

MATH 542 Final Exam

Saket Choudhary
skchoudh@usc.edu

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Problem 1

Problem 1a

$$\begin{aligned} Y_i &= \beta_0 + \beta_1 x_{i1} + \beta_2 x_{i2} + \cdots + \beta_{p-1} x_{i,p-1} + \epsilon_i \\ Y_i &= \alpha + \beta_1(x_{i1} - \bar{x}_1) + \beta_2(x_{i2} - \bar{x}_2) + \cdots + \beta_{p-1}(x_{i,p-1} - \bar{x}_{p-1}) \\ &= \alpha - \beta_1 \bar{x}_1 - \beta_2 \bar{x}_2 - \cdots - \beta_{p-1} \bar{x}_{p-1} + \beta_1 x_{i1} + \beta_2 x_{i2} + \cdots + \beta_{p-1} x_{i,p-1} + \epsilon_i \\ \implies \beta_0 &= \alpha - \beta_1 \bar{x}_1 - \beta_2 \bar{x}_2 - \cdots - \beta_{p-1} \bar{x}_{p-1} \\ \implies \alpha &= \beta_0 + \beta_1 \bar{x}_1 + \beta_2 \bar{x}_2 + \cdots + \beta_{p-1} \bar{x}_{p-1} \end{aligned}$$

Problem 1.2

$$\begin{aligned}
\bar{x}_j &= \sum_{i=1}^n \frac{x_{ij}}{n} \\
&= \frac{1}{n} \mathbf{1}_n' \mathbf{X}_i \\
Y &= \begin{pmatrix} 1 & x_{11} - \bar{x}_1 & x_{12} - \bar{x}_2 & \dots & x_{1,p-1} - \bar{x}_{p-1} \\ 1 & x_{21} - \bar{x}_1 & x_{22} - \bar{x}_2 & \dots & x_{2,p-1} - \bar{x}_{p-1} \\ \vdots & & & & \\ 1 & x_{n1} - \bar{x}_1 & x_{n2} - \bar{x}_2 & \dots & x_{n,p-1} - \bar{x}_{p-1} \end{pmatrix} \begin{pmatrix} \alpha \\ \beta_1 \\ \vdots \\ \beta_{p-1} \end{pmatrix} + \epsilon \\
&= \begin{pmatrix} 1 & x_{11} & x_{12} & \dots & x_{1,p-1} \\ 1 & x_{21} & x_{22} & \dots & x_{2,p-1} \\ \vdots & & & & \\ 1 & x_{n1} & x_{n2} & \dots & x_{n,p-1} \end{pmatrix} - \begin{pmatrix} 0 & \bar{x}_1 & \bar{x}_2 & \dots & \bar{x}_{p-1} \\ 0 & \bar{x}_1 & \bar{x}_2 & \dots & \bar{x}_{p-1} \\ \vdots & & & & \\ 0 & \bar{x}_1 & \bar{x}_2 & \dots & \bar{x}_{p-1} \end{pmatrix} \begin{pmatrix} \alpha \\ \beta_1 \\ \vdots \\ \beta_{p-1} \end{pmatrix} + \epsilon \\
&= \left((\mathbf{1}_n \quad \mathbf{X}) - \begin{pmatrix} \mathbf{0}_n & \frac{1}{n} \mathbf{1}_n \mathbf{1}_n' \mathbf{X} \end{pmatrix} \right) \begin{pmatrix} \alpha \\ \beta_1 \\ \vdots \\ \beta_{p-1} \end{pmatrix} + \epsilon \\
&= \left(\mathbf{1}_n \quad \mathbf{X} - \frac{1}{n} \mathbf{1}_n \mathbf{1}_n' \mathbf{X} \right) \begin{pmatrix} \alpha \\ \beta_1 \\ \vdots \\ \beta_{p-1} \end{pmatrix} + \epsilon \\
&= (\mathbf{1}_n \quad \mathbf{X}_c) \begin{pmatrix} \alpha \\ \beta_1 \\ \vdots \\ \beta_{p-1} \end{pmatrix} + \epsilon \\
\Rightarrow \mathbf{X}_c &= \left(\mathbf{I} - \frac{\mathbf{1}_n \mathbf{1}_n'}{n} \right) \mathbf{X}
\end{aligned}$$

Problem 1.3

$$\begin{aligned}
\mathbf{1}_n' \mathbf{X}_c &= (1 \quad 1 \quad 1 \quad \dots \quad 1) \left(\mathbf{I} - \frac{1}{n} \mathbf{1}_n \mathbf{1}_n' \right) \mathbf{X} \\
&= \left(1 - \frac{1}{n} \right) - \frac{1}{n} * (n-1) \\
&= 0
\end{aligned}$$

Problem 1.4.a

$$Y_1 = \alpha + \beta_1(x_{11} - \bar{x}_1) + \beta_2(x_{12} - \bar{x}_2) + \cdots + \beta_{p-1}(x_{1,p-1} - \bar{x}_{p-1})$$

$$Y_2 = \alpha + \beta_1(x_{21} - \bar{x}_1) + \beta_2(x_{22} - \bar{x}_2) + \cdots + \beta_{p-1}(x_{2,p-1} - \bar{x}_{p-1})$$

$$\vdots$$

$$Y_n = \alpha + \beta_1(x_{n1} - \bar{x}_1) + \beta_2(x_{n2} - \bar{x}_2) + \cdots + \beta_{p-1}(x_{n,p-1} - \bar{x}_{p-1})$$

$$\sum_{i=1} Y_i = n\alpha + \beta_1\left(\sum_{i=1} x_{i1} - n\bar{x}_1\right) + \beta_2\left(\sum_{i=1} x_{i2} - n\bar{x}_2\right) + \cdots + \beta_{p-1}\left(\sum_{i=1} x_{i,p-1} - n\bar{x}_{p-1}\right)$$

$$\sum_{i=1} x_{i,j} - n\bar{x}_j = 0$$

$$\implies \hat{\alpha} = \frac{\sum_{i=1} Y_i}{n}$$

Now, we perform $Y_i - \bar{Y}$ eliminating α . Let $Z = Y_i - \bar{Y}$ in the problem then reduces to the following form:
 $Z = \mathbf{X}_c \beta + \epsilon$ where $\beta = (\beta_1 \quad \beta_2 \quad \dots \quad \beta_{p-1})$ and hence simply re-using OLS results, $\hat{\beta} = \mathbf{X}_c' \mathbf{X}_c^{-1} \mathbf{X}_c' Y$
 More rigorously:

$$\hat{\beta} = (X'X)^{-1}X'Y$$

$$X'X\hat{\beta} = X'Y$$

$$\mathbf{X}'\mathbf{X} = \begin{pmatrix} \mathbf{1}_n' \\ \mathbf{X}_c' \end{pmatrix} \begin{pmatrix} \mathbf{1}_n & \mathbf{X}_c \end{pmatrix}$$

$$= \begin{pmatrix} \mathbf{1}_n' \mathbf{1}_n & \mathbf{1}_n' \mathbf{X}_c \\ \mathbf{X}_c' \mathbf{1}_n & \mathbf{X}_c' \mathbf{X}_c \end{pmatrix}$$

$$\begin{pmatrix} \mathbf{1}_n' \mathbf{1}_n & \mathbf{1}_n' \mathbf{X}_c \\ \mathbf{X}_c' \mathbf{1}_n & \mathbf{X}_c' \mathbf{X}_c \end{pmatrix} \begin{pmatrix} \alpha \\ \beta_1 \end{pmatrix} = \begin{pmatrix} \mathbf{1}_n' \\ \mathbf{X}_c' \end{pmatrix} Y$$

$$\implies n\hat{\alpha} + \mathbf{1}_n' \mathbf{X}_c \beta_1 = \sum_{i=1} Y_i = n\bar{Y} \text{ using first row}$$

$$\hat{\alpha} + 0 = \bar{Y} \text{ since } \mathbf{X}_c' \mathbf{1}_n \mathbf{1} = 0$$

$$\implies \hat{\alpha} = \bar{Y}$$

$$\mathbf{X}_c' \mathbf{1}_n \mathbf{1}_n' + \mathbf{X}_c' \mathbf{X}_c \hat{\beta}_1 = \mathbf{X}_c' Y \text{ using second row}$$

$$\implies \mathbf{X}_c' \mathbf{X}_c \hat{\beta}_1 = \mathbf{X}_c' Y \text{ since } \mathbf{X}_c' \mathbf{1}_n \mathbf{1} = 0$$

$$\implies \hat{\beta}_1 = (\mathbf{X}_c' \mathbf{X}_c)^{-1} \mathbf{X}_c' Y$$

Problem 1.4.b

Inverse of a block matrix

$$\begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}^{-1} = \begin{pmatrix} (A_{11} - A_{12}A_{22}^{-1}A_{21})^{-1} & -(A_{11} - A_{12}A_{22}^{-1}A_{21})^{-1}A_{12}A_{22}^{-1} \\ -(A_{22} - A_{21}A_{11}^{-1}A_{12})^{-1} & (A_{22} - A_{21}A_{11}^{-1}A_{12})^{-1} \end{pmatrix}$$

$$\begin{aligned}
\mathbf{X}'\mathbf{X} &= \begin{pmatrix} \mathbf{1}_n' \\ \mathbf{X}_c' \end{pmatrix} \begin{pmatrix} \mathbf{1}_n & \mathbf{X}_c \end{pmatrix} \\
&= \begin{pmatrix} \mathbf{1}_n' \mathbf{1}_n & \mathbf{1}_n' \mathbf{X}_c \\ \mathbf{X}_c' \mathbf{1}_n & \mathbf{X}_c' \mathbf{X}_c \end{pmatrix} \\
&= \begin{pmatrix} \mathbf{1}_n' \mathbf{1}_n & 0 \\ 0 & \mathbf{X}_c' \mathbf{X}_c \end{pmatrix} \\
\mathbf{X}'\mathbf{X}^{-1} &= \begin{pmatrix} \mathbf{1}_n' \mathbf{1}_n & \mathbf{1}_n' \mathbf{X}_c \\ \mathbf{X}_c' \mathbf{1}_n & \mathbf{X}_c' \mathbf{X}_c \end{pmatrix}^{-1} \\
&= \begin{pmatrix} n & 0 \\ 0 & (\mathbf{X}_c' \mathbf{X}_c)^{-1} \end{pmatrix} \\
(X'X)^{-1}X'Y &= \begin{pmatrix} \frac{1}{n} & 0 \\ 0 & (\mathbf{X}_c' \mathbf{X}_c)^{-1} \end{pmatrix}_{(p+1) \times (p+1)} \begin{pmatrix} \mathbf{1}_n' \\ \mathbf{X}_c' \end{pmatrix}_{(p+1) \times n} \mathbf{Y}_{n \times 1} \\
&= \begin{pmatrix} \frac{1}{n} & 0 \\ 0 & (\mathbf{X}_c' \mathbf{X}_c)^{-1} \end{pmatrix} \begin{pmatrix} \mathbf{1}_n' Y \\ \mathbf{X}_c' Y \end{pmatrix} \\
&= \begin{pmatrix} \frac{1}{n} \mathbf{1}_n' Y_{1 \times 1} \\ (\mathbf{X}_c' \mathbf{X}_c)^{-1} \mathbf{X}_c' Y_{p-1 \times 1} \end{pmatrix} \\
&= \begin{pmatrix} \bar{Y} \\ (\mathbf{X}_c' \mathbf{X}_c)^{-1} \mathbf{X}_c' Y \end{pmatrix}_{p \times 1} \\
\begin{pmatrix} \alpha \\ \beta_1 \\ \beta_2 \\ \vdots \\ \beta_{p-1} \end{pmatrix} &= \begin{pmatrix} \bar{y} \\ (\mathbf{X}_c' \mathbf{X}_c)^{-1} \mathbf{X}_c' Y_{p-1 \times 1} \end{pmatrix}
\end{aligned}$$

Problem 1.5

Yes, column space of X is identical to $\begin{pmatrix} \mathbf{1}_n & \mathbf{X}_c \end{pmatrix}$ since $X_c = (I - \frac{J}{n})X$

Problem 1.6

$$\begin{aligned}
\mathbf{P} &= \mathbf{X}(\mathbf{X}'\mathbf{X})^{-1}\mathbf{X}'\mathbf{Y} \\
&= \begin{pmatrix} \mathbf{1}_n & \mathbf{X}_c \end{pmatrix} \begin{pmatrix} \frac{1}{n} & 0 \\ 0 & (\mathbf{X}_c' \mathbf{X}_c)^{-1} \end{pmatrix} \begin{pmatrix} \mathbf{1}_n' \\ \mathbf{X}_c' \end{pmatrix} \\
\Rightarrow \mathbf{P} &= \mathbf{X}_c(\mathbf{X}_c' \mathbf{X}_c)^{-1} \mathbf{X}_c' + \frac{\mathbf{1}_n \mathbf{1}_n'}{n}
\end{aligned}$$

$$\begin{aligned}
SSE &= \epsilon' \epsilon \\
&= (Y - X\hat{\beta})'(Y - X\hat{\beta}) \\
&= Y'Y - \hat{\beta}'X'Y - Y'X\hat{\beta} + \hat{\beta}'X'X\hat{\beta} \\
&= Y'Y - 2Y'X\hat{\beta} + \hat{\beta}'X'X\{(X'X)^{-1}X'Y\} \\
&= Y'Y - 2Y'X\hat{\beta} + \hat{\beta}'IX'Y \\
&= Y'Y - 2Y'X\hat{\beta} + \hat{\beta}'X'Y \\
&= Y'Y - 2Y'X\hat{\beta} + Y'X\hat{\beta}' \\
&= Y'Y - Y'X\hat{\beta} \\
&= Y'Y - Y' \begin{pmatrix} 1_n & X_c \end{pmatrix} \begin{pmatrix} \bar{y} \\ (\mathbf{X}_c' \mathbf{X}_c)^{-1} \mathbf{X}_c' Y_{p-1 \times 1} \end{pmatrix} \\
&= Y'Y - Y' \begin{pmatrix} 1_n & X_c \end{pmatrix} \begin{pmatrix} \frac{1}{n} \mathbf{1}_n' \mathbf{Y} \\ (\mathbf{X}_c' \mathbf{X}_c)^{-1} \mathbf{X}_c' Y_{p-1 \times 1} \end{pmatrix} \\
&= Y'(I - \frac{1_n \mathbf{1}_n'}{n})Y - \mathbf{Y}' \mathbf{X}_c (\mathbf{X}_c' \mathbf{X}_c)^{-1} \mathbf{X}_c' \mathbf{Y} \\
&= \sum_{i=1}^n (Y_i - \bar{Y})^2 - \mathbf{Y}' \mathbf{X}_c (\mathbf{X}_c' \mathbf{X}_c)^{-1} \mathbf{X}_c' \mathbf{Y} \\
&= \sum_{i=1}^n (Y_i - \bar{Y})^2 - \mathbf{Y}' \mathbf{P}_c \mathbf{Y}
\end{aligned}$$

Problem 1.7

$$\begin{aligned}
Y_i^* &= Y_i - \bar{Y} \\
\mathbf{Y}^* &= \mathbf{Y} - \frac{1}{n} \mathbf{1}_n \mathbf{1}_n' \mathbf{Y} \\
\sum_{i=1}^n (Y_i - \bar{Y})^2 &= \mathbf{Y}'^* \mathbf{Y}^* \\
\implies SSE &= \mathbf{Y}'^* \mathbf{Y}^* - \mathbf{Y}'^* \mathbf{P}_c \mathbf{Y}^*
\end{aligned}$$