MATH-505A: Homework # 3

Due on Friday, September 12, 2014

Saket Choudhary
2170058637

Contents

Exercise # 1.7

(1)

Given: Two roads $r1_{AB}$, $r2_{AB}$ connectinng points A and B and $s1_{BC}$, $s2_{BC}$ connecting B and C.
Let $p(AB)$ denote the probability that path between A$\longrightarrow$B is open and let $p(AB^c)$ denote the probability that there is(are) no path open b/w A and B.
To find:$Y = P(AB|AC^c)$.
Y is equal to the probability that path between A and B is open AND(given that) the path between A and C is closed $\implies$ Path between B and C is closed AND between A and B is open
p(AB) = Path b/w A,B is open = 1 - Path b/w A,B is closed = $1 - p * p$
$p(AC^C|AB)$ = Path b/w A,C is closed given A,B is open = Path b/w B,C is closed given A,B is open. = $p(BC^C)$
Thus

$$p(AB) = 1 - p^2 \tag{1}$$

Also,

$$p(AB) = p(BC) \tag{2}$$

$p(AC^C)$ = 1 - Probability A,C is open = 1 - Probability AB is open AND BC is open. Thus,

$$p(AC^c) = 1 - p(AB)p(AC) = 1 - (1 - p^2)^2 \tag{3}$$

$$p(AB \cap AC^C) = p(AC^C|AB)p(AB) = p(BC^C)p(AB) = p^2(1 - p^2) \tag{4}$$

$$p(AB|AC^c) = \frac{P(AB \cap AC^c)}{p(AC^c)} = \frac{p(AC^C|AB)p(AB)}{p(AC^c)} = \frac{p^2(1 - p^2)}{1 - (1 - p^2)^2} \tag{5}$$

Part 2: Additional direct road from A to C. Find $p(AB|AC^c)$:
$p(AC^c|AB)$ = Probability that path b/w A,C is closed given path b/w A,B are open = Probability path b/w A,C(direct) are closed AND path b/w B,C are closed

$$p(AC^C|AB) = p * p(BC^c)p(AB) \tag{6}$$

where the extra p in 6 as compared to 4 is because the direct path A,C should be blocked too.

$$p(AC^c) = 1 - (p(1 - p^2)^2 - (1 - p)) \tag{7}$$

where the second term is the probability that direct A $\longrightarrow$ C is blocked, but A $\longleftrightarrow$ B and $B \longrightarrow$ is open(both). The third term represents the probaility that A $\longrightarrow$ C is open directly(does not matter if the other paths are open or blocked),
Thus, for part 2:

$$p(AB|AC^c) = \frac{p^3(1 - p^2)}{1 - (1 - p^2)^2 - (1 - p)} = \frac{p^2(1 - p^2)}{1 - (1 - p^2)^2} \tag{8}$$

(2)

$$p(2K \cap 1A) = \frac{\binom{4}{2} * \binom{4}{1} * \binom{52-4-4}{10}}{\binom{52}{13}} \quad (9)$$

$p(1A|2K) = \frac{p(1A \cap 2K)}{p(2K)}$

$$p(2K) = \frac{\binom{4}{2} * \binom{52-4}{11}}{\binom{52}{13}} = \frac{6 * 48! * 13!}{52! * 11!} = \frac{6 * 12 * 13}{49 * 50 * 51 * 52} = 2.52 * 10^-4 \quad (10)$$

Thus,

$p(1A|2K) = \frac{\binom{4}{2} * \binom{4}{1} * \binom{44}{10}}{\binom{4}{2} * \binom{48}{11}} = \frac{4*11*37!}{46*47*48*34!} = 0.0399$

(4)

To prove/disprove: $p(x|C) > p(y|C) AND p(x|C^c) > p(y|C^c) \implies p(x) > p(y)$

$$p(x|C) - p(y|C) > 0 \quad (11)$$

$$p(x|C^c) - p(y|C^c) > 0 \quad (12)$$

$$p(x) = p(x|C)p(C) + p(x|C^c)p(C^c) \quad (13)$$

Also,

$$p(y) = p(y|C)p(C) + p(y|C^c)p(C^c) \quad (14)$$

Consider $p(x) - p(y)$:

$$p(x) - p(y) = (p(x|C) - p(y|C))p(C) + (p(x|C^c) - p(y|C^c))p(C^c) \quad (15)$$

From the 12, **??** and 15:

$$p(x) - p(y) > 0 \forall x, y \quad (16)$$

Thus, x is always prefered over y.

(5)

Let X_i represent the i^{th} ccard draw
Given: $X_k > X_i,\ \forall i \in [1, k-1] and k \in [1, m]$
$p(X_k = m|X_k > X_i) = \frac{p(X_k = m \cap X_k > X_i)}{p(X_k > X_i)} = \frac{p(X_k = m)}{p(X_k > X_i)} = \frac{\frac{1}{m}}{\frac{1}{k}}$
Where the equality in the last step comes from the fact that the probability of choosing cards such that $p(X_k > X_i)$ is simply to choose the largest card, i.e. k among the rest i.
Thus $p(X_k = m|X - k > X_i) = \frac{k}{m}$.

[

Exercise # 1.8]

1

(a)

Six turns up exactly once The dice on which 6 should appear is choosen in $\binom{2}{1}$ way and has just one option(6) while the other has $6 - 1 = 5$ options. Total outcomes $= 6 * 6 = 36$
Thus:
$p = \frac{1*5}{36} = \frac{5}{36}$

(b)

Both numbers are odd. 3 out of 6 numbers are odd and the outcome of odd on both dice are independent. Thus,

$p = \frac{3*3}{36} = \frac{1}{12}$

(c)

Sum of scores is 4.
Possible configurations: $(1, 3); (2, 2); (3, 1)$
Thus:
$p = \frac{3}{36} = \frac{1}{12}$

(d)

Sum of scores is divisible by 3.
Possible choices for sum to be divisible by 3:
$3, 6, 9, 12$:
3: $(1, 2), (2, 1)$
6: $(1, 5), (2, 4), (3, 3), (4, 2), (5, 1)$
9: $(3, 6), (4, 5), (5, 4), (6, 3)$
12: $(6, 6)$
Thus $p = \frac{2+5+4+1}{36} = \frac{1}{3}$

2

(a)

Head appears the first time on n^{th} throw. Thus a series of $n - 1$ consecutive tails followed by head.
$p = \binom{n}{1}\frac{1}{2^{n-1}} * \frac{1}{2} = \frac{n}{2^n}$

(b)

The number of heads and dates to date are equal:
Case1: n is odd.
Clearly the probability of heads and tails being equal in this case is zero!
Case2: n is even.
A series of H,T such that $|H| = |T|$. Since the probability of occurence of either a H or a T is $\frac{1}{2}$, the result will still be:
$p = \binom{n}{n/2}\frac{1}{2^n}$

(c)

Exactly two heads have appeared altogether.
$p = \binom{n}{2} * \frac{1}{2^2} * \frac{1}{2^{n-2}} = \binom{n}{2} * \frac{1}{2^n}$

(d)

At least two heads have appeared.
This is same as the one minus probability that 0 or 1 heads have appeared so far:
$p = 1 - p(0\ heads) - p(1\ head\ only)$ $p = 1 - \binom{n}{0}\frac{1}{2^0}\frac{1}{2^n} - \binom{n}{1}\frac{1}{2}\frac{1}{2^n-1} = 1 - \frac{1}{2^n} - \frac{n}{2^n}$

4

(a)

Biased coin tossed three times:
Sample space: HHH,HHT,HTH,HTT,THH,THT,TTH,TTT

(b)

Balls drawn without replacement from 2U,2V balls
Sample space: $(U1, V1), (U2, V1), (V1, U1), (V1, U2), (U1, V2), (U2, V2), (V2, U1), (V2, U2)$

(c)

Biased coin tossed till H tunrs up
Sample Space:
$\{H\}, \{T, H\}, \{T, T, H\}, \{T, T, T, H\}.....\{T, T, T, ..., T, H\}$

5

To find: Probablity that exactly one of A,B occurs

Probability that exactly one of A,B occurs is equal to the probability that either **only A** occurs or **only B** occurs. This is simply given by:
$P(A \cup B) - P(A \cap B)$. That is either of A,B occurs removing the portion when both A,B occur.
Thus, the probability that exactly one of A,B occurs is :
$p(A \cup b) - p(A \cap B) = p(A) + p(B) - p(A \cap B) - p(A \cap B) = p(A) + p(B) - 2p(A \cap B)$

6

To Prove: $P(A \cup B \cup C) = 1 - P(A^C|B^C \cap C^C)p(B^C|C^C)p(C^C)$ $RHS = 1 - P(A^C|B^C \cap C^C)P(B^C|C^C)p(C^C)$
$RHS = 1 - P(A^C|B^C \cap C^C)\frac{P(B^C \cap C^C)}{p(C^C)}p(C^C)$
Thus, expanding further $p(A^C|B^C \cap C^C) = \frac{p(A^C \cap B^C \cap C^C)}{p(B^C \cap C^C)}$
Thus,
$RHS = 1 - \frac{p(A^C \cap B^C \cap C^C)}{p(B^C \cap C^C)} * \frac{P(B^C \cap C^C)}{p(C^C)} * p(C^C)$
$RHS = 1 - P(A^C \cap B^C \cap C^C)$ which is same as LHS

14

Consider only two events: A,B_j.
From the definition of conditional probability:

$$p(A_j|B) = \frac{p(A_j \cap B)}{p(B)} \tag{17}$$

Also:

$$p(B|A_j) = \frac{p(A_j \cap B)}{p(A_j)} \tag{18}$$

Substituing for $p(A_j \cap B)$ from 17 in 18:

$$p(B|A_j) = \frac{p(A_j|B)p(B)}{p(A_j)} \tag{19}$$

or equivalently,

$$p(A_j|B) = \frac{p(B|A_j)p(A_j)}{p(B)} \tag{20}$$

where p(B) can be expressed as(using the law of partition):

$$p(B) = p(B|A_1)P(A_1) + P(B|A_2)P(A_2) + ... + P(B|A_n)P(A_n) \tag{21}$$

Thus $p(B) = \sum_1^n p(B|A_i)p(A_i)$
Ans hence,

$$p(A_j|B) = \frac{p(B|A_j)p(A_j)}{\sum_1^n p(B|A_i)p(A_i)}$$

continued on next page...