CSCI-567: Assignment #1

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Contents

Problem # 1

Problem # 1: (a) 1

Given: $X_i \sim Beta(\alpha, 1)$ MLE for α:
Consider $X = (X_1, X_2, \dots, X_n)$ Likelihood function: $L(\alpha|X)$ $L(\alpha|X) = \Pi_{i=1}^{n} f(x_i)$ where

$$f(x_i) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1} = \frac{\Gamma(\alpha+1)}{\Gamma(\alpha)\Gamma(1)} x^{\alpha-1} \tag{1}$$

$$= \frac{\alpha\Gamma(\alpha)}{\Gamma(\alpha)} x^{\alpha-1} \qquad = \alpha x^{\alpha-1} \tag{2}$$

$$L(\alpha|X) = (\frac{\Gamma(\alpha+1)}{\Gamma(\alpha)\Gamma(1)})^n \Pi_{i=1}^{n} (x_i)^{\alpha-1} \tag{3}$$

$$LL = \log(L(\alpha|X)) = n\log(\alpha) + (\alpha-1)\sum_{i=1}^{n} x_i \tag{4}$$

$$\frac{dLL}{d\alpha} = \frac{n}{\alpha} + \sum_{i=1}^{n} \log(x_i) \tag{5}$$

$$\frac{dLL}{d\alpha} = 0 \implies \hat{\alpha} = \frac{n}{\sum_{i=1}^{n} log(1/x_i)} \tag{6}$$

Minima at $\hat{\alpha} = \frac{n}{\sum_{i=1}^{n} log(1/x_i)}$ is guaranteed due to log being a concave function.

Problem # 1: (a) 1

Given: $x_i \sim N(\theta, \theta)$ i.e $f(x_i) = (2\pi\theta)^{-\frac{1}{2}} e^{-\frac{(x_i-\theta)^2}{2\theta}}$ MLE estimate for θ:

$$L(\theta|X) = (2\pi\theta)^{-\frac{N}{2}} e^{-\sum_{i=1}^{n} \frac{(x_i-\theta)^2}{2\theta}} \tag{7}$$

$$LL = \log(L(\theta|X)) = -\frac{N}{2}\log((2\pi\theta)) - \sum_{i=1}^{n} \frac{(x_i-\theta)^2}{2\theta} \tag{8}$$

$$\frac{dLL}{d\theta} = -\frac{N}{2}(\frac{1}{\theta}) + \frac{\sum_{i=1}^{n} x_i^2}{2\theta^2} - \frac{N\theta}{2} \tag{9}$$

$$\frac{dLL}{d\theta} = 0 \implies N\theta^2 + N\theta - \sum_{i=1}^{n} x_i^2 = 0 \tag{10}$$

The above equation is a quadratic and will have two solutions, Since, $\theta \geq 0$ (a constraint that comes from θ being the variance), the
$\theta = \frac{-N \pm \sqrt{N^2 + 4N\sum_{i=1}^{n} x_i^2}}{2N}$
Since, $\hat{\theta} \geq 0$, $\hat{\theta} = \frac{-N + \sqrt{N^2 + 4N\sum_{i=1}^{n} x_i^2}}{2N}$

Problem # 1: (b) 1

Given: $f(\hat{x}) = \frac{1}{n}\sum_{i=1}^{n}\frac{1}{h}K(\frac{x-X_i}{h})$ To show: $E_{X_1,X_2,\dots X_n}[f(\hat{x})] = \frac{1}{h}\int K(\frac{x-t}{h})f(t)dt$
Proof:

$$E[f(\hat{x})] = E[\frac{1}{n}\sum_{i=1}^{n}\frac{1}{h}K(\frac{x-X_i}{h})] \tag{11}$$

$$= \frac{1}{nh}E[K(\frac{x-X_i}{h})] \tag{12}$$

$$= \frac{1}{h}E[K(\frac{x-X_1}{h})] \qquad = \frac{1}{h}E[K(\frac{x-t}{h})] \tag{13}$$

where the penultimate equality comes from the fact that X_i are iid for all $i \in [1,n]$. and t X and hence.

$$E[f(\hat{x})] = \frac{1}{h}E[K(\frac{x-X_1}{h})] \qquad = \frac{1}{h}\int K(\frac{x-t}{h})f(t)dt = RHS \tag{14}$$

Problem # 1: (b) 2

Consider $z = \frac{x-t}{h} \implies t = x - hu$
Then,

$$E[f(\hat{x})] = \frac{1}{h}\int K(z)f(x-hz)dz \tag{15}$$

$$\tag{16}$$

$f(x-hz) = f(x) - f'(x)hz + \frac{1}{2}f''(x)\frac{(hz)^2}{2} - \frac{1}{3}f'''(x)\frac{(hz)^3}{3!} + \dots + (-1)^n\frac{1}{n!}f^{(n)}(x)(\frac{(hu)^n}{n!})$
By definition, $\int k(z)dz = 1$. Also define an auxilllary variable $M_j = \int k(z)z^j dz$ for the j^{th} moment of the kernel function, and hence, $\int K(z)f(x-hz)dz = f(x) - hf'(x)M_1 + \frac{1}{2}(h^2)f^{''}(x)M_2 + \dots + (-1)^n\frac{1}{n!}f^{(n)}M_n$
Now, $Bias = E[f(\hat{x})] - f(x) = -hf'(x)M_1 + \frac{1}{2}(h^2)f^{''}(x)M_2 + \dots + (-1)^n\frac{1}{n!}f^{(n)}M_n$
And as $h \longrightarrow 0$, $Bias \longrightarrow 0$

Problem 2

Problem 2: (c)

Total points: N
Total points with label class c: N_c
Given: $p(x|Y=c) = \frac{K_c}{N_cV}$ and $\sum K_c = K$ Class prior: $p(Y=c) = \frac{N_c}{N}$
Unconditional density $p(x) = \sum_c p(x|Y=c)p(Y=c) = \sum_c \frac{K_c}{N_cV} \times \frac{N_c}{N} = \sum_c \frac{K_c}{NV} = \frac{K}{NV}$
Posterior $P(Y=c|x) = \frac{P(x|Y=c)\times P(Y=c)}{P(x)} = \frac{\frac{K_c}{N_cV}\times\frac{N_c}{N}}{\frac{K}{NV}} = \frac{K_c}{K}$