\frac{\partial f}{\partial x} dx + \frac{\partial f}{\partial y} dy + \frac{\partial f}{\partial z} dz = 0$ ...(2)

Total differentiation of equation (1) is

$$\frac{\partial \phi}{\partial x} dx + \frac{\partial \phi}{\partial y} dy + \frac{\partial \phi}{\partial z} dz = 0 \qquad ...(3)$$

Multiply equation (3) by λ and adding to equation (2), we get the required condition

$$\left.\begin{aligned}\frac{\partial f}{\partial x}+\lambda\frac{\partial \phi}{\partial x}&=0\\ \frac{\partial f}{\partial y}+\lambda\frac{\partial \phi}{\partial y}&=0\\ \frac{\partial f}{\partial z}+\lambda\frac{\partial \phi}{\partial z}&=0\end{aligned}\right\}$$

Solving the above three equations, the values of x, y, z and λ are find out for which $f(x, y, z)$ is maximum or minimum.

Questions-Answers

Long Answer Type and Medium Answer Type Questions

Que 2.25. **Discuss the maximum or minimum values of u where $u = 2a^2 xy - 3ax^2 y - ay^3 + x^3y + xy^3$.**

Answer

We have, $$\frac{\partial u}{\partial x} = 2a^2y - 6axy + 3x^2 y + y^3$$

$$\frac{\partial u}{\partial y} = 2a^2x - 3ax^2 - 3ay^2 + x^3 + 3xy^2$$

Also $$r = \frac{\partial^2 u}{\partial x^2} = -6ay + 6xy$$

$$s = \frac{\partial^2 u}{\partial x \partial y} = 2a^2 - 6ax + 3x^2 + 3y^2$$

and $$t = \frac{\partial^2 u}{\partial y^2} = -6ay + 6xy$$

For a maximum or minimum of u, we have

$$\frac{\partial u}{\partial x} = 0 \text{ and } \frac{\partial u}{\partial y} = 0$$

Thus we have $y(2a^2 - 6ax + 3x^2 + y^2) = 0$

$$2a^2 x - 3ax^2 - 3ay^2 + x^3 + 3xy^2 = 0$$

Therefore we have to consider pair of equations,

$$\left.\begin{aligned}y&=0\\ 2a^2x-3ax^2-3ay^2+x^3+3xy^2&=0\end{aligned}\right\} \quad ...(2.25.1)$$

$$\left.\begin{aligned}2a^2-6ax+3x^2+y^2&=0\\ 2a^2x-3ax^2-3ay^2+x^3+3xy^2&=0\end{aligned}\right\} \quad ...(2.25.2)$$

Putting $y = 0$ in the second equation of pair (2.25.1), we get

$$2a^2x - 3ax^2 + x^3 = 0$$

i.e., $x(x^2 - 3ax + 2a^2) = 0$

$$x(x - a)(x - 2a) = 0$$

$$x = 0, a, 2a$$

Thus the pair (2.25.1) gives the following values of x and y :

$$x = 0, y = 0 ; x = a, y = 0 ; x = 2a, y = 0$$

Multiplying the first equation of the pair (2.25.2) by x and subtracting it from the second equation of the pair, we get

$$3ax^2 - 3ay^2 - 2x^3 + 2xy^2 = 0$$

or $(x^2 - y^2)(3a - 2x) = 0$

$\therefore$ $x = \dfrac{3a}{2}$ and , $x = \pm y$.

When $x = \dfrac{3a}{2}$, the first equation of the pair (2.25.2) gives

$$y = \pm \frac{a}{2}$$

When $x = y$, we have $2a^2 - 6ay + 4y^2 = 0$

i.e., $y = a, \dfrac{a}{2}$

Also when $x = -y$, we have $2a^2 - 6ay + 4y^2 = 0$

i.e., $y = -a, -\dfrac{a}{2}$

Thus in all we get the following pairs of values of x and y which make the function u stationary.

$(0, 0), (a, 0), (2a, 0),$

$\left(\dfrac{3a}{2}, \dfrac{a}{2}\right), \left(\dfrac{3a}{2}, -\dfrac{a}{2}\right) (a, a), \left(\dfrac{a}{2}, \dfrac{a}{2}\right), (a, -a), \left(\dfrac{a}{2}, \dfrac{-a}{2}\right).$

For (0, 0)

$r = 0, s = 2a^2, t = 0$, so that $rt - s^2$ is negative.

Therefore u at (0, 0) is neither maximum nor minimum. Similarly we can show u at $(a, 0), (2a, 0), (a, a), (a, -a)$.

For $\left(\dfrac{3a}{2}, \dfrac{a}{2}\right)$ $r = \dfrac{3a^2}{2}, s = \dfrac{a^2}{2}, t = \dfrac{3a^2}{2}$, so $rt - s^2$ is positive.

Since r is positive therefore u has minimum at this point.

Similarly u has minimum at $\left(\dfrac{a}{2}, -\dfrac{a}{2}\right)$.

For $\left(\dfrac{3a}{2}, \dfrac{a}{2}\right)$ $r = -\dfrac{3a^2}{2}, s = -\dfrac{a^2}{2}, t = -\dfrac{3a^2}{2}$, so that $rt - s^2$ is positive. Since r is negative, therefore u has a maximum at this point.

Similarly u is maximum at $\left(\dfrac{a}{2}, \dfrac{a}{2}\right)$.

Que 2.26. Find the semi vertical angle of the cone of maximum volume and of a given slant height.

Answer

Let us assume l is the slant height of cone and θ be the semi vertical angle of it.

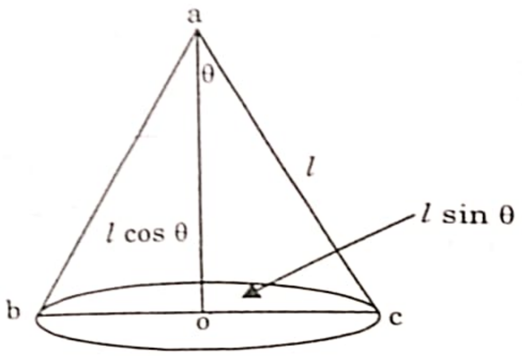

Fig. 2.26.1.

The radius of cone, $r = oc = l \sin \theta$ and height, $oa = h = l \cos \theta$. Then volume of the cone is :

$$V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi \ (l \sin \theta)^2 \, l \cos \theta = \frac{1}{3}\pi \ l^3 \sin^2 \theta \cos \theta$$

$$\therefore \quad \frac{dV}{d\theta} = \frac{1}{3}\pi \, l^3 \{\sin^2 \theta \, (- \sin \theta) + \cos \theta \; 2 \sin \theta \cos \theta\}$$

$$= \frac{1}{3}\pi \, l^3 \sin \theta \, (2 \cos^2 \theta - \sin^2 \theta)$$

and
$$\frac{d^2V}{d\theta^2} = \frac{1}{3}\pi \, l^3 \{\sin \theta \, (- 4 \cos \theta \sin \theta - 2 \sin \theta \cos \theta) + (2 \cos^2 \theta - \sin^2 \theta). \cos \theta\}$$

$$= \frac{1}{3}\pi \, l^3 \{- 6 \sin^2 \theta \cos \theta + \cos \theta \, (2 \cos^2 \theta - \sin^2 \theta)\}$$

For maximum or minimum value of V,

$$\frac{dV}{d\theta} = 0, \text{ that give } \sin \theta = 0$$

or
$$\tan^2 \theta = 2 \Rightarrow \theta = 0 \text{ or } \theta = \pm \tan^{-1} \sqrt{2}$$

When $\theta = 0$, the volume of cone becomes zero, which is not considerable.

When, $\theta = \tan^{-1} \sqrt{2}$, we have

$$\frac{d^2V}{d\theta^2} = \frac{1}{3}\pi \, l^3 \left[-6 \, . \frac{2}{3} \frac{1}{\sqrt{3}} + 0\right] = \text{negative}$$

Hence the volume V is maximum when $\theta = \tan^{-1} \sqrt{2}$

When, $\theta = -\tan^{-1}\sqrt{2}$, the volume is negative, which is not considerable.

Que 2.27. **Find the volume of the largest rectangular parallelopiped that can be inscribed in the ellipsoid,**

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2} = 1.$$

Answer

Consider (x, y, z) be a vertex of parallelopiped, then it lies on the ellipsoid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2} = 1$$

Dimensions are $2x$, $2y$ and $2z$, then the volume V is given by

$$V = 2x \,.\, 2y \,.\, 2z = 8xyz$$

$$V^2 = 64\, x^2 y^2 z^2 = 64\, x^2\, y^2\, c^2\left(1-\frac{x^2}{a^2}-\frac{y^2}{b^2}\right)$$

[By converting the problem of three variables into two variables]

$$= 64c^2\left(x^2y^2-\frac{x^4y^2}{a^2}-\frac{x^2y^4}{b^2}\right) = f(x, y)$$

$$\frac{\partial f}{\partial x} = 64c^2\left(2xy^2-\frac{4x^3y^2}{a^2}-\frac{2xy^4}{b^2}\right)$$

$$\frac{\partial f}{\partial y} = 64c^2\left(2x^2y-\frac{2x^4y}{a^2}-\frac{4x^2y^3}{b^2}\right)$$

$$\therefore \qquad r = \frac{\partial^2 f}{\partial x^2} = 64c^2\left(2y^2-\frac{12x^2y^2}{a^2}-\frac{2y^4}{b^2}\right)$$

$$s = \frac{\partial^2 f}{\partial x \partial y} = 64c^2\left(4xy-\frac{8x^3y}{a^2}-\frac{8xy^3}{b^2}\right)$$

$$t = \frac{\partial^2 f}{\partial y^2} = 64c^2\left(2x^2-\frac{2x^4}{a^2}-\frac{12x^2y^2}{b^2}\right)$$

Now $\dfrac{\partial f}{\partial x} = 0$ and $\dfrac{\partial f}{\partial y} = 0$ for maximum and minimum of V.

$$128\, c^2\, xy^2\left(1-\frac{2x^2}{a^2}-\frac{y^2}{b^2}\right) = 0 \text{ and } 128\, c^2\, x^2y\left(1-\frac{x^2}{a^2}-\frac{2y^2}{b^2}\right) = 0$$

$$1-\frac{2x^2}{a^2}-\frac{y^2}{b^2} = 0 \qquad \text{...(2.27.1)}$$

$$1-\frac{x^2}{a^2}-\frac{2y^2}{b^2} = 0 \qquad \text{...(2.27.2)}$$

Subtracting equation (2.27.2) from equation (2.27.1),

$$-\frac{x^2}{a^2}+\frac{y^2}{b^2} = 0 \text{ or } y = \frac{bx}{a}$$

$\therefore$ From equation (2.27.1) $1-\frac{2x^2}{a^2}-\frac{x^2}{a^2} = 0$

$$\Rightarrow \qquad x = \frac{a}{\sqrt{3}}$$

$$\therefore \qquad y = \frac{b}{a}\cdot\frac{a}{\sqrt{3}} = \frac{b}{\sqrt{3}}$$

and $$z^2 = c^2\left(1-\frac{x^2}{a^2}-\frac{y^2}{b^2}\right) = c^2\left(1-\frac{1}{3}-\frac{1}{3}\right)$$

$$= \frac{c^2}{3}$$

Thus $x = \frac{a}{\sqrt{3}}, y = \frac{b}{\sqrt{3}}$ is a stationary point. At this point,

$$r = 64\,c^2\left\{\frac{2b^2}{3}-\frac{12}{a^2}\cdot\frac{a^2}{3}\cdot\frac{b^2}{3}-\frac{2}{b^2}\cdot\frac{b^4}{9}\right\}$$

$$r = \frac{-512}{9}b^2c^2 < 0$$

$$s = 64\,c^2\left\{4\cdot\frac{a}{\sqrt{3}}\cdot\frac{b}{\sqrt{3}}-\frac{8}{a^2}\cdot\frac{a^3}{3\sqrt{3}}\cdot\frac{b}{\sqrt{3}}-\frac{8}{b^2}\cdot\frac{a}{\sqrt{3}}\cdot\frac{b^3}{3\sqrt{3}}\right\}$$

$$= -\frac{256}{9}abc^2$$

$$t = 64c^2\left\{2\cdot\frac{a^2}{3}-\frac{2}{a^2}\frac{a^4}{9}-\frac{12}{b^2}\frac{a^2}{3}\cdot\frac{b^2}{3}\right\}$$

$$= -\frac{512}{9}a^2c^2$$

$$rt-s^2 = \left(\frac{512}{9}\right)^2 a^2b^2c^4-\left(\frac{256}{9}\right)^2 a^2b^2c^4$$

$$= \left(\frac{256}{9}\right)^2 a^2b^2c^4\,(4-1) > 0$$

Also $r < 0$.

$\therefore$ V^2 is maximum, hence V is maximum when $x = \frac{a}{\sqrt{3}}, y = \frac{b}{\sqrt{3}}, z = \frac{c}{\sqrt{3}}$ and its maximum value is

$$V = 8\frac{a}{\sqrt{3}}\frac{b}{\sqrt{3}}\frac{c}{\sqrt{3}} = \frac{8\,abc}{3\sqrt{3}}$$

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Que 2.28. **A rectangular box, open at the top is to have a volume of 32 cubic feet. Determine the dimensions of the box requiring least material for its construction.**

UPTU 2011-12, 2014-2015 Marks 05

Answer

Let x, y and z are the dimensions of the box.

$$\text{Volume, } V = 32$$

$$xyz = 32$$

$$y = \frac{32}{xz} \qquad \text{...(2.28.1)}$$

Surface area $(S) = 2(x + y)\, z + xy$

Putting the value of y, we get

$$S = 2\left(x + \frac{32}{xz}\right) z + \frac{32}{z} \qquad \text{...(2.28.2)}$$

$$S = 2xz + \frac{32}{x} \times 2 + \frac{32}{z}$$

$$S = 2xz + \frac{64}{x} + \frac{32}{z}$$

$$\frac{\partial S}{\partial x} = 2z - \frac{64}{x^2}$$

$$\frac{\partial S}{\partial z} = 2x - \frac{32}{z^2}$$

For least material,

$$\frac{\partial S}{\partial x} = 0 \quad i.e., \quad z = \frac{32}{x^2} \qquad \text{...(2.28.3)}$$

$$\frac{\partial S}{\partial z} = 0 \quad i.e., \quad x = \frac{16}{z^2} \qquad \text{...(2.28.4)}$$

From equation (2.28.3), equation (2.28.4) and equation (2.28.1),

$$x = 4, y = 4, z = 2$$

$$\frac{\partial^2 S}{\partial x^2} = \frac{128}{x^3} = 2, \frac{\partial^2 S}{\partial x \partial z} = 2, \frac{\partial^2 S}{\partial z^2} = \frac{64}{z^3} = 8$$

$$\frac{\partial^2 S}{\partial x^2} \cdot \frac{\partial^2 S}{\partial z^2} - \left(\frac{\partial^2 S}{\partial x \partial z}\right)^2 = 2 \times 8 - (2)^2 = 16 - 4 = 12$$

$$\frac{\partial^2 S}{\partial x^2} = +2, \text{ so } S \text{ is minimum for } x = 4, y = 4, z = 2.$$

Que 2.29. **Find the extreme values of : $f(x, y) = x^3 + y^3 - 3axy$.**

UPTU 2011-12, Marks 05

Answer

$$f(x, y) = x^3 + y^3 - 3axy$$
$$p = \frac{\partial f}{\partial x} = 3x^2 - 3ay,\ q = \frac{\partial f}{\partial y} = 3y^2 - 3ax$$
$$r = 6x,\ s = -3a,\ t = 6y$$

For extreme values,

$$p = 0, q = 0$$
$$x^2 = ay = 0 \text{ and } y^2 = ax$$

On solving, the stationary points are $(0, 0)$, (a, a).

At (0, 0) : $r = 0,\ s = -3a,\ t = 0$

$$rt - s^2 = -9a^2 \text{ (negative)}$$

Thus there is no extreme value at $(0, 0)$.

At (a, a) : $r = 6a,\ s = -3a,\ t = 6a$

$$rt - s^2 = 27a^2 \text{ (positive)}$$

Thus, $rt - s^2 > 0$, $r = 6a$ and if $a > 0$, $f(x, y)$ is minimum.

$$f_{\min}(x, y) = a^3 + a^3 - 3a.a.a = -a^3$$

And $rt - s^2 > 0$, $r = 6a$ and if $a < 0$, $f(x, y)$ is maximum.

$$f_{\max}(x, y) = (-a)^3 + (-a)^3 - 3(-a)(-a)(-a) = a^3$$

Que 2.30. **Find the maximum and minimum distance of the point (1, 2, – 1) from the sphere $x^2 + y^2 + z^2 = 24$.**

UPTU 2012-13, Marks 10

OR

The shortest distance from the point (1, 2, –1) to the sphere $x^2 + y^2 + z^2 = 24$ shall be..........

UPTU 2009-10, Marks 02

Answer

Let the coordinates of the point on the sphere be (x, y, z) whose distance from $(1, 2, -1)$ is

$$D = \sqrt{(x-1)^2 + (y-2)^2 + (z+1)^2}$$

Let

$$f(x, y, z) = D^2 = (x-1)^2 + (y-2)^2 + (z+1)^2$$
$$\phi(x, y, z) = x^2 + y^2 + z^2 - 24$$

Let us form a Lagrange's function,

$$F(x, y, z) = f(x, y, z) + \lambda\phi(x, y, z) = (x-1)^2 + (y-2)^2 + (z+1)^2 + \lambda(x^2 + y^2 + z^2 - 24)$$

For maximum or minimum distances $dF = 0$

i.e., $2(x-1) + 2\lambda x = 0$

or
$$x = \frac{1}{1+\lambda}$$

Similarly, $2(y-2) + 2\lambda y = 0$

$$y = \frac{2}{1+\lambda}$$

and $\quad 2(z+1)+2\lambda z = 0$

$$z = \frac{-1}{1+\lambda}$$

Putting the values of x, y and z in $x^2+y^2+z^2 = 24$

$$\left(\frac{1}{1+\lambda}\right)^2 + \left(\frac{2}{1+\lambda}\right)^2 + \left(\frac{-1}{1+\lambda}\right)^2 = 24$$

On solving, $\quad \lambda = -\frac{1}{2}, -\frac{3}{2}$

Thus, when $\quad \lambda = -\frac{1}{2}$

$$x = 2, y = 4, z = -2$$

and when $\quad \lambda = -\frac{3}{2}$

$$x = -2, y = -4, z = 2$$

Thus the required points are (2, 4, – 2) and (– 2, – 4, 2).
Thus the distance for point (2, 4, – 2) is

$$D = \sqrt{(2-1)^2+(4-2)^2+(-2+1)^2} = \sqrt{6}$$

and for (– 2, – 4, 2) is

$$D = \sqrt{(-2-1)^2+(-4-2)^2+(2+1)^2} = \sqrt{54}$$

Thus, shortest distance = $\sqrt{6}$

and $\quad$ longest distance = $\sqrt{54}$

Que 2.31. Find the maximum value of $u = x^p y^q z^r$ when the variables x, y, z are related to $ax + by + cz = p + q + r$.

Answer

Lagrange's equation for the given function is :

$$\frac{\partial u}{\partial x} + \lambda \frac{\partial \phi}{\partial x} = 0 \qquad \text{...(2.31.1)}$$

$$\frac{\partial u}{\partial y} + \lambda \frac{\partial \phi}{\partial y} = 0 \qquad \text{...(2.31.2)}$$

$$\frac{\partial u}{\partial z} + \lambda \frac{\partial \phi}{\partial z} = 0 \qquad \text{...(2.31.3)}$$

Where, $\quad \phi(x, y, z) = ax + by + cz - p - q - r$

$$u = x^p y^q z^r$$

$$\log u = p \log x + q \log y + r \log z$$

$$\frac{1}{u}\frac{\partial u}{\partial x} = \frac{p}{x}$$

$$\frac{\partial u}{\partial x} = \frac{up}{x}, \quad \frac{\partial u}{\partial y} = \frac{qu}{y} \text{ and } \frac{\partial u}{\partial z} = \frac{ru}{z}$$

Also $\qquad \dfrac{\partial\phi}{\partial x} = a, \dfrac{\partial\phi}{\partial y} = b, \dfrac{\partial\phi}{\partial z} = c$

Putting in equation (2.31.1), equation (2.31.2) and equation (2.31.3), we get

$$\frac{pu}{x} + a\lambda = 0$$

$$x = \frac{pu}{-a\lambda} \text{ and } \frac{qu}{y} + b\lambda = 0$$

$$y = \frac{qu}{-b\lambda}$$

Similarly, $\qquad z = \dfrac{ru}{-c\lambda}$

Putting the values of x, y and z in given relation

$$ax + by + cz = p + q + r$$

$$-\frac{pu}{\lambda} - \frac{qu}{\lambda} - \frac{rz}{\lambda} = p + q + r$$

$$-\frac{u}{\lambda}(p + q + r) = p + q + r$$

$$\lambda = -u$$

Thus, $\qquad x = \dfrac{p}{a}, y = \dfrac{q}{b}$ and $z = \dfrac{r}{c}$

Maximum value of u is $\left(\dfrac{p}{a}\right)^p \left(\dfrac{q}{b}\right)^q \left(\dfrac{r}{c}\right)^r$.

Que 2.32. **Use the method of Lagrange's multipliers to find the volume of the largest rectangular parallelopiped that can be inscribed in the ellipsoid $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} + \dfrac{z^2}{c^2} = 1$.**

Answer

Let $2x$, $2y$, $2z$ are the length, breadth and height of the rectangular parallelopiped.

$$V = 8xyz$$

$$\frac{\partial V}{\partial x} = 8yz, \frac{\partial V}{\partial y} = 8xz, \frac{\partial V}{\partial z} = 8xy$$

$$\phi(x, y, z) = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} - 1 = 0$$

Thus Lagrange's equations are :

$$8yz + \lambda\frac{2x}{a^2} = 0 \qquad \text{...(2.32.1)}$$

and $$8xz + \lambda \frac{2y}{b^2} = 0 \quad ...(2.32.2)$$

and $$8xy + \lambda \frac{2z}{c^2} = 0 \quad ...(2.32.3)$$

Multiplying equation (2.32.1), (2.32.2) and (2.32.3) by x, y and z respectively and on adding

$$24xyz + 2\lambda \left[\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2}\right] = 0$$

$$24xyz + 2\lambda\,(1) = 0$$

$$\lambda = -12xyz$$

From equation (2.32.1)

$$8yz + (-12xyz)\frac{2x}{a^2} = 0$$

$$x = \frac{a}{\sqrt{3}}$$

Similarly, from equation (2.32.2) and equation (2.32.3)

$$y = \frac{b}{\sqrt{3}}$$

$$z = \frac{c}{\sqrt{3}}$$

Thus the volume of the largest rectangular parallelopiped is $8xyz$.

$$V = 8\left(\frac{a}{\sqrt{3}}\right)\left(\frac{b}{\sqrt{3}}\right)\left(\frac{c}{\sqrt{3}}\right) = \frac{8abc}{3\sqrt{3}}$$

Que 2.33. **A tent of a given volume has a square base of side $2a$, has its four sides, vertical of length b and is surmounted by a regular pyramid of height h. Find the values of a and b in terms of h such that the canvas required for its construction is minimum.**

Answer

Let V is the volume and S is the surface area of the tent

$$V = 4a^2b + \frac{1}{3}(4a^2)\,h$$

$$S = 8ab + 4a\sqrt{a^2 + h^2}$$

$$\frac{\partial S}{\partial a} + \lambda\frac{\partial V}{\partial a} = 0$$

$$8b + 4\sqrt{a^2 + h^2} + \frac{4a^2}{\sqrt{a^2 + h^2}} + \lambda\left[8ab + \frac{8ah}{3}\right] = 0 \quad ...(2.33.1)$$

$$\frac{\partial S}{\partial b} + \lambda\frac{\partial V}{\partial b} = 0$$

$$8a + 4\lambda a^2 = 0 \quad ...(2.33.2)$$

$$\frac{\partial S}{\partial h} + \lambda \frac{\partial V}{\partial h} = 0$$

$$\frac{4ah}{\sqrt{a^2 + h^2}} + \frac{4}{3}\lambda a^2 = 0 \quad ...(2.33.3)$$

From equation (2.33.2), $\lambda a + 2 = 0$ ...(2.33.4)

From equation (2.33.3),

$$12ah + 4\lambda a^2\sqrt{a^2 + h^2} = 0$$

$$3h + \lambda a\sqrt{a^2 + h^2} = 0 \quad ...(2.33.5)$$

From equation (2.33.4) and equation (2.33.5),

$$a = \frac{\sqrt{5}}{2}h$$

From equation (2.33.2),

$\lambda a = -2$ and $a = \frac{\sqrt{5}}{2}h$, putting in equation (2.33.1),

we get

$$b = \frac{h}{2}$$

Thus,
$$a = \frac{\sqrt{5}}{2}h$$
$$b = \frac{h}{2}$$

Que 2.34. **Find the dimension of rectangular box of maximum capacity whose surface area is given when (*a*) box is open at the top (*b*) box is closed.**

OR

Find the dimension of rectangular box of maximum capacity whose surface area is given, when box is open at the top.

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OR

Using the Lagrange's method find the dimension of rectangular box of maximum capacity whose surface area is given when (*a*) box is open at the top (*b*) box is closed. **UPTU 2015-16, Marks 10**

Answer

Let x, y, z be the length, breadth and height of the rectangular box. If box is open at the top capacity V is to be maximized, surface area S is given

$$S = nxy + 2yz + 2zx \quad ...(2.34.1)$$
$$V = xyz \quad ...(2.34.2)$$

Lagrange's method of undetermined multiplier

$$F = V + \lambda (S)$$

$$F = xyz + \lambda\,(nxy + 2yz + 2zx)$$
$$dF = 0$$

i.e., $$\frac{\partial F}{\partial x} = \frac{\partial F}{\partial y} = \frac{\partial F}{\partial z} = 0$$

$$yz + \lambda\,(ny + 2z) = 0 \qquad ...(2.34.3)$$
$$xz + \lambda\,(nx + 2z) = 0 \qquad ...(2.34.4)$$
$$xy + \lambda\,(2y + 2x) = C \qquad ...(2.34.5)$$

Multiply equation (2.34.3) by x, equation (2.34.4) by y and equation (2.34.5) by z,

$$xyz + \lambda\,(nxy + 2zx) = 0 \qquad ...(2.34.6)$$
$$xyz + \lambda\,(nxy + 2zy) = 0 \qquad ...(2.34.7)$$
$$xyz + \lambda\,(2yz + 2xz) = 0 \qquad ...(2.34.8)$$

On adding above three equations, we get

$$3xyz + 2\lambda(nxy + 2zx + 2yz) = 0$$
$$3V + 2\lambda S = 0$$
$$\lambda = \frac{-3V}{2S}$$

Substituting λ in equation (2.34.3), equation (2.34.4) and equation (2.34.5),

$$yz - \frac{3V}{2S}(ny + 2z) = 0$$
$$yz - \frac{3xyz}{2S}(ny + 2z) = 0$$
$$nxy + 2xz = \frac{2S}{3} \qquad ...(2.34.9)$$

Similarly, $$nxy + 2yz = \frac{2S}{3} \qquad ...(2.34.10)$$
$$2yz + 2zx = \frac{2S}{3} \qquad ...(2.34.11)$$

Equation (2.35.9) – equation (2.35.10),

$$x = y$$

Equation (2.35.10) – equation (2.35.11),

$$ny = 2z$$

Thus, from equation (2.34.1)

$$S = n\,x\,.\,x + 4\,x\,.\,\frac{nx}{2}$$
$$S = 3nx^2$$
$$x^2 = \frac{S}{3n}$$

(*a*) When box is open at top, $n = 1$

$$x = y = \sqrt{\frac{S}{3}}$$