Probability and Statistics

3 – Discrete Random Variables

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Discrete Random Variables

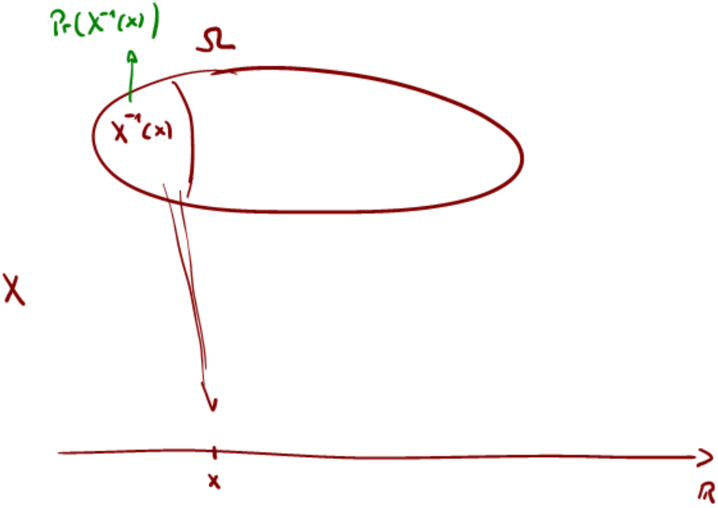
Definition (3.1)

Let Ω be a sample space and $\mathcal{A} \subseteq \mathcal{P}(\Omega)$ a set of events with a probability measure $\Pr : \mathcal{A} \to \mathbb{R}$. Given $(\Omega, \mathcal{A}, \Pr)$, a mapping $X : \Omega \to \mathbb{R}$ is called a *discrete random variable* if

(1) $$X^{-1}(x) \in \mathcal{A} \qquad \text{for all } x \in \mathbb{R}$$

and there exist countable (finite or infinite) many distinct real numbers $(x_i)_{i \in I}$ such that:

(2) $$\sum_{i \in I} \Pr(X^{-1}(x_i)) = 1$$



$$P_X(x) := Pr(X^{-1}(x))$$

Probability Distribution (Probability Mass Function)

Definition (3.1)

If X is a discrete random variable, its *distribution* (also called *probability mass function* (pmf)) is defined by:

$$p_X : \mathbb{R} \to [0,1], \qquad p_X(x) \;:=\; \Pr(X = x) \;:=\; \Pr(X^{-1}(x)) \quad \text{for all} \;\; x \in \mathbb{R}$$

Probabilities for subsets of $\mathbb{R}$

Let $\{x_i \mid i \in I\} \subseteq \mathbb{R}$ be the set of all $x \in \mathbb{R}$ with $p_X(x) > 0$

Definition

$$\Omega = \underbrace{\dot{\bigcup_{i \in I}} X^{-1}(x_i)}_{1} \;\dot\cup\; \underbrace{N}_{0}$$

Lemma (3.2)

Let $(\Omega, \mathcal{A}, \Pr)$ be as in (3.1) and $X : \Omega \to \mathbb{R}$ a discrete random variable. Then, for any subset $S \subseteq \mathbb{R}$ with $X^{-1}(S) \in \mathcal{A}$:

$$\Pr(X \in S) \;:=\; \Pr(X^{-1}(S)) \;=\; \Pr(\{\omega \in \Omega \mid X(\omega) \in S\}) \;=\; \sum_{\substack{x \in S \\ p_X(x) \neq 0}} p_X(x)$$

Ex

$S = [a,b]$

$\Pr(X \in S)$

$= \Pr\left(\dot\bigcup X^{-1}(x_i) \;\dot\cup\; X^{-1}(S \setminus \{x_i \mid i \in I\}) \right)$

$= \Pr\left(\dot\bigcup X^{-1}(x_i) \right) + \Pr(\text{—})$

$= \sum_{\substack{x \in S \\ p_X(x) > 0}} p_X(x) \;+\; 0$

Probabilities for subsets of $\mathbb{R}$

Notation (3.3)

If $X : \Omega \to \mathbb{R}$ is a discrete random variable, then for every subset $S \subseteq \mathbb{R}$

$$\sum_{x \in S} p_X(x) \quad := \quad \sum_{\substack{x \in S \\ p_X(x) \neq 0}} p_X(x)$$

is well defined and will be denoted by $\Pr(X \in S)$, even if $X^{-1}(S) \notin \mathcal{A}$.

Probabilities for subsets of $\mathbb{R}$

Remark (3.4)

Even if $X^{-1}(S) \notin \mathcal{A}$, "the event" $X^{-1}(S)$ may be defined. It is the smallest set in $\mathcal{A}$ containing $X^{-1}(S)$, i.e.:

$$\left(X^{-1}(S)\right)_{\mathcal{A}} \;:=\; \bigcap_{\substack{A \in \mathcal{A} \\ X^{-1}(S) \subseteq A}} A$$

Note: If N is defined as in the proof of (3.2), then

$$(X^{-1}(S))_{\mathcal{A}} \;\subseteq\; X^{-1}(S) \cup N \;\in\; \mathcal{A}$$

and:

$$\Pr(X \in S) \;=\; \Pr\left(\left(X^{-1}(S)\right)_{\mathcal{A}}\right) \;=\; \Pr\left(X^{-1}(S) \cup N\right)$$

Probabilities for intervals of $\mathbb{R}$

Notation (3.5)

In line with the introduced notations $\Pr(X = x)$ and $\Pr(X \in S)$, the probability that $X(\omega)$ belongs to some interval $S = (a, b]$ is denoted by:

$$\Pr(a < X \leq b) \quad := \quad \Pr(X \in (a, b])$$

$\hookrightarrow \; X^{-1}((a,b])$

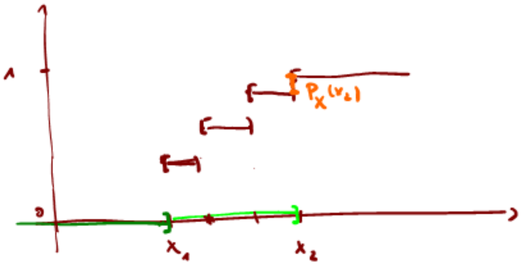
Similar notations are used for other types of intervals.

Cumulative Distribution Function (cdf)

Definition (3.6)

The *cumulative distribution function* (cdf) of a random variable X is defined by:

$$F_X : \mathbb{R} \to [0,1] \,, \qquad F_X(x) \;=\; \Pr(X \leq x)$$



Discrete sample spaces of real numbers

If Ω is a countable subset of $\mathbb{R}$, i.e. $\Omega = \{\omega_i | i \in I\} \subseteq \mathbb{R}$ for some countable set I, and Pr is a probability on $\mathcal{A} = \mathcal{P}(\Omega)$ (i.e. $p_i = \Pr(\omega_i)$ is known for all $i \in I$), then the embedding $X : \Omega \to \mathbb{R}$ with $X(\omega) = \omega$ for all $\omega \in \Omega$ is a random variable. By virtue of such embeddings, all notions related to random variables (like expectation, variance, etc., which will be defined later) can also be applied to discrete sample spaces of real numbers.

Sample spaces defined by a probability mass function

Remark (3.8)

Any function $p : \mathbb{R} \to [0, 1]$ with countable support $\Omega := \{\omega \in \mathbb{R} \mid p(\omega) \neq 0\}$ and $\sum_{\omega \in \Omega} p(\omega) = 1$ is a probability mass function $p = p_X$, defined by:

- sample space $\Omega := \{\omega \in \mathbb{R} \mid p(\omega) \neq 0\}$,
- probability $\Pr : \mathcal{P}(\Omega) \to [0, 1]$ with $\Pr(\{\omega\}) = p(\omega)$ for all $\omega \in \Omega$, and
- random variable $X : \Omega \hookrightarrow \mathbb{R}$ with $X(\omega) = \omega$ for all $\omega \in \Omega$

Rolling two dice

Example (3.9)

Consider the sample space

$$\Omega \quad = \quad \{(i,j) \mid i,j \in \{1,2,3,4,5,6\}\}$$

with a uniform probability measure, i.e.

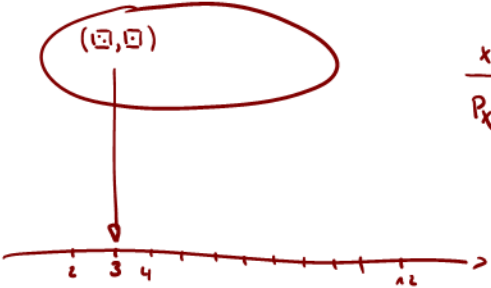
$$\Pr(\{\omega\}) \quad = \quad \frac{1}{36}$$

for every $\omega \in \Omega$.

Rolling two dice

Example (3.9)

If $X : \Omega \to \mathbb{R}$ is defined to be the sum of the two components of ω, i.e. $X((i,j)) = i + j$, its pmf is as follows:



x	2	3	4	5	6	7	8	9	10	11	12
$p_X(x)$	1/36	2/36	3/36	4/36	5/36	6/36	5/36	4/36	3/36	2/36	1/36

Expectations

Definition (3.10)

Let X be discrete random variable with pmf $p_X(x)$. The *expectation* (*mean*) of X is defined to be

$$E(X) := \sum_{\substack{x \in \mathbb{R} \\ p_X(x) > 0}} x \cdot p_X(x)$$

if at least one of the sums $\sum_{x \in \mathbb{R}^-} x \cdot p_X(x)$ or $\sum_{x \in \mathbb{R}^+} x \cdot p_X(x)$ is finite.

Expectations (countable sample spaces)

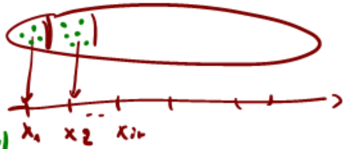
Lemma (3.11)

Let $X : \Omega \to \mathbb{R}$ be a random variable with countable sample space Ω and $\mathcal{A} = \mathcal{P}(\Omega)$. Then:

$$E(X) \;=\; \sum_{\omega\in\Omega} X(\omega)\cdot \Pr(\omega) \qquad \leftarrow$$

$$E(X) = \sum_{x\in\mathbb{R}} x\cdot P_X(x) = \sum_{x\in\mathbb{R}} x\cdot \Pr(\{\omega\in\Omega \mid X(\omega)=x\}) \qquad X^{-1}(x) = \dot{\bigcup_{\omega\in X^{-1}(x)}}\{\omega\}$$

$$= \sum_{\substack{x\in\mathbb{R}\\ P_X(x)>0}} x\cdot\Big(\sum_{\omega\in X^{-1}(x)} \Pr(\omega)\Big)$$

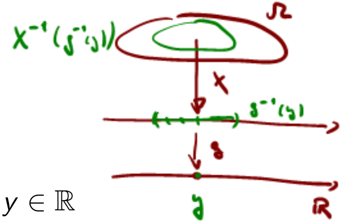


$$= \sum_{P_X(x)>0}\ \sum_{\omega\in X^{-1}(x)} \overset{X(\omega)}{x}\cdot \Pr(\omega) = \sum_{\omega\in\Omega} X(\omega)\Pr(\omega)$$

Lemma (3.12)

Let X be a discrete random variable. Then, every function $g : \mathbb{R} \to \mathbb{R}$ defines a random variable:

$$Y \;=\; g(X) \;:=\; g \circ X$$



(i) *The pmf of Y is given by:*

$$p_Y(y) \;=\; \sum_{x \in g^{-1}(y)} p_X(x) \qquad \text{for all } y \in \mathbb{R}$$

(ii) *If $E(Y)$ exists, then:*

$$E(Y) \;=\; \sum_{x \in \mathbb{R}} g(x) \cdot p_X(x)$$

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