MATH-505A: Homework # 1

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10:30am

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Contents

Exercise # 1.2

(3)

At the start of the tournament we have 2^n players to begin with. At each round there will be **one** winner emerging from each of the pairs while the other gets 'knocked out'. One possible configuration for the first round of the tournament would be: $Player_1$ v/s $Player_2$; $Player_3$ v/s $Player_4$;...;, $Player_(2^n-1)$ v/s $Player(2^n)$. At the end of first round, there are exactly $\frac{2^n}{2} = 2^{n-1}$ winners and an equal number of knocked out players.

At round 1 the set of $2^n - 1$ pairs can be represented as :$P_1, P_2, P_3, P_4, .., P_{2^n-1}$. The total number of such pairs is 2^n divided by 2 since each pair has 2 players. The outcome of first round can generate two values for each of these pairs depending on who amongst the two players is the winner. For e.g. $Player_1$ can win while playing in P_1 or $Player_2$ can, Thus total number of such configurations for the round 1 would be $2*2*2*...*(2^n-1)$ times which is equal to $2^{2^{n-}}$. Now at round 2 we would have 2^{n-2} pairs of players to play with and the possible configuration for choosing a winner of such a configuration is $2^{2^{n-2}}$

Thus, the sample space representing how the winners are chosen (or the knocked out persons are knocked out) can be calculated by multiplying configurations as obtained in $round_1$ $round_2$, ... $round_n$ by the **rule of product** as: $2^{2^{n-1}} * 2^{2^{n-2}} * .... * 2^1 = X$

$$log_2 X = 2^{n-1} + 2^{n-2} + ... + 1$$
$$log_2 X = \frac{2^{n-1+1}-1}{2-1}$$

Thus $X = 2^{2^n-1}$

5

5: (a)

$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

Let $x \in A \cup (B \cap C) \implies x \in A \ OR \ x \in B \cap C$. Case 1: $xinA$ Then $x \in (A \cup B) \ AND \ x \in (A \cup C)$. That is given x is contained in A it is for sure contained in union of A and B, and also in the union of A with C. From the definition of intersection, this would imply: $x \in (A \cup B) \cap (A \cup C)$

Case 2: $x \in (B \cap C)$ Then $x \in B \ AND \ x \in C \implies x \in (A \cap B) \ AND \ x \in (A \cap C)$ where A can be any set, since $B \subseteq (A \cap B)$

Thus from both the cases we get: $x \in (A \cup B) \cap (A \cup C)$

This implies

$$A \cup (B \cap C) \subseteq (A \cup B) \cap (A \cup C) \tag{1}$$

Now consider a $y \in (A \cup B) \cap (A \cup C) \implies y \in (A \cup B) \ AND \ y \in (A \cup C)$. This implies x belongs to A $OR \ B \ AND \ A \ OR \ C$ Two cases again: Case 1: $x \in A \implies x \in A \cup (B \cap C)$ as $A \subseteq (A \cup (B \cap C))$

Case 2: $x \in B \ AND \ x \in C \implies x \in (B \cap C) \cup A$ as $(B \cap C) \subseteq (A \cup (B \cap C))$

Thus from both the cases we draw the same conclusion: $x \in (A \cup (B \cap C)) \implies (A \cup B) \cap (A \cup C) \subseteq A \cup (B \cap C)$

From 1 and , it is implied that:

$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ **Ans. TRUE**

5 continued on next page...

5: (b)

$A \cap (B \cap C) = (A \cap B) \cap C$
Let $x \in (A \cap (B \cap C)) \implies x \in A\ AND\ x \in B\ AND\ xinC$, which can be easily regrouped as $()x \in A\ AND\ x \in B)\ AND\ x \in C$, which is same as $x \in (A \cap B) \cap C$.
Another approach would be what we used in part (a) above to show that the $L.H.S$ and $R.H.S$ are subsets of each other. However the AND solution is straight forward, since there are no $OR's$ involved.
Ans. TRUE

5: (c)

$(A \cup B) \cap C = A \cup (B \cap C)$
From part (a) of this problem, we proved that the following equation is true:
$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$
Substituting the $R.H.S$ of as the $L.H.S$ of we get:
$(A \cup B) \cap C = (A \cup B) \cap (A \cup C)$
Comparing and we see, that for to be always true, the following should hold:
$C = A \cup C$
which will only be true if $A \subseteq C$.

5: (d)

$(A \backslash (B \cap C)) = (A \backslash B) \cup (A \backslash C)$
$X \backslash Y = X$ but not $Y \implies X \cap (X \cap (X \cap Y)^C)$
Thus can be expanded as follows: $A \cap (A \cap (B \cap C))^C = (A \cap (A \cap B)^C) \cup (A \cap (A \cap C)^C)$
Expanding $L.H.S$ using results from above probles (5b) we get $L.H.S = A \cap (A \cap (B \cap C))^C = A \cap (A \cap B \cap C)^C = A \cap (A^C \cup B^C \cup C^C)$
Again using distribution of union over intersection property : $L.H.S = (A \cap (A^C)) \cup (A \cap (B^C)) \cup (A \cap (C^C)) = (A \cap B^C) \cup (A \cap C^C)$
Expanding $R.H.S$ similarly we get: $R.H.S = (A \cap (A \cap B)^C) \cup (A \cap (A \cap C)^C) = (A \cap (A^C \cup B^C)) \cup (A \cap (A^C \cup C^C)) = (A \cap A^C \cup A \cap B^C) \cup (A \cap A^C \cup A \cap C^C) = (A \cap B^C) \cup (A \cap C^C)$
Thus $L.H.S = R.H.S = (A \cap B^C) \cup (A \cap C^C)$

Exercise # 1.3

1

Given: $P(A) = \frac{3}{4}$ $P(B) = \frac{1}{3}$
To Prove: $\frac{1}{12} \leq P(A \cap B) \leq \frac{1}{3}$

Solution: $P(A \cap B)$ has an upper bound coming from either A or B depending on whichever is a smaller set. Thus :
$P(A \cap B) \leq max(P(A), P(B))$ where $max()$ represents the maximum function. $max(P(A), P(B)) = P(B) = \frac{1}{3} \implies P(A \cap B) \leq \frac{1}{3}$
Now consider $P(A \cup B)$: $P(A \cup B) = P(A) + P(B) - P(A \cap B)$. Also, from the law of probability: $P(A \cup B) \leq 1 \implies P(A \cap B) \geq P(A) + P(B) - 1 \implies P(A \cap B) \geq \frac{13}{12} - 1$. Thus $P(A \cap B) \geq \frac{1}{12}$
From and
$\frac{1}{12} \leq P(A \cap B) \leq \frac{1}{3}$
Now for $P(A \cup B)$:
By law of probability, the upper bound is: $P(A \cup B) \leq 1$.
For lower bound consider: $P(A \cup B) = P(A) + P(B) - P(A \cap B)$ and $\frac{-1}{12} \geq -P(A \cap B) \geq \frac{-1}{3}$
From and : $P(A \cup B) \geq P(A) + P(B) - \frac{1}{3}$
$\implies$: $P(A \cup B) \geq \frac{3}{4}$
$1 \geq P(A \cup B) \geq \frac{3}{4}$

3

2red, 2white, 2stars pairs of (cup, saucer). Probablity that no cup is on the saucer of the same pan. E.g configuration: (Cup Color, Saucer Color): (R,W), (R,S), (W,R),(W,S), (S,R), (S,W)
Here 'corresponding saucer'=
P(no cup on same color saucer) = 1 - P(one cup on same color saucer) + P(two cups on two corresponding colored saucers) - P(three cups on three corresponding saucers) + P(four cups on four correpsonding saucers)

4

To Prove: $P(\cup_{i=1}^{n} A_i) = \sum_i^n P(A_i) - \sum_{i<j} P(A_i \cap A_j) + \sum_{i<j<k} P(A_i \cap A_j \cap A_k))... + (-1)^{n+1} P(A_1 \cap A_2 ... \cap A_n)$ clearly holds for $n = 1$. Also for $n = 2$: $P(A \cup B) = P(A) + P(B) - P(A \cap B)$
Assume holds for $n = s$:
$P(\cup_{i=1}^{s} A_i) = \sum_i^s P(A_i) - \sum_{i<j} P(A_i \cap A_j) + \sum_{i<j<k} P(A_i \cap A_j \cap A_k))... + (-1)^{s+1} P(A_1 \cap A_2 ... \cap A_n)$
Now for $n = s + 1$, using results from :
$P(\cup_{i=1}^{s+1} A_i) = P(\cup_{i=1}^{s} A_i \cup A_{s+1}) = P(\cup_{i=1}^{s} A_i) + P(A_{s+1}) - P(\cup_{i=1}^{s} A_i \cap A_{s+1})$
Consider $P(\cup_{i=1}^{s} A_i \cap A_{s+1}) = P(\cup_{i=1}^{s}(A_i \cap A_{s+1}))$
Expanding using we get: $P(\cup_{i=1}^{s}(A_i \cap A_{s+1})) = \sum_{i=1}^{n} P(A_i \cap A_{s+1}) - \sum_{i<j}^{n} P(A_i \cap A_j \cap A_{s+1})$

Now e

Exercise # 1.4

2

To Prove: $P(A_1 \cap A_2 \cap A_3 .. \cap A_n) = P(A_1)P(A_2|A_1)P(A_3|A_1 \cap A_2).....P(A_n|A_1 \cap A_2 \cap A_3 .. \cap A_{n-1})$
From the definition of conditional probability: $P(A_1 \cap A_2) = P(A_1|A_2)P(A_2)$
Expanding the LHS of using results from we get: $P(A_1 \cap A_2 \cap A_3 .... \cap A_n) = P(X \cap A_n) = P(A_n|X)P(X)$
where $X = A_1 \cap A_2 \cap A_3 ... \cap A_{n-1}$ Thus from and definition of X we get:
$P(A_1 \cap A_2 \cap A_3 .... \cap A_n) = P(A_1 \cap A_2 ... \cap A_{n-1})P(A_n|A_1 \cap A_2 ... \cap A_{n-1})$
The RHS of can be similarly expanded as: $P(A_1 \cap A_2 \cap A_3 .... \cap A_{n-1}) = P(A_1 \cap A_2 ... \cap A_{n-2})P(A_{m-1}|A_1 \cap A_2 ... \cap A_{n-2})$
Hence combining and and doing similar such operations we get:

$P(A_1 \cap A_2 \cap A_3 .. \cap A_n) = P(A_1)P(A_2|A_1)P(A_3|A_1 \cap A_2).....P(A_n|A_1 \cap A_2 \cap A_3 .. \cap A_{n-1})$