MATH-505A: Homework # 5

Due on Friday, September 26, 2014

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Contents

Exercise # 2.7

(1)

Coin toss shows head with probability $= p$ **To Find:** $P(X > m)$
By definition $P(X > m) = 1 - P(X < m)$
$P(X < m) = P(head\ comes\ on\ toss\ 1) + P(head\ comes\ on\ toss\ 2) + .. + P(head\ comes\ on\ toss\ m-1) =$
$p + (1-p)p + (1-p)^2 p + ... + (1-p)^(m-1)p = p(\frac{1-(1-p)^{m-1+1}}{1-(1-p)}) = 1 - (1-p)^m$
Thus $P(X > m) = 1 - 1 - (1-p)^m = (1-p)^m$
The distribution function of X is given by:
$$F(x) = \begin{cases} 1 - (1-p)^x & x \leq 0 \\ 0 & otherwise \end{cases}$$
given that x takes only integer values[X is a discrete random variable]

(2 a)

X is a random variable. Let $\Omega = \{x1, x2, ..., x_n\}$ Define a indicator random variables $I_j(x_j)$ such that:
$$I_j(x) = \begin{cases} 1 & if x = x_j \\ 0 & otherwise \end{cases}$$
Thus X can be now expressed as :
$X = \sum_{j=1}^{n} x_j I_j(x_j)$

(2 c)

Consider the set of events $X = \{X_1, X_2, X_3, ..., X_n\}$ such that $X_1(\omega) < X_2(\omega) < ... < X_m(\omega) \forall \omega in \Omega$
In order to prove if X is a random variable we consider $\{X(\omega) \leq x\}$ which is equivalent to $\{X_i(\omega) \leq x\} \forall i \in [1, n]]$ which is equivalent to $\{min X(\omega) \leq x\}$ where $min X$ refers to $min(X_i)$

Exercise # 2.7 continued on next page...

(4)

$F(x) = \begin{cases} 0 & if x < 0 \\ \frac{1}{2}x & if 0 < x < 2, \\ 1 & if x > 2 \end{cases}$

Thus $f(x) = F'(x)$:

$f(x) = \begin{cases} 0 & if x < 0 \\ \frac{1}{2} & if 0 < x < 2, \\ 0 & if x > 2 \end{cases}$

Part a: $P(\frac{1}{2} \leq X \leq \frac{3}{2})$ $P(\frac{1}{2} \leq X \leq \frac{3}{2}) = \int_{\frac{1}{2}}^{\frac{3}{2}} f(x)dx = \int_{\frac{1}{2}}^{\frac{3}{2}} \frac{1}{2}dx = \frac{1}{2}$

Part b: $P(1 \leq X \leq 2)$ $P(1 \leq X \leq 2) = \int_1 2\frac{1}{2}dx = \frac{1}{2}$

Part c: $P(Y \leq X)$ $P(Y \leq X) = P(X^2 \leq X) = P(X^2 - X \leq 0) = P(X^2 - X \leq 0) = P(X(X-1) \leq 0)$
Since $X(X-1) \leq 0 \implies 0 \leq X \leq 1$

Thus, $P(Y \leq X) = P(0 \leq X \leq 1) = \int_0^1 \frac{1}{2}dx = \frac{1}{2}$

Part d: $P(Y \leq 2X)$ $P(Y \leq 2X) = p(X \leq 2X^2) = P(X(1-2X) \leq 0) = P(X(2X-1) \geq 0) = P(X \geq \frac{1}{2}) \cup P(X \leq 0) = P(X \geq \frac{1}{2}) + 0 = \int_{\frac{1}{2}}^2 \frac{1}{2}dx + \int_2^\infty 0dx = \frac{3}{4}$

Exercise # 2.7 continued on next page...

(5)

$$F(x) = \begin{cases} 0 & if x < -1 \\ 1-p & if 0 \leq x < 1, \\ 1-p+\frac{1}{2}xp & if 0 \geq x < 2 \\ 1 & x \geq 2 \end{cases}$$

Thus $f(x) = F'(x)$:

$$f(x) = \begin{cases} 0 & if x < -1 \\ 0 & if 0 \leq x < 1, \\ \frac{1}{2}p & if 0 \geq x < 2 \\ 0 & x \geq 2 \end{cases}$$

Part a $P(X=-1)$
$P(X=-1) = f(x=-1) = 0$

Part b $P(X=0)$
$P(X=0) = f(x=0) = 0$

Part c$P(X \geq 1)$
$P(X \geq 1) = \int_1^2 (\frac{1}{2}p)dx + \int_1 0dx = \frac{1}{2}p$

(7)

p(Teeny Weeny is overbooked) = p(All 10 passengers turn up) = $(\frac{9}{10})^1 0 = 0.34$
p(Blockbuster airways is overbooked) = p(19 or 20 passengers turn up) = $\binom{20}{19}(\frac{1}{10})(\frac{9}{10})^1 9 + \binom{20}{2}(\frac{1}{10})^2(\frac{9}{10})^1 8 = .39$
Thus blockbuster ariways is overbooked on average.

(9)

Part a
$X^+ = max(0, X)$:
As $F(x)$ is positive definite:
$P(X^+ \leq x) = \begin{cases} 0 & x \leq 0 \\ F(x) & x \geq 0 \end{cases}$ **Part b**
For $X^- = -min(0, X)$:
$P(X^- \leq x) = P((0 \leq x) \cup (-X \leq 0)) = P((0 \leq x) \cup (X \geq -x)) = P(0 \leq x) + P(X \geq -x) = P(0 \leq x) + (1 - P(X \leq -x))$
Thus, $P(X^- \leq x) = \begin{cases} 0 & x \leq 0 \\ 1 - F(-x) & x \geq 0 \end{cases}$
Part c

$|X| = X^+ + X^-$
Thus $P(|X| \leq x) = P((X^+ \leq x) \cup P(X^- \leq -x)) = P(X^+ \leq x) + P(X^- \leq -x) - P((X^+ \leq x) \cap P(X^- \leq -x))$
Part d
$-X$
$P(-X \leq x) = P(-X \geq x) = 1 - P(X \leq -x) = 1 - F(-x)$

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Given: Median $m = \lim_{y \to \infty} F(y) \leq \frac{1}{2} \leq F(m)$ **To Prove:** Every distribution of F has a median and it is a closed interval
Since F is a continuous function with range $[0, 1]$ applying intermediate values theorem:
F will take all values between its extremum. so every distribution of F has a median and is closed under $[0, 1]$

(12)

Consider the outcomes of the dice to be $X1, X2$ and the sum to be $S = X1 + X2$. Let us assume the outcomes that $S = 2$ to $S = 12$ are equally likely Then let:
$S = 2 : A = \{(1, 1)\}$
$S = 9 : A = \{(3, 6), (4, 5), (5, 4), (6, 3)\}$
Thus for $S = 2, 9$, $A = 1, 4$ the number of favourable events are unequal $\implies$ the initial assumption is false!

(15)

If the user is searching for book B_x the