

MATH-505A: Homework # 4

Due on Friday, September 19, 2014

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Exercise # 2.1

(1)

Given: X is a random variable \implies

$$\{\omega \in \Omega : X(\omega) \leq x\} \in \mathcal{F} \quad \forall x \in \mathbb{R} \quad (1)$$

Part A) To Prove: aX is a random variable

Consider $Y = aX$, then since equation 1 holds:

Case1: $a \geq 0$

Then $\{\omega \in \Omega : aX(\omega) \leq x'\} \in \mathcal{F} \quad \forall x' \in \mathbb{R}$ where $x' = ax$

Case2: $a \leq 0$

Then $\{\omega \in \Omega : aX(\omega) \geq x'\} \forall x' \in \mathbb{R}$ where $x' = ax \implies \cup \{\{\omega \in \Omega : aX(\omega) \leq x''\}\}^c \in \mathcal{F}$ where $x'' = x'$

Case3: a is 0

Then, $aX = 0$

Case i: $x < 0$

$$\{\omega \in \Omega : aX(\omega) = \phi\} \in \mathcal{F}$$

Case ii: $x \geq 0$

$$\{\omega \in \Omega : aX(\omega) = \Omega\} \in \mathcal{F}$$

Thus from all the above cases.

Part (b):

Consider $Y = X - X$, Then:

$$Y = X(\omega) - X(\omega) \forall \omega \in \Omega \implies Y = 0$$

Consider $Y = X + X$, Then $Y = X(\omega) + X(\omega) \forall \omega \in \Omega \implies Y = 2X(\omega) \forall \omega \in \Omega$ Thus $Y = 2X$.

(2)

For part 1, $Y' = aX$ is also a random variable:

To Find: Distribution function of $Y = aX + b$

$$\text{Consider } P(Y \leq y) = P(aX + b \leq y) = P(X \leq \frac{y-b}{a}) \implies P(Y \leq y) = P(X \leq \frac{y-b}{a})$$

(3)

$$p(H) = p; p(T) = 1 - p$$

Tossing a coin n times is a binomial process (each individual toss is a Bernoulli process) and let A be the event such that k out of n tosses are heads and this can occur in $\binom{n}{k}$ ways with probability p^k . There would also be $n - k$ tails and the probability for that is $(1 - p)^{n-k}$. Thus,:

$$p(A) = \binom{n}{k} p^k * (1 - p)^{n-k}$$

$$\text{For a fair coin, } p = \frac{1}{2} \text{ and hence } p(A) = \binom{n}{k} \left(\frac{1}{2}\right)^k \left(\frac{1}{2}\right)^{n-k} = \binom{n}{k} \left(\frac{1}{2}\right)^n$$

(4)

A distribution function satisfies the following set of properties:

a) $\lim_{x \rightarrow -\infty} F(x) = 0, \lim_{x \rightarrow \infty} F(x) = 1$

b) if $x < y$ then $F(x) \leq F(y)$,

c) F is right continuous, $c < x < c + \delta$ then $|F(x) - F(c)| < \epsilon$ for $\epsilon > 0, \delta > 0$

Consider $Y = \lambda F + (1 - \lambda)G$, Both G,F satisfy a, b, c Then $\lim_{x \rightarrow -\infty} Y(x) = \lambda \lim_{x \rightarrow -\infty} F(x) + (1 - \lambda) \lim_{x \rightarrow -\infty} G(x) \implies \lim_{x \rightarrow -\infty} Y(x) = 0$

Similarly considering limit as $x \rightarrow \infty$: Then $\lim_{x \rightarrow \infty} Y(x) = \lambda \lim_{x \rightarrow \infty} F(x) + (1 - \lambda) \lim_{x \rightarrow \infty} G(x) \implies \lim_{x \rightarrow \infty} Y(x) = \lambda * 1 + (1 - \lambda) * 1 = 1$

Since for $x < y$, then $F(x) < F(y); G(x) < G(y) \implies \lambda F(x) < \lambda F(y); (1 - \lambda)G(x) < (1 - \lambda)G(y)$ since $0 \leq \lambda \leq 1$

Adding the two inequalities we get:

$$\lambda F(x) + (1 - \lambda)G(x) < \lambda F(y) + (1 - \lambda)G(y) \implies Y(x) < Y(y).$$

Since F,G are right continuous, any linear combination of these would be right continuous too.

Hence $Y = \lambda F + (1 - \lambda)G$ satisfies all the 3 required properties and is a distribution function.

(5)

Since F is a distribution function:

(i) $\lim_{x \rightarrow -\infty} F(x) = 0; \lim_{x \rightarrow \infty} F(x) = 1$

(ii) If $x < y$ then, $F(x) < F(y)$

(iii) F is right continuous

Part a) $F(x)^r$ (i) $\lim_{x \rightarrow -\infty} F(x)^r = 0$ since $\lim_{x \rightarrow -\infty} F(x) = 0$ and $r > 0$

(ii) If $x < y$ as $F(x) < F(y)$ and $r > 0 \implies F(x)^r < F(y)^r$

(iii) Since $r > 0$ and $F(x)$ is right-continuous $F(x)^r$ is right continuous. (One possible case where $F(x)^r$ would not have been right continuous is for $r < 0$ say $r = -1$ where $F(x)^{-1}$ is not right continuous at all x_0 such that $F(x_0) = 0$.)

Part b) $1 - (1 - F(x))^r$

(i) ; $\lim_{x \rightarrow -\infty} (1 - (1 - F(x))^r) = 1 - \lim_{x \rightarrow -\infty} (1 - F(x))^r = 1 - (1 - 0)^r = 0$

Similarly for ; $\lim_{x \rightarrow \infty} (1 - (1 - F(x))^r) = 1 - (1 - 1)^r = 1$

(ii) If $x < y$, $F(x) < F(y) \implies -F(x) > -F(y) \implies 1 - F(x) > 1 - F(y) \implies (1 - F(x))^r > (1 - F(y))^r \forall r > 0$ Thus, $1 - (1 - F(x))^r < 1 - (1 - F(y))^r$

(iii) Since $F(x)$ is right continuous, $1 - F(x)$ is right continuous $\implies (1 - F(x))^r$ is right continuous (since $r > 0$) implies $1 - (1 - F(x))^r$ is right continuous

Part c) $F(x) + (1 - F(x))\log(1 - F(x))$

(i) $\lim_{x \rightarrow -\infty} (F(x) + (1 - F(x))\log(1 - F(x))) = \lim_{x \rightarrow -\infty} F(x) + \lim_{x \rightarrow -\infty} (1 - F(x))\log(1 - F(x)) = 0 + (1 - 0)\log(1 - 0) = 0$

Consider $\lim_{x \rightarrow \infty} (F(x) + (1 - F(x))\log(1 - F(x))) = \lim_{x \rightarrow \infty} F(x) + \lim_{x \rightarrow \infty} (1 - F(x))\log(1 - F(x)) = 1 + (1 - 1)\log(1 - 1) = 0$

(ii) If $x < y$ then $F(x) < F(y) \implies 1 - F(x) > 1 - F(y)$. Since \log is a monotonic non-increasing function in $[0, 1]$ and is in fact negative definite: $\log(1 - F(x)) < \log(1 - F(y))$ and $1 - F(x) > 1 - F(y) \implies -F(x)\log(1 - F(x)) < -F(y)\log(1 - F(y))$ (This holds only because \log is negative definite in $[0, 1]$) Thus, $F(x) - F(x)\log(1 - F(x)) < F(y) - F(y)\log(1 - F(y))$

(iii) $F(x) - F(x)\log(1 - F(x))$ is right continuous as $\log(1 - F(x))$ is right continuous

Part d) $(F(x) - 1)e + \exp(1 - F(x))$

(i) $\lim_{x \rightarrow -\infty} (F(x) - 1)e + \exp(1 - F(x)) = \lim_{x \rightarrow -\infty} (F(x) - 1)e + \exp(\lim_{x \rightarrow -\infty} (1 - F(x))) = (0 - 1)e + \exp(1 - 0) = -e + e = 0$

$\lim_{x \rightarrow \infty} (F(x) - 1)e + \exp(1 - F(x)) = \lim_{x \rightarrow \infty} (F(x) - 1)e + \exp(\lim_{x \rightarrow \infty} (1 - F(x))) = (1 - 1)e + \exp(1 - 1) = 0 + 1 = 1$

(ii) if $x < y$, $F(x) < F(y) \implies F(x) - 1 < F(y) - 1 \implies (F(x) - 1)e < (F(y) - 1)e$ Also, $1 - F(x) > 1 - F(y)$ Since \exp is a non-increasing function in $[0, 1]$ $\exp(1 - F(x)) < \exp(1 - F(y))$

Thus,

$(F(x) - 1)e + \exp(1 - F(x)) < (F(y) - 1)e + \exp(1 - F(y))$

(iii) $(F(x) - 1)e + \exp(1 - F(x))$ is right continuous as \exp is right continuous.

FG is also a density function since it satisfies:

(i) $\lim_{x \rightarrow -\infty} F(x)G(x) = \lim_{x \rightarrow -\infty} F(x) * \lim_{x \rightarrow -\infty} G(x) = 0$

And $\lim_{x \rightarrow \infty} F(x)G(x) = \lim_{x \rightarrow \infty} F(x) * \lim_{x \rightarrow \infty} G(x) = 1$

(ii) If $x < y$, $F(x) < F(y)$ and $G(x) < G(y) \implies F(x)G(x) < F(y)G(y)$

(iii) Since $F(x), G(x)$ are right continuous

Exercise # 2.3

(1)

Given: $\lim_{m \rightarrow -\infty} a_m \rightarrow -\infty$ and $\lim_{m \rightarrow \infty} a_m \rightarrow \infty$; $G(x) = P(X \leq a_m)$ when $a_{m-1} \leq x \leq a_m$; a_m is a strictly increasing sequence.

Sequence a is chosen so that $\sup_m |a_m - a_{m-1}|$ becomes smaller and smaller so even though the sequence is increasing the successive difference between the terms keep on decreasing essentially indicating a_m saturates as $m \rightarrow \infty$

(2)

Given: $g(x)$ is continuous and strictly increasing, X is a random variable:

Since $g(X)$ is continuous and strictly increasing $\implies g^{-1}$ exists.

Consider $\{Y \leq y\} \implies \{g(X) \leq y\}$. Since g^{-1} exists, such a set is equivalent to: $\{X \leq g^{-1}(y)\}$ which belongs to \mathcal{F} as $g : R \rightarrow R$

(3)

$F(x) = P(X \leq x) = \begin{cases} 0 & \text{if } x \leq 0, \\ x & \text{if } 0 < x \leq 1, \\ 1 & \text{if } x > 1, \end{cases}$ **To Prove:** $Y = F^{-1}(x)$ is a random variable: Consider $\{Y \leq y\} = \{F^{-1}(x) \leq y\}$. Since F is continuous and strictly increasing $\implies F^{-1}(x)$ exists in R so $\{Y \leq y\} = \{F^{-1}(x) \leq y\} = \{x \leq F(y)\} = \{x \leq P(X \leq y)\} \in \mathcal{F}$
 F should necessarily be continuous and monotonic for the inverse to exist!

(4)

f, g are density functions:

$$\int_{-\infty}^{\infty} f(x) dx = 1 \text{ and } \int_{-\infty}^{\infty} g(x) dx = 1$$

Consider $y(x) = \lambda f(x) + (1 - \lambda)g(x)$ Thus, $\int_{-\infty}^{\infty} y(x) dx = \int_{-\infty}^{\infty} \lambda f(x) dx + \int_{-\infty}^{\infty} (1 - \lambda)g(x) dx = \lambda * \int_{-\infty}^{\infty} f(x) dx + (1 - \lambda) * \int_{-\infty}^{\infty} g(x) dx = \lambda * 1 + (1 - \lambda) * 1 = 1$

Thus $\lambda f(x) + (1 - \lambda)g(x)$ is a density function too.

Now consider $y(x) = f(x)g(x)$, then :

$\int_{-\infty}^{\infty} y(x) dx = \int_{-\infty}^{\infty} f(x)g(x) dx$ Clearly this is not necessarily equal to 1 so fg is not a density function!

(5)

Part a) $f(x) = \begin{cases} cx^{-d} & x > 1, \\ 0 & \text{otherwise,} \end{cases}$

$$F(x) = \int_{-\infty}^{\infty} f(c) dx = \int_1^{\infty} cx^{-d} = 1 \implies \frac{-c}{-d+1} = 1 \implies c = d - 1$$

Part b) $f(x) = ce^x(1 + e^x)^{-2} \in R$

$F(x) = \int_{-\infty}^{\text{inf}} ce^x(1 + e^x)^{-2}$ Let $t = e^x + 1$ then $e^x dx = dt$ $F(x) = \int_1^{\infty} ct^{-2} dt$ $F(x = 1) = -(c * 0 - c) = 1$
 Thus $c = 1$.