

# **MATH-505A: Homework # 2**

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## Exercise # 1.5

(1)

**Given:**  $A, B$  are independent

**To Prove:**  $A^C, B$ ;  $A^C, B^C$  are independent

Since  $A, B$  are independent:

$$P(A \cap B) = P(A)P(B) \quad (1)$$

Thus,

$$P(A \cap B) = (1 - P(A^C))P(B) = P(B) - P(B)P(A^C) \quad (2)$$

Rearranging 2:

$$P(B)P(A^C) = P(B) - P(A \cap B) \quad (3)$$

$$P(B) - P(A \cap B) = P(A^C \cap B) = P(A^C)P(B) \quad (4)$$

From 3 and 4 :  $A^C, B$  are independent.

Similiary to prove  $A^C, B^C$  are independent, we perform substitute  $B^C$  in  $P(B \cap A^C)$

(2)

$A_{ij} = i^{th}$  and  $j^{th}$  rolls produce the same number For any  $i \neq j$ , total outcome are  $6*6 = 36$  and favourable outcomes are  $\binom{6}{1} * 1 = 6$ , thus  $p(A_{ij}) = \frac{6}{36} = \frac{1}{6}$

Consider  $P(A_{ij} \cap A_{kj})$ , such that  $i \neq j \neq k$ , then :

$$P(A_{ij} \cap A_{kj}) = \frac{\binom{6}{1} * 1 * 1}{6 * 6 * 6} = \frac{1}{36} = P(A_{ij})P(A_{kj})$$

Thu,  $A_{ij}$  are pairwise independent.

Consider:

$$P(A_{ij} \cap A_{jk} \cap A_{kl}) = \frac{\binom{6}{1} * 1 * 1 * 1}{6 * 6 * 6} = \frac{1}{36} \neq P(A_{ij})P(A_{jk})P(A_{kl})$$

Since  $P(A_{ij} \cap A_{jk} \cap A_{kl}) \neq P(A_{ij})P(A_{jk})$ , it will not be true in general consider other  $A_{lm}$ .

(3)

To Prove:

(a) outcomes of coin tosses are independent

(b) Given a sequence of length  $m$  of heads and tails the chance of it occurring in first  $m$  tosses is  $2^{-m}$ .

In order to prove (a) and (b) are equivalent it is sufficient to prove that if  $a \implies b$  and  $b \implies a$ .

If the outcomes are independent, probability of a head or tail in a sequence is  $\frac{1}{2}$ . Consider  $m$  tosses, since they are independent the probability of seeing any string of H and T is given by  $\frac{1}{2} * \frac{1}{2} * \dots * (m) \text{ times} = \frac{1}{2^m} = 2^{-m}$ . Hence  $\implies b$ .

Now consider if  $b$  is true, then :  $P(m) = 2^{-m} \implies P(m+1) = 2^{-(m+1)}$ , . The  $P(m+1)$  case is similar to  $P(m)$  with an extra toss.  $P(m+1) = 2^{-m} * \frac{1}{2}$ . The extra half factor is accounted by the extra toss that is performed which must be independent with respect to the  $m$  tosses for yielding such a relation for  $P(m+1) \implies a$  is true

and hence  $a \iff b$  and  $a \implies b$

(4)

**Given:**  $\omega = \{1, 2, 3, \dots, p\}$  where  $p$  is prime.  $F$  is set of all subsets of  $\omega$ ;  $P(A) = \frac{|A|}{p}$  **To Prove:**  $A$  'or'  $B$  is a null set or is the set *omega*

$$P(A) = \frac{|A|}{p}$$

$$P(B) = \frac{|B|}{p}$$

Now, by definition:

$$P(A \cap B) = \frac{|A \cap B|}{p} \quad (5)$$

Since  $A, B$  are independent:

$$P(A \cap B) = P(A)P(B) = \frac{|A \cap B|}{p} = \frac{|A|}{p} \frac{|B|}{p} \quad (6)$$

Thus:

$$p|A \cap B| = |A||B| \implies |A||B| \bmod p = 0 \quad (7)$$

and  $0 \leq |A|, |B| \leq p \implies |A| \text{ or } |B| = p \text{ OR } |A|, |B| = 0 \implies A, B \text{ are either null or full sets.}$

(5)

$$P(A, B|C) = \frac{P(A, B, C)}{P(C)} = P(A|B, C) * \frac{P(B, C)}{P(C)} = P(A|B, C) * P(B|C)$$

For which event  $C$ , are  $A$  and  $B$  (for all  $A, B$ ) independent iff they are conditionally independent given  $C$ :  
?

If  $A, B$  are independent :  $P(A|B) = P(A)$

Given:  $P(A, B|C) = P(A|C)P(B|C)$  for all  $A, B$ .

For  $P(A|B) = P(A)$  can be made true if we set  $P(C)P(A, B|C) = P(A|C)P(B|C)$

$$\text{Thus } P(C) = 1$$

(7)

 $A = \{ \text{all children of same sex} \}$ 
 $B = \{ \text{there is at most one boy} \}$ 
 $C = \{ \text{one boy and one girl included} \}$ 

$$P(A) = P(\text{all boys}) + P(\text{all girls}) = \frac{1}{2} * \frac{1}{2} * \frac{1}{2} * 2 = \frac{1}{4}$$

$$P(B) = P(0 \text{ boys}) + P(1 \text{ boy}) = \frac{1}{8} + 3 * \frac{1}{8} = \frac{1}{2}$$

$$P(C) = P(1 \text{ boy} + 1 \text{ girl}) = 2 * \frac{1}{2} * \frac{1}{2} * 3 = \frac{3}{4}$$

**Part a):** A is independent of B and B is independent of C

$P(A \cap B) = \text{all children are of same sex AND there is at most one boy} \implies \text{all children are boys}$

$$\implies P(A \cap B) = \frac{1}{2} * \frac{1}{2} * \frac{1}{2} = \frac{1}{8} = P(A) * P(B).$$

Hence A,B are independent

$P(B \cap C) = \text{there is at most one boy AND there is one boy and a girl} \implies \text{there is one boy and two girls.}$

$$P(B \cap C) = \frac{1}{2} * \frac{1}{2} * \frac{1}{2} * 3 = \frac{3}{8} = P(B) * P(C)$$

Hence B,C are independent

**Part b):** Is A independent of C?

$P(A \cap C) = \text{the family includes boy and girl AND all children are of same sex}$

Clearly  $P(A \cap C) = 0$  and hence A,B are not necessarily independent!

**Part c):** Do the results hold if boys and girls are not equally likely?

NO. Since

**Part d):** Do these results hold if there are 4 children?

Yes, the calculations are independent of the number of children since independence relations are not dependent on the number.