MATH-505A: Homework # 4

Due on Friday, September 19, 2014

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Contents

Exercise # 2.1

(1)

Given: X is a random variable $\implies$

$$\{\omega \in \Omega : X(\omega) \leq x\} \in \mathcal{F} \ \forall x \in R \tag{1}$$

Part A) To Prove: aX is a random variable
Consider $Y = aX$, then since equation 1 holds:
Case1: $a \geq 0$
Then $\{\omega \in \Omega : aX(\omega) \leq x'\} \in \mathcal{F} \ \forall x' \in R$ where $x' = ax$
Case2: $a \leq 0$
Then $\{\omega \in \Omega : aX(\omega) \geq x'\} \forall x' \in R$ where $x' = ax \implies \cup\{\{\omega \in \Omega : aX(\omega) \leq x''\}\}^c \in \mathcal{F}$ where $x'' = x'$
Case3: $a \ is \ 0$
Then, $aX = 0$
Case i: $x < 0$
$\{\omega \in \Omega : aX(\omega) = \phi\} \in \mathcal{F}$
Case ii: $x \geq 0$
$\{\omega \in \Omega : aX(\omega) = \Omega\} \in \mathcal{F}$
Thus from all the above cases.
Part (b)):
Consider $Y = X - X$, Then:
$Y = X(\omega) - X(\omega) \forall \omega \in R \implies Y = 0$
Consider $Y = X + X$, Then $Y = X(\omega) + X(\omega) \forall \omega in \Omega \implies Y = 2X(\omega) \forall \omega in \Omega$ Thus $Y = 2X$.

(2)

For part 1, $Y' = aX$ is also a random variable:
To Prove: $Y = Y' + b$ is a ranodm variable where Y' is a random variable and b is a constant.
Since Y' is a random variable: $\{\omega \in \Omega : Y(\omega) \leq y\} \in \mathcal{F} \ \forall y \in R$ and so, $\{\omega \in \Omega : Y(\omega) + b \leq y'\} \in \mathcal{F} \ \forall y' \in R$ where $y' = y + b$
Since $\{\omega \in \Omega : Y(\omega) + b \leq y'\} \in \mathcal{F} \ \forall y' \in R$, $Y' + b$ is a random variable $\implies aX + b$ is a random variable

(3)

$p(H) = p$; $p(T) = 1 - p$
Tossing a coin n times is a binomial process(each individual toss is a bernoulli process) and let A be the event such that k out of n tosses are heads and this can occur in $\binom{n}{k}$ ways with probability p^k. There would also be $n - k$ tails and the probability for that is $(1-p)^{n-k}$. Thus,:
$p(A) = \binom{n}{k} p^k * (1-p)^{n-k}$
For a fair coin, $p = \frac{1}{2}$ and hence p(A) = $\binom{n}{k}(\frac{1}{2})^k(\frac{1}{2})^{n-k} = \binom{n}{k}(\frac{1}{2})^n$

(4)

A distribution function satisfies the following set of properties:

a) $\lim_{x\to-\infty} \ F(x) = 0$, $\lim_{x\to\infty} \ F(x) = 1$
b) if $x < y$ then $F(x) \le F(y)$,
c) F is right continuous, $c < x < c + \delta$ then $|F(x) - F(c)| < \epsilon$ for $\epsilon > 0, \delta > 0$
Consider $Y = \lambda F + (1-\lambda)G$, Both G,F satisfy a, b, c Then $\lim_{x\to-\infty} Y(x) = \lambda \lim_{x\to-\infty} F(x) + (1-\lambda) \lim_{x\to-\infty} G(x) \implies \lim_{x\to-\infty} Y(x) = 0$
Similarly considering limit as $x \to \infty$: Then $\lim_{x\to\infty} Y(x) = \lambda \lim_{x\to\infty} F(x) + (1-\lambda) \lim_{x\to\infty} G(x) \implies \lim_{x\to-\infty} Y(x) = \lambda * 1 + (1-\lambda) * 1 = 1$
Since for $x < y$, then $F(x) < F(y)$; $G(x) < G(y) \implies \lambda F(x) < \lambda F(y)$; $(1-\lambda)G(x) < (1-\lambda)G(y)$ since $0 \le \lambda \le 1$
Adding the two inequalities we get:

$\lambda F(x) + (1-\lambda)G(x) < \lambda F(y) + (1-\lambda)G(y) \implies Y(x) < Y(y)$.
Since F,G are right continuous, any linear combination of these would be right continuous too.
Hence $Y = \lambda F + (1-\lambda)G$ satisfies all the 3 required properties and is a distribution function.

(5)

Exercise # 2.3

(1)

(2)

(3)

(4)

(5)