# Probability and Statistics

## 4 – Continuous Random Variables

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December 19, 2023

# Gamma Function

different $\lim\limits_{x \to 0+} f(x)$

for $\alpha \begin{cases} \in (0,1) \\ = 1 \\ > 1 \end{cases}$

### Definition (4.55)

For $\alpha > 0$ the *gamma function* is defined by: $\quad \Gamma(\alpha) \;=\; \int_0^\infty x^{\alpha-1} e^{-x}\, dx$

### Theorem (4.56)

(i) $\Gamma(\alpha) \;=\; (\alpha - 1)\,\Gamma(\alpha - 1) \qquad$ *for all* $\underline{\alpha > 1}$

(ii) $\Gamma(n) \;=\; (n-1)! \qquad$ *for all* $n \in \mathbb{N}$

(iii) $\Gamma(\frac{1}{2}) \;=\; \sqrt{\pi}$ ✓

(iv) $\displaystyle\int_0^\infty x^{\alpha-1}\, e^{-\beta x}\, dx \;=\; \frac{\Gamma(\alpha)}{\beta^\alpha} \qquad$ *for all* $\alpha, \underline{\underline{\beta > 0}}$ $\quad \beta = 1: \quad \Gamma(\alpha)$

# Proof of Theorem (4.56)

(i) $$\Gamma(\alpha) = \int_0^\infty \underbrace{x^{\alpha-1}}_{f} \cdot \underbrace{e^{-x}}_{g'} \,dx = x^{\alpha-1}\cdot(-e^{-x})\Big|_0^\infty - \int_0^\infty (\alpha-1)x^{\alpha-2}\cdot(-e^{-x})\,dx$$
$$= 0 - 0 + (\alpha-1)\int_0^\infty x^{\alpha-2}\cdot e^{-x}\,dx = (\alpha-1)\cdot\Gamma(\alpha-1)$$

(ii) pf by induction $n=1$: $\Gamma(1) = \int_0^\infty e^{-x}\,dx = -e^{-x}\Big|_0^\infty = -(-1) = 1 = 0!$ ✓

Ind. step $n \to n+1$: $\Gamma(n+1) \overset{(i)}{=} n\cdot\Gamma(n) \overset{\text{Ind.}}{=} n\cdot(n-1)! = n!$

(iii) $$\Gamma(1/2) = \int_0^\infty x^{-1/2}\cdot e^{-x}\,dx \underset{u=\sqrt{x}=x^{1/2}}{=} 2\cdot\int_0^\infty e^{-u^2}\left(\tfrac{1}{2}x^{-1/2}\,dx\right) = 2\cdot\int_0^\infty e^{-u^2}\,du \overset{(4.48)(i)}{=} \sqrt{\pi}$$

$$\frac{du}{dx} = \frac{1}{2}x^{-1/2}$$

(iv) $$\int_0^\infty x^{\alpha-1}e^{-\beta x}\,dx \underset{u=\beta x}{=} \int_0^\infty \left(\frac{u}{\beta}\right)^{\alpha-1} e^{-u}\,\frac{1}{\beta}\underbrace{(\beta\,dx)}_{du} = \frac{1}{\beta^\alpha}\cdot\int_0^\infty u^{\alpha-1}\cdot e^{-u}\,du = \frac{\Gamma(\alpha)}{\beta^\alpha}$$

$$du/dx = \beta$$

# Gamma Distributions

**Definition (4.57)**

A random variable has a *gamma distribution* $\Gamma(\alpha, \beta)$ for some parameters $\alpha, \beta \in \mathbb{R}^+$ if its pdf is defined by:

$$f_X(x) = \begin{cases} \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x} & \text{for } x \geq 0 \\ 0 & \text{for } x < 0 \end{cases}$$

# Gamma Distributions

**Theorem (4.58)**

*If $X$ has a gamma distribution with $X \sim \Gamma(\alpha, \beta)$, then:*

(i) $E(X) = \dfrac{\alpha}{\beta}$

(ii) $Var(X) = \dfrac{\alpha}{\beta^2}$

(iii) $\phi_X(t) = \left(\dfrac{\beta}{\beta - t}\right)^{\alpha}$ for $\underline{t < \beta}$

(iv) $b \cdot X \sim \Gamma(\alpha, \beta/b)$ for $b \in \mathbb{R}^+$

# Proof of Theorem (4.58)

HW : Test 10

# Gamma Distributions

**Remark (4.59)**

MATLAB provides implementations of pdf's, cdf's and the inverses of cdf's of gamma distributions under the names `gampdf()`, `gamcdf()` and `gaminv()`, respectively.

# Sum of independent random variables $X_i \sim \Gamma(\alpha_i, \beta)$

**Theorem (4.60)**

*If $X_1, \ldots, X_n$ are independent random variables with $X_i \sim \Gamma(\alpha_i, \underline{\beta})$, then*

$$X = X_1 + \cdots + X_n$$

*has a gamma distribution with* $X \sim \Gamma\left(\sum_{i=1}^{n} \alpha_i,\ \beta\right)$.

$$\phi_X(t) = \prod_{i=1}^{n} \phi_{X_i}(t) \overset{(4.58)(iii)}{=} \prod_{i=1}^{n} \left(\frac{\beta}{\beta - t}\right)^{\alpha_i} = \left(\frac{\beta}{\beta - t}\right)^{\sum_{i=1}^{n} \alpha_i} \longleftarrow$$

moment generating function of a $\Gamma(\sum \alpha_i, \beta)$ distr.

$\square$

# Erlang distributions

## Definition (4.61)

A random variable has an *Erlang distribution* for some parameters $n \in \mathbb{N}, \beta \in \mathbb{R}^+$ if its pdf is defined by:

$$f_X(x) = \begin{cases} \frac{\beta^n}{(n-1)!} x^{n-1} e^{-\beta x} & \text{for } x \geq 0 \\ 0 & \text{for } x < 0 \end{cases}$$

I.e.:

$$\text{Erlang}(n, \beta) := \Gamma(n, \beta) \qquad n \in \mathbb{N}$$

## Remark (4.62)

$$f_X = \lambda e^{-\lambda x}$$

$$X \sim \exp(\lambda) = \text{Erlang}(1, \lambda) = \Gamma(1, \lambda)$$

# Erlang distributions

$\Gamma(n,\beta)$
$\shortparallel$

Special case of (4.58)

**Theorem (4.63)**

*If $X$ has an Erlang distribution with $X \sim$ Erlang$(n, \beta)$, then:*

(i) $\phi_X(t) = \left(\frac{\beta}{\beta - t}\right)^n$ ✓ for $t < \beta$

(ii) $E(X) = \frac{n}{\beta}$ ✓

(iii) $Var(X) = \frac{n}{\beta^2}$ ✓

(iv) $2\beta \cdot X \sim$ Erlang$(n, 1/2) = \chi^2_{2n}$ $= \Gamma\left(2\frac{n}{2}, \frac{1}{2}\right)$

$\underset{\Gamma(\alpha,\beta)}{b \cdot X} \sim \Gamma(\alpha, \beta/b)$

# Sum of independent random variables $X_i \sim \exp(\lambda)$

**Theorem (4.64)**

*If $X_1, \ldots, X_n$ are independent random variables with $X_i \sim \exp(\lambda)$, then*

$$X = X_1 + \cdots + X_n$$

*has an Erlang distribution with $X \sim$ Erlang$(n, \lambda)$.*

$$X_i \sim \Gamma(1, \lambda) \underset{(4.60)}{\Rightarrow} X_1 + \cdots + X_n \sim \Gamma(n, \lambda)$$

# PDF's of Erlang distributions

**Lemma (4.65)**

*Let $f_X$ be the pdf of a random variable $X \sim$ Erlang$(n, \lambda)$. Then*

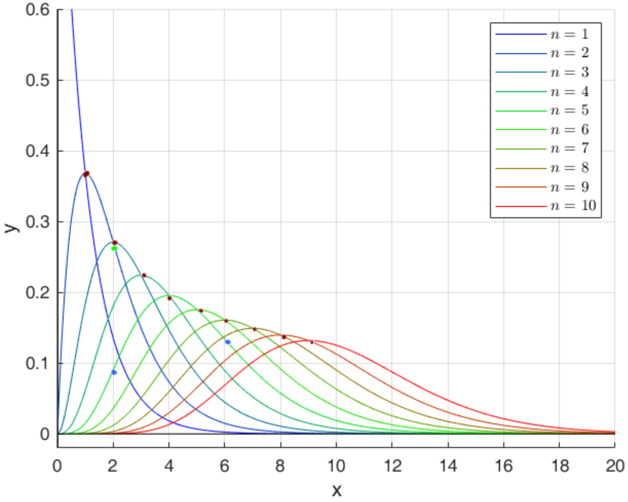
(i) *$f_X$ has a maximum at: $m_n = \dfrac{n-1}{\lambda}$*

(ii) *If $n = 1$, then $f_X$ is concave upward.*

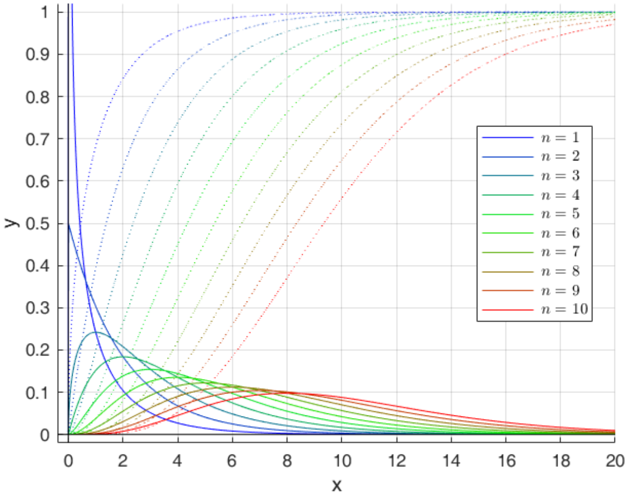
(iii) *If $n = 2$, then $f_X$ is concave downward and then upward, with inflection point at: $\dfrac{2}{\lambda}$*

(iv) *If $n > 2$, then $f_X$ is concave upward, then downward, then upward again with inflection points at: $m_n \pm \dfrac{\sqrt{n-1}}{\lambda}$*

# PDF's of $\mathrm{Erlang}(n, 1) = \Gamma(n, 1)$ distributions for $n = 1, 2, \ldots, 10$



# Pdf's and cdf's of $\chi_n^2 = \Gamma\left(\frac{n}{2}, \frac{1}{2}\right)$ distributions for $n = 1, 2, \ldots, 10$



# CDF's of Erlang distributions

**Theorem (4.66)**

*If $X$ has an Erlang distribution with $X \sim$ Erlang$(n, \lambda)$, then its cdf is given by:*

$$F_X(x) = \begin{cases} 1 - \sum_{k=0}^{n-1} \frac{(\lambda x)^k}{k!} e^{-\lambda x} & \text{for } x \geq 0 \\ 0 & \text{for } x < 0 \end{cases}$$

pf by induction

$n=1$ : $F_X(x) = \int_0^x \lambda \cdot e^{-\lambda \xi} d\xi = -e^{-\lambda \xi} \Big|_0^x = 1 - e^{-\lambda x} = 1 - \sum_{k=0}^{0} \frac{(\lambda x)^0}{0!} e^{-\lambda x}$ ✓

# CDF's of Erlang distributions

induction step: $n \to n+1$

$X \sim \text{Erlang}(n+1, \lambda)$

$$F_X(x) = \frac{\lambda^{n+1}}{n!} \int_0^x \underset{\substack{\uparrow \\ f}}{\xi^n} \cdot \underset{\substack{\uparrow \\ g'}}{e^{-\lambda\xi}} \, d\xi = \frac{\lambda^{n+1}}{n!} \left( \xi^n \cdot \left(-\frac{1}{\lambda} e^{-\lambda\xi}\right) \Big|_0^x + \int_0^x n \cdot \xi^{n-1} \cdot \left(+\frac{1}{\lambda} e^{-\lambda\xi}\right) d\xi \right)$$

$$= \frac{\lambda^{n+1}}{n!} \left( -\frac{1}{\lambda} x^n \cdot e^{-\lambda x} + n \cdot \frac{1}{\lambda} \int_0^x \xi^{n-1} e^{-\lambda\xi} \, d\xi \right)$$

$$= \underline{-\frac{\lambda^n}{n!} x^n \cdot e^{-\lambda x}} + \underbrace{\frac{\lambda^n}{(n-1)!} \int_0^x \xi^{n-1} e^{-\lambda\xi} \, d\xi}_{F_Y(x)} \qquad Y \sim \text{Erlang}(n, \lambda)$$

Ind

$$= 1 - \left( \sum_{k=0}^{n-1} \frac{(\lambda x)^k}{k!} \right) e^{-\lambda x} - \cancel{\frac{(\lambda x)^n}{n!}} \cdot e^{-\lambda x}$$