MATH-505A: Homework # 4

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Saket Choudhary
2170058637

Contents

Exercise # 2.1

(1)

Given: X is a random variable $\implies$

$$\{\omega \in \Omega : X(\omega) \leq x\} \in \mathcal{F}\ \forall x \in R \tag{1}$$

Part A) To Prove: aX is a random variable
Consider $Y = aX$, then since equation 1 holds:
Case1: $a \geq 0$
Then $\{\omega \in \Omega : aX(\omega) \leq x'\} \in \mathcal{F}\ \forall x' \in R$ where $x' = ax$
Case2: $a \leq 0$
Then $\{\omega \in \Omega : aX(\omega) \geq x'\} \forall x' \in R$ where $x' = ax \implies \cup\{\{\omega \in \Omega : aX(\omega) \leq x''\}\}^c \in \mathcal{F}$ where $x'' = x'$
Case3: $a\ is\ 0$
Then, $aX = 0$
Case i: $x < 0$
$\{\omega \in \Omega : aX(\omega) = \phi\} \in \mathcal{F}$
Case ii: $x \geq 0$
$\{\omega \in \Omega : aX(\omega) = \Omega\} \in \mathcal{F}$
Thus from all the above cases.
Part (b)):
Consider $Y = X - X$, Then:
$Y = X(\omega) - X(\omega) \forall \omega \in R \implies Y = 0$
Consider $Y = X + X$, Then $Y = X(\omega) + X(\omega) \forall \omega in \Omega \implies Y = 2X(\omega) \forall \omega in \Omega$ Thus $Y = 2X$.

(2)

For part 1, $Y' = aX$ is also a random variable:
To Find: Distribution fucntion of $Y = aX + b$
Consider $P(Y \leq y) = P(aX + b \leq y) = P(X \leq \frac{y-b}{a}) \implies P(Y \leq y) = P(X \leq \frac{x-b}{a})$

(3)

$p(H) = p$; $p(T) = 1 - p$
Tossing a coin n times is a binomial process(each individual toss is a bernoulli process) and let A be the event such that k out of n tosses are heads and this can occur in $\binom{n}{k}$ ways with probability p^k. There would also be $n - k$ tails and the probability for that is $(1 - p)^{n-k}$. Thus,:
$p(A) = \binom{n}{k} p^k * (1 - p)^{n-k}$
For a fair coin, $p = \frac{1}{2}$ and hence p(A) = $\binom{n}{k}(\frac{1}{2})^k(\frac{1}{2})^{n-k} = \binom{n}{k}(\frac{1}{2})^n$

(4)

A distribution function satisfies the following set of properties:

a) $\lim_{x\to-\infty} F(x) = 0$, $\lim_{x\to\infty} F(x) = 1$
b) if $x < y$ then $F(x) \leq F(y)$,
c) F is right continuous, $c < x < c + \delta$ then $|F(x) - F(c)| < \epsilon$ for $\epsilon > 0, \delta > 0$
Consider $Y = \lambda F + (1-\lambda)G$, Both G,F satisfy a, b, c Then $\lim_{x\to-\infty} Y(x) = \lambda \lim_{x\to-\infty} F(x) + (1-\lambda)\lim_{x\to-\infty} G(x) \implies \lim_{x\to-\infty} Y(x) = 0$
Similarly considering limit as $x \to \infty$: Then $\lim_{x\to\infty} Y(x) = \lambda \lim_{x\to\infty} F(x) + (1-\lambda)\lim_{x\to\infty} G(x) \implies \lim_{x\to-\infty} Y(x) = \lambda * 1 + (1-\lambda) * 1 = 1$
Since for $x < y$, then $F(x) < F(y)$; $G(x) < G(y) \implies \lambda F(x) < \lambda F(y)$; $(1-\lambda)G(x) < (1-\lambda)G(y)$ since $0 \leq \lambda \leq 1$
Adding the two inequalities we get:

$\lambda F(x) + (1-\lambda)G(x) < \lambda F(y) + (1-\lambda)G(y) \implies Y(x) < Y(y)$.
Since F,G are right continuous, any linear combination of these would be right continuous too.
Hence $Y = \lambda F + (1-\lambda)G$ satisfies all the 3 required properties and is a distribution function.

Exercise # 2.1 continued on next page...

(5)

Since F is a distribution function:
(i) $\lim_{x\to-\infty} F(x) = 0; \lim_{x\to\infty} = 1$
(ii) If $x < y$ then, $F(x) < F(y)$
(iii) F is right continuous
Part a) $F(x)^r$ (i) $\lim_{x\to-\infty} F(x)^r = 0$ since $\lim_{x\to-\infty} F(x) = 0$ and $r > 0$
(ii) If $x < y$ as $F(x) < F(y) and r > 0 \implies F(x)^r < F(y)^r$
(iii) Since $r > 0$ and $F(x)$ is right-continuous $F(x)^r$ is right continuous. (One possible case ehere $F(x)^r$ would not have been right continuous is for $r < 0$ say r = -1 where $F(x)^-1$ is not right continuous at all x_0 such that $F(x_0) = 0$.
Part b) $1 - (1 - F(x))^r$
(i) $; \lim_{x\to-\infty}(1 - (1 - F(x))^r)) = 1 - \lim_{x\to-\infty}(1 - F(x))^r = 1 - (1 - 0)^r = 0$
Similarly for $; \lim_{x\to\infty}(1 - (1 - F(x))^r) = 1 - (1 - 1)^r = 1$
(ii) If $x < y$, $F(x) < F(y) \implies -F(x) > -F(y) \implies 1 - F(x) > 1 - F(y) \implies (1 - F(x))^r > (1 - F(y))^r \forall r > 0$ Thus, $1 - (1 - F(x))^r < 1 - (1 - F(y))^r$
(iii) Since F(x) is right continuous, $1 - F(x)$ is right continuous $\implies (1 - F(x))^r$ is right continuous(since $(r > 0)$ *implies* $1 - (1 - F(x))^r$ is right continuous
Part c $F(x) + (1 - F(x))log(1 - F(x))$
(i) $\lim_{x\to-\infty}(F(x) + (1 - F(x))log(1 - F(x))) = \lim_{x\to-\infty} F(x) + \lim_{x\to-\infty}(1 - F(x))log(1 - F(x)) = 0 + (1 - 0)log(1 - 0) = 0$
Consider $\lim_{x\to\infty}(F(x) + (1 - F(x))log(1 - F(x))) = \lim_{x\to\infty} F(x) + \lim_{x\to\infty}(1 - F(x))log(1 - F(x)) = 1 + (1 - 1)log(1 - 1) = 0$
(ii) If $x < y$ then $F(x) < F(y) \implies 1 - F(x) > 1 - F(y)$. Since log is a monotonic non-increasing function in $[0, 1]$ and is in fact negative definite: $log(1 - F(x)) < log(1 - F(y))$ and $1 - F(x) > 1 - F(y)$ $\implies -F(x)log(1 - F(x)) < -F(y)log(1 - F(y))$ (This holds only because log is negative definite in $[0, 1]$)
Thus, $F(x) - F(x)log(1 - F(x)) < F(y) - F(y)log(1 - F(y))$
(iii) $F(x) - F(x)log(1 - F(x))$ is right continuous as log(1-F(x)) is right continuous
Part d) $(F(x) - 1)e + exp(1 - F(x))$
(i) $\lim_{x\to-\infty}(F(x) - 1)e + exp(1 - F(x)) = \lim_{x\to-\infty}(F(x) - 1)e + exp(\lim_{x\to-\infty}(1 - F(x))) = (0 - 1)e + exp(1 - 0) = -e + e = 0$
$\lim_{x\to\infty}(F(x)-1)e+exp(1-F(x)) = \lim_{x\to\infty}(F(x)-1)e+exp(\lim_{x\to\infty}(1-F(x))) = (1-1)e+exp(1-1) = 0 + 1 = 1$
(ii) if $x < y$, $F(x) < F(y) \implies F(x)-1 < F(y)-1 \implies (F(x)-1)e < (F(y)-1)e$ Also, $1-F(x) > 1-F(y)$
Since exp is a non-increasing function in $[0, 1]$ $exp(1 - F(x)) < exp(1 - F(y))$
Thus,
$(F(x) - 1)e + exp(1 - F(x)) < (F(y) - 1)e + exp(1 - F(y))$
(iii) $(F(x) - 1)e + exp(1 - F(x))$ is right continuous as exp is right continuous.
FG is also a density function since it satisfies:
(i) $\lim_{x\to-\infty} F(x)G(x) = \lim_{x\to-\infty} F(x) * \lim_{x\to-\infty} G(x) = 0$
And $\lim_{x\to\infty} F(x)G(x) = \lim_{x\to\infty} F(x) * \lim_{x\to\infty} G(x) = 1$
(ii) If $x < y$, $F(x) < F(y)$ and $G(x) < G(y) \implies F(x)G(x) < F(y)G(y)$
(iii) Since $F(x), G(x)$ are right continuous

Exercise # 2.3

(1)

Given: $\lim_{m\to-\infty} a_m \to -\infty$ and $\lim_{m\to\infty} a_m \to \infty$; $G(x) = P(X \leq a_m)$ when $a_{m-1} \leq x \leq a_m$; a_m is a strictly increasing sequence.
Sequence a is chosen so that $sup_m|a_m - a_{m-1}|$ becomes smaller and smaller so even though the sequence is increasing the successive difference between the terms keep on decreasing essentially indicating a_m saturates as $m \to \infty$
so $F(x)$ approaches $G(x)$ $\forall x$

(2)

Given: $g(x)$ is continuous and strictly increasing, X is a random variable:

Since g(X) is continuous and strictly increasing $\implies g^{-1}$ exists.
Consider $\{Y \leq y\} \implies \{g(X) \leq y\}$. Since g^{-1} exists, such a set is equivalent to: $\{X \leq g^{-1}(y)\}$ which belongs to $\mathcal{F}$ as $g : R \to R$. Since the last equations hold true and hence $\{Y \leq y\} \in \mathcal{F}$ so $g(X)$ is a RV

(3)

$F(x) = P(X \leq x) = \begin{cases} 0 & if x \leq 0, \\ x & if 0 < x \leq 1, \\ 1 & if x > 1, \end{cases}$ **To Prove:** $Y = F^{-1}(x)$ is a random variable: Consider $\{Y \leq y\} = \{F^{-1}(x) \leq y\}$. Since F is continuous and strictly increasing $\implies F^{-1}(x)$ exists in R so $\{Y \leq y\} = \{F^{-1}(x) \leq y\} = \{(x) \leq F(y)\} = \{x \leq P(X \leq y)\} \in \mathcal{F}$
Since the last equation holds true, it $\implies \{Y \leq y\} \in \mathcal{F}$
F should necessarily be continuous and monotonic for the inverse to exist!

(4)

f, g are density functions:
$\int_{-\infty}^{\infty} f(x) = 1$ and $\int_{-\infty}^{\infty} g(x) = 1$
Consider $y(x) = \lambda f(x) + (1-\lambda)g(x)$ Thus, $\int_{-\infty}^{\infty} y(x)dx = \int_{-\infty}^{\infty} \lambda f(x)dx + \int_{-\infty}^{\infty}(1-\lambda)g(x)dx = \lambda * \int_{-\infty}^{\infty} f(x)dx + (1-\lambda) * \int_{-\infty}^{\infty} g(x)dx = \lambda * 1 + (1-\lambda) * 1 = 1$
Thus $\lambda f(x) + (1-\lambda)g(x)$ is a density function too.
Now consider $y(x) = f(x)g(x)$, then :
$\int_{-\infty}^{\infty} y(x)dx = \int_{-\infty}^{\infty} f(x)g(x)dx$ Clearly this is nor neceesarily equal to 1 so fg is not a density function!

Exercise # 2.3 continued on next page. . .

(5)

Part a) $f(x) = \begin{cases} cx^{-d} & x > 1, \\ 0 & otherwise, \end{cases}$
$F(x) = \int_{-\infty}^{\infty} f(c)dx = \int_1^{\infty} cx^{-d} = 1 \implies \frac{-c}{-d+1} = 1 \implies c = d - 1$ and $-d + 1 < 1$ i.e $d > 0$ else the integral blows up to ∞
Part b) $f(x) = ce^x(1 + e^x)^{-2} x \in R$
$F(x) = \int_{-\infty}^{infty} ce^x(1+e^x)^{-2}$ Let $t = e^x + 1$ then $e^x dx = dt$ $F(x) = \int_1^{\infty} ct^{-2}dt$ $F(x = 1) = -(c * 0 - c) = 1$
Thus $c = 1$.