MATH-578B: Midterm

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Contents

Problem 1

Problem 1: (a)

$$P = \begin{bmatrix} 1-\alpha & \alpha \\ \beta & 1-\beta \end{bmatrix}$$

Let the stationary state be given by $\pi = (\pi_1, \pi_2)$, then:

$$\pi.P = \pi$$
$$\pi_1 + \pi_2 = 1$$

Solving which gives:

$$(1-\alpha)\pi_1 + \pi_2 = 1$$
$$\pi_1 + \pi_2 = 1$$
$$\implies (\pi_1, \pi_2) = (\frac{\beta}{\alpha+\beta}, \frac{\alpha}{\alpha+\beta})$$

Problem 1: (b)

$\mathbf{w} = 101$

$$\beta_{w,w}(0) = 1$$
$$\beta_{w,w}(1) = 0$$
$$\beta_{w,w}(2) = 1$$

$$P_u(0) = 1$$
$$P_u(1) = p_{w_2 w_3} = p_{01} = \alpha$$
$$P_U(2) = p_{w_1 w_2} p_{w_2 w_3} = p_{10} p_{01} = \beta\alpha$$

Now,

$$G_{w,w}(t) = \sum_{j=0}^{2} t^j \beta_{w,w}(j) P_{w,w}(j)$$

Thus,

$$G_{w,w}(t) = 1 \times 1 \times 1 + t \times 0 \times \alpha + t^2 \times 1 \times \beta\alpha$$
$$= 1 + \alpha\beta t^2$$

Problem 1: (c)

X_n : Number of occurrences(overlaps allowed) in $A_1 A_2 A_3 \ldots A_n$ Using Theorem 12.1:

$$\lim_{n\to\infty} \frac{1}{n} E(X_n) = \pi_w = \pi_1 \times p_{10} \times p_{01} = 1 \times \beta \times \alpha$$

Thus $\lim_{n\to\infty} \frac{X_n}{n} = \alpha\beta$

Problem 1: (d)

Spectral decomposition of P:

$$det \begin{bmatrix} \alpha-\lambda & 1-\alpha \\ 1-\beta & \beta-\lambda \end{bmatrix} = 0$$

$$\lambda^2 + (\alpha+\beta-2)\lambda + (1-\alpha-\beta) = 0$$

Thus, $\lambda_1 = 1$ and $\lambda_2 = 1-\alpha-\beta$
Eigenvectors are given by:
$v_1^T = \begin{pmatrix} x_1 & x_1 \end{pmatrix} \ \forall \ x_1 \in R$
and for λ_2 , $v_2 = \begin{pmatrix} x_1 & \frac{-\beta x_1}{\alpha} \end{pmatrix}$
Now using Markov property: $P(X_n = 1|X_0 = 0) = (P^n)_{01}$
Now,
$P^n = VD^nV^{-1}$
where:

$$V = \begin{bmatrix} 1 & 1 \\ 1 & \frac{-\beta}{\alpha} \end{bmatrix}$$

and

$$D = \begin{bmatrix} 1 & 0 \\ 0 & (1-\alpha-\beta) \end{bmatrix}$$

$$V^{-1} = \frac{-1}{\frac{\beta}{\alpha}+1} \begin{bmatrix} -\frac{\beta}{\alpha} & -1 \\ -1 & 1 \end{bmatrix}$$

Thus,

$$P^n = \begin{bmatrix} 1 & 1 \\ 1 & \frac{-\beta}{\alpha} \end{bmatrix} \times \begin{bmatrix} 1 & 0 \\ 0 & (1-\alpha-\beta)^n \end{bmatrix} \times \frac{-1}{\frac{\beta}{\alpha}+1} \begin{bmatrix} -\frac{\beta}{\alpha} & -1 \\ -1 & 1 \end{bmatrix}$$

$$P^n = \frac{1}{\alpha+\beta} \begin{bmatrix} \beta+\alpha(1-\alpha-\beta)^n & \alpha-\alpha(1-\alpha-\beta)^n \\ \beta-\beta(1-\alpha-\beta)^n & \alpha+\beta(1-\alpha-\beta)^n \end{bmatrix}$$

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We consider the following identity: $P^{n+1} = PP^n$
then:

$$\begin{bmatrix} p_{00}^{n+1} & p_{01}^{n+1} \\ p_{10}^{n+1} & p_{11}^{n+1} \end{bmatrix} = \begin{bmatrix} p_{00}^{n} & p_{01}^{n} \\ p_{10}^{n} & p_{11}^{n} \end{bmatrix} \times \begin{bmatrix} 1-\alpha & \alpha \\ \beta & 1-\beta \end{bmatrix}$$

$\implies$

$$\begin{aligned} p_{11}^{n+1} &= p_{10}^n(\alpha) + p_{11}^n(1-\beta) \\ &= (1-p_{11}^n)(\alpha) + (p_{11}^n)(1-\beta) \\ &= \alpha + (1-\alpha-\beta)p_{11}^n \end{aligned} \tag{1d.1}$$

Problem 1 [Problem 1: (d)] continued on next page...

On similar lines:

$$p_{00}^{n+1} = (1-\alpha-\beta)p_{00}^n + \beta \tag{1d.2}$$

In order to solve equations of type $1d.1$ and $1d.2$ we take the following strategy:
By substituting $p_{00}^{n+1} = p_{00}^n$ (and thus obtaining the stationary solution at $\frac{\beta}{\alpha+\beta}$), $1d.2$ can be reduced to the following form:

$$p_{00}^{n+1} = \frac{\beta}{\alpha+\beta} = (1-\alpha-\beta)(p_{00}^n - \frac{\beta}{\alpha+\beta})$$

Let's call $y^(n) = p_{00}^n - \frac{\beta}{\alpha+\beta}$
Then $1d.3$ is similar to:

$$y^{(n+1)} = (1-\alpha-\beta)y^(n)$$
$$y^{(n+1)} = (1-\alpha-\beta)^n y^(0)$$

$y^(0) = p_{00}^{(0)} - \frac{\beta}{\alpha+\beta}$
Assume $p_{00}^{(0)} = 1 \implies y^{(0)} = \frac{\alpha}{\alpha+\beta}$
Which gives:

$$p_{00}^n = (1-\alpha-\beta)^n \frac{\alpha}{\alpha+\beta} + \frac{\beta}{\alpha+\beta} \text{if } \alpha+\beta > 0$$

Similarly,

$$p_{11}^n = (1-\alpha-\beta)^n \frac{\beta}{\alpha+\beta} + \frac{\alpha}{\alpha+\beta} \text{if } \alpha+\beta > 0$$

NOTE: If $\alpha+\beta = 0$, we get:

$$P^n = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Problem 1: (e)

$\pi_w = \pi_i p_{10} p_{01} = 1 \times \beta\alpha = \alpha\beta$

$$\begin{aligned}
\lim_{n\to\infty} \frac{Var(X_n)}{n} &= 2\pi_w\big((\beta_{w,w}(0)P_w(0) - \pi_w) + (\beta_{w,w}(1)P_w(1) - \pi_w) + (\beta_{w,w}(2)P_w(2) - \pi_w)\big) \\
&+ 2\pi_w P_w(2) \sum_{j=0}^{\infty} \{p_{11}^{j+1} - \pi_1\} + \pi_w^2 - \pi_w \\
&= 2\alpha\beta(1 + \alpha\beta - 3\alpha\beta) + 2\alpha\beta \sum_{j=0}^{\infty} \{\beta^{j+1} - 1\} - \alpha\beta + (\alpha\beta)^2 \\
&= \alpha\beta\big(2(1-2\alpha\beta) + 2\sum_{j=0}^{\infty}\{\beta^{j+1} - 1\} - 1 + \alpha\beta\big)
\end{aligned}$$

: (f)

Y_n: Number of occurences of word w in all words, thus, $Y_n \approx cX_n$
NOTE: I assume the problem should be to estimate $\lim_{n\to\infty} \frac{Y_n}{n}$ and $\lim_{n\to\infty} \frac{Var(Y_n)}{n}$ In the problem it mentions X_n instead of Y_n
Thus,

$$\begin{aligned}
\lim_{n\to\infty} \frac{Y_n}{n} &= c \times \lim_{n\to} \frac{X_n}{n} \\
&= c\alpha\beta \\
\lim_{n\to\infty} \frac{Var(Y_n)}{n} &= c^2 \times \lim_{n\to\infty} \frac{Var(X_n)}{n} \\
&= c^2 \times \big(\alpha\beta\big(2(1-2\alpha\beta) + 2\sum_{j=0}^{\infty}\{\beta^{j+1}-1\} - 1 + \alpha\beta\big)\big)
\end{aligned}$$

Problem 2

Problem 2

Problem 2: (a)

Expected number of squares of side length t such that all X_v are 1 in the square: We simply choose a position on the positive lattice x axis and then construct a square around it $[(x_0, y_0)(x_0+t, y_0+t)]$ so we have $n-t$ choices for x_0 and $n-t$ choices for y_0, the constraint being that all points inside are all 1, there are approximately t^2 integer points inside

$$E(\#\text{number of squares of side length } t \text{ such that all } X_v \text{ are 1 inside}) = (n-t) \times (n-t)p^{t^2} = (n-t)^2 p^{t^2}$$

Problem 2: (b)