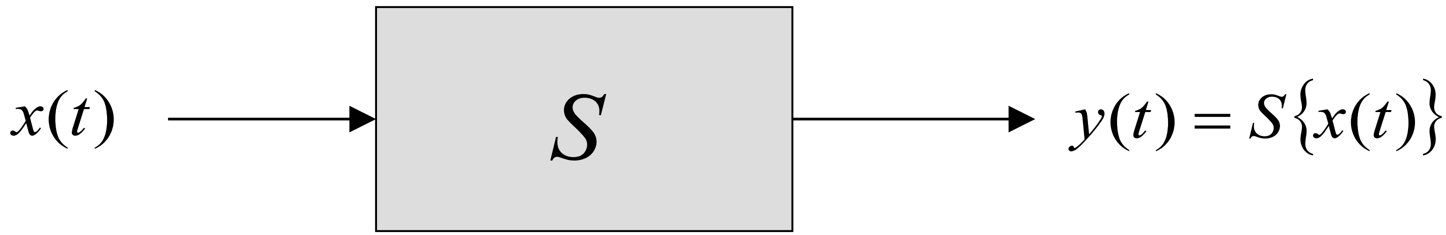
Chapter 3

Continuous-Time Systems

- 3.1 System Definition and Properties
- 3.2 Time-Domain Representation
- 3.3 Frequency-Domain Representation
- 3.4 HILBERT Transformer
- 3.5 LAPLACE Transform (LT)
- 3.6 s-Domain Representation of LTI Systems
- 3.7 Block Diagram Representations

3.1 System Definition and Properties



- A system is described by its **input and output signals** or by its **system function**, respectively.
- **Continuous-time system**: A continuous-time input signal results in a continuous-time output signal
- Ideal test signals are used for stimulation in order to simplify the mathematical description.
- Differential equations can be used for the system function.
- Without loss of generality we focus on systems with one input and one output: twoport, 1D system, SISO system

- Mainly **linear time invariant (LTI) systems** are considered in this course
  - **Time invariance**: the system properties don't change with time

$$x(t) \rightarrow y(t) \;\; ; \;\; x(t-t_0) \rightarrow y(t-t_0)$$

  - **Linearity**:

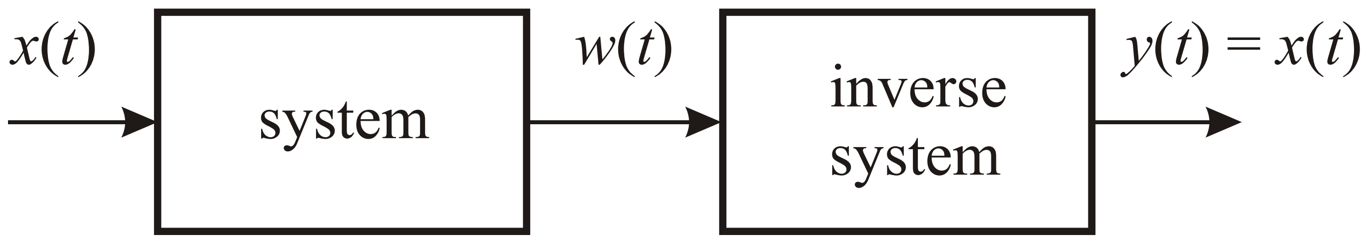
$$x_1(t) \rightarrow y_1(t), \quad x_2(t) \rightarrow y_2(t)$$

$$x_1(t) + x_2(t) \rightarrow y_1(t) + y_2(t)$$

$$K \cdot x(t) \rightarrow K \cdot y(t), \quad K \in \mathrm{R}$$

→ This feature requires **source free** systems.

- Systems with and without **memory**
  - **Memoryless system**: The output value at a specific time depends only on the input value at that same time. → no energy storage
  - **Memory systems**: The output value at a specific time depends on the history. → energy storage
- **Invertibility** and **inverse systems**: If a system is invertible, an inverse system exists that inverts the system function if cascaded with the original system.



→ Important application: encoding and decoding of signals

- **Causality**: A system is causal if the output response doesn't appear before the input stimulus appears. I.e., the output at any time depends only on values at the input at the present time and in the past. → non-anticipative system, non-predictive system
- **Stability**:
  - General definition: Small input stimuli lead to system responses that do not diverge.
  - More formally: If the input signal is bounded (limited magnitude), then the output must also be bounded and thus cannot diverge.
  - These systems are called **BIBO** (bounded input bounded output) stable.

$$x(t) \rightarrow y(t) \;\; ; \;\; |x(t)| < B_x \rightarrow |y(t)| < B_y$$

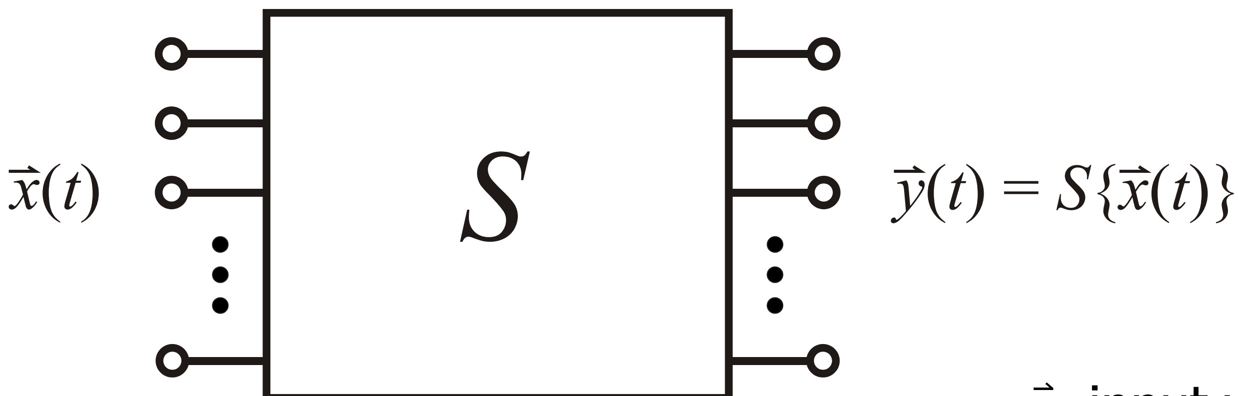
- **Passive and lossless systems**
  - A system is passive, if for every finite-energy / finite-power input signal $x(t)$ the output signal $y(t)$ has at most the same energy / power.

$$E_{y,\infty} \leq E_{x,\infty} < \infty \qquad P_{y,\infty} \leq P_{x,\infty} < \infty$$

  - For a lossless system, the energy / power of the input and output signals are identical.

$$E_{y,\infty} = E_{x,\infty} < \infty \qquad P_{y,\infty} = P_{x,\infty} < \infty$$

- **Systems with multiple inputs and outputs**
  - SISO: single input single output $\quad y(t) = S\{x(t)\}$
  - SIMO: single input multiple output $\quad \vec{y}(t) = S\{x(t)\}$
  - MISO: multiple input single output $\quad y(t) = S\{\vec{x}(t)\}$
  - MIMO: multiple input multiple output $\quad \vec{y}(t) = S\{\vec{x}(t)\}$



$\vec{x}$ input vector
$\vec{y}$ output vector

3.2 Time-Domain Representation
3.2.1 Convolution Integral, Impulse Response

$$x(t) \longrightarrow \boxed{\text{LTI}} \longrightarrow y(t) = S\{x(t)\}$$

- The input-output relation is given by the convolution operation.

$$y(t) = \int_{-\infty}^{\infty} x(\tau) \cdot h(t-\tau)\,\mathrm{d}\tau = x(t) * h(t)$$

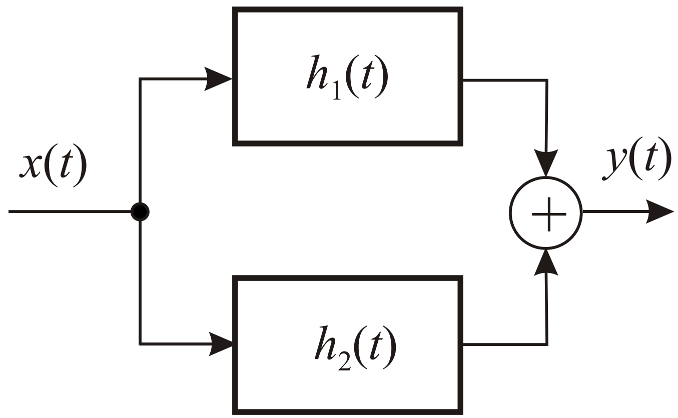
- $h(t)$ is the **unit impulse response** of the system for $x(t) = \delta(t)$

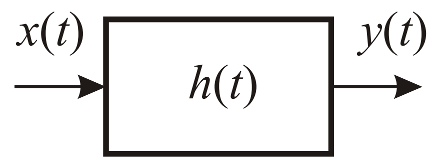
$$y(t) = x(t) * h(t) = \delta(t) * h(t) = h(t)$$

- The characteristics of an LTI system are completely determined by its unit impulse response.

Interconnection of Systems

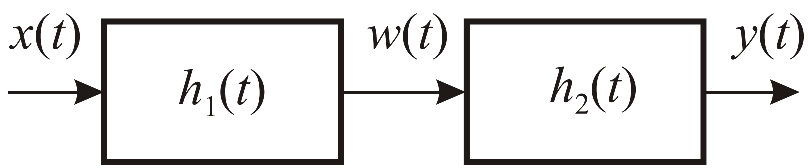
- Parallel connection

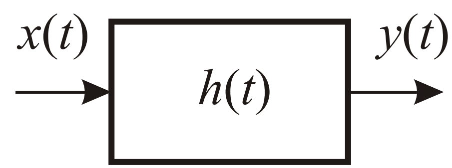




$$h(t) = h_1(t) + h_2(t)$$

- Chain or cascade connection {sometimes called 'serial connection' **:-(** }





$$h(t) = h_1(t) * h_2(t)$$

- **Memoryless system**

  $$h(t) = 0 \quad \text{for} \quad t \neq 0$$

- **Inverse system** $h_{\mathrm{Inv}}(t)$

  $$h(t) * h_{\mathrm{Inv}}(t) = \delta(t)$$ → Identity system

- **Causality**

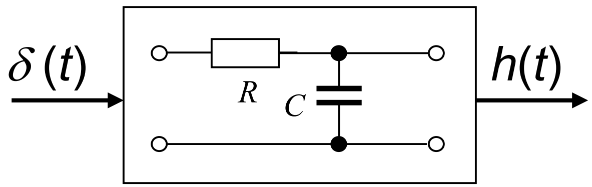
  $$h(t) = 0 \quad \text{for} \quad t < 0$$

- **Stability**

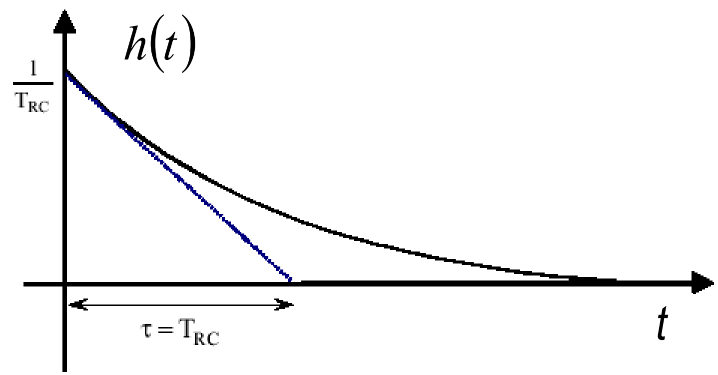
  $$\int_{-\infty}^{\infty} \left| h(t) \right| \mathrm{d}\,t < \infty$$ → Absolutely integrable

Example

1st order *RC* lowpass stimulated by unit impulse

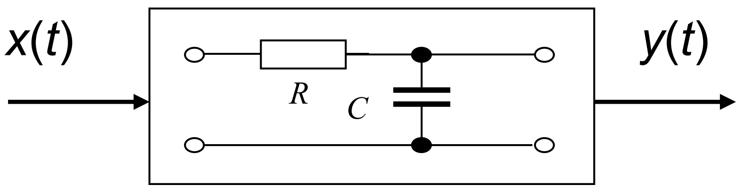


$$h(t) = u(t) \cdot \frac{1}{\tau} \cdot \mathrm{e}^{-\frac{t}{\tau}} \,, \quad \tau = T_{RC} = RC$$

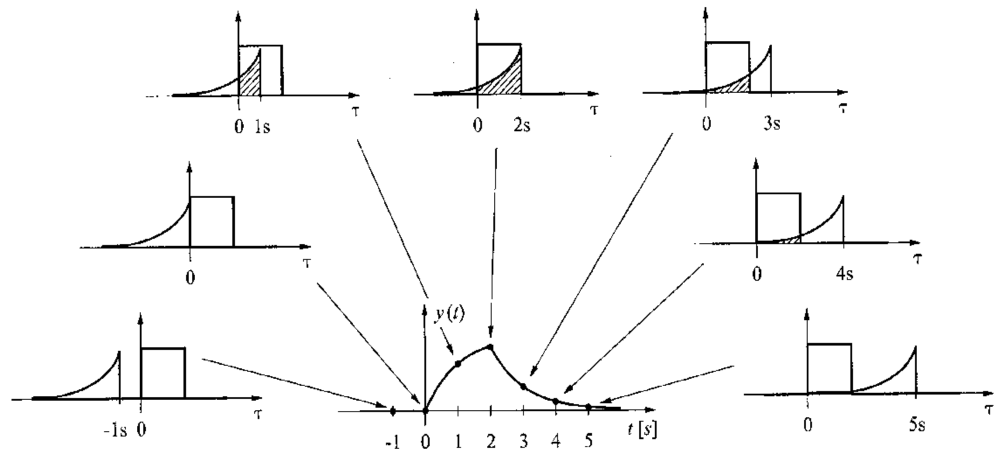


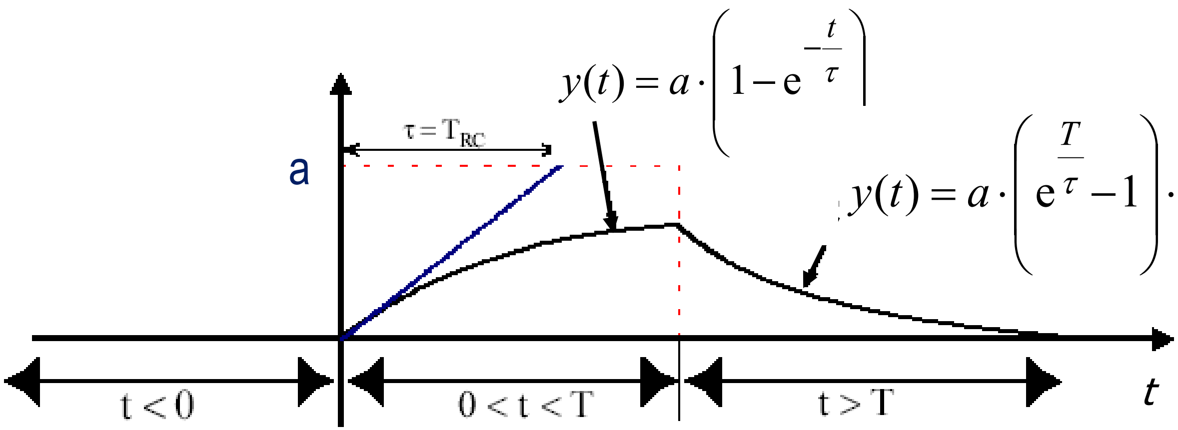
Example

1st order *RC* lowpass stimulated by rect impulse

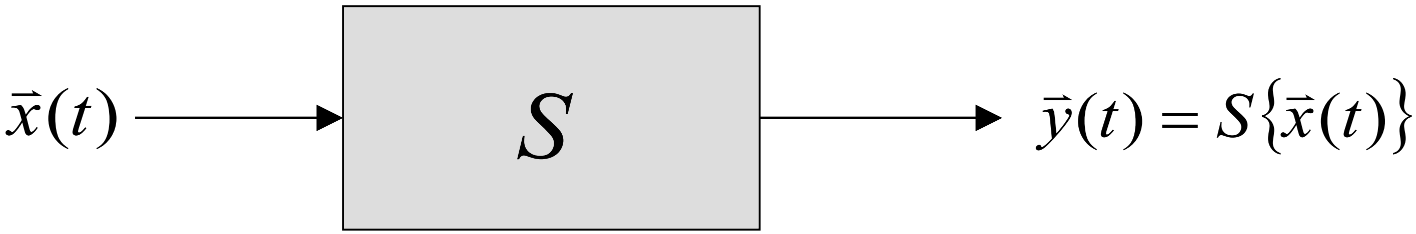


$$y(t) = \mathrm{rect}\left(\frac{t - T/2}{T}\right) * h(t)$$





3.2.2 Differential Equations



- Input and output of a system are related through a **differential equation**

$$\vec{y}(t) = f\left\{\frac{\mathrm{d}\,\vec{y}(t)}{\mathrm{d}\,t}, \frac{\mathrm{d}^2\,\vec{y}(t)}{\mathrm{d}\,t^2}, \ldots, \vec{x}(t), \frac{\mathrm{d}\,\vec{x}(t)}{\mathrm{d}\,t}, \frac{\mathrm{d}^2\,\vec{x}(t)}{\mathrm{d}\,t^2}, \ldots, t\right\}$$

$$x(t) \longrightarrow \boxed{\mathrm{LTI}} \longrightarrow y(t) = S\{x(t)\}$$

- Input and output of a LTI system are related through a **linear differential equation with constant coefficients**. Example: SISO

$$a_0 \cdot y + a_1 \cdot \frac{\mathrm{d}\, y}{\mathrm{d}\, t} + \ldots + a_n \cdot \frac{\mathrm{d}^n\, y}{\mathrm{d}\, t^n} = b_0 \cdot x + b_1 \cdot \frac{\mathrm{d}\, x}{\mathrm{d}\, t} + \ldots + b_m \cdot \frac{\mathrm{d}^m\, x}{\mathrm{d}\, t^m}$$

$$\sum_{k=0}^{n} a_k \cdot \frac{\mathrm{d}^k\, y(t)}{\mathrm{d}\, t^k} = \sum_{k=0}^{m} b_k \cdot \frac{\mathrm{d}^k\, x(t)}{\mathrm{d}\, t^k}$$

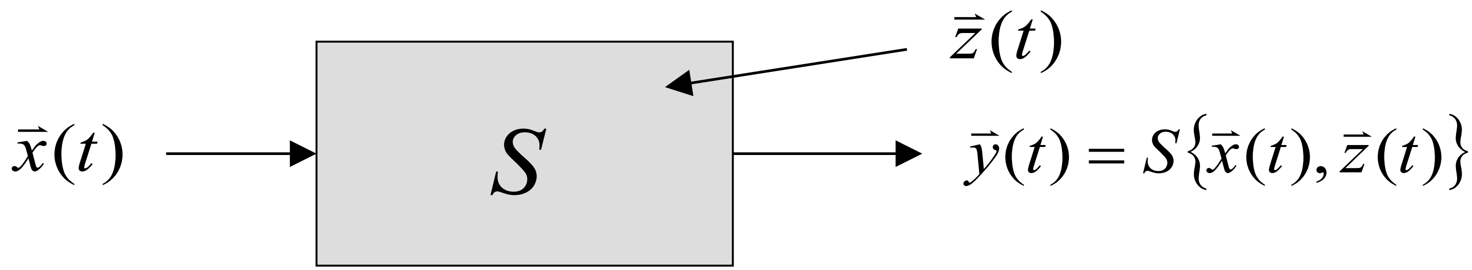
Auxiliary conditions are necessary to solve this equation for given $x(t)$.

- **Auxiliary condition**: If $x(t) = 0$ for $t < 0$ the condition of **initial rest** requires $y(t) = 0$ for $t < 0$. → **Causal system**
- General solution: $y(t) = y_\mathrm{h}(t) + y_\mathrm{p}(t)$
- **Homogeneous solution** $y_\mathrm{h}(t)$ solves

$$a_0 \cdot y + a_1 \cdot \frac{\mathrm{d}\,y}{\mathrm{d}\,t} + \ldots + a_n \cdot \frac{\mathrm{d}^n\,y}{\mathrm{d}\,t^n} = 0 \quad \text{with} \quad y_\mathrm{h}(t) = A \cdot \mathrm{e}^{pt}$$

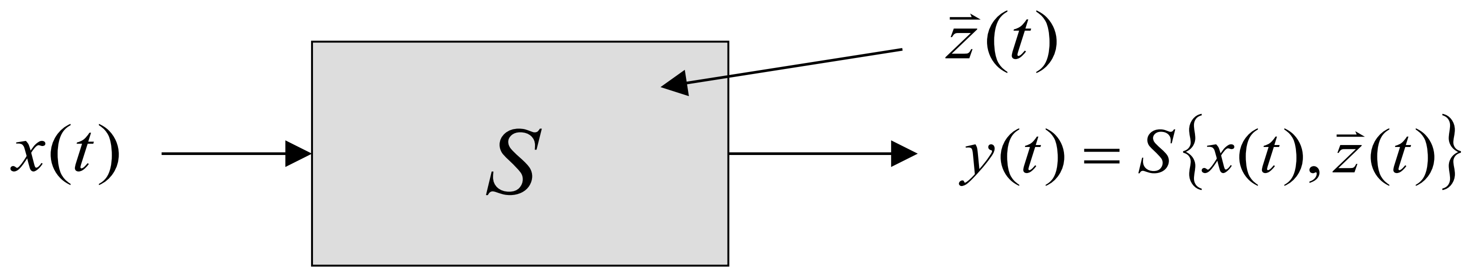
  - **Roots** of characteristic system equation: $p_1, p_2, \ldots, p_n$
  - Homogeneous solution is the natural response of the system.
- **Particular solution** $y_\mathrm{p}(t)$ depends on $x(t)$

3.2.3 State Space



- **State variables** $\vec{z}(t)$ as internal signals are additionally considered.
  - Inner system instabilities can be detected
  - Advantageous in control theory and for simulation purposes
  - Suitable for nonlinear systems
  - System order: Minimal number of required state variables
- **State equation** $$\frac{\mathrm{d}\,\vec{z}(t)}{\mathrm{d}\,t} = f\{\vec{z}(t), \vec{x}(t), t\}$$
- **Output equation** $$\vec{y}(t) = g\{\vec{z}(t), \vec{x}(t), t\}$$

Example: SISO LTI System



- **State equation**

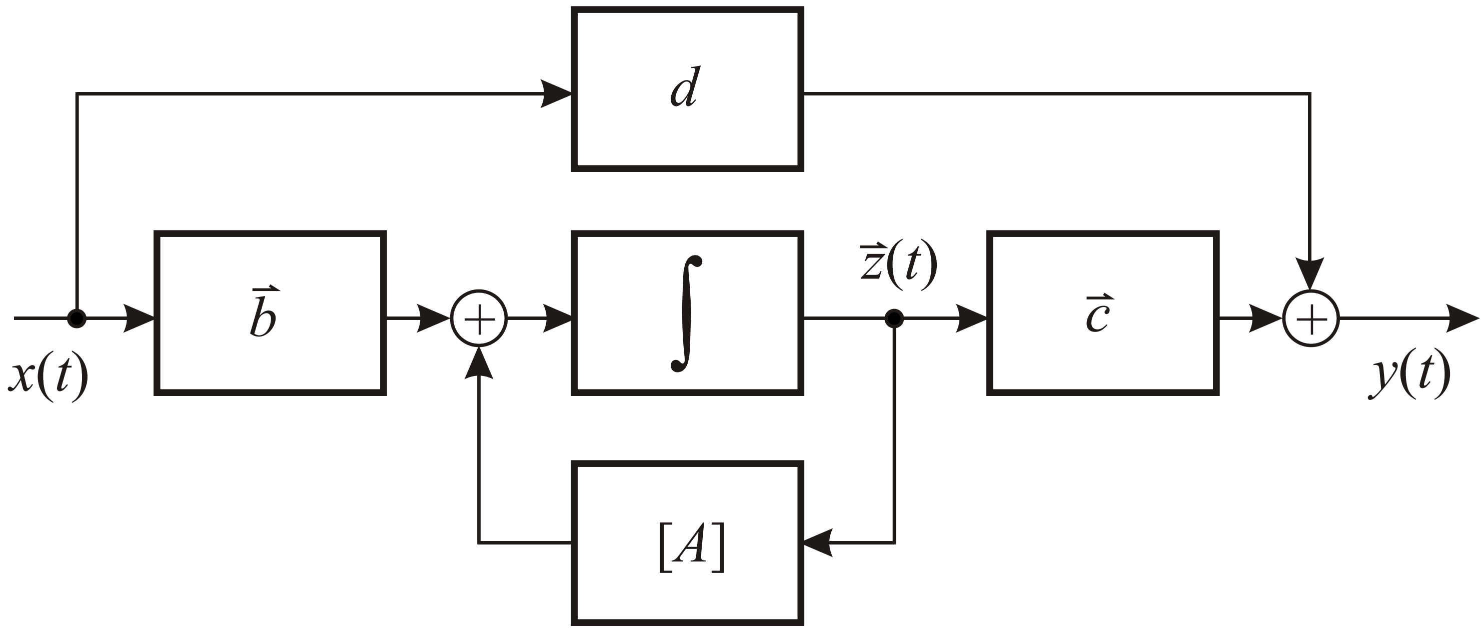
$$\frac{\mathrm{d}\,\vec{z}(t)}{\mathrm{d}\,t} = f\{\vec{z}(t), x(t), t\} = [A]\cdot\vec{z}(t) + \vec{b}\cdot x(t)$$

- **Output equation**

$$y(t) = g\{\vec{z}(t), x(t), t\} = \vec{c}^{\,\mathrm{T}}\cdot\vec{z}(t) + d\cdot x(t)$$

$\rightarrow$ Differential equation of order n is transformed into set of n first order differential equations.

Block Diagram of State-Space Representation (SISO)

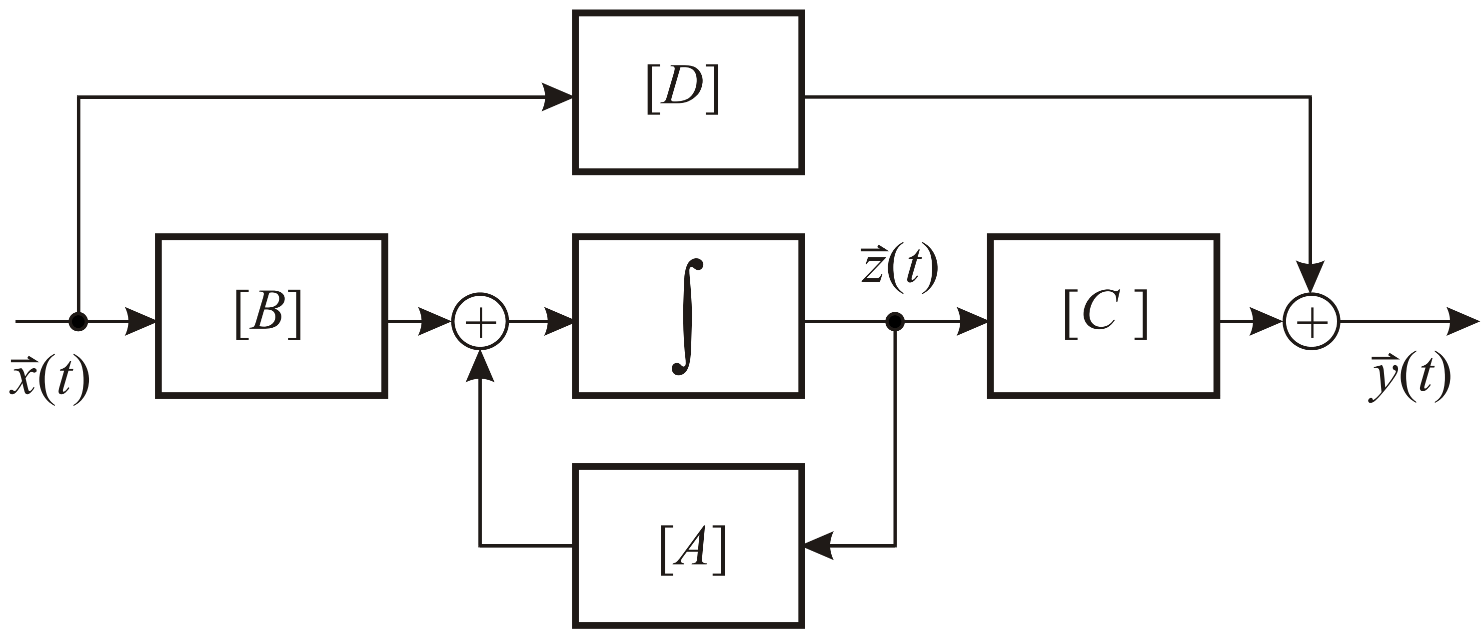


$[A]$ system matrix
$\vec{b}$ control vector
$\vec{c}$ monitor vector
d transition coefficient

$$\frac{\mathrm{d}\,\vec{z}(t)}{\mathrm{d}\,t} = [A] \cdot \vec{z}(t) + \vec{b} \cdot x(t)$$

$$y(t) = \vec{c}^{\,\mathrm{T}} \cdot \vec{z}(t) + d \cdot x(t)$$

Block Diagram of State-Space Representation (MIMO)



$[A]$ system matrix
$[B]$ control matrix
$[C]$ monitor matrix
$[D]$ transition matrix

$$\frac{\mathrm{d}\,\vec{z}(t)}{\mathrm{d}\,t} = [A]\cdot\vec{z}(t) + [B]\cdot\vec{x}(t)$$

$$\vec{y}(t) = [C]\cdot\vec{z}(t) + [D]\cdot\vec{x}(t)$$

3.3 Frequency-Domain Representation

$$x(t) \longrightarrow \boxed{\text{LTI}} \longrightarrow y(t) = S\{x(t)\}$$

- Stimulation with an **eigenfunction**

$$x(t) = A \cdot \mathrm{e}^{\mathrm{j}\omega t} \quad ; \quad y(t) = A \cdot H(\mathrm{j}\omega) \cdot \mathrm{e}^{\mathrm{j}\omega t}$$

- $H(\mathrm{j}\omega)$ is the **eigenvalue** or **frequency response**

$$h(t) \;\circ\!-\!\bullet\; H(\mathrm{j}\omega) = |H(\omega)| \cdot \mathrm{e}^{\mathrm{j}\varphi(\omega)}$$

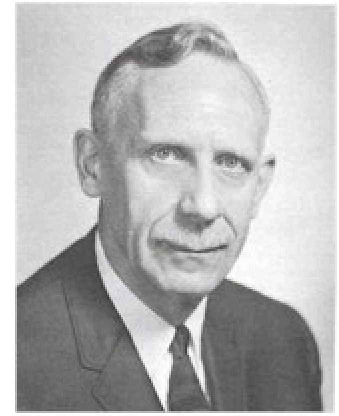
  - $|H(\omega)|$ is the **magnitude** (amplitude) **response**
  - $\varphi(\omega)$ is the **phase response**

- **Bode plot**

$$\left|H(\omega)\right|_{\mathrm{dB}} = 20\ \mathrm{dB} \cdot \log_{10}\left|H(\omega)\right| \text{ and } \varphi(\omega)$$

versus logarithmic frequency.

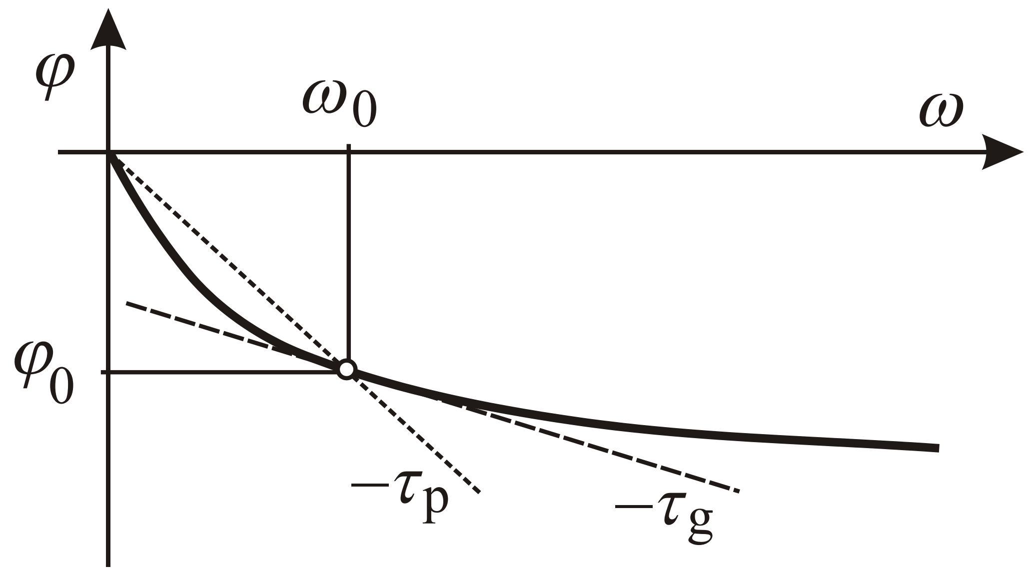
Hendrik Wade Bode
1905 - 1982



- **Phase and group delay**

$$\text{phase delay}\ \ \tau_{\mathrm{p}} = -\frac{\varphi}{\omega}$$

$$\text{group delay}\ \ \tau_{\mathrm{g}} = -\frac{\mathrm{d}\varphi}{\mathrm{d}\omega}$$



LTI System: Time domain versus frequency domain

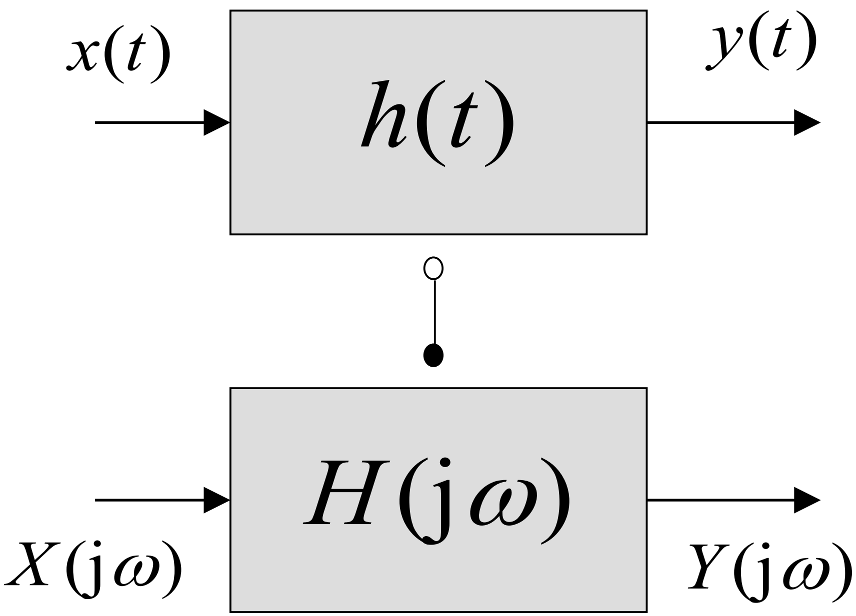
$$x(t) \longrightarrow \boxed{h(t)} \longrightarrow y(t) = x(t) * h(t)$$

$$\circ\!\!-\!\!\bullet \qquad \circ\!\!-\!\!\bullet \qquad \circ\!\!-\!\!\bullet$$

$$X(\mathrm{j}\omega) \longrightarrow \boxed{H(\mathrm{j}\omega)} \longrightarrow Y(\mathrm{j}\omega) = X(\mathrm{j}\omega) \cdot H(\mathrm{j}\omega)$$

$$H(\mathrm{j}\omega) = \frac{Y(\mathrm{j}\omega)}{X(\mathrm{j}\omega)}$$

LTI System: Time domain versus frequency domain

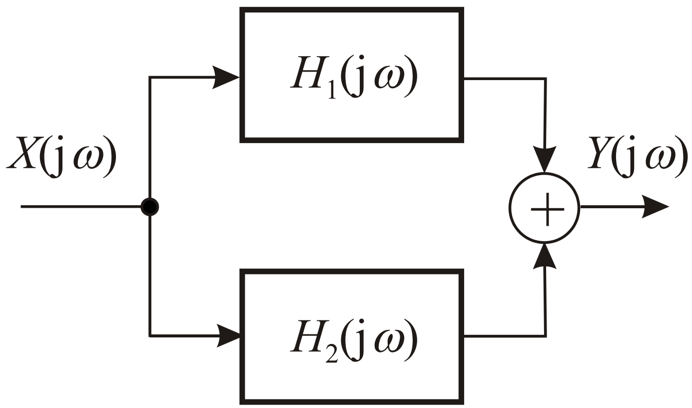


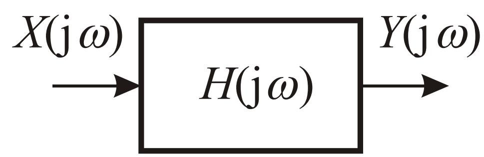
$$\sum_{k=0}^{n} a_k \cdot \frac{\mathrm{d}^k\, y(t)}{\mathrm{d}\,t^k} = \sum_{k=0}^{m} b_k \cdot \frac{\mathrm{d}^k\, x(t)}{\mathrm{d}\,t^k}$$

$$H(\mathrm{j}\omega) = \frac{Y(\mathrm{j}\omega)}{X(\mathrm{j}\omega)} = \frac{\sum_{k=0}^{m} b_k \cdot (\mathrm{j}\omega)^k}{\sum_{k=0}^{n} a_k \cdot (\mathrm{j}\omega)^k}$$

Interconnection of Systems

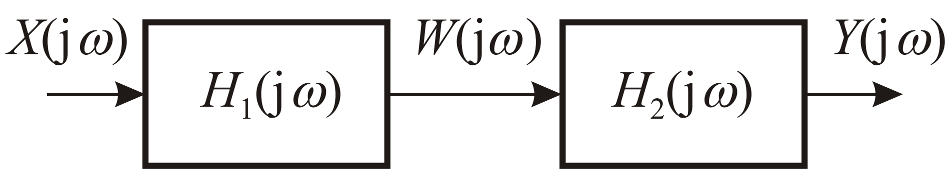
- Parallel connection

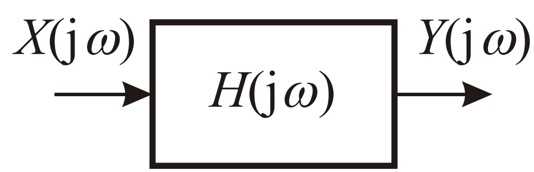




$$H(j\omega) = H_1(j\omega) + H_2(j\omega)$$

- Chain or cascade connection {sometimes 'serial connection' :-(}





$$H(j\omega) = H_1(j\omega) \cdot H_2(j\omega)$$

3.4 HILBERT Transformer

$$y(t) = \frac{1}{\pi} \cdot \int_{-\infty}^{\infty} \frac{x(\tau)}{t-\tau} \mathrm{d}\tau = x(t) * \frac{1}{\pi \cdot t}$$

$$h(t) = \frac{1}{\pi \cdot t} \quad \circ\!-\!\bullet \quad H(f) = \begin{cases} -\mathrm{j} \;, & f > 0 \\ \mathrm{j} \;, & f < 0 \end{cases} \;;\quad H(\mathrm{j}\omega) = \begin{cases} -\mathrm{j} \;, & \omega > 0 \\ \mathrm{j} \;, & \omega < 0 \end{cases}$$

- The HILBERT transformer is a non-causal frequency selective filter.
- $x(t)$ and $y(t)$ are orthogonal.

- The HILBERT transformer can be used to create **analytical signals**. If $x(t)$ is a real signal, the derived analytical signal is:

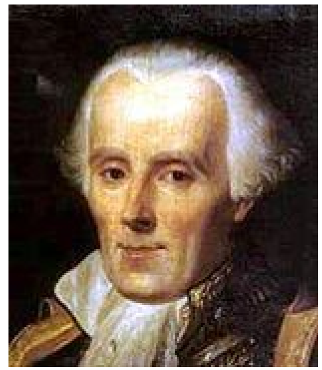
$$y(t) = x(t) + \mathrm{j} \cdot \mathrm{H}\{x(t)\}$$

- The analytical signal has only positive spectral components:

$$Y(f) = 0 \quad \text{for} \quad f < 0$$

3.5 LAPLACE Transform (LT)
3.5.1 Definitions and Convergence

- CTFT improve the signal characterization (harmonic content) and are well suited for LTI systems ($h(t) \circ\!\!-\!\!\bullet\ H(\mathrm{j}\omega)$).
- LT is a generalization of CTFT: CTFT ∈ LT
- LT improves and extents the system description
  - unstable systems can be handled
  - stability and instability can be investigated
  - better for feedback systems
  - simplify algebraic properties of CTFT
  - better convergence properties than CTFT



PIERRE-SIMON LAPLACE
1749 - 1827

- **Bilateral LT**

$$X(s) = \int_{-\infty}^{\infty} x(t) \cdot \mathrm{e}^{-st} \mathrm{d}t \, ; \; s = \sigma + \mathrm{j}\omega \qquad x(t) \circ\!-\!\bullet \; X(s) = \mathrm{L_B}\{x(t)\}$$

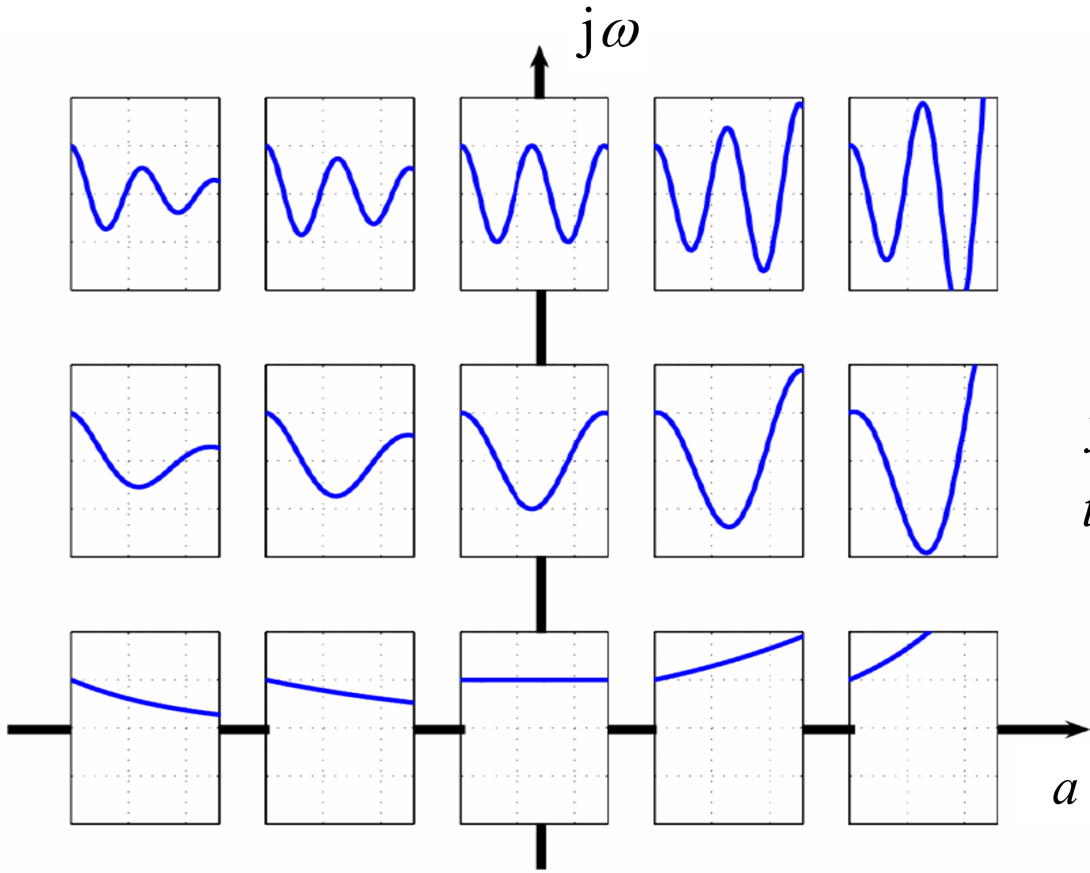
- **Unilateral LT** (causal LT)

$$X(s) = \int_{0_-}^{\infty} x(t) \cdot \mathrm{e}^{-st} \mathrm{d}t \, ; \; s = \sigma + \mathrm{j}\omega \qquad x(t) \circ\!-\!\bullet \; X(s) = \mathrm{L_U}\{x(t)\}$$

- The unilateral LT is identical to the bilateral LT for causal signals ($x = 0$ for $t < 0$)
- Without further notice: In the context of this course we will only consider the unilateral LT:

$$X(s) = \mathrm{L_U}\{x(t)\} = \mathrm{L}\{x(t)\}$$

- **s-Plane**



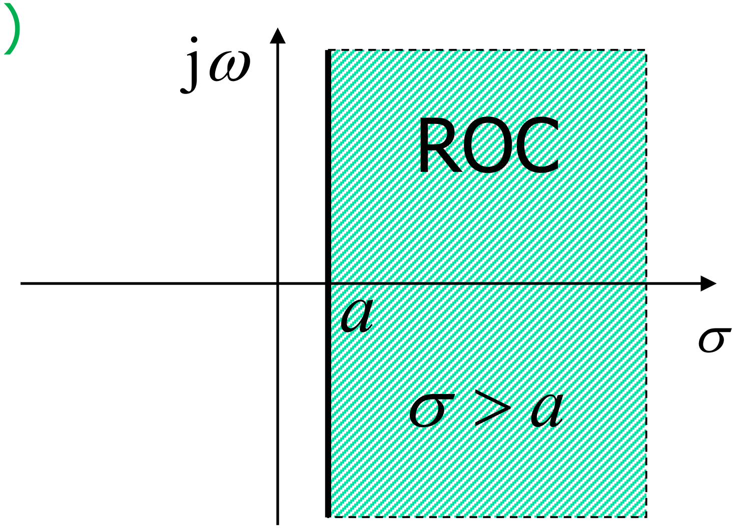
$$y(t) = u(t) \cdot \cos(\omega t) \cdot e^{a \cdot t}$$
$$t \geq 0$$

- **Inverse LT**

$$x(t)=\frac{1}{2\pi\,\mathrm{j}}\cdot\int_{\sigma-\mathrm{j}\infty}^{\sigma+\mathrm{j}\infty} X(s)\cdot \mathrm{e}^{st}\,\mathrm{d}s\ ;\ \ s=\sigma+\mathrm{j}\omega\in\mathrm{ROC}$$

$$X(s)\ \bullet\!-\!\circ\ \ x(t)=\mathrm{L}^{-1}\{X(s)\}$$

- **ROC: region of convergence**, depends on $x(t)$
- The integration operation can be avoided in most of the cases when $X(s)$ is a rational polynomial. :-)



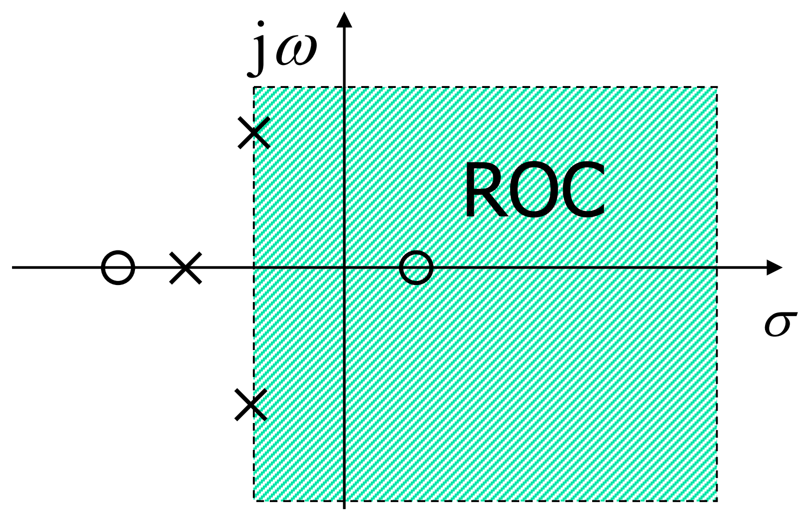
3.5.2 Properties of the LAPLACE Transform

- $X(s)$ is rational, whenever $x(t)$ is a linear combination of real or complex exponentials.

$$X(s) = \frac{N(s)}{D(s)} = \frac{(s-z_1)\cdot(s-z_2)\cdots}{(s-p_1)\cdot(s-p_2)\cdots}$$

  - The roots z_i of $N(s)$ are referred to as the finite **zeros** z_i of $X(s)$
  - The roots p_i of $D(s)$ are referred to as the finite **poles** p_i of $X(s)$

pole-zero plot
finite zeros: ○
finite poles: ×



- If $X(s)$ has real coefficients: zeros and poles are real or conjugate complex
- If {order of $N(s)$} < {order of $D(s)$}: $X(s)$ has *additional* zeros at infinity
- If {order of $N(s)$} > {order of $D(s)$}: $X(s)$ has *additional* poles at infinity

- **A complete specification of the LAPLACE transform requires $\boldsymbol{X(s)}$ and additionally the associated ROC.**

- For rational $X(s)$ the ROC doesn't contain any poles.
- If $x(t)$ is of finite duration and is absolutely integrable, then the ROC is the entire s-plane.

3.5.3 Theorems of the Unilateral LAPLACE Transform

$$x(t) \circ\!-\!\bullet\; X(s)\,;\;\; \mathrm{ROC} = R$$

- Linearity

$$a \cdot x(t) + b \cdot y(t) \circ\!-\!\bullet\; a \cdot X(s) + b \cdot Y(s)\,;\;\; \mathrm{ROC} = R_1 \cap R_2$$

- Time shifting

$$x(t - t_0) \circ\!-\!\bullet\; \mathrm{e}^{-st_0} \cdot X(s)\,;\;\; \mathrm{ROC} = R$$

- Shifting in the s-domain

$$\mathrm{e}^{s_0 t} \cdot x(t) \circ\!-\!\bullet\; X(s - s_0)\,;\;\; \mathrm{ROC} = R + \mathrm{Re}\{s_0\}$$

- Time and frequency scaling

$$x(a \cdot t) \circ\!-\!\bullet \frac{1}{|a|} \cdot X\left(\frac{s}{a}\right);\ \mathrm{ROC} = \frac{R}{a}$$

- Differentiation

$$\frac{\mathrm{d}\,x(t)}{\mathrm{d}\,t} \circ\!-\!\bullet\ s \cdot X(s) - x(0\_)\,;\ R \in \mathrm{ROC}\ (\mathrm{ROC\ contains}\ R)$$

$$\frac{\mathrm{d}^n}{\mathrm{d}\,t^n} x(t) \quad \circ\!-\!\bullet \quad s^n \cdot X(s) - \sum_{i=1}^{n} s^{n-i} x^{(i-1)}(0\_)$$

- Integration

$$\int_0^t x(\tau)\,\mathrm{d}\,\tau \circ\!-\!\bullet \frac{1}{s} \cdot X(s);\ (R \cap \sigma > 0) \in \mathrm{ROC}$$

- Convolution

$$x(t) * y(t) \circ\!-\!\bullet\; X(s) \cdot Y(s);\; (R_1 \cap R_2) \in \mathrm{ROC}$$

- Multiplication

$$x(t) \cdot y(t) \circ\!-\!\bullet\; X(s) * Y(s)\;;\; (R_1 \cap R_2) \in \mathrm{ROC}$$

- Conjugation

$$x^*(t) \circ\!-\!\bullet\; X^*(s^*)\;;\; \mathrm{ROC} = R$$

- Initial- and final-value theorems

1) $\lim\limits_{t \to 0_+} x(t) = \lim\limits_{s \to \infty} s \cdot X(s)$

2) $\lim\limits_{t \to \infty} x(t) = \lim\limits_{s \to 0} s \cdot X(s)$

Note on 2) for $X(s)$: Only single pole at $s = 0$ allowed. No poles in the right half plane.

3.5.4 Basic LAPLACE Transform Pairs

- **Unit impulse function - DIRAC impulse**

$$x(t) = \delta(t) \quad \circ\!-\!\bullet \quad X(s) = 1 \; ; \; \sigma \text{ arbitrary}$$

- **Unit step function**

$$x(t) = u(t) \quad \circ\!-\!\bullet \quad X(s) = \frac{1}{s} \; ; \quad \sigma > 0$$

- **Exponentially weighted unit step function**

$$x(t) = u(t) \cdot \mathrm{e}^{-at} \quad \circ\!-\!\bullet \quad X(s) = \frac{1}{s+a} \; ; \quad \sigma > -a$$

- **Switched cosine signal**

$$x(t) = u(t) \cdot \cos\left(\omega_0 t\right) \quad \circ\!-\!\bullet \quad X(s) = \frac{s}{s^2 + \omega_0^2} \;;\; \sigma > 0$$

- **Switched sine signal**

$$x(t) = u(t) \cdot \sin\left(\omega_0 t\right) \quad \circ\!-\!\bullet \quad X(s) = \frac{\omega_0}{s^2 + \omega_0^2} \;;\; \sigma > 0$$

3.5.5 Inverse LAPLACE Transform of Rational Functions

- Apply **partial-fraction expansion**
- Assumption: {order $N(s)$} < {order $D(s)$} $\rightarrow$ **proper fraction**
- Single poles

$$X(s)=\frac{N(s)}{D(s)}=\frac{A_1}{(s-p_1)}+\frac{A_2}{(s-p_2)}+\ldots+\frac{A_n}{(s-p_n)}$$

- Multiple poles

$$X(s)=\frac{N(s)}{D(s)}=\frac{A_1}{(s-p_i)}+\frac{A_2}{(s-p_i)^2}+\ldots+\frac{A_k}{(s-p_i)^k}+\widetilde{X}(s)$$

Pole p_i has multiplicity k

only single poles

Residual calculus

- Single poles

$$A_j = \lim_{s \to p_j} \left\{ (s - p_j) \cdot X(s) \right\}$$

- Multiple poles

$$A_j = \lim_{s \to p_i} \frac{1}{(k-j)!} \left[\frac{\mathrm{d}^{k-j}}{\mathrm{d}\, s^{k-j}} \left((s - p_i)^k \cdot X(s) \right) \right], \quad j = 1 \ldots k$$

Linear set of equations (recommended by Dorota)

Example

$$X(s) = \frac{N(s)}{(s-p_1)^2 \cdot (s-p_2)} = \frac{A_1}{(s-p_1)} + \frac{A_2}{(s-p_1)^2} + \frac{A_3}{(s-p_2)}$$

$$N(s) = A_1 \cdot (s-p_1) \cdot (s-p_2) + A_2 \cdot (s-p_2) + A_3 \cdot (s-p_1)^2$$

3.6 s-Domain Representation of LTI Systems
3.6.1 System Function

- Linear differential equation with constant coefficients.

$$a_0 \cdot y + a_1 \cdot \frac{\mathrm{d}\,y}{\mathrm{d}\,t} + \ldots + a_n \cdot \frac{\mathrm{d}^n\,y}{\mathrm{d}\,t^n} = b_0 \cdot x + b_1 \cdot \frac{\mathrm{d}\,x}{\mathrm{d}\,t} + \ldots + b_m \cdot \frac{\mathrm{d}^m\,x}{\mathrm{d}\,t^n}$$

- Auxiliary condition for causal LTI systems

$$x(0\_) = y(0\_) = 0 \;\; ; \;\; x^{(k)}(0\_) = y^{(k)}(0\_) = 0$$

- Apply LAPLACE transform

$$H(s) = \frac{Y(s)}{X(s)} = \frac{b_0 + b_1 s + b_2 s^2 + \ldots + b_m s^m}{a_0 + a_1 s + a_2 s^2 + \ldots + a_n s^n} = \frac{N_m(s)}{D_n(s)} \qquad m \le n$$

$H(s)$ is the **system function** or **transfer function**

3.6.2 State-Space Representation in the *s*-Domain

- Time domain

$$\frac{\mathrm{d}\,\vec{z}(t)}{\mathrm{d}\,t} = [A]\cdot\vec{z}(t) + [B]\cdot\vec{x}(t)$$

$$\vec{y}(t) = [C]\cdot\vec{z}(t) + [D]\cdot\vec{x}(t)$$

- *s*-domain ($\vec{x}(t)$ causal, condition of initial rest)

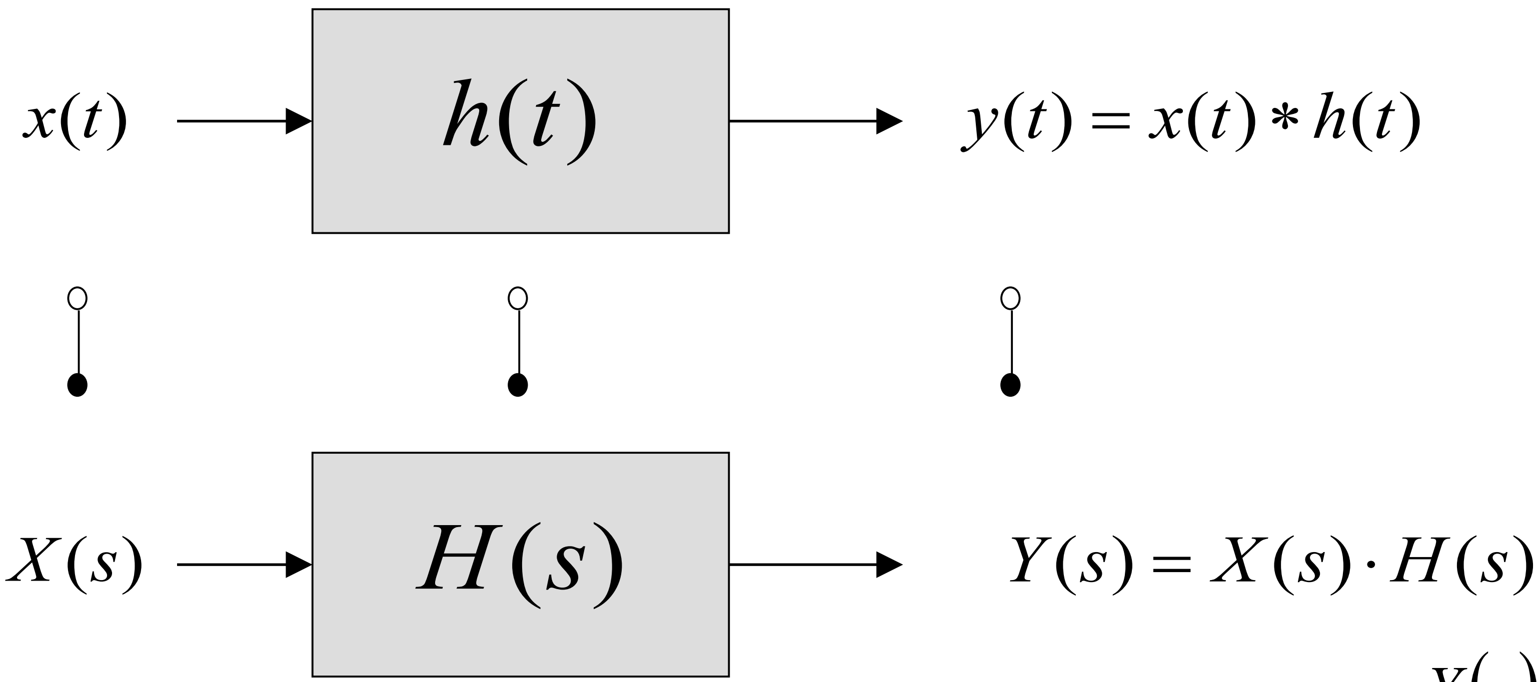
$$\vec{Y}(s) = [H(s)]\cdot\vec{X}(s)$$

$$[H(s)] = [C]\cdot\{s\cdot[I]-[A]\}^{-1}\cdot[B]+[D]$$

$[H(s)]$ is the **system matrix** or **transfer matrix**

$[I]$ is the unity matrix

3.6.3 Time Domain Versus s-Domain of LTI Systems



$$H(s) = \frac{Y(s)}{X(s)}$$

The **frequency response** $H(j\omega)$ can be derived from the **system function** with $s = j\omega$.

- **Causality**

$$h(t) = 0 \quad \text{for} \quad t < 0$$

  - The ROC of $H(s)$ is a right-half plane.
  - If $H(s)$ is a rational function, the ROC is the right-half plane to the right of the rightmost pole.

- **Stability**

$$\int_{-\infty}^{\infty} \left| h(t) \right| \mathrm{d}\, t < \infty$$

  - An LTI system is stable if all poles of its system function $H(s)$ are in the left-half plane, i.e. $\mathrm{Re}\{p_k\} < 0$
  - Degree m of the numerator of $H(s) \leq$ degree n of the denominator of $H(s)$.

3.6.4 Frequency Response Derived from Pole-Zero Plot

- LTI system with rational transfer function

$$H(s)=\frac{b_0+b_1\cdot s+b_2\cdot s^2+\ldots b_m\cdot s^m}{a_0+a_1\cdot s+a_2\cdot s^2+\ldots a_n\cdot s^n} \qquad m\le n$$

$$H(s)=H_0\cdot\frac{(s-z_1)\cdot(s-z_2)\cdots}{(s-p_1)\cdot(s-p_2)\cdots}=H_0\cdot\frac{\prod_{i=1}^{m}(s-z_i)}{\prod_{i=1}^{n}(s-p_i)} \qquad H_0=\frac{b_m}{a_n}$$

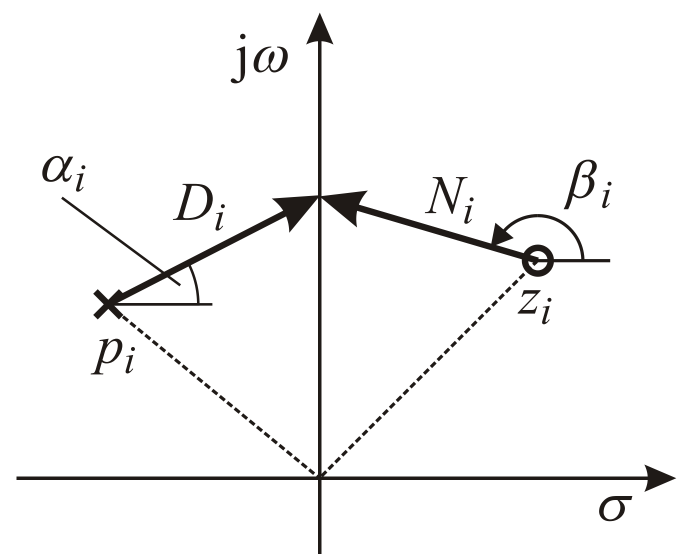
  - m finite zeros at $s=z_i$
  - n finite poles at $s=p_i$
  - $n-m$ additional zeros at infinity
  - H_0: gain constant

- The frequency response of a stable LTI system is obtained for

$$s = \mathrm{j}\omega$$

$$H(\mathrm{j}\omega) = H_0 \cdot \frac{\prod_{i=1}^{m} (\mathrm{j}\omega - z_i)}{\prod_{i=1}^{n} (\mathrm{j}\omega - p_i)} = H_0 \cdot \frac{\prod_{i=1}^{m} N_i \cdot \mathrm{e}^{\mathrm{j}\beta_i}}{\prod_{i=1}^{n} D_i \cdot \mathrm{e}^{\mathrm{j}\alpha_i}}$$

  - Zero close to or on $\mathrm{j}\omega$ axis:
    Attenuate signal components
  - Pole close to $\mathrm{j}\omega$ axis:
    Emphasize signal components



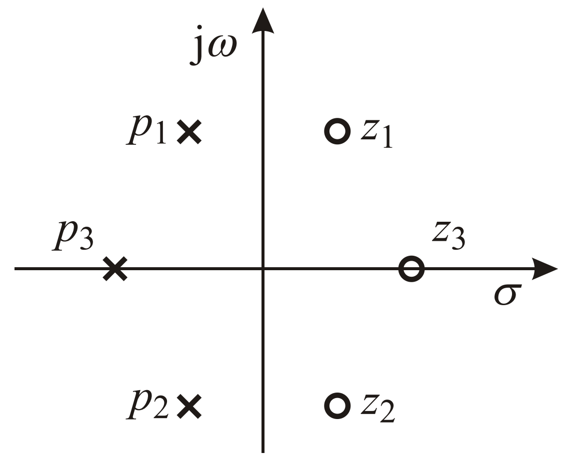
3.6.5 Allpass and Minimum-Phase Transfer Function

- **Allpass transfer function**

$$\left|H_{\mathrm{A}}(\omega)\right| = 1$$

$$H_{\mathrm{A}}(s) = \frac{\prod_{i=1}^{m}(s - z_i)}{\prod_{i=1}^{n}(s - p_i)} \;;\; m = n \;;\; z_i = -p_i^*$$

$$H_{\mathrm{A}}(s) = \frac{N_n(s)}{D_n(s)} = (-1)^n \cdot \frac{D_n(-s)}{D_n(s)}$$



real coefficients of H_{A}

$D_n(s)$ and $N_n(s)$ are mirror-image polynomials.

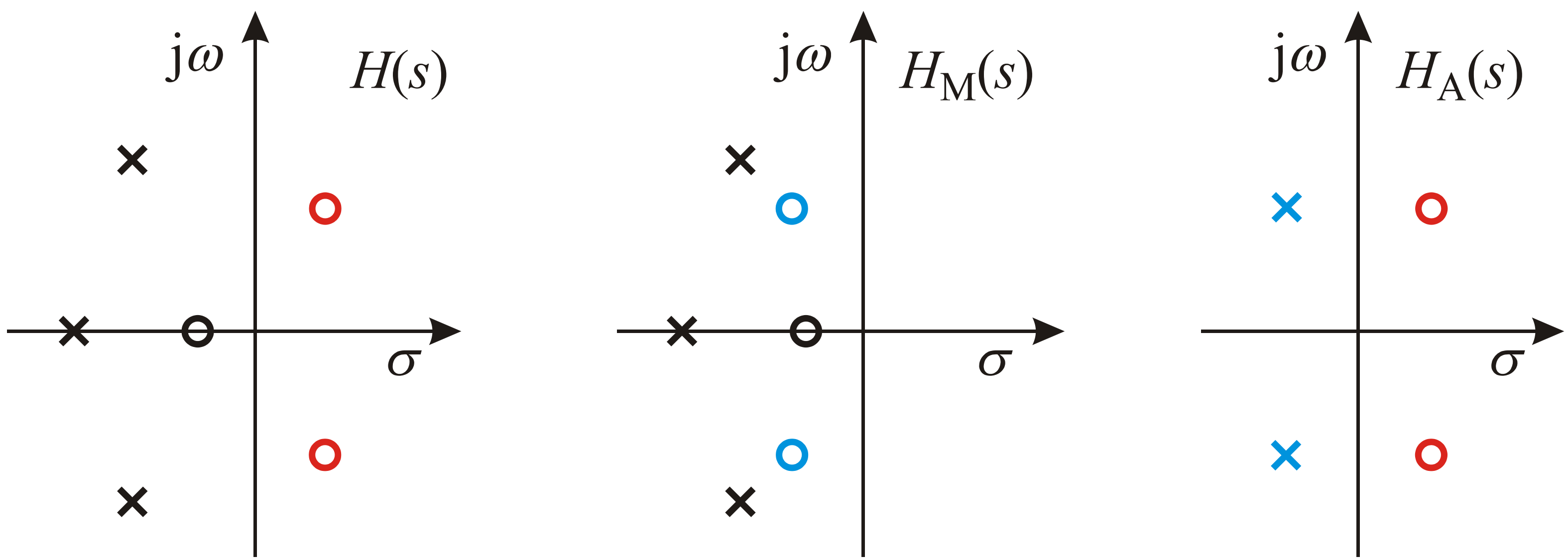
- **Minimum-phase transfer function**

A causal stable rational transfer function with $\mathrm{Re}\{z_i\} \leq 0$ is called a minimum-phase transfer function $H_{\mathrm{M}}(s)$.

- A transfer function with zeros left and right of the $j\omega$ axis is called a **mixed-phase transfer function**.

- A causal stable LTI system can always be splitted into an allpass and a minimum-phase system.

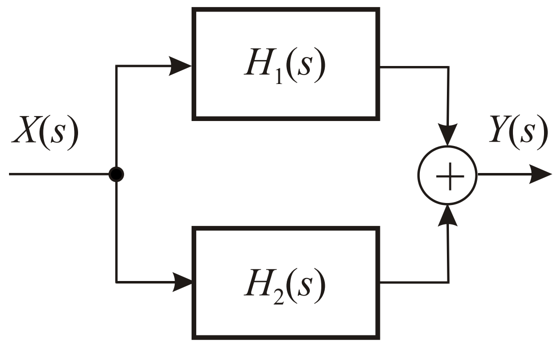
$$H(s) = H_{\mathrm{M}}(s) \cdot H_{\mathrm{A}}(s)$$



3.7 Block Diagram Representations

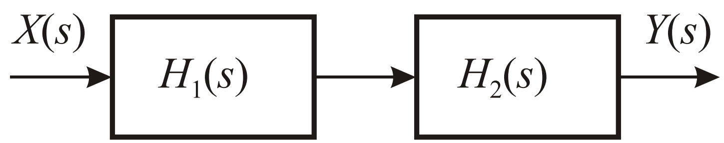
- Parallel connection

$$H(s) = H_1(s) + H_2(s)$$



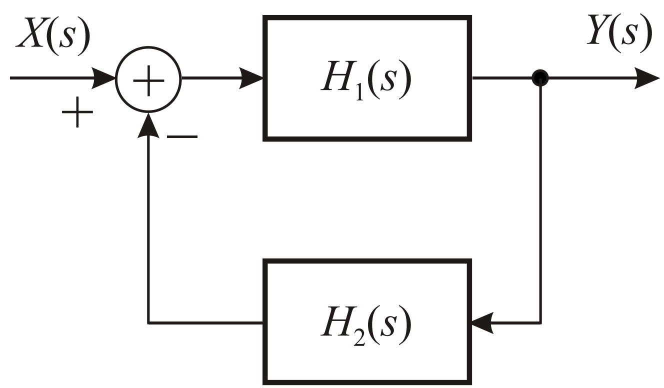
- Cascade (chain) connection

$$H(s) = H_1(s) \cdot H_2(s)$$



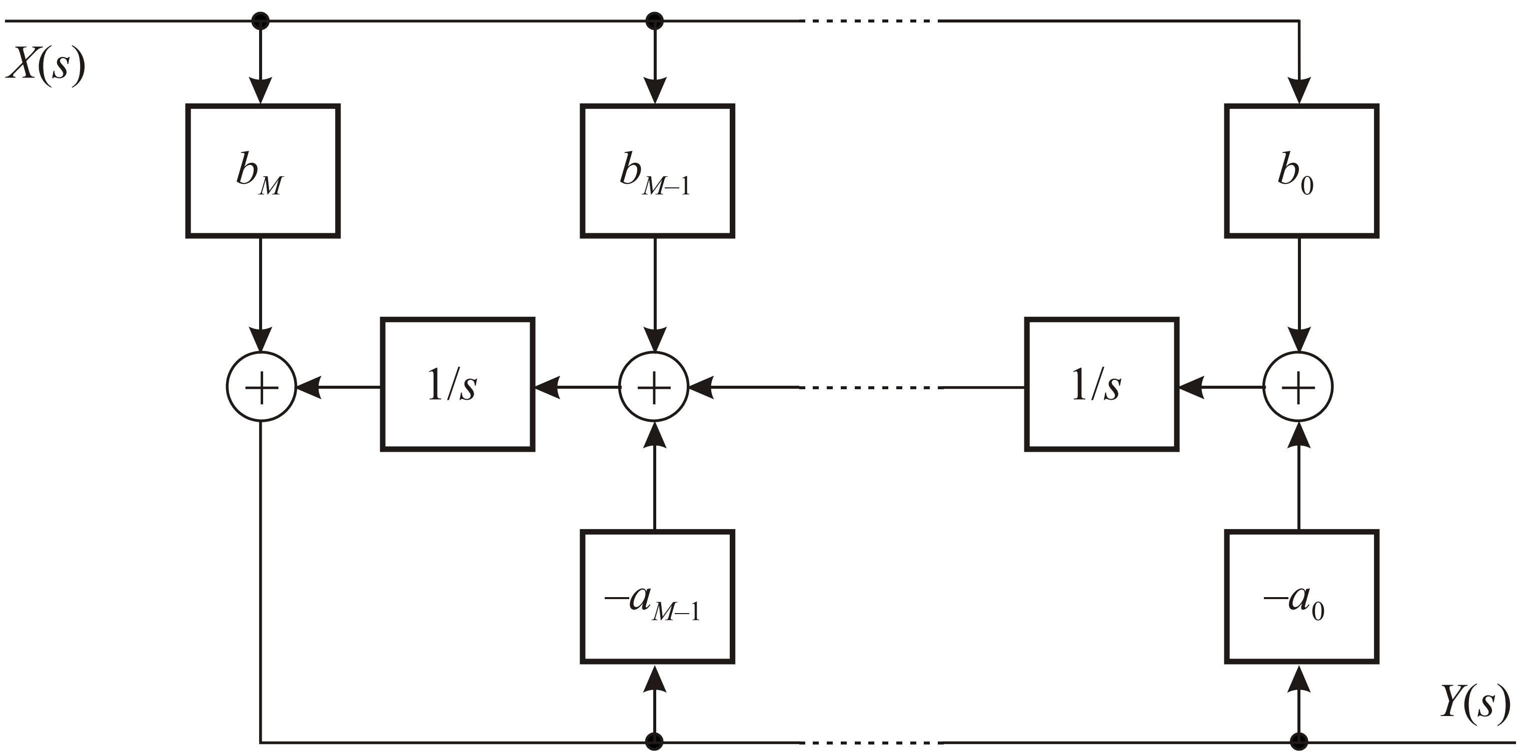
- Feedback connection

$$H(s) = \frac{H_1(s)}{1 + H_1(s) \cdot H_2(s)}$$



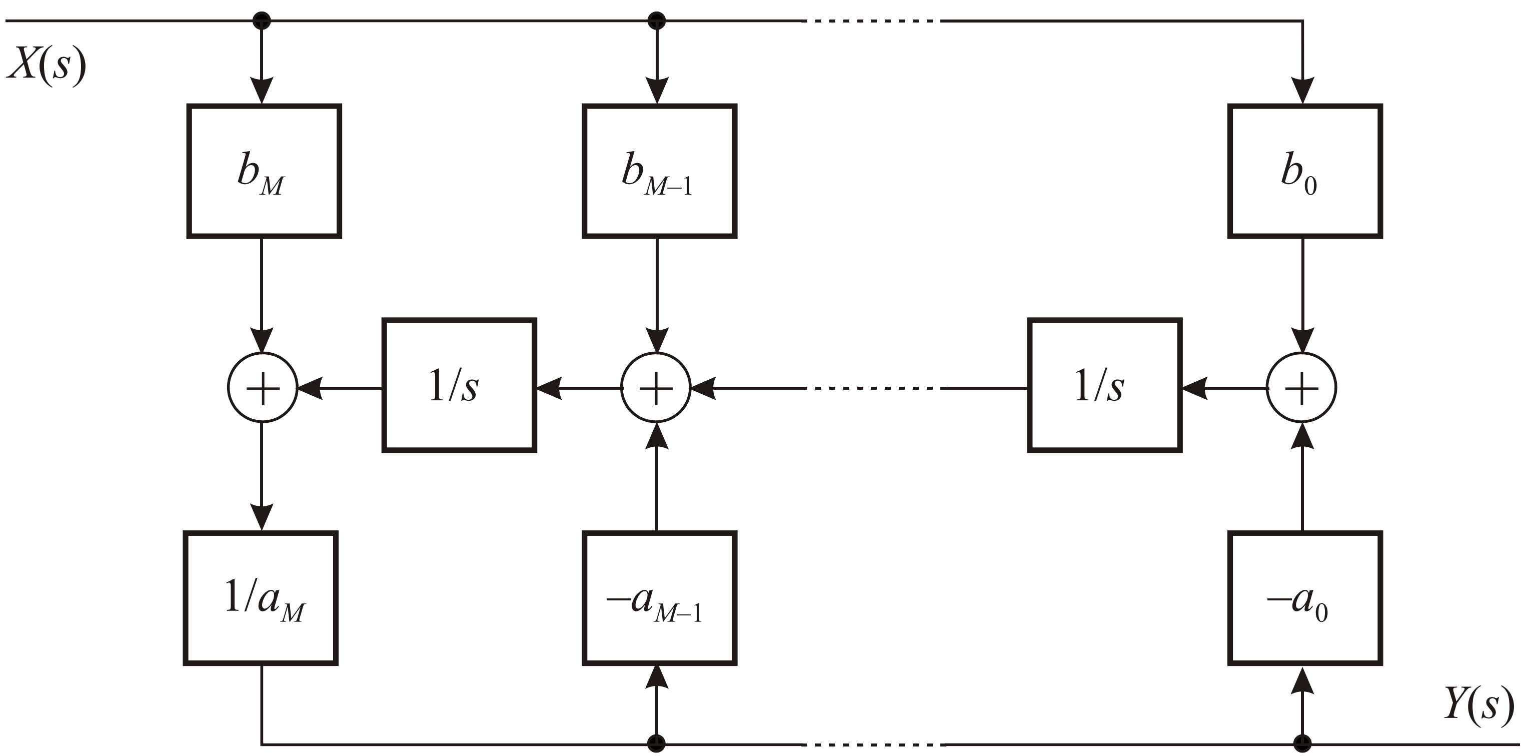
ARMA Topology (auto regressive, moving average)

$$H(s) = \frac{Y(s)}{X(s)} = \frac{N_m(s)}{D_n(s)} = \frac{b_0 + b_1 s + b_2 s^2 + \ldots + b_M s^M}{a_0 + a_1 s + a_2 s^2 + \ldots + s^M} \qquad M = \max\{m, n\}$$



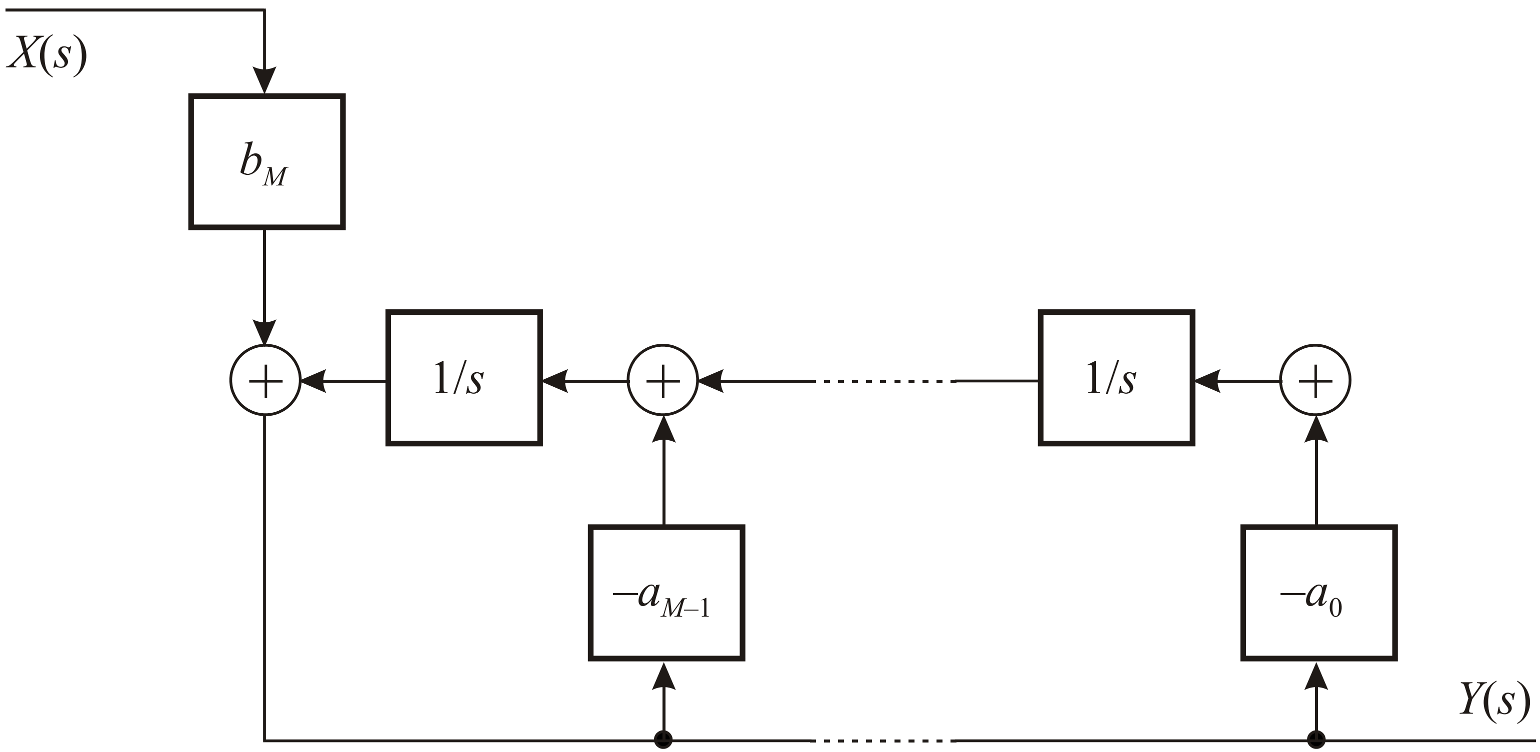
ARMA Topology (auto regressive, moving average)

$$H(s)=\frac{Y(s)}{X(s)}=\frac{N_m(s)}{D_n(s)}=\frac{b_0+b_1 s+b_2 s^2+\ldots+b_M s^M}{a_0+a_1 s+a_2 s^2+\ldots+a_M s^M} \qquad M=\max\{m,n\}$$



AR Topology (auto regressive)

$$H(s) = \frac{Y(s)}{X(s)} = \frac{N_m(s)}{D_n(s)} = \frac{b_M s^M}{a_0 + a_1 s + a_2 s^2 + \ldots + s^M}$$



MA Topology (moving average)

$$H(s)=\frac{Y(s)}{X(s)}=\frac{N_m(s)}{D_n(s)}=\frac{b_0+b_1s+b_2s^2+\ldots+b_Ms^M}{s^M}$$