

# **MATH-505A: Homework # 6**

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## Exercise # 3.1

1

**Part a:**  $f(x) = C2^{-x}$

For  $f(x)$  to be a mass function  $\sum_1^\infty C2^{-i} = 1$   $C(\frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \dots) = C\frac{1}{2} * \frac{1}{1-\frac{1}{2}} = 1 \implies C = 1$

**Part b:**  $f(x) = \frac{C2^{-x}}{x} \sum_1^\infty \frac{C}{2^i i} = 1$

Notice  $\ln(1+x) = x + \frac{x^2}{2} + \frac{x^3}{3} + \dots$

Hence  $C \sum_1^\infty \frac{(\frac{1}{2})^i}{i} = C \ln(1+1/2) = 1 \implies C = \frac{1}{\ln 1.5}$

**Part c:**  $f(x) = Cx^{-2} \sum_1^\infty \frac{C}{x^2} = 1$

Besel sum:

$$\sum_1^n \frac{1}{i^2} =$$

Using Taylor expansion of  $\sin x$  and the fact that  $\frac{\sin x}{x}$  has roots at  $x = \pi, 2\pi, 3\pi, \dots$ :

$$\frac{\sin x}{x} = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} + \dots = (1 - \frac{x}{\pi})(1 - \frac{x}{2\pi})(1 - \frac{x}{3\pi}) \dots (1 - \frac{x}{\pi})$$

The product of productions of  $\frac{\sin x}{x}$  is given the coefficient of  $x^2$  in the original and hence  $-\frac{1}{3!} = -\pi^2 \sum_1^\infty i^2$

Thus  $\sum_1^\infty \frac{C}{x^2} = C * \frac{\pi^2}{6}$ . Thus  $C = \frac{6}{\pi^2}$

**Part d:**  $C2^x/x! \sum_1^\infty C2^i/i! = C \sum_1^\infty 2^i/i! = Ce^2 \implies C = \frac{1}{e^2}$

2(i)

**Part a**  $P(X > 1) = \sum_2^\infty 2^{-i} = \frac{1}{4} * 2 = \frac{1}{2}$

**Part b**  $P(X > 1) = \frac{1}{1.5} \sum_2^\infty \frac{(\frac{1}{2})^i}{i} = 1 - \ln(1.5)/2$

**Part c**  $P(X > 1) = 1 - \frac{6}{\pi^2} 1^{-2} = 1 - \frac{6}{\pi^2}$

**Part d**  $P(X > 1) = 1 - \frac{1}{e^2} 2 = 1 - \frac{2}{e^2}$

2 (iii)

Probability that  $X$  is even =  $P(X = 2k)$  for  $k = 1, 2, 3, \dots$

**Part a**  $P(X = 2k) = 2^{-2k}$  Summing up over all  $k$ :  $P = \frac{1}{2^2} + \frac{1}{2^4} + \frac{1}{2^6} + \dots = \frac{1}{4} \frac{1}{1-\frac{1}{4}} = \frac{1}{3}$

**Part b**  $P(X = 2k) = \ln(3.5) \frac{1}{2^k} = \ln(3.5) \frac{1}{2} \frac{(\frac{1}{4})^k}{k} = \frac{\ln(3.5)}{2} e^{1.25}$

**Part c**

$$P(X = 2k) = \frac{6}{\pi^2} 4k^2 = \frac{3}{2\pi^2} * \frac{\pi^2}{6} = \frac{1}{4}$$

**Part d**

$$P(X = 2k) = \frac{1}{e^2} \frac{2^{2k}}{(2k)!}$$

3

Since the coin tosses are independent, the choice can be represented by two successive coin tosses with probability of heads being  $p * p$ . Thus  $P(X = k) = \binom{n}{k} p^{2k} (1 - p^2)^{n-k}$

## 5a

**For Binomial:**  $f(k) = \binom{n}{k} p^k (1-p)^{n-k}$  Consider  $LHS = f(k-1) * f(k+1)$   $LHS = \binom{n}{k-1} p^{k-1} (1-p)^{n-k+1} + \binom{n}{k+1} p^{k+1} (1-p)^{n-k-1} = \binom{n}{k-1} \binom{n}{k+1} (p^k (1-p)^{n-k})^2$   $RHS = \binom{n}{k}^2$  We now focus on  $\binom{n}{k-1} \binom{n}{k+1}$   
 Let  $y = \frac{\binom{n}{k-1} \binom{n}{k+1}}{\binom{n}{k}^2}$  Expanding:  $y = \frac{n!n!(n-k)!(n-k)!k!k!}{(k-1)!(k+1)!(n-k+1)!(n-k-1)!} = \frac{k(n-k)}{(k+1)(n-k+1)} \frac{k}{k+1} \leq 1$  and  $\frac{n-k}{n-k+1} \leq 1$   
 $\forall k$   
 Hence  $y \leq 1$   
 Thus  $LHS \leq RHS$   
**For Poisson**  $f(k) = e^{-\lambda} \frac{\lambda^k}{k!}$   $LHS = f(k-1)f(k+1) = \frac{e^{-2\lambda} \lambda^{2k}}{(k+1)!(k-1)!}$   
 Thus  $LHS \leq RHS$

## 5b

$f(k) = \frac{90}{(\pi k)^4}$   $LHS = f(k+1)f(k-1) = \frac{90^2}{\pi^8 (k+1)^4 (k-1)^4}$   $RHS = f(k+1)^2 = \frac{90^2}{\pi^8 (k)^8}$   
 $y = LHS/RHS = \frac{k^8}{(k+1)^4 (k-1)^4} = \left(\frac{k}{k+1}\right)^4 \left(\frac{k}{k-1}\right)^4 = \left(\frac{k^2}{k^2-1}\right)^4 \geq 1$   
 Thus  $LHS \geq RHS$

## 5c

Any function of the form  $P(x=k) = \frac{1}{n}$  satisfies  $f(k)f(k-1) = f(k)^2$  Note: We aren't explicitly talking about countably many case.

## Exercise # 3.2

## 4

**Part a**  $P(\min(X, Y) \leq x)$

## Exercise # 3.3

## 4

$E(1$   
 $X) = \sum p(x)(1$   
 $x)$   
 $1$   
 $E(X) = \sum p(x)(x)$   
 For  $E(X) = E(1$   
 $X): \sum(p(x)(x-1$   
 $x) = 0$  The above equation might not true be in general. However one possible case where this is true is for this distribution:  
 $p(x) = \begin{cases} 1/2 & x = 1 \text{ or } x = -1 \\ 0 & \text{otherwise} \end{cases}$