Problem 1 Proof, if the following systems are linear, time invariant, causal, and memoryless.

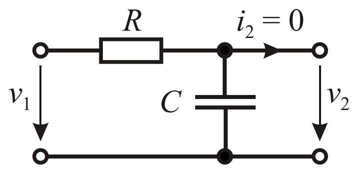
a) $y(t)=\sin\left[x(t-1)\right]$; x,t real   b) $y(t)=a\cdot x(t)-b\cdot\frac{\mathrm{d}\,y(t)}{\mathrm{d}\,t}$; a,b,x,t real

c) $y(t)=a\cdot t^2+x(t+3)$; a,x,t real

Problem 2 Which of the given unit impulse responses characterizes a stable LTI system?

a) $h(t)=\cos(t)\cdot u(t)$ b) $h(t)=\mathrm{e}^{at}\cdot u(t)$; a real

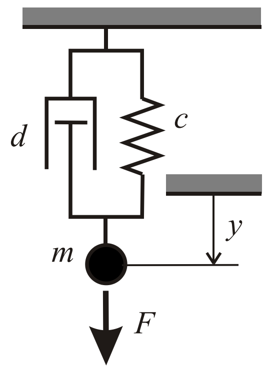
Problem 3 The given RC circuit shall be analyzed by solving the system's differential equation under the initial rest condition. The capacitor requires a steady voltage v_2.



a) Compute $y(t)=v_2(t)$ for $x(t)=v_1(t)=\frac{1\,\mathrm{Vs}}{T}\cdot\mathrm{rect}\left(\frac{t-T/2}{T}\right)$

b) Compute $x(t)=v_1(t)$ and $y(t)=v_2(t)$ for $T\rightarrow 0$ and interpret the result.

Problem 4 A mechanical system shall be analyzed. Fixed at a ceiling is a spring (c), a damper (d) and a mass (m). An external force F stimulates the system. The spring is characterized by the relation $F_c=c\cdot y$. The distance y is measured from the relaxed position of the spring. A spring is able to store energy according to $E_c=c\cdot y^2/2$. The damper is characterized by $F_d=d\cdot v$, where v is the velocity of the moving part. The damper dissipates energy.



a) Derive a differential equation between the stimulation force F and the mass position y. Is the system linear?

b) Derive a state-space representation.

Problem 5 Let $x(t)$ be an input signal whose FOURIER transform is

$$X(\mathrm{j}\omega)=\delta(\omega)+\delta(\omega-\pi)+\delta(\omega-5)$$

and let $h(t)=u(t)-u(t-2)$

be the unit impulse response of a system.

a) Is $x(t)$ periodic?

b) Compute the output signal $y(t)$. Is $y(t)$ periodic?

Problem 6 Derive the frequency response of the circuit in problem 3 from the differential equation.

Problem 7 Consider a causal LTI system with frequency response

$$H(\mathrm{j}\omega)=\frac{1}{3+\mathrm{j}\omega}$$

For a particular input $x(t)$ this system is observed to produce the output signal

$$y(t)=\mathrm{e}^{-3t}\cdot u(t)-\mathrm{e}^{-4t}\cdot u(t)$$

a) Determine and plot the magnitude and the phase response.

b) Which properties of $h(t)$ can be derived from the magnitude and the phase response?

c) Determine the input signal $x(t)$.

Answers

Problem 1

linear, time invariant, causal, memoryless

a) no, yes, yes, no ; b) yes, yes, yes, no ; c) no, no, no, no

Problem 2

a) not stable ; b) stable for $a < 0$

Problem 3

a) $v_2 = 0$ for $t < 0$; $v_2 = \frac{1\ \mathrm{Vs}}{T} \cdot \left(1 - \mathrm{e}^{-t/\tau}\right)$ for $0 \le t < T$; $v_2 = \frac{1\ \mathrm{Vs}}{T} \cdot \left(\mathrm{e}^{T/\tau} - 1\right) \cdot \mathrm{e}^{-t/\tau}$ for $t > T$

b) $x(t) = 1\ \mathrm{Vs} \cdot \delta(t)$; $y(t) = 1\ \mathrm{Vs} \cdot h(t) = u(t) \cdot \frac{1\ \mathrm{Vs}}{\tau} \cdot \mathrm{e}^{-t/\tau}$

Problem 4

a) $c \cdot y + d \cdot \frac{\mathrm{d}\,y}{\mathrm{d}\,t} + m \cdot \frac{\mathrm{d}^2\,y}{\mathrm{d}\,t^2} = m \cdot g + F$; no

b) $\begin{bmatrix} \frac{\mathrm{d}\,z_1}{\mathrm{d}\,t} \\ \frac{\mathrm{d}\,z_2}{\mathrm{d}\,t} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\frac{c}{m} & -\frac{d}{m} \end{bmatrix} \cdot \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{m} \end{bmatrix} \cdot x$; $x = m \cdot g + F$; $\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} y \\ \frac{\mathrm{d}\,y}{\mathrm{d}\,t} \end{bmatrix}$

$y = \begin{bmatrix} 1 & 0 \end{bmatrix} \cdot \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} + 0 \cdot x$

Problem 5

a) no ; b) $y(t) = \frac{1}{\pi}\left(1 + \frac{\sin 5}{5} \cdot \mathrm{e}^{\mathrm{j}5(t-1)}\right)$; yes

Problem 6

$$H(\omega) = \frac{1}{1 + \mathrm{j}\omega RC}$$

Problem 7

a) $\left|H(\omega)\right| = \frac{1}{\sqrt{9 + \omega^2}}$; $\varphi(\omega) = -\arctan\left(\frac{\omega}{3}\right)$

b) $h(t)$ real with $h_\mathrm{e} \ne 0$ and $h_\mathrm{o} \ne 0$

c) $x(t) = \mathrm{e}^{-4t} \cdot u(t)$

Problem 1 Compute the LAPLACE transforms, determine the ROC and the zero-pole plots for each of the following functions.

a) $x(t) = \mathrm{rect}\left(\dfrac{t - T/2}{T}\right)$

b) $x(t) = u(t) \cdot \left(\dfrac{2}{3} - \dfrac{1}{5} \cdot \mathrm{e}^{2t} - \dfrac{7}{15} \cdot \mathrm{e}^{-3t}\right)$

c) $x(t) = u(t) \cdot \mathrm{e}^{-2t} + u(t) \cdot \mathrm{e}^{-t} \cdot \cos(3t)$

Problem 2 Compute the causal time functions of the given LAPLACE transforms.

a) $X(s) = \dfrac{s-4}{s^3 + s^2 - 6s}$ b) $X(s) = \dfrac{1}{s^2 + s + 1}$ c) $X(s) = \dfrac{s}{s^2 + s + 1}$

Plot and compare the zero-pole characteristics. Check the initial- and final-value theorems.

Problem 3 The input to and the output of an LTI system are given by

$$x(t) = u(t) \cdot \mathrm{e}^{-3t} \ ; \quad y(t) = u(t) \cdot \left[\mathrm{e}^{-t} - \mathrm{e}^{-2t}\right]$$

Determine the systems differential equation under the condition of initial rest.

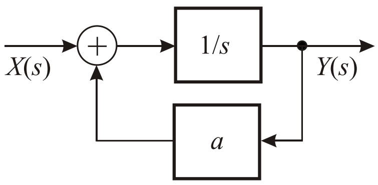
Problem 4 Determine the system function $H(s)$ of the system with the impulse response

$$h(t) = u(t) \cdot (2 - t) \cdot 4 \cdot t \cdot \mathrm{e}^{-2t}.$$

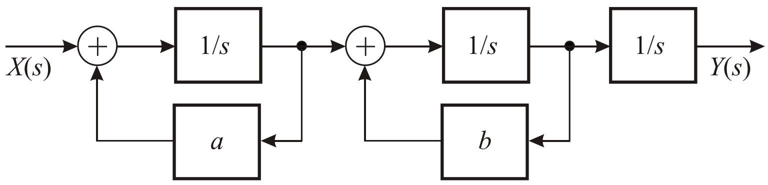
Plot the zero-pole diagram.

Problem 5 Consider a stable system with a real impulse response. Its system function has two poles and a) no, b) one, c) two zeros. Discuss and plot possible impulse responses.

Problem 6 The block diagrams of two systems are given. a, b are real.



System 1



System 2

Determine for each of the systems:

a) system function, b) zero-pole plot, c) stability, d) impulse response, e) ARMA topology.

Problem 7 The differential equation of an LTI system is given by

$$\tau \cdot \frac{\mathrm{d}\,y(t)}{\mathrm{d}t} + y(t) = x(t)$$

where τ denotes the time constant. Determine $y(t)$ for $x(t) = 8 \cdot u(t)$ under the condition of initial rest. Solve the differential equation in the time domain and apply LAPLACE transform as well.

Answers

Problem 1

a) $X(s) = \frac{1}{s} \cdot \left(1 - \mathrm{e}^{-sT}\right)$; ROC : entire s - plane ; $X(s)$ not rational

b) $X(s) = \frac{(s - z_1)}{(s - p_1) \cdot (s - p_2) \cdot (s - p_3)}$; $\sigma > 2$; $z_1 = 4$; $p_1 = 0$; $p_2 = 2$; $p_3 = -3$

c) $X(s) = \frac{2 \cdot (s - z_1) \cdot (s - z_2)}{(s - p_1) \cdot (s - p_2) \cdot (s - p_3)}$; $\sigma > -1$; $z_{1/2} = -1{,}25 \pm \mathrm{j}\,2{,}11$; $p_1 = -2$; $p_{2/3} = -1 \pm \mathrm{j}3$

Problem 2

a) $x(t) = u(t) \cdot \left(\frac{2}{3} - \frac{1}{5} \cdot \mathrm{e}^{2t} - \frac{7}{15} \cdot \mathrm{e}^{-3t} \right)$

b) $x(t) = u(t) \cdot \frac{2}{\sqrt{3}} \cdot \mathrm{e}^{-t/2} \cdot \sin\left(\frac{\sqrt{3}}{2} t \right)$

c) $x(t) = u(t) \cdot \mathrm{e}^{-t/2} \cdot \left(\cos\left(\frac{\sqrt{3}}{2} t \right) - \frac{1}{\sqrt{3}} \sin\left(\frac{\sqrt{3}}{2} t \right) \right)$

Problem 3

$$2 \cdot y(t) + 3 \cdot \frac{\mathrm{d}y}{\mathrm{d}t} + \frac{\mathrm{d}^2 y}{\mathrm{d}t^2} = 3 \cdot x(t) + \frac{\mathrm{d}x}{\mathrm{d}t}$$

Problem 4

$$H(s) = \frac{8 \cdot (s - z)}{(s - p)^3} \; ; z = -1 ; \; p = -2$$

Problem 6

$$H_1(s) = \frac{1}{s - a} \quad ; \quad H_2(s) = \frac{1}{s - a} \cdot \frac{1}{s - b} \cdot \frac{1}{s}$$

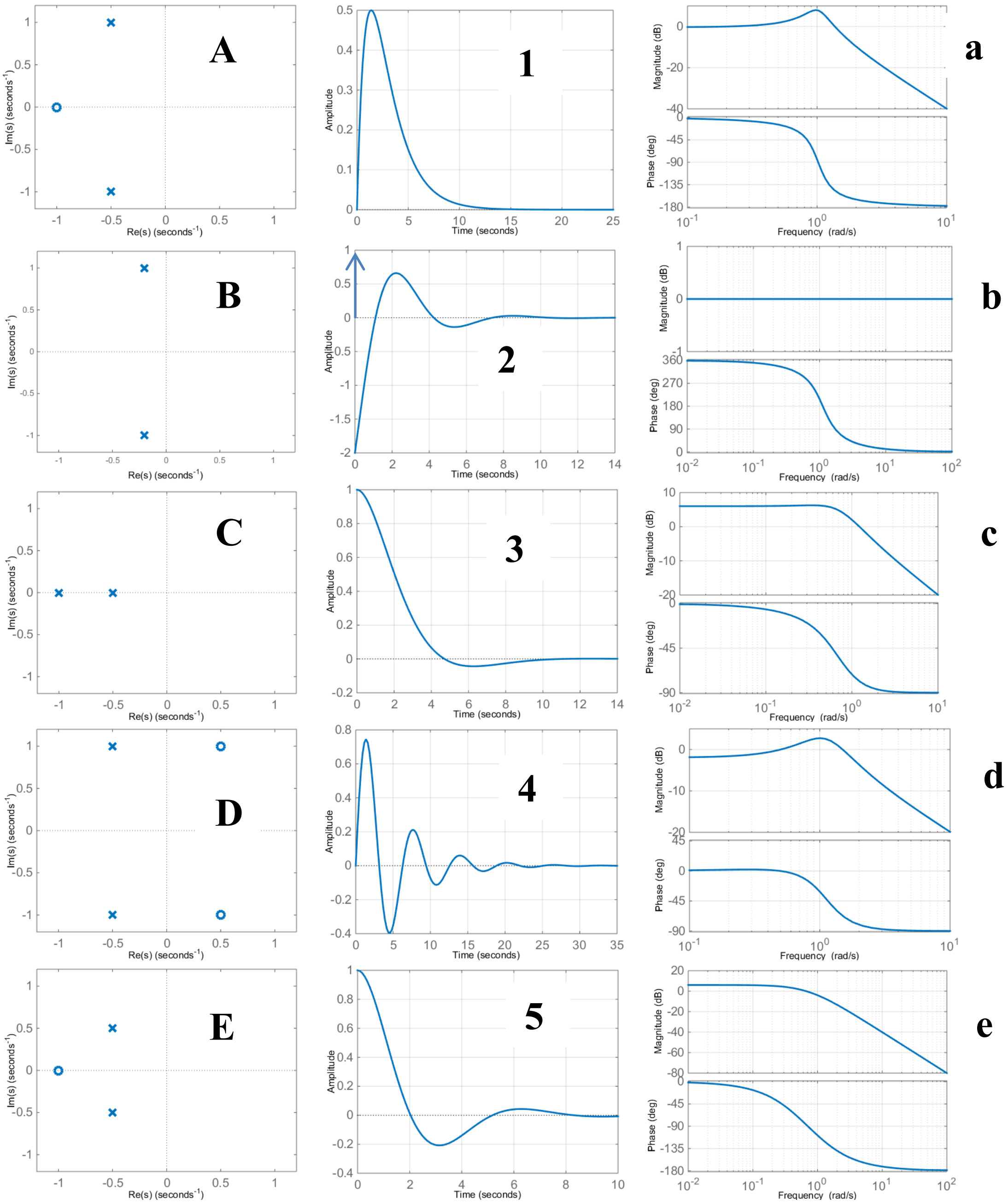
System 1: Stable for $a < 0$; System 2: Not stable

Problem 7

$$y(t) = 8 \cdot \left(1 - \mathrm{e}^{-t/\tau}\right) \cdot u(t)$$

Mapping task: Continuous-time systems

Different continuous-time systems shall be considered. Unfortunately, the given relation of the zero-pole plots to the impulse responses and to the BODE plots are erroneous. Find the correct relation and give the answer in the form (A, 2, c), if you think that this is a correct constellation.



Answers

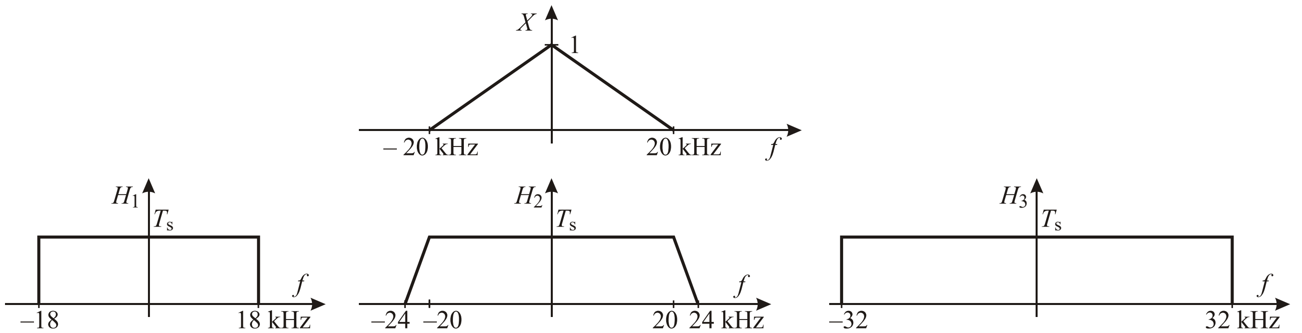
A5d, B4a, C1e, D2b, E3c

Problem 1 Is ideal impulse train sampling a linear operation? Is it a time-invariant operation?

Problem 2 Determine the sampling period T_s for a successful sampling and reconstruction of the signal

$$x(t) = 1 + \cos(2\ \mathrm{Hz} \cdot \pi \cdot t) + 2 \cdot \sin(40\ \mathrm{Hz} \cdot \pi \cdot t)$$

Problem 3 A time-continuous signal $x(t)$ shall be sampled with one of the given sampling frequencies: 36 kHz, 44 kHz, 64 kHz. Subsequently the original signal shall be reconstructed with one of the given low-pass filters H_1, H_2, H_3.



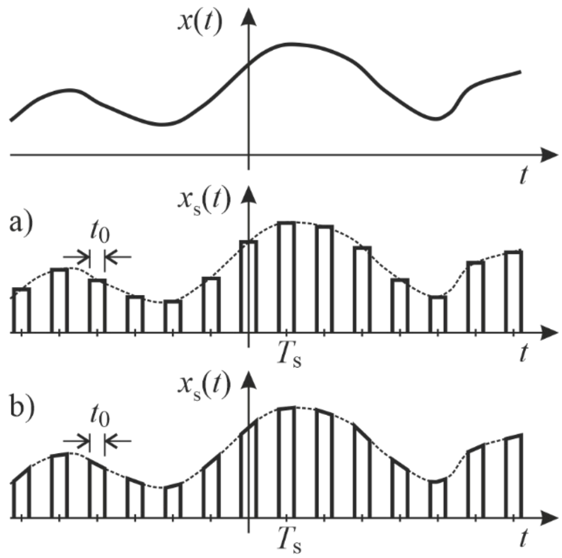
a) Select the minimal sampling frequency for a successful reconstruction.

b) Select one of the given low-pass filters: H_1, H_2, H_3

Problem 4 A real sampling system uses impulses of finite width t_0. Circuit a) is referred as sample-and-hold circuit. Circuit b) is referred as linear-gate circuit.

a) Determine and plot the spectra of the sampled signals $x_s(t)$.

b) Is a perfect reconstruction possible?



Answers

Problem 1 yes, no

Problem 2 < 25 ms

Problem 3 44 kHz; H_2

Problem 4

a) $X_s(f) = \frac{t_0}{T_s} \cdot \mathrm{si}\left(\pi \cdot f \cdot t_0\right) \cdot \sum_{k=-\infty}^{+\infty} X\left(f - k \cdot f_s\right)$

b) $X_s(f) = \frac{t_0}{T_s} \cdot \sum_{k=-\infty}^{+\infty} \mathrm{si}\left(\pi \cdot k \cdot f_s \cdot t_0\right) \cdot X\left(f - k \cdot f_s\right)$

Problem 1 Consider the finite-length sequence of length 7 defined for $-3 \le n \le 3$:

$$\{x[n]\} = \{0,\ 1+\mathrm{j}4,\ -2+\mathrm{j}3,\ 4-\mathrm{j}2,\ -5-\mathrm{j}6,\ -\mathrm{j}2,\ 3\}\ .$$

Determine the conjugate-symmetric and conjugate-antisymmetric sequences.

Problem 2 Consider the causal sequences defined by:

a) $x[n] = \begin{cases} 3\cdot(-1)^n\ , & n \ge 0 \\ 0\ , & \text{otherwise} \end{cases}$ b) $x[n] = \begin{cases} 1/n\ , & n \ge 1 \\ 0\ , & \text{otherwise} \end{cases}$

Are these sequences energy or power signals? If yes, compute the energy or power, respectively.

Problem 3 Compute the N-point DFT $X[k]$ of the length-N sequences with $0 \le k \le N-1$

a) $\{x[n]\} = \{2, 0, 1, 0, 0, 0, 1, 0\}$; b) $\{x[n]\} = \{1, 2, 3, 4\}$

Check for symmetry relations.

Problem 4 Compute the N-point DFT $X[k]$ of the length-N sequences with $0 \le k \le N-1$

a) $x[n] = \mathrm{e}^{\mathrm{j}2\pi rn/N}\ ;\ 0 \le n \le N-1\ ,$

b) $x[n] = \cos\left(2\pi rn/N\right)\ ;\ 0 \le n \le N-1\ ,$

where r is an integer in the range $0 < r \le N-1$.

Problem 5 Building resonances between 0.8 Hz and 1.0 Hz must be avoided in skyscrapers, because they cause braking window glasses. Thus, civil engineers need to know exactly the resonance frequencies. The building is equipped with sensors and their signals are being processed digitally. After AD conversion an FFT algorithm is applied.

a) How many FFT points are necessary for the sampling frequency 5.4 Hz in order to achieve a frequency resolution of 0.001 Hz?

b) Determine the observation time.

Problem 6 Use the equation

$$\sum_{n=0}^{N-1} \mathrm{e}^{-\mathrm{j}2\pi kn/N} = \begin{cases} N\ , & k = l\cdot N\ ,\ l \text{ integer} \\ 0\ , & \text{otherwise} \end{cases}$$

for the computation of the length-N DFT.

a) Compute the continuous-time FOURIER transform $X(f)$ of $x(t) = \sin(2\pi f_0 t)$ and plot $|X(f)|$.

b) Compute the continuous-time FOURIER transform $X(f)$ of

$$x(t) = \begin{cases} \sin(2\pi f_0 t)\ , & 0 \le t \le 1/f_0 \\ 0\ , & \text{otherwise} \end{cases}$$

c) Compute $X[k]$ of $x[n] = \sin(2\pi f_0 n T_\mathrm{s})$ with $T_\mathrm{s} = 1/(8 f_0)$ and $N = 8$. Plot $|X[k]|$.

d) Compute $X[k]$ of $x[n] = \sin(2\pi f_0 n T_\mathrm{s})$ with $T_\mathrm{s} = 0.9/(8 f_0)$ and $N = 8$. Plot $|X[k]|$.

e) Perform suitable Matlab simulations and check the results.

Answers

Problem 1

$$\{x_{\mathrm{cs}}[n]\}=\{1.5,\ \ 0.5+\mathrm{j}3,\ \ -3.5+\mathrm{j}4.5,\ \ 4,\ \ -3.5-\mathrm{j}4.5,\ \ 0.5-\mathrm{j}3,\ \ 1.5\}$$

$$\{x_{\mathrm{ca}}[n]\}=\{-1.5,\ \ 0.5+\mathrm{j},\ \ 1.5-\mathrm{j}1.5,\ \ -\mathrm{j}2,\ \ -1.5-\mathrm{j}1.5,\ \ -0.5+\mathrm{j},\ \ 1.5\}$$

Problem 2

a) Power signal: $P_\infty = 4.5$; $E_\infty \to \infty$

b) Energy signal: $E_\infty = \pi^2/6 = 1.645$; $P_\infty = 0$

Problem 3

a) $\{X[k]\}=\{4,\ \ 2,\ \ 0,\ \ 2,\ \ 4,\ \ 2,\ \ 0,\ \ 2\}$

b) $\{X[k]\}=\{10,\ \ -2+\mathrm{j}2,\ \ -2,\ \ -2-\mathrm{j}2\}$

Problem 4

a) $X[k]=\begin{cases} N & \text{for } k=r \\ 0 & \text{otherwise} \end{cases}$; b) $X[k]=\begin{cases} N/2 & \text{for } k=r \\ N/2 & \text{for } k=N-r \\ 0 & \text{otherwise} \end{cases}$

Problem 5

a) 8192 b) 1517 s

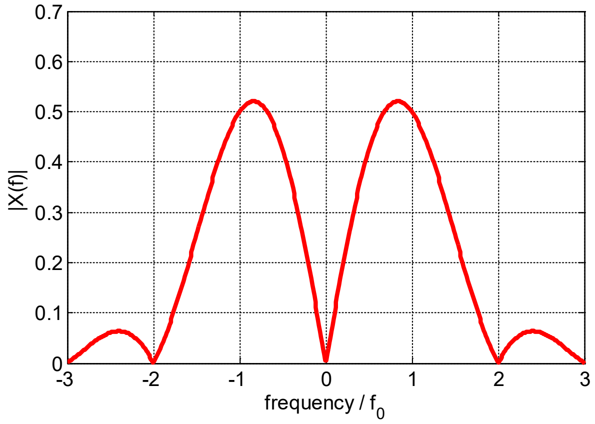
Problem 6

a) $X(f)=\dfrac{\mathrm{j}}{2}\big(\delta(f+f_0)-\delta(f+f_0)\big)$

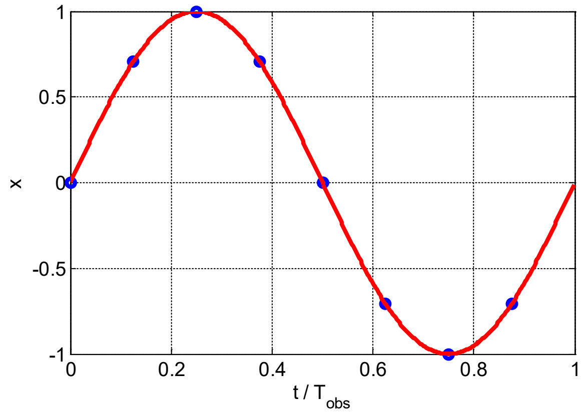
b) $X(f)=\mathrm{j}\dfrac{T}{2}\Big(\mathrm{si}\big(\pi(f+f_0)T\big)\cdot \mathrm{e}^{-\mathrm{j}\pi(f+f_0)T}-\mathrm{si}\big(\pi(f-f_0)T\big)\cdot \mathrm{e}^{-\mathrm{j}\pi(f-f_0)T}\Big)$

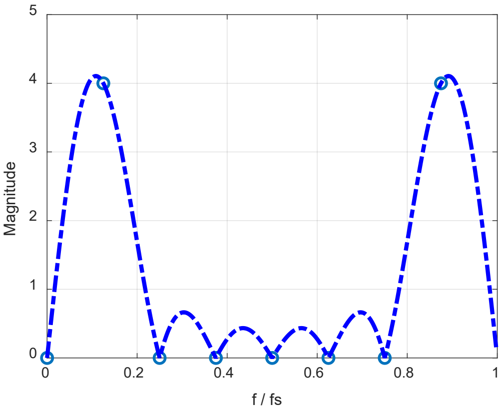
c) $X[k]=\begin{cases} -4\,\mathrm{j}\ , & k=1 \\ 4\,\mathrm{j}\ , & k=7 \\ 0\ , & k=0,\ 2,\ 3,\ 4,\ 5,\ 6 \end{cases}$

b)

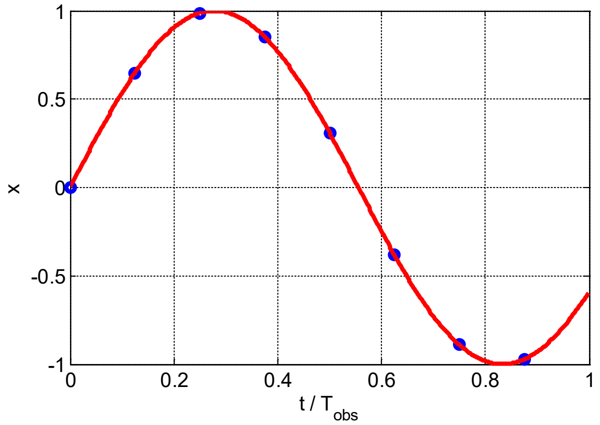


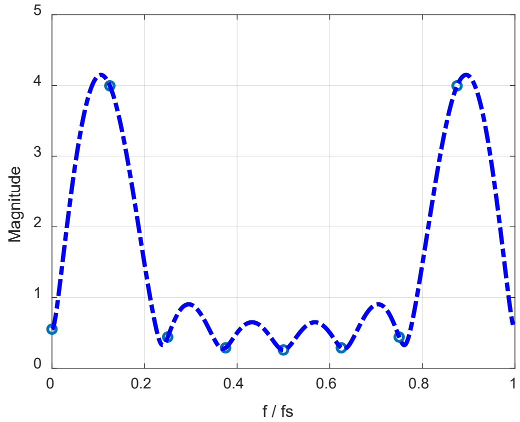
c) $T_s = 1/(8 f_0)$ and $N = 8$





d) $T_s = 0.9/(8 f_0)$ and $N = 8$





Problem 1 Is the down-sampler a linear and time-invariant system?

Problem 2 Is the M-point moving average filter a stable system? Its input-output relation is given by:

$$y[n]=\frac{1}{M}\sum_{i=0}^{M-1}x[n-i]$$

Problem 3 Consider a causal LTI discrete-time system with the impulse response $h[n]=\beta^n\cdot u[n]$, where $|\beta|<1$. Determine the output sequence $y[n]$ for the causal input sequence $x[n]=\alpha^n\cdot u[n]$ with $|\alpha|<1$. $u[n]$ is the unit step sequence.

Problem 4 A discrete-time system is characterized by the following difference equation:

$$y[n]-2^n\cdot y[n+1]+3\cdot y^2[n+2]=4\cdot x[n]-2\cdot x[n+1]$$

Check, whether the system is linear, time-invariant, causal, and memoryless.

Problem 5 A causal discrete-time LTI system is characterized by the following difference equation:

$$y[n]+y[n-1]-6\cdot y[n-2]=x[n]$$

Compute the impulse response. Is the system stable?

Problem 6 A causal discrete-time LTI system is characterized by the following difference equation:

$$y[n]-\frac{1}{2}\cdot y[n-1]=x[n]$$

a) Compute the impulse response and check for stability.

b) Determine the solution for $n\geq 0$ for the step input $x[n]=8\cdot u[n]$ with initial condition $y[0]=0$.

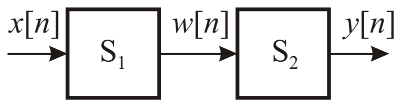
c) Determine the solution for $n\geq 0$ for $x[n]=\left(\frac{1}{2}\right)^n\cdot u[n]$ with initial condition $y[0]=0$.

Problem 7 Consider the cascade of the causal LTI systems S_1 and S_2.

$$S_1:\ w[n]=\frac{1}{2}\cdot w[n-1]+x[n]\quad;\quad S_2:\ y[n]=\alpha\cdot y[n-1]+\beta\cdot w[n]$$

The difference equation relating $x[n]$ and $y[n]$ is:

$$y[n]=-\frac{1}{8}\cdot y[n-2]+\frac{3}{4}\cdot y[n-1]+x[n]$$



Determine α and β and the impulse response $h[n]$.

Problem 8 a) Show that the sequence $x[n]=z^n$, where z is a complex constant, is an eigenfunction of an LTI discrete-time system. b) Is the sequence $x[n]=z^n\cdot u[n]$ with $u[n]$ being the unit step sequence also an eigenfunction of an LTI discrete-time system?

Answers

Problem 1
linear: yes ; time-invariant: no

Problem 2
yes

Problem 3

$$y[n] = \sum_{k=0}^{n} \alpha^k \cdot \beta^{n-k}$$

Problem 4
linear: no ; time-invariant: no ; causal: yes; memoryless: no

Problem 5
$h[n] = 0.4 \cdot 2^n + 0.6 \cdot (-3)^n$ with $n \geq 0$; not stable

Problem 6

a) $h[n] = \left(\frac{1}{2}\right)^n \cdot u[n]$

b) $y[n] = 16 \cdot \left\{1 - \left(\frac{1}{2}\right)^n\right\} \cdot u[n]$

c) $y[n] = n \cdot \left(\frac{1}{2}\right)^n \cdot u[n]$

Problem 7

$\alpha = 1/4$; $\beta = 1$; $h[n] = \left\{2 \cdot \left(\frac{1}{2}\right)^n - \left(\frac{1}{4}\right)^n\right\} \cdot u[n]$

Problem 8
a) yes; b) no

Problem 1 Compute the frequency response of a moving average filter. Its impulse response is given by

$$h[n] = \begin{cases} 1/M & , 0 \le n \le M-1 \\ 0 & , \text{otherwise} \end{cases}$$

Problem 2 Consider a causal LTI discrete-time system with real impulse response $h[n]$. Compute the steady-state response for the sinusoidal input

$$x[n] = A \cdot \cos(\Omega_0 n + \phi)$$

where $\Omega_0 = \omega_0 \cdot T_\text{s}$ is the normalized radian frequency.

Problem 3 Consider a causal and stable LTI discrete-time system with real impulse response $h[n]$. Compute the response to the causal exponential sequence

$$x[n] = \mathrm{e}^{\mathrm{j}n\Omega} \cdot u[n]$$

where $\Omega = \omega \cdot T_\text{s}$ is the normalized radian frequency and $u[n]$ is the unit step sequence. Identify and discuss the different parts of the solution.

Problem 4 Prove the bilateral z-transforms and the region of convergence as listed in appendix B:

a) $x[n] = \alpha^n \cdot u[n]$

b) $x[n] = \cos(\Omega_0 \cdot n) \cdot u[n]$

c) $x[n] = r^n \cdot \cos(\Omega_0 \cdot n) \cdot u[n]$

Problem 5 Determine the bilateral z-transform in a closed form and the region of convergence of the following series:

a) $x[n] = \begin{cases} \alpha^n & , M \le n \le N-1 \\ 0 & , \text{otherwise} \end{cases}$

b) $x[n] = 7 \cdot \left(\frac{1}{3}\right)^n \cdot u[n] - 6 \cdot \left(\frac{1}{2}\right)^n \cdot u[n]$

Check the initial- and final-value theorems if possible.

Problem 6 Determine the inverse causal z-transforms:

a) $X(z) = \dfrac{z \cdot (z+2)}{(z-0.2) \cdot (z+0.6)}$

b) $X(z) = \dfrac{2 + 0.8 \cdot z^{-1} + 0.5 \cdot z^{-2} + 0.3 \cdot z^{-3}}{1 + 0.8 \cdot z^{-1} + 0.2 \cdot z^{-2}}$

c) $X(z) = \dfrac{3 \cdot z^2 + 2 \cdot z - 10}{z^3 - 5 \cdot z^2 + 8z - 4}$ with $p_1 = 1$

Check the initial- and final-value theorems.

Answers

Problem 1

$$H\left(\mathrm{e}^{\mathrm{j}\Omega}\right)=\frac{1}{M}\cdot\frac{\sin(M\Omega/2)}{\sin(\Omega/2)}\cdot\mathrm{e}^{-\mathrm{j}(M-1)\Omega/2}$$

Problem 2

$$y[n]=A\cdot\left|H\left(\mathrm{e}^{\mathrm{j}\Omega_0}\right)\right|\cdot\cos\left(\Omega_0 n+\phi+\varphi(\Omega_0)\right)\quad;\quad\varphi(\Omega_0):\text{phase response}$$

Problem 3

$$y[n]=H\left(\mathrm{e}^{\mathrm{j}\Omega}\right)\cdot\mathrm{e}^{\mathrm{j}n\Omega}-\left(\sum_{k=n+1}^{\infty}h[k]\cdot\mathrm{e}^{-\mathrm{j}k\Omega}\right)\cdot\mathrm{e}^{\mathrm{j}n\Omega}\quad;\quad n\geq 0$$

Problem 4

see appendix B

Problem 5

a) $X(z)=\dfrac{\alpha^M\cdot z^{-M}-\alpha^N\cdot z^{-N}}{1-\alpha\cdot z^{-1}}$

ROC is the entire z-plane, except: 1) $z=0$ for $0\leq M<N$; 2) $z=0$ and $z=\infty$ for $M<0<N$; 3) $z=\infty$ for $M<N<0$.

b) $X(z)=\dfrac{z\left(z-\frac{3}{2}\right)}{\left(z-\frac{1}{3}\right)\cdot\left(z-\frac{1}{2}\right)}\quad;\quad\text{ROC}:|z|>\dfrac{1}{2}$

Problem 6

a) $x[n]=2.75\cdot(0.2)^n\cdot u[n]-1.75\cdot(-0.6)^n\cdot u[n]$

b) $x[n]=-3.5\cdot\delta[n]+1.5\cdot\delta[n-1]+$
$(2.75+\mathrm{j}0.25)\cdot(-0.4+\mathrm{j}0.2)^n\cdot u[n]+(2.75-\mathrm{j}0.25)\cdot(-0.4-\mathrm{j}0.2)^n\cdot u[n]$

c) $x[n]=2.5\cdot\delta[n]-5\cdot u[n]+2.5\cdot 2^n\cdot u[n]+1.5\cdot n\cdot 2^n\cdot u[n]$
$\{x[n]\}=\{0,3,17,51,\ldots\}$ with $n=0,1,2,3,\ldots$

or $x[n]=-5\cdot u[n-1]+4\cdot n\cdot 2^n\cdot u[n]-5\cdot(n-1)\cdot 2^{n-1}\cdot u[n-1]$

Problem 1 The impulse response of a moving average filter is given by

$$h[n]=\sum_{i=0}^{M-1}\frac{1}{M}\cdot\delta[n-i].$$

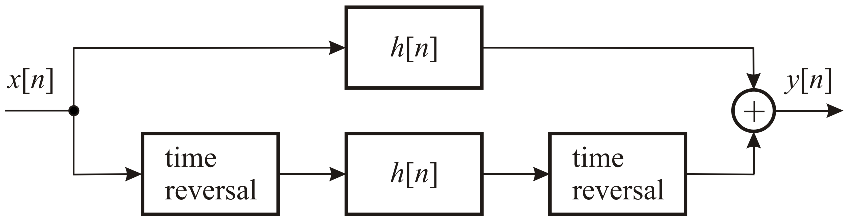
Compute the transfer function and the pole-zero location for $M > 1$. Is the system stable?

Problem 2 A causal, finite-dimensional, linear, time-invariant IIR filter is characterized by the constant coefficient difference equation

$$y[n]=x[n-1]-1.2\cdot x[n-2]+x[n-3]+1.3\cdot y[n-1]-1.04\cdot y[n-2]+0.222\cdot y[n-3].$$

Determine the transfer function and the pole-zero plot. Is the system stable?

Problem 3 Let a causal LTI discrete-time system be characterized by a real impulse response $h[n]$ with the DTFT $H\left(e^{j\Omega}\right)$. Consider the given system with a complex finite-length sequence $x[n]$. Determine the frequency response $G\left(e^{j\Omega}\right)$ of the overall system and characterize the system.



Problem 4 The frequency response of an LTI discrete-time system is given:

$$H\left(e^{j\Omega}\right)=\begin{cases} e^{-j6\Omega} & , \ 0\le|\Omega|\le\frac{\pi}{4} \\ 0 & , \ \frac{\pi}{4}<|\Omega|\le\pi \end{cases}$$

a) Plot the magnitude and phase response and characterize the system.

b) Determine the impulse response.

c) Is the system realizable?

d) Truncate the impulse response to $0 \le n \le 12$ and discuss the realizability and the truncation effect.

Problem 5 Consider the following causal IIR transfer function:

$$H(z)=\frac{3\cdot z^3+2\cdot z^2+5}{(0.5\cdot z+1)\cdot(z^2+z+0.6)}$$

a) Is $H(z)$ a stable transfer function?

b) If it is not stable, find a stable transfer function $G(z)$ such that $\left|G\left(e^{j\Omega}\right)\right|=\left|H\left(e^{j\Omega}\right)\right|$.

c) Is there any other transfer function having the same magnitude response as that of $H(z)$?

Problem 6 Consider the problem 7 from ex3-2 again: Discretize the differential equation with $T_s=\tau/10$ and compute $y[n]$ in the time domain and additionally by applying the z-transform.

Answers

Problem 1

$$H(z)=\frac{1}{M}\cdot\frac{1}{z^{M-1}}\cdot\frac{z^M-1}{z-1}=\frac{1}{M}\cdot\frac{1}{z^{M-1}}\cdot(z-z_1)\cdot(z-z_2)\cdots(z-z_{M-1})$$

$M-1$ finite zeros at $z_k=\mathrm{e}^{\mathrm{j}2\pi k/M}$ with $k=1,2,\ldots,M-1$

$M-1$ poles at $z=0$. Stable.

Problem 2

$$H(z)=\frac{z^2-1.2\cdot z+1}{z^3-1.3\cdot z^2+1.04\cdot z-0.222}$$

$z_{1,2}=0.6\pm\mathrm{j}0.8$; $p_1=0.3$; $p_{2,3}=0.5\pm\mathrm{j}0.7$; stable

Problem 3

$G\left(\mathrm{e}^{\mathrm{j}\Omega}\right)=2\cdot\mathrm{Re}\left\{H\left(\mathrm{e}^{\mathrm{j}\Omega}\right)\right\}$; zero-phase transfer function

Problem 4

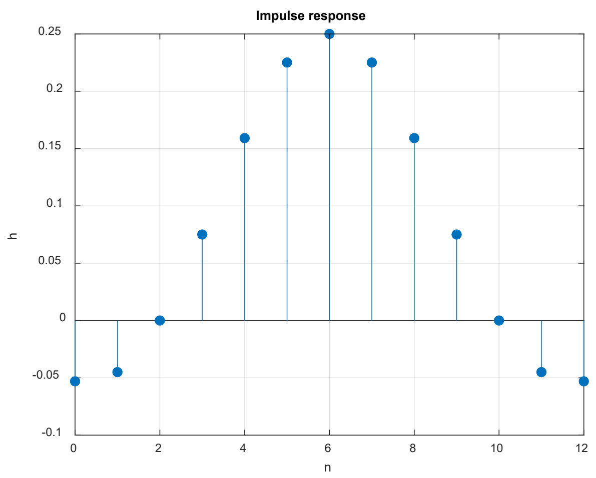
a) Linear-phase lowpass filter

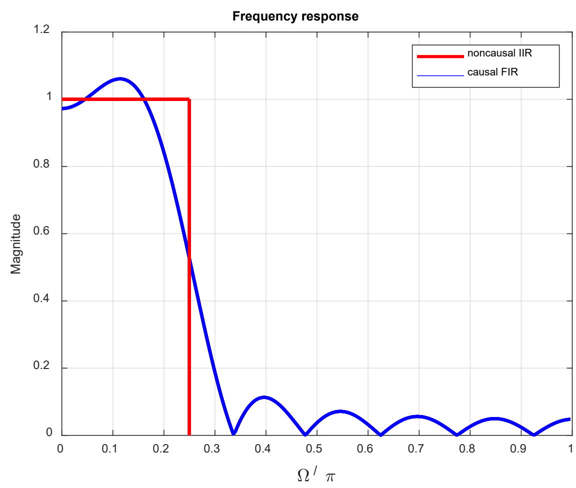
b) $h[n]=\dfrac{1}{4}\cdot\dfrac{\sin(\pi\cdot(n-6)/4)}{\pi\cdot(n-6)/4}$

c) Noncausal with $-\infty<n<\infty$. Not realizable.

d) Causal FIR filter with length 13. Rectangular magnitude response is modified.

$$H_{\mathrm{T}}\left(\mathrm{e}^{\mathrm{j}\Omega}\right)=\left\{\mathrm{rect}\left(\frac{\Omega}{\pi/2}\right)*\frac{\sin(6.5\cdot\Omega)}{\sin(\Omega/2)}\right\}\cdot\mathrm{e}^{-\mathrm{j}6\Omega}$$





Problem 5

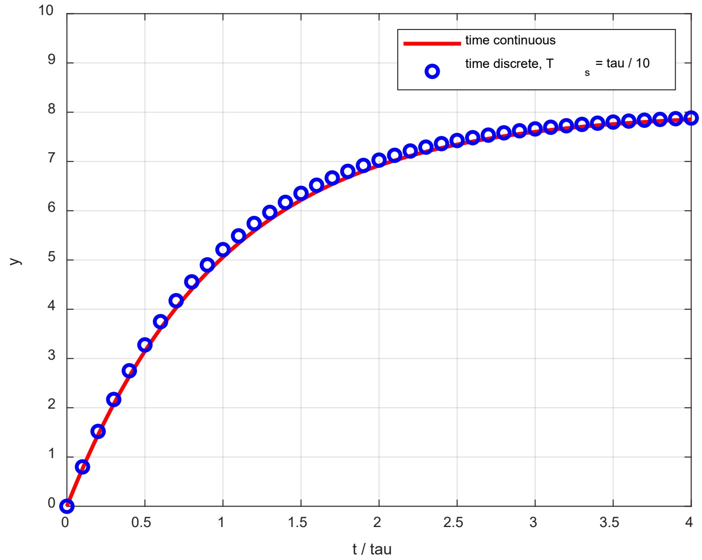
a) No

b) $G(z) = \dfrac{3 \cdot z^3 + 2 \cdot z^2 + 5}{(z+0.5)\cdot(z^2+z+0.6)}$

c) Yes, infinitely many

Problem 6

$$y[n] = 8 \cdot \left(1 - \left(\frac{9}{10}\right)^n\right) \cdot u[n]$$



Problem 1 Determine the location of the notch frequency of the FIR notch filter given by

$$H(z) = 1 + \sqrt{2} \cdot z^{-1} + z^{-2}.$$

Compute the response to the input signal $x[n] = \cos(n \cdot \pi / 4)$.

Problem 2 Design a first-order highpass filter with a normalized 3-dB cutoff frequency at 0.25 radian/sample.

Problem 3 Show that the causal FIR transfer function

$$H(z) = \frac{1}{1+\alpha} \cdot \left(1 + \alpha \cdot z^{-1}\right);\ \alpha > 0$$

is a bounded-real (BR) function.

Problem 4 The frequency response of a causal digital filter is given

$$H(z) = 1 - 2 \cdot z^{-1} + 3 \cdot z^{-2} - 3 \cdot z^{-4} + 2 \cdot z^{-5} - z^{-6}.$$

Determine the magnitude and phase responses.

Problem 5 A causal LTI FIR discrete-time system is characterized by an impulse response

$$h[n] = a_1 \cdot \delta[n] + a_2 \cdot \delta[n-1] + a_3 \cdot \delta[n-2] + a_4 \cdot \delta[n-3] + a_5 \cdot \delta[n-4] + a_6 \cdot \delta[n-5].$$

For what values of the impulse response samples will its frequency response have a constant group delay?

Problem 6 A type-II real-coefficient FIR filter (symmetric impulse response, N odd) with the transfer function $H(z)$ has the following zeros: $z_1 = 1$, $z_2 = -1$, $z_3 = 0.5$, $z_4 = 0.8 + j$.

a) Determine the locations of the remaining zeros of $H(z)$ having the lowest order.

b) Determine the transfer function of the filter.

Problem 7 Estimate the order and the group delay of a linear-phase lowpass FIR filter with the following specifications: passband edge $f_{pass} = 1.8$ kHz, stopband edge $f_{stop} = 2$ kHz, peak passband ripple $a_{pass} = 0.1$ dB, minimum stopband attenuation $a_{stop} = 35$ dB, and sampling rate $f_s = 12$ kHz.

Problem 8 Estimate the order of a linear-phase bandpass FIR filter with the following specifications: passband edges $f_{pass,1} = 0.35$ kHz and $f_{pass,2} = 1$ kHz, stopband edges $f_{stop,1} = 0.3$ kHz and $f_{stop,2} = 1.1$ kHz, passband ripple $\delta_{pass} = 0.002$, stopband ripple $\delta_{stop} = 0.01$, and sampling rate $f_s = 10$ kHz.

Answers

Problem 1

$\Omega_0 = 0.75 \cdot \pi = 135°$

$y[n] = 2 \cdot \sqrt{2} \cdot \cos\left((n-1) \cdot \pi / 4\right)$

Problem 2

$$H(z) = \frac{0.8884 \cdot (1 - z^{-1})}{1 - 0.7767 \cdot z^{-1}}$$

Problem 3

Problem 4

$\tilde{H}(\Omega) = 6 \cdot \sin(\Omega) - 4 \cdot \sin(2 \cdot \Omega) + 2 \cdot \sin(3 \cdot \Omega)$

$\varphi(\Omega) = -3 \cdot \Omega \pm \dfrac{\pi}{2}$

Problem 5

$a_1 = a_6, a_2 = a_5, a_3 = a_4$

or

$a_1 = -a_6, a_2 = -a_5, a_3 = -a_4$

Problem 6

a) $z_5 = 1$, $z_6 = 2$, $z_7 = 0.8 - j$, $z_8 = 0.488 - j \cdot 0.610$, $z_9 = 0.488 + j \cdot 0.610$,

b) $H(z) = \prod_{i=1}^{9} \left(1 - z_i \cdot z^{-1}\right)$

Problem 7

KAISER: order ≈ 99, group delay ≈ 4.13 ms

BELLANGER: order ≈ 108, group delay ≈ 4.5 ms

Problem 8

KAISER: order ≈ 466

Problem 1 Check, if the signals

a) $x(t)=\mathrm{e}^{-2t}\cdot u(t)$; b) $x(t)=\mathrm{e}^{\mathrm{j}(2t+\pi/4)}$; c) $x(t)=\cos(t)$

are energy or power signals. $u(t)$ is the unit step function. If possible, determine the energy E_∞ and the average power P_∞.

Problem 2 Which of the following signals is periodic? $u(t)$ is the unit step function.

a) $x(t)=2\cdot\mathrm{e}^{\mathrm{j}(t+\pi/4)}\cdot u(t)$; b) $x(t)=\mathrm{j}\cdot\mathrm{e}^{\mathrm{j}10t}$; c) $x(t)=\mathrm{e}^{(-1+\mathrm{j})t}$

If a signal is periodic, specify the period T.

Problem 3 Express the real part of each of the following signals in the form

$A\cdot\mathrm{e}^{-at}\cdot\cos(\omega t+\phi)$,

where A, a, ω, and ϕ are real numbers with $A>0$ and $-\pi<\varphi\le\pi$:

a) $x(t)=-2$ b) $x(t)=\sqrt{2}\cdot\mathrm{e}^{\mathrm{j}\pi/4}\cos\left(3t+2\pi\right)$

c) $x(t)=\mathrm{e}^{-t}\cdot\sin\left(3t+\pi\right)$ d) $x(t)=\mathrm{j}\cdot\mathrm{e}^{(-2+\mathrm{j}100)t}$

Problem 4 Compute the convolution of the signals:

a) $x_1(t)=\mathrm{e}^{2t}\cdot u(-t)$; $x_2(t)=u(t-3)$

b) $x_1(t)=\begin{cases}1+t\ ; & 0\le t\le 1\\ 2-t\ ; & 1<t\le 2\\ 0\ ; & \text{elsewhere}\end{cases}$; $x_2(t)=\delta(t+2)+2\cdot\delta(t+1)$

c) $x_1(t)=u(t-3)-u(t-5)$; $x_2(t)=\mathrm{e}^{-3t}\cdot u(t)$ d) $x(t)*u(t)$

Problem 5 Derive the following FOURIER transform properties:

a) Time and frequency shifting b) Scaling

c) Differentiation and integration d) Convolution and product

Problem 6 Compute the FOURIER transforms of the signals:

a) $x(t)=\begin{cases}\cos(\omega_0 t)\ ; & t>0\\ 0\ ; & t<0\end{cases}$ b) $x(t)=\begin{cases}\sin(\omega_0 t)\ ; & t>0\\ 0\ ; & t<0\end{cases}$ c) $x(t)=\begin{cases}1-t\ ; & 0<t<1\\ 1+t\ ; & -1<t<0\\ 0\ ; & \text{elsewhere}\end{cases}$

d) $x(t)=1+\cos\left(6\pi t+\dfrac{\pi}{8}\right)$ e) $x(t)=t\cdot\left(\dfrac{\sin t}{\pi t}\right)^2$

Problem 7 The medium duration D and the medium bandwidth B of a signal are given by

$$D=\frac{1}{x_{\max}}\int_{-\infty}^{\infty}x(t)\mathrm{d}t\ ,\quad B_\omega=\frac{1}{X_{\max}}\int_{-\infty}^{\infty}X(\omega)\mathrm{d}\omega\ ,\quad B_\omega=2\pi\cdot B$$

Calculate the product $D\cdot B_\omega$ for a rectangular impulse of width T. *Aid:* $\int_0^\infty \mathrm{si}(ax)\cdot\mathrm{d}x=\dfrac{\pi}{2a}$

Answers

Problem 1 a) $P_\infty = 0;\ E_\infty = \frac{1}{4}$; b) $P_\infty = 1;\ E_\infty = \infty$; c) $P_\infty = \frac{1}{2};\ E_\infty = \infty$

Problem 2 a) No ; b) $T = \frac{\pi}{5}$; c) No

Problem 3

a) $A = 2$; $a = 0$; $\omega = 0$; $\varphi = \pi$ b) $A = 1;\ a = 0;\ \omega = 3;\ \phi = 0$

c) $A = 1$; $a = 1$; $\omega = 3$; $\varphi = \frac{\pi}{2}$ d) $A = 1$; $a = 2$; $\omega = 100$; $\varphi = \frac{\pi}{2}$

Problem 4

a) $y(t) = \begin{cases} \frac{1}{2}\cdot e^{2(t-3)} & \text{for } t \le 3 \\ \frac{1}{2} & \quad\ \ t \ge 3 \end{cases}$ ... b) $y(t) = \begin{cases} 3+t & -2 < t \le -1 \\ 4+t & -1 < t \le 0 \\ 2-2t & \ \ 0 < t \le 1 \\ 0 & \text{elsewhere} \end{cases}$

c) $y(t) = \begin{cases} 0 & -\infty < t \le 3 \\ \frac{1}{3}\left[1 - e^{-3(t-3)}\right] & \text{for } 3 < t \le 5 \\ \frac{1}{3}\left(1 - e^{-6}\right)\cdot e^{-3(t-5)} & \quad\ \ 5 < t < \infty \end{cases}$ d) $\int_{-\infty}^{t} x(\tau)\,d\tau$

Problem 5 See lecture

Problem 6

a) $X(j\omega) = \frac{\pi}{2}\left(\delta(\omega+\omega_0) + \delta(\omega-\omega_0)\right) - \frac{j\omega}{\omega^2 - \omega_0^2},\ X(f) = \frac{1}{4}\left(\delta(f+f_0) + \delta(f-f_0)\right) - \frac{jf}{2\pi\cdot\left(f^2 - f_0^2\right)}$

b) $X(j\omega) = \frac{j\pi}{2}\left(\delta(\omega+\omega_0) - \delta(\omega-\omega_0)\right) - \frac{\omega_0}{\omega^2 - \omega_0^2},\ X(f) = \frac{j}{4}\left(\delta(f+f_0) - \delta(f-f_0)\right) - \frac{f_0}{2\pi\cdot\left(f^2 - f_0^2\right)}$

c) $X(j\omega) = \mathrm{si}^2\left(\frac{\omega}{2}\right) = \mathrm{sinc}^2\left(\frac{\omega}{2\pi}\right)$ e) $X(j\omega) = \begin{cases} j/2\pi & ; \ -2 \le \omega < 0 \\ -j/2\pi & ; \ \ 0 \le \omega < 2 \\ 0 & ; \ \text{elsewhere} \end{cases}$

d) $X(j\omega) = 2\pi\delta(\omega) + \pi\left[e^{j\pi/8}\cdot\delta(\omega - 6\pi) + e^{-j\pi/8}\cdot\delta(\omega + 6\pi)\right]$

Problem 7 $D\cdot B_\omega = 2\pi$, $D\cdot B = 1$