Probability and Statistics

4 – Continuous Random Variables

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Random Variables

Definition (4.1)

Let Ω be a sample space and $\mathcal{A} \subseteq \mathcal{P}(\Omega)$ a set of events with a probability measure $\Pr : \mathcal{A} \to \mathbb{R}$. A mapping $X : \Omega \to \mathbb{R}$ is called a *random variable* if

$$X^{-1}((-\infty, x]) \;=\; \{\omega \in \Omega \mid X(\omega) \leq x\} \;\in\; \mathcal{A}$$

for all $x \in \mathbb{R}$.

Note: Discrete Random variables are random variables in the sense of the definition given above and the definition of a *cumulative distribution function* (cdf) given for discrete random variables generalizes to all random variables:

$$F_X : \mathbb{R} \to [0,1], \qquad \boxed{F_X(x) \;:=\; \Pr(X \leq x)}$$

Lemma (4.2)

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From Definition (4.1) it follows, that $X^{-1}(S) \in \mathcal{A}$ *for all intervals* $S \subseteq \mathbb{R}$.

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(i) $S = \mathbb{R} = (-\infty, \infty)$ $\quad X^{-1}(\mathbb{R}) = \Omega \in \mathcal{A}$

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(i) $S = \mathbb{R} = (-\infty, \infty)$

(ii) $S = (-\infty, x)$

$$X^{-1}(-\infty, x) = X^{-1}\left(\bigcup_{i=1}^{\infty} (-\infty, x - \tfrac{1}{i}]\right)$$

$$= \underbrace{\bigcup_{i=1}^{\infty} \underbrace{X^{-1}(-\infty, x - \tfrac{1}{i}]}_{\in \mathcal{A}}}_{\in \mathcal{A}}$$

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(iii) $S = [x, \infty)$

(iv) $S = (x, \infty)$

$\mathbb{R} = (-\infty, x) \,\dot\cup\, [x, \infty)$

$X^{-1}([x,\infty)) = X^{-1}(\mathbb{R} \setminus (-\infty, x)) = \underbrace{\Omega \setminus \overbrace{X^{-1}((-\infty,x))}^{\in \mathcal{A}\ (ii)}}_{\in \mathcal{A}}$

Lemma (4.2)

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From Definition (4.1) it follows, that $X^{-1}(S) \in \mathcal{A}$ for all intervals $S \subseteq \mathbb{R}$. $x_1, x_2 \in \mathbb{R}$

(i) $S = \mathbb{R} = (-\infty, \infty)$

(ii) $S = (-\infty, x)$

(iii) $S = [x, \infty)$

(iv) $S = (x, \infty)$

(v) $S = (x_1, x_2)$

(vi) $S = (x_1, x_2]$ $\leftarrow X^{-1}\left(\mathbb{R} \setminus \left((x_2,\infty) \,\dot\cup\, (-\infty,x_1]\right)\right) = \underbrace{\Omega \setminus \underbrace{\left(\underbrace{X^{-1}((x_2,\infty))}_{\in\mathcal{A}\ \text{(iv)}} \cup \underbrace{X^{-1}((-\infty,x_1])}_{\in\mathcal{A}\ \text{Def.}}\right)}_{\in\mathcal{A}}}_{\in\mathcal{A}}$

(vii) $S = [x_1, x_2)$

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Lemma (4.3)

Lemma

(i) *If* $A_i \in \mathcal{A}$ *with* $A_1 \subseteq A_2 \subseteq \ldots$, *then:* $$\Pr\left(\bigcup_{i=1}^{\infty} A_i\right) = \lim_{n\to\infty} \Pr(A_n)$$

Ex. $A_i = X^{-1}((-\infty, x - \frac{1}{i}])$

in pf of (4.2)(ii)

(ii) *If* $A_i \in \mathcal{A}$ *with* $A_1 \supseteq A_2 \supseteq \ldots$, *then:* $$\Pr\left(\bigcap_{i=1}^{\infty} A_i\right) = \lim_{n\to\infty} \Pr(A_n)$$

Proof of Lemma (4.3)

(i) If $A_i \in \mathcal{A}$ with $A_1 \subseteq A_2 \subseteq \dots$, then: $\Pr\left(\bigcup_{i=1}^{\infty} A_i\right) = \lim_{n\to\infty} \Pr(A_n)$

$B_1 = A_1$, $B_i = A_i \setminus A_{i-1}$ $(i \geq 2)$

$$\Pr\left(\bigcup_{i=1}^{\infty} A_i\right) = \Pr\left(\dot{\bigcup_{i=1}^{\infty}} B_i\right) = \sum_{i=1}^{\infty} \Pr(B_i) = \lim_{n\to\infty} \sum_{i=1}^{n} \Pr(B_i)$$

$$= \lim_{n\to\infty} \Pr\left(\dot{\bigcup_{i=1}^{n}} B_i\right) = \lim_{n\to\infty} \Pr(A_n)$$

$\square$

Proof of Lemma (4.3)

(ii) If $A_i \in \mathcal{A}$ with $A_1 \supseteq A_2 \supseteq \ldots$, then: $$\Pr\left(\bigcap_{i=1}^{\infty} A_i\right) = \lim_{n\to\infty} \Pr(A_n)$$

HW

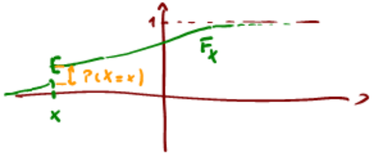
Cumulative Distribution Functions

Lemma (4.4)

If F_X is a cumulative distribution function, then the following holds:

(i) $\Pr(a < X \leq b) \;=\; F_X(b) - F_X(a)$ *for all* $a, b \in \mathbb{R}$ *with* $a < b$

(ii) F_X *is monotonically increasing.*



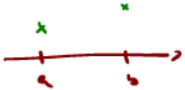
(iii) $\lim\limits_{x \to -\infty} F_X(x) \;=\; 0\,, \quad \lim\limits_{x \to \infty} F_X(x) \;=\; 1$

(iv) $F_X(x+) \;:=\; \lim\limits_{\xi \to x+} F_X(\xi) \;=\; F_X(x)$ *for every* $x \in \mathbb{R}$

(v) $\Pr(X = x) \;=\; F_X(x) \;-\; F_X(x-)$ *for every* $x \in \mathbb{R}$

Proof of Lemma (4.4)

(i) $\overset{\geq 0}{\Pr(a < X \leq b)} = F_X(b) - F_X(a)$ for all $a, b \in \mathbb{R}$ with $a < b$



$\Downarrow$

(ii) F_X is monotonically increasing.

$$
\begin{aligned}
\text{(i)}: \quad F_X(b) - F_X(a) &= \Pr(X \in (-\infty, b]) - \Pr(X \in (-\infty, a]) \\
&= \Pr(X^{-1}(-\infty, b]) - \Pr(X^{-1}(-\infty, a]) \\
&= \Pr(X^{-1}(-\infty, b] \setminus X^{-1}(-\infty, a]) \\
&= \Pr(X^{-1}((-\infty, b] \setminus (-\infty, a])) \\
&= \Pr(X^{-1}((a, b])) = \Pr(a < X \leq b)
\end{aligned}
$$

Proof of Lemma (4.4)

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(iii) $\lim_{x \to -\infty} F_X(x) = 0\,, \quad \lim_{x \to \infty} F_X(x) = 1$

$A_i := X^{-1}((-\infty, i]) \in \mathcal{A}\,, \quad A_1 \subseteq A_2 \subseteq A_3$

$$\lim_{x \to \infty} F_X(x) = \lim_{\substack{i \to \infty \\ (i \in \mathbb{N})}} F_X(i) = \lim_{i \to \infty} \Pr(X \in (-\infty, i]) = \lim_{i \to \infty} \Pr(A_i)$$

$$\overset{(4.3)(i)}{=} \Pr\left(\bigcup_{i=1}^{\infty} A_i\right) = \Pr(\Omega) = 1$$

Proof of Lemma (4.4)

(iv) $\quad F_X(x+) := \lim\limits_{\xi \to x+} F_X(\xi) = F_X(x) \quad$ for every $x \in \mathbb{R}$

Proof. Let $(x_i)_{i\in\mathbb{N}}$ be a sequence of real numbers with $x_i > x$ for all $i \in \mathbb{N}$, $x_1 \geq x_2 \geq x_3 \geq \dots$ and $\lim_{i\to\infty} x_i = x$. Then:

e.g. $x_i = x + \frac{1}{i}$

$$F_X(x+) = \lim_{i\to\infty} F_X(x_i) = \lim_{i\to\infty} \Pr(X \leq x_i)$$

$A_i = X^{-1}((-\infty, x_i])$

$$\overset{(4.3)(\mathrm{ii})}{=} \Pr\left(\bigcap_{i\to\infty} \{\omega \in \Omega \mid X(\omega) \leq x_i\}\right)$$

$$= \Pr\left(\{\omega \in \Omega \mid X(\omega) \leq x\}\right) = F_X(x)$$

Proof of Lemma (4.4)

(v) $\Pr(X = x) \;=\; F_X(x) - F_X(x-)$ for every $x \in \mathbb{R}$

Proof. Put $x_i := x - \frac{1}{i}$. Then

$$
\begin{aligned}
F_X(x-) &= \lim_{i\to\infty} F_X(x_i) = \lim_{i\to\infty} \Pr(X \le x_i) \\
&\overset{(4.3)(\mathrm{i})}{=} \Pr\left(\bigcup_{i\to\infty} \{\omega \in \Omega \mid X(\omega) \le x_i\} \right) \\
&= \Pr\left(\{\omega \in \Omega \mid X(\omega) < x\}\right) = \Pr(X < x)
\end{aligned}
$$

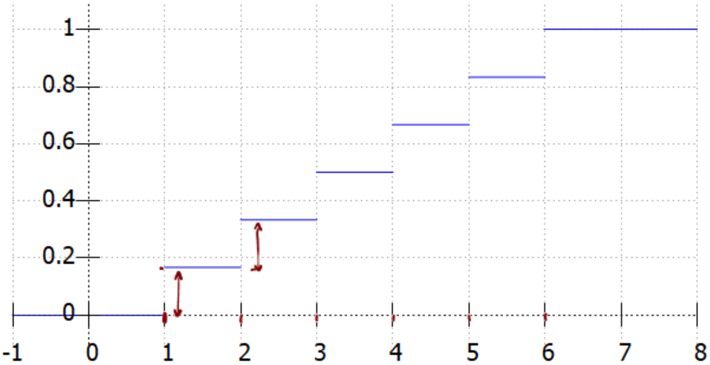
and:

$$\Pr(X = x) = \Pr(X \le x) - \Pr(X < x) = F_X(x) - F_X(x-)$$

Cumulative Distribution Functions

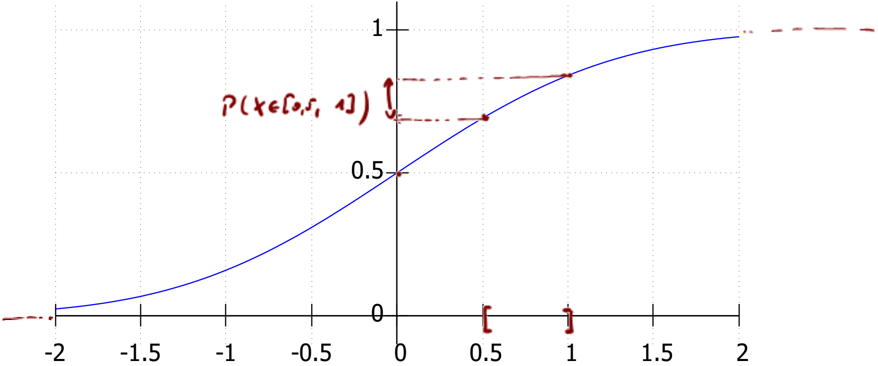
cdf

fair die

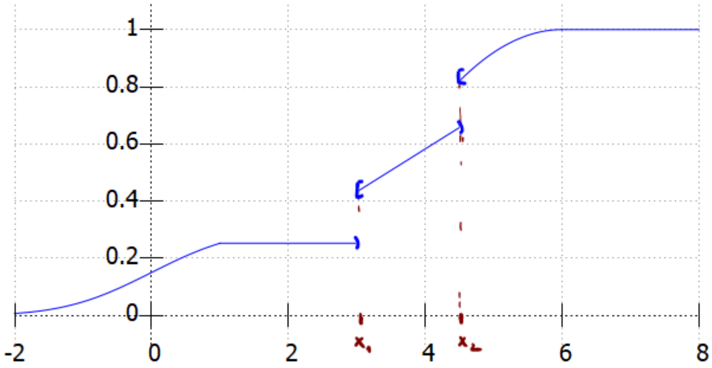


$P(X=1) = \frac{1}{6}$

Cumulative Distribution Functions



Cumulative Distribution Functions



Quantile Functions

Definition (4.5)

Let X be a random variable with cdf F_X. The *quantile function* of X is defined for all $p \in (0,1)$ by:

$$F_X^{-1}(p) \quad := \quad \min\{x \mid F_X(x) \geq p\}$$

First quartile, *median* and *third quartile* are defined to be: $F_X^{-1}\left(\frac{1}{4}\right)$, $F_X^{-1}\left(\frac{1}{2}\right)$ and $F_X^{-1}\left(\frac{3}{4}\right)$

Note: If F_X is continuous and strictly increasing, then restricting the codomain of F_X to $F_X(\mathbb{R}) = (0,1)$ yields a bijective mapping with the quantile function as the inverse mapping.

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Note: If F_X is continuous and strictly increasing, then restricting the codomain of F_X to $F_X(\mathbb{R}) = (0,1)$ yields a bijective mapping with the quantile function as the inverse mapping. Moreover:

(i) $F_X(F_X^{-1}(p)) \;\geq\; p$ for all $p \in (0,1)$

(ii) $F_X(F_X^{-1}(p)) \;=\; p$ for all $p \in (0,1)$ if F_X is continuous

(iii) F_X^{-1} is strictly increasing if F_X is continuous

(iv) $F_X^{-1}(F_X(x)) \;\leq\; x$ for all $x \in \mathbb{R}$ with $F_X(x) \in (0,1)$

Quantile Functions

$F_X^{-1}(p) := \min\{x \mid F_X(x) \geq p\}$

(i) $F_X(F_X^{-1}(p)) \geq p$ for all $p \in (0,1)$

(ii) $F_X(F_X^{-1}(p)) = p$ for all $p \in (0,1)$ if F_X is continuous

(iii) F_X^{-1} is strictly increasing if F_X is continuous

(iv) $F_X^{-1}(F_X(x)) \leq x$ for all $x \in \mathbb{R}$ with $F_X(x) \in (0,1)$