

MATH-578B: Assignment # 2

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Problem 1

Define $h(w)$ to be an indicator function:

$$h(x) = \begin{cases} 1 & x \in A \\ -1 & x \notin A \end{cases}$$

Now consider $E[h(W)]$:

$$\begin{aligned} E[h(W)] &= P(W \in A) \times 1 + P(W \notin A) \times -1 \\ &= P(W \in A) - (1 - P(W \in A)) \\ &= 2P(W \in A) - 1 \end{aligned} \tag{1.1}$$

Similarly,

$$E[h(Z)] = 2P(Z \in A) - 1 \tag{1.2}$$

where $A \in \mathcal{Z}^+$

From (1.1), (1.2)

$$\begin{aligned} E(h(W)) - E(h(Z)) &= 2P(W \in A) - 2P(Z \in A) \\ |E(h(W)) - E(h(Z))| &= 2|P(W \in A) - P(Z \in A)| \\ \max_{h: \|h\|=1} |E(h(W)) - E(h(Z))| &= 2 \max_{A \in \mathcal{Z}^+} |P(W \in A) - P(Z \in A)| \\ &= 2|P(W = 0) - P(Z = 0) + P(W = 1) - P(Z = 1) + \dots| \\ &= \sum_{k \geq 0} |P(W = k) - P(Z = k)| \end{aligned}$$

Problem 2

$w = 11011011$

Periods: $P(w) = \{p \in \{1, 2, \dots, w-1\} : w_i = w_{i+p} \forall i \in \{1, 2, \dots, w-p\}\}$

$w_1 = w_4; w_2 = w_5; w_3 = w_6; \dots w_5 = w_8$

$w_1 = w_7$

$w_1 = w_8$

Thus $P(w) = \{3, 6, 7\}$

$P(w) = 6$ can be written as multiple of $P(w) = 3$ and hence the principal period $P(w') = \{3, 7\}$

The mean $(n - w + 1)\mu(w)$ using poisson approximation for the number of clumps is given by: $P(w) - \sum_{p \in P(w')} P(w^{(p)}w)$

$w = 11011011$

$P(w) = \{3, 6, 7\}$

$P(w') = \{3, 7\}$

$$\begin{aligned}\mu(w) &= P(11011011) - P(110w) - P(1101101w) \\ &= p^6 q^2 - p^2 q(p^6 q^2) - p^5 q^2(p^6 q^2) \\ &= p^6 q^2(1 - p^2 q - p^5 q^2)\end{aligned}$$

And hence the mean of the poisson approximation is given by $(n - 7)p^6 q^2(1 - p^2 q - p^5 q^2)$

The number of occurrences of w can be approximated by:

$$P(L_i = j) = \frac{u_j(w)}{\sum_{i \geq 1} \mu_j(w)}$$

where C_j is the event that there are j consecutive occurrences of w starting at that particular location.

$C_j = \{w^{(p_1)}w^{(p_2)} \dots w^{(p_{j-1})}w\}$ where p_i is a principal period of w . and the mean of this process is given by $\mu_j(w) = P(C_j) - 2P(C_{j+1}) + P(C_{j+2})$

$P(C_j)$ calculation:

Let there be $kw^{(3)}$ and $j - 1 - k$ with $w^{(7)}$ among the first $j - 1$ occurrences of w starting at i . The probability of this is $\binom{j-1}{k}(p^2 q)^k (p^5 q^2)^{j-1-k}$

Thus,

$$\begin{aligned}P(C_j) &= P(w) \sum_{k=0}^{j-1} (P(w^{(3)})^k (P(w^{(7)}))^{j-1-k}) \\ &= p^6 q^2 (p^2 q + p^5 q^2)^{j-1} \\ &= p^{2j+4} q^{j+1} (1 + p^3 q)^{j-1} \\ \mu_j(w) &= p^{2j+4} q^{j+1} (1 + p^3 q)^{j-1} (1 - 2p^2 q(1 + p^3 q) + p^4 q^2(1 + p^3 q)^2)\end{aligned}$$

Exercise 6.1

Expected number of apparent islands $= Ne^{-c(1-\theta)} = cge^{-c(1-\theta)}$
 where $g = \frac{G}{L}$ (since $cg \rightarrow \infty$) as $g \rightarrow \infty$

$$\begin{aligned} f(c) &= cge^{-c(1-\theta)} \\ \frac{\partial f(c)}{\partial c} &= g(e^{-c(1-\theta)} + -c(1-\theta)e^{-c(1-\theta)}) = 0 \\ \implies 1 - c(1-\theta) &= 0 \\ c^* &= (1-\theta)^{-1} \end{aligned}$$

and the maximum number of apparent islands is $Ne^{-c^*(1-\theta)} = Ne^{-1} = \frac{G}{L}e^{-1}(1-\theta)^{-1}$

Exercise 6.5

From our solution for *Excercise6.1* we see that the the maximum number of apparent islands occur at $c^* = (1-\theta)^{-1}$ Now c was defined to be the expected number of clones covering a point. and c^* is the maximum number of clones arranged such that no two clones overlap.(overlap if at all is less than θ) Then the maximum number of islands at any point would be $ceil((1-\theta)^{-1}) = 1 + \max\{k : k \text{ integer}, k < (1-\theta)^{-1}\}$

Excercise 11.14

Stein equation: $(Lf)(x) = \lambda f(x+1) - xf(x)$ and $W = \sum_{i=1}^n X_i$

$E((Lf)(W)) = 0$

And $Z \sim \text{Poisson}(E(W))$

Since X_i is iid.

$$b_1 = \sum_{i \in I} \sum_{j \in J_i} E(X_i)E(X_j) = \sum_{i=1}^n p^2 = np^2$$

$$b_2 = \sum_{i \in I} \sum_{i \neq j \in J_i} E(X_i X_j) = \sum_i \sum_{i \neq j \in I_i} E(X_i X_j) = 0$$

And hence, by Theorem 11.22:

$$\|W - Z\| \leq 2np^2 \frac{1 - e^{-\lambda}}{\lambda} \leq 2np^2$$

where $EZ = EW = \lambda$

Exercise 11.10

$$X_i = \prod_{j=1}^{i+t-1} D_j$$

$$W = \sum_{i \in I} X_i$$

$I = \{1, 2, \dots, n - t + 1\}$ and $J_i = \{j \in I : |i - j| < t\}$

$EW = \lambda = (n - t + 1)p^t$

Since $p < 1$ define $q = 1/p$

Now, $EW = \frac{n-t+1}{q^t}$ and as $n \rightarrow \infty$ In order to prevent $EW \rightarrow \infty$, we make $t \rightarrow n$ when $n \rightarrow \infty$

$i - (t - 1) \leq j \leq i + (t - 1)$

$b_1 = \sum_i p^t \times ((2t - 2) + 1)p^t = (n - t + 1)(2t - 1)p^{2t}$

$b_2 = \sum_i p^{(2t-2)+1} = (n - t + 1)p^{2t-1}$