MATH-578B: Midterm

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Contents

Problem 1

Problem 1: (a)

$$P = \begin{bmatrix} 1-\alpha & \alpha \\ \beta & 1-\beta \end{bmatrix}$$

Let the stationary state be given by $\pi = (\pi_1, \pi_2)$, then:

$$\begin{aligned} \pi.P &= \pi \\ \pi_1 + \pi_2 &= 1 \end{aligned}$$

Solving which gives:

$$\begin{aligned} (1-\alpha)\pi_1 + \pi_2 &= 1 \\ \pi_1 + \pi_2 &= 1 \\ \implies (\pi_1, \pi_2) &= (\frac{\beta}{\alpha+\beta}, \frac{\alpha}{\alpha+\beta}) \end{aligned}$$

Problem 1: (b)

$\mathbf{w} = 101$

$$\begin{aligned} \beta_{w,w}(0) &= 1 \\ \beta_{w,w}(1) &= 0 \\ \beta_{w,w}(2) &= 1 \end{aligned}$$

$$\begin{aligned} P_u(0) &= 1 \\ P_u(1) = p_{w_2 w_3} &= p_{01} = \alpha \\ P_U(2) = p_{w_1 w_2} p_{w_2 w_3} &= p_{10} p_{01} = \beta\alpha \end{aligned}$$

Now,

$$G_{w,w}(t) = \sum_{j=0}^{2} t^j \beta_{w,w}(j) P_{w,w}(j)$$

Thus,

$$\begin{aligned} G_{w,w}(t) &= 1 \times 1 \times 1 + t \times 0 \times \alpha + t^2 \times 1 \times \beta\alpha \\ &= 1 + \alpha\beta t^2 \end{aligned}$$

Problem 1: (c)

X_n : Number of occurrences(overlaps allowed) in $A_1 A_2 A_3 \dots A_n$ Using Theorem 12.1:

$$\lim_{n\to\infty} \frac{1}{n} E(X_n) = \pi_w = \pi_1 \times p_{10} \times p_{01} = 1 \times \beta \times \alpha$$

Thus $\lim_{n\to\infty} \frac{X_n}{n} = \alpha\beta$

Problem 1: (d)

Spectral decomposition of P:

$$det \begin{bmatrix} \alpha - \lambda & 1-\alpha \\ 1-\beta & \beta - \lambda \end{bmatrix} = 0$$

$$\lambda^2 + (\alpha+\beta-2)\lambda + (1-\alpha-\beta) = 0$$

Thus, $\lambda_1 = 1$ and $\lambda_2 = 1-\alpha-\beta$
Eigenvectors are given by:
$v_1^T = \begin{pmatrix} x_1 & x_1 \end{pmatrix} \;\forall\; x_1 \in R$
and for λ_2 , $v_2 = \begin{pmatrix} x_1 & \frac{-\beta x_1}{\alpha} \end{pmatrix}$
Now using Markov property: $P(X_n = 1|X_0 = 0) = (P^n)_{01}$
Now,
$P^n = VD^nV^{-1}$
where:

$$V = \begin{bmatrix} 1 & 1 \\ 1 & \frac{-\beta}{\alpha} \end{bmatrix}$$

and

$$D = \begin{bmatrix} 1 & 0 \\ 0 & (1-\alpha-\beta) \end{bmatrix}$$

$$V^{-1} = \frac{-1}{\frac{\beta}{\alpha}+1} \begin{bmatrix} -\frac{\beta}{\alpha} & -1 \\ -1 & 1 \end{bmatrix}$$

Thus,

$$P^n = \begin{bmatrix} 1 & 1 \\ 1 & \frac{-\beta}{\alpha} \end{bmatrix} \times \begin{bmatrix} 1 & 0 \\ 0 & (1-\alpha-\beta)^n \end{bmatrix} \times \frac{-1}{\frac{\beta}{\alpha}+1} \begin{bmatrix} -\frac{\beta}{\alpha} & -1 \\ -1 & 1 \end{bmatrix}$$

$$P^n = \frac{1}{\alpha+\beta} \begin{bmatrix} \beta + \alpha(1-\alpha-\beta)^n & \alpha - \alpha(1-\alpha-\beta)^n \\ \beta - \beta(1-\alpha-\beta)^n & \alpha + \beta(1-\alpha-\beta)^n \end{bmatrix}$$

ALITER
We consider the following identity: $P^{n+1} = PP^n$
then:

$$\begin{bmatrix} p_{00}^{n+1} & p_{01}^{n+1} \\ p_{10}^{n+1} & p_{11}^{n+1} \end{bmatrix} = \begin{bmatrix} p_{00}^{n} & p_{01}^{n} \\ p_{10}^{n} & p_{11}^{n} \end{bmatrix} \times \begin{bmatrix} 1-\alpha & \alpha \\ \beta & 1-\beta \end{bmatrix}$$

$\Longrightarrow$

$$\begin{aligned} p_{11}^{n+1} &= p_{10}^n(\alpha) + p_{11}^n(1-\beta) \\ &= (1-p_{11}^n)(\alpha) + (p_{11}^n)(1-\beta) \\ &= \alpha + (1-\alpha-\beta)p_{11}^n \end{aligned} \tag{1d.1}$$

On similar lines:

$$p_{00}^{n+1} = (1-\alpha-\beta)p_{00}^{n} + \beta \tag{1d.2}$$

In order to solve equations of type $1d.1$ and $1d.2$ we take the following strategy:
By substituting $p_{00}^{n+1} = p_{00}^{n}$ (and thus obtaining the stationary solution at $\frac{\beta}{\alpha+\beta}$), $1d.2$ can be reduced to the following form:

$$p_{00}^{n+1} = \frac{\beta}{\alpha+\beta} = (1-\alpha-\beta)(p_{00}^{n} - \frac{\beta}{\alpha+\beta})$$

Let's call $y^(n) = p_{00}^{n} - \frac{\beta}{\alpha+\beta}$
Then $1d.3$ is similar to:

$$y^{(n+1)} = (1-\alpha-\beta)y^(n)$$
$$y^{(n+1)} = (1-\alpha-\beta)^{n}y^(0)$$

$y^(0) = p_{00}^{(0)} - \frac{\beta}{\alpha+\beta}$
Assume $p_{00}^{(0)} = 1 \implies y^{(0)} = \frac{\alpha}{\alpha+\beta}$
Which gives:

$$p_{00}^{n} = (1-\alpha-\beta)^{n}\frac{\alpha}{\alpha+\beta} + \frac{\beta}{\alpha+\beta}\text{if } \alpha+\beta > 0$$

Similarly,

$$p_{11}^{n} = (1-\alpha-\beta)^{n}\frac{\beta}{\alpha+\beta} + \frac{\alpha}{\alpha+\beta}\text{if } \alpha+\beta > 0$$

NOTE: If $\alpha+\beta = 0$, we get:

$$P^{n} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Problem 1: (e)

$\pi_w = \pi_i p_{10} p_{01} = 1 \times \beta\alpha = \alpha\beta$

$$\begin{aligned}
\lim_{n\to\infty} \frac{Var(X_n)}{n} &= 2\pi_w\big((\beta_{w,w}(0)P_w(0) - \pi_w) + (\beta_{w,w}(1)P_w(1) - \pi_w) + (\beta_{w,w}(2)P_w(2) - \pi_w)\big) \\
&+ 2\pi_w P_w(2)\sum_{j=0}^{\infty}\{p_{11}^{j+1} - \pi_1\} + \pi_w^2 - \pi_w \\
&= 2\alpha\beta(1 + \alpha\beta - 3\alpha\beta) + 2\alpha\beta\sum_{j=0}^{\infty}\{\beta^{j+1} - 1\} - \alpha\beta + (\alpha\beta)^2 \\
&= \alpha\beta\big(2(1-2\alpha\beta) + 2\sum_{j=0}^{\infty}\{\beta^{j+1} - 1\} - 1 + \alpha\beta\big)
\end{aligned}$$

: (f)

Y_n: Number of occurences of word w in all words, thus, $Y_n \approx cX_n$

NOTE: I assume the problem should be to estimate $\lim_{n\to\infty} \frac{Y_n}{n}$ and $\lim_{n\to\infty} \frac{Var(Y_n)}{n}$ In the problem it mentions X_n instead of Y_n

Thus,

$$\begin{aligned}
\lim_{n\to\infty} \frac{Y_n}{n} &= c \times \lim_{n\to} \frac{X_n}{n} \\
&= c\alpha\beta \\
\lim_{n\to\infty} \frac{Var(Y_n)}{n} &= c^2 \times \lim_{n\to\infty} \frac{Var(X_n)}{n} \\
&= c^2 \times \big(\alpha\beta\big(2(1-2\alpha\beta) + 2\sum_{j=0}^{\infty}\{\beta^{j+1} - 1\} - 1 + \alpha\beta\big)\big)
\end{aligned}$$

Problem 2

Problem 2