Laplace transforms

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Intro to Laplace transforms :

from signals and systems we have a transform that transforms from time domain to frequency domain

Fourier transform : $x(\omega) = \int_{t=0}^{\infty} x(t) e^{-j\omega t} dt$

assume there is a signal $e^{-at} = x(t)$

$$\Rightarrow x(\omega) = \int_{0}^{\infty} e^{-(a+j\omega)t} dt$$

$$= \left[\frac{-e^{-(a+j\omega)t}}{a+j\omega} \right]_0^{\infty} \Rightarrow \left[\frac{1}{a+j\omega} \right]$$

$$|x(\omega)| = \frac{1}{\sqrt{a^2+\omega^2}} \qquad \phi = \tan^{-1}\left[\frac{\omega}{a}\right]$$

now assume a signal $e^{at} = x(t)$

$$\Rightarrow x(\omega) = \int_{0}^{\infty} e^{(a-j\omega t)} dt$$

$$= \left[\frac{e^{(a-j\omega)t}}{(a-j\omega)} \right]_0^{\infty} = \text{this func } \underline{\text{doesnt}} \text{ converge}$$

hence we add a component $e^{-\sigma t}$ such that it limits the transform from going out of bound.

$$x(s) = \int_0^\infty x(t)\, e^{-st} \quad \Big| \quad s = (\sigma + j\omega)$$

Some laplace transforms :

i) $f(t) = u(t)$

$$= x(s) = \int_0^\infty u(t)\, e^{-st} = \int_0^\infty e^{-st}\, dt$$

$$= \left[\frac{-e^{-st}}{s} \right]_0^\infty = \underline{\underline{\frac{1}{s}}}$$

ii) $f(t) = e^{\pm at}\, u(t)$

$$= x(s) = \int_0^\infty e^{\pm at}\, u(t)\, e^{-st} \Rightarrow \int_0^\infty e^{-(s \mp a)t}$$

$$= \left[\frac{-e^{-(s \mp a)t}}{(s \mp a)} \right]_0^\infty = \left[\frac{1}{s \mp a} \right]$$

Laplace transform Table

$f(t)$	$F(s)$
$f(t) = 1$	$F(s) = 1/s$
$f(t) = t^n$	$F(s) = \frac{n!}{s^{(n+1)}}$
$f(t) = e^{\pm at}$	$F(s) = \frac{1}{s \mp a}$
$f(t) = \sin(at)$	$F(s) = \frac{a}{s^2 + a^2}$
$f(t) = \cos(at)$	$F(s) = \frac{s^2}{a^2 + s^2}$
$f(t) = \sinh(at)$	$F(s) = \frac{a}{s^2 - a^2}$
$f(t) = \cosh(at)$	$F(s) = \frac{s}{s^2 - a^2}$
$f(t) = t^n e^{at}$	$F(s) = \frac{n!}{(s-a)^{n+1}}$

Laplace Properties :

property	f(t)	f(s)
Linearity	$a_1 f_1(t) + a_2 f_2(t)$	$a_1 F_1(s) + a_2 F_2(s)$
Scaling	$f(at)$	$\frac{1}{a} F(\frac{s}{a})$
time shift	$f(t-a)$	$e^{-as} F(s)$
freq shift	$e^{-at} f(t)$	$F(s+a)$
time diff	$\frac{df}{dt}$	$sF(s) - f(0^-)$
	$\frac{d^2f}{dt^2}$	$s^2 F(s) - s' f(0) - f'(0)$
	$\frac{d^nf}{dt^n}$	$s^n(F(s)) - s^n f(0) - s^{n-1} f' \dots s f^{n-1} - f^n(0)$
time Integration	$\int f(n)\,dt$	$\frac{1}{s} F(s)$
Initial value	$f(0)$	$\lim_{s\to\infty} sF(s)$
Final value	$f(\infty)$	$\lim_{s\to\infty} sF(s)$
Convolution	$f_1 * f_2$	$= F_1(s) F_2(s)$