MATH-547: Assignment

Due on Monday, September 21, 2015

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Contents

Problem 1

$(X, Y) \in R^d \times \{\pm 1\}$

$$P(Y = 1) = \pi_+$$
$$P(Y = -1) = \pi_- P(dx|Y = 1) \qquad = p_+(x)$$
$$P(dx|Y = -1) = p_-(x)$$

Problem 1: (a)

Expression for bayes classifier:

$$P(Y = 1|x) = \frac{P(x|Y = 1)P(Y = 1)}{P(x|Y = 1)P(Y = 1) + P(x|Y = -1)P(Y = -1)}$$
$$P(Y = 1|x) = \frac{\pi_+ p_+(x)}{\pi_+ p_+(x) + \pi_- p_-(x)}$$

$$P(Y = -1|x) = \frac{P(x|Y = -1)P(Y = 1)}{P(x|Y = -1)P(Y = -1) + P(x|Y = -1)P(Y = -1)}$$
$$P(Y = -1|x) = \frac{\pi_- p_-(x)}{\pi_+ p_+(x) + \pi_- p_-(x)}$$

Define $\eta(x) = E[Y|x] = 1 \times P(Y = 1|x) + -1 \times P(Y = 1|x)$
Now,

$$P(error|x) = \begin{cases} P(Y = 1|x) & \text{if actual class for x is -1} \\ P(Y = -11|x) & \text{if actual class for x is 1} \end{cases}$$

Intuitively to minimise the error, we choose class '1' for x when $P(Y = 1|x) > P(Y = -1|X)$ and hence, the bayes classifier $g_*(x)$ is given by:

$$g_*(x) = \begin{cases} 1 & \pi_+ p_+(x) \geq \pi_- p_-(x) \\ -1 & \text{otherwise} \end{cases}$$

and the total probability of making an error is given by: $P(error|x) = min(P(Y = 1|x), P(Y = -1|x))$
The loss function $L(\alpha_i, \omega_j)$ is the 'loss' incurred for taking action α_i instead of ω_j

Problem 1: (b)

Bayes risk

Problem 4

Assume the sufficient condition exists, i.e.: There exist $a_1, a_2, \dots a_k \geq 0$ and binary classifiers $g_1, g_2, \dots g_k$ such that $\forall 1 \leq i \leq n$: $Y_i \sum_{j=1}^{k} a_j g_j(X_i) \geq 2\gamma$
Y_i is given by the widghted sum of predictions g_j : $Y_i = sign(\sum_{j=1}^{k} a_j g_j(X_i))$ Taking expectations:

$$E[Y_i \sum_{j=1}^{k} a_j g_j(X_i)] \geq 2\gamma$$
$$\sum_{j=1}^{k} a_j E[Y_i g_j(X_i)] \geq 2\gamma$$

Since, $\sum_j a_j = 1$ and $a_j \geq 0$ for $j = \{1, 2, \dots, k\}$ and $\sum_{j=1}^{k} a_j E[Y_i g_j(X_i)] \geq 2\gamma$ then there exists a g_j such that:
$E[Y_i g_j(X_i)] \geq 2\gamma$

$$\begin{aligned} E[Y_i g_j(X_i)] &= 1 \times P[Y_i = g_j(X_i)] + -1 \times P(Y_i \neq g_j(X_i)) \\ &= 1 - 2P(Y_i \neq g_j(X_i)) \\ \implies P(Y_i \neq g_j(X_i)) &= \frac{1 - E[Y_i g_j(X_i)]}{2} \\ P(Y_i \neq g_j(X_i)) &\leq \frac{1 - 2\gamma}{2} \end{aligned}$$

Now for weights, $w_1, w_2, \dots w_j$ such that $\sum_j w_j = 1$:

$$\sum_{j=1}^{n} P(Y_j \neq g(X-j)) \leq \frac{1}{2} - \gamma$$
$$\sum_{j=1}^{n} E[I(Y_j \neq g(X-j))] \leq \frac{1}{2} - \gamma$$
$$\sum_{j=1}^{n} I(Y_j \neq g(X-j)) \leq \frac{1}{2} - \gamma$$