MATH 542 Homework 4

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Problem 1

$$
\begin{aligned}
E[(X-a)(X-a)'] &= E[(X-a)(X'-a')] \\
&= E[XX' - Xa' - aX' + aa'] \\
&= E[XX'] - E[Xa'] - E[aX'] + E[aa'] \\
&= E[XX'] - E[X]a' - aE[X'] + E[aa'] \\
&= (Var[X] + E[X]E[X]') - E[X]a' - aE[X'] + aa' \\
&= Var[X] + E[X]E[X]' - E[X]a' - aE[X]' + aa' \;\; \text{since } E[X'] = E[X]' \\
&= Var[X] + (E[X] = a)(E[X]' - a') \\
&= Var[X] + (E[X] = a)(E[X] - a)'
\end{aligned}
$$

$Var[X] = \sum = (\sigma_{ij})$

$||X-a||^2_{1\times 1} = (X-a)'_{1\times r}(X-a)_{1\times r}$ and hence(replace X with X' and a with a'):

$$
\begin{aligned}
E[||X-a||^2] &= E[(X-a)'(X-a)] \\
&= E[X'X] - E[X']a - a'E[X] + a'a \\
&= \sum_i E[X_i^2] - E[X']a - a'E[X] + a'a \\
&= \sum_i (Var[X_i] + E[X_i]^2) - E[X']a - a'E[X] + a'a \\
&= \sum_i Var[X_i] + E[X'X] - E[X']a - a'E[X] + a'a \text{ since } \sum_i E[X_i]^2 = E[X'X] \\
&= \sum_i \sigma_i + ||E[X] - a||^2
\end{aligned}
$$

1 Problem 2

Fact: $X - a - E[X-a] = X - E[X]$

$$
\begin{aligned}
Cov[X-a, Y-b] &= E[(X - a - E[X-a])(Y - b - E[Y-b])']) \\
&= E[(X - E[X])(Y - E[Y])'] \\
&= Cov[X, Y]
\end{aligned}
$$

2 Problem 3

$Y_i = X_i - X_{i-1}$

$Cov[Y_i, Y_j] = 0$ for $i \neq j$

Consider the vector $(Y_1, Y_2, Y_3, \ldots, Y_n)' = (X_1, X_2 - X_1, X_3 - X_2, \ldots, X_n - X_{n-1})'$

We make use of $Var(AX) = AVar(X)A'$.

To find A, consider the vectors $(Y_1, Y_2, Y_3, \ldots, Y_n)' = (X_1, X_2 - X_1, X_3 - X_2, \ldots, X_n - X_{n-1})'$

$$\begin{aligned}
\begin{pmatrix} Y_1 \\ Y_2 \\ Y_3 \\ \vdots \\ Y_n \end{pmatrix} &= \begin{pmatrix} X_1 \\ X_2 - X_1 \\ X_3 - X_2 \\ \vdots \\ X_n - X_{n-1} \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 & 0 & \ldots & 0 & \\ -1 & 1 & 0 & \ldots & 0 & 0 \\ 0 & -1 & 1 & \ldots & 0 & 0 \\ \vdots & & & & & \\ 0 & 0 & 0 & \ldots & -1 & 1 \end{pmatrix} \times \begin{pmatrix} X_1 \\ X_2 \\ X_3 \\ \vdots \\ X_n \end{pmatrix}
\end{aligned}$$

Hence $A = \begin{pmatrix} 1 & 0 & 0 & \ldots & 0 & \\ -1 & 1 & 0 & \ldots & 0 & 0 \\ 0 & -1 & 1 & \ldots & 0 & 0 \\ \vdots & & & & & \\ 0 & 0 & 0 & \ldots & -1 & 1 \end{pmatrix}$

Now using $Var(Y) = AVar(X)A'$ we get $Var(X) = A^{-1}Var(Y)A'^{-1}$

Also $Var(Y) = I_{n \times n}$ and hence $Var(X) = A^{-1}A'^{-1} = BB^T$ where $B = A^{-1}$

3 Problem 4

$X_{i+1} = \rho X_i$

Consider:

$$\begin{aligned}
\begin{pmatrix} X_1 \\ X_2 \\ X_3 \\ \vdots \\ X_n \end{pmatrix} &= \begin{pmatrix} X_1 \\ \rho X_1 \\ \rho X_2 \\ \vdots \\ \rho X_{n-1} \end{pmatrix} \\
&= \begin{pmatrix} 1 \\ \rho \\ \rho^2 \\ \vdots \\ \rho^{n-1} \end{pmatrix} X_1
\end{aligned}$$

Let $A = \begin{pmatrix} 1 & \rho & \rho^2 & \vdots & \rho^{n-1} \end{pmatrix}'$ and hence variance $Var[X] = AVar(X_1)A' = \sigma^2 AA'$

$$Var[X] = \begin{pmatrix} 1 & \rho & \rho^2 & \dots & \rho^{n-1} \\ \rho & \rho^2 & \rho^3 & \dots & \rho^n \\ \rho^2 & \rho^3 & \rho^4 & \dots & \rho^{n+1} \\ \vdots \\ \rho^n & \rho^{n+1} & \rho^{n+2} & \dots & \rho^{2n-2} \end{pmatrix}$$

1b. Problem 1

$$\begin{aligned} X_1^2 + 2X_1X_2 - 4X_2X_3 + X_3^2 &= (X_1 + X_2)X_1 + (X_1 - 2X_3)X_2 + (-2X_2 + X_3)X_3 \\ &= \begin{pmatrix} X_1 & X_2 & X_2 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & -2 \\ 0 & -2 & 1 \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \\ X_3 \end{pmatrix} \end{aligned}$$

$X = \begin{pmatrix} X_1 & X_2 & X_3 \end{pmatrix}$

$A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & -2 \\ 0 & -2 & 1 \end{pmatrix}$

$AS = \begin{pmatrix} 1 & 1 & \frac{1}{4} \\ 1 & -\frac{1}{2} & -2 \\ 0 & -\frac{7}{4} & \frac{1}{2} \end{pmatrix}$

Thus, $E[X'AX] = tr(A\sum) + \mu' A\mu = 1 + \mu' A\mu$

1b. Problem 2

$\sum_i (X_i - \bar{X})^2 = \sum_i (X_i^2 - 2X_i\bar{X} + \bar{X}^2) = \sum_i X_i^2 - 2\sum_i X_i\bar{X} + \bar{X}^2 = \sum_i X_i^2 - n\bar{X}^2$

Now $\sum_i X_i^2 = X'X$ and $\bar{X} = \frac{1}{n}\sum_i X_i = \frac{1}{n}\mathbf{1}'X = \frac{1}{n}X'\mathbf{1}$

Hence,

$$\begin{aligned} \sum_i (X_i - \bar{X})^2 &= \sum_i X_i^2 - n\bar{X}^2 \\ &= X'X - n\frac{1}{n^2}(X'\mathbf{1}\mathbf{1}'X) \\ &= X'X - \frac{1}{n}(X'\mathbf{1}\mathbf{1}'X) \\ &= X'(\mathbf{I} - \frac{1}{n}\mathbf{1}\mathbf{1}')X \end{aligned}$$

Let $A = (\mathbf{I} - \frac{1}{n}\mathbf{1}\mathbf{1}')$

Now using $E[X'AX] = tr(A\sum) + \mu' A\mu$ we have:

$A\sum = \begin{pmatrix} 1 - 1/n & -1/n & -1/n & \dots & -1/n \\ -1/n & 1 - 1/n & -1/n & \dots & -1/n \\ \vdots \\ -1/n & -1/n & -1/n & \dots & 1 - 1/n \end{pmatrix} \times diag(\sigma_1^2, \sigma_2^2, \dots, \sigma_n^2) = (1 - \frac{1}{n})\sum_i \sigma_i^2$

Also, $\mu' A = \begin{pmatrix}\mu & \mu & \dots & \mu\end{pmatrix} \times \begin{pmatrix} 1-1/n & -1/n & -1/n & \dots & -1/n \\ -1/n & 1-1/n & -1/n & \dots & -1/n \\ \vdots \\ -1/n & -1/n & -1/n & \dots & 1-1/n \end{pmatrix} =$ 0 and hence $\mu' A \mu = 0$

$$E[\sum_i (X_i - \bar{X})^2] = (1 - \frac{1}{n}) \sum_i \sigma_i^2$$
$$E[\frac{1}{n(n-1)} \sum_i (X_i - \bar{X})^2] = \frac{1}{n^2} \sum_i \sigma_i^2$$

Finally,

$$var(\bar{X}) = Var(\frac{1}{n} \sum_i X_i)$$
$$= \frac{1}{n^2} \sum_i Var(X_i)$$

since X_i are mutually independent

$$= \frac{1}{n^2} \sum_i \sigma_i^2$$
$$= E[\frac{1}{n(n-1)} \sum_i (X_i - \bar{X})^2]$$

1b. Problem 3

Given: $\bar{X_w} = \sum_i w_i X_i$ and $\sum w_i = 1$

$$\begin{aligned} Var(\bar{X_w}) &= Var(\sum w_i X_i) \\ &= \sum w_i^2 Var(X_i) \quad \text{since } X_i \text{ are mutually independent} \\ &= \sum w_i^2 \sigma_i^2 \end{aligned}$$

Now consider,

$$\text{minimize} \sum w_i^2 \sigma_i^2 \text{ subject to } \sum_i w_i = 1$$

We consider the following lagrange formulation $min_w f(\mathbf{w}) = \sum_i w_i^2 \sigma_i^2 + \lambda(\sum_i w_i - 1)$

Now to find optimal λ, we solve $\frac{\partial f(\mathbf{w})}{\partial w_i} = 0$

$$\frac{\partial f()}{\partial w_i} = 2 w_i \sigma_i^2 + \lambda = 0$$
$$\implies w_i = -\frac{\lambda}{2\sigma_i^2}$$

Thus, $w_i = -\frac{\lambda}{2\sigma_i^2}$ or $w_i \propto \frac{1}{\sigma_i^2}$
Using $\sum_i w_i = 1$ we get:

$$\begin{aligned}
\sum_i w_i &= 1 \\
\sum_i \frac{\lambda}{-2\sigma_i^2} &= 1 \\
\implies \lambda &= \frac{-2}{\sum_i 1/\sigma_i^2} \\
\implies w_i &= \frac{1}{\sigma_i^2 \sum_i (1/\sigma_i^2)} \\
\implies f_min(w) &= \sum_i (\frac{1}{\sigma_i^2 \sum_i (1/\sigma_i^2)})\sigma_i^2 \\
&= \frac{1}{\sum_i (1/\sigma_i^2)}
\end{aligned}$$

Part b

$$\begin{aligned}
\sum_i w_i(X_i - \bar{X_w})^2 &= \sum_i w_i(X_i^2 - 2X_i\bar{X_w} + \bar{X_w}^2) \\
&= \sum_i w_i X_i^2 - 2\bar{X_w}\sum_i w_i X_i + \bar{X_w^2} \\
&= \sum_i w_i X_i^2 - 2\bar{X_w}^2 + \bar{X_w}^2 \\
&= \sum_i w_i X_i^2 - \bar{X_w}^2
\end{aligned}$$

Now, we rewrite $\sum_i w_i X_i^2 = X'\Lambda X$ where $\Lambda = diag(w_1, w_2, \ldots, w_n)$
and $\bar{X_w} = \sum_i w_i X_i = X'w = w'X$

$$\begin{aligned}
\bar{X_w}^2 &= (\sum_i w_i X_i)^2 \\
&= X'ww'X
\end{aligned}$$

and hence $\sum_i w_i(X_i - \bar{X_w})^2 = X'(\Lambda - ww')X$
Define $A = \Lambda - ww'$
so that $E[\sum_i w_i(X_i - \bar{X_w})^2] = E[X'AX] = tr(A\sum) + \mu' A\mu$
$tr(A\sum) = \sum_i (w_i - w_i^2)\sigma_i^2 = \sum_i w_i\sigma_i^2 - w_i(w_i\sigma_i^2) = \sum_i (a - aw_i) = na - a$
and $\mu' A_i = \mu(w_i - w_i^2 - w_i(1 - w_i)) = 0$
Also, $v_min = \frac{1}{\sum_i 1/\sigma_i^2} = \frac{1}{\sum_i \frac{w_i}{a}} = a$ Thus,

$$\begin{aligned}
E[S_w^2] &= \frac{1}{n-1} tr(A\sum) + \mu' A\mu \\
&= \frac{1}{n-1}(na - a) + 0 \\
&= a \\
&= v_min
\end{aligned}$$