# MATH-505A: Homework # 3

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# Exercise # 1.7

## (1)

**Given:** Two roads $r1_{AB}$, $r2_{AB}$ connectinng points A and B and $s1_{BC}$, $s2_{BC}$ connecting B and C.
Let $p(AB)$ denote the probability that path between A$\longrightarrow$B is open and let $p(AB^c)$ denote the probability that there is no open road b/w A and B. Alternativels $p(AB)$ denotes that road(s) between A and B are open. **To find:**$Y = P(AB|AC^c)$.
Y is equal to the probability that road between A and B is open AND still the path between A and C is closed $\implies$ Path between B and C is closed AND between A and B is open
p(AB) = Path b/w A,B is open = 1 - Path b/w A,B is closed = $1 - p * p$
Thus

$$p(AB) = 1 - p^2 \tag{1}$$

Also,

$$p(AB) = p(BC) \tag{2}$$

$p(AC^C)$ = 1 - Probability A,C is open = 1 - Probability AB is open AND BC is open. Thus,

$$p(AC^c) = 1 - p(AB)p(AC) = 1 - (1 - p^2)^2 \tag{3}$$

$$p(AB \cap AC^C) = p(AC^C|AB)p(AB) = p(BC^C)p(AB) = p^2(1 - p^2) \tag{4}$$

$$p(AB|AC^c) = \frac{P(AB \cap AC^c)}{p(AC^c)} = \frac{p(AC^C|AB)p(AB)}{p(AC^c)} = \frac{p^2(1 - p^2)}{1 - (1 - p^2)^2} \tag{5}$$

**Part 2:** Additional direct road from A to C. Find $p(AB|AC^c)$:
$p(AC^c|AB)$ = Probability that A,C is closed given A,B are open = Probability A,C(direct) are closed AND B,C are closed

$$p(AC^C|AB) = p * p(BC^c)p(AB) \tag{6}$$

where the extra p in 6 as compared to 4 is because the direct path A,C should be blocked too.

$$p(AC^c) = 1 - (1 - p^2)^2(1 - p) \tag{7}$$

where the extra $(1 - p)$ factor in 7 as compared to 3 accounts for the fact that direct path AC is open.
Thus, for part 2:

$$p(AB|AC^c) = \frac{p^3(1 - p^2)}{1 - (1 - p^2)^2(1 - p)} \tag{8}$$

## (2)

$$p(2K \cap 1A) = \frac{\binom{4}{2} * \binom{4}{1} * \binom{52-4-4}{10}}{\binom{52}{13}} = \frac{24 * 44! * 13!}{10! * 52!} = 1.357 * 10^{-9} \tag{9}$$

$p(1A|2K) = \frac{p(1A \cap 2K)}{p(2K)}$

$$p(2K) = \frac{\binom{4}{2} * \binom{52-4}{11}}{\binom{52}{13}} \tag{10}$$

### (4)

**To prove/disprove:** $p(x|C) > p(y|C) AND p(x|C^c) > p(y|C^c) \implies p(x) > p(y)$

$$p(x|C) - p(y|C) > 0 \tag{11}$$

$$p(x|C^c) - p(y|C^c) > 0 \tag{12}$$

$$p(x) = p(x|C)p(C) + p(x|C^c)p(C^c) \tag{13}$$

Also,

$$p(y) = p(y|C)p(C) + p(y|C^c)p(C^c) \tag{14}$$

Consider $p(x) - p(y)$ :

$$p(x) - p(y) = (p(x|C) - p(y|C))p(C) + (p(x|C^c) - p(y|C^c))p(C^c) \tag{15}$$

From the 12, **??** and 15:

$$p(x) - p(y) > 0 \forall x, y \tag{16}$$

**Thus, x is always prefered over y.**

### (5)

Let $X_i$ represent the $i^{th}$ ccard draw
**Given:** $X_k > X_i, \ \forall i \in [1, k-1] and k \in [1, m]$
$p(X_k = m|X_k > X_i) = \frac{p(X_k = m \cap X_k > X_i)}{p(X_k > X_i)} = \frac{p(X_k = m)}{p(X_k > X_i)} = \frac{\frac{1}{m}}{\frac{1}{k}}$
Where the equality in the last step comes from the fact that the probability of choosing cards such that $p(X_k > X_i)$ is simply to choose the largest card, i.e. $k$ among the rest $i$.
Thus $p(X_k = m|X - k > X_i) = \frac{k}{m}$.

[

Exercise # 1.8]

## 1

### (a)

**Six turns up exactly once** The dice on which 6 should appear is choosen in $\binom{2}{1}$ way and has just one option(6) while the other has $6 - 1 = 5$ options. Total outcomes $= 6 * 6 = 36$
Thus:
$p = \frac{\binom{2}{1} * 1 * 5}{36} = \frac{5}{18}$

### (b)

**Both numbers are odd.** 3 out of 6 numbers are odd and the outcome of odd on both dice are independent. Thus,

$p = \frac{3*3}{36} = \frac{1}{12}$

### (c)

**Sum of scores is 4**.
Possible configurations: $(1,3); (2,2); (3,1)$
Thus:
$p = \frac{3}{36} = \frac{1}{12}$

### (d)

**Sum of scores is divisible by 3**.
Possible choices for sum to be divisible by 3:
$3, 6, 9, 12$:
3: $(1,2), (2,1)$
6: $(1,5), (2,4), (3,3), (4,2), (5,1)$
9: $(3,6), (4,5), (5,4), (6,3)$
12: $(6,6)$
Thus $p = \frac{2+5+4+1}{36} = \frac{1}{3}$

## 2

### (a)

Head appears the first time on $n^{th}$ throw. Thus a series of $n-1$ consecutive tails followed by head.
$p = \frac{1}{2^{n-1}} * \frac{1}{2} = \frac{1}{2^n}$

### (b)

The number of heads and dates to date are equal:
Case1: $n$ is odd.
Clearly the probability of heads and tails being equal in this case is zero!
Case2: $n$ is even.
A series of H,T such that $|H| = |T|$. Since the probability of occurence of either a H or a T is $\frac{1}{2}$, the result will still be:
$p = \frac{1}{2^n}$

### (c)

Exactly two heads have appeared together.
So exactly at two consecutive positions there is a H,H. The rest positions either alternate with H,T or only T.
$p = \frac{1}{2} * \frac{1}{2} * \frac{1}{2^{n-2}}$

### (d)

At least two heads have appeared.
This is same as the one minus probability that 0 or 1 heads have appeared so far:
$p = 1 - p(0\ heads) - p(1\ head\ only)$ $p = 1 - \binom{n}{0}\frac{1}{2^0}\frac{1}{2^n} - \binom{n}{1}\frac{1}{2}\frac{1}{2^n-1} = 1 - \frac{1}{2^n} - \frac{n}{2^n}$

## 4

### (a)

**Biased coin tossed three times: Sample space**: HHH,HHT,HTH,HTT,THH,THT,TTH,TTT

### (b)

**Balls drawn without replacement from 2U,2V balls Sample space**: $(U1, V1), (U2, V1), (V1, U1), (V1, U2), (U1, V2), (U2, V2), (V2, U1), (V2, U2)$

### (c)

**Biased coin tossed till H tunrs up** Sample Space:
$\{H\}, \{T, H\}, \{T, T, H\}, \{T, T, T, H\}.....\{T, T, T, ...., T, H\}$

## 5

**To find:** Probablity that exactly one of A,B occurs

Probability that exactly one of A,B occurs is equal to the probability that either **only A** occurs or **only B** occurs. This is simply given by:
$P(A \cup B) - P(A \cap B)$. That is either of A,B occurs removing the portion when both A,B occur.
Thus, the probability that exactly one of A,B occurs is :
$p(A \cup b) - p(A \cap B) = p(A) + p(B) - p(A \cap B) - p(A \cap B) = p(A) + p(B) - 2p(A \cap B)$

## 6

**To Prove:** $P(A \cup B \cup C) = 1 - P(A^C|B^C \cap C^C)p(B^C|C^C)p(C^C)$ $RHS = 1 - P(A^C|B^C \cap C^C)P(B^C|C^C)p(C^C)$
$RHS = 1 - P(A^C|B^C \cap C^C)\frac{P(B^C \cap C^C)}{p(C^C)}p(C^C)$
Thus, expanding further $p(A^C|B^C \cap C^C) = \frac{p(A^C \cap B^C \cap C^C)}{p(B^C \cap C^C)}$
Thus,
$RHS = 1 - \frac{p(A^C \cap B^C \cap C^C)}{p(B^C \cap C^C)} * \frac{P(B^C \cap C^C)}{p(C^C)} * p(C^C)$
$RHS = 1 - P(A^C \cap B^C \cap C^C)$ which is same as $LHS$

## 14

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