Chapter 6

Applications

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  - Window Functions
  - Properties
- 6.2 Frequency-Selective Filters
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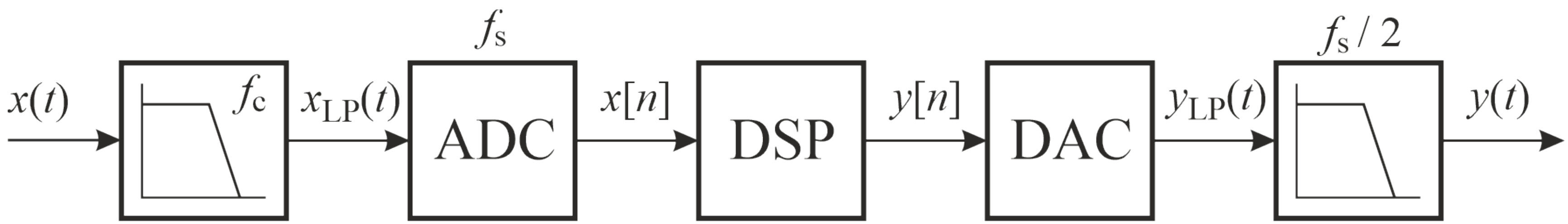
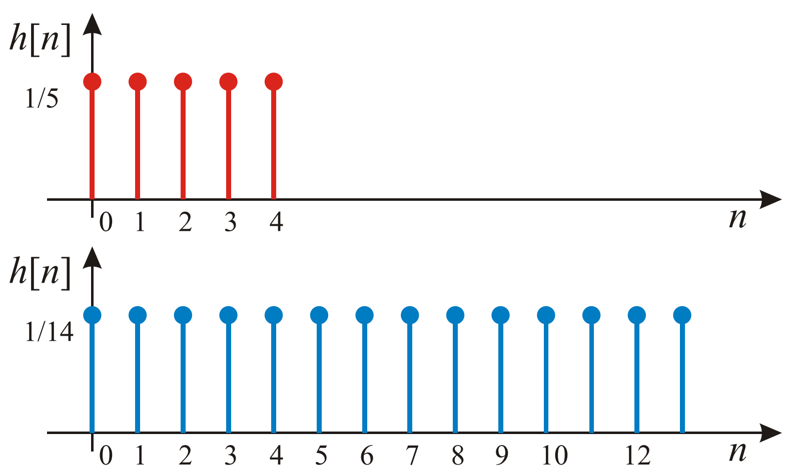
  - Sampled-Data Feedback Systems
  - Root-Locus Analysis, NYQUIST Stability Criterion

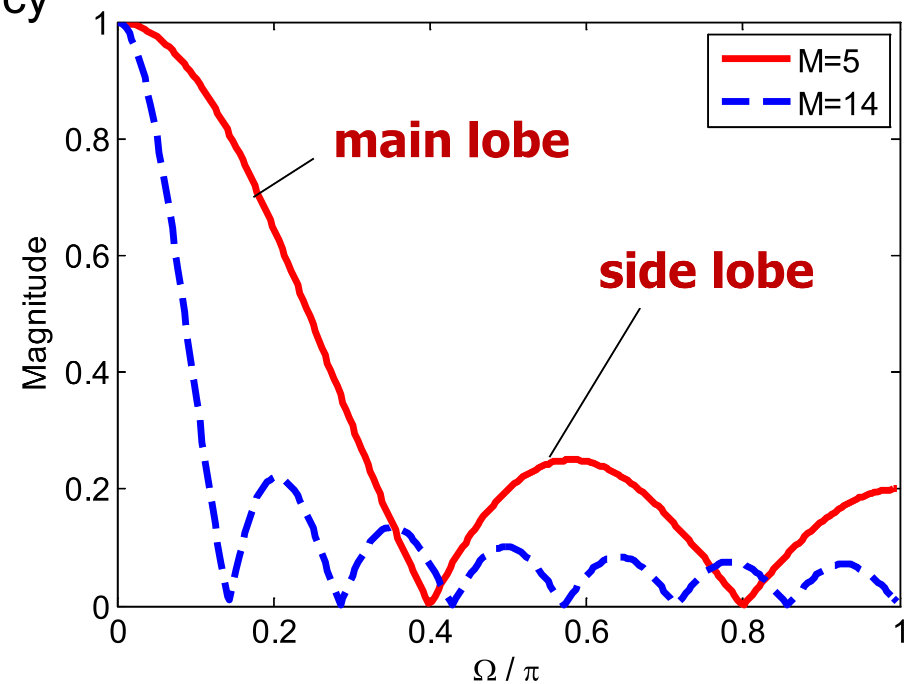
- Time-discrete processing of time-continuous signals



6.1 Time-Selective Filters

- FIR filters: The truncation of the impulse response is defined by a **window function**.
- Example: Moving average filter with rectangular window
  - Filter order $M-1 \rightarrow$ cutoff frequency
  - Window type $\rightarrow$ stopband ripple



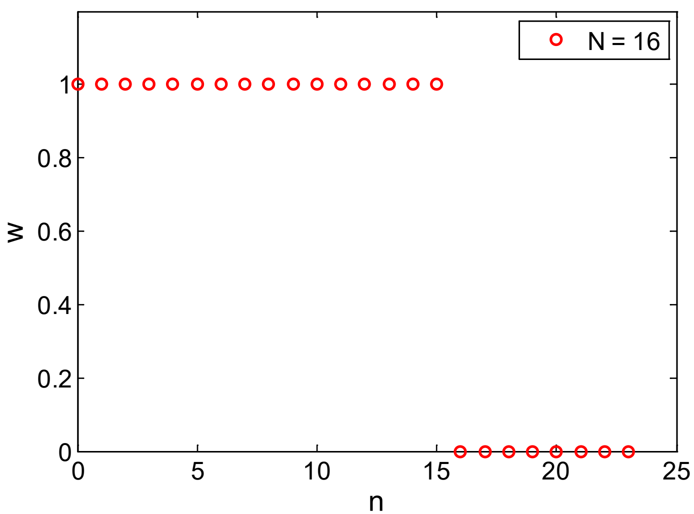


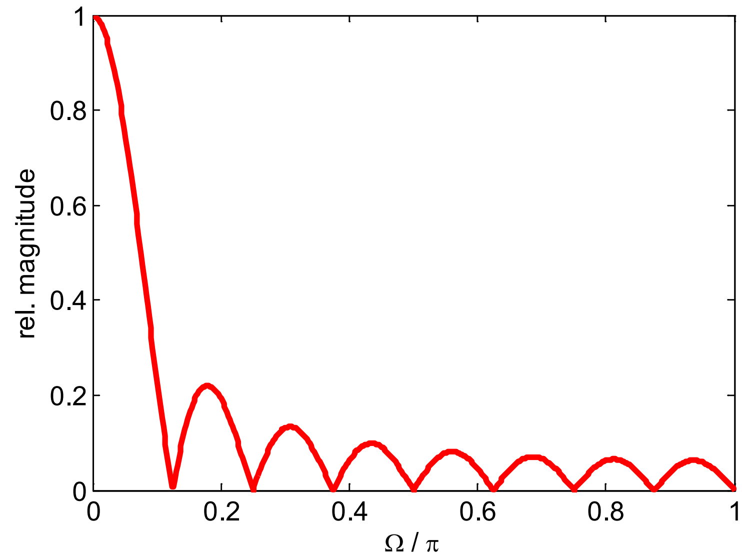
6.1.1 Window Functions

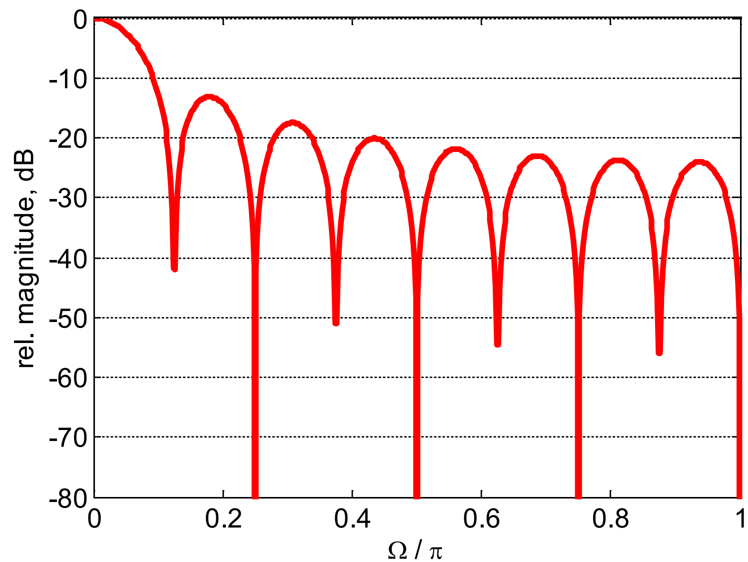
Rectangular Window

$$w[n]=\begin{cases}1 \ , \ n=0,\ldots,N-1 \\ 0 \ , \ \text{otherwise}\end{cases}$$

$$W\left(\mathrm{e}^{\mathrm{j}\Omega}\right)=\frac{\sin(\Omega\cdot N/2)}{\sin(\Omega/2)}\cdot \mathrm{e}^{-\mathrm{j}\Omega\cdot(N-1)/2}$$



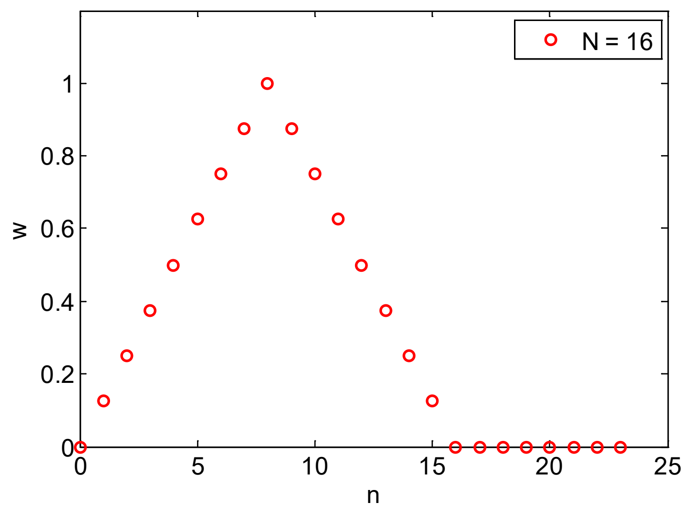


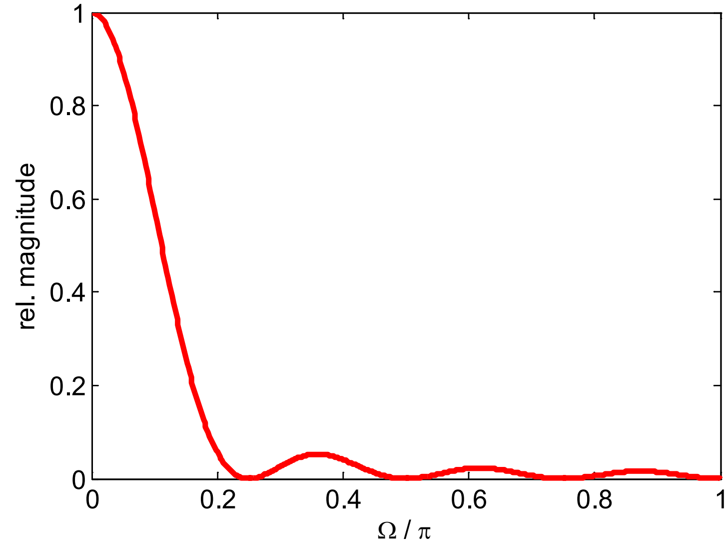


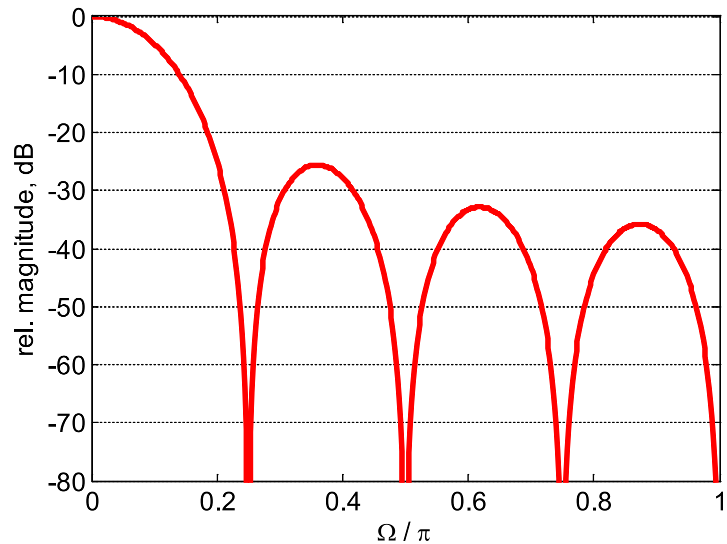
BARLETT Window - triangular window

$$w[n]=\begin{cases} 2n/N & ,\ 0\le n\le N/2-1 \\ 2(N-n)/N & ,\ N/2\le n\le N-1 \\ 0 & ,\ \text{otherwise} \end{cases}$$

$$W\left(e^{j\Omega}\right)=\frac{\sin^2(\Omega\cdot N/4)}{\sin^2(\Omega/2)}\cdot e^{-j\Omega\cdot(N-1)/2}$$

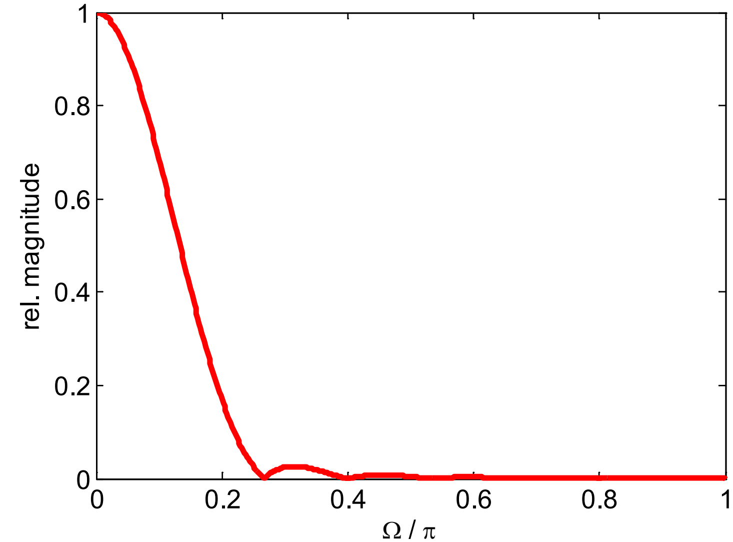


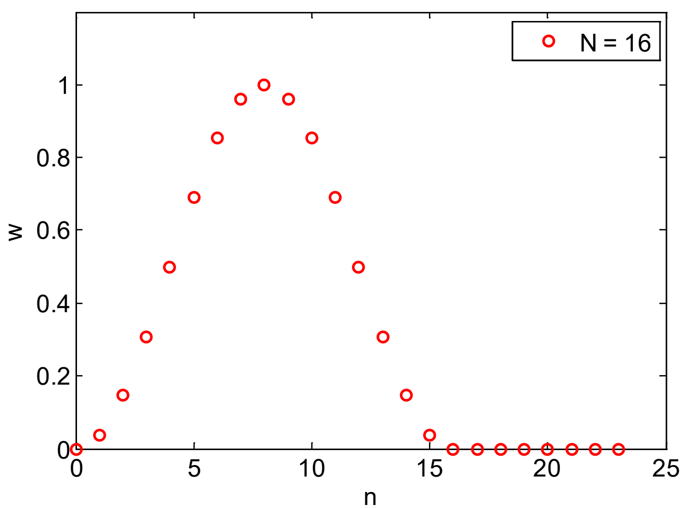


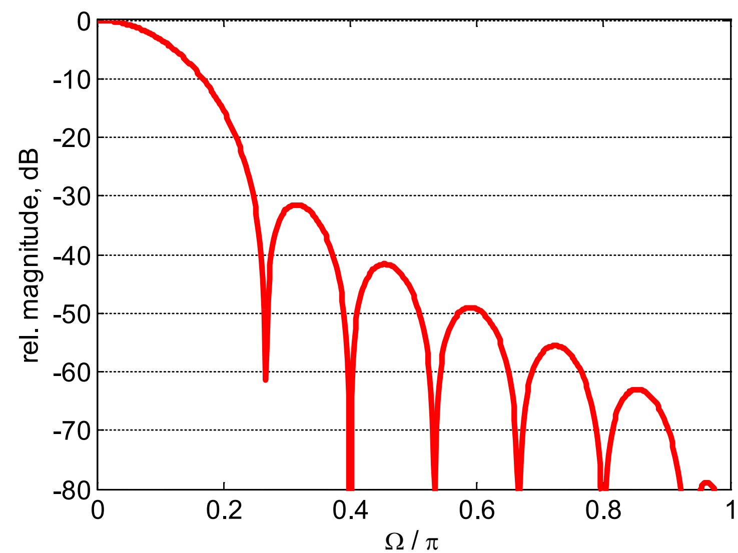


HANNING Window

$$w[n]=\begin{cases}0.5\cdot\left(1-\cos\left(\dfrac{2\cdot\pi\cdot n}{N}\right)\right) & ,\; n=0,\ldots,N-1\\ 0 & ,\; \text{otherwise}\end{cases}$$







Hamming Window

$$w[n] = \begin{cases} 0.54 - 0.46 \cdot \cos\left(\dfrac{2 \cdot \pi \cdot n}{N}\right) & , \; n = 0, \ldots, N-1 \\ 0 & , \; \text{otherwise} \end{cases}$$

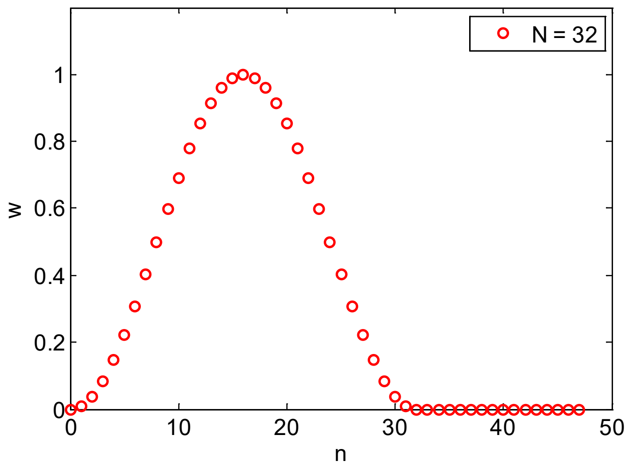
Blackmann Window

$$w[n] = \begin{cases} 0.42 - 0.5 \cdot \cos\left(\dfrac{2 \cdot \pi \cdot n}{N}\right) + 0.08 \cdot \cos\left(\dfrac{4 \cdot \pi \cdot n}{N}\right) & , \; n = 0, \ldots, N-1 \\ 0 & , \; \text{otherwise} \end{cases}$$

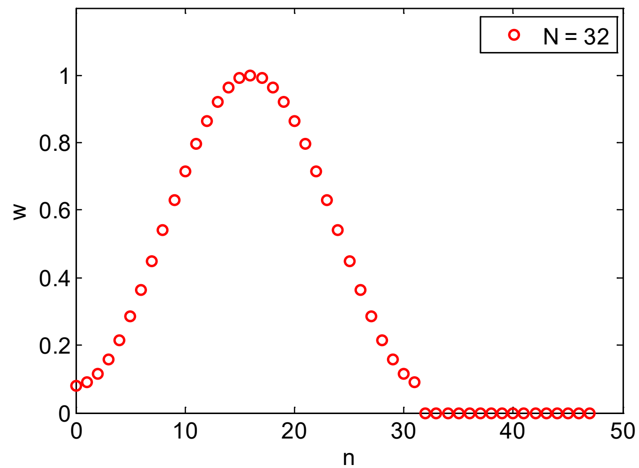
Gauss Window

$$w[n] = \begin{cases} e^{-a\left(n-(N-1)/2\right)^2} & , \; n = 0, \ldots, N-1 \\ 0 & , \; \text{otherwise} \end{cases}$$

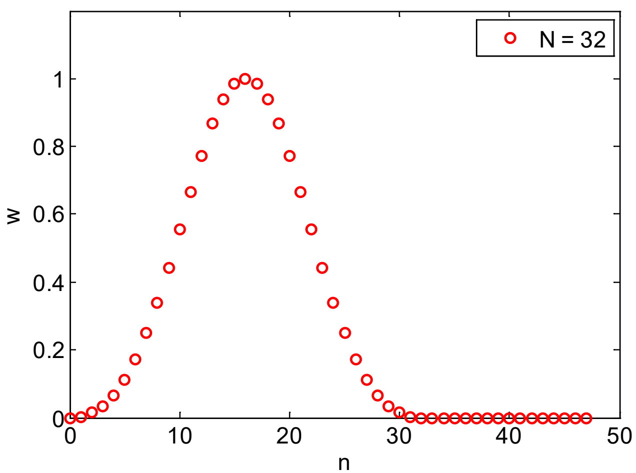
Hanning Window



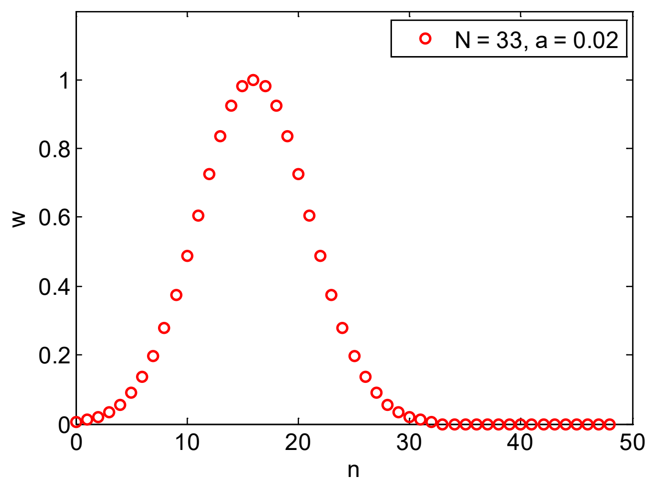
Hamming Window



Blackman Window



Gauss Window



- Adjustable windows
  - GAUSS window
  - KAISER window
  - DOLPH-CHEBYSHEV window

  They allow to adjust the level of the ripple in the stopband.

6.1.2 Properties

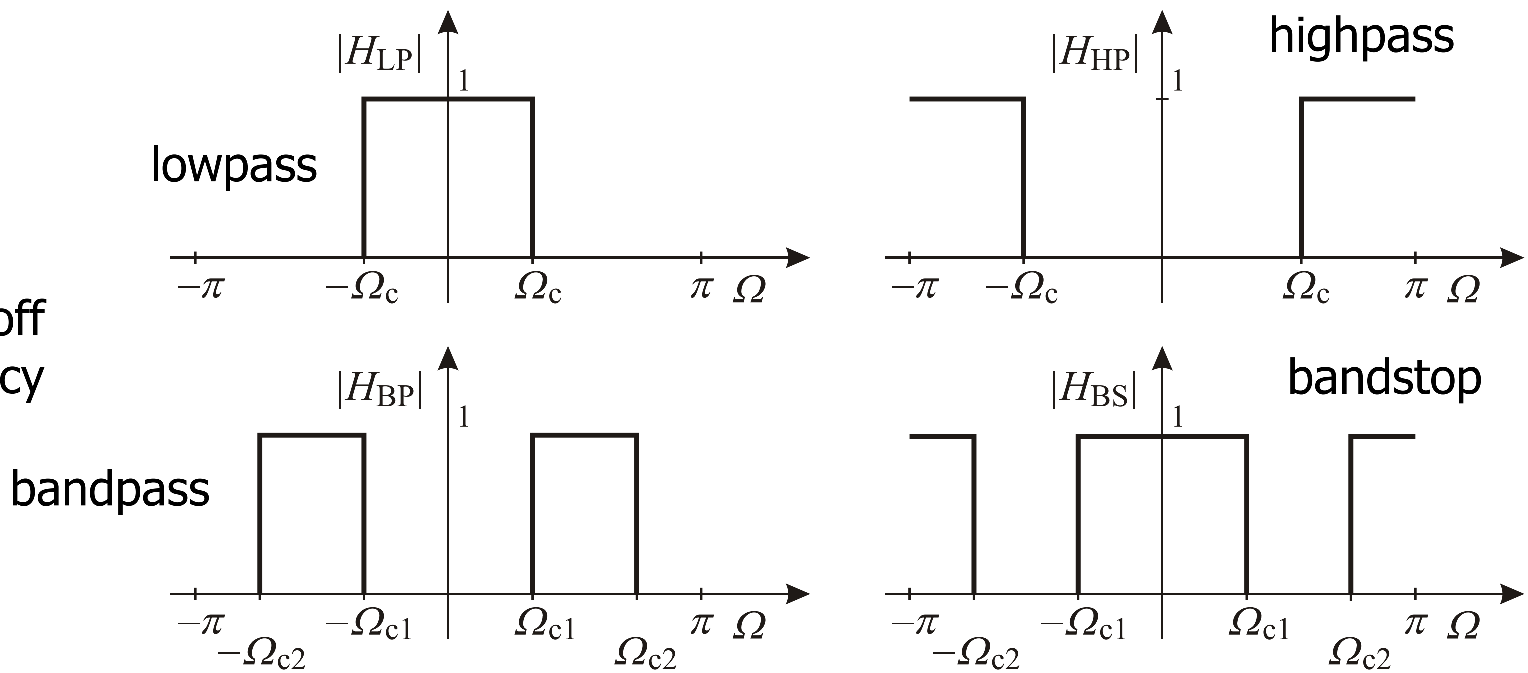
Window	**Main Lobe Width**	**Rel. Sidelobe Level**
Rectangular	$4 \cdot \pi / N$	–13.3 dB
BARLETT	$8 \cdot \pi / (N+1)$	–26.5 dB
HANNING	$8 \cdot \pi / N$	–31.5 dB
HAMMING	$8 \cdot \pi / N$	–42.7 dB
BLACKMAN	$12 \cdot \pi / N$	–58.1 dB

- The smooth transition of the window edge is called a **taper**.
- The GAUSS window has the smallest duration-bandwidth product of all window types.
- HANNING, HAMMING, and BLACKMAN are the most popular window types.

6.2 Frequency-Selective Filters

- Ideal frequency-selective filters have a brickwall-type frequency response.
  - Doubly infinite, not causal impulse response.
  - Not realizable.

Ω_c: cutoff frequency



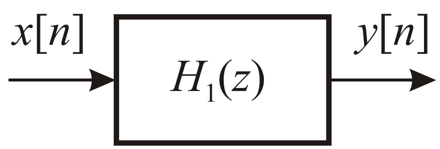
6.2.1 Simple Digital Filters

- These filters have very simple impulse response coefficients and are employed in a number of practical applications because of their simplicity.

- **Lowpass FIR digital filters**

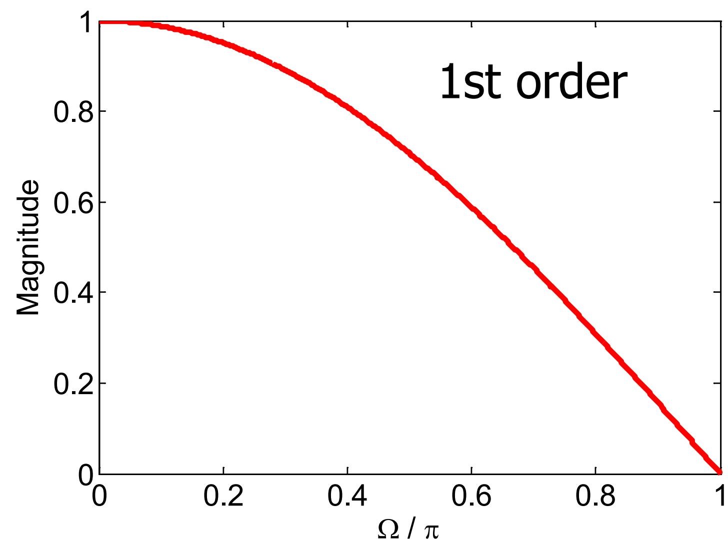
$$h[n] = \frac{1}{2} \cdot \left(\delta[n] + \delta[n-1]\right) \circ\!-\!\bullet \; H_{\mathrm{LP},1}(z) = \frac{1}{2} \cdot \left(1 + z^{-1}\right) = \frac{z+1}{2z}$$

$$H_{\mathrm{LP},1}\left(\mathrm{e}^{\mathrm{j}\Omega}\right) = \cos\left(\frac{\Omega}{2}\right) \cdot \mathrm{e}^{-\mathrm{j}\Omega/2}$$

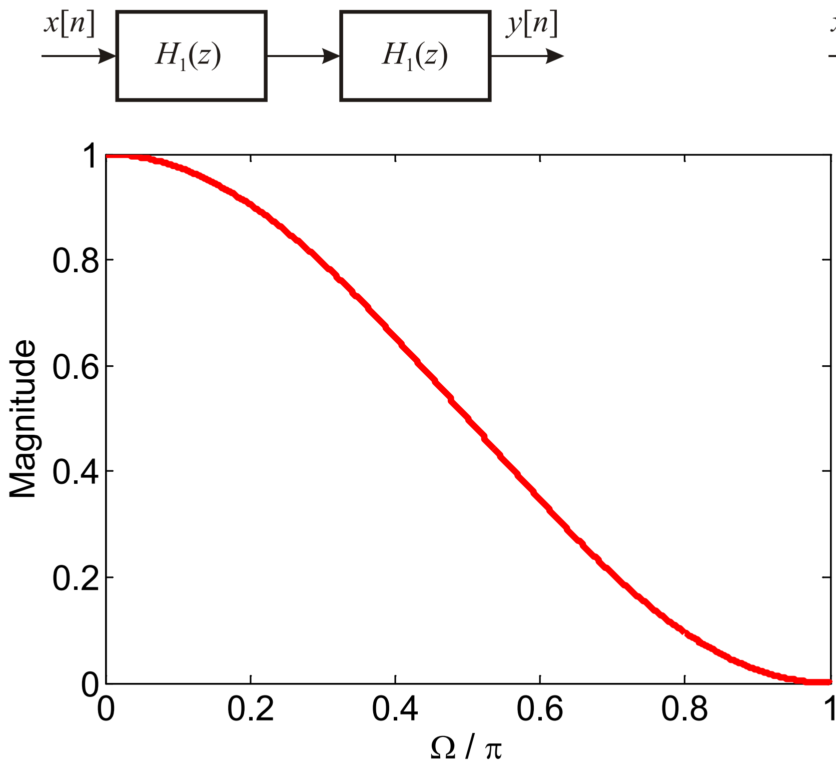


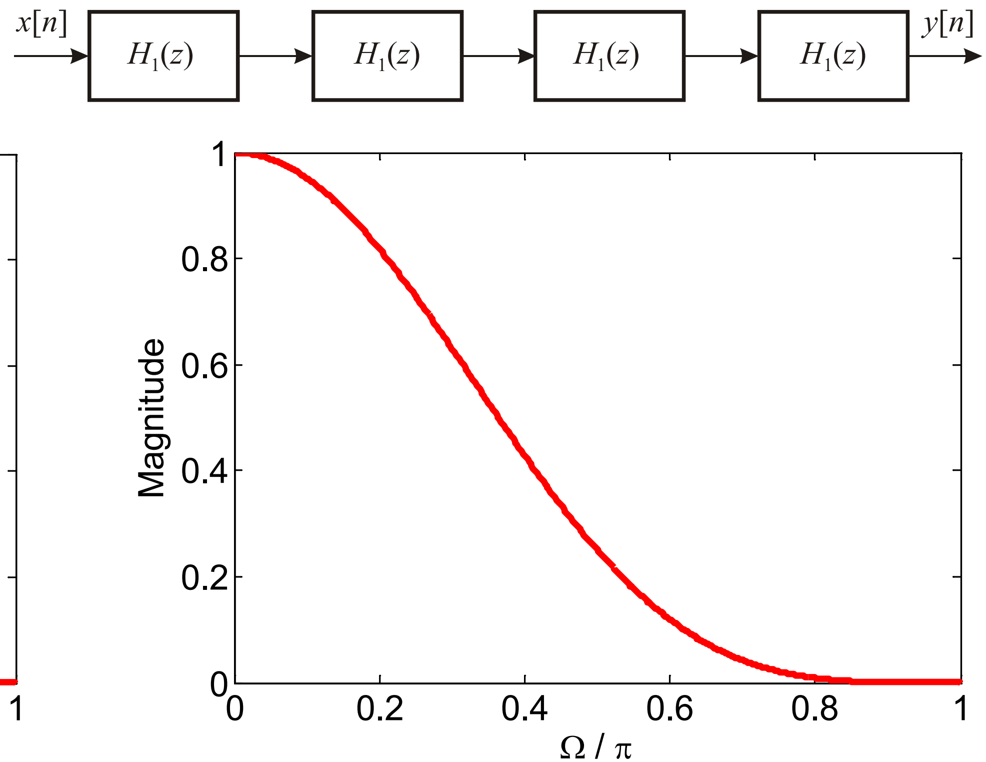
3-dB-cutoff frequency

$$\Omega_{\mathrm{c}} = \frac{\pi}{2} = 90 \text{ degree}$$



- Cascading filters improves the frequency selectivity.





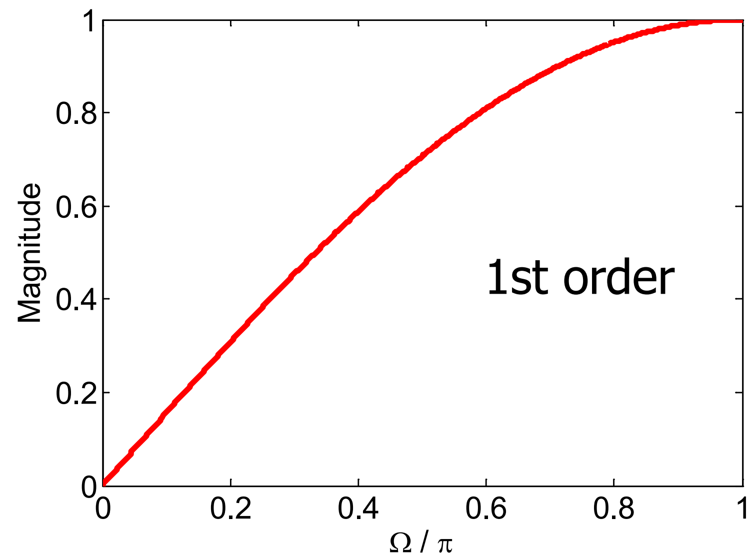
- Higher-order moving-average filters show a different magnitude response.

$$H_{\mathrm{LP},M-1}(z) = \frac{1}{M}\sum_{i=0}^{M-1} z^{-i}$$

Highpass FIR digital filters

$$h[n]=\frac{1}{2}\cdot\left(\delta[n]-\delta[n-1]\right)\quad\circ\!-\!\bullet\quad H_{\mathrm{HP},1}(z)=\frac{1}{2}\cdot\left(1-z^{-1}\right)=\frac{z-1}{2z}$$

$$H_{\mathrm{HP},1}\left(\mathrm{e}^{\mathrm{j}\Omega}\right)=\sin\left(\frac{\Omega}{2}\right)\cdot\mathrm{e}^{\mathrm{j}(\pi-\Omega)/2}$$



3 - dB cutoff frequency

$$\Omega_{\mathrm{c}}=\frac{\pi}{2}=90\ \mathrm{degree}$$

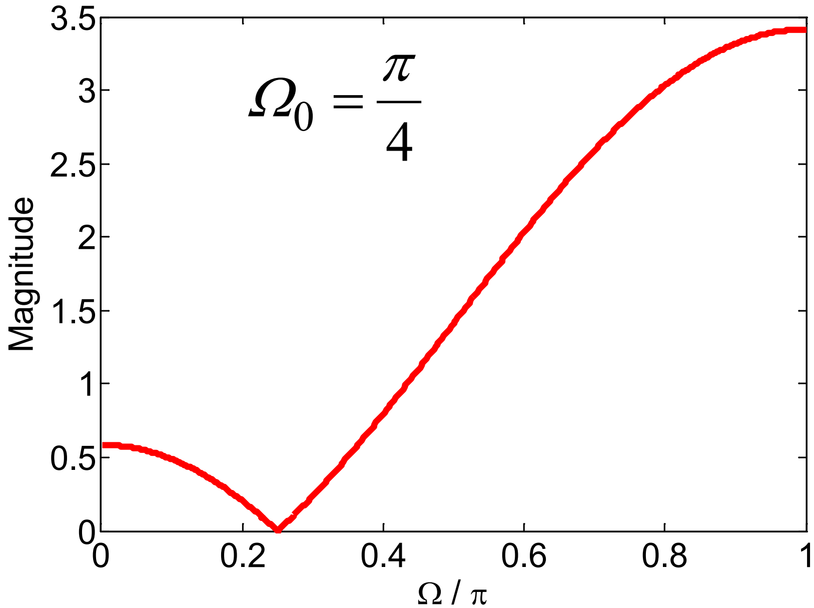
- Higher-order filter

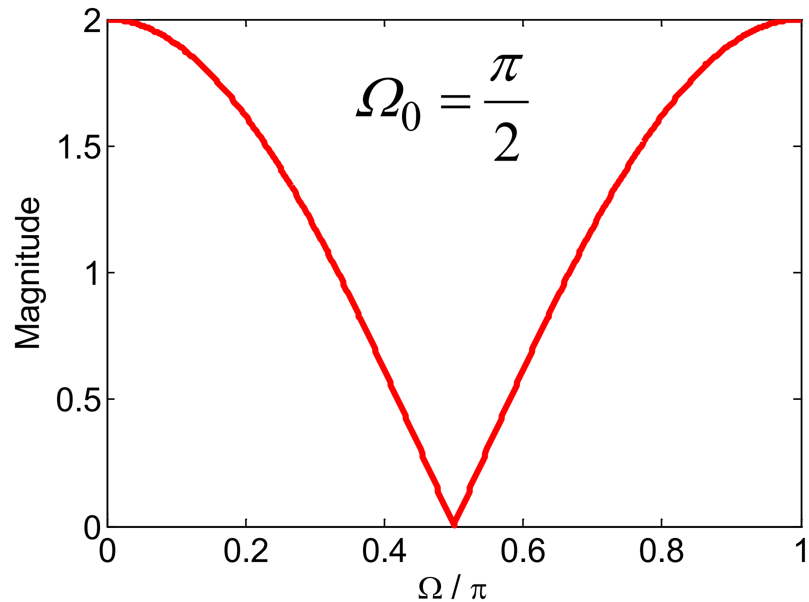
$$H_{\mathrm{HP},M-1}(z)=\frac{1}{M}\sum_{i=0}^{M-1}(-1)^{i}\cdot z^{-i}$$

- **FIR notch digital filters** are used to suppress a particular sinusoidal component at Ω_0.

$$H_{\text{Notch}}(z) = 1 - 2\cdot\cos(\Omega_0)\cdot z^{-1} + z^{-2}$$

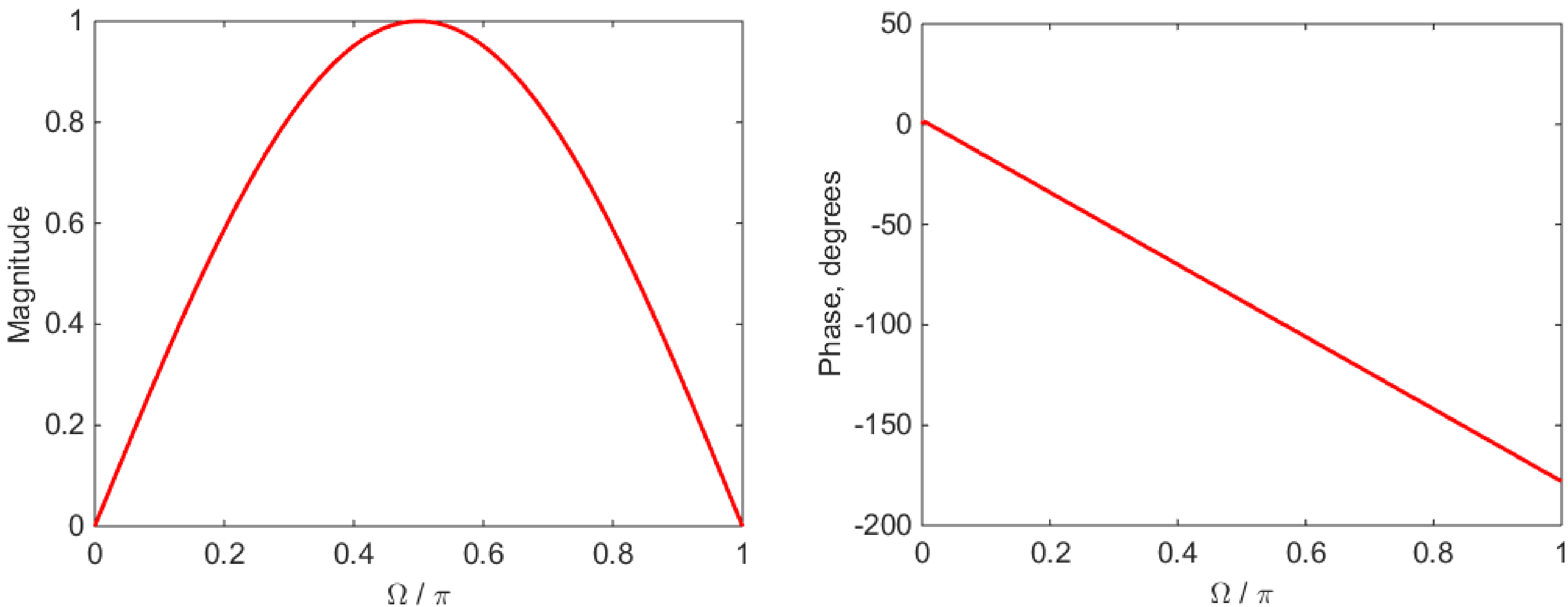
$$H_{\text{Notch}}\left(e^{j\Omega}\right) = 1 - e^{j(\Omega_0-\Omega)} - e^{-j(\Omega_0+\Omega)} + e^{-j2\Omega}$$





- **FIR bandpass digital filters** ... are not popular

$$H_{\mathrm{BP},2}(z) = \frac{1}{2}\cdot\left(1 - z^{-2}\right) = \frac{1}{2}\cdot\frac{z^2 - 1}{z^2}$$

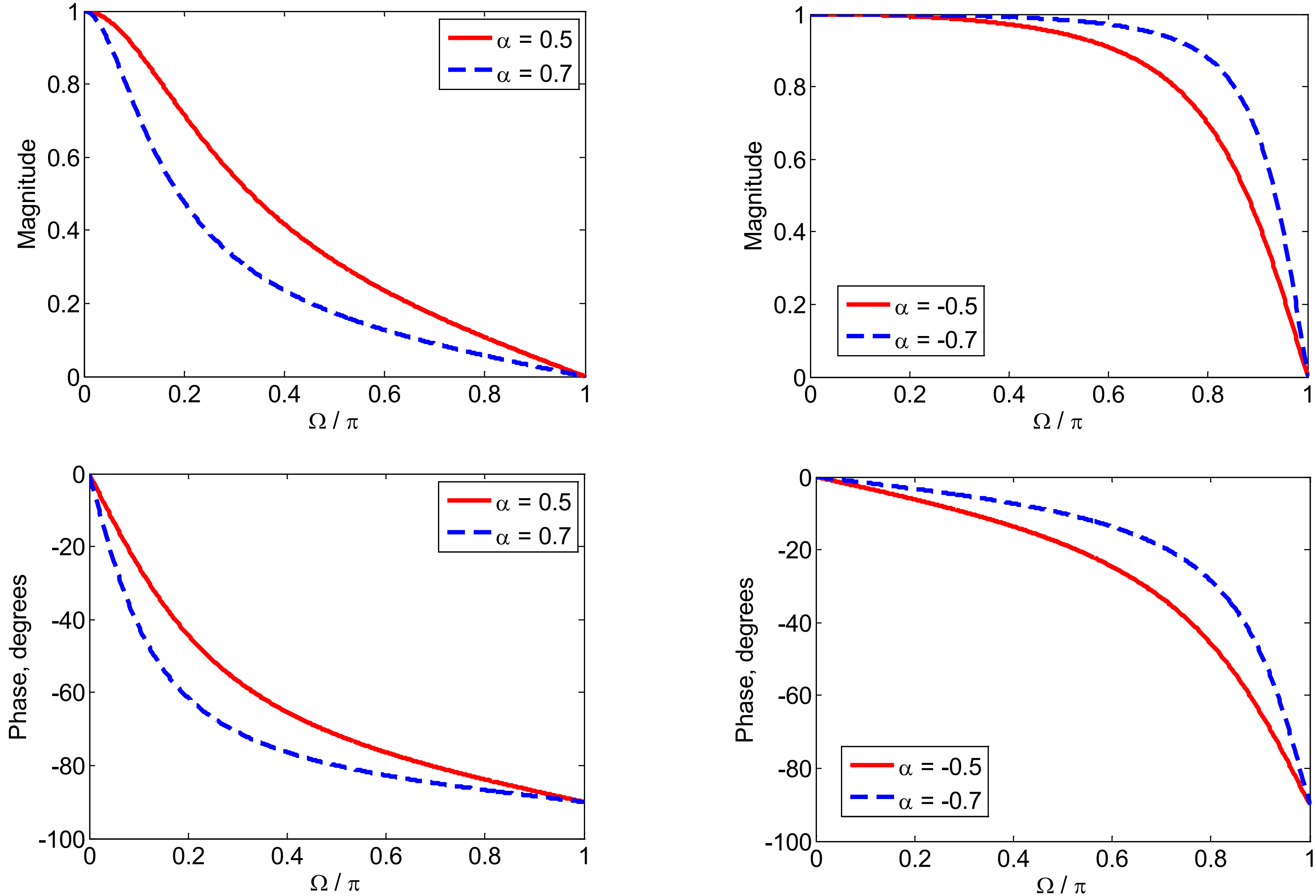


- All FIR digital filters have their poles at the origin. → Only the zeros determine the filter characteristic.
- The filter characteristic of IIR digital filters are determined by zeros and poles.

- **Lowpass IIR digital filters**

$$H_{\mathrm{LP},1}(z) = \frac{1-\alpha}{2} \cdot \frac{1+z^{-1}}{1-\alpha \cdot z^{-1}} = \frac{1-\alpha}{2} \cdot \frac{z+1}{z-\alpha}$$

$$H_{\mathrm{LP},1}\left(\mathrm{e}^{\mathrm{j}\Omega}\right) = \frac{1-\alpha}{2} \cdot \frac{\mathrm{e}^{\mathrm{j}\Omega}+1}{\mathrm{e}^{\mathrm{j}\Omega}-\alpha}$$

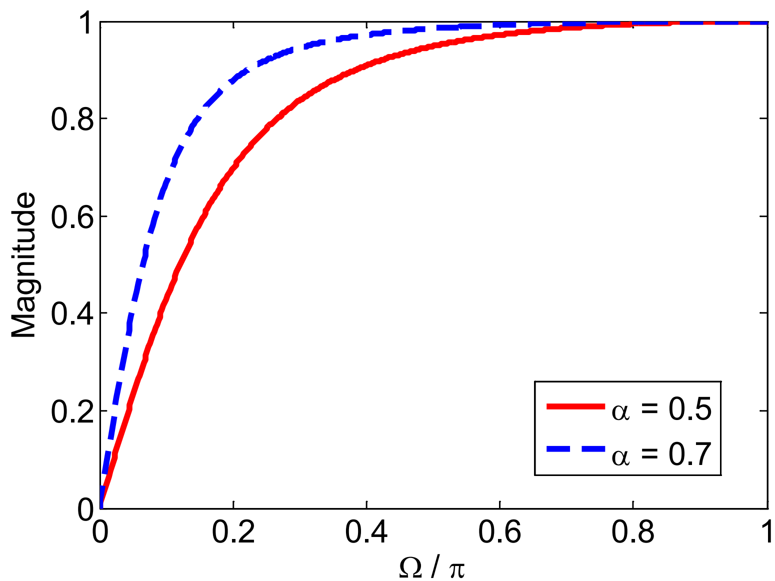


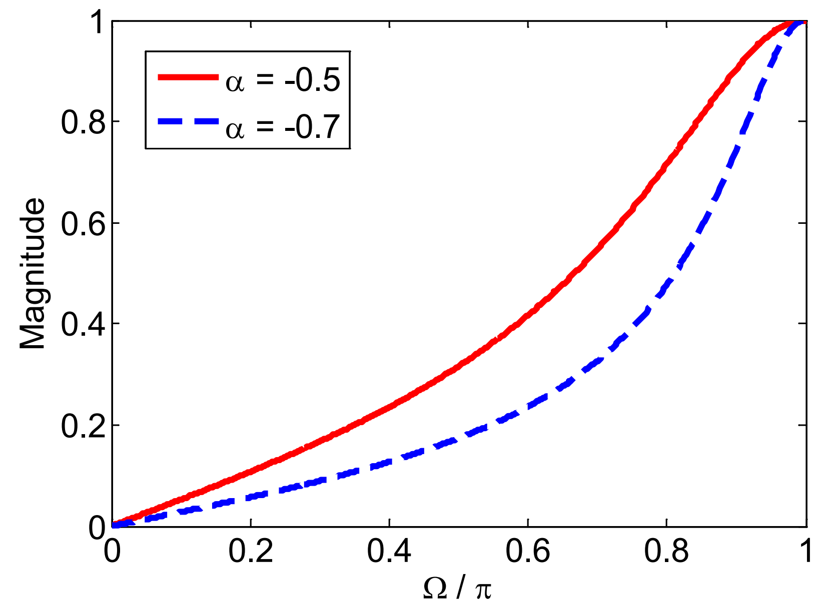
α = 0.5
α = 0.7
Magnitude
Ω / π
α = -0.5
α = -0.7
Magnitude
Ω / π
α = 0.5
α = 0.7
Phase, degrees
Ω / π
α = -0.5
α = -0.7
Phase, degrees
Ω / π

- **Highpass IIR digital filters**

$$H_{\mathrm{HP},1}(z) = \frac{1+\alpha}{2} \cdot \frac{1-z^{-1}}{1-\alpha \cdot z^{-1}} = \frac{1+\alpha}{2} \cdot \frac{z-1}{z-\alpha}$$

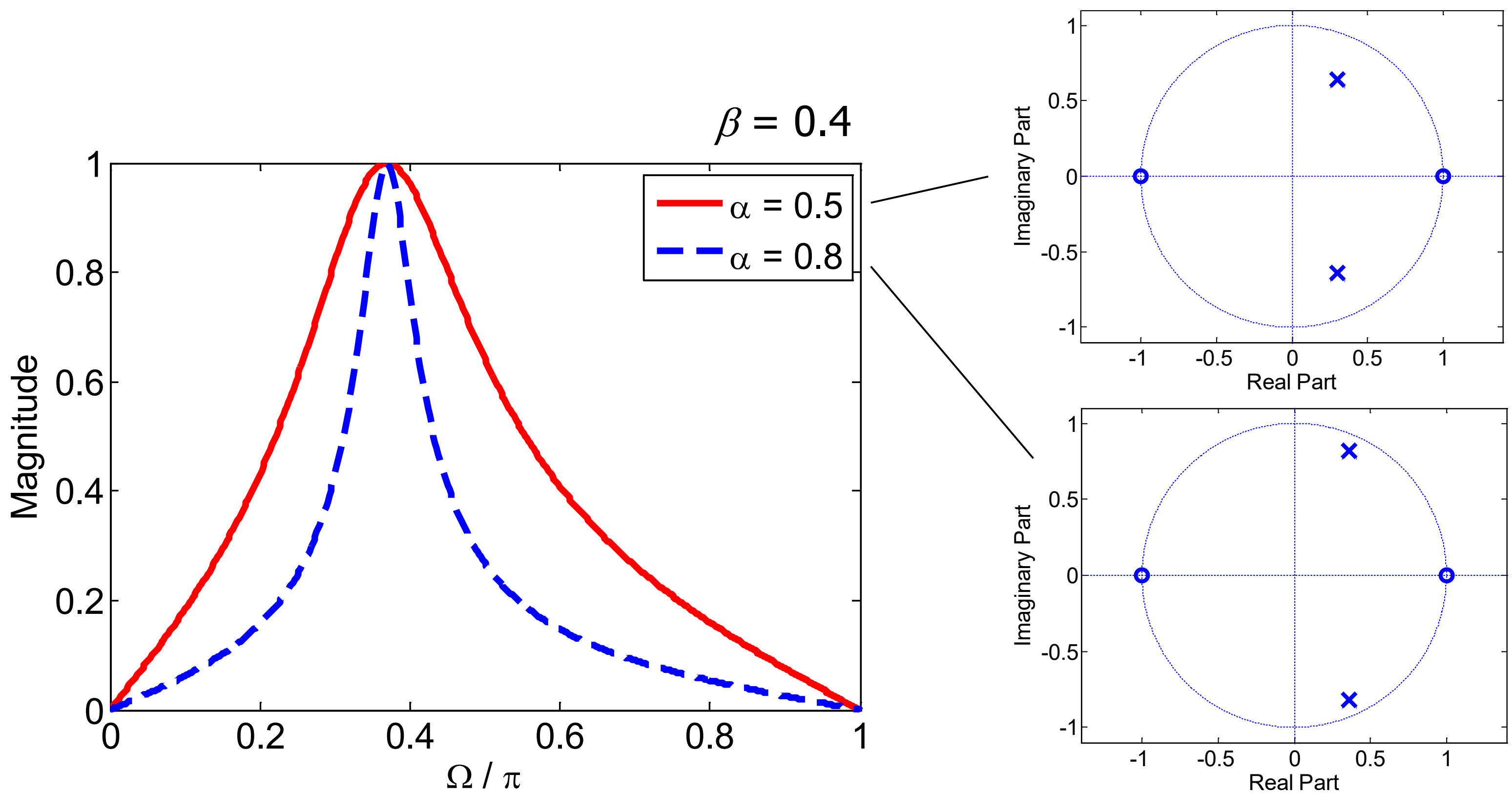
$$H_{\mathrm{HP},1}\left(\mathrm{e}^{\mathrm{j}\Omega}\right) = \frac{1+\alpha}{2} \cdot \frac{\mathrm{e}^{\mathrm{j}\Omega}-1}{\mathrm{e}^{\mathrm{j}\Omega}-\alpha}$$





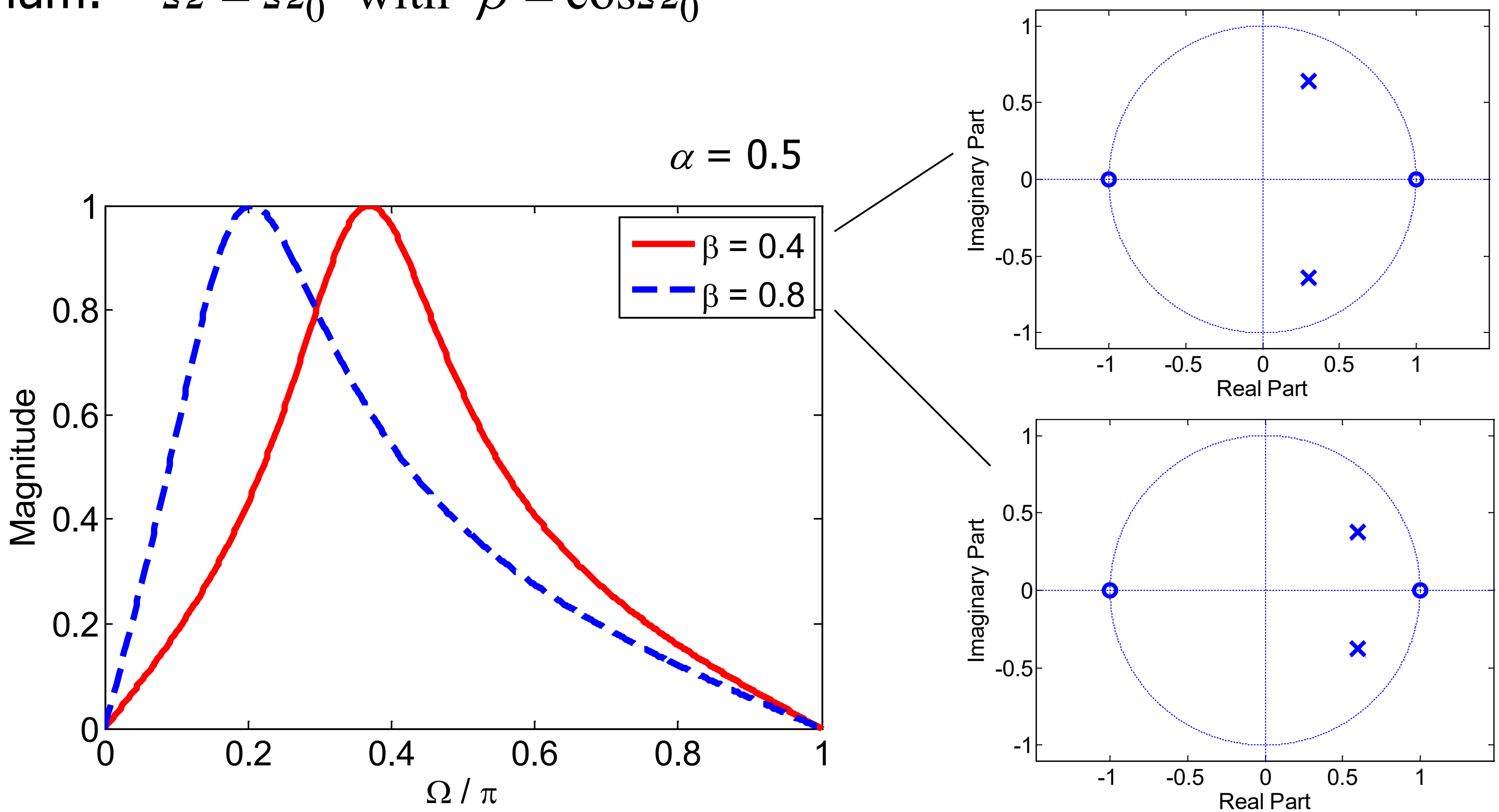
Bandpass IIR digital filters

$$H_{\mathrm{BP},2}(z) = \frac{1-\alpha}{2} \cdot \frac{1-z^{-2}}{1-\beta\cdot(1+\alpha)\cdot z^{-1}+\alpha\cdot z^{-2}} = \frac{1-\alpha}{2} \cdot \frac{z^2-1}{z^2-\beta\cdot(1+\alpha)\cdot z+\alpha}$$



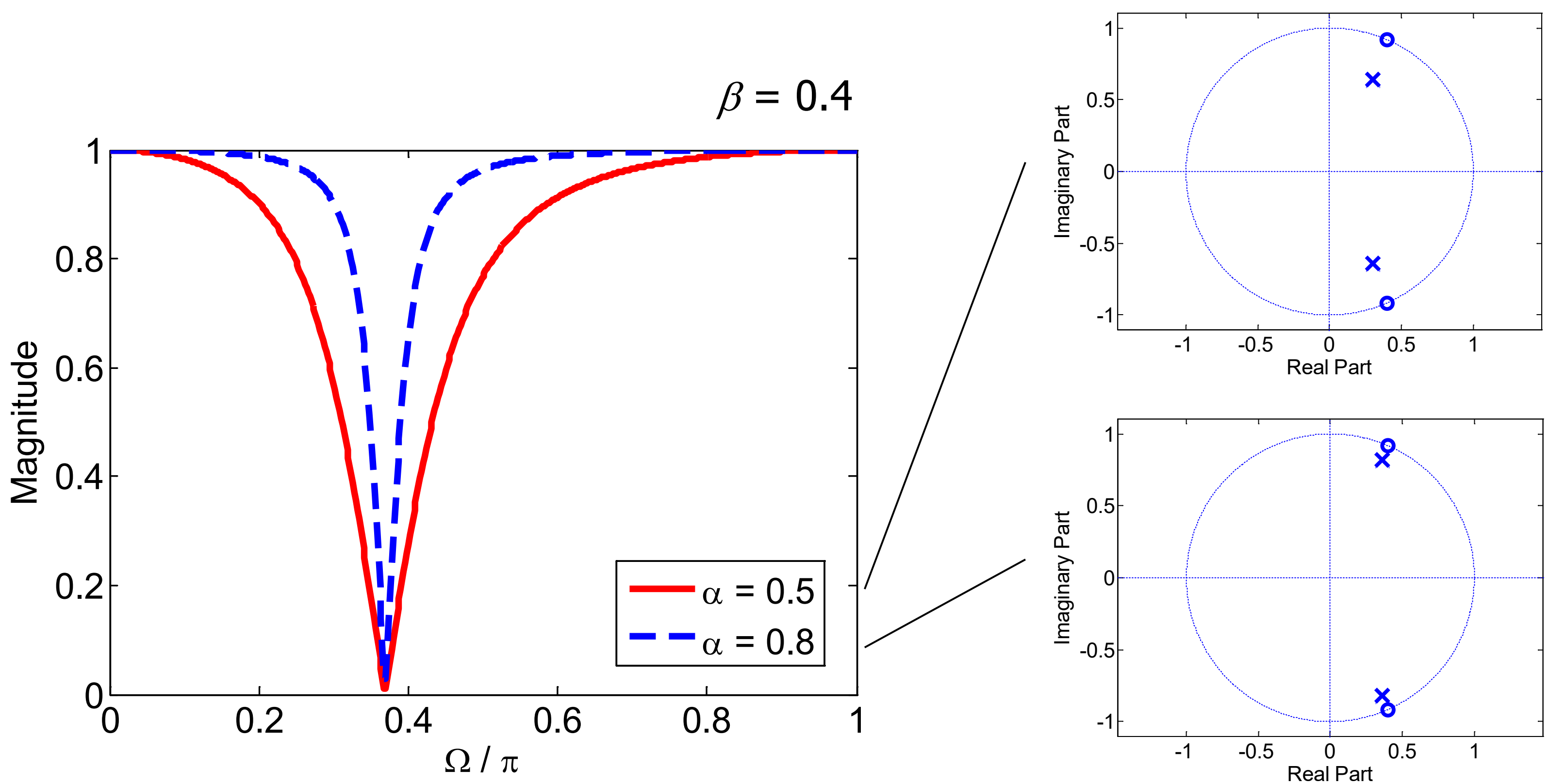
- Poles: $p_{1/2} = r \cdot \mathrm{e}^{\pm \mathrm{j}\phi}\ ;\ r = \sqrt{\alpha}\ ;\ \cos\phi = \dfrac{\beta \cdot (1+\alpha)}{2 \cdot \sqrt{\alpha}}$

- Maximum: $\Omega = \Omega_0 \ \text{ with } \ \beta = \cos\Omega_0$

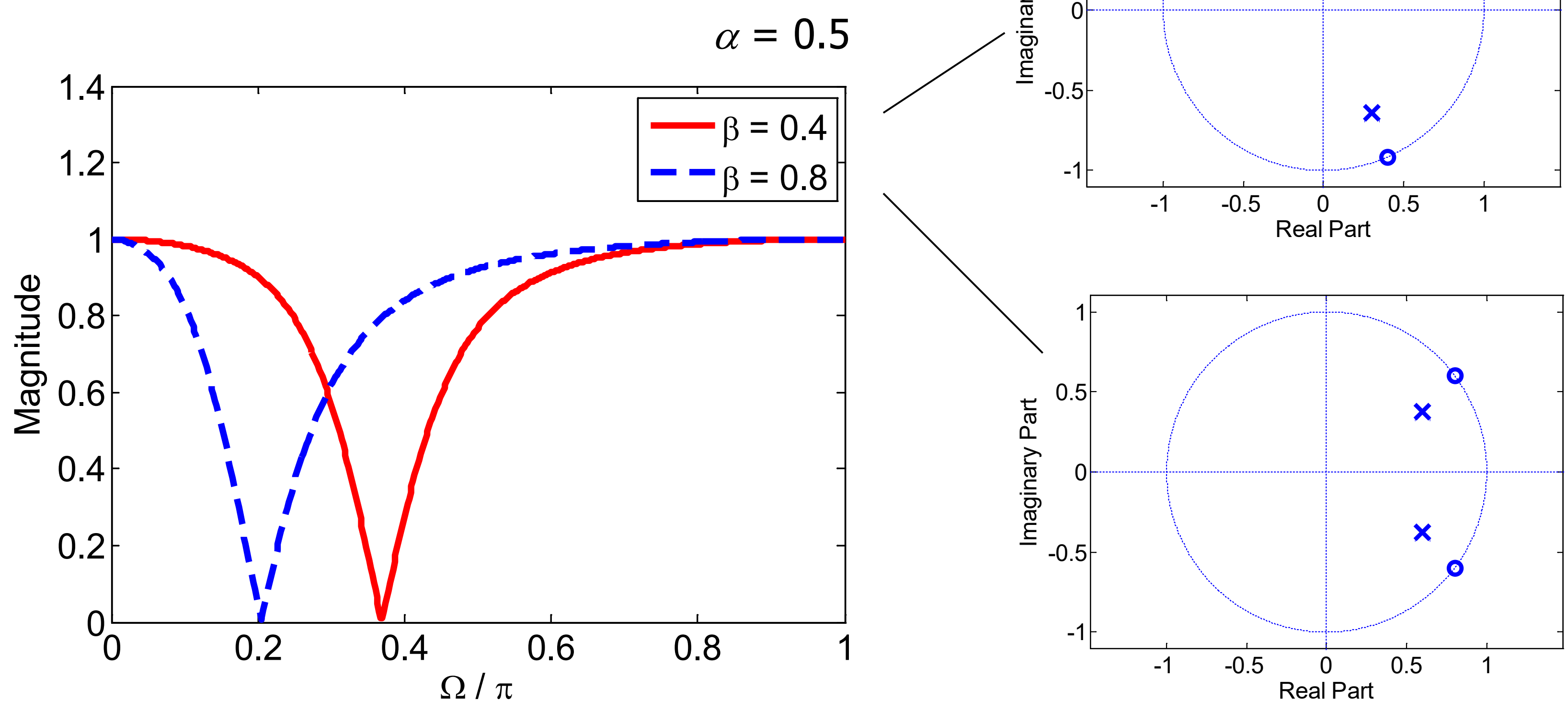


- **IIR notch digital filters**

$$H_{BP,2}(z) = \frac{1+\alpha}{2} \cdot \frac{1 - 2 \cdot \beta \cdot z^{-1} + z^{-2}}{1 - \beta \cdot (1+\alpha) \cdot z^{-1} + \alpha \cdot z^{-2}} = \frac{1-\alpha}{2} \cdot \frac{z^2 - 2 \cdot \beta \cdot z + 1}{z^2 - \beta \cdot (1+\alpha) \cdot z + \alpha}$$



- **Zeros:** $z_{1/2} = r \cdot \mathrm{e}^{\pm \mathrm{j}\phi}\ ;\ r = 1;\ \cos\phi = \beta$
- **Poles:** $p_{1/2} = r \cdot \mathrm{e}^{\pm \mathrm{j}\phi}\ ;\ r = \sqrt{\alpha}\ ;\ \cos\phi = \dfrac{\beta \cdot (1+\alpha)}{2 \cdot \sqrt{\alpha}}$
- **Notch:** $\Omega = \Omega_0$ with $\beta = \cos\Omega_0$



6.2.2 FIR Digital Filter Design
6.2.2.1 Linear-Phase FIR Transfer Functions

- **Extended Definition**

$$H(\mathrm{e}^{\mathrm{j}\Omega}) = \tilde{H}(\Omega)\cdot \mathrm{e}^{\mathrm{j}\varphi} \text{ with } \varphi = a\cdot\Omega + b$$

$$\tilde{H}(\Omega)\text{: Zero-phase response}$$

$$\text{phase delay } \tau_{\mathrm{p}} = -\frac{\varphi}{\Omega} = -a - \frac{b}{\Omega} \qquad \text{group delay } \tau_{\mathrm{g}} = -\frac{\mathrm{d}\varphi}{\mathrm{d}\Omega} = -a$$

- $H(z) = \sum_{n=0}^{N} h[n]\cdot z^{-n}$ with $h[n]$ real.

Order N of transfer function can be even or odd.

- $\tilde{H}(\Omega)$ even requires a symmetric impulse response.

  $$h[n] = h[N-n] \ \text{with} \ 0 \le n \le N\,;\ b = 0 \text{ or } \pi$$

- $\tilde{H}(\Omega)$ odd requires an antisymmetric impulse response.

  $$h[n] = -h[N-n] \ \text{with} \ 0 \le n \le N\,;\ b = \pm\frac{\pi}{2}$$

- General form of frequency response

  $$H(\mathrm{e}^{\mathrm{j}\Omega}) = \tilde{H}(\Omega)\cdot \mathrm{e}^{\mathrm{j}b} \cdot \mathrm{e}^{-\mathrm{j}N\Omega/2}$$

  $$\tau_{\mathrm{g}} = \frac{N}{2}$$

- **Type I:** Symmetric impulse response, N even

$$\widetilde{H}(\Omega) = h\left[\frac{N}{2}\right] + 2 \cdot \sum_{n=1}^{N/2} h\left[\frac{N}{2} - n\right] \cdot \cos(n \cdot \Omega)$$

- **Type II:** Symmetric impulse response, N odd

$$\widetilde{H}(\Omega) = 2 \cdot \sum_{n=1}^{(N+1)/2} h\left[\frac{N+1}{2} - n\right] \cdot \cos\left(\left(n - \frac{1}{2}\right) \cdot \Omega\right)$$

- **Type III:** Antisymmetric impulse response, N even

$$\tilde{H}(\Omega) = 2 \cdot \sum_{n=1}^{N/2} h\left[\frac{N}{2} - n\right] \cdot \sin(n \cdot \Omega)$$

- **Type IV:** Antisymmetric impulse response, N odd

$$\tilde{H}(\Omega) = 2 \cdot \sum_{n=1}^{(N+1)/2} h\left[\frac{N+1}{2} - n\right] \cdot \sin\left(\left(n - \frac{1}{2}\right) \cdot \Omega\right)$$

6.2.2.2 Zero Locations of Linear-Phase FIR Filter

- **Type I and II**

$$H(z) = z^{-N} \cdot H\left(z^{-1}\right)$$ → mirror-image polynomial (MIP)

- **Type III and IV**

$$H(z) = -z^{-N} \cdot H\left(z^{-1}\right)$$ → antimirror-image polynomial (AIP)

- **Requirements for zeros**
  - Real zero ($z \neq \pm 1$): $z = r$ <u>and</u> $z = 1/r$ (2 zeros)
  - Zero on unit circle ($z \neq \pm 1$): $z = \mathrm{e}^{\mathrm{j}\phi}$ <u>and</u> $z = \mathrm{e}^{-\mathrm{j}\phi}$ (2 zeros)
  - Complex zero ($|z| \neq 1$, not real): $z = r \cdot \mathrm{e}^{\pm\mathrm{j}\phi}$ <u>and</u> $z = \frac{1}{r} \cdot \mathrm{e}^{\pm\mathrm{j}\phi}$ (4 zeros)
- **Additional zero requirements**
  - Filter type I: None or even number at $z = \pm 1$.
  - Filter type II: Odd number at $z = -1$. None or even number at $z = 1$.
    $\rightarrow$ no highpass
  - Filter type III: Odd number at $z = 1$ and at $z = -1$. $\rightarrow$ only bandpass
  - Filter type IV: Odd number at $z = 1$. None or even number at $z = -1$.
    $\rightarrow$ no lowpass

6.2.3 FIR Filter Order

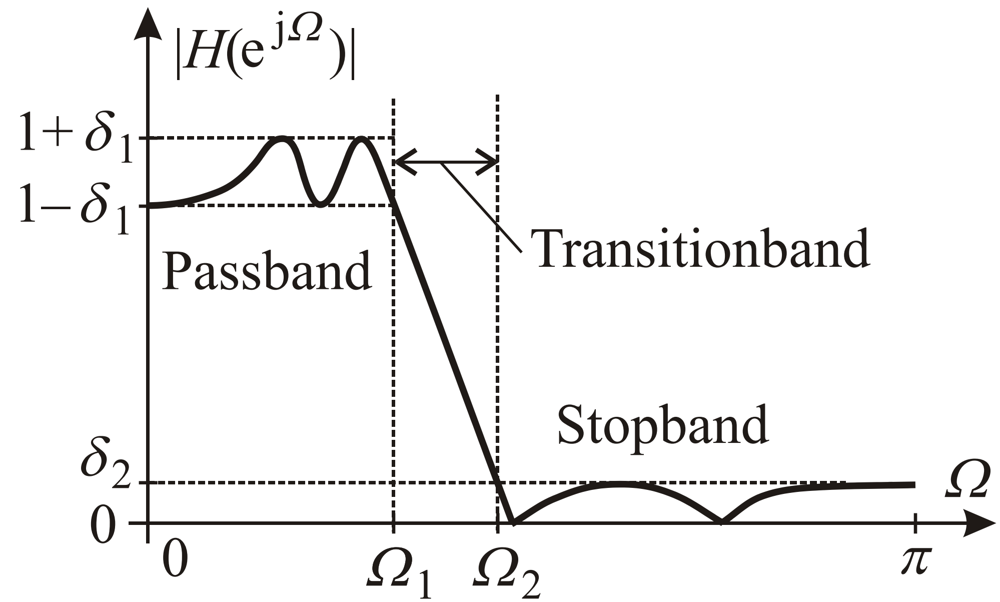
- KAISER's formula

$$N \approx \frac{-20 \cdot \log_{10}\left(\sqrt{\delta_1 \cdot \delta_2}\right) - 13}{14.6 \cdot (\Omega_2 - \Omega_1)/(2\pi)}$$

- BELLANGER's formula

$$N \approx \frac{-2 \cdot \log_{10}(10 \cdot \delta_1 \cdot \delta_2)}{3 \cdot (\Omega_2 - \Omega_1)/(2\pi)} - 1$$

- The formulas can also be used for highpass, bandpass, and bandstop filters. Dominant is the smallest transition band.



Typical magnitude specifications [MIT]

Ω_1, Ω_2: passband, stopband edge frequencies

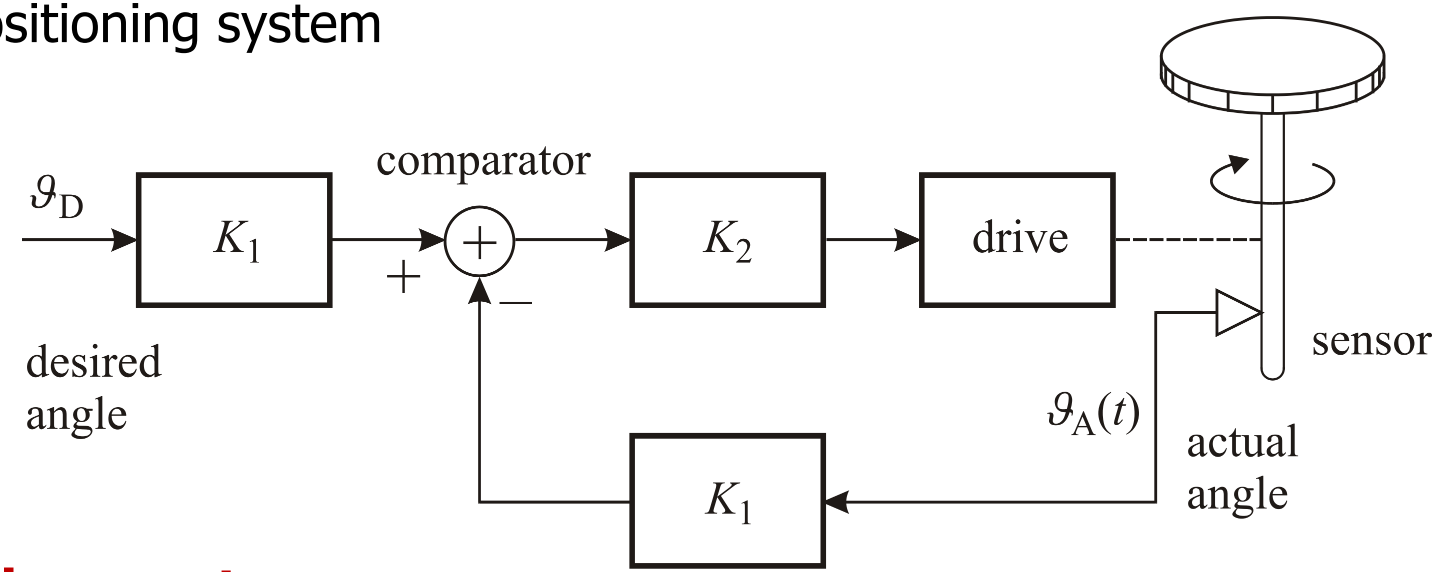
δ_1, δ_2: peak ripple values

6.2.4 Design Concepts

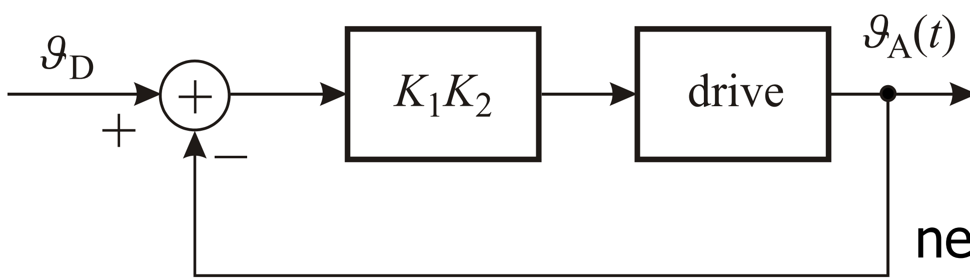
- Contrary to IIR filters, the design of FIR filters doesn't has any relations to the design of analog filters.
- The order of the FIR transfer functions is usually much higher compared to IIR filters.
- Design goal
  - Linear-phase design for constant group delay.
  - Minimum-phase design for minimal group delay.
  - Implementation efficient approaches with minimum number of multipliers.
- Design approach
  - Frequency sampling approach.
  - Windowed FOURIER series approach.

6.3 Control Systems

Example: Positioning system



closed-loop system

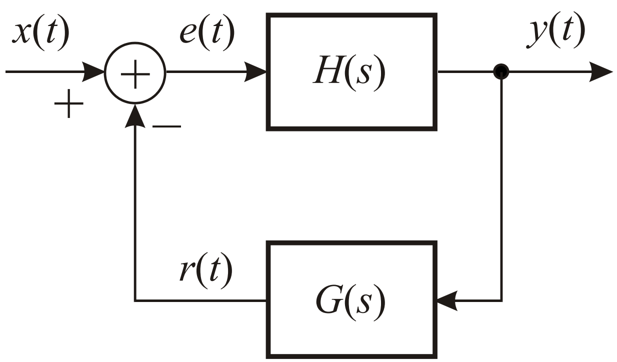


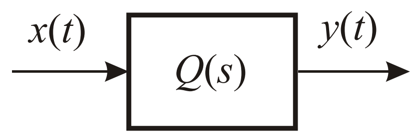
simplified block diagram

negative feedback

6.3.1 Linear Feedback Systems

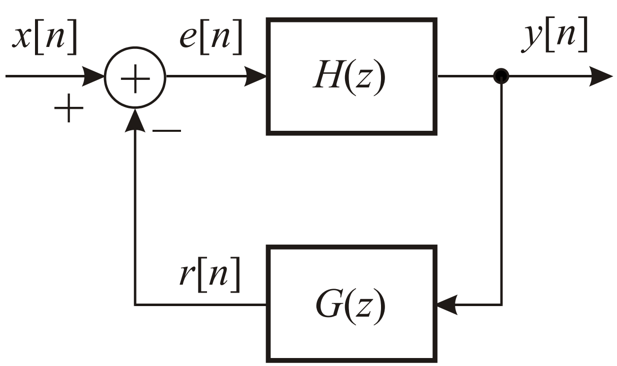
- **Continuous-time LTI feedback system**

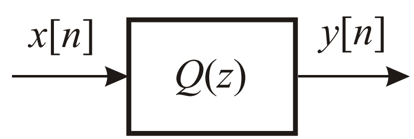




$$Q(s) = \frac{H(s)}{1 + G(s) \cdot H(s)}$$

- **Discrete-time LTI feedback system**





$$Q(z) = \frac{H(z)}{1 + G(z) \cdot H(z)}$$

- $H(s)$, $H(z)$: System function of the forward path
- $G(s)$, $G(z)$: System function of the feedback path
- $Q(s)$, $Q(z)$: Closed-loop system function

- **Open-loop system**
  - $G(s) \cdot H(s)$
  - $G(z) \cdot H(z)$

6.3.2 Stabilization of Unstable Systems

- One use of feedback is to stabilize systems, which are unstable without feedback.
- Continuous-time system

$$H(s) = \frac{b}{s-a} \;\; ; \;\; G(s) = K \;\; ; \quad a, b, K \text{ real}$$

$$r(t) = K \cdot y(t) \qquad \rightarrow \text{proportional feedback system}$$

Feedback system is stable for $K > \frac{a}{b}$ with $b > 0$
or $K < \frac{a}{b}$ with $b < 0$

- Discrete-time system: **population growth**

$$y[n] = 2 \cdot y[n-1] + e[n] \quad \leftarrow \text{unstable}$$

$y[n]$: population at generation n

$e[n]$: external influences (migration, disease, ...)

$$e[n] = x[n] - r[n]; \; r[n] = 2 \cdot \beta \cdot y[n-1] \quad \leftarrow \text{feedback approach}$$

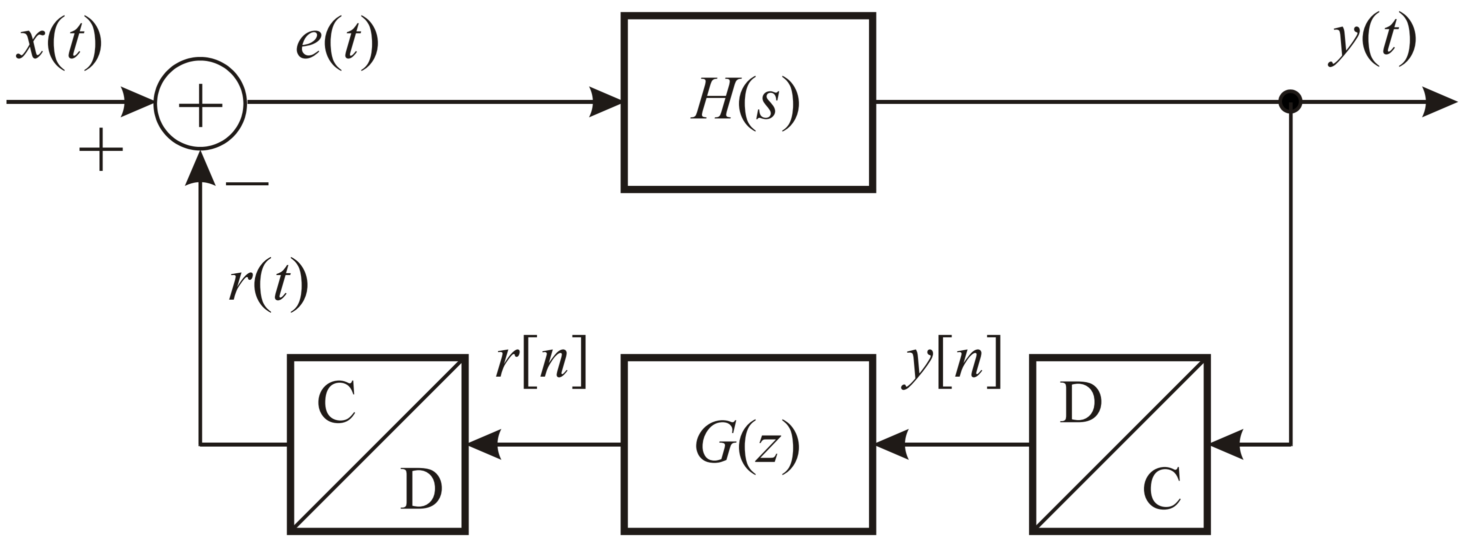
$r[n]$: regulative influences ; β real

$x[n]$: external effects (migration, disease, ...)

Feedback system is stable for $\frac{1}{2} < \beta < \frac{3}{2}$

6.3.3 Sampled-Data Feedback Systems

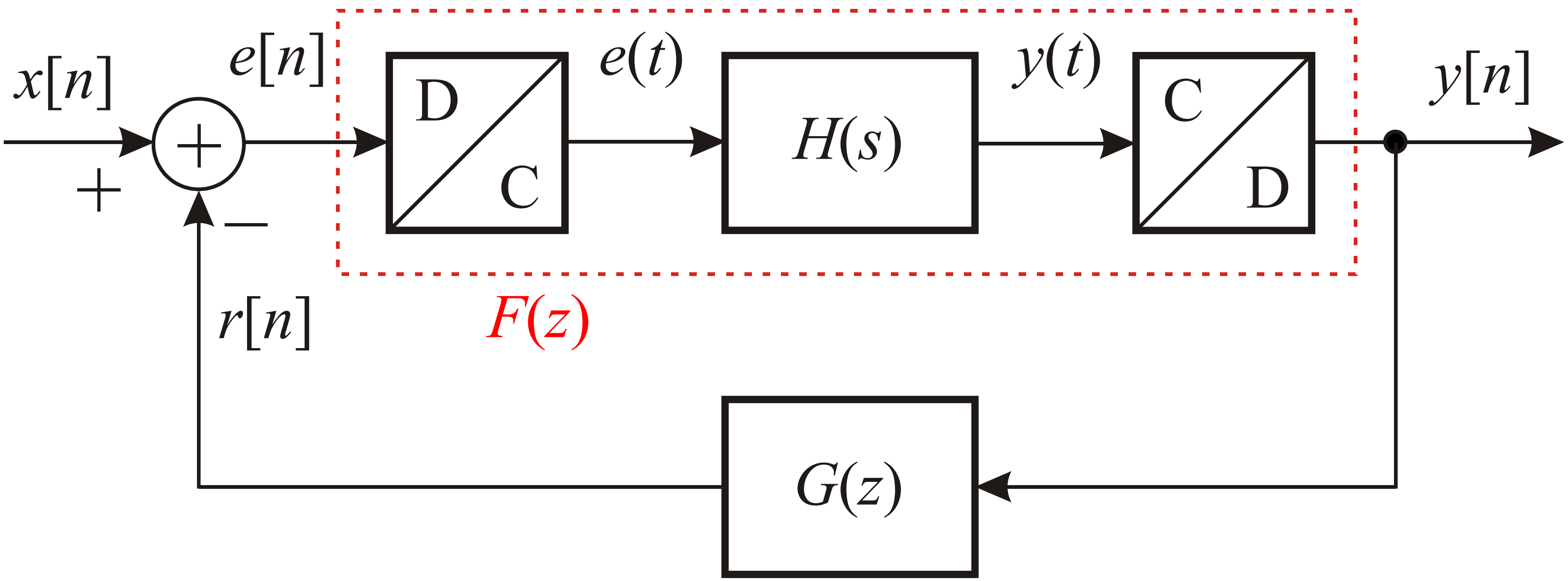
- Discrete-time feedback systems apply discrete processing in the feedback path and consider a real-world continuous-time system in the forward path



- Assumption

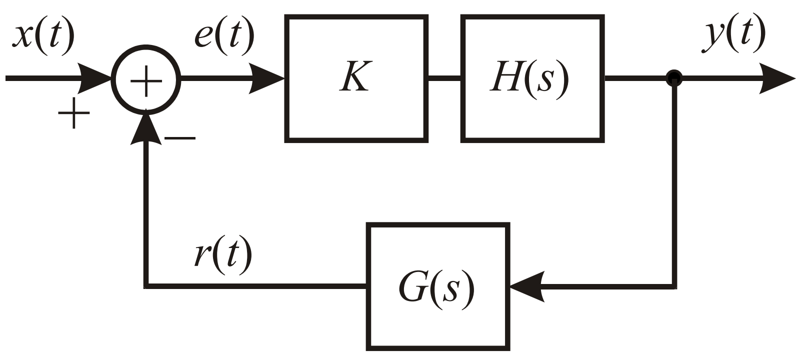
$$x(t) = x[n] \ \text{ for } \ n \cdot T_s \leq t < (n+1) \cdot T_s$$

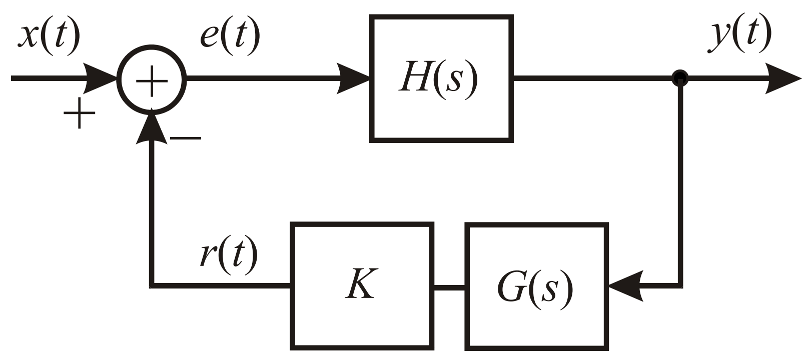
- Resultant is a completely discrete-time feedback system.



6.3.4 Root-Locus Analysis

- The feedback systems are supplemented by an adjustable gain element.
- **Continuous-time systems**





$$Q(s) = \frac{K \cdot H(s)}{1 + K \cdot G(s) \cdot H(s)} \qquad Q(s) = \frac{H(s)}{1 + K \cdot G(s) \cdot H(s)}$$

- **Discrete-time systems**

$$Q(z) = \frac{K \cdot H(z)}{1 + K \cdot G(z) \cdot H(z)} \qquad Q(z) = \frac{H(z)}{1 + K \cdot G(z) \cdot H(z)}$$

Closed-loop poles

$$G(\ldots)\cdot H(\ldots) = -\frac{1}{K} \quad \text{with } K \text{ real}, \quad '\ldots' = s \text{ or } z$$

- $K \to 0 \Longrightarrow G(\ldots)\cdot H(\ldots) \to -\infty$ $\leftarrow$ poles of $G(\ldots)\cdot H(\ldots)$
- $K \to \pm\infty \Longrightarrow G(\ldots)\cdot H(\ldots) \to 0$ $\leftarrow$ zeros of $G(\ldots)\cdot H(\ldots)$
- $\arg\{G(\ldots)\cdot H(\ldots)\} = n\cdot\pi$; n: integer
  - $\arg\{G(\ldots)\cdot H(\ldots)\} = k\cdot\pi$; k: odd

$$K = \frac{1}{|G(\ldots)\cdot H(\ldots)|} > 0$$

  - $\arg\{G(\ldots)\cdot H(\ldots)\} = k\cdot\pi$; k: even

$$K = \frac{-1}{|G(\ldots)\cdot H(\ldots)|} < 0$$

The **root-locus curve** is a plot of the closed-loop poles with varying gain element.

The root-locus curve

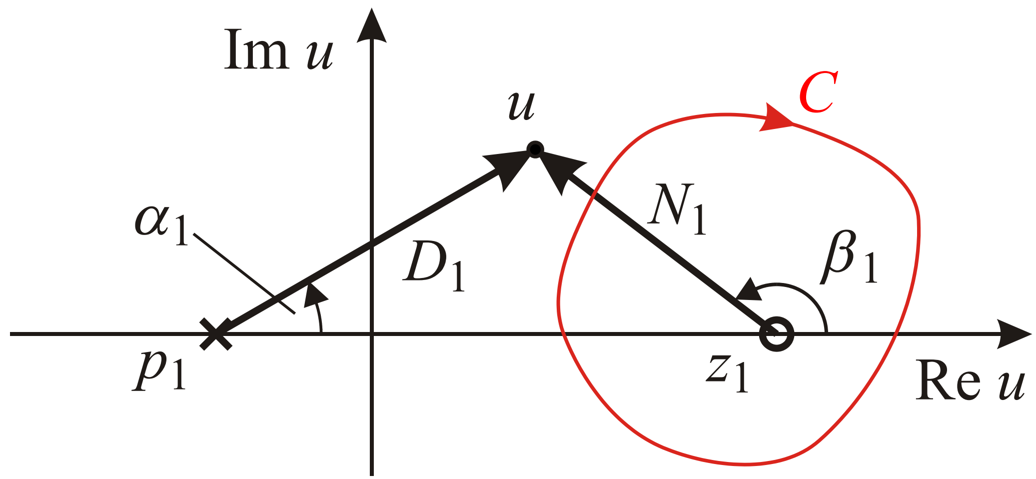
- starts at zeros of $G(\dots)\cdot H(\dots)$ for $K \to -\infty$,
- equals poles of $G(\dots)\cdot H(\dots)$ for $K = 0$,
- ends at zeros of $G(\dots)\cdot H(\dots)$ for $K \to \infty$.
- Branches between two real poles must break off into the complex plane for $|K| > 0$.

6.3.5 NYQUIST Stability Criterion

- The root−locus analysis requires the analytic description of the forward and feedback system functions in a rational form.
- The NYQUIST criterion
  - determines whether or not the closed-loop system is stable for a given value of the adjustable gain parameter K,
  - can be applied to nonrational system functions and in situations in which to analytic descriptions of the system functions are available.

Encirclement Property

$$W(u) = \frac{u - z_1}{u - p_1} = \frac{N_1 \cdot e^{j\beta_1}}{D_1 \cdot e^{j\alpha_1}}$$



If u traverses along a closed contour C in the u-plane once in the clockwise direction,

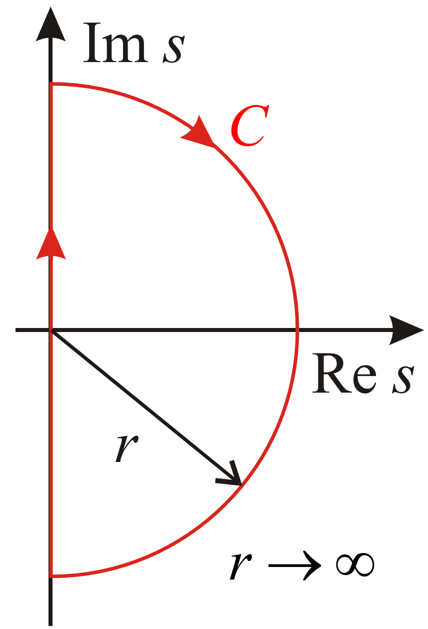
the corresponding plot of $W(u)$ will encircle the origin $M - N$ times in the clockwise direction, where

M is the number of zeros within C and

N is the number of poles within C.

Continuous-Time NYQUIST Stability Criterion

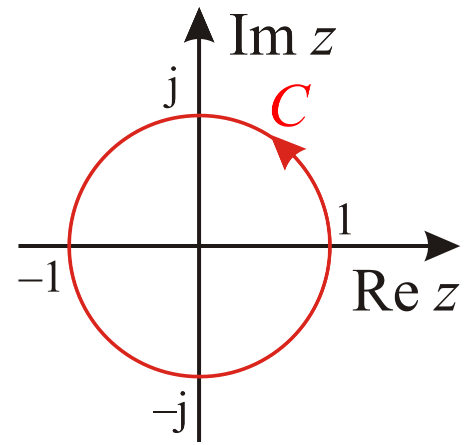
- **Requirement for stability:** No zero of $W(s) = \frac{1}{K} + H(s) \cdot G(s)$ lies in the right half of the s-plane.
- The plot of $H(\mathrm{j}\omega) \cdot G(\mathrm{j}\omega)$ as ω varies from $-\infty$ to $+\infty$ is called the NYQUIST plot.
- The number of clockwise encirclements of the point $-1/K$ by the NYQUIST plot equals the number of right-half plane closed-loop poles minus the number of right-half plane poles of $H(s) \cdot G(s)$.



- **Requirement for stability:** The number of <u>counterclockwise</u> encirclements of the point $-1/K$ by the NYQUIST plot of $H(\mathrm{j}\omega) \cdot G(\mathrm{j}\omega)$ must equal the number of right-half-plane poles of $H(s) \cdot G(s)$.

Discrete-Time NYQUIST Stability Criterion

- **Requirement for stability:** No zero of $W(z) = \frac{1}{K} + H(z) \cdot G(z)$ lies outside the unit circle of the z-plane.



- The plot of $H\left(e^{j\Omega}\right) \cdot G\left(e^{j\Omega}\right)$ as Ω varies from 0 to 2π is called the NYQUIST plot.
- The number of clockwise encirclements of the point $-1/K$ by the NYQUIST plot equals the number of closed-loop poles outside the unit circle minus the number of poles of $H(z) \cdot G(z)$ outside the unit circle.

- **Requirement for stability:** The number of <u>counterclockwise</u> encirclements of the point $-1/K$ by the NYQUIST plot of $H\left(e^{j\Omega}\right) \cdot G\left(e^{j\Omega}\right)$ as Ω varies from 0 to 2π must equal the number of poles of $H(z) \cdot G(z)$ outside the unit circle.