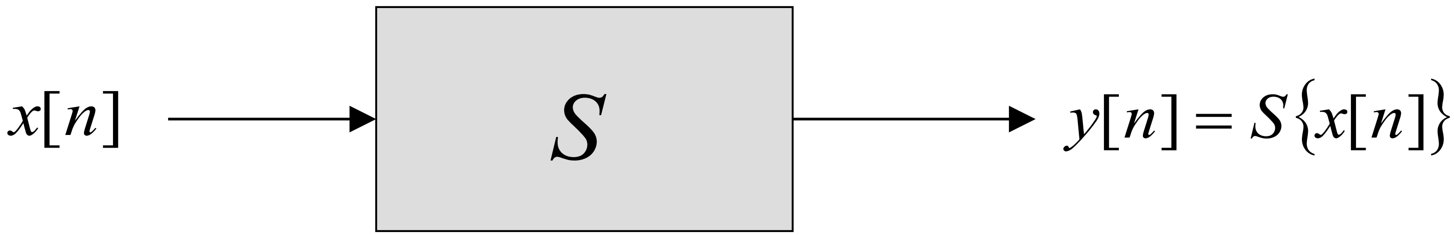
Chapter 5

Discrete-Time Systems

5.1 System Definition and Properties



- A **discrete-time system** processes a given input sequence $x[n]$ and generates an output sequence $y[n]$.
- $S\{\dots\}$ is the discrete-time **operator equation**.

- **Linearity** enables superposition principle

  $$x_1[n] \rightarrow y_1[n], \quad x_2[n] \rightarrow y_2[n]$$

  $$x_1[n] + x_2[n] \rightarrow y_1[n] + y_2[n]$$

  $$K \cdot x[n] \rightarrow K \cdot y[n], \quad K \in \mathrm{R}$$

  → This feature requires **source free** systems.

- **Shift invariance** is equivalent to time invariance, if n is related to a discrete instant of time

  $$x[n] \rightarrow y[n] \;;\;\; x[n-n_0] \rightarrow y[n-n_0] \;;\;\; n_0 : \text{integer}$$

- **Causality:** The output sample $y[n_0]$ depends only on input samples $x[n]$ for $n \leq n_0$.
- Systems with and without **memory**
  - **Memoryless system**: The output value at a specific time depends only on the input value at that same time. → No energy storage. Such systems are non-dynamic systems.
  - **Memory systems**: The output value at a specific time depends on the history. → Energy storage. Such systems are dynamic systems. A dynamic system has a memory of length N.

- **Stability:** A bounded input results in a bounded output. → Bounded-input, bounded-output (BIBO) stability.

$$x[n] \rightarrow y[n] \ ; \ \ |x[n]| < B_x \rightarrow |y[n]| < B_y$$

- **Passive and lossless systems**
  - A system is passive, if for every finite-energy input sequence $x[n]$ the output sequence $y[n]$ has at most the same energy.

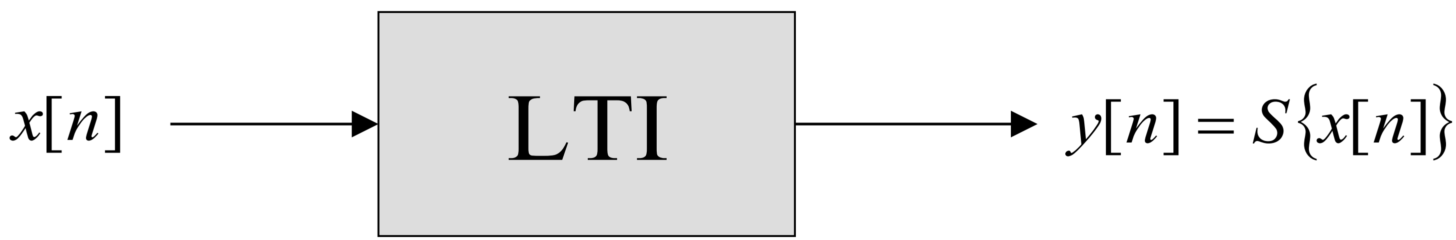
$$\sum_{n=-\infty}^{\infty} |y[n]|^2 \leq \sum_{n=-\infty}^{\infty} |x[n]|^2 < \infty$$

  - For a lossless system, the energy of the input and output sequences are identical.

$$\sum_{n=-\infty}^{\infty} |y[n]|^2 = \sum_{n=-\infty}^{\infty} |x[n]|^2 < \infty$$

5.2 Time-Domain Representation

5.2.1 Impulse and Step Response



- Unit sample response or simply: **impulse response**

$$x[n] = \delta[n] \quad \rightarrow \quad y[n] = S\{\delta[n]\} = h[n]$$

- Unit step response or simply: **step response**

$$x[n] = u[n] \quad \rightarrow \quad y[n] = S\{u[n]\} = s[n]$$

- A linear time-invariant (LTI) discrete-time system is completely characterized by its impulse or step response.

- **Causality** of a discrete-time LTI system

$$h[n] = 0 \quad \text{for} \quad n < 0$$

- **Stability** of a discrete-time LTI system

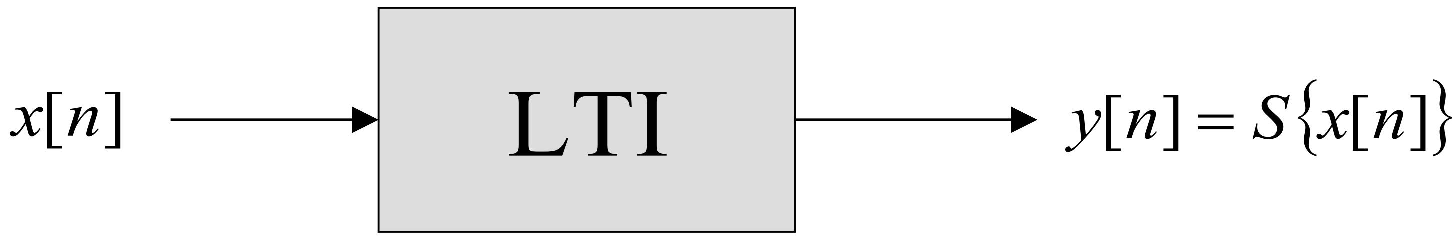
$$\sum_{n=-\infty}^{\infty} \left| h[n] \right| < \infty$$

- **Memoryless** discrete-time LTI system

$$h[n] = 0 \quad \text{for} \quad n \neq 0$$

- **Finite-impulse response (FIR) systems**: The impulse response is a finite-length sequence.
- **Infinite-impulse response (IIR) systems**: The impulse response is an infinite-length sequence.
- Systems with real or complex impulse responses are defined as **real** or **complex system**.

5.2.2 Convolution Sum



- The response $y[n]$ of the discrete-time system to the input sequence $x[n]$ is given by

$$y[n] = \sum_{k=-\infty}^{\infty} x[k] \cdot h[n-k] = \sum_{k=-\infty}^{\infty} x[n-k] \cdot h[k]$$

$$y[n] = x[n] * h[n] = h[n] * x[n]$$

- **FIR systems**

$$h[n]=0 \;\text{ for }\; n<N_1 \;\text{ and }\; n>N_2 \;\text{ with }\; N_1<N_2$$

$$y[n]=\sum_{k=N_1}^{N_2} h[k]\cdot x[n-k]=\sum_{k=n-N_2}^{n-N_1} x[k]\cdot h[n-k]$$

  - Examples: moving-average filter, linear interpolator
  - The input sequence $x[n]$ can be of finite or infinite length.

- **IIR systems**

  For a causal IIR discrete-time system with a causal length-N input $x[n]$, the convolution sum can be expressed in the form

$$y[n] = \sum_{k=0}^{N-1} x[k] \cdot h[n-k] = \sum_{k=n-N+1}^{n} h[k] \cdot x[n-k]$$

  - Examples: accumulator, exponentially weighted running average filter
  - The input sequence $x[n]$ must be of finite length N.

5.2.3 Difference Equations

$$x[n] \longrightarrow \boxed{S} \longrightarrow y[n] = S\{x[n]\}$$

- The derivative is approximated by a difference equation

$$\frac{\mathrm{d}\,y(t)}{\mathrm{d}\,t} \approx \frac{y(nT_\mathrm{s}) - y((n-1)\cdot T_\mathrm{s})}{T_\mathrm{s}} = \frac{y[n] - y[n-1]}{T_\mathrm{s}}$$

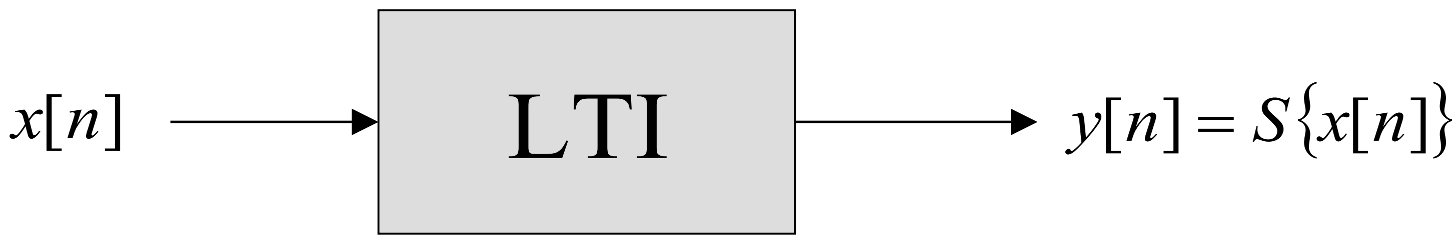
Backward EULER

$$\frac{\mathrm{d}\,y(t)}{\mathrm{d}\,t} \approx \frac{y((n+1)\cdot T_\mathrm{s}) - y(nT_\mathrm{s})}{T_\mathrm{s}} = \frac{y[n+1] - y[n]}{T_\mathrm{s}}$$

Forward EULER

- Input-output relation for **causal systems** (only present and past samples)

$$y[n] = f\{y[n-1], \ldots, y[n-N], x[n], x[n-1], \ldots, x[n-M], n\}$$



- Input and output of an LTI system are related through a **linear constant coefficient difference equation**

$$a_0 \cdot y[n] + a_1 \cdot y[n-1] + \ldots + a_N \cdot y[n-N] =$$

$$b_0 \cdot x[n] + b_1 \cdot x[n-1] + \ldots + b_M \cdot x[n-M]$$

$$\sum_{k=0}^{N} a_k \cdot y[n-k] = \sum_{k=0}^{M} b_k \cdot x[n-k]$$

order = $\max\{N, M\}$

- **Recursive computation** of output sequence $y[n]$

$$y[n] = -\frac{a_1}{a_0}\cdot y[n-1] - \ldots - \frac{a_N}{a_0}\cdot y[n-N] + \frac{b_0}{a_0}\cdot x[n] + \ldots + \frac{b_M}{a_0}\cdot x[n-M]$$

$$y[n] = -\sum_{k=1}^{N}\frac{a_k}{a_0}\cdot y[n-k] + \sum_{k=0}^{M}\frac{b_k}{a_0}\cdot x[n-k]$$

  - The output $y[n]$ can be computed for all $n \geq n_0$, knowing $x[n]$ and the initial conditions $y[n_0-1]$, $y[n_0-2]$, ..., $y[n_0-N]$. These systems usually result in IIR systems.

- If $N = 0$ the above equation reduces to a **nonrecursive type**:

$$y[n] = \frac{b_0}{a_0}\cdot x[n] + \ldots + \frac{b_M}{a_0}\cdot x[n-M] = \sum_{k=0}^{M}\frac{b_k}{a_0}\cdot x[n-k]$$

  - Auxiliary conditions are not needed in order to compute $y[n]$. These systems are FIR systems.

Calculation of Total Solution

- Auxiliary condition: If $x[n] = 0$ for $n < n_0$ the condition of **initial rest** (zero initial condition) requires $y[n] = 0$ for $n < n_0$. $\rightarrow$ **Causal system**
- General solution: $y[n] = y_{\mathrm{h}}[n] + y_{\mathrm{p}}[n]$

- **Homogeneous** (complementary, natural) **solution** $y_\mathrm{h}[n]$ solves

$$a_0 \cdot y[n] + a_1 \cdot y[n-1] + \ldots + a_N \cdot y[n-N] = 0 \text{ with } y_\mathrm{h}[n] = \alpha \cdot p^n$$

  - **Characteristic polynomial:** $\sum_{k=0}^{N} a_k \cdot p^{N-k} = 0$
  - **Roots:** $p_1, p_2, \ldots, p_N$
  - Distinct roots: $y_\mathrm{h}[n] = \alpha_1 \cdot p_1^n + \alpha_2 \cdot p_2^n + \ldots + \alpha_N \cdot p_N^n$
  - p_1 is of multiplicity L; $p_2, \ldots, p_{N-L}$ are distinct.

$$y_\mathrm{h}[n] = \left(\alpha_1 + \alpha_2 \cdot n + \ldots + \alpha_L \cdot n^{L-1}\right) \cdot p_1^n + \alpha_{L+1} \cdot p_2^n + \ldots + \alpha_N \cdot p_{N-L}^n$$

- **Particular solution** $y_\mathrm{p}[n]$ is of the same form as the input or forcing function $x[n]$.

Impulse Response Calculation

- The impulse response $h[n]$ of a **causal LTI** discrete-time system is the output observed with input $x[n] = \delta[n]$. Because $x[n] = 0$ for $n > 0$, the particular solution is zero: $y_\mathrm{p}[n] = 0$.
- $h[n] = y_\mathrm{h}[n]$ with coefficients which meet the **zero initial condition**, i.e. $y[n] = 0$ for $n < 0$.
- The impulse response of a finite-dimensional LTI system characterized by a difference equation of order N or M is usually of **infinite length**.
- There exist LTI discrete-time systems with an infinite impulse response that cannot be characterized by a difference equation of the form

$$\sum_{k=0}^{N} a_k \cdot y[n-k] = \sum_{k=0}^{M} b_k \cdot x[n-k]$$

Stability

- A **causal LTI** discrete-time system characterized by a linear constant coefficient difference equation of the form

$$\sum_{k=0}^{N} a_k \cdot y[n-k] = \sum_{k=0}^{M} b_k \cdot x[n-k]$$

is **stable** if the magnitude of each of the roots of the characteristic polynomial is less than one.

Classification Based on Output Calculation

- **Nonrecursive discrete-time systems**: The output samples can be calculated sequentially, knowing only the present and past input samples. Example: FIR system

$$y[n] = \sum_{k=0}^{M} \frac{b_k}{a_0} \cdot x[n-k]$$

- **Recursive discrete-time systems**: The computation of the output samples involves past output samples in addition to the present and past input samples. Example:

$$y[n] = -\sum_{k=1}^{N} \frac{a_k}{a_0} \cdot y[n-k] + \sum_{k=0}^{M} \frac{b_k}{a_0} \cdot x[n-k]$$

- **Moving average (MA) model**: It is a generalization of the moving average filter and is a FIR system.

$$y[n] = \sum_{k=0}^{M} b_k \cdot x[n-k]$$

- **Autoregressive (AR) model**: It is an IIR system.

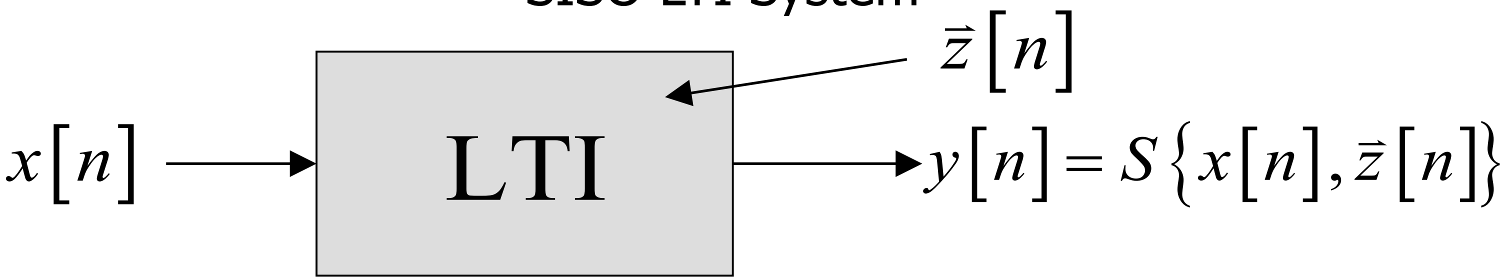
$$y[n] = x[n] - \sum_{k=1}^{N} a_k \cdot y[n-k]$$

- **Autoregressive moving average (ARMA) model**: It is an IIR system.

$$y[n] = \sum_{k=0}^{M} b_k \cdot x[n-k] - \sum_{k=1}^{N} a_k \cdot y[n-k]$$

5.2.4 State Space

SISO LTI System



- ...

5.3 Frequency-Domain Representation

$$x[n] \longrightarrow \boxed{\text{LTI}} \longrightarrow y[n] = S\{x[n]\}$$

- Stimulation with an **eigenfunction**

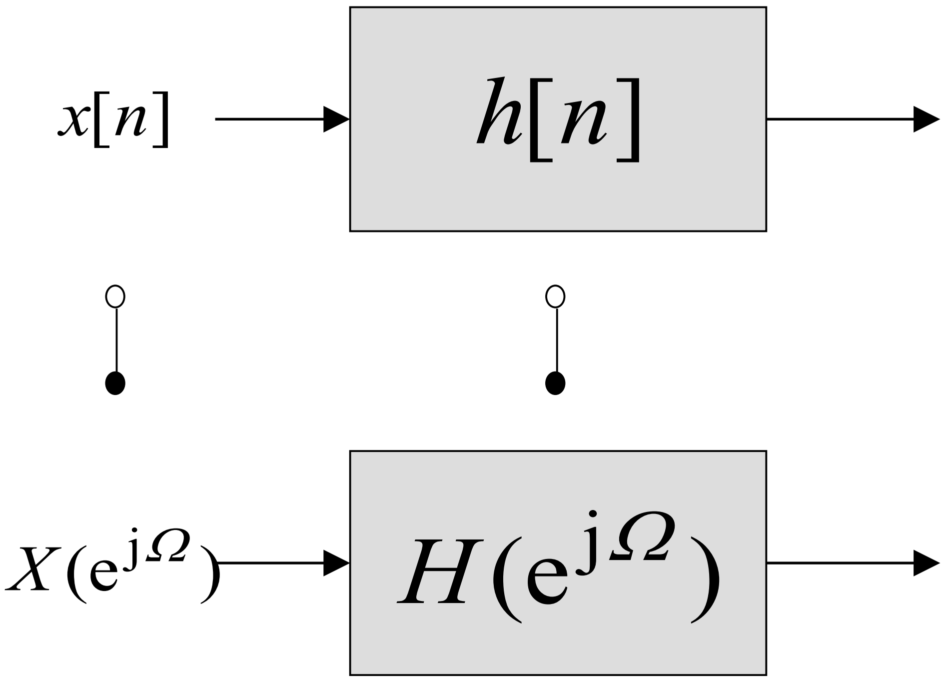
$$x[n] = A \cdot \mathrm{e}^{\mathrm{j}n\Omega} \;\; ; \;\; y[n] = A \cdot H\left(\mathrm{e}^{\mathrm{j}\Omega}\right) \cdot \mathrm{e}^{\mathrm{j}n\Omega} \;\; \text{with} \;\; \Omega = \omega T_\mathrm{s}$$

- $H(\mathrm{e}^{\mathrm{j}\Omega})$ is the **eigenvalue** or **frequency response**

$$h[n] \;\circ\!-\!\bullet\; H\left(\mathrm{e}^{\mathrm{j}\Omega}\right) = \left|H(\mathrm{e}^{\mathrm{j}\Omega})\right| \cdot \mathrm{e}^{\mathrm{j}\varphi(\Omega)}$$

  - $\left|H(\mathrm{e}^{\mathrm{j}\Omega})\right|$ is the **magnitude** (amplitude) **response**
  - $\varphi(\Omega)$ is the **phase response**

LTI System: Time domain versus frequency domain



$$y[n] = x[n] * h[n]$$

$$Y(e^{j\Omega}) = X(e^{j\Omega}) \cdot H(e^{j\Omega})$$

$$H\left(e^{j\Omega}\right) = \frac{Y\left(e^{j\Omega}\right)}{X\left(e^{j\Omega}\right)}$$

Frequency Responses of LTI Discrete-Time Systems

- **Recursive type, IIR systems**

$$H\left(e^{j\Omega}\right) = \frac{Y\left(e^{j\Omega}\right)}{X\left(e^{j\Omega}\right)} = \frac{\sum\limits_{k=0}^{M} b_k \cdot e^{-jk\Omega}}{\sum\limits_{k=0}^{N} a_k \cdot e^{-jk\Omega}}$$

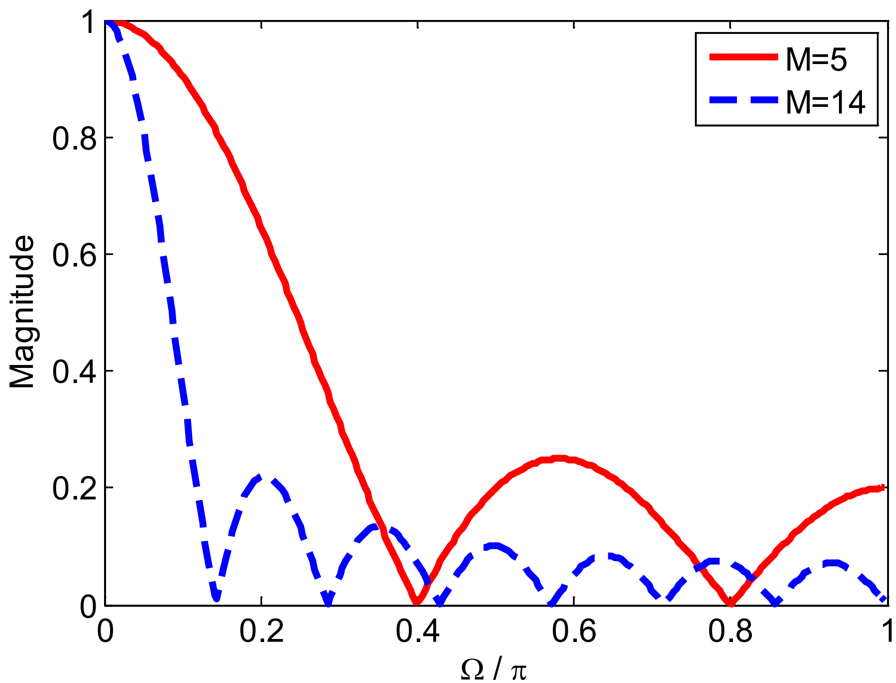
- **Nonrecursive type, FIR systems**

$$H(e^{j\Omega}) = \sum_{k=0}^{M} \frac{b_k}{a_0} \cdot e^{-jk\Omega} = \sum_{k=N_1}^{N_2} h[k] \cdot e^{-jk\Omega}$$

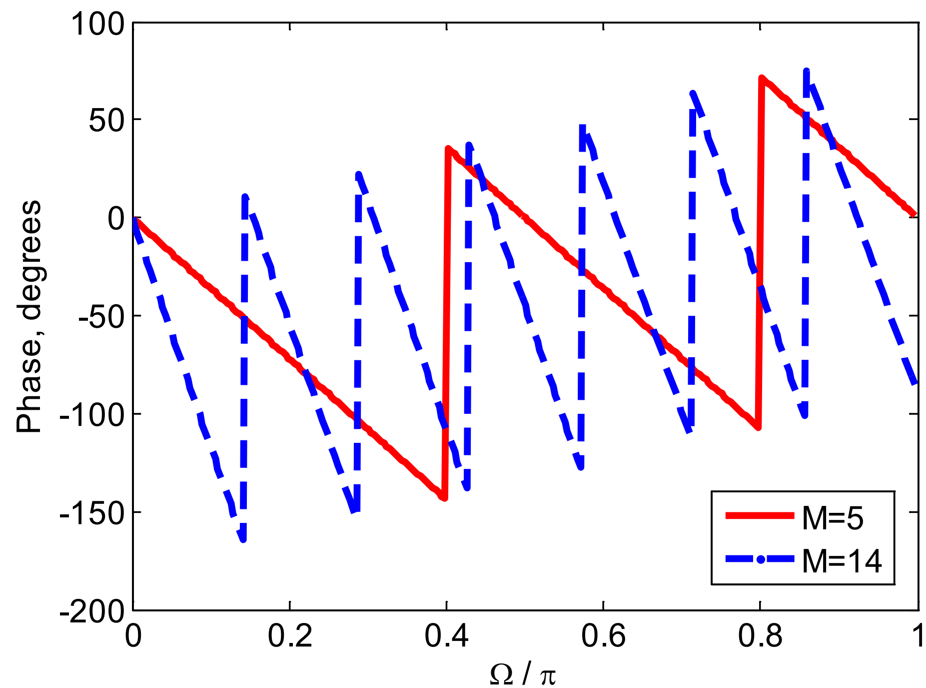
Example: Moving Average Filter

$$h[n] = \frac{1}{M} \sum_{i=0}^{M-1} \delta[n-i]$$

magnitude response



phase response



→ Use MATLAB command "freqz(h,w)"

5.4 z-Transform
5.4.1 Definition and Convergence

- The z-transform is a generalization of the DTFT: DTFT ∈ z-transform
- The z-transform is similar to the LAPLACE transform of time-continuous systems:
  - better convergence properties,
  - better system description.
- **Bilateral z-transform** (LAURENT series)

$$X(z) = \sum_{n=-\infty}^{\infty} x[n] \cdot z^{-n} \; ; \; z = r \cdot \mathrm{e}^{\mathrm{j}\Omega} \; ; \; \Omega = \omega \cdot T_{\mathrm{s}}$$

$$x[n] \circ\!-\!\bullet \; X(z) = \mathrm{Z}\{x[n]\}$$

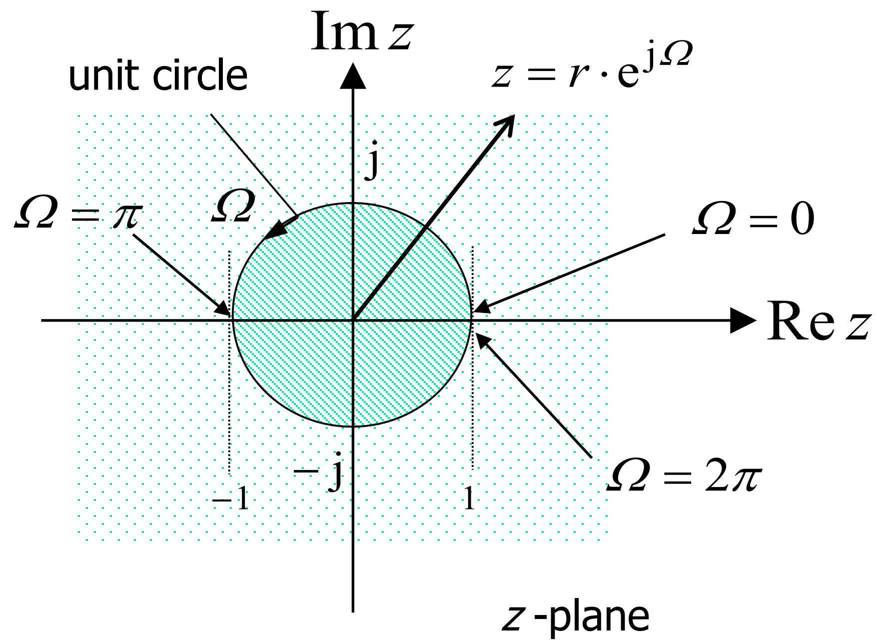
- For $r = 1$ the z-transform reduces to the DTFT.

- The z-transform converges if:

$$\sum_{n=-\infty}^{\infty} \left| x[n] \cdot r^{-n} \right| < \infty$$

- In general the **region of convergence** (ROC) is an annular region:

$$0 \le r_{\min} < |z| < r_{\max} < \infty$$



- The DTFT of a sequence $x[n]$ converges uniformly if and only if the ROC of its z-transform includes the unit circle.

- **Unilateral z-transform**

$$X_{\mathrm{U}}(z) = \sum_{n=0}^{\infty} x[n] \cdot z^{-n}$$

  - The bilateral and unilateral z-transform are identical for *causal sequences*.
  - The unilateral z-transform is particularly useful in analyzing causal systems specified by linear constant-coefficient difference equations with *nonzero initial conditions*.

→ **This course considers only the bilateral z-transform.**

5.4.2 Basic z-Transform Pairs

- **Unit impulse sequence**

$$x[n] = \delta[n] \quad \circ\!-\!\bullet \quad X(z) = 1 \; ; \; \text{All values of } z$$

- **Unit step sequence**

$$x[n] = u[n] \quad \circ\!-\!\bullet \quad X(z) = \frac{1}{1 - z^{-1}} \; ; \; |z| > 1$$

- **Causal exponential sequence**

$$x[n] = \alpha^{n} \cdot u[n] \quad \circ\!-\!\bullet \quad X(z) = \frac{1}{1 - \alpha \cdot z^{-1}} \; ; \; |z| > |\alpha|$$

- **Further pairs ...**

$$x[n] = n \cdot \alpha^n \cdot u[n] \quad \circ\!-\!\bullet \quad X(z) = \frac{\alpha \cdot z^{-1}}{\left(1 - \alpha \cdot z^{-1}\right)^2} \; ; \quad |z| > |\alpha|$$

$$x[n] = (n+1) \cdot \alpha^n \cdot u[n] \quad \circ\!-\!\bullet \quad X(z) = \frac{1}{\left(1 - \alpha \cdot z^{-1}\right)^2} \; ; \quad |z| > |\alpha|$$

5.4.3 Properties of the z-Transform

- $X(z)$ is rational, whenever $x[n]$ is a linear combination of real or complex exponentials.

polynomial in z^{-1}

$$X(z) = \frac{N_M(z)}{D_N(z)} = \frac{b_0 + b_1 \cdot z^{-1} + \ldots + b_M \cdot z^{-M}}{a_0 + a_1 \cdot z^{-1} + \ldots + a_N \cdot z^{-N}}$$

polynomial in z

$$X(z) = \frac{N_M(z)}{D_N(z)} = z^{(N-M)} \cdot \frac{b_0 \cdot z^M + b_1 \cdot z^{M-1} + \ldots + b_{M-1} \cdot z + b_M}{a_0 \cdot z^N + a_1 \cdot z^{N-1} + \ldots + a_{N-1} \cdot z + a_N}$$

$$X(z) = \frac{N_M(z)}{D_N(z)} = z^{(N-M)} \cdot \frac{b_0}{a_0} \cdot \frac{(z-z_1)\cdot(z-z_2)\cdots}{(z-p_1)\cdot(z-p_2)\cdots}$$

  - M roots z_i of $N(z)$ are referred to as the *finite* **zeros** of $X(z)$
  - N roots p_i of $D(z)$ are referred to as the *finite* **poles** of $X(z)$
  - $N > M$: $N-M$ *add.* **zeros** at $z = 0$. $M > N$: $M-N$ *add.* **poles** at $z = 0$.

- If $x[n]$ is of finite duration, than the ROC is the entire z-plane, except possibly $z = 0$ and/or $z = \infty$.
- If $x[n]$ is a causal (right-sided) sequence, and if the circle $|z| = r$ is in the ROC, then all finite values of z with $|z| > r$ will also be in the ROC.
- If $x[n]$ is an anticausal (left-sided) sequence, and if the circle $|z| = r$ is in the ROC, then all finite values of z with $|z| < r$ will also be in the ROC.
- If $x[n]$ is two sided, and if the circle $|z| = r$ is in the ROC, than the ROC is a ring in the z-plane that includes the circle $|z| = r$.
- The ROC does not contain any poles.
- If $X(z)$ is rational and $x[n]$ is causal, then the ROC is the region outside the outermost pole and includes $z = \infty$.

5.4.4 Theorems of the Bilateral z-Transform

$$x[n] \circ\!-\!\bullet\ X(z);\ \mathrm{ROC} = R_x \qquad y[n] \circ\!-\!\bullet\ Y(z);\ \mathrm{ROC} = R_y$$

- **Conjugation**

$$x^*[n] \circ\!-\!\bullet\ X^*(z^*);\ \mathrm{ROC} = R_x$$

- **Time reversal**

$$x[-n] \circ\!-\!\bullet\ X(1/z);\ \mathrm{ROC} = 1/R_x$$

- **Linearity**

$$a \cdot x[n] + b \cdot y[n] \circ\!-\!\bullet\ a \cdot X(z) + b \cdot Y(z);\ \mathrm{ROC} = R_x \cap R_y$$

- **Time shifting**

$$x[n-n_0] \circ\!-\!\bullet\ z^{-n_0} \cdot X(z)$$

$\mathrm{ROC} = R_x$ excluding possibly the point $z = 0$ or $z = \infty$

- **Multiplication by an exponential sequence**

$$\alpha^n \cdot x[n] \circ\!-\!\bullet\; X(z/\alpha);\ \mathrm{ROC} = |\alpha| \cdot R_x$$

- **Differentiation**

$$n \cdot x[n] \circ\!-\!\bullet\; -z \cdot \frac{\mathrm{d}\,X(z)}{\mathrm{d}\,z}$$

ROC $= R_x$ excluding possibly the point $z = 0$ or $z = \infty$

- **Convolution**

$$x[n] * y[n] \circ\!-\!\bullet\; X(z) \cdot Y(z);\ (R_1 \cap R_2) \in \mathrm{ROC}$$

- **Multiplication**

$$x[n] \cdot y^*[n] \circ\!-\!\bullet\; \frac{1}{2\pi\,\mathrm{j}} \cdot \oint_{\mathrm{C}} X(v) \cdot Y^*\left(z^*/v^*\right) \cdot v^{-1}\,\mathrm{d}\,v;\ (R_1 \cdot R_2) \in \mathrm{ROC}$$

- **PARSEVAL's theorem**

$$\sum_{n=-\infty}^{\infty} x[n]\cdot y^*[n] = \frac{1}{2\pi\,\mathrm{j}}\cdot\oint_{\mathrm{C}} X(v)\cdot Y^*\left(\frac{1}{v^*}\right)\cdot v^{-1}\,\mathrm{d}\,v$$

- **Initial-value theorem**

$$x[0] = \lim_{z\to\infty} X(z) \ \text{ if } \ x[n]=0 \ \text{ for } \ n<0$$

- **Final-value theorem**

$$\lim_{n\to\infty} x[n] = \lim_{z\to 1}(z-1)\cdot X(z) < \infty$$

5.4.5 Inverse z-Transform

- Result of CAUCHYs integral theorem:

$$x[n] = \frac{1}{2\pi\,\mathrm{j}} \cdot \oint_{\mathrm{C}} X(z) \cdot z^{n-1}\,\mathrm{d}\,z$$

- Alternative ways to compute the inverse z-transform:
  - Rational z-transforms: apply CAUCHYs residue theorem

$$x[n] = \sum \left\{ \text{residues of } X(z) \cdot z^{n-1} \text{ at the poles inside C} \right\}$$

  - Partial-fraction expansion
  - Power series expansion (Polynomial division)

Partial-Fraction Expansion

- Assumption: {order $N_M(z)$} < {order $D_N(z)$}. This is called a **proper fraction**.
- An **improper fraction** with {order $N_M(z)$} ≥ {order $D_N(z)$} can be converted into a proper fraction plus residual polynomial.
- **Single poles**: p_1, p_2, …, p_N

Residues

$$X(z) = \frac{N_M(z)}{D_N(z)} = \sum_{l=1}^{N} \frac{A_l}{1 - p_l \cdot z^{-1}}$$

$$A_l = \left(1 - p_l \cdot z^{-1}\right) \cdot X(z)\Big|_{z = p_l}$$

$$x[n] = \sum_{l=1}^{N} A_l \cdot \left(p_l\right)^n \cdot u[n]$$

Assumption: ROC given by $|z| > |p_l|$

- **Multiple poles**

Pole p_{mul} is of multiplicity L. p_1, p_2, ..., p_{N-L} are single Poles.

$$X(z) = \frac{N_M(z)}{D_N(z)} = \sum_{l=1}^{N-L} \frac{A_l}{1 - p_l \cdot z^{-1}} + \sum_{i=1}^{L} \frac{B_i}{\left(1 - p_{\mathrm{mul}} \cdot z^{-1}\right)^i}$$

$$B_i = \frac{1}{(L-i)! \cdot (-p_{\mathrm{mul}})^{L-i}} \cdot \frac{\mathrm{d}^{L-i}}{\mathrm{d}\left(z^{-1}\right)^{L-i}} \left\{ \left(1 - p_{\mathrm{mul}} \cdot z^{-1}\right)^L \cdot X(z) \right\} \Big|_{z = p_{\mathrm{mul}}}$$

with $1 \leq i \leq L$

Power Series Expansion

- For causal sequences $x[n]$ with rational z-transforms $X(z)$ the power series expansion in z^{-1} can be obtained by applying long division.

$$X(z)=\frac{N_M(z)}{D_N(z)}=\frac{b_0+b_1\cdot z^{-1}+\ldots+b_M\cdot z^{-M}}{a_0+a_1\cdot z^{-1}+\ldots+a_N\cdot z^{-N}}=\frac{b_0}{a_0}+P_1\cdot z^{-1}+P_2\cdot z^{-2}+\ldots$$

$$x[n]=\frac{b_0}{a_0}\cdot\delta[n]+P_1\cdot\delta[n-1]+P_2\cdot\delta[n-2]+\ldots$$

M, N arbitrary

- **Initial-value theorem**:

$$\lim_{z\to\infty} X(z)=\frac{b_0}{a_0}=x[0]$$

5.5 z-Domain Representation of LTI Systems

- An LTI discrete-time system is characterized by a linear difference equation with constant coefficients:

$$\sum_{k=0}^{N} a_k \cdot y[n-k] = \sum_{k=0}^{M} b_k \cdot x[n-k]$$

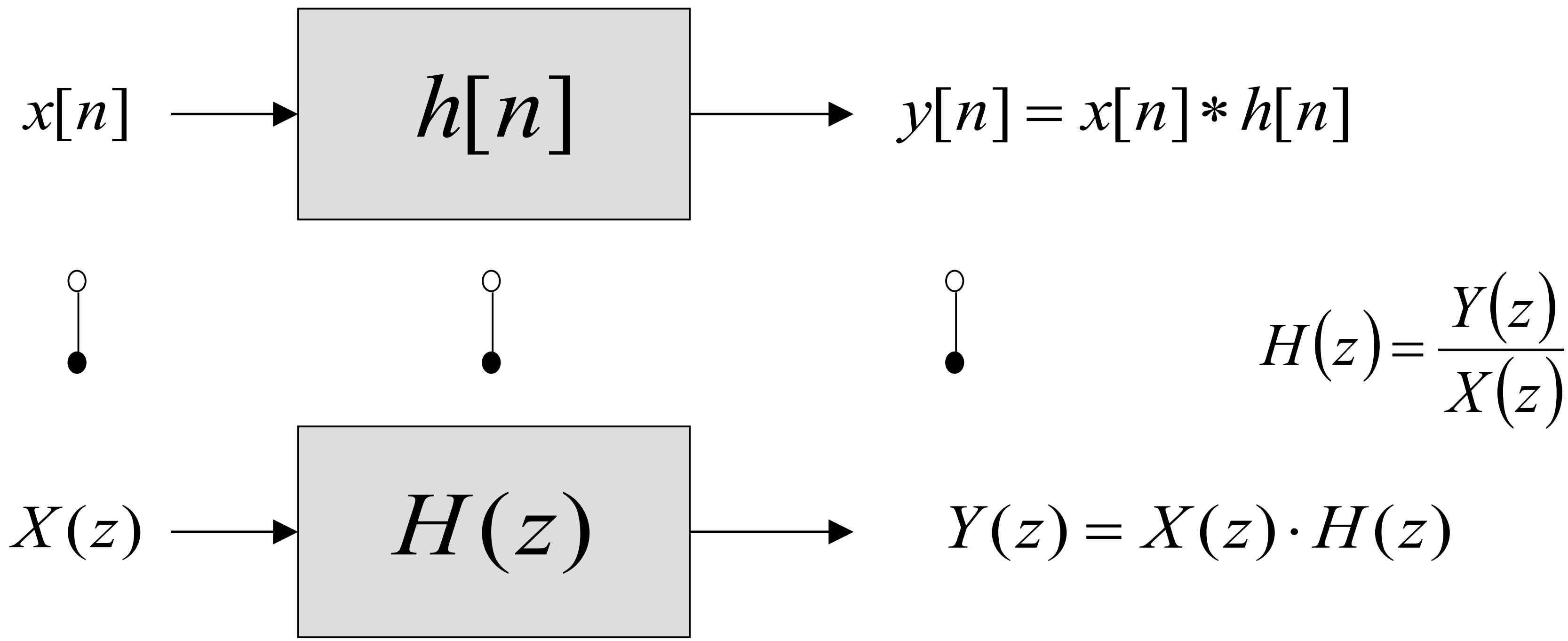
- Condition for causal LTI systems: $x[n] = 0$ and $y[n] = 0$ for $n < 0$
- Apply z-transform:

$$\sum_{k=0}^{N} a_k \cdot z^{-k} \cdot Y(z) = \sum_{k=0}^{M} b_k \cdot z^{-k} \cdot X(z)$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{b_0 + b_1 \cdot z^{-1} + b_2 \cdot z^{-2} + \ldots + b_M \cdot z^{-M}}{a_0 + a_1 \cdot z^{-1} + a_2 \cdot z^{-2} + \ldots + a_N \cdot z^{-N}}$$

$H(z)$ is the **system function** or **transfer function**

LTI System: Time domain versus z-domain



The **frequency response** $H\left(\mathrm{e}^{\mathrm{j}\Omega}\right)$ can be derived from the **system function** with $z = \mathrm{e}^{\mathrm{j}\Omega}$, if the ROC includes the unit circle.

- **Causality**

$$h[n]=0 \text{ for } n<0$$

  - The ROC of $H(z)$ is the exterior of a circle, including infinity.
  - If $H(z)$ is a rational function, the ROC is the exterior of a circle outside the outermost pole.

- **Stability**

$$\sum_{n=-\infty}^{\infty} |h[n]| < \infty$$

  - An LTI system is stable if all poles of its system function $H(z)$ lie inside the unit circle, i.e. $|p_k| < 1$.
  - The ROC of its system function $H(z)$ includes the unit circle $|z| = 1$.
  - The DTFT of $h[n]$ exists.

Transfer Functions of LTI Discrete-Time Systems

- **Recursive type, IIR systems**

$$H(z)=\frac{b_0+b_1\cdot z^{-1}+b_2\cdot z^{-2}+\ldots+b_M\cdot z^{-M}}{a_0+a_1\cdot z^{-1}+a_2\cdot z^{-2}+\ldots+a_N\cdot z^{-N}}$$

ROC: $|z| > \max\{|p_k|\}$. $\rightarrow$ Stable, if $|p_k| < 1$

- **Nonrecursive type, FIR systems**

$$H(z)=\frac{b_0}{a_0}+\frac{b_1}{a_0}\cdot z^{-1}+\ldots+\frac{b_M}{a_0}\cdot z^{-M} = h[0]+h[1]\cdot z^{-1}+\ldots+h[M]\cdot z^{-M}$$

ROC: All z, excluding $z = 0$.
All poles are at the origin $z = 0 \rightarrow$ always stable.

Convolution Using Polynomial Multiplication

- **Linear convolution** of two causal finite-length sequences

$$y[n] = h[n] * x[n] \circ\!-\!\bullet \; Y(z) = H(z) \cdot X(z)$$

$$X(z) = x[0] + x[1] \cdot z^{-1} + x[2] \cdot z^{-2} + \ldots + x[L] \cdot z^{-L}$$

$$H(z) = h[0] + h[1] \cdot z^{-1} + h[2] \cdot z^{-2} + \ldots + h[M] \cdot z^{-M}$$

$$Y(z) = y[0] + y[1] \cdot z^{-1} + y[2] \cdot z^{-2} + \ldots + y[L+M] \cdot z^{-(L+M)}$$

$$y[n] = \sum_{k=0}^{L+M} h[k] \cdot x[n-k]$$

Frequency Response Derived from Pole-Zero Plot

- LTI system with rational transfer function

$$H(z)=\frac{N_M(z)}{D_N(z)}=z^{(N-M)}\cdot\frac{b_0\cdot z^M+b_1\cdot z^{M-1}+\ldots+b_{M-1}\cdot z+b_M}{a_0\cdot z^N+a_1\cdot z^{N-1}+\ldots+a_{N-1}\cdot z+a_N}$$

$$=z^{(N-M)}\cdot\frac{b_0}{a_0}\cdot\frac{(z-z_1)\cdot(z-z_2)\cdots}{(z-p_1)\cdot(z-p_2)\cdots}=z^{(N-M)}\cdot\frac{b_0}{a_0}\cdot\frac{\prod_{i=1}^{M}(z-z_i)}{\prod_{i=1}^{N}(z-p_i)}$$

gain constant

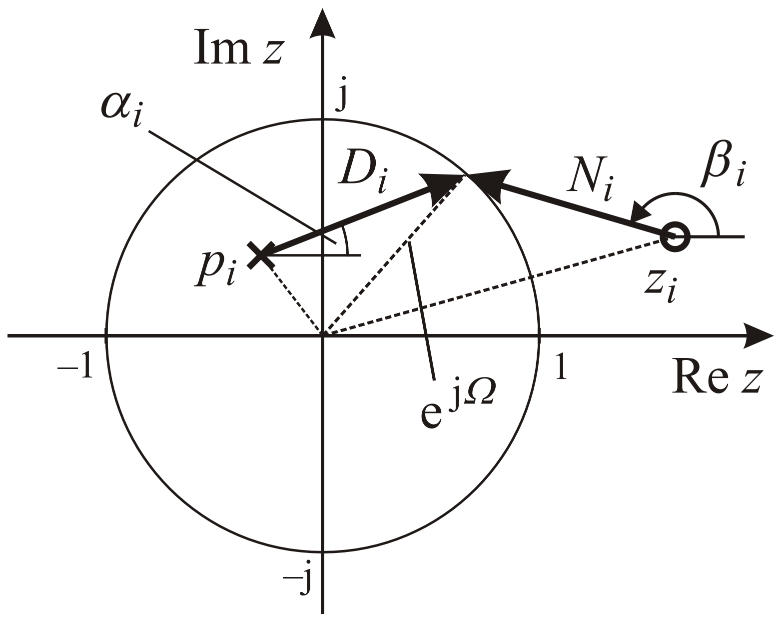
  - M finite zeros at $z=z_i$
  - N finite poles at $z=p_i$
  - $N-M$ additional zeros at $z=0$ if $N>M$
  - $M-N$ additional poles at $z=0$ if $M>N$

- The frequency response of a stable LTI system is obtained for

$$z = \mathrm{e}^{\mathrm{j}\Omega}$$

$$H(\mathrm{e}^{\mathrm{j}\Omega}) = \mathrm{e}^{\mathrm{j}\Omega(N-M)} \cdot \frac{b_0}{a_0} \cdot \frac{\prod_{i=1}^{M} (\mathrm{e}^{\mathrm{j}\Omega} - z_i)}{\prod_{i=1}^{N} (\mathrm{e}^{\mathrm{j}\Omega} - p_i)} = \mathrm{e}^{\mathrm{j}\Omega(N-M)} \cdot \frac{b_0}{a_0} \cdot \frac{\prod_{i=1}^{M} N_i \cdot \mathrm{e}^{\mathrm{j}\beta_i}}{\prod_{i=1}^{N} D_i \cdot \mathrm{e}^{\mathrm{j}\alpha_i}}$$

  - Zero close to or on unit circle:
    Attenuate signal components
  - Pole close to unit circle:
    Emphasize signal components



5.6 Transfer Function Classification
5.6.1 Magnitude Based Classification

- **Frequency selectivity**: low pass, high pass, band pass, band stop
- **Bounded-real (BR) transfer function**

$$\left|H\left(\mathrm{e}^{\mathrm{j}\Omega}\right)\right| \leq 1 \text{ with } H(z) \text{ causal, stable and real coefficients}$$

- **Allpass transfer function**

$$\left|H\left(\mathrm{e}^{\mathrm{j}\Omega}\right)\right| = 1$$

- M-th order causal real-coefficient allpass transfer function

$$H_{\mathrm{A}}(z) = \pm \frac{\widetilde{D}_M(z)}{D_M(z)} = \pm \frac{a_M + a_{M-1} \cdot z^{-1} + \ldots + a_1 \cdot z^{-M+1} + z^{-M}}{1 + a_1 \cdot z^{-1} + \ldots + a_{M-1} \cdot z^{-M+1} + a_M \cdot z^{-M}}$$

$$D_M(z) = 1 + a_1 \cdot z^{-1} + \ldots + a_{M-1} \cdot z^{-M+1} + a_M \cdot z^{-M}$$

$$\tilde{D}_M(z) = z^{-M} \cdot D_M(z^{-1}) \;\text{ if }\; D_M(z) \;\text{ has real coefficients}$$

$$H_{\mathrm{A}}(z) = \pm \prod_{i=1}^{M} \frac{-p_i^* + z^{-1}}{1 - p_i \cdot z^{-1}}$$

  - Numerator and denominator polynomials are mirror-image polynomials.
  - If $z = r \cdot \mathrm{e}^{\mathrm{j}\phi}$ is a pole, $\frac{1}{z^*} = \frac{1}{r} \cdot \mathrm{e}^{\mathrm{j}\phi}$ is a zero.
  - If poles lie inside the unit circle, zeros lie outside the unit circle.

$$H_A(z) \cdot H_A(z^{-1}) = 1$$

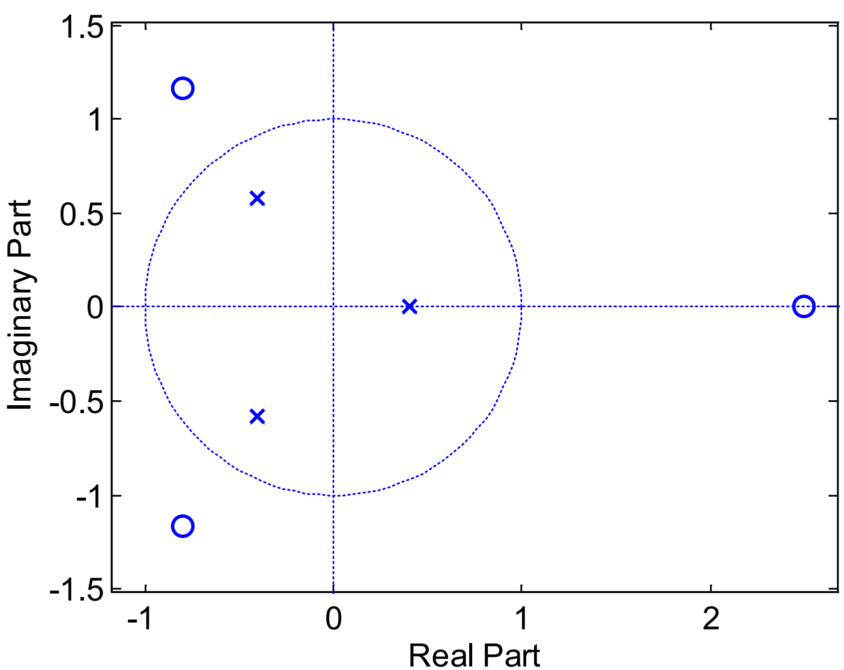
- A causal stable real-coefficient allpass transfer function is a lossless bounded real (LBR) transfer function. → A causal stable allpass filter is a lossless structure.
- Magnitude of the transfer function

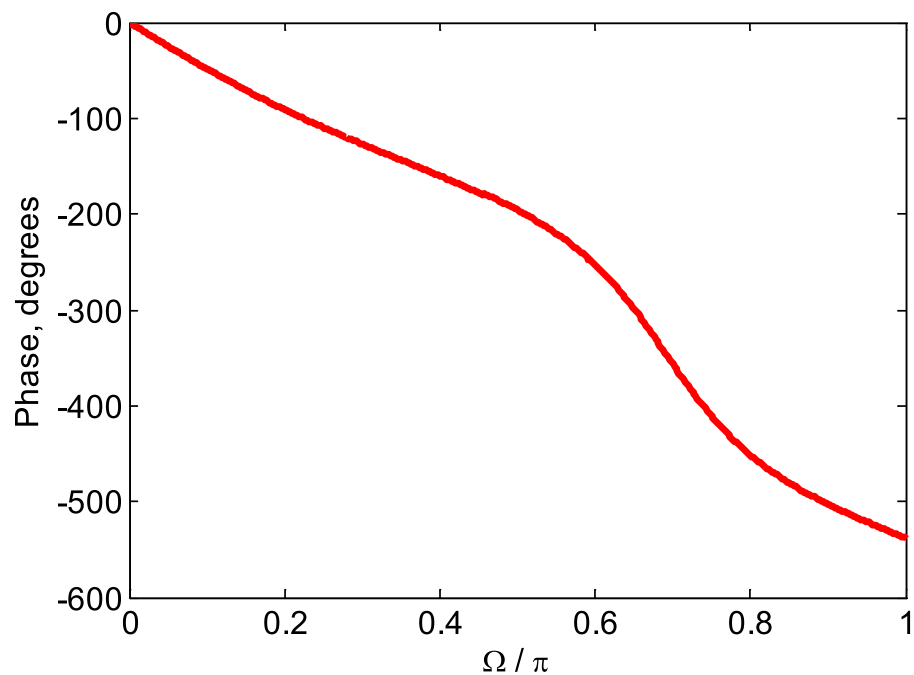
$$\left|H_A(z)\right| = \begin{cases} < 1 \ , & |z| > 1 \\ = 1 \ , & |z| = 1 \\ > 1 \ , & |z| < 1 \end{cases}$$

- The phase decreases monotonically from 0 to $-M \cdot \pi$ when Ω varies from 0 to π.

- Example

$$H_{\mathrm{A}}(z) = \frac{-0.2 + 0.18 \cdot z^{-1} + 0.4 \cdot z^{-2} + z^{-3}}{1 + 0.4 \cdot z^{-1} + 0.18 \cdot z^{-2} - 0.2 \cdot z^{-3}} \quad \text{with} \quad M = 3$$

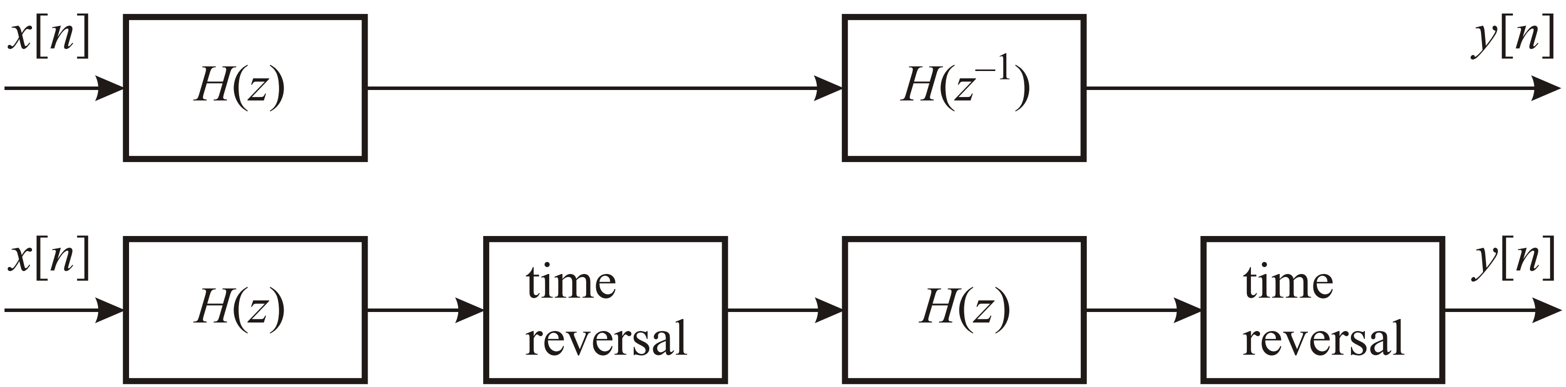




5.6.2 Phase Based Classification

- **Zero-phase transfer function**

$$F(z) = \frac{Y(z)}{X(z)} = H(z) \cdot H(z^{-1}) \;\; ; \;\; F\left(e^{j\Omega}\right) = \left|H\left(e^{j\Omega}\right)\right|^2$$



It is not possible to implement a causal digital filter with zero phase.

- **Linear-phase transfer function**

$$H(\mathrm{e}^{\mathrm{j}\Omega}) = \frac{Y(\mathrm{e}^{\mathrm{j}\Omega})}{X(\mathrm{e}^{\mathrm{j}\Omega})} = \left|H\left(\mathrm{e}^{\mathrm{j}\Omega}\right)\right| \cdot \mathrm{e}^{\mathrm{j}\varphi}$$

with $\varphi = -\Omega \cdot D$

phase delay $\tau_{\mathrm{p}} = -\dfrac{\varphi}{\Omega} = D$ group delay $\tau_{\mathrm{g}} = -\dfrac{\mathrm{d}\,\varphi}{\mathrm{d}\,\Omega} = D$

- It is always possible to design an FIR transfer function with an exact linear-phase response.
- It is not possible to design a stable causal IIR transfer function with exact linear phase.

- **Minimum-phase transfer function**
  - A causal stable transfer function with all zeros inside the unit circle is called a **minimum-phase transfer function** $H_{\mathrm{M}}(z)$.
  - A transfer function with zeros inside and outside the unit circle is called a **mixed-phase transfer function**.

$$H(z)=\frac{N(z)}{D(z)}=\frac{N_1(z)\cdot N_2(z)}{D(z)}=\frac{N_1(z)\cdot \widetilde{N}_2(z)}{D(z)}\cdot\frac{N_2(z)}{\widetilde{N}_2(z)}=H_{\mathrm{M}}(z)\cdot H_{\mathrm{A}}(z)$$

$N_1(z)$: only zeros inside the unit circle

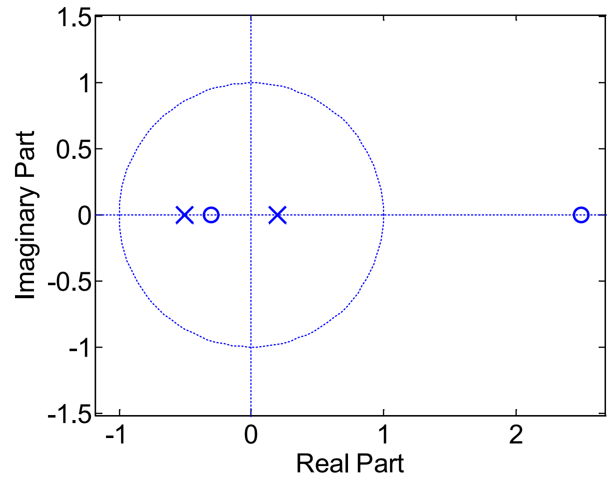
$N_2(z)$: only zeros outside the unit circle

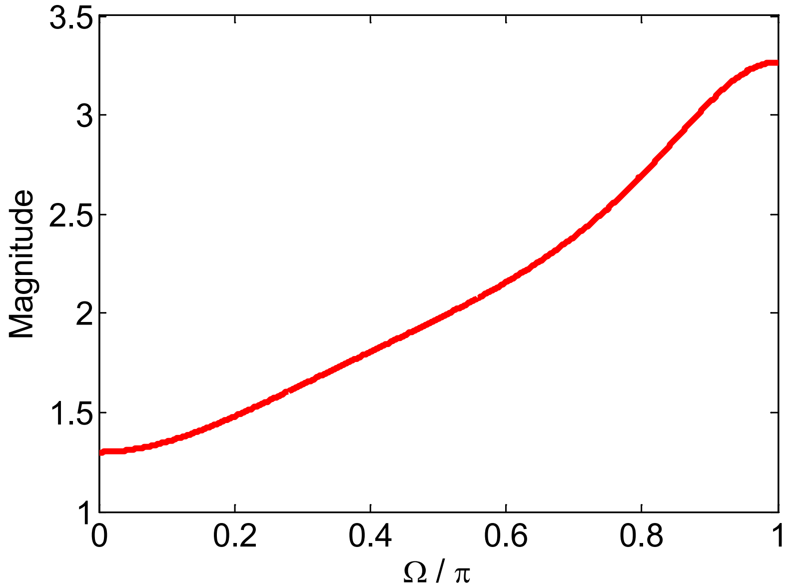
$\widetilde{N}_2(z)$: mirror-image polynomial of $N_2(z)$

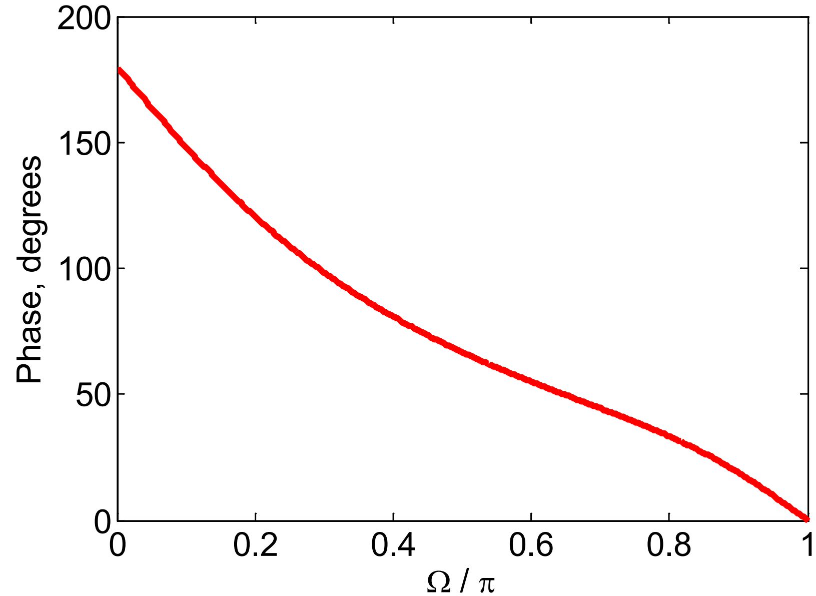
$\rightarrow$ only roots inside the unit circle

- Example: Mixed-phase system

$$H(z)=\frac{2\cdot\left(1+0.3\cdot z^{-1}\right)\cdot\left(0.4-z^{-1}\right)}{\left(1-0.2\cdot z^{-1}\right)\cdot\left(1+0.5\cdot z^{-1}\right)}$$

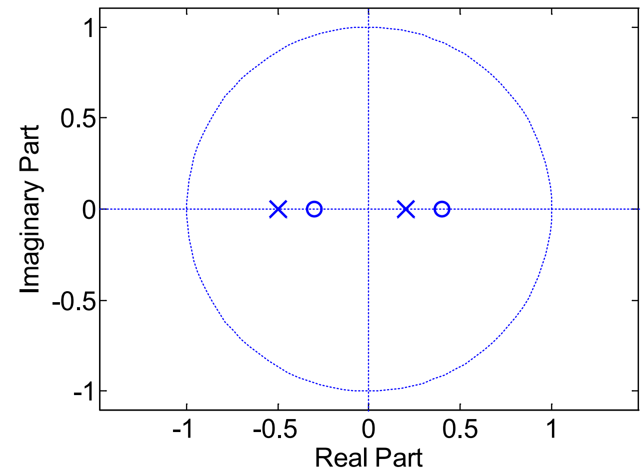


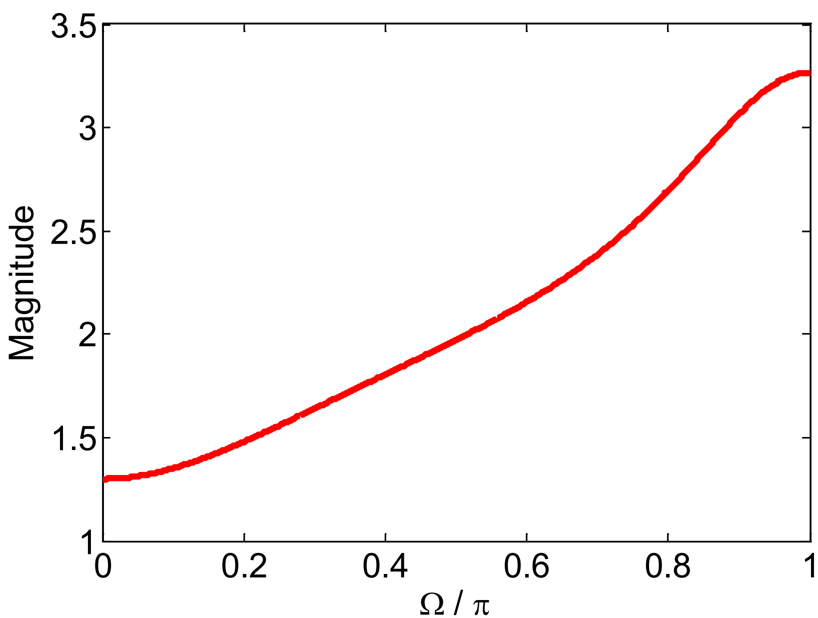


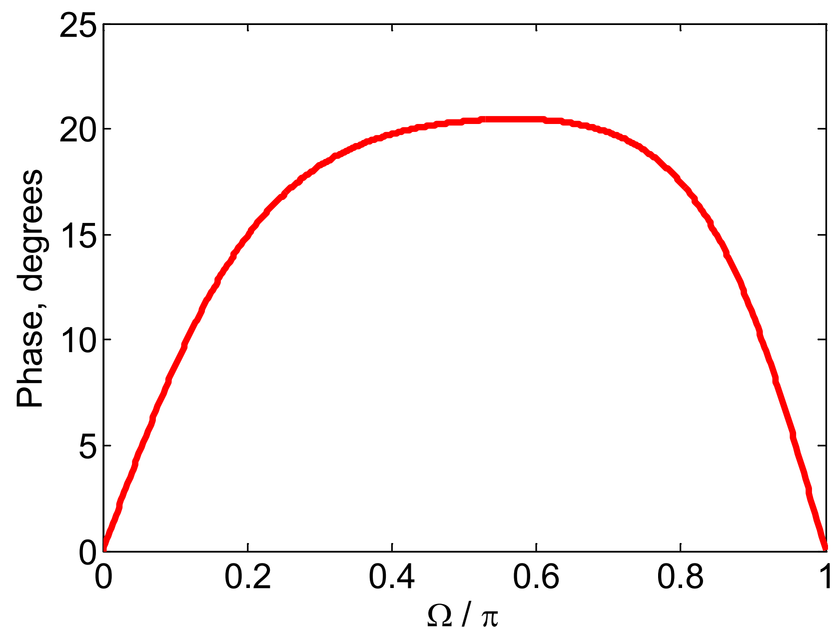


- Minimum-phase part

$$H_M(z) = \frac{2 \cdot \left(1 + 0.3 \cdot z^{-1}\right) \cdot \left(1 - 0.4 \cdot z^{-1}\right)}{\left(1 - 0.2 \cdot z^{-1}\right) \cdot \left(1 + 0.5 \cdot z^{-1}\right)}$$

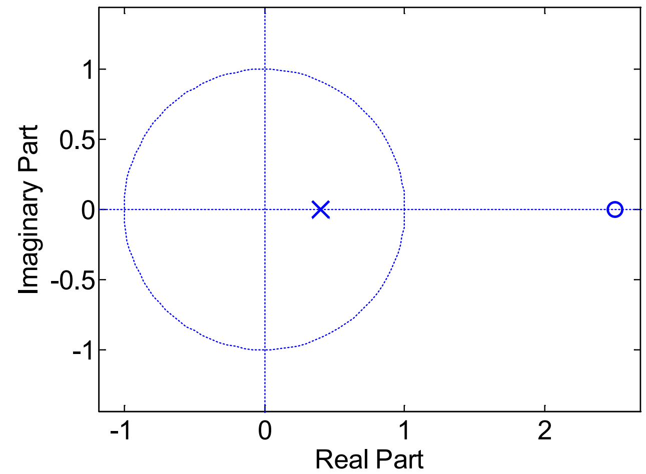


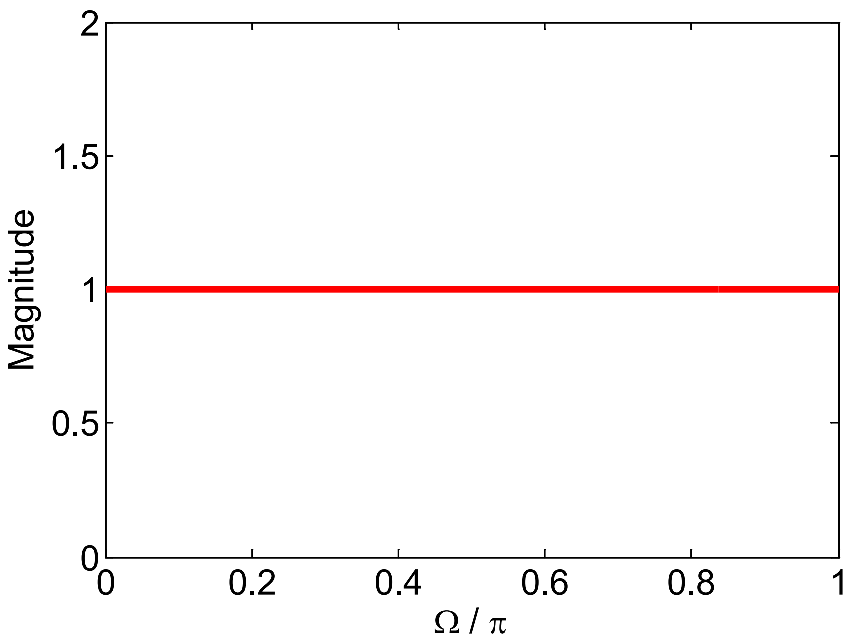


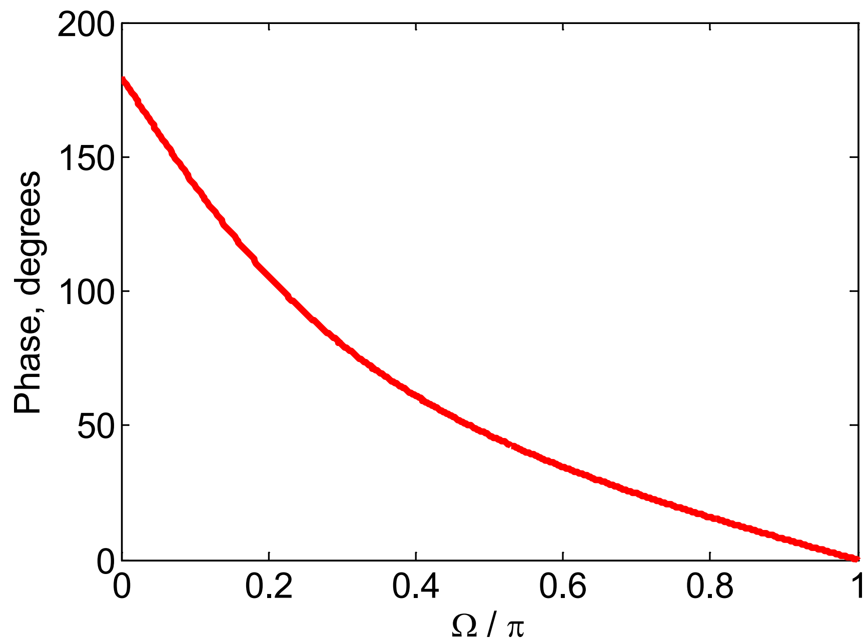


- Allpass part

$$H_A(z) = \frac{0.4 - z^{-1}}{1 - 0.4 \cdot z^{-1}}$$







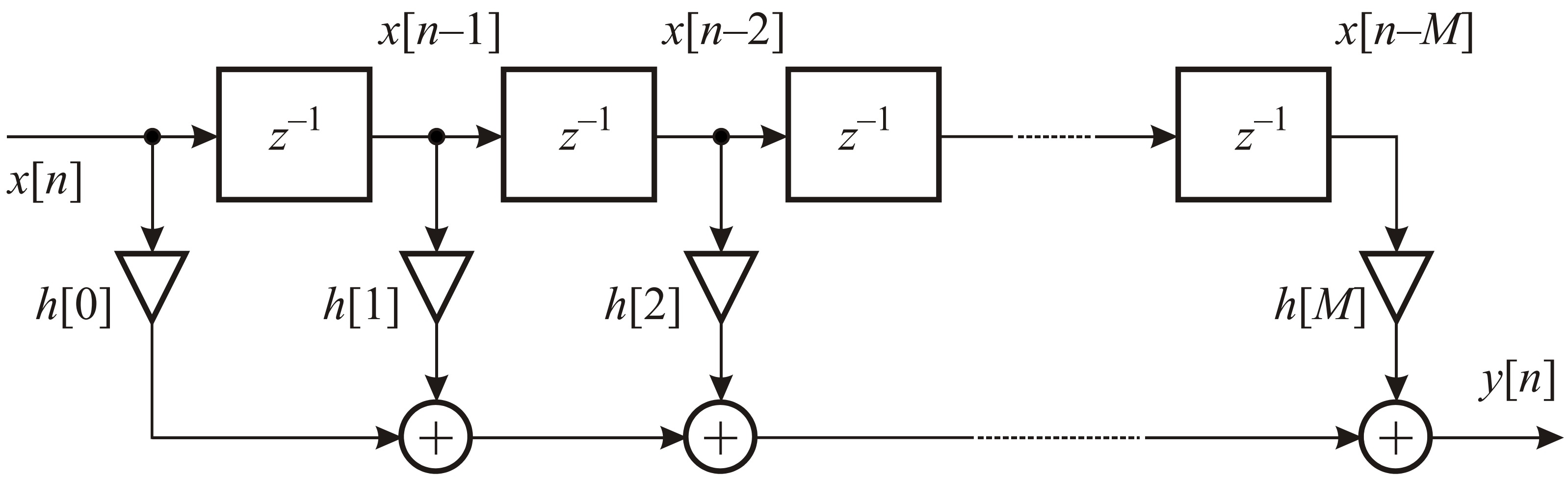
5.7 Block Diagram Representations

- **Elements**: delay block, multiplier, adder, pick-off node
- **Direct-form structure**: The multiplier coefficients are precisely the coefficients of the transfer function.
- **Canonic and noncanonic structures**: A topology is canonic if the number of delays is equal to the order of the transfer function, i.e. the order of the difference equation.
- **Equivalent structures** have the same transfer function.
- A **transpose operation** generates an equivalent structure:
  - Reverse all paths.
  - Replace pick-off nodes with adders, and vice versa.
  - Interchange the input and output nodes.

Nonrecursive discrete-time systems - FIR structures

$$y[n] = \sum_{k=0}^{M} \frac{b_k}{a_0} \cdot x[n-k] = \sum_{k=0}^{M} h[k] \cdot x[n-k]$$

$$Y(z) = \left(h[0] + h[1] \cdot z^{-1} + h[2] \cdot z^{-2} + \ldots + h[M] \cdot z^{-M} \right) \cdot X(z)$$

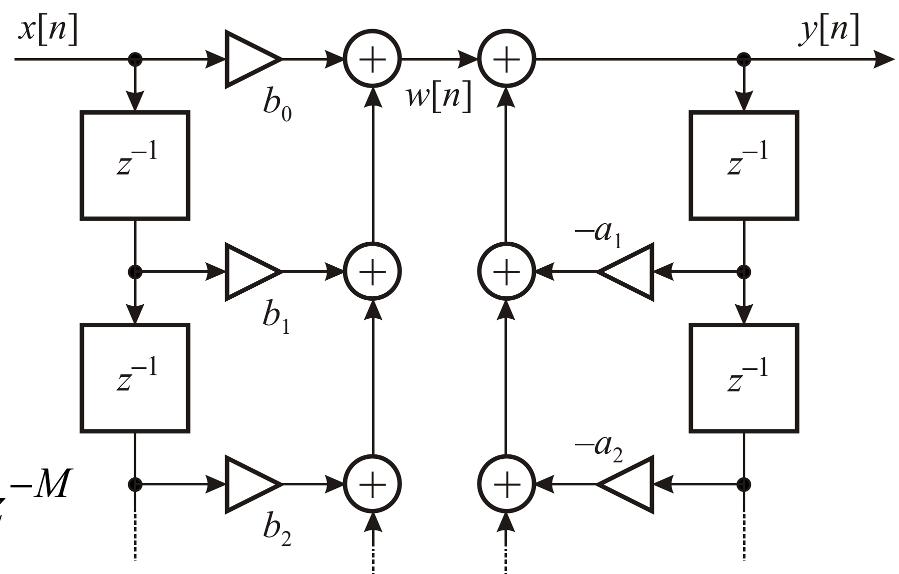


This topology is called **tapped delay line**.

- **Recursive discrete-time systems - IIR structures**

$$H(z)=\frac{b_0+b_1\cdot z^{-1}+b_2\cdot z^{-2}+\ldots+b_M\cdot z^{-M}}{1+a_1\cdot z^{-1}+a_2\cdot z^{-2}+\ldots+a_N\cdot z^{-N}}$$

direct form, noncanonic



$$H(z)=H_1(z)\cdot H_2(z)$$

$$H_1(z)=\frac{W(z)}{X(z)}=b_0+b_1\cdot z^{-1}+\ldots+b_M\cdot z^{-M}$$

$$H_2(z)=\frac{Y(z)}{W(z)}=\frac{1}{1+a_1\cdot z^{-1}+\ldots+a_N\cdot z^{-N}}$$

ARMA Topology (auto regressive, moving average)

$$H(z) = \frac{b_0 + b_1 \cdot z^{-1} + b_2 \cdot z^{-2} + \ldots + b_M \cdot z^{-M}}{1 + a_1 \cdot z^{-1} + a_2 \cdot z^{-2} + \ldots + a_N \cdot z^{-N}}$$

$$L = \max\{M, N\}$$

direct form, canonic