

MATH-505A: Homework # 2

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Exercise # 1.5

(1)

Given: A, B are independent

To Prove: $(A^C, B); (A^C, B^C)$ are independent

Since A, B are independent:

$$P(A \cap B) = P(A)P(B) \quad (1)$$

Thus,

$$P(A \cap B) = (1 - P(A^C))P(B) = P(B) - P(B)P(A^C) \quad (2)$$

Rearranging 2:

$$P(B)P(A^C) = P(B) - P(A \cap B) \quad (3)$$

$P(B) - P(A \cap B)$ signifies 'in B but not in A AND B '. Thus, it should belong to A^C AND B

$$P(B) - P(A \cap B) = P(A^C \cap B) = P(A^C)P(B) \quad (4)$$

From 3 and 4 : A^C, B are independent.

Similiary to prove A^C, B^C are independent, we perform substitute B^C in $P(B \cap A^C)$, since 4 is true to obtain:

$$P(B^C \cap A^C) = P(A^C)P(B^C) \quad (5)$$

(2)

$A_{ij} = i^{th}$ and j^{th} rolls produce the same number.

For any $i \neq j$, total outcome are $6 * 6 = 36$ and number of favourable outcomes are $\binom{6}{1} * 1 = 6$, thus

$$p(A_{ij}) = \frac{6}{36} = \frac{1}{6}$$

Consider $P(A_{ij} \cap A_{kj})$, such that $i \neq j \neq k$, then :

$P(A_{ij} \cap A_{jk})$ refers to the probability when i^{th}, j^{th} and k^{th} rolls show the same number, which can be caluclated as: $P(A_{ij} \cap A_{kj}) = \frac{\binom{6}{1} * 1 * 1}{6 * 6 * 6} = \frac{1}{36} = P(A_{ij})P(A_{jk})$

Thus, A_{ij} are pairwise independent as it is true for any choice of i, j, k as long as $i \neq j \neq k$

Consider:

$$P(A_{ij} \cap A_{jk} \cap A_{kl}) = \frac{\binom{6}{1} * 1 * 1 * 1}{6 * 6 * 6} = \frac{1}{36} \neq P(A_{ij})P(A_{jk})P(A_{kl})$$

Since $P(A_{ij} \cap A_{jk} \cap A_{kl}) \neq P(A_{ij})P(A_{jk})P(A_{kl})$, it will not be true in general.

And since the independence criterion is not satisfied for the above case, it will not be true for the case when A_{ij} are considered together for all values of i, j .

(3)

To Prove:

(a) outcomes of coin tosses are independent

(b) Given a sequence of length m of heads and tails the chance of it occurring in first m tosses is 2^{-m} .In order to prove (a) and (b) are equivalent it is sufficient to prove that if $a \implies b$ and $b \implies a$.If the outcomes are independent, probability of a head or tail in a sequence is $\frac{1}{2}$. Consider m tosses, since they are independent the probability of seeing any string of H and T is given by $\frac{1}{2} * \frac{1}{2} * \dots * (m) \text{ times} = \frac{1}{2^m} = 2^{-m}$. Hence $\implies b$.Now consider if b is true, then : $P(m) = 2^{-m} \implies P(m+1) = 2^{-(m+1)}$, . The $P(m+1)$ case is similar to $P(m)$ with an extra toss. $P(m+1) = 2^{-m} * \frac{1}{2}$. The extra half factor is accounted by the extra toss that is performed which must be independent with respect to the m tosses for yielding such a relation for $P(m+1) \implies a$ is true and this is true for any m . This inductive hypothesis proves that the tosses must be independent ($P(1) = \frac{1}{2}$ is trivially true)and hence $a \iff b$ and $a \implies b$ so a and b are equivalent statements.

(4)

Given: $\Omega = \{1, 2, 3, \dots, p\}$ where p is prime. F is set of all subsets of ω ; $P(A) = \frac{|A|}{p}$ **To Prove:** A 'or' B is a null set or is the set Ω

$$P(A) = \frac{|A|}{p}$$

$$P(B) = \frac{|B|}{p}$$

Now, from the definition of $P(A)$:

$$P(A \cap B) = \frac{|A \cap B|}{p} \quad (6)$$

Since A, B are independent:

$$P(A \cap B) = P(A)P(B) = \frac{|A \cap B|}{p} = \frac{|A|}{p} \frac{|B|}{p} \quad (7)$$

Thus:

$$p * |A \cap B| = |A||B| \implies |A||B| \bmod p = 0 \quad (8)$$

where \bmod operator gives the remainder.

and

$$0 \leq |A|, |B| \leq p \implies (|A| \text{ or } |B| = p) \text{ OR } (|A| \text{ OR } |B| = 0) \implies A, B \text{ are either null or complete sets (with } |A|, |B| = |\Omega|).$$

(5)

Given:

$$P(A, B|C) = P(A|C)P(B|C) \quad (9)$$

for all A,B.If A, B are independent :

$$P(A|B) = P(A) \quad (10)$$

$$P(A, B|C) = \frac{P(A, B, C)}{P(C)} = P(A|B, C) * \frac{P(B, C)}{P(C)} = P(A|B, C) * P(B|C) \quad (11)$$

Hence the conditional independence of A and B given C is dependent on conditional independence of A given B and C and B given C and neither implies $P(A|B) = P(A)$ nor is implied by this.

Part2:

For which event C, are A and B(for all A,B) independent **iff** they are conditionally independent given C: ?
 If $P(A, B|C) = P(A|C) * P(B|C) \implies P(A|B) = P(A)$ then what is C?. This relation can be set to true for all values of A,B if $P(C)$ is set to 1. because that would automatically imply $P(A, B) = P(A)P(B)$

$$\text{Thus } P(C) = 1$$

(7)

 $A = \{ \text{all children of same sex} \}$
 $B = \{ \text{there is at most one boy} \}$
 $C = \{ \text{one boy and one girl included} \}$

$$P(A) = P(\text{all boys}) + P(\text{all girls}) = \left(\frac{1}{2} * \frac{1}{2} * \frac{1}{2}\right) * 2 = \frac{1}{4}$$

$$P(B) = P(0 \text{ boys}) + P(1 \text{ boy}) = \left(\frac{1}{8}\right) + 3 * \left(\frac{1}{8}\right) = \frac{1}{2}$$

$$P(C) = P(1 \text{ boy} + 1 \text{ girl}) = 2 * \left(\frac{1}{2} * \frac{1}{2} * \frac{1}{2}\right) * 3 = \frac{3}{4}$$

Part a): A is independent of B and B is independent of C

$P(A \cap B) = \text{all children are of same sex AND there is at most one boy} \implies \text{all children are boys}$

$$\implies P(A \cap B) = \frac{1}{2} * \frac{1}{2} * \frac{1}{2} = \frac{1}{8} = P(A) * P(B).$$

Hence A,B are independent

$P(B \cap C) = \text{there is at most one boy AND there is one boy and a girl} \implies \text{there is one boy and two girls.}$

$$P(B \cap C) = \frac{1}{2} * \frac{1}{2} * \frac{1}{2} * 3 = \frac{3}{8} = P(B) * P(C)$$

Hence B,C are independent

Part b): Is A independent of C?

$P(A \cap C) = \text{the family includes boy and girl AND all children are of same sex}$

Clearly $P(A \cap C) = \phi$ and hence A,B are NOT necessarily independent!(null intersection does not imply independence)

Part c): Do the results hold if boys and girls are not equally likely?

NO. if the girls and boys are not likely these events will have different probabilities

Part d): Do these results hold if there are 4 children?

No, the results might not necessarily be valid for 4 children as the number of permutations would be the