

MATH 542 Homework 8

Saket Choudhary
skchoudh@usc.edu

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Problem 3a.2

Consider $\|Y - X\beta\|^2$. Since X is full rank $\hat{\beta} = (X'X)^{-1}X'Y$. This involves $\frac{\partial \|Y - X\beta\|^2}{\partial \beta_i} = 0$. Also $\hat{Y}_i = \beta_0 + x_{i1}\beta_1 + \cdots + x_{ip-1}\beta_{p-1}$. For $i = 0$:

$$\begin{aligned}\frac{\partial \|Y - X\beta\|^2}{\partial \beta_i} &= 0 \\ \frac{\partial \sum (Y_i - (\beta_0 + x_{i1}\beta_1 + \cdots + x_{ip-1}\beta_{p-1}))^2}{\partial \beta_0} &= 0 \\ \sum (Y_i - (\beta_0 + x_{i1}\beta_1 + \cdots + x_{ip-1}\beta_{p-1})) &= 0 \\ \sum (Y_i - \hat{Y}_i) &= 0\end{aligned}$$

Problem 3a.3

$$\begin{pmatrix} Y_1 \\ Y_2 \\ Y_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 2 & -1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} \theta \\ \phi \end{pmatrix}$$

Thus $X = \begin{pmatrix} 1 & 0 \\ 2 & -1 \\ 1 & 2 \end{pmatrix}$, $\beta = X'X$ and hence using R:

$$\begin{aligned}\theta &= 0.167Y_1 + 0.333Y_2 + 0.167Y_3 \\ \phi &= -0.2Y_2 + 0.4Y_3\end{aligned}$$

Problem 3a.4

$$\begin{pmatrix} Y_1 \\ Y_2 \\ Y_3 \end{pmatrix} = \begin{pmatrix} 1 & -1 & 1 \\ 1 & 0 & -2 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \end{pmatrix}$$

$$\begin{pmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \end{pmatrix} = (X'X)^{-1}X'Y$$

$$\begin{pmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \end{pmatrix} = \begin{pmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ -0.5 & 0 & 0.5 \\ \frac{1}{6} & -\frac{1}{3} & \frac{1}{6} \end{pmatrix} \begin{pmatrix} Y_1 \\ Y_2 \\ Y_3 \end{pmatrix}$$

Consider $\beta_2 = 0$:

$$\begin{pmatrix} Y_1 \\ Y_2 \\ Y_3 \end{pmatrix} = \begin{pmatrix} 1 & -1 \\ 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \beta_0 \\ \beta_1 \end{pmatrix}$$

Then using R:

$$\begin{pmatrix} \beta_0 \\ \beta_1 \end{pmatrix} = (X'X)^{-1}X'Y$$

$$\begin{pmatrix} \beta_0 \\ \beta_1 \end{pmatrix} = \begin{pmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ -0.5 & 0 & 0.5 \end{pmatrix} \begin{pmatrix} Y_1 \\ Y_2 \\ Y_3 \end{pmatrix}$$

Problem 3a.7

$$\hat{Y} = X\hat{\beta} = PY$$

$$\begin{aligned} \sum \hat{Y}_i(Y_i - \hat{Y}_i) &= \hat{Y}'(Y - \hat{Y}) \\ &= Y'P'(Y - PY) \\ &= Y'P(I_n - P)Y \\ &= Y'(P - P^2)Y \\ &= 0 \text{ Since } P \text{ is idempotent} \end{aligned}$$

Problem 3b.3

Consider $\bar{Y} = \frac{\sum_i Y_i}{n}$

$E[\bar{Y}] = \theta$ so \bar{Y} is unbiased estimate.

Also using Rao's minimum variance lower bound, $\alpha'\beta$ is a minimum variance estimate for $\mathcal{N}(X\beta, \sigma^2)$, Thus \bar{Y} is both unbiased and minimum variance.

Problem 3b.4

$$\begin{aligned}
 Y_i &= \beta_0 + \beta_1(x_{i1} - \bar{x}_1) + \beta_2(x_{i2} - \bar{x}_2) + \epsilon_i \\
 \begin{pmatrix} Y_1 \\ Y_2 \\ Y_3 \\ \vdots \\ Y_n \end{pmatrix} &= \begin{pmatrix} 1 & x_{11} - \bar{x}_1 & x_{12} - \bar{x}_2 \\ 1 & x_{21} - \bar{x}_1 & x_{22} - \bar{x}_2 \\ \vdots & x_{n1} - \bar{x}_1 & x_{n2} - \bar{x}_2 \end{pmatrix} \begin{pmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \end{pmatrix} \\
 X &= \begin{pmatrix} 1 & x_{11} - \bar{x}_1 & x_{12} - \bar{x}_2 \\ 1 & x_{21} - \bar{x}_1 & x_{22} - \bar{x}_2 \\ \vdots & x_{n1} - \bar{x}_1 & x_{n2} - \bar{x}_2 \end{pmatrix} \\
 X'X &= \begin{pmatrix} n & \sum x_{i1} - n\bar{x}_1 & \sum x_{i2} - n\bar{x}_2 \\ \sum x_{i1} - n\bar{x}_1 & \sum (x_{i1} - \bar{x}_1)^2 & \sum (x_{i1} - \bar{x}_1)(x_{i2} - \bar{x}_2) \\ \sum x_{i2} - n\bar{x}_2 & \sum (x_{i1} - \bar{x}_1)(x_{i2} - \bar{x}_2) & \sum (x_{i2} - \bar{x}_2)^2 \end{pmatrix} \\
 &= \begin{pmatrix} n & 0 & 0 \\ 0 & \sigma_1^2 & r\sigma_1\sigma_2 \\ 0 & r\sigma_1\sigma_2 & \sigma_2^2 \end{pmatrix} \\
 (X'X)^{-1} &= \frac{1}{\sigma_1^2\sigma_2^2(1-r^2)} \begin{pmatrix} \frac{1}{n(\sigma_1^2\sigma_2^2(1-r^2))} & 0 & 0 \\ 0 & \sigma_2^2 & r\sigma_1\sigma_2 \\ 0 & r\sigma_1\sigma_2 & \sigma_1^2 \end{pmatrix}
 \end{aligned}$$

$$\text{Thus } \text{Var}(\beta_1) = \sigma^2(X'X)^{-1} = \frac{\sigma^2\sigma_2^2}{\sigma_1^2\sigma_2^2(1-r^2)} = \frac{\sigma^2}{\sigma_1^2(1-r^2)}$$