CSCI-567: Assignment #4

Due on Monday, November 9, 2015

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Contents

Problem 1

Problem 1: (a) Gradient Calculation

$$L(y_i, \hat{y_i}) = \log(1 + \exp(-y_i\hat{y_i}))$$

$$g_i = \frac{\partial L(y_i, \hat{y_i})}{\partial \hat{y_i}}$$

$$\boxed{g_i = \frac{-y_i \exp(-y_i\hat{y_i})}{1 + \exp(-y_i\hat{y_i})}}$$

Problem 1: (b) Weak Learner Section

$$h^* = \min_{h \in H} \big(\min_{\gamma \in R} \sum_{i=1}^{n} (-g_i - \gamma h(x_i))^2 \big)$$

$$\implies \frac{\partial h^*}{\partial \gamma} = 0$$

$$\implies 2\sum_{i=1}^{n}(-g_i - \gamma h(x_i))(-h(x_i)) = 0$$

$$\boxed{\hat{h} = -\frac{\sum_{i=1}^{n} g_i h(x_i)}{\sum_{i=1}^{n} h(x_i)^2}}$$

Also check if it is indeed minimum with a second derivative test:

$$\frac{\partial^2 h^*}{\partial \gamma^2} = 2\sum_{i=1}^{n} h(x_i)^2 > 0$$

Since the second derivative is positive definite, $ga\hat{m}ma$ is indeed where the minima occurs.

Problem 1 continued on next page. . .

Problem 1: (c) Step Size Selection

$$\alpha^* = \arg\min_{\alpha \in R} \sum_{1}^{n} L(y_i, \hat{y}_i + \alpha h^*(x_i))$$

Newton's approximation:

$$\alpha_1 = \alpha_0 - \frac{f'(\alpha_0)}{f''(\alpha_0)}$$

We start from $\alpha_0 = 0$ and hence:

$$\begin{aligned} f(\alpha_0) &= \sum_{i=1}^{n} \log(1 + \exp(-y_i \hat{y}_i)) \\ f'(\alpha) &= \sum_{i=1}^{n} \frac{\partial L}{\partial \alpha} \\ &= \sum_{i=1}^{n} \frac{-y_i h^*(x_i) \exp\left(-y_i(\hat{y}_i + \alpha h^*(x_i))\right)}{1 + \exp\left(-y_i(\hat{y}_i + \alpha h^*(x_i))\right)} \end{aligned}$$

$$\boxed{f'(\alpha = \alpha_0) = -\sum_{i=1}^{n} \frac{y_i h^*(x_i) \exp(-y_i \hat{y}_i)}{1 + \exp(-y_i \hat{y}_i)}}$$

And,

$$\begin{aligned} f''(\alpha) &= \sum_{i=1}^{n} \frac{\partial^2 L}{\partial \alpha^2} \\ &= \sum_{i=1}^{n} \frac{\left\{\left(1 + \exp\left(-y_i(\hat{y}_i + \alpha h^*(x_i))\right)\right)(y_i h^*(x_i))^2 + y_i h^*(x_i)\right\} \exp\left(-y_i(\hat{y}_i + \alpha h^*(x_i))\right)}{\left(1 + \exp\left(-y_i(\hat{y}_i + \alpha h^*(x_i))\right)\right)^2} \end{aligned}$$

$$\boxed{f''(\alpha_0) = \sum_{i=1}^{n} \frac{\left\{\left(1 + \exp\left(-y_i \hat{y}_i\right)\right)(y_i h^*(x_i))^2 + y_i h^*(x_i)\right\} \exp\left(-y_i \hat{y}_i\right)}{\left(1 + \exp\left(-y_i \hat{y}_i\right)\right)^2}}$$

Thus,

$$\boxed{\alpha_1 = \frac{\sum_{i=1}^{n} \frac{y_i h^*(x_i) \exp(-y_i \hat{y}_i)}{1 + \exp(-y_i \hat{y}_i)}}{\sum_{i=1}^{n} \frac{\left\{\left(1 + \exp\left(-y_i \hat{y}_i\right)\right)(y_i h^*(x_i))^2 + y_i h^*(x_i)\right\} \exp\left(-y_i \hat{y}_i\right)}{\left(1 + \exp\left(-y_i \hat{y}_i\right)\right)^2}}}$$

Problem 2

Problem 2: (a)

Primal form:

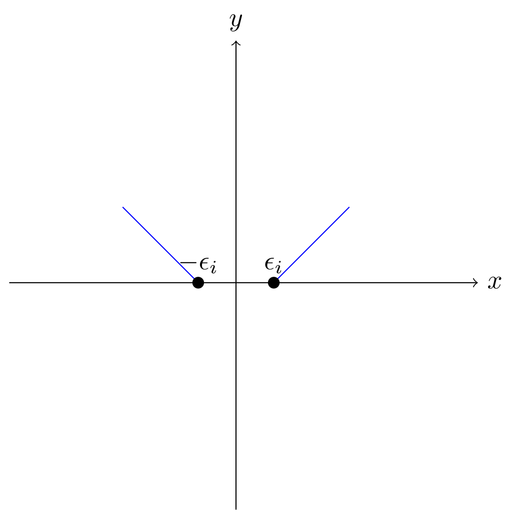
$$\min_{w} ||w||^2$$
$$\text{such that } |y_i - (w^T x_i + b)| \leq \epsilon$$

Problem 2: (b)

$$\min_{w,\epsilon_i} \frac{1}{2}||w||^2 + C\sum_i \epsilon_i$$
$$\text{such that } (w^T x_i + b) - y_i \leq n_i + \epsilon_i \text{ (positive deviation)}$$
$$\text{and } y_i - (w^T x_i + b) \leq p_i + \epsilon_i \text{ (negative deviation)}$$
$$n_i \geq 0$$
$$p_i \geq 0$$

Also, the slackness loss needs further constraints:

$$n_i = \begin{cases} 0 & |n_i| < \epsilon_i, \\ |n_i| - \epsilon & otherwise \end{cases}$$
$$p_i = \begin{cases} 0 & |p_i| < \epsilon_i, \\ |p_i| - \epsilon & otherwise \end{cases}$$



So essentially n_i, p_i are non zero, only above the two blue lines

Problem 2: (c)

$$
\begin{aligned}
L = \frac{1}{2}||w||^2 + C\sum_i (p_i + n_i) \\
- \sum_i (\lambda_i p_i + \lambda_i' n_i) \\
- \sum_i \alpha_i(\epsilon + p_i - (y_i - (w^T x_i + b))) \\
- \sum_i \beta_i(\epsilon + n_i + (y_i - (w^T x_i + b)))
\end{aligned}
$$

Conditions :

$$
\begin{aligned}
\alpha_i \geq 0 \\
\beta_i \geq 0 \\
\lambda_i \geq 0 \\
\lambda_i' \geq 0
\end{aligned}
$$

Dual Form(all summations are from 1 to n)::

$$
\begin{aligned}
\Delta_w L &= 0 \\
&= w - \sum_i \alpha_i x_i + \sum_i \beta_i x_i = 0 \\
&= w - \sum_i (\alpha_i - \beta_i) x_i = 0 \\
\Delta_b L &= 0 \\
&= \sum_i \alpha_i - \sum_i \beta_i = 0 \\
\Delta_{p_i} L &= 0 \\
&= C - \sum \lambda_i - \sum_i \alpha_i = 0 \\
\Delta_{n_i} L &= 0 \\
&= C - \sum \lambda_i' - \sum_i \beta_i = 0
\end{aligned}
$$

Thus, w is given by:

$$w = \sum_i \alpha_i x_i - \sum_i \beta_i x_i$$

depends only on the support vectors.
This reduces the optimisation to:

$$\begin{aligned}
\max f &= \frac{1}{2}\sum_{i,j}(\alpha_i - \beta_i)x_i^T x_j(\alpha_j - \beta_j) + p_i(C - \sum_i \lambda_i - \sum_i \alpha_i) \\
&+ n_i(C + \sum_i \lambda_i' - \sum_i \beta_i) \\
&+ \epsilon(-\sum_i \alpha_i - \sum_i \beta_i) \\
&+ \sum_i y_i(\alpha_i - \beta_i) - \sum_i(\alpha_i w^T x_i - \beta_i w^T x_i) \\
&= -\frac{1}{2}\sum_{i,j}(\alpha_i - \beta_i)x_i^T x_j(\alpha_j - \beta_j) - \epsilon(\sum_i(\alpha_i + \beta_i)) \\
&+ \sum_i y_i(\alpha_i - \beta_i) \\
\text{such that } &\sum_i(\alpha_i - \beta_i) = 0 \\
\text{and } &\alpha_i, \beta_i \in [0, C]
\end{aligned}$$

Problem 2: (d)

Using Kernel transformation, we simply replace $x_i^T x_j$ with $k(x_i, x_j)$:

$$w = \sum_i(\alpha_i - \beta_i)\phi(x_i)$$

this happens because $x_i' x_j$ gets mapped onto by an equivalent kernel function $k(x_i, x_j) = \phi^T(x_i)\phi(x_j)$ and the objective function is:

$$\max_f = \frac{1}{2}\sum_{i,j}(\alpha_i - \beta_i)k(x_i, x_j)(\alpha_j - \beta_j) - \epsilon(\sum_i(\alpha_i + \beta_i)) + \sum_i y_i(\alpha_i - \beta_i)$$

Problem 3.3

Problem 3.3: (a)

C	Tr. Dataset 1(s)	Tr. Dataset 2(s)	Training Dataset 3(s)	CV Accuracy	Avg. time(s)
4^{-6}=0.000244	0.606156	0.400791	0.346078	0.578976	0.451
4^{-5}=0.000977	0.379495	0.477559	0.498054	0.907001	0.451
4^{-4}=0.003906	0.529940	0.495841	0.534946	0.926001	0.520
4^{-3}=0.015625	0.516236	0.577324	0.561874	0.935501	0.551
4^{-2}=0.062500	0.512287	0.529517	0.554510	0.945006	0.532
4^{-1}=0.250000	0.630195	0.663459	0.657651	0.943010	0.650
4^{0}=1.000000	0.746649	0.601710	0.563063	0.939003	0.637
4^{1}=4.000000	0.633370	0.572011	0.595552	0.942501	0.600
4^{2}=16.000000	0.674677	0.681010	0.698041	0.943503	0.684

The time shown(t) is in seconds for three partitions, the last column being the average time

As seen from the table. the time seems to increase with C and the CV increases too.

- **Increasing C leads to increase in runtime**
  The larger the value of C, the more is the penalisation and hence smaller the ϵ_i would be, this causes the lower bound in input 'quadprog' to increase in size (since $A.x \leq b$ and in this case ϵ is included in vector x, so the search space for A increases, since $|x|$ is small, when C is large!)

- **Increasing C leads to higher training accuracy**
  C determines the tradeoff between objective function complexity and the overall loss. When C is small, there are chances of overfitting, this is evident from low CV values for lower C(because the generalisation error is high, and this is where cross validation is helpful) and hence increasing C helps overcome the problem of over-fitting and the generalization error lowers(training accuracy increases)

Problem 3.3: (b)

Based on **highest cross validation accuracy**. $C = 4^2$ (maximum training accuracy = 94.3503%)

Problem 3.3: (c)

With $C = 16$, test accuracy = 94.35% (it's pretty close to the training accuracy itself)

Problem 3.4

Problem 3.4: (a)

Platform Used: *Ubuntu* 12.04, $x86\_64$
'libsvm' gives 96.55% as its accuracy which is pretty close to 94.35% that my code gives. **Hence the cross validation accuracy is not exactly the same, and 'libsvm' performs better, by a margin of 2.2786%**

C	Avg. Training Time	CV Accuracy
4^{-6}	0.829624	0.5575
4^{-5}	0.827846	0.5575
4^{-4}	0.831586	0.5575
4^{-3}	0.831599	0.7295
4^{-2}	0.649459	0.9195
4^{-1}	0.425499	0.934
4^{0}	0.281219	0.949
4^{1}	0.210071	0.955
4^{2}	0.201989	0.9655

Problem 3.4: (b)

'libsvm' is around 3 times faster in the worst case.

Problem 3.5

Problem 3.5: (a) Polynomial Kernel

C	*degree*	Avg. Training Time(s)	CV Accuracy(%)
0.015625	1.000000	0.652999	70.100000
0.015625	2.000000	0.658651	55.750000
0.015625	3.000000	0.658636	55.750000
0.062500	1.000000	0.513652	91.750000
0.062500	2.000000	0.587962	86.550000
0.062500	3.000000	0.655689	76.900000
0.250000	1.000000	0.334359	92.800000
0.250000	2.000000	0.417291	92.550000
0.250000	3.000000	0.504703	91.800000
1.000000	1.000000	0.230170	93.850000
1.000000	2.000000	0.273486	94.700000
1.000000	3.000000	0.335671	94.350000
4.000000	1.000000	0.181141	94.400000
4.000000	2.000000	0.197570	95.100000
4.000000	3.000000	0.230603	95.700000
16.000000	1.000000	0.165130	94.450000
16.000000	2.000000	0.172796	96.100000
16.000000	3.000000	0.197574	96.450000
64.000000	1.000000	0.186566	94.150000
64.000000	2.000000	0.170041	96.600000
64.000000	3.000000	0.184937	96.550000
256.000000	1.000000	0.288991	94.050000
256.000000	2.000000	0.173684	96.000000
256.000000	3.000000	0.184054	96.300000
1024.000000	1.000000	1.013039	94.350000
1024.000000	2.000000	0.223741	95.700000
1024.000000	3.000000	0.184358	96.300000
4096.000000	1.000000	4.086584	94.400000
4096.000000	2.000000	0.204906	95.400000
4096.000000	3.000000	0.183501	96.250000
16384.000000	1.000000	21.828099	94.400000
16384.000000	2.000000	0.206865	95.400000
16384.000000	3.000000	0.183685	96.250000

Polynomial Kernel maximum train accuracy: 96.600000
Polynomial Kernel optimal C: 64
Polynomial Kernel optimal degree: 2
Polynomal Kernel test accuracy(%): 95.150000

Problem 3.5 continued on next page...

Problem 3.5: (b) RBF Kernel

C	γ	Training Time(s)	CV (%)
0.015625	0.000061	0.799739	55.750000
0.015625	0.000244	0.799145	55.750000
0.015625	0.000977	0.798638	55.750000
0.015625	0.003906	0.801114	55.750000
0.015625	0.015625	0.802529	64.750000
0.015625	0.062500	0.805714	73.750000
0.015625	0.250000	0.822912	55.750000
0.062500	0.000061	0.799229	55.750000
0.062500	0.000244	0.798608	55.750000
0.062500	0.000977	0.799373	55.750000
0.062500	0.003906	0.800940	83.350000
0.062500	0.015625	0.648832	91.200000
0.062500	0.062500	0.634587	91.850000
0.062500	0.250000	0.824124	63.600000
0.250000	0.000061	0.798746	55.750000
0.250000	0.000244	0.799292	55.750000
0.250000	0.000977	0.799286	86.350000
0.250000	0.003906	0.587110	92.000000
0.250000	0.015625	0.426160	93.200000
0.250000	0.062500	0.423444	94.750000
0.250000	0.250000	0.713477	92.300000
1.000000	0.000061	0.800186	55.750000
1.000000	0.000244	0.797172	86.900000
1.000000	0.000977	0.571743	91.850000
1.000000	0.003906	0.381565	93.050000
1.000000	0.015625	0.281917	94.650000
1.000000	0.062500	0.285756	96.150000
1.000000	0.250000	0.568394	96.100000
4.000000	0.000061	0.798443	86.950000
4.000000	0.000244	0.569461	91.850000
4.000000	0.000977	0.371507	93.000000
4.000000	0.003906	0.259033	94.250000
4.000000	0.015625	0.203123	95.550000
4.000000	0.062500	0.230975	96.650000
4.000000	0.250000	0.539952	96.100000
16.000000	0.000061	0.570222	91.850000
16.000000	0.000244	0.369962	93.050000
16.000000	0.000977	0.261383	93.950000
16.000000	0.003906	0.203001	95.050000
16.000000	0.015625	0.195201	96.500000
16.000000	0.062500	0.225650	96.900000
16.000000	0.250000	0.541735	95.950000

C	γ	Training Time(s)	CV(%)
64.000000	0.000061	0.369279	93.050000
64.000000	0.000244	0.262142	93.950000
64.000000	0.000977	0.210888	94.750000
64.000000	0.003906	0.191527	94.950000
64.000000	0.015625	0.183208	96.650000
64.000000	0.062500	0.229521	96.550000
64.000000	0.250000	0.541598	95.950000
256.000000	0.000061	0.257230	93.900000
256.000000	0.000244	0.214556	94.700000
256.000000	0.000977	0.191176	94.900000
256.000000	0.003906	0.188024	96.450000
256.000000	0.015625	0.186326	96.500000
256.000000	0.062500	0.224757	96.350000
256.000000	0.250000	0.542700	95.950000
1024.000000	0.000061	0.212776	94.600000
1024.000000	0.000244	0.191017	94.700000
1024.000000	0.000977	0.208193	95.050000
1024.000000	0.003906	0.225442	96.600000
1024.000000	0.015625	0.194065	96.400000
1024.000000	0.062500	0.224793	96.350000
1024.000000	0.250000	0.542050	95.950000
4096.000000	0.000061	0.192989	94.550000
4096.000000	0.000244	0.210952	94.750000
4096.000000	0.000977	0.271062	96.250000
4096.000000	0.003906	0.277291	96.250000
4096.000000	0.015625	0.187257	96.350000
4096.000000	0.062500	0.224741	96.350000
4096.000000	0.250000	0.541835	95.950000
16384.000000	0.000061	0.241238	94.350000
16384.000000	0.000244	0.310596	94.950000
16384.000000	0.000977	0.401110	96.600000
16384.000000	0.003906	0.330598	96.250000
16384.000000	0.015625	0.187273	96.350000
16384.000000	0.062500	0.224825	96.350000
16384.000000	0.250000	0.542722	95.950000

RBF Kernel maximum train accuracy: $\boxed{96.900000}$
RBF Kernel optimal γ: 0.062500
RBF Kernel optimal C: 16
RBF Kernel test accuracy(%): 96.500000

Summary:

Polynomial Kernel maximum train accuracy: 96.600000
Polynomial Kernel optimal C: 64
Polynomial Kernel optimal degree: 2
Polynomal Kernel test accuracy(%): 95.150000

RBF Kernel maximum train accuracy: $\boxed{96.900000}$
RBF Kernel optimal γ: 0.062500
RBF Kernel optimal C: 16
RBF Kernel test accuracy(%): 96.500000

Hence, Best Kernel: $\boxed{RBF}$**(based on maximum train accuracy) and the best choice for RBF kernels 'C' and γ are 16 and 0.0625 respectively**(incidentally, also gives a higher test accuracy)