MATH-578B: Midterm

Due on Thursday, November 5, 2015

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Contents

Problem 1 **3**
Problem 1: (a) 3
Problem 1: (b) 3
Problem 1: (d) 4

Problem 1

Problem 1: (a)

$$P = \begin{bmatrix} 1-\alpha & \alpha \\ \beta & 1-\beta \end{bmatrix}$$

Let the stationary state be given by $\pi = (\pi_1, \pi_2)$, then:

$$\begin{aligned} \pi.P &= \pi \\ \pi_1 + \pi_2 &= 1 \end{aligned}$$

Solving which gives:

$$\begin{aligned} (1-\alpha)\pi_1 + \pi_2 &= 1 \\ \pi_1 + \pi_2 &= 1 \\ \implies (\pi_1, \pi_2) &= (\frac{\beta}{\alpha+\beta}, \frac{\alpha}{\alpha+\beta}) \end{aligned}$$

Problem 1: (b)

$\mathbf{w} = 101$

$$\begin{aligned} \beta_{w,w}(0) &= 1 \\ \beta_{w,w}(1) &= 0 \\ \beta_{w,w}(2) &= 1 \end{aligned}$$

$$\begin{aligned} P_u(0) &= 1 \\ P_u(1) = p_{w_2w_3} = p_{01} &= \alpha \\ P_U(2) = p_{w_1w_2}p_{w_2w_3} = p_{10}p_{01} &= \beta\alpha \end{aligned}$$

Now,

$$G_{w,w}(t) = \sum_{j=0}^{2} t^j \beta_{w,w}(j) P_{w,w}(j)$$

Thus,

$$\begin{aligned} G_{w,w}(t) &= 1 \times 1 \times 1 + t \times 0 \times \alpha + t^2 \times 1 \times \beta\alpha \\ &= 1 + \alpha\beta t^2 \end{aligned}$$

Problem 1: (c)

X_n : Number of occurrences(overlaps allowed) in $A_1A_2A_3\dots A_n$ Using Theorem 12.1:

$$\lim_{n\to\infty} \frac{1}{n} E(X_n) = \pi_w = \pi_1 \times p_{10} \times p_{01} = 1 \times \beta \times \alpha$$

Thus $\lim_{n\to\infty} \frac{X_n}{n} = \alpha\beta$

Problem 1: (d)

Spectral decomposition of P:

$$det \begin{bmatrix} \alpha-\lambda & 1-\alpha \\ 1-\beta & \beta-\lambda \end{bmatrix} = 0$$

$$\lambda^2 + (\alpha+\beta-2)\lambda + (1-\alpha-\beta) = 0$$

Thus, $\lambda_1 = 1$ and $\lambda_2 = 1-\alpha-\beta$
Eigenvectors are given by:
$v_1^T = \begin{pmatrix} x_1 & x_1 \end{pmatrix} \; \forall \; x_1 \in R$
and for λ_2 , $v_2 = \begin{pmatrix} x_1 & \frac{-\beta x_1}{\alpha} \end{pmatrix}$
Now using Markov property: $P(X_n = 1|X_0 = 0) = (P^n)_{01}$
Now,
$P^n = VD^nV^{-1}$
where:

$$V = \begin{bmatrix} 1 & 1 \\ 1 & \frac{-\beta}{\alpha} \end{bmatrix}$$

and

$$D = \begin{bmatrix} 1 & 0 \\ 0 & (1-\alpha-\beta) \end{bmatrix}$$

$$V^{-1} = \frac{-1}{\frac{\beta}{\alpha}+1} \begin{bmatrix} -\frac{\beta}{\alpha} & -1 \\ -1 & 1 \end{bmatrix}$$

Thus,

$$P^n = \begin{bmatrix} 1 & 1 \\ 1 & \frac{-\beta}{\alpha} \end{bmatrix} \times \begin{bmatrix} 1 & 0 \\ 0 & (1-\alpha-\beta)^n \end{bmatrix} \times \frac{-1}{\frac{\beta}{\alpha}+1} \begin{bmatrix} -\frac{\beta}{\alpha} & -1 \\ -1 & 1 \end{bmatrix}$$

$$P^n = \frac{1}{\alpha+\beta} \begin{bmatrix} \beta+\alpha(1-\alpha-\beta)^n & \alpha-\alpha(1-\alpha-\beta)^n \\ \beta-\beta(1-\alpha-\beta)^n & \alpha+\beta(1-\alpha-\beta)^n \end{bmatrix}$$