1.) Let X be a random variable with cdf $F_X(x)$. For $a, b \in \mathbb{R}, a \neq 0$ consider the random variable

$$Y = aX + b$$

and show that its cdf is given by:

$$F_Y(x) = \begin{cases} F_X(\frac{x-b}{a}) & \text{if } a > 0 \\ 1 - F_X\left(\frac{x-b}{a}\right) + \Pr\left(X = \frac{x-b}{a}\right) & \text{if } a < 0 \end{cases}$$

2.) Let X and Y be as in exercise 1. Furthermore assume that X is continuous with pdf $f_X(x)$. Show that Y is continuous with pdf:

$$f_Y(x) = \frac{1}{|a|} \cdot f_X\left(\frac{x-b}{a}\right)$$

3.) Let X be a continuous random variable with cdf $F_X(x)$ and pdf $f_X(x)$. For $a \in \mathbb{R}^+$ and $b \in \mathbb{R}$ consider the random variable

$$Y = \frac{X-b}{a}$$

and show that its cdf and pdf are given by:

$$F_Y(x) = F_X(a\,x + b) = F_X\left(a\left(x + \frac{b}{a}\right)\right)$$

$$f_Y(x) = a \cdot f_X(a\,x + b) = a \cdot f_X\left(a\left(x + \frac{b}{a}\right)\right)$$

In particular, if X has finite expectation $\mu = E(X)$ and variance $\sigma^2 = \text{Var}(X)$ then the normalized version of X given by $Y = \dfrac{X-\mu}{\sigma}$ possesses the pdf:

$$f_Y(x) = \sigma \cdot f_X(\sigma\,x + \mu) = = \sigma \cdot f_X\left(\sigma\left(x + \frac{\mu}{\sigma}\right)\right)$$

4.) Provide a MatLab implementation of a simulation of a random variable having a normal distribution with $\mu = 1500$ and $\sigma = 300$. The simulation shall be based on the simple random number generating function `rand` and the inverse function of the cdf $\Phi(x)$ of the standard normal distribution.

Generate a plot of 1000 values from the simulation. (The plot should be similar to Fig. 4 in the first chapter of the script.)

5.) Packet transmission times on a certain Internet link are independent identical random variables with mean m and variance σ^2. Suppose n packets are transmitted. Then the total expected transmission time for the n packets is nm. Use the central limit theorem to approximate the probability that the total transmission time for the n packets exceeds twice the expected transmission time.

6.) Solve problem 28 and problems 30–33 from [Ross, p.199].

7.) Solve problems 13–16 from [Ross, chapter 6, p.225].

8.) Let X be a random variable with $X \sim \text{Poisson}(\lambda)$ and X^o be the normalized version of X. The following figure shows the cdf F_{X^o} of X^o for some $\lambda \in \mathbb{N}$ and the cdf Φ of the standard normal distribution.

Determine λ:

Could you justify why F_{X^o} and Φ are so similar?

Now assume $X \sim \text{Poisson}(101.3)$ and calculate:

9.) Prove: If X and Y are independent continous random variables with $Y, X \sim \mathcal{N}(0, \sigma^2)$, then $\frac{X}{Y} \sim \text{Cauchy}(1)$.

10.) Provide a MatLab script that generates a series of plots for the pdf's of gamma distributions $\Gamma(\frac{n}{2}, \frac{1}{2})$ for a suitable selection of numbers $n \in \mathbb{N}$. (In particular plots should be generated for $n = 1, ..., 10$ and some numbers greater than 10.) Compare some of these graphs with the graphs of the pdf's of the normal distributions having the same expectations and variances.

11.) For $\beta, b \in \mathbb{R}^+$, $n \in \mathbb{N}$:

(i) $\quad X \sim \Gamma(\alpha, \beta) \quad \Longrightarrow \quad b \cdot X \sim \Gamma(\alpha, \beta/b)$

(ii) $\quad X \sim \text{ Erlang}\,(n, \beta) \quad \Longrightarrow \quad 2\beta \cdot X \sim \chi^2_{2n}$

12.) Prove: If X and Y are independent continous random variables with $X \sim \Gamma(\alpha_1, \beta)$ and $Y \sim \Gamma(\alpha_2, \beta)$, then a pdf of $Z = \frac{X}{Y}$ is gven by:

$$f_Z(z) = \frac{1}{B(\alpha_1, \alpha_2)} \cdot \frac{z^{\alpha_1 - 1}}{(1+z)^{\alpha_1+\alpha_2}} \cdot I_{(0,\infty)}$$

13.) Use (4.86) to provide a new proof of:

$$\int_{-\infty}^{\infty} e^{-x^2}\, dx = \sqrt{\pi}$$

Values of `normcdf(x)`:

x	$\Phi(x)$	x	$\Phi(x)$	x	$\Phi(x)$
0.1	0.53983	1.1	0.86433	2.1	0.98214
0.2	0.57926	1.2	0.88493	2.2	0.98610
0.3	0.61791	1.3	0.90320	2.3	0.98928
0.4	0.65542	1.4	0.91924	2.4	0.99180
0.5	0.69146	1.5	0.93319	2.5	0.99379
0.6	0.72575	1.6	0.94520	2.6	0.99534
0.7	0.75804	1.7	0.95543	2.7	0.99653
0.8	0.78814	1.8	0.96407	2.8	0.99744
0.9	0.81594	1.9	0.97128	2.9	0.99813
1.0	0.84134	2.0	0.97725	3.0	0.99865

Values of `norminv(p)`:

$p = .95$	$p = .975$	$p = .99$	$p = .995$	$p = .999$	$p = .9995$
1.64485	1.95996	2.32635	2.57583	3.09023	3.29053