Probability and Statistics

4 – Continuous Random Variables

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Standard Normal Distribution

Definition (4.50)

The normal distribution $\mathcal{N}(0,1)$ is called the *standard normal distribution*. Its pdf and cdf are denoted by

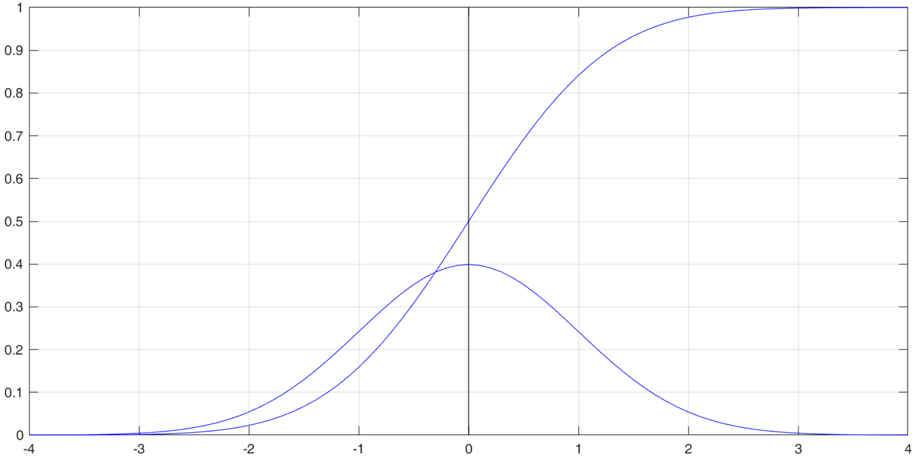
$$\varphi(x) = \frac{1}{\sqrt{2\pi}} \cdot e^{-\frac{1}{2}x^2}$$

and

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \cdot \int_{-\infty}^{x} e^{-\frac{1}{2}u^2}\, du$$

respectively.

Standard Normal Distribution



Normal Distributions

Lemma (4.51)

If $X \sim \mathcal{N}(\mu, \sigma)$*, then:*

(i) $E(X) = \mu$

(ii) $Var(X) = \sigma^2$

(iii) $\phi_X(t) = e^{\mu t + \frac{1}{2}\sigma^2 t^2}$ $\quad$ *(for all* $t \in \mathbb{R}$*)*

(iv) $aX + b \sim \mathcal{N}(a\mu + b, |a| \cdot \sigma)$ $\quad$ *if* $a \neq 0$

(v) $F_X(x) = \Phi\left(\frac{x - \mu}{\sigma}\right)$

(vi) $\Phi(-x) = 1 - \Phi(x)$ $\quad$ *for all* $x \in \mathbb{R}$

(vii) $\Phi^{-1}(p) = -\Phi^{-1}(1 - p)$ $\quad$ *for all* $p \in (0, 1)$

$X \sim \mathcal{N}(\mu, \sigma)$

Proof of (4.51)(iv): $aX + b \sim \mathcal{N}(a\mu + b, |a| \cdot \sigma)$ if $a \neq 0$

$Y = aX + b$

Exc. 6-2: $f_Y(x) = \frac{1}{|a|} f_X\left(\frac{x-b}{a}\right) = \frac{1}{|a|} \frac{1}{\sqrt{2\pi} \cdot \sigma} \cdot e^{-\frac{1}{2}\left(\frac{\frac{x-b}{a} - \mu}{\sigma}\right)^2}$

$\frac{x - b - a\mu}{a \cdot \sigma}$

$$= \underbrace{\frac{1}{\sqrt{2\pi} \cdot |a| \cdot \sigma} \cdot e^{-\frac{1}{2}\left(\frac{x - (b + a\mu)}{|a| \cdot \sigma}\right)^2}}_{\text{pdf of } \mathcal{N}(a\mu + b,\ |a| \cdot \sigma)}$$

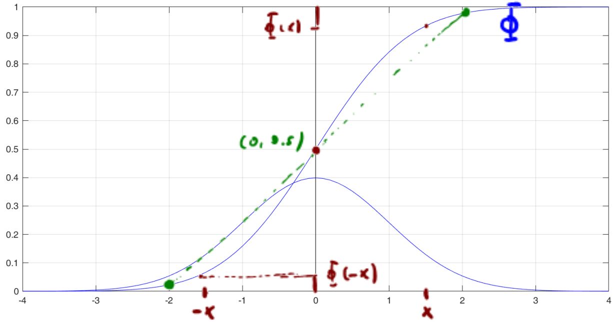
Proof of (4.51)(v): $F_X(x) = \Phi\left(\frac{x-\mu}{\sigma}\right)$

$X \sim \mathcal{N}(\mu, \sigma)$

$$F_X(x) = \Pr(X \le x) = \Pr\left(\underbrace{\frac{X-\mu}{\sigma}}_{Y} \le \frac{x-\mu}{\sigma}\right) = \Phi\left(\frac{x-\mu}{\sigma}\right)$$

(4.51)(iv) $\quad Y = \underbrace{\frac{1}{\sigma}}_{a} \cdot X - \underbrace{\frac{\mu}{\sigma}}_{b} \sim \mathcal{N}\left(\frac{1}{\sigma}\cdot\mu - \frac{\mu}{\sigma}, \frac{1}{\sigma}\cdot\sigma\right) = \mathcal{N}(0,1)$

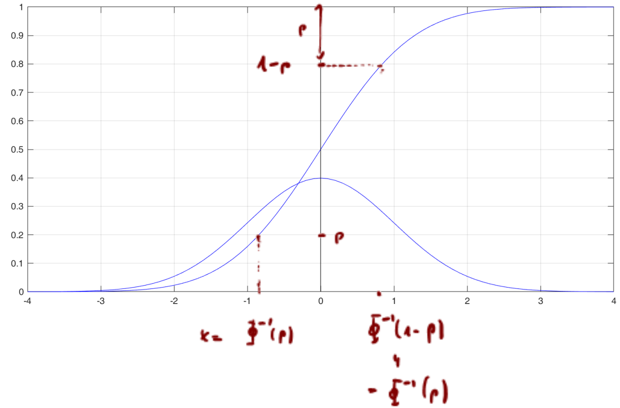
Proof of (4.51)(vi): $\Phi(-x) = 1 - \Phi(x)$ for all $x \in \mathbb{R}$



formal proof: $X \sim \mathcal{N}(0,1) \Rightarrow -X \sim \mathcal{N}(0,1)$ because pdf $f(x)$ is an even function

$$\Phi(-x) = \Pr(X \leq -x) = \Pr(-X \geq x) = \Pr(X \geq x) = 1 - \Pr(X \leq x) = 1 - \Phi(x)$$

Proof of (4.51)(vii): $\Phi^{-1}(p) = -\Phi^{-1}(1-p)$ for all $p \in (0,1)$



$X := \Phi^{-1}(p)$

$\Rightarrow \Phi(x) = p$

$\overset{(vi)}{\Rightarrow} \quad 1-p = 1-\Phi(x) = -\Phi(-x)$

$\downarrow$

$-\Phi^{-1}(1-p) = -\Phi^{-1}(\Phi(-x))$

$= -(-x) = x = \Phi^{-1}(p)$

Normal Distributions

Remark (4.52)

MATLAB provides implementations of pdf's, cdf's and the inverses of cdf's of normal distributions under the names `normpdf()`, `normcdf()` and `norminv()`, respectively.

φ Φ Φ^{-1}

Sum of independent random variables $X_i \sim \mathcal{N}(\mu_i, \sigma_i)$

Theorem (4.53)

If $X_1, \ldots, X_n$ are independent random variables with

$$X_i \sim \mathcal{N}(\mu_i, \sigma_i)$$

then

$$X = X_1 + \cdots + X_n \sim \mathcal{N}(\mu, \sigma)$$

has a normal distribution with mean $\mu = \mu_1 + \cdots + \mu_n$ and variance $\sigma^2 = {\sigma_1}^2 + \cdots + {\sigma_n}^2$.

moment generating fct. of X

$$\phi_X(t) = \prod_{i=1}^{n} e^{\mu_i \cdot t + \frac{1}{2}\sigma_i^2 t^2} = e^{\sum_{i=1}^{n}(\mu_i t + \frac{1}{2}\sigma_i^2 t^2)} = e^{(\sum \mu_i)\cdot t + \frac{1}{2}(\sum \sigma_i^2)\cdot t^2}$$

(4.51)(iii)

is the moment generating fct of $\mathcal{N}(\sum \mu_i, \sqrt{\sum \sigma_i^2})$

Central Limit Theorem

$X_1, X_2, X_3, \ldots$ "mean" of $X_1, X_2, \ldots, X_n$: $\frac{1}{n}\sum_{i=1}^{n} X_i = \bar{X}_n$

$E(\bar{X}_n) = \mu$

Theorem (4.54)

Let $(X_i)_{i\in\mathbb{N}}$ be a sequence of independent, identically distributed random variables with finite mean μ and finite variance σ^2. Let F_{Y_n} be the cdf of

$Var(\bar{X}_n) = (\frac{1}{n})^2 \cdot n \cdot \sigma^2 = \frac{\sigma^2}{n}$

$$Y_n = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \left(\frac{X_i - \mu}{\sigma} \right) \overset{!}{=} \frac{\overbrace{\left(\sum_{i=1}^{n} X_i\right)/n}^{\bar{X}_n} - \mu}{\sigma/\sqrt{n}} = \left(\bar{X}_n\right)^0$$

Then $E(Y_n) \overset{\checkmark}{=} 0$, $Var(Y_n) \overset{\checkmark}{=} 1$ for all $n \in \mathbb{N}$ and:

$$\lim_{n\to\infty} F_{Y_n} = \Phi$$

Proof of Theorem (4.54)

Put $Z_i = \dfrac{X_i - \mu}{\sigma}$. Then:

$E(Z_i) = 0, \quad Var(Z_i) = 1 = E(Z_i^2) - \cancel{(E(Z_i))^2}$

$v \in \mathbb{R}$:

$$\varphi_{Y_n}(v) = E\left(e^{jvY_n}\right) = E\left(e^{\frac{jv}{\sqrt{n}}(Z_1+\cdots+Z_n)}\right) = E\left(e^{\frac{jv}{\sqrt{n}}Z_1}\cdots e^{\frac{jv}{\sqrt{n}}Z_n}\right) = \prod_{i=1}^{n} E\left(e^{\frac{jv}{\sqrt{n}}Z_i}\right)$$

$$= \left(E\left(e^{\frac{jv}{\sqrt{n}}Z_1}\right)\right)^n = \left(E\left(1 + \frac{jv}{\sqrt{n}}Z_1 + \frac{1}{2}\left(\frac{jv}{\sqrt{n}}Z_1\right)^2 + \frac{1}{6}\left(\frac{jv}{\sqrt{n}}Z_1\right)^3 + \ldots\right)\right)^n$$

$$= \left(E(1) + \cancel{E\left(\frac{jv}{\sqrt{n}}Z_1\right)} + E\left(\frac{1}{2}\left(\frac{jv}{\sqrt{n}}Z_1\right)^2\right) + E\left(\frac{1}{6}\left(\frac{jv}{\sqrt{n}}Z_1\right)^3\right) + \ldots\right)^n$$

$$= \left(1 - \frac{1}{2}\frac{v^2}{n} + \frac{-jv^3}{6n^{3/2}}E({Z_1}^3) + \ldots\right)^n$$

Proof of Theorem (4.54)

$$\varphi_{Y_n}(v) = \ldots = \left(1 - \frac{1}{2}\frac{v^2}{n} + \frac{-jv^3}{6n^{3/2}}E({Z_1}^3) + \ldots\right)^n$$

$$= \left(1 + \frac{1}{n}\cdot\underbrace{\left(-\frac{1}{2}v^2 + \underbrace{\frac{jv^3}{6n^{1/2}}E({Z_1}^3) + \ldots}_{\longrightarrow 0 \text{ for } n\to\infty}\right)}_{x_i \longrightarrow -\frac{1}{2}v^2 \text{ for } n\to\infty}\right)^n \xrightarrow{n\to\infty} e^{-\frac{1}{2}v^2}$$

$$x_i \to x_0 \implies \left(1 + \frac{x_i}{n}\right)^n \to e^{x_0}$$

Proof of Theorem (4.54)

$$\varphi_{Y_n}(v) = \ldots = \left(1 - \frac{1}{2}\frac{v^2}{n} + \frac{-jv^3}{6n^{3/2}} E({Z_1}^3) + \ldots\right)^n$$

$$= \left(1 + \frac{1}{n} \cdot \left(-\frac{1}{2}v^2 + \frac{-jv^3}{6n^{1/2}} E({Z_1}^3) + \ldots\right)\right)^n \xrightarrow{n\to\infty} e^{-\frac{1}{2}v^2}$$

characteristic fct. of $\mathcal{N}(0,1)$

Hence, the sequence of characteristic functions $\varphi_{Y_n}(v)$ converges to the characteristic function of the standard normal distribution and this implies the convergence of $(Y_n)_{n\in\mathbb{N}}$ to the standard normal distribution.