Probability and Statistics

4 – Continuous Random Variables

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Joint Random Variables

Definition (4.20)

Let X, Y be random variables with respect to the same probability measure Pr, i.e. with respect to the same triple $(\Omega, \mathcal{A}, \Pr)$. The *joint cumulative distribution function* of X and Y, $F_{XY} : \mathbb{R}^2 \to [0,1]$, is defined by:

$$\begin{aligned} F_{XY}(x,y) &:= \Pr(X \leq x, Y \leq y) \\ &= \Pr(X^{-1}((-\infty,x]) \cap Y^{-1}((-\infty,y])) \qquad \text{for all } x,y \in \mathbb{R} \end{aligned}$$

Joint Cumulative Distribution Functions

Lemma (4.21)

Let X, Y be random variables with respect to the same probability measure $\Pr$ and $(x_1, x_2] \times (y_1, y_2]$ be a rectangle. Then:

$$\begin{aligned}&\Pr(x_1 < X \leq x_2,\ y_1 < Y \leq y_2)\\ &\quad = F_{XY}(x_2, y_2) - F_{XY}(x_1, y_2) - F_{XY}(x_2, y_1) + F_{XY}(x_1, y_1)\end{aligned}$$

Marginal Cumulative Distribution Functions

Lemma (4.22)

Let F_{XY} be the joint cumulative distribution function of two random variables X and Y. The (marginal) cumulative distribution functions of X and Y are determined by F_{XY} as follows:

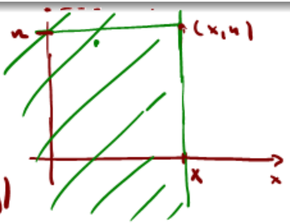
$$F_X(x) = \lim_{y\to\infty} F_{XY}(x,y)$$

$$F_Y(y) = \lim_{x\to\infty} F_{XY}(x,y)$$

$$F_X(x) = Pr(X \le x) = Pr\Big(\bigcup_{n\in\mathbb{N}} \underbrace{\{\omega \mid X(\omega)\le x, Y(\omega)\le n\}}_{A_n}\Big)$$

$A_1 \subseteq A_2 \subseteq \ldots$

$$\overset{4.3}{=} \lim_{n\to\infty} Pr(X\le x, Y\le n) = \lim_{n\to\infty} F_{XY}(x,n) = \lim_{y\to\infty} F_{XY}(x,y)$$



Independent Random Variables

Definition (4.23)

Two random variables X and Y, defined with respect to the same probability measure Pr, are called *independent*, if for all intervals $S, T \subseteq \mathbb{R}$:

$$\Pr(X \in S,\ Y \in T) \;=\; \Pr(X \in S) \cdot \Pr(Y \in T)$$

$A = X^{-1}(S)$, $B = Y^{-1}(T)$

$Pr(A \cap B) = Pr(A) \cdot Pr(B)$

Independent Random Variables

Lemma (4.24)

Let X, Y be random variables with respect to the same probability measure Pr. *Then X and Y are independent, if and only if one of the following holds:*

(i) *For all $x_1, x_2, y_1, y_2 \in \mathbb{R}$ with $x_1 < x_2$ and $y_1 < y_2$:* $S = (x_1, x_2], T = (y_1, y_2]$

$$\Pr(x_1 < X \leq x_2,\ y_1 < Y \leq y_2) \;=\; (F_X(x_2) - F_X(x_1)) \cdot (F_Y(y_2) - F_Y(y_1))$$

(ii) *For all $x, y \in \mathbb{R}$:*

$$F_{XY}(x, y) \;=\; F_X(x) \cdot F_Y(y)$$

$S = (-\infty, x], T = (-\infty, y]$

Proof of Lemma (4.24)

Show: $\Pr((X, Y) \in S \times T) = \Pr(X \in S) \cdot \Pr(Y \in T)$ for all intervals $S, T \subseteq \mathbb{R}$

(i) $S = \mathbb{R} = (-\infty, \infty)$

(ii) $S = (-\infty, x)$ see (4.2), (4.3)

(iii) $S = [x, \infty)$

(iv) $S = (x, \infty)$

(v) $S = (x_1, x_2)$

(vi) $S = (x_1, x_2]$

(vii) $S = [x_1, x_2)$

(viii) $S = [x_1, x_2]$

Proof of Lemma (4.24)

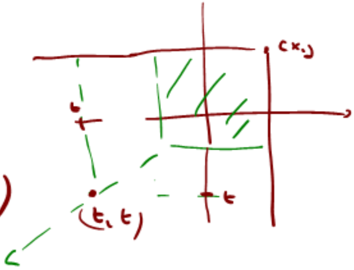
E.g. (i) $\Rightarrow$ (ii)

$$F_{XY}(x,y) \overset{(4.3)}{=} \lim_{t \to -\infty} \Pr(t < X \le x,\ t < Y \le y)$$

$$= \lim_{t \to -\infty} \left(F_X(x) - F_X(t)\right) \cdot \left(F_Y(y) - F_Y(t)\right)$$

$$= \lim (\qquad) \cdot \lim (\qquad)$$

$$= \left(F_X(x) - 0\right) \cdot \left(F_Y(y) - 0\right) = F_X(x) \cdot F_Y(y)$$



Proof of Lemma (4.24)

(ii) $\Rightarrow$ (i)

$$\Pr(x_1 < X \le x_2,\ y_1 < Y \le y_2)$$

$$\overset{(4.21)}{=} F_{XY}(x_2, y_2) - F_{XY}(x_1, y_2) - F_{XY}(x_2, y_1) + F_{XY}(x_1, y_1)$$

$$\overset{(ii)}{=} F_X(x_2)\, F_Y(y_2) - F_X(x_1)\, F_Y(y_2) - F_X(x_2)\, F_Y(y_1) + F_X(x_1) \cdot F_Y(y_1)$$

$$= \big(F_X(x_2) - F_X(x_1)\big)\big(F_Y(y_2) - F_Y(y_1)\big)$$

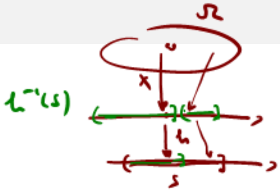
$\square$

Independent Random Variables

Lemma (4.25)

Let X, Y be joint independent random variables and $h, k : \mathbb{R} \to \mathbb{R}$ be piecewise continuous functions, then $h(X)$ and $k(Y)$ are independent.

Proof of Lemma (4.25)



$S, T \subseteq \mathbb{R}$ intervals $\quad h^{-1}(S) = \dot{\bigcup}_{i=1}^{m} S_i \,, \quad k^{-1}(T) = \dot{\bigcup}_{j=1}^{n} T_j$

$$\Pr(h(X) \in S, k(Y) \in T) = \Pr(X \in h^{-1}(S), Y \in k^{-1}(T))$$

$$= \Pr(X \in \dot{\bigcup} S_i, Y \in \dot{\bigcup} T_j)$$

$$= \Pr((X,Y) \in \dot{\bigcup_{i,j}} S_i \times T_j) = \sum_{i,j} \Pr((X,Y) \in S_i \times T_j)$$

assumption: X, Y are independent!

$$= \sum_{i,j} \Pr(X \in S_i) \cdot \Pr(Y \in T_j) = \sum_{i=1}^{m} \Pr(X \in S_i) \cdot \sum_{j=1}^{n} \Pr(Y \in T_j)$$

$$= \Pr(X \in \dot{\bigcup} S_i) \cdot \Pr(Y \in \dot{\bigcup} T_j)$$

$$= \Pr(X \in h^{-1}(S)) \cdot \Pr(Y \in k^{-1}(T))$$

$$= \Pr(h(X) \in S) \cdot \Pr(k(Y) \in T)$$

Jointly Continuous Random Variables

Definition (4.26)

Two random variables X and Y are *jointly continuous random variables*, if there exists a function $f_{XY} : \mathbb{R}^2 \to \mathbb{R}_0^+$, such that

$$F_{XY}(x, y) = \int_{-\infty}^{y} \int_{-\infty}^{x} f_{XY}(s, t)\, ds\, dt$$

for all $x, y \in \mathbb{R}$. The function f_{XY} is called a *joint probability density function* of X and Y.

Jointly Continuous Random Variables

Lemma (4.27)

Let f_{XY} be a joint probability density function of two random variables X and Y. The (marginal) probability density functions of X and Y are determined by f_{XY} as follows:

$$f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x,t)\,dt \qquad \checkmark$$

$$f_Y(y) = \int_{-\infty}^{\infty} f_{XY}(s,y)\,ds$$

$$F_X(x) = \lim_{y \to \infty} F_{XY}(x,y) = \lim_{y \to \infty} \int_{-\infty}^{x} \left(\int_{-\infty}^{y} f_{XY}(s,t)\,dt \right) ds = \int_{-\infty}^{x} \Bigg(\underbrace{\int_{-\infty}^{+\infty} f_{XY}(s,t)\,dt}_{f(s)} \Bigg) ds$$

is a pdf for X

Jointly Continuous Random Variables

Lemma (4.28)

Let f_{XY} be a joint probability density function of two random variables X and Y. If f_{XY} is continuous, then for all $x, y \in \mathbb{R}$:

$$\frac{\partial^2}{\partial y \partial x} F_{XY}(x,y) = \frac{\partial^2}{\partial x \partial y} F_{XY}(x,y) = f_{XY}(x,y)$$

$$\frac{\partial^2}{\partial y \partial x} F_{XY}(x,y) = \frac{\partial}{\partial x}\left(\frac{\partial}{\partial y}\left(\int_{-\infty}^{y}\left(\int_{-\infty}^{x} f_{XY}(s,t)\, ds\right) dt\right)\right)$$

$$= \frac{\partial}{\partial x}\left(\int_{-\infty}^{x} f_{XY}(s,y)\, ds\right) = f_{XY}(x,y)$$