# MATH 542 Homework 3

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## Problem 1

Given: $A_{n\times n}$ is idempotent $\implies AA = A$; $P$ is non-singular and $C_{n\times n}$ is orthogonal $\implies CC^T = C^TC = I$

### 1.a

$$\begin{aligned}(I-A)(I-A) &= I - A - A + AA \\ &= I - 2A + A \quad \text{using } AA = A \\ &= I - A\end{aligned}$$

Hence $I - A$ is idempotent

### 1.b

$$\begin{aligned}A(I-A) &= A - AA \\ &= A - A \quad \text{using } AA = A \\ &= 0\end{aligned}$$

Similarly,

$$\begin{aligned}(I-A)A &= A - AA \\ &= A - A \quad \text{using } AA = A \\ &= 0\end{aligned}$$

### 1.c

$$\begin{aligned}(P^{-1}AP)(P^{-1}AP) &= P^{-1}APP^{-1}AP \\ &= P^{-1}AAP \text{ since } PP^{-1} = I \\ &= P^{-1}AP \quad \text{using } AA = A\end{aligned}$$

Hence $P^{-1}AP$ is idempotent

### 1.d

$$\begin{aligned}(C'AC)(C'AC) &= C'ACC'AC \\ &= C'AAC \text{ since } CC' = C'C = I \\ &= C'AC \;\text{ using } AA = A\end{aligned}$$

### 1.e

$A$ is a projection matrix and $C'C = C'C = I$ For $C'AC$ to be a projection matrix $C'ACz = z \;\forall z \in S$ for some vector space $S$

For some $z \in S$:

$$\begin{aligned}C'ACz &= C'Ay \text{ where } y = Az \\ &= C'Ay \\ &= C'y \text{ since } A \text{ is projection matrix} \\ &= C'Cz \\ &= z \;\text{ since } C'C = I\end{aligned}$$

Thus, $C'ACz \in S$ and $C'ACz = z$ and hence $C'AC$ is a projection matrix

## Problem 2

### 2.a

$$\begin{aligned}Q(x_1, x_2, x_3) &= 12x_1^2 + 3x_2^2 + 3x_3^2 + 2x_1x_2 - 10x_1x_3 + 4x_2x_3 \\ &= 3(2x_1 + x_2 + x_3)^2 - (12x_1x_2 + 6x_2x_3 + 12x_1x_3) + 2x_1x_2 - 10x_1x_3 + 4x_2x_3 \\ &= 3(2x_1 + x_2 + x_3)^2 - 10x_1x_2 - 2x_2x_3 - 22x_1x_3 \\ &= (x_1 + x_2)^2 + 2(x_2 + x_3)^2 + (5x_1 - x_3)^2 - 14x_1^2 \\ &= (x_1 + x_2)^2 + 2(x_2 + x_3)^2 + x_3^2 + 11x_1^2 - 10x_1x_3 \\ &= (x_1 + x_2)^2 + 2(x_2 + x_3)^2 + (5x_1 - x_3)^2 - 14x_1^2\end{aligned}$$

and hence $Q(x_1, x_2, x_3)$ is neither positive definite nor positive semidefinite.

### 2.b

$$x'Ax = \begin{pmatrix} x_1 & x_2 & x_3 \end{pmatrix} \begin{pmatrix} 2 & 1 & 2 \\ 1 & 2 & 1 \\ 2 & 1 & 4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$$\begin{aligned}&= (2x_1 + x_2 + x_3)x_1 + (x_1 + 2x_2 + x_3)x_2 + (2x_1 + x_2 + 4x_3)x_3 \\ &= 2x_1^2 + 2x_2^2 + 4x_3^2 + 2x_1x_2 + 3x_1x_3 + 2x_2x_3 \\ &= (x_1 + x_2)^2 + (x_2 + x_3)^2 + (x_1 + \frac{3}{2}x_2)^2 + \frac{3}{4}x_2^2 \\ &\geq 0\end{aligned}$$

$A$ is positive semidefinite

### 2.c

$$
\begin{aligned}
x'Ax = \begin{pmatrix} x_1 & x_2 & x_3 \end{pmatrix} \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 1 \\ 3 & 1 & -2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \\
&= (x_1 + 2x_2 + 3x_3)x_1 + (2x_1 + x_2 + x_3)x_2 + (3x_1 + x_2 - 2x_3)x_3 \\
&= x_1^2 + x_2^2 - 2x_3^2 + 4x_1x_2 + 6x_1x_3 + 2x_2x_3 \\
&= (x_2 + x_3)^2 + 3(x_1 + x_3)^2 + 2(x_1 + x_2)^2 - 6x_1^2 - 6x_3^2 - 4x_1^2 - 2x
\end{aligned}
$$

$A$ is neither positive definite nor positive semidefinite.

## Problem 3

$$
\begin{pmatrix} a & b \\ c & d \end{pmatrix} \times \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}
$$

$$
\begin{pmatrix} a^2 + b^2 & ac + bd \\ ac + bd & c^2 + d^2 \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}
$$

$$
\begin{aligned}
a^2 + b^2 &= 2 \\
ac + bd &= -1 \\
c^2 + d^2 &= 2
\end{aligned}
$$

Let $a = 1, b = 1 \implies c + d = 1$

$$
\begin{aligned}
c^2 + d^2 &= 2 \\
c^2 + (-1 - c)^2 &= 2 \\
2c^2 + 2c - 1 &= 0 \\
c &= \frac{-2 \pm \sqrt{4 + 8}}{4} \\
c &= \frac{-1 \pm \sqrt{3}}{2} \\
d &= \frac{-1 \pm \sqrt{3}}{2}
\end{aligned}
$$

and hence one possible $B$ is:

$$
B = \begin{pmatrix} 1 & 1 \\ \frac{-1-\sqrt{3}}{2} & \frac{-1+\sqrt{3}}{2} \end{pmatrix}
$$

## Problem 4

$G = (X'X)^-$

Let $Y = (X'X)$ so that $G = Y^-$

$$
\begin{aligned}
G(X'X)G &= GYG \\
&= Y^-YY^- \\
&= Y^-
\end{aligned}
$$

Note that $Y$ is symmetric, $Y = Y^T$ (Proof: $Y = X^TX$, $Y^T = (X^TX)^T = X^TX$)

Since $Y$ is symmetric, $Y^-$ is symmetric too $\implies G(X'X)G$ is symmetric.

To prove $P = G(X'X)G$ is a generalised inverse of $Y = X'X$ we need to show $YPY = P$

$$
\begin{aligned}
YPY &= YGYGY \\
&= Y(Y^-)Y \text{ using } GYG = Y^- \text{ from the previous result} \\
&= Y \quad \text{since } YY^- = I
\end{aligned}
$$

## Problem 5

Given: $E[X] = 1$ and $Var(X) = E[X^2] - E[X]^2 = 5$ Using $E[X] = 1$ we get, $E[X^2] = 5 + E[X]^2 = 6$

### 5.a

$$
\begin{aligned}
E[(1+2X)^2] &= E[1 + 4X + 4X^2] \\
&= E[1] + E[4X] + E[4X^2] \\
&= 1 + 4E[X] + 4E[X^2] \\
&= 1 + 4 + 4(6) \\
&= 30
\end{aligned}
$$

### 5.b

$$
\begin{aligned}
var(3 + 4X) &= Var(3) + Var(4X) + 2Cov(3, 4X) \\
&= 0 + 16Var(X) + 2(0) \\
&= 80
\end{aligned}
$$