A Mathematical Basics

Greek Symbols
Decimal Factors
Series
Differential Calculus
Integral Calculus
Trigonometry
Complex Calculus
Matrix Calculus
Linear Differential Equations

Greek Symbols

Small	Large	Pronunciation	Latin
α		Alfa	a
β		Beta	b
γ		Gamma	g
δ	Δ	Delta	d
ε		Epsilon	e
η		Eta	h
ϑ	Θ	Teta	
λ	Λ	Lambda	l
μ		Mü	m
ν		Nü	n
τ		Tau	t
ϕ	Φ	Fi	f
	Ψ	Psi	
ω	Ω	Omega	o
π	Π	Pi	p
ρ		Ro	r
σ	Σ	Sigma	s

Decimal Factors

Value	**Shortcut**	**Pronunciation**
10^{12}, 10^{15}, 10^{18}	T, P, E	Tera, Peta, Exa
10^{3}, 10^{6}, 10^{9}	k, M, G	Kilo, Mega, Giga
10^{1}, 10^{2}	da, h	Deka, Hekto
1		
10^{-1}, 10^{-2}	d, c	Dezi, Centi
10^{-3}, 10^{-6}, 10^{-9}	m, µ, n	Milli, Mikro, Nano
10^{-12}, 10^{-15}, 10^{-18}	p, f, a	Piko, Femto, Atto
10^{-2}	%	Percent

Series

Geometric series

$$\sum_{i=0}^{I-1} x^i = 1 + x + x^2 + \ldots + x^{I-1} = \frac{x^I - 1}{x - 1}$$

$$\sum_{i=0}^{\infty} x^i = 1 + x + x^2 + \ldots = \frac{1}{1-x} \quad \text{if } |x| < 1$$

Miscellaneous

$$\sum_{i=1}^{I} i = 1 + 2 + 3 + \ldots + I = \frac{I \cdot (I-1)}{2}$$

$$\sum_{i=1}^{\infty} \left(\frac{1}{n}\right)^2 = \frac{\pi^2}{6}$$

$$1 + 2\sum_{n=1}^{N} \cos(n \cdot x) = \frac{\sin\left((0.5+N)x\right)}{\sin\frac{x}{2}} \quad \text{for} \quad x \neq k \cdot 2\pi \quad (k \in \mathbb{Z})$$

$$\sum_{n=0}^{N-1} \sin\left[(2n+1)x\right] = \frac{\left(\sin(n \cdot x)\right)^2}{\sin x} \quad \text{for} \quad x \neq k \cdot \pi \quad (k \in \mathbb{Z})$$

Differential Calculus

Given is a function $f(x)$. Its **derivative** is $f'(x)=\dfrac{\mathrm{d}f}{\mathrm{d}x}$.

Geometrical interpretation
The derivative $f'(x)$ is the slope or gradient of the function $f(x)$ at the position x.

Important examples

$f(x)$	$f'(x)$
x	1
x^2	$2x$
x^k	$k \cdot x^{k-1}$
$\ln\lvert x\rvert \quad (x \neq 0)$	$\dfrac{1}{x}$
e^x	e^x
$\mathrm{e}^{a \cdot x}$	$a \cdot \mathrm{e}^{a \cdot x}$
$\sin ax$	$a \cdot \cos ax$
$\cos ax$	$-a \cdot \sin ax$

Basic rules

	$f(x)=\mathrm{const}\cdot u(x)$	$f'(x)=\mathrm{const}\cdot u'(x)$
Product rule	$f(x)=u(x)\cdot v(x)$	$f'(x)=u'(x)\cdot v(x)+u(x)\cdot v'(x)$
Quotient rule	$f(x)=\dfrac{u(x)}{v(x)}$	$f'(x)=\dfrac{u'(x)\cdot v(x)-u(x)\cdot v'(x)}{v^2(x)}$
Chain rule	$f(x)=u(v)$ with $v(x)$	$f'(x)=\dfrac{\mathrm{d}u}{\mathrm{d}v}\cdot\dfrac{\mathrm{d}v}{\mathrm{d}x}$

Integral Calculus

Given is a function $g(x)$. Its derivative is $g'(x) = \frac{\mathrm{d}\,g}{\mathrm{d}\,x} = f(x)$. This can be computed with *differential calculus*. Vice versa a function $f(x)$ is given and a function $g(x)$ with its derivative $f(x)$ shall be determined. This can be computed with *integral calculus*. $g(x)$ is the **antiderivative** of $f(x)$.

Important examples

$f(x)$	$g(x)$
1	x
x	$\frac{x^2}{2}$
$x^k \quad (k \neq -1$ ganz; falls $k < 0$: $x \neq 0)$	$\frac{x^{k+1}}{k+1}$
$\frac{1}{x}$	$\ln \lvert x \rvert \quad (x \neq 0)$
e^x	e^x
$\mathrm{e}^{a \cdot x}$	$\frac{\mathrm{e}^{a \cdot x}}{a}$
$\sin ax$	$-\frac{\cos ax}{a}$
$\cos ax$	$\frac{\sin ax}{a}$

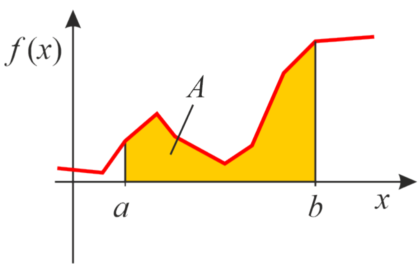
Indefinite integral of the function $f(x)$

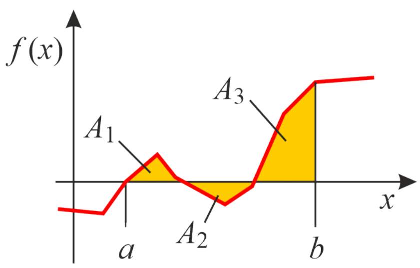
$$\int f(x) \cdot \mathrm{d}\,x = g(x) + \text{const}$$

Definite integral of the function $f(x)$

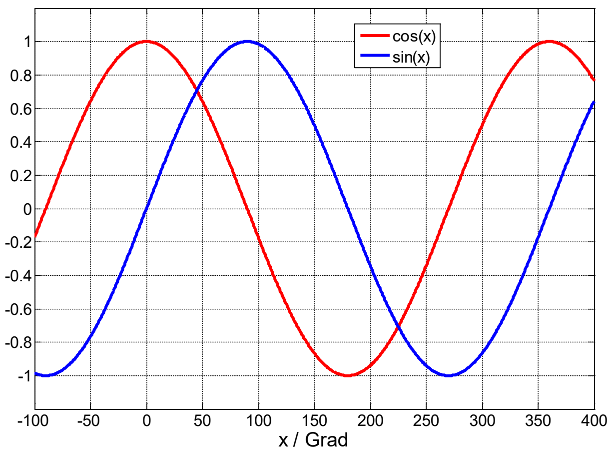
$$A = \int_a^b f(x) \cdot \mathrm{d}\,x = g(x) \Big|_a^b = g(b) - g(a)$$

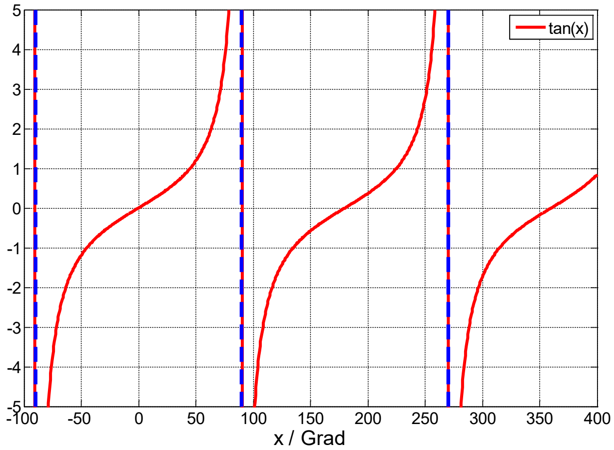
A is the area under the curve $f(x)$ between the lower limit a and the upper limit b. The area A in the right picture is $A = A_1 + A_2 + A_3$ with $A_2 < 0$.





Trigonometry





x	$\sin x$	$\cos x$	$\tan x$
0	0	1	0
$30° = \pi / 6$	$1/2 = 0.5$	$\sqrt{3}/2 = 0.866$	0.577
$45° = \pi / 4$	$1/\sqrt{2} = 0.707$	$1/\sqrt{2} = 0.707$	1
$60° = \pi / 3$	$\sqrt{3}/2 = 0.866$	$1/2 = 0.5$	1.732
$90° = \pi / 2$	1	0	$\to \infty$

$$\sin^2 x + \cos^2 x = 1$$

$$\tan x = \sin x / \cos x$$

$$\sin(x + 90°) = \cos x$$

$$\sin(x - 90°) = -\cos x$$

$$\cos(x + 90°) = -\sin x$$

$$\cos(x - 90°) = \sin x$$

$$\sin^2 x = 0.5 \cdot (1 - \cos 2x)$$

$$\cos^2 x = 0.5 \cdot (1 + \cos 2x)$$

$$\cos(x \pm y) = \cos x \cos y \mp \sin x \sin y$$

$$\sin(x \pm y) = \sin x \cos y \pm \cos x \sin y$$

$$\cos x \cdot \cos y = 0.5 \cdot \cos(x - y) + 0.5 \cdot \cos(x + y)$$

$$\sin x \cdot \sin y = 0.5 \cdot \cos(x - y) - 0.5 \cdot \cos(x + y)$$

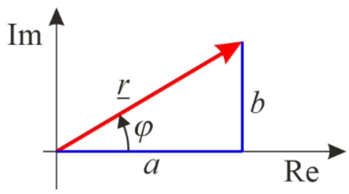
$$\sin x \cdot \cos y = 0.5 \cdot \sin(x - y) + 0.5 \cdot \sin(x + y)$$

Complex Calculus

Imaginary unit

$$\mathrm{j}^2 = -1$$

Complex number $\underline{r}$ written in the **R type**



$$\underline{r} = a + \mathrm{j}b$$

$a = \mathrm{Re}\{\underline{r}\} = r \cdot \cos\varphi$	Real part
$b = \mathrm{Im}\{\underline{r}\} = r \cdot \sin\varphi$	Imaginary part

Please note: The imaginary part b is a real number. jb is an imaginary number.

Complex number $\underline{r}$ written in the **P type (Polar type)**

$$\underline{r} = r \cdot \mathrm{e}^{\mathrm{j}\varphi} = r \underline{/\varphi}$$

$r = \lvert\underline{r}\rvert = \sqrt{a^2 + b^2}$	Amount
$\varphi = \arctan\dfrac{b}{a}$	Angle
$r \underline{/\varphi}$	say: r versor φ

a	b	φ
>0	>0	$0 < \varphi < 90°$
<0	>0	$90° < \varphi < 180°$
<0	<0	$-180° < \varphi < -90°$
>0	<0	$-90° < \varphi < 0°$
>0	0	0
<0	0	$180°$
0	>0	$90°$
0	<0	$-90°$

EULER's equation

$$\mathrm{e}^{\mathrm{j}\varphi} = \cos\varphi + \mathrm{j} \cdot \sin\varphi$$

Sum, difference

$$\underline{r}_1 \pm \underline{r}_2 = (a_1 \pm a_2) + \mathrm{j}\cdot(b_1 \pm b_2)$$

Product, ratio

$$\underline{r}_1 \cdot \underline{r}_2 = r_1 \cdot r_2 \cdot \mathrm{e}^{\mathrm{j}(\varphi_1 + \varphi_2)} = r_1 \cdot r_2 \cdot \underline{/\varphi_1 + \varphi_2}$$

$$\frac{\underline{r}_1}{\underline{r}_2} = \frac{r_1}{r_2} \cdot \mathrm{e}^{\mathrm{j}(\varphi_1 - \varphi_2)} = \frac{r_1}{r_2} \cdot \underline{/\varphi_1 - \varphi_2}$$

Inversion

$$\frac{1}{\underline{r}} = \frac{1}{r} \cdot \mathrm{e}^{-\mathrm{j}\cdot\varphi} = \frac{1}{r} \cdot \underline{/-\varphi}$$

Negative number

$$-\underline{r} = -a - \mathrm{j}\cdot b = r \cdot \mathrm{e}^{\mathrm{j}\cdot(\varphi + 180°)} = r \cdot \underline{/\varphi + 180°}$$

Complex conjugate number

$$\underline{r}^* = a - \mathrm{j}\cdot b = r \cdot \mathrm{e}^{-\mathrm{j}\cdot\varphi} = r \cdot \underline{/-\varphi}$$

Important equations

$$\underline{r} \cdot \underline{r}^* = r^2$$

$$\underline{r} + \underline{r}^* = 2 \cdot a$$

$$\underline{r} - \underline{r}^* = \mathrm{j}\cdot 2 \cdot b$$

$$\underline{r}^2 = r^2 \cdot \mathrm{e}^{\mathrm{j}2\varphi} = r^2 \cdot \underline{/2\varphi}$$

$$\sqrt{\underline{r}} = \begin{cases} \sqrt{r} \cdot \underline{/\varphi/2} \\ \sqrt{r} \cdot \underline{/(\varphi/2) + 180°} \end{cases}$$

Matrix Calculus

Matrix

$$[a]=\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

Sum, difference

$$[a]\pm[b]=\begin{bmatrix} a_{11}\pm b_{11} & a_{12}\pm b_{12} \\ a_{21}\pm b_{21} & a_{22}\pm b_{22} \end{bmatrix}$$

Product of matrix with a factor

$$k\cdot[a]=\begin{bmatrix} k\cdot a_{11} & k\cdot a_{12} \\ k\cdot a_{21} & k\cdot a_{22} \end{bmatrix}$$

Product oft wo matrices

$$[a]\cdot[b]=\begin{bmatrix} a_{11}\cdot b_{11}+a_{12}\cdot b_{21} & a_{11}\cdot b_{12}+a_{12}\cdot b_{22} \\ a_{21}\cdot b_{11}+a_{22}\cdot b_{21} & a_{21}\cdot b_{12}+a_{22}\cdot b_{22} \end{bmatrix}$$

$$\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\cdot\begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}=\begin{bmatrix} \boxed{c_{11}} & \boxed{c_{12}} \\ \boxed{c_{21}} & \boxed{c_{22}} \end{bmatrix}$$

Inversion

$$[a]^{-1}=\frac{1}{\det[a]}\begin{bmatrix} a_{22} & -a_{12} \\ -a_{21} & a_{11} \end{bmatrix} \text{ mit } \det[a]=a_{11}a_{22}-a_{12}a_{21}$$

Linear Differential Equations

Linear differential equation of order *n* with constant coefficients

$$A_n \cdot \frac{\mathrm{d}^n a(t)}{\mathrm{d}t^n} + \ldots + A_2 \cdot \frac{\mathrm{d}^2 a(t)}{\mathrm{d}t^2} + A_1 \cdot \frac{\mathrm{d}a(t)}{\mathrm{d}t} + A_0 \cdot a(t) = e(t)$$

$e(t)$	known time dependent input signal
$a(t)$	unknown time dependent output signal
$A_0 \ldots A_n$	constant coefficients

If $e(t) = 0$, the differential equation is **homogenous**. Otherwise it is **inhomogenous** with $e(t) \neq 0$.

Total solution

$$a(t) = a_\mathrm{H}(t) + a_\mathrm{P}(t)$$

$a_\mathrm{H}(t)$ is the **homogeneous solution** for $e(t) = 0$. The homogeneous solution is also called *eigen solution* or in case of an oscillation *eigen oscillation*. A *stable system* requires $a_\mathrm{H}(t) \to 0$ for $t \to \infty$.

$a_\mathrm{P}(t)$ is the **particular solution** for $e(t) \neq 0$. This is the solution fort the **steady state** regime. A *stable system* requires $a(t) \to a_\mathrm{P}(t)$ for $t \to \infty$.

The homogenous differential equation is always solved with the following approach:

$$a_\mathrm{H}(t) = K \cdot \mathrm{e}^{\lambda \cdot t}$$

K and λ are cofficients which need to be defined.

1. Computation of λ

$a_\mathrm{H}(t)$ is inserted into the homogenous differential equation. One obtains the *characteristic equation*

$$A_n \cdot \lambda^n + \ldots A_2 \cdot \lambda^2 + A_1 \cdot \lambda + A_0 = 0\,.$$

The characteristic equation has n roots λ_1, λ_2, ... λ_n as solution. Due to linearity superposition of the individual solutions is possible:

$$a_\mathrm{H}(t) = K_1 \cdot \mathrm{e}^{\lambda_1 \cdot t} + K_2 \cdot \mathrm{e}^{\lambda_2 \cdot t} + \ldots + K_n \cdot \mathrm{e}^{\lambda_n \cdot t}\,.$$

If two roots are identical, a seperate determination is necessary.

2. Computation of K_1, K_2, ... K_n

The coefficients K_1, K_2, ... K_n are determined with respect to given constraints for the total solution with $e(t) \neq 0$. I. e. steady constraints for memory elements need to be considered.