

# **CSCI-567: Assignment #1**

Due on Tuesday, September 23, 2015

**Saket Choudhary**  
**2170058637**

## Contents

<b>Problem # 1</b>	<b>3</b>
Problem # 1: (a) . . . . .	3
Problem # 1: (b) . . . . .	3
<b>Problem 2</b>	<b>4</b>
Problem 2: (a) . . . . .	4
Problem 2: (b) . . . . .	4
<b>Prob. III</b>	<b>4</b>

## Problem # 1

### Problem # 1: (a)

Given:  $X_i \sim \text{Beta}(\alpha, 1)$  MLE for  $\alpha$ :

Consider  $X = (X_1, X_2, \dots, X_n)$  Likelihood function:  $L(\alpha|X)$   $L(\alpha|X) = \prod_{i=1}^n f(x_i)$  where

$$f(x_i) = \frac{\Gamma(\alpha + \beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1} (1-x)^{\beta-1} = \frac{\Gamma(\alpha + 1)}{\Gamma(\alpha)\Gamma(1)} x^{\alpha-1} \quad (1)$$

$$= \frac{\alpha\Gamma(\alpha)}{\Gamma(\alpha)} x^{\alpha-1} = \alpha x^{\alpha-1} \quad (2)$$

$$L(\alpha|X) = \left( \frac{\Gamma(\alpha + 1)}{\Gamma(\alpha)\Gamma(1)} \right)^n \prod_{i=1}^n (x_i)^{\alpha-1} \quad (3)$$

$$LL = \log(L(\alpha|X)) = n \log(\alpha) + (\alpha - 1) \sum_{i=1}^n \log(x_i) \quad (4)$$

$$\frac{dLL}{d\alpha} = \frac{n}{\alpha} + \sum_{i=1}^n \log(x_i) \quad (5)$$

$$\frac{dLL}{d\alpha} = 0 \implies \hat{\alpha} = \frac{n}{\sum_{i=1}^n \log(1/x_i)} \quad (6)$$

Minima at  $\hat{\alpha} = \frac{n}{\sum_{i=1}^n \log(1/x_i)}$  is guaranteed due to log being a concave function.

### Problem # 1: (b)

Given:  $x_i \sim N(\theta, \theta)$  i.e  $f(x_i) = (2\pi\theta)^{-\frac{1}{2}} e^{-\frac{(x_i-\theta)^2}{2\theta}}$  MLE estimate for  $\theta$ :

$$L(\theta|X) = (2\pi\theta)^{-\frac{N}{2}} e^{-\sum_{i=1}^n \frac{(x_i-\theta)^2}{2\theta}} \quad (7)$$

$$LL = \log(L(\theta|X)) = -\frac{N}{2} \log((2\pi\theta)) - \sum_{i=1}^n \frac{(x_i - \theta)^2}{2\theta} \quad (8)$$

$$\frac{dLL}{d\theta} = -\frac{N}{2} \left( \frac{1}{\theta} \right) + \frac{\sum_{i=1}^n x_i^2}{2\theta^2} - \frac{N\theta}{2} \quad (9)$$

$$\frac{dLL}{d\theta} = 0 \implies N\theta^2 + N\theta - \sum_{i=1}^n x_i^2 = 0 \quad (10)$$

The above equation is a quadratic and will have two solutions, Since,  $\theta \geq 0$  (a constraint that comes from  $\theta$  being the variance), the

$$\theta = \frac{-N \pm \sqrt{N^2 + 4N \sum_{i=1}^n x_i^2}}{2N}$$

$$\text{Since, } \hat{\theta} \geq 0, \hat{\theta} = \frac{-N + \sqrt{N^2 + 4N \sum_{i=1}^n x_i^2}}{2N}$$

## Problem 2

### Problem 2: (a)

Given:  $\hat{f}(x) = \frac{1}{n} \sum_{i=1}^n \frac{1}{h} K(\frac{x-X_i}{h})$  To show:  $E_{X_1, X_2, \dots, X_n}[\hat{f}(x)] = \frac{1}{h} \int K(\frac{x-t}{h}) f(t) dt$   
 Proof:

$$E[\hat{f}(x)] = E[\frac{1}{n} \sum_{i=1}^n \frac{1}{h} K(\frac{x-X_i}{h})] \quad (11)$$

$$= \frac{1}{nh} E[K(\frac{x-X_i}{h})] \quad (12)$$

$$= \frac{1}{h} E[K(\frac{x-X_1}{h})] = \frac{1}{h} E[K(\frac{x-t}{h})] \quad (13)$$

where the penultimate equality comes from the fact that  $X_i$  are iid for all  $i \in [1, n]$ . and  $t \sim X$  and hence.

$$E[\hat{f}(x)] = \frac{1}{h} E[K(\frac{x-X_1}{h})] = \frac{1}{h} \int K(\frac{x-t}{h}) f(t) dt = RHS \quad (14)$$

### Problem 2: (b)

Consider  $z = \frac{x-t}{h} \implies t = x - hu$   
 Then,

$$E[\hat{f}(x)] = \frac{1}{h} \int K(z) f(x - hz) dz \quad (15)$$

$$(16)$$

$$f(x - hz) = f(x) - f'(x)hz + \frac{1}{2}f''(x)\frac{(hz)^2}{2} - \frac{1}{3}f'''(x)\frac{(hz)^3}{3!} + \dots + (-1)^n \frac{1}{n!} f^{(n)}(x) \left(\frac{hz}{n!}\right)^n$$

By definition,  $\int k(z) dz = 1$ . Also define an auxillary variable  $M_j = \int k(z) z^j dz$  for the  $j^{th}$  moment of the kernel function, and hence,  $\int K(z) f(x - hz) dz = f(x) - hf'(x)M_1 + \frac{1}{2}(h^2)f''(x)M_2 + \dots + (-1)^n \frac{1}{n!} f^{(n)}(x)M_n$

Now,  $Bias = E[\hat{f}(x)] - f(x) = -hf'(x)M_1 + \frac{1}{2}(h^2)f''(x)M_2 + \dots + (-1)^n \frac{1}{n!} f^{(n)}(x)M_n$

And as  $h \rightarrow 0$ ,  $Bias \rightarrow 0$

## Prob. III

Suspendisse vitae elit. Aliquam arcu neque, ornare in, ullamcorper quis, commodo eu, libero. Fusce sagittis erat at erat tristique mollis. Maecenas sapien libero, molestie et, lobortis in, sodales eget, dui. Morbi ultrices rutrum lorem. Nam elementum ullamcorper leo. Morbi dui. Aliquam sagittis. Nunc placerat. Pellentesque tristique sodales est. Maecenas imperdiet lacinia velit. Cras non urna. Morbi eros pede, suscipit ac, varius vel, egestas non, eros. Praesent malesuada, diam id pretium elementum, eros sem dictum tortor, vel consectetur odio sem sed wisi.