Probability and Statistics

4 – Continuous Random Variables

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HW (due next lecture on friday) : Exc. 6 , Q1-3

Exponential Distributions

Definition (4.39)

A random variable X has a *exponential distribution*, $X \sim \exp(\lambda)$, for some parameter $\lambda > 0$, if it has a pdf defined by:

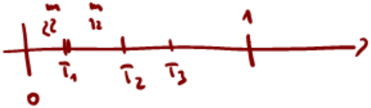
$$f_X(x) \;=\; \begin{cases} \lambda e^{-\lambda x} & \text{for } x \geq 0 \\ 0 & \text{for } x < 0 \end{cases}$$

$$\left(\text{Note:} \quad \int_0^\infty \lambda e^{-\lambda x}\, dx \;=\; -e^{-\lambda x}\Big|_0^\infty \;=\; 1\right)$$

Exponential Distributions

$m = E(X)$, $\lambda := \frac{1}{m}$ [success] rate

$t := x$ (time)



Lemma (4.40)

If $X \sim \exp(\lambda)$, *then:*

(i) $$F_X(x) = \begin{cases} 1 - e^{-\lambda x} & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}$$

(ii) $$E(X^n) = \frac{n!}{\lambda^n} \quad \text{for } n \in \mathbb{N}$$

(iii) $$E(X) = \frac{1}{\lambda}, \quad Var(X) = \frac{1}{\lambda^2}, \quad \mathrm{skew}(X) = 2, \quad \mathrm{kurt}(X) = 9$$

(iv) $$\phi_X(t) = \frac{\lambda}{\lambda - t} \quad \text{for } t < \lambda$$

(v) $$F_X^{-1}(p) = -\frac{\ln(1-p)}{\lambda} \quad \text{for } p \in (0,1)$$

(vi) *first quartile, median and third quartile are:*

$$\ln(4/3)/\lambda \approx 0.288/\lambda, \qquad \ln(2)/\lambda \approx 0.693/\lambda, \qquad \ln(4)/\lambda \approx 1.386/\lambda$$

Memorylessness of Exponential Distributions

Theorem (4.42)

X has an exponential distribution, if and only if X has the memoryless property:

$$\Pr(X > t + h \mid X > t) = \Pr(X > h) \qquad \text{for all } t, h \geq 0$$

$$\Pr(T \leq t + \Delta t \mid T > t) = \Pr(T \leq \Delta t) = F_T(\Delta t)$$

Exponential Distributions

Theorem (4.44)

Let T be a continous random variable with a continous pdf f_T with constant success rate λ, i. e. for all $t > 0$, we have:

$$\lambda \quad = \quad \lim_{\Delta t \to 0} \frac{\Pr(T \leq t + \Delta t \mid T > t)}{\Delta t}$$

Then $T \sim \exp(\lambda)$.

$\parallel$ (4.42)

trivial: Assume $T \sim \exp(\lambda)$

$$\lim_{\Delta t \to 0} \frac{F_T(\Delta t)}{\Delta t} = \lim_{\Delta t \to 0} \frac{1 - e^{-\lambda \Delta t}}{\Delta t} = \frac{d}{dt}\left(1 - e^{-\lambda t}\right)\Big|_{t=0} = \lambda e^{-\lambda t}\Big|_{t=0} = \lambda$$

Exponential Distributions

$$\lambda = \lim_{\Delta t \to 0} \frac{\Pr(T \leq t + \Delta t \mid T > t)}{\Delta t} = \lim_{\Delta t \to 0} \frac{\Pr(t \leq T \leq t + \Delta t)}{\Pr(T > t) \cdot \Delta t}$$

$$= \lim_{\Delta t \to 0} \frac{\int_t^{t+\Delta t} f_T(\tau)\, d\tau}{(1 - F_T(t)) \cdot \Delta t} = \frac{1}{1 - F_T(t)} f_T(t)$$

$$\Rightarrow \quad f_T(t) = \lambda (1 - F_T(t)) \quad \Rightarrow \quad \frac{d}{dt} f_T(t) = -\lambda f_T(t)$$

differential equation for $f_T(t)$

$$\Rightarrow \quad \left.\begin{array}{l} f_T(t) = c \cdot e^{-\lambda t} \\ \int \ldots = 1 \end{array}\right\} \Rightarrow \quad c = \lambda, \quad f_T \text{ is the p.d.f. of } \exp(\lambda)\text{-dist.}$$

Minimum of independent $X_i \sim \exp(\lambda_i)$

Theorem (4.41)

Let $X_1, \dots, X_n$ be independent random variables with $X_i \sim \exp(\lambda_i)$ for $i = 1, \dots, n$. Then

$$X \quad = \quad \min\{X_1, \dots, X_n\}$$

is a random variable with $X \sim \exp(\lambda_1 + \dots + \lambda_n)$.

$$F_X(x) = \Pr(X \le x) = 1 - \Pr(X > x) = 1 - \Pr(X_1 > x, X_2 > x, \dots, X_n > x)$$

$$= 1 - \Pr(X_1 > x) \cdot \Pr(X_2 > x) \cdot \dots \cdot \Pr(X_n > x)$$

$$= 1 - e^{-\lambda_1 x} \cdot e^{-\lambda_2 x} \cdot \dots \cdot e^{\lambda_n \cdot x} = 1 - e^{-(\lambda_1 + \dots + \lambda_n) x}$$

cdf of $\exp(\lambda_1 + \dots + \lambda_n)$-dist.

$$\Pr(X_i > x) = 1 - F_{X_i}(x)$$
$$= 1 - (1 - e^{-\lambda_i x}) = e^{-\lambda_i x}$$

$\square$

Exponential Distributions

$n=3$

$i_1 = 2$
$i_2 = 1$
$i_3 = 3$

Theorem (4.43)

Let $\lambda > 0$ and let $X_1, \ldots, X_n$ be independent random variables with $X_i \sim \exp(\lambda)$ for $i = 1, \ldots, n$. Furthermore, let $X_{i_1}, \ldots, X_{i_n}$ be a reordering of $X_1, \ldots, X_n$, such that:

$$X_{i_1} \leq \ldots \leq X_{i_n}$$

Then

$$Y_k := X_{i_k} - X_{i_{k-1}} \qquad (X_{i_0} := 0)$$

is random variable with $Y_k \sim \exp((n+1-k)\,\lambda)$. $[Y_n \sim \exp(\lambda)]$

$k=1$: $Y_1 \sim \exp(n \cdot \lambda)$: true by (4.41)

Y_2: $X_{i_2}, \ldots, X_{i_n}$ $\qquad Pr(X_{i_j} > X_{i_1} + t) = e^{-\lambda t}$ $\qquad$ because of the memoryless property

$\rightarrow$ Result follows by induction

Reliability theory

Definition (4.45)

Let T be a random variable, describing the *lifetime* or *time-to-failure* of a device or system and let f_T and F_T be its density functions.

(i) The *reliability function* is defined by:

$$R(t) \; := \; \Pr(T > t) \; = \; 1 \; - \; F_T(t)$$

(ii) The *mean time to failure* ($MTTF$) is defined to be the expected lifetime:

$$\mathrm{MTTF} \; := \; E(T)$$

(iii) The *failure rate* is defined to be

$$\lambda(t) \; := \; \lim_{\Delta t \to 0} \frac{\Pr(T \leq t + \Delta t \mid T > t)}{\Delta t}$$

Reliability theory

$R(t) = 1 - F_T(t)$

Lemma (4.46)

$$\mathrm{MTTF} = \int_0^\infty R(t)\,dt$$

$$\mathrm{MTTF} = E(T) = \int_0^\infty t \cdot f_T(t)\,dt = -t \cdot \underbrace{(1 - F_T(t))}_{R(t)}\Big|_0^\infty + \int_0^\infty 1 \cdot R(t)\,dt$$

$-t \downarrow$, $(1-F_T(t)) \downarrow_{t\to\infty}$

add. assumption : $\infty \cdot 0 = 0$

Reliability theory

$R(t) = 1 - F_T(t) \Rightarrow R'(t) = -f_T(t)$

Lemma (4.47)

(i)

$$\lambda(t) \overset{\checkmark}{=} \frac{f_T(t)}{\underbrace{1 - F_T(t)}_{R(t)}} \underset{\checkmark}{=} \frac{f_T(t)}{\int_t^\infty f_T(\tau)\, d\tau} \underset{\checkmark}{=} -\frac{R'(t)}{R(t)}$$

(ii)

$$F_T(t) = 1 - e^{-\int_0^t \lambda(\tau)\, d\tau}, \qquad f_T(t) = \lambda(t)\, e^{-\int_0^t \lambda(\tau)\, d\tau}$$

(iii) *If $\lambda(t) = \lambda$ is constant, then $T \sim \exp(\lambda)$.*

(i) $\lambda(t) = \lim_{\Delta t \to 0} \frac{P_r(t < T < t + \Delta t)}{P_r(T > t) \cdot \Delta t} = \frac{1}{P_r(T > t)} \frac{F_T(t + \Delta t) - F_T(t)}{\Delta t} = \frac{f(t)}{P_r(T > t)} = \frac{f_T(t)}{1 - F_T(t)}$

(ii), (iii): HW