2-1

Mittwoch, 5. Oktober 2016 13:55

Problem 1 Check, if the signals

a) $x(t) = e^{-2t} \cdot u(t)$; b) $x(t) = e^{j(2t+\pi/4)}$; c) $x(t) = \cos(t)$

are energy or power signals. $u(t)$ is the unit step function. If possible, determine the energy E_∞ and the average power P_∞.

a)
$$E_\infty = \int_0^\infty e^{-4t}\, dt < \infty \qquad P_\infty = \lim_{T\to\infty} \frac{1}{T} \int_0^{T/2} e^{-4t}\, dt$$

$$\Leftrightarrow -\frac{1}{4} e^{-4t}\Big|_0^\infty < \infty \qquad \Leftrightarrow \lim_{T\to\infty} \frac{1}{T}\left(-\frac{1}{4} e^{-2T} - 1\right) = 0 \checkmark$$

$$\Leftrightarrow \frac{1}{4} < \infty \checkmark$$

Energy

b)
$$E_\infty = \int_{-\infty}^{\infty} e^{j(2t+\pi/4)} \cdot e^{-j(2t+\pi/4)}\, dt < \infty \qquad P_\infty = \lim_{T\to\infty} \frac{1}{T}\left(t \Big|_{-T/2}^{T/2} \right) < \infty$$

$$= \int_{-\infty}^{\infty} \cancel{e^{j2t + j\pi/4 - j2t - j\pi/4}}\, 1\, dt < \infty \qquad = \lim_{T\to\infty} \frac{1}{T}(T) = 1 < \infty$$

$$= t\Big|_{-\infty}^{\infty} < \infty = \infty - (-\infty) = \infty < \infty$$

Power

Problem 2 Which of the following signals is periodic? $u(t)$ is the unit step function.

a) $x(t) = 2 \cdot e^{j(t+\pi/4)} \cdot u(t)$; b) $x(t) = j \cdot e^{j10t}$; c) $x(t) = e^{(-1+j)t}$

If a signal is periodic, specify the period T.

a) none-periodic $\rightarrow x(t) = 0 \quad t < 0$

b) $x(t) = j \cdot e^{j10t} = j\cos(10t) - \sin(10t) = j\cos\left(2\pi \cdot \frac{10}{2\pi} t\right) - \sin\left(2\pi \cdot \frac{10}{2\pi} t\right)$

$f = \frac{10}{2\pi} \qquad T = \frac{2\pi}{10}$

c) $x(t) = e^{(-1+j)t} = e^{-t+jt} = e^{-t} \cdot e^{jt} \quad \lim_{t\to\infty} e^{-t} \Rightarrow 0$ non periodic

Problem 3 Express the real part of each of the following signals in the form

$A \cdot e^{-at} \cdot \cos(\omega t + \phi)$,

where A, a, ω, and ϕ are real numbers with $A > 0$ and $-\pi < \phi \le \pi$:

a) $x(t) = -2$ b) $x(t) = \sqrt{2} \cdot e^{j\pi/4} \cos(3t + 2\pi)$

c) $x(t) = e^{-t} \cdot \sin(3t + \pi)$ d) $x(t) = j \cdot e^{(-2+j100)t}$

a) $A = 2$, $a = 0$, $\omega = 0$, $\phi = \pi$

b) $e^{j\frac{\pi}{4}} = \cos\left(\frac{\pi}{4}\right) + j\sin\left(\frac{\pi}{4}\right) \rightarrow x(t) = \sqrt{2}\cos\left(\frac{\pi}{4}\right) \cdot \cos(3t+2\pi) + j\sin\left(\frac{\pi}{4}\right)\cos(3t+2\pi)$

$= \cancel{\sqrt{2}} \cdot \frac{\cancel{\sqrt{2}}}{\cancel{2}\,1} \cdot \cos(3t+2\pi) + j\cos(3t+2\pi)$

$= \cos(3t+2\pi)$ .....

$A = 1$, $a = 1$, $\omega = 3$, $\phi = 0$

c) $x(t) = e^{-t} \cdot \cos(3t + \pi - \frac{1}{2}\pi)$

$= e^{-t} \cos(3t + \frac{\pi}{2})$

$A = 1, \quad a = 1, \quad \omega = 3, \quad \phi = \frac{\pi}{2}$

d) $x(t) = j \cdot e^{-2t} \cdot e^{j100t} = j\, e^{-2t} \cdot \cos(100t) - e^{-2t} \sin(100t)$

$= \ldots\ldots + e^{-2t} \sin(100t - \frac{\pi}{2} + \pi)$

$+ e^{-2t} \cos(100t + \frac{\pi}{2})$

$A = 1, \quad a = -2, \quad \omega = 100, \quad \phi = \frac{\pi}{2}$

Problem 4 Compute the convolution of the signals:

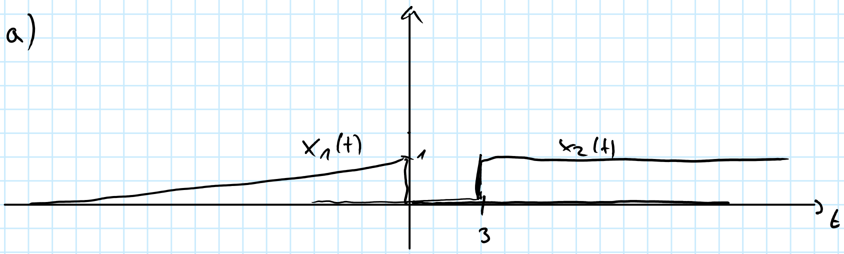
$$y(t) = \int_{-\infty}^{\infty} x_1(\tau) \cdot x_2(t-\tau)\, d\tau = x_1(t) * x_2(t)$$

a) $x_1(t) = e^{2t} \cdot u(-t)$; $x_2(t) = u(t-3)$

b) $x_1(t) = \begin{cases} 1+t \ ; & 0 \le t \le 1 \\ 2-t \ ; & 1 < t \le 2 \\ 0 \ ; & \text{elsewhere} \end{cases}$; $x_2(t) = \delta(t+2) + 2 \cdot \delta(t+1)$

c) $x_1(t) = u(t-3) - u(t-5)$; $x_2(t) = e^{-3t} \cdot u(t)$ d) $x(t) * u(t)$

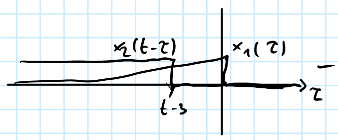
a)



$$y(t) = \int_{-\infty}^{\infty} e^{2\tau} \cdot u(-\tau) \cdot u(t-\tau-3)\, d\tau$$

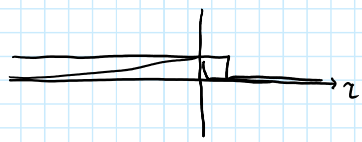
$$= \int_{-\infty}^{\infty} e^{2\tau} \cdot u(-\tau) \cdot u(-\tau + t - 3)\, d\tau$$

$$= \int_{-\infty}^{t-3} e^{2\tau}\, d\tau = \frac{1}{2} e^{2\tau} \Big|_{-\infty}^{t-3} = \frac{1}{2} e^{2t-6}$$



$-\infty < t \le 3$

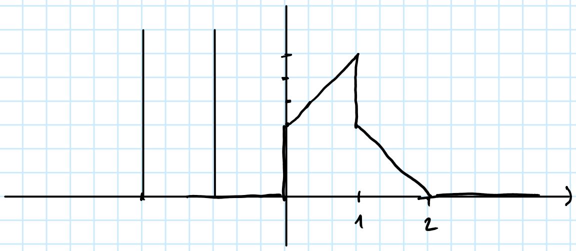
$$= \int_{-\infty}^{0} e^{2\tau}\, d\tau = \frac{1}{2} e^{2\tau} \Big|_{-\infty}^{0} = \frac{1}{2}$$



$3 < t < \infty$

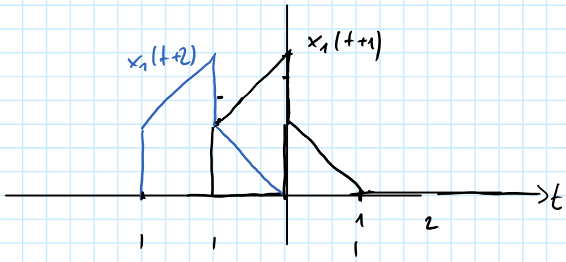
b)

b)



$$x_1(t) * \delta(t+2) + 2 \cdot x_1(t) * \delta(t+1)$$

$$x_1(t+2) + 2 \cdot x_1(t+1)$$



1) $-2 < t < 1$: $\quad y(t) = 1 + (t+2) = 3 + t$

2) $-1 < t < 0$: $\quad y(t) = 2 - (t+2) + 2 \cdot [1 + (t+1)]$

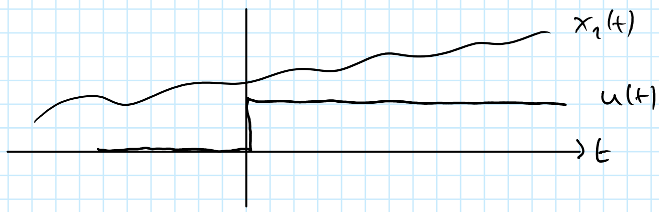
$$y(t) = -t + 4 + 2t = 4 + t$$

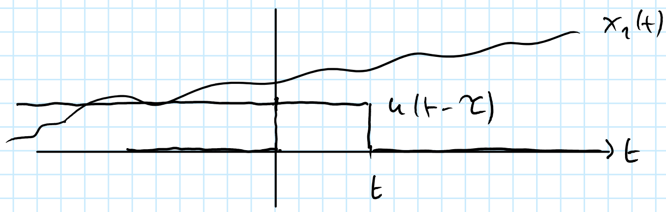
3) $0 < t < 1$: $\quad y(t) = 2 \cdot (2 - (t+1)) = 2 - 2t$

4) $y(t) = 0$: $\quad$ elsewere

d)

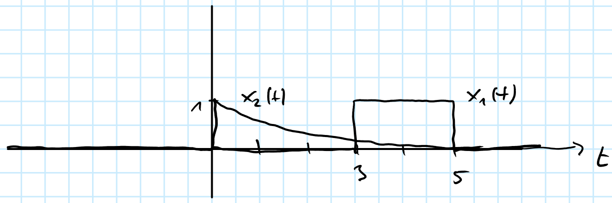
$$x(t) * u(t) = \int_{-\infty}^{+\infty} x(t) \cdot u(t-\tau) \, d\tau$$





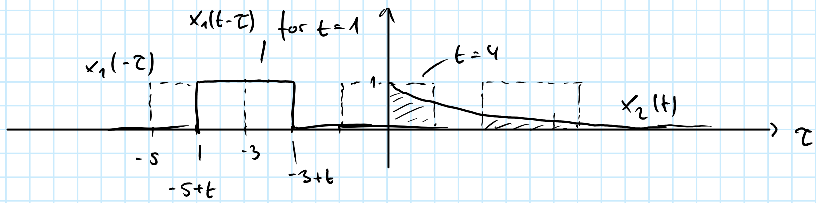
$$= \int_{-\infty}^{t} x(\tau) \, d\tau$$

c) $\quad$ c) $x_1(t) = u(t-3) - u(t-5)$; $\quad x_2(t) = e^{-3t} \cdot u(t)$



$$x_1(t) * x_2(t) = x_2(t) * x_1(t) = y(t)$$

$$y(t) = \int^{\infty} x_2(\tau) \cdot x_1(t-\tau)\, d\tau$$



1) $t \leq 3$: $y(t) = 0$

2) $3 \leq t \leq 5$: $y(t) = \int_0^{-3+t} x_2(\tau) \cdot d\tau$

$$y(t) = \int_0^{-3+t} e^{-3\tau}\, d\tau = \frac{1}{-3} e^{-3\tau} \Big|_0^{-3+t}$$

$$= -\frac{1}{3}\left(e^{-3(-3+t)} - 1\right) = -\frac{1}{3}\left(e^{3-3t} - 1\right)$$

3) $t \geq 5$: $y(t) = \int_{-5+t}^{-3+t} e^{-3\tau}\, d\tau = \dots$

Problem 5 Derive the following FOURIER transform properties:

a) Time and frequency shifting b) Scaling

c) Differentiation and integration d) Convolution and product

Scaling	$x(at)$	$\frac{1}{\lvert a\rvert} \cdot X\left(\frac{f}{a}\right)$

b)

$x(at) \circ\!\!-\!\!\bullet\ ?$ a: real

$$X(f) = \int_{-\infty}^{\infty} x(a \cdot t)\, e^{-j2\pi f t}\, dt$$

$u = a \cdot t$; $du = a \cdot dt$

$t \to \infty$: $u \to \infty$ for $a > 0$

$\to -\infty$ $a < 0$

$t \to -\infty$: $u \to -\infty$: $a > 0$

$\to +\infty$: $a < 0$

1) $a > 0$

$$X(f) = \frac{1}{a} \cdot \int^{\infty} x(u)\, e^{-j2\pi f \frac{u}{a}}\, du$$

1) $a>0$

$$X(f) = \frac{1}{a} \cdot \int_{-\infty}^{\infty} x(u)\, e^{-j2\pi f \frac{u}{a}}\, du$$

$$= \frac{1}{a} \cdot X\left(\frac{f}{a}\right)$$

2) $a<0$: $-\infty$

$$X(f) = \frac{1}{a} \int_{\infty}^{-\infty} x(u)\, e^{-j2\pi f \frac{u}{a}}\, du = -\frac{1}{a} \cdot X\left(\frac{f}{a}\right)$$

c) $\frac{dx(t)}{dt}$ o—● ?

$$= \frac{d}{dt}\left\{ \int_{-\infty}^{+\infty} X(f) \cdot e^{j2\pi f t}\, df \right\}$$

$$= \int_{-\infty}^{+\infty} \underbrace{X(f)\,(j2\pi f)}_{\bullet\!-\!\circ\ \frac{dx(t)}{dt}} \cdot e^{j2\pi f t}\, df$$

d)

Time convolution	$x(t) * y(t)$	$X(f) \cdot Y(f)\ ;\ X(j\omega) \cdot Y(j\omega)$

$x(t) * y(t)$ o—● ?

$$= \int_{-\infty}^{+\infty} (x * y)\, e^{-j2\pi f t}\, dt$$

$$= \int_{-\infty}^{+\infty} \int_{-\infty}^{+\infty} x(\tau) \cdot y(t-\tau)\, d\tau \cdot e^{-j2\pi f t}\, dt$$

$$= \int_{-\infty}^{\infty} x(t) \cdot \underbrace{\int_{-\infty}^{\infty} y(t-\tau)\, e^{-j2\pi f t}\, dt}_{e^{-j2\pi f \cdot \tau} \cdot Y(f)} \cdot d\tau$$

$$= Y(f) \cdot \underbrace{\int_{-\infty}^{\infty} x(\tau) \cdot e^{-j2\pi f \tau}\, d\tau}_{X(f)}$$

a) $x(t-t_0)$ o—● ?

$$X(\mathrm{j}\omega)=\int_{-\infty}^{\infty} x(t)\cdot \mathrm{e}^{-\mathrm{j}\omega t}\,\mathrm{d}t$$

$t-t_0$

$$\tau = t - t_0 \qquad = \qquad X(j\omega)=\int_{-\infty}^{\infty} x(\tau)\cdot e^{-j\omega(\tau+t_0)}\,d\tau$$

$$= e^{-j\omega t_0}\underbrace{\int_{-\infty}^{\infty} x(\tau)\, e^{-j\omega \tau}\,d\tau}_{X(j\omega)}$$

Time shifting	$x(t-t_0)$	$\mathrm{e}^{-\mathrm{j}2\pi f t_0}\cdot X(f)$

Problem 6 Compute the FOURIER transforms of the signals:

a) $x(t)=\begin{cases}\cos(\omega_0 t)\,; & t>0\\ 0 & ; t<0\end{cases}$ b) $x(t)=\begin{cases}\sin(\omega_0 t)\,; & t>0\\ 0 & ; t<0\end{cases}$ c) $x(t)=\begin{cases}1-t\,; & 0<t<1\\ 1+t\,; & -1<t<0\\ 0\,; & \text{elsewhere}\end{cases}$

d) $x(t)=1+\cos\left(6\pi t+\dfrac{\pi}{8}\right)$ e) $x(t)=t\cdot\left(\dfrac{\sin t}{\pi t}\right)^2$

e) $x(t)=t\cdot\left(\dfrac{\sin t}{\pi t}\right)^2 \circ\!-\!\bullet\; X(j\omega)=?$

$$=\frac{1}{\pi^2}\, t\, \underbrace{\left(\frac{\sin t}{t}\right)^2}_{y(t)} \circ\!-\!\bullet\; \frac{1}{\pi^2}\cdot j\,\frac{dY}{d\omega}$$

$$-j2\pi t\cdot x(t) \circ\!-\!\bullet\; \frac{dX}{df}=2\pi\cdot\frac{dX}{d\omega}$$

$$\cancel{-}j\cdot t\cdot x(t) \circ\!-\!\bullet\; \frac{dX}{d\omega}\cdot\frac{1}{-j}$$

$$\boxed{t\cdot x(t) \circ\!-\!\bullet\; j\cdot\frac{dX}{d\omega}}$$

$$y(t)=\left(\frac{\sin(t)}{t}\right)^2$$

$$\boxed{B\cdot\frac{\sin(\pi B t)}{\pi B t} \circ\!-\!\bullet\; \mathrm{rect}\left(\frac{\omega}{2\pi B}\right)}$$

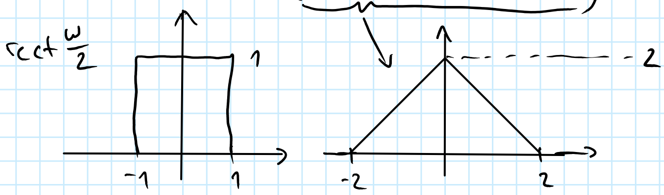
$$\frac{\sin t}{t}=\pi\left\{\frac{1}{\pi}\,\frac{\sin t}{t}\right\} \circ\!-\!\bullet\; \pi\cdot\mathrm{rect}\,\frac{\omega}{2\pi\frac{1}{\pi}}=\pi\cdot\mathrm{rect}\,\frac{\omega}{2}$$

$$B=\frac{1}{\pi}$$

$$\frac{\sin t}{t}\cdot\frac{\sin t}{t} \circ\!-\!\bullet\; \frac{1}{2\pi}\left\{\pi\cdot\mathrm{rect}\,\frac{\omega}{2} * \pi\,\mathrm{rect}\,\frac{\omega}{2}\right\}$$

$$\circ\!-\!\bullet\; \frac{\pi^{\cancel{2}}}{2\cancel{\pi}}\cdot\left\{\mathrm{rect}\,\frac{\omega}{2} * \mathrm{rect}\,\frac{\omega}{2}\right\}$$

$$\circ\!\!-\!\!\bullet \quad \frac{\pi^{\cancel{2}}}{2\cancel{\pi}} \cdot \Big\{ \underbrace{\operatorname{rect}\frac{\omega}{2} * \operatorname{rect}\frac{\omega}{2}} \Big\}$$



$$Y(\omega) = \frac{\pi}{2}\left\{ \operatorname{rect}\frac{\omega}{2} * \operatorname{rect}\frac{\omega}{2} \right\}$$

$$X(j\omega) = \begin{cases} j/\pi^{\cancel{2}} \cdot 1 \cdot \frac{\cancel{\pi}}{2} & \text{for } -2 \le \omega \le 0 \\ j/\pi^{\cancel{2}} \cdot (-1)\frac{\cancel{\pi}}{2} & 0 \le \omega \le 2 \\ 0 & \text{elsewhere} \end{cases}$$

d) $x(t) = 1 + \cos\left(6\pi t + \frac{\pi}{8}\right)$

$$= 1 + \cos\Big[\underbrace{6\pi}_{\omega_0}\underbrace{\Big(t + \frac{\pi}{\underset{48}{\underbrace{8\cdot 6\cdot \pi}}}\Big)}_{t+t_0}\Big] \qquad t_0 = \frac{1}{48}$$

$$\circ\!\!-\!\!\bullet$$

$$X(j\omega) = 2\pi\cdot\delta(\omega) + e^{+j\frac{2\pi f}{\omega}\frac{1}{48}} \cdot \pi\left\{\delta(\omega + 6\pi) + \delta(\omega - 6\pi)\right\}$$

$$= 2\pi\,\delta(\omega)\cdot\pi\cdot\left\{e^{-j6\pi/48}\cdot\delta(\omega+6\pi) + e^{j6\pi/48}\cdot\delta(\omega-6\pi)\right\}$$

a) $x(t) = \begin{cases} \cos(\omega_0 t)\,; & t > 0 \\ 0 & ; \; t < 0 \end{cases}$

$$x(t) = \cos(\omega_0 t)\cdot u(t)$$

Time multiplication	$x(t)\cdot y(t)$	$X(f) * Y(f)\,;\; \frac{1}{2\pi}(X(j\omega) * Y(j\omega))$

$$x(t) \;\circ\!\!-\!\!\bullet\; X(j\omega) = \frac{1}{2\pi}\left(\pi\cdot[\delta(\omega+\omega_0)+\delta(\omega-\omega_0)] * \frac{1}{j\omega} + \pi\cdot\delta(\omega)\right)$$

$$= \frac{1}{2}\delta(\omega+\omega_0) * \frac{1}{j\omega} + \frac{1}{2}\delta(\omega+\omega_0) * \pi\,\delta(\omega) +$$

$$\frac{1}{2}\delta(\omega-\omega_0) * \frac{1}{j\omega} + \frac{1}{2}\delta(\omega-\omega_0) * \pi\delta(\omega)$$

$$= \frac{1}{2j(\omega+\omega_0)} + \frac{1}{2j(\omega-\omega_0)} + \frac{\pi}{2}\delta(\omega+\omega_0) + \frac{\pi}{2}\delta(\omega-\omega_0)$$

b) $x(t) = \begin{cases} \sin(\omega_0 t)\,; & t > 0 \\ 0 & ; \; t < 0 \end{cases}$

$$x(t) \;\circ\!\!-\!\!\bullet\; X(j\omega) = \frac{1}{2\pi}\left(j\pi\cdot[\delta(\omega+\omega_0)-\delta(\omega-\omega_0)] * \frac{1}{j\omega} + \pi\cdot\delta(\omega)\right)$$

$$= \frac{j}{2}\delta(\omega+\omega_0) * \frac{1}{j\omega} + \frac{j}{2}\delta(\omega+\omega_0) * \pi\,\delta(\omega) -$$

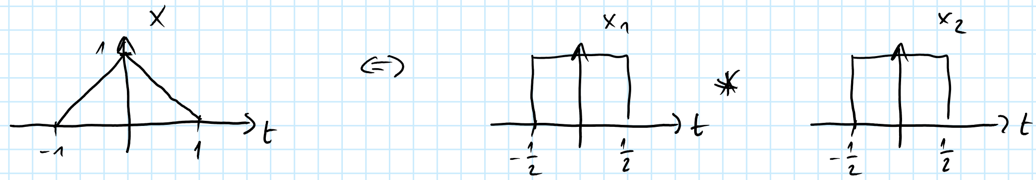
$$\frac{\cancel{j}}{2}\,\delta(\omega-\omega_0) * \frac{1}{j\omega} - \frac{j}{2}\,\delta(\omega-\omega_0) * \pi\,\delta(\omega)$$

$$= \frac{1}{2(\omega+\omega_0)} + \frac{\pi j}{2j(\omega+\omega_0)} - \frac{1}{2(\omega-\omega_0)} - \frac{\pi j}{2j(\omega-\omega_0)}$$

$$= \frac{j\pi}{2}\left(\frac{1}{j(\omega+\omega_0)} - \frac{1}{j(\omega-\omega_0)}\right) + \frac{\cancel{1}(\omega-\omega_0) - \cancel{1}(\omega+\omega_0)}{2\cancel{1}(\omega+\omega_0)(\omega-\omega_0)}$$

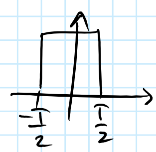
$$= \frac{j\pi}{2}\left(\frac{1}{j(\omega+\omega_0)} - \frac{1}{j(\omega-\omega_0)}\right) - \frac{\cancel{2}\omega_0}{\cancel{2}(\omega^2-\omega_0^2)}$$

c) $x(t) = \begin{cases} 1-t\,; & 0<t<1 \\ 1+t\,; & -1<t<0 \\ 0\,; & \text{elsewhere} \end{cases}$



$x_1(t) * x_2(t) \quad ○\!\!-\!\!\!-\!\!● \quad X_1(j\omega)\cdot X_2(j\omega)$ $\qquad \mathrm{rect}\left(\frac{t}{T}\right)$

$\mathrm{rect}(t) * \mathrm{rect}(t) \quad ○\!\!-\!\!\!-\!\!● \quad \left(\cancel{1}\cdot\frac{\sin(\omega T/2)}{\omega T/2}\right)^2$



Problem 7 The medium duration D and the medium bandwidth B of a signal are given by

$$D = \frac{1}{x_{\max}}\int_{-\infty}^{\infty} x(t)\,\mathrm{d}t\;, \quad B = \frac{1}{X_{\max}}\int_{-\infty}^{\infty} X(\omega)\,\mathrm{d}\omega$$

Calculate the product $D{\cdot}B$ for a rectangular impulse of width T. *Aid:* $\int_0^{\infty} \mathrm{si}(ax)\cdot\mathrm{d}x = \frac{\pi}{2a}$

$D = T$

$B = \;\ldots\ldots = \frac{2\pi}{T}$

$x(t) = \mathrm{rect}\left(\frac{t}{T}\right) \quad ○\!\!-\!\!\!-\!\!● \quad X(f) = T\,\underbrace{\frac{\sin(\pi f T)}{\pi f T}}_{\mathrm{si}(\pi f T)}$

$X(j\omega) = T\cdot \mathrm{si}\left(\frac{\omega T}{2}\right)$ si gerade (da $\int_0^{\infty}$, muss $\times 2$)

$B = \frac{1}{\cancel{T}}\,2\cdot\int_0^{\infty} \cancel{T}\,\mathrm{si}\left(\frac{\omega T}{2}\right)\mathrm{d}\omega$ $\quad a$

$= \cancel{2}\cdot\frac{\pi\cdot 2}{\cancel{2}T} = \frac{2\pi}{T}$

3-1

Freitag, 14. Oktober 2016 08:12

Problem 1 Proof, if the following systems are linear, time invariant, causal, and memoryless.

a) $y(t) = \cos[x(t-1)]$; x, t real b) $y(t) = a \cdot x(t) - b \cdot \frac{\mathrm{d}\,y(t)}{\mathrm{d}\,t}$; a, b, x, t real

c) $y(t) = a \cdot t^2 + x(t+3)$; a, x, t real

a) linear : $k \cdot x(t-1) \rightarrow \cos(k \cdot x(t-1)) \neq k \cdot \cos(x(t-1))$ $k \in \mathbb{R}$

$\Rightarrow$ non-linear

time invariant: $y(t) = \cos(x(t-1)) \rightarrow y(t-t_0) = \cos(x(t-t_0-1))$

$\Rightarrow$ time-invariant

causal : yes, because the output response doesn't appear before the input appears

memoryless : no, because $y(t)$ output depends on the history.

b) linear : $b \cdot \dot{y}(t) + y(t) = a \cdot x(t)$

$k \cdot x(t) \rightarrow a \cdot k \cdot x(t) = k \cdot a \cdot x(t)$ $k \in \mathbb{R}$

$\Rightarrow$ linear

time invariant :

$y(t-t_0) = a \cdot x(t-t_0) - b \cdot \dot{y}(t-t_0)$

$\rightarrow$ time invariant

causal : : yes, the output depends on the input and itself.

memoryless : : No, because of the differentation (Ableitungen $\rightarrow$ Energiespeicher)

c) linear : $k \cdot x(t+3) \rightarrow a \cdot t^2 + k \cdot x(t+3) \neq k(a \cdot t^2 + x(t+3))$ $k \in \mathbb{R}$

$\Rightarrow$ non-linear

time invariant: $a \cdot (t-t_0)^2 + x(t-t_0+3) \neq a \cdot t^2 + x(t-t_0+3)$

$\Rightarrow$ not time invariant

causal : not, because it depends on t^2 and not just on $x(t+3)$

memoryless : not, because $x(t)$ is a time shifted function,

$t+3 = t'$ $t = t'-3$

$y(t'-3) = a \cdot (t'-3)^2 + x(t')$

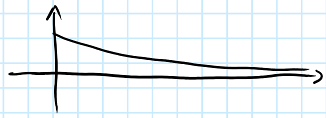
Problem 2 Which of the given unit impulse responses characterizes a stable LTI system?

a) $h(t) = \cos(t) \cdot u(t)$ b) $h(t) = e^{at} \cdot u(t)$; a real

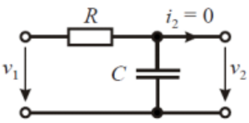
a) $\int_{-\infty}^{\infty} |h(t)|\, dt = \int_{0}^{\infty} |\cos(t)|\, dt < \infty$? $u(t) \begin{cases} 1 & t \geq 0 \\ 0 & t < 0 \end{cases}$

$\int_{0}^{\infty} |\cos(t)|\, dt = \infty$ $\underline{\underline{\text{not stable}}}$

b) $\int_{0}^{\infty} |e^{at}|\, dt$ stable for $a < 0$ $\longrightarrow$



Problem 3 The given *RC* circuit shall be analyzed by solving the system's differential equation under the initial rest condition. The capacitor requires a steady voltage v_2.



a) Compute $y(t) = v_2(t)$ for $x(t) = v_1(t) = \frac{1\,\text{Vs}}{T} \cdot \text{rect}\left(\frac{t - T/2}{T}\right)$

b) Compute $x(t) = v_1(t)$ and $y(t) = v_2(t)$ for $T \to 0$ and interpret the result.

a) $v_1 = i_1 \cdot R + v_2$; $i_1 = C \cdot \frac{dv_2}{dt}$ initial rest: $v_2(t) = 0$ for $t < 0$

$\underbrace{v_2}_{y} + RC \cdot \underbrace{\frac{dv_2}{dt}}_{y'} = \underbrace{v_1}_{x}$

$v_2 = v_{2h} + v_{2p}$

$v_{2h} = A \cdot e^{pt}$; $v_1 = 0$

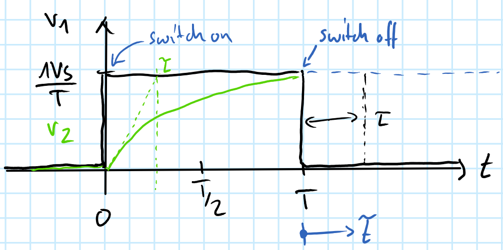
$\cancel{A} \cdot \cancel{e^{pt}}_{1} + RC \cdot p \cdot \cancel{A} \cdot \cancel{e^{pt}} = 0$

$\boxed{1 + RC \cdot p = 0}$ characteristic system equation

root : $p = -\frac{1}{RC} = -\frac{1}{\tau}$ τ: time constant

$\boxed{v_{2h} = \underbrace{A}_{?} \cdot e^{-t/\tau}}$

$x(t) = v_1(t) = \frac{1\,\text{Vs}}{T}\, \text{rect}\left(\frac{t - T/2}{T}\right)$



$t > 0$: $v_2 = A \cdot e^{-t/\tau} + v_{2p}$ $v_{2h} = A \cdot e^{-t/\tau}$; $v_{2h} \to 0$ for $t \to \infty$

stable system

$t \to \infty$: $v_2 = v_{2p} = 1 \frac{\text{Vs}}{T}$ $v_{2p} = v_2 \big|_{t \to \infty} = 1 \frac{\text{Vs}}{T}$

$v_2 = A \cdot e^{-t/\tau} + \frac{1\text{Vs}}{T}$

auxiliary condition :

$v_2(t<0) = 0$ initial rest

$\llcorner$ state variable

steady voltage v_2 around 0

at $t=0$: $\boxed{v_2(0-0) = 0 = v_2(0+0)}$

$$0 = A\cdot e^0 + \frac{1Vs}{T} \Leftrightarrow A = -\frac{1Vs}{T}$$

$$\Rightarrow \boxed{v_2 = \frac{1Vs}{T}\left(1-e^{-t/\tau}\right)} \quad 0<t<T$$

$$v_2(T) = \frac{1Vs}{T}\left(1-e^{-T/\tau}\right) = B$$

$\underline{t \geq T}$ $\quad v_2 = A\cdot e^{-\tilde{t}/\tau} + v_{2p} \qquad \tilde{t} = t - T$

$\tilde{t} \to \infty$: $\quad v_2 = v_{2p} = 0$

aux. cond:

$$v_2(\tilde{t}-0) = B = v_2(\tilde{t}+0) = A\cdot e^0 + 0 = A = B$$

$$v_2 = B\cdot e^{-\tilde{t}/\tau} = \frac{1Vs}{T}\left(1-e^{-T/\tau}\right)\left(e^{-\frac{t-T}{\tau}}\right) \qquad e^{-\frac{t}{\tau}}\cdot e^{+\frac{T}{\tau}}$$

$$v_2 = \frac{1Vs}{T}\left(e^{T/\tau}-1\right)\cdot e^{-t/\tau}$$

$$v_2(T-0) = 1\frac{Vs}{T}\left(1-e^{-T/\tau}\right)$$

$$v_{2h} = B\cdot e^{-\tilde{t}/\tau}$$

$$v_{2p} = 0$$

$$\Rightarrow v_2 = B\cdot e^{-\tilde{t}/\tau}$$

$$v_2(\tilde{t}=0-0) = v_2(\tilde{t}=0+0)$$

$$1\frac{Vs}{T}\left(1-e^{-T/\tau}\right) = B$$

$$v_2 = 1\frac{Vs}{T}\left(1-e^{-T/\tau}\right)\cdot \underbrace{e^{-\tilde{t}/\tau}}_{e^{-t/\tau}\cdot e^{+T/\tau}}$$

$$v_2 = 1\frac{Vs}{T}\left(e^{T/\tau}-1\right)\cdot e^{-t/\tau}$$

b) $\quad T \to 0$

$0 \leq t \leq T$: $\quad t \ll \tau$

$$e^{-t/\tau} \approx 1 - \frac{t}{\tau}$$

$$v_2(t) \approx \frac{1Vs}{T}\left(\cancel{1} - \cancel{1} + \frac{t}{\tau}\right)$$

$$= \frac{1Vs}{T}\cdot\frac{t}{\tau}$$

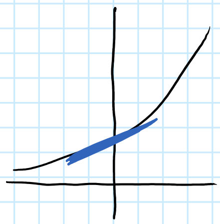
$$v_{2,max} = v_2(T) = \frac{1Vs}{\cancel{T}}\cdot\frac{\cancel{T}}{\tau} = \frac{1Vs}{\tau}$$

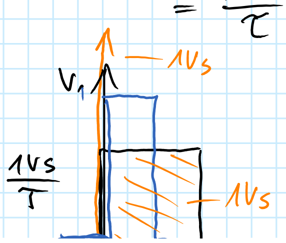
$t \geq T$:

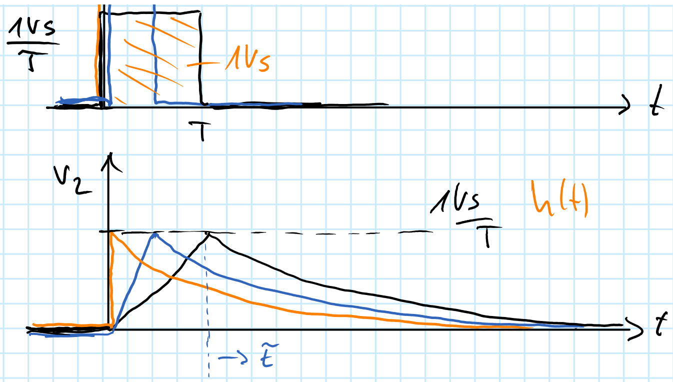
$$v_2(t) \approx \frac{1Vs}{T}\left(\cancel{1}+\frac{T}{\tau}-\cancel{1}\right)\cdot e^{-\tilde{t}/\tau}$$

$$= \frac{1Vs}{\tau}\, e^{-\tilde{t}/\tau}$$

$e^z \approx 1 + z$; $|z| \ll 1$



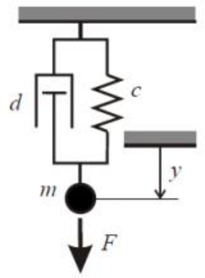




Problem 4 A mechanical system shall be analyzed. Fixed at a ceiling is a spring (c), a damper (d) and a mass (m). An external force F stimulates the system. The spring is characterized by the relation $F_c = c{\cdot}y$. The distance y is measured from the relaxed position of the spring. A spring is able to store energy according to $E_c = c{\cdot}y^2/2$. The damper is characterized by $F_d = d{\cdot}v$, where v is the velocity of the moving part. The damper dissipates energy.

a) Derive a differential equation between the stimulation force F and the mass position y. Is the system linear?

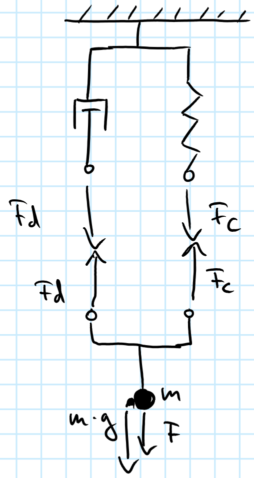
b) Derive a state-space representation.



a) force balance

$$F_d + m a + F_c = m \cdot g + F$$

$$m \cdot \frac{dy^2}{dt^2} + d\frac{dy}{dt} + c \cdot y = \underbrace{m \cdot g}_{\text{source}} + \underset{x}{\underset{\|}{F}}$$



system: non-linear

$F >> m \cdot g$: $\approx$ linear

linear, if $(m \cdot g + F) = x$

b) state variables:

1) y for the spring

kinetic energy: $E_{kin} = \frac{1}{2} m \cdot v^2$

2) $\frac{dy}{dt}$

Problem 5 Let $x(t)$ be an input signal whose FOURIER transform is

$$X(j\omega) = \delta(\omega) + \delta(\omega - \pi) + \delta(\omega - 5)$$

and let $h(t) = u(t) - u(t-2)$

be the unit impulse response of a system.

a) Is $x(t)$ periodic?

b) Compute the output signal $y(t)$. Is $y(t)$ periodic?

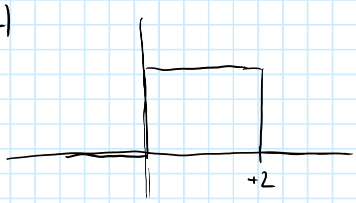
a) $X(j\omega) \circ\!\!-\!\!\bullet\, x(t) = \frac{1}{2\pi} + e^{j\pi t}\frac{1}{2\pi} + \frac{1}{2\pi} e^{j5t}$ (Appendix B Korresp)

→ not periodic because $n \cdot 5 \neq m \cdot \pi$ $n, m \in \mathbb{Z}$

b) $h(t) \circ\!\!-\!\!\bullet \; H(j\omega) = \frac{1}{j\omega} + \pi\cdot\delta(\omega) - \left|\frac{1}{j\omega} + \pi\cdot\delta(\omega)\right| e^{-j2\omega}$

b) $y(t) = \frac{1}{\pi}\left(1 + \frac{\sin 5}{5}\cdot e^{j5(t-1)}\right)$; yes

$u(t) - u(t-2)$



$\text{rect}\left(\frac{t-1}{2}\right) \circ\!\!-\!\!\bullet \; e^{-j\omega} 2\cdot\frac{\sin(\omega)}{\omega}$

$$H(j\omega) = e^{-j\omega}\cdot 2\,\frac{\sin(\omega)}{(\omega)} = 2\cdot si(\omega)\cdot e^{-j\omega}$$

$$X(j\omega)\cdot H(j\omega) = Y(j\omega) = e^{-j\omega}\cdot 2\,\frac{\sin(\omega)}{\omega}\cdot\left(\delta(\omega) + \delta(\omega-\pi) + \delta(\omega-5)\right)$$

$$= e^{-j\cdot 0}\cdot 2\cdot si(0) + e^{-j\pi}\cdot 2\,\frac{\sin(\pi)}{\pi} + e^{-j5}\,\frac{\sin(5)}{5}\cdot 2$$

$$= 1\cdot 2\cdot 1 + 0 + e^{-j5}\cdot si(5)\cdot 2$$

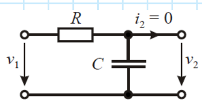
$$= \delta(\omega)\, 2 + e^{-j5}\, si(5)\cdot\delta(\omega-5)\cdot 2$$

$$= \frac{1}{\pi} + \cancel{2}\, si(5)\cdot e^{j5t}\,\frac{1}{\cancel{2}\pi}\, e^{-j5} = \frac{1}{\pi} + si(5)\cdot e^{j5t-j5}\,\frac{1}{\pi}$$

$$= \frac{1}{\pi}\left(\frac{\sin(5)}{5}\, e^{j5(t-1)}\right)$$

$\rightarrow$ periodic because only one t

Problem 6 Derive the frequency response of the circuit in problem 3 from the differential equation.



$$v_2 + RC\cdot\frac{dv_2}{dt} = v_1$$

$v_2(j\omega)\cdot 1 + RC\cdot(j\omega)\cdot v_2(j\omega) = v_1(j\omega)$ $\qquad \frac{v_2}{v_1} = H(j\omega) = \frac{1}{RC\cdot j\omega}$

Problem 7 Consider a causal LTI system with frequency response

$$H(j\omega) = \frac{1}{3 + j\omega}$$

For a particular input $x(t)$ this system is observed to produce the output signal

$$y(t) = e^{-3t}\cdot u(t) - e^{-4t}\cdot u(t)$$

a) Determine and plot the magnitude and the phase response.

b) Which properties of $h(t)$ can be derived from the amplitude and the phase response?

c) Determine the input signal $x(t)$.

a) $|H(j\omega)| = \frac{1}{\sqrt{9+\omega^2}}$ $\qquad \varphi(\omega) = \cancel{\arctan(0)} - \arctan\left(\frac{\omega}{3}\right)$

b) c)

$$H(j\omega) = \frac{1}{3+j\omega} = \frac{1}{\sqrt{9+\omega^2}} e^{-j\arctan\frac{\omega}{3}}$$

angle = phase = $\arctan\left(\frac{Im}{Re}\right)$

$$|H(j\omega)| = \frac{1}{\sqrt{9+\omega^2}} \quad : \text{even}$$

$$\varphi(\omega) = -\arctan\left(\frac{\omega}{3}\right) : \text{odd}$$

$$H(-j\omega) = H(j\omega)^*$$

conjugate complex

$$H(j\omega) = \frac{3-j\omega}{9+\omega^2} = \underbrace{\frac{3}{9+\omega^2}}_{H_{real}} + j\underbrace{\left\{-\frac{\omega}{9+\omega^2}\right\}}_{H_{Im}}$$

$$H_{real} = H_{even}, \quad H_{Im} = H_{odd}$$

$H_{even} + H_{odd}$

$H(j\omega)$ ●—○ $h(t) = h_{ev} + h_{od}$

↑ real

$$y(t) = e^{-3t} \cdot u(t) - e^{-4t} \cdot u(t)$$

$$y(t) = y_{ev} + y_{od} = \frac{e^{-3|t|} u(|t|)}{2}$$

↑ real

$e^{j2\pi f_0 t} = e^{j\omega_0 t}$	$\delta(f - f_0)$	$2\pi \cdot \delta(\omega - \omega_0)$

$H(j\omega)$ ●—○ $h(t) = h_{ev} + h_{odd}$

↑ real

3-2

Freitag, 28. Oktober 2016 09:01

Problem 1 Compute the LAPLACE transforms, determine the ROC and the zero-pole plots for each of the following functions.

a) $x(t)=\mathrm{rect}\left(\frac{t-T/2}{T}\right)$

b) $x(t)=u(t)\cdot\left(\frac{2}{3}-\frac{1}{5}\cdot e^{2t}-\frac{7}{15}\cdot e^{-3t}\right)$

c) $x(t)=u(t)\cdot e^{-2t}+u(t)\cdot e^{-t}\cdot\cos(3t)$

1c) $x(t) = u(t)\cdot e^{-2t} + u(t)\, e^{-t}\cos(3t)$

$$= u(t)\cdot e^{-2t} + \underbrace{u(t)\, e^{-t}\,\tfrac{1}{2}\left(e^{j3t}+e^{-j3t}\right)}_{u(t)\,\frac{1}{2}\left(e^{(-1+j3)\cdot t}+e^{-(1+j3)t}\right)}$$

$$X(s) = \frac{1}{s+2} + \frac{1}{2}\cdot\frac{1}{s+(1-j3)} + \frac{1}{2}\cdot\frac{1}{s+(1+j3)}$$

$\downarrow$ $\sigma > -2$ $\quad\downarrow$ $\sigma > -1$ $\quad\downarrow$ $\sigma > -1$

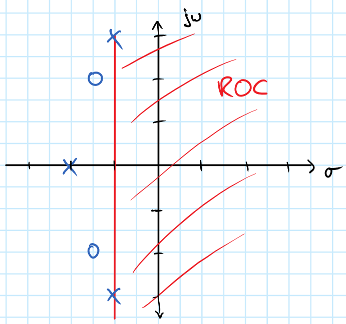
ROC : $\sigma > -1$

$$X(s) = \frac{2\cdot(s+1-j3)(s+1+j3)+(s+2)(s+1+j3)\,(s+2)(s+1-j3)}{2\cdot(s+2)(s+1-j3)(s+1+j3)}$$

$$X(s) = \frac{2\cdot(s-s_1)(s-s_2)}{(s-p_1)(s-p_2)(s-p_3)}$$

$p_1=-2$; $p_{2/3}=-1\pm j3$ $\Leftarrow$ oscillation

$z_{1/2} = -1.25 \pm j\cdot 2.11$



Problem 2 Compute the causal time functions of the given LAPLACE transforms.

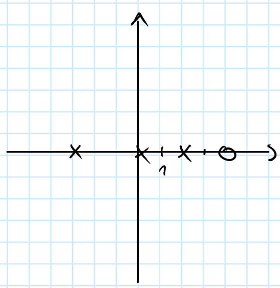
a) $X(s)=\frac{s-4}{s^3+s^2-6s}$ b) $X(s)=\frac{1}{s^2+s+1}$ c) $X(s)=\frac{s}{s^2+s+1}$

Plot and compare the zero-pole characteristics. Check the initial- and final-value theorems.

a)

$$\frac{s-4}{(s+3)(s-2)\,s} = \frac{A}{s} + \frac{B}{s+3} + \frac{C}{s-2}$$

$$s-4 = A(s+3)(s-2) + B\,s(s-2) + C\,s(s+3)$$



$s=0$: $-4=-A6$ $\Leftrightarrow$ $A=\frac{2}{3}$

$s=-3$: $-7 = B(-3)(-5)$ $\Leftrightarrow$ $B=\frac{-7}{15}$

$s=2$: $-2 = C\cdot 10$ $\Leftrightarrow$ $C=\frac{-1}{5}$

$$X(s) = \frac{2/3}{s} - \frac{7/15}{s+3} + \frac{-1/5}{s-2} \;\bullet\!\!-\!\!\circ\; \frac{2}{3}u(t) - e^{-3t}\frac{7}{15}\cdot u(t) - e^{+2t}\frac{1}{5}\cdot u(t)$$

Initial-value theorem	$\lim\limits_{t\to 0_+} x(t) = \lim\limits_{s\to\infty} s\cdot X(s)$
Final-value theorem *Note:* Only single pole at $s = 0$ allowed for $X(s)$. No poles in the right half plane.	$\lim\limits_{t\to\infty} x(t) = \lim\limits_{s\to 0} s\cdot X(s)$

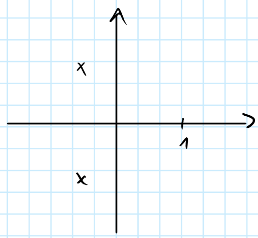
initial:

$$\lim_{t\to 0} x(t) = \frac{2}{3} - \frac{7}{15} - \frac{1}{5} = 0 \quad \checkmark$$

$$\lim_{s\to\infty} s\cdot X(s) = \frac{\cancel{s}(s-4)}{(s+3)(s-2)\cancel{s}} = \frac{1-\frac{4}{s}}{(1+\frac{3}{s})(1-\frac{2}{s})} = 0$$

final: pole in the right half plane

b) $X(s) = \dfrac{1}{s^2+s+1}$ no zeros $p_{1,2} = \dfrac{-1 \pm \sqrt{3}\,j}{2}$



$$X(s) = \frac{1}{(s-p_1)(s-p_2)} = \frac{A}{s-p_1} + \frac{B}{s-p_2}$$

$$1 = A(s-p_2) + B(s-p_1)$$

$s = p_1$: $1 = A(p_1 - p_2) \Leftrightarrow A = \frac{j}{\sqrt{3}}$

$s = p_2$: $1 = B(p_2 - p_1) \Leftrightarrow B = \frac{-j}{\sqrt{3}}$

$$X(s) = \frac{j/\sqrt{3}}{s-p_1} - \frac{j/\sqrt{3}}{s-p_2} \quad \bullet\!\!-\!\!\circ \quad x(t) = \frac{j}{\sqrt{3}}\,u(t)\cdot e^{p_1 t} - \frac{j}{\sqrt{3}}\,u(t)\cdot e^{p_2 t}$$

$$= \frac{j}{\sqrt{3}}\cdot u(t)\, e^{-t(\frac{1}{2}+\frac{\sqrt{3}}{2}j)} - \frac{j}{\sqrt{3}}\cdot u(t)\cdot e^{-t(\frac{1}{2}-\frac{\sqrt{3}}{2}j)}$$

$$= \frac{j}{\sqrt{3}}\cdot e^{-\frac{1}{2}t}\cdot e^{-\frac{\sqrt{3}}{2}jt}\cdot u(t) - \frac{j}{\sqrt{3}}\cdot e^{-\frac{1}{2}t}\cdot e^{+\frac{\sqrt{3}}{2}jt}\cdot u(t)$$

$\frac{1}{2j}\left(e^{ix} - e^{-ix}\right) = \sin$

$$= u(t)\, e^{-\frac{1}{2}t}\,\frac{j}{\sqrt{3}}\left(e^{-\frac{\sqrt{3}}{2}jt} - e^{+\frac{\sqrt{3}}{2}jt}\right)$$

$$= -u(t)\, e^{-\frac{1}{2}t}\,\frac{j}{\sqrt{3}}\left(e^{\frac{\sqrt{3}}{2}jt} - e^{-\frac{\sqrt{3}}{2}jt}\right)$$

$$= u(t)\, e^{-\frac{1}{2}t}\,\frac{2}{\sqrt{3}}\,\sin\left(\frac{\sqrt{3}}{2}t\right)$$

Initial: $\lim\limits_{t\to 0_+} x(t) = 0$

$$\lim_{s\to\infty} X(s) = \frac{\frac{1}{s^2}}{1+\frac{1}{s}+\frac{1}{s^2}} = 0 \quad \checkmark$$

Final: $\lim\limits_{t\to\infty} x(t) = 0$

$$\lim_{s\to 0} s\cdot X(s) = \frac{0}{1} = 0 \quad \checkmark$$

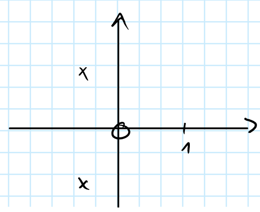
c) $X(s) = \dfrac{s}{s^2+s+1} = \dfrac{s}{(s-p_1)(s-p_2)}$

$$X(s) = \frac{1}{(s-p_1)(s-p_2)} = \frac{A}{s-p_1} + \frac{B}{s-p_2}$$

$$s = A(s-p_2) + B(s-p_1)$$

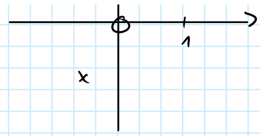
$s = p_1$: $p_1 = A(p_1 - p_2) = A = \frac{p_1}{p_1 - p_2} = \frac{1}{2} + \frac{\sqrt{3}}{6}j$

$s = p_2$: $p_2 = B(p_2 - p_1) = B = \frac{p_2}{p_2 - p_1} = \frac{1}{2} - \frac{\sqrt{3}}{6}j$



$X(s) = \frac{A}{\ldots} + \frac{B}{\ldots}$ $\bullet\!\!-\!\!\circ$ $x(t) = u(t)\cdot A\cdot e^{p_1 t} + B\cdot u(t)\cdot e^{p_2 t}$

$s = p_2: \quad p_2 = B(p_2 - p_1) \quad = B = \frac{p_2}{p_2 - p_1} \quad = \frac{1}{2} - \frac{\sqrt{3}}{6} j$



$X(s) = \frac{A}{s-p_1} + \frac{B}{s-p_2} \quad \bullet\!\!-\!\!\circ \quad x(t) = u(t) \cdot A \cdot e^{p_1 t} + B \cdot u(t) \cdot e^{p_2 t}$

$$x(t) = \left(\frac{1}{2} + \frac{\sqrt{3}}{6} j\right) \cdot e^{-\frac{1}{2}t} \cdot e^{-\frac{\sqrt{3}}{2} jt} \cdot u(t) + \left(\frac{1}{2} - \frac{\sqrt{3}}{6}\right) j\, e^{-\frac{1}{2}t} \cdot e^{+\frac{\sqrt{3}}{2} jt} \cdot u(t)$$

$$= u(t)\, e^{-\frac{1}{2}t} \left(+\frac{1}{2} e^{-\frac{\sqrt{3}}{2} jt} + \frac{\sqrt{3}}{6} j\, e^{-\frac{\sqrt{3}}{2} jt} + \frac{1}{2} e^{+\frac{\sqrt{3}}{2} jt} - \frac{\sqrt{3}}{6} j\, e^{+\frac{\sqrt{3}}{2} jt} \right) \quad \text{(Vorzeichenfehler i-wo)}$$

$$= u(t)\, e^{-\frac{1}{2}t} \cdot \left(\cos\left(\frac{\sqrt{3}}{2} t\right) - \frac{1}{\sqrt{3}} \sin\left(\frac{\sqrt{3}}{2} t\right) \right) \quad \checkmark$$

Einfacher:

$\frac{d^n}{dt^n} x(t) \qquad s^n \cdot X(s) \qquad$ ($x(t)$ von b))

Initial:

$\lim_{t \to 0} x(t) = 1 \quad \checkmark$

$\lim_{s \to \infty} s \cdot X(s) = \frac{1}{1 + \frac{1}{s} + \frac{1}{s^2}} = 1$

final

$\lim_{t \to \infty} x(t) = 0 \quad \checkmark$

$\lim_{s \to 0} s \cdot X(s) = 0$

Problem 3 The input to and the output of an LTI system are given by

$$x(t) = u(t) \cdot e^{-3t} \;; \quad y(t) = u(t) \cdot \left[e^{-t} - e^{-2t}\right]$$

Determine the systems differential equation under the condition of initial rest.

$x(t) \quad \circ\!\!-\!\!\bullet \quad X(s) = \frac{1}{s+3}$

$y(t) \quad \circ\!\!-\!\!\bullet \quad Y(s) = \frac{1}{s+1} - \frac{1}{s+2} = \frac{(s+2)-(s+1)}{(s+1)(s+2)}$

$$\frac{Y(s)}{X(s)} = \frac{\overbrace{(\cancel{s+2}) - (\cancel{s+1})}^{1}(s+3)}{(s+1)(s+2)} = \frac{s+3}{s^2 + 2 + 3s}$$

$Y(s) \cdot (s^2 + 3s + 2) = X(s)\,(s+3)$

$\ddot{y}(t) + 3\dot{y}(t) + 2y(t) = \dot{x}(t) + x(t)\, 3$

Problem 4 Determine the system function $H(s)$ of the system with the impulse response

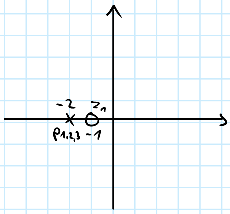
$$h(t) = u(t) \cdot (2-t) \cdot 4 \cdot t \cdot e^{-2t}. \quad = u(t)\left(8t\, e^{-2t} - 4t^2 e^{-2t}\right)$$

Plot the zero-pole diagram.

$u(t) \quad \circ\!\!-\!\!\bullet \quad \frac{8}{(s+2)^2} - \frac{8}{(s+2)^3} \qquad \frac{1}{n!} \cdot t^n \cdot e^{-at} \qquad \frac{1}{(s+a)^{n+1}}$

$H(s) = \frac{(s+2)8 - 8}{(s+2)^3} = \frac{8s+8}{(s+2)^3} \qquad z_1 = -1$

$p_{1,2,3} = -2$



Problem 5 Consider a stable system with a real impulse response. Its system function has two poles and a) no, b) one, c) two zeros. Discuss and plot possible impulse responses.

$$H(s) = \frac{1}{(s-p_1)(s-p_2)} = \frac{A_1}{s-p_1} + \frac{A_2}{s-p_2}$$

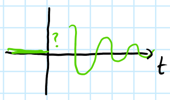
$$h(t) = A_1 \cdot e^{p_1 \cdot t} + A_2 \cdot e^{p_2 t}$$

a) p_1, p_2 real : $h(t)$

b) p_1, p_2 complex $p_2 = p_1^*$: $h(t)$

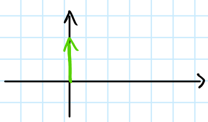
$p_1 = a + jb$ oscillators

$$h(t) = -\frac{1}{b} \cdot e^{a \cdot t} \cdot \sin(b \cdot t)$$



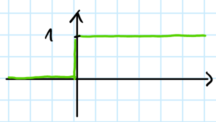
Laplace transforms

1) $x(t) = \delta(t)$ $\circ\!\!-\!\!\bullet$ $X(s) = 1$



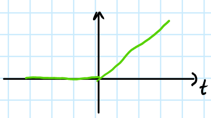
$\lim_{t \to 0_+} x(t) = \to \infty$ $\circ\!\!-\!\!\bullet$ $\lim_{s \to \infty} s \cdot X(s) \to \infty$

2) $x(t) = u(t)$ $\circ\!\!-\!\!\bullet$ $X(s) = \frac{1}{s}$



$\lim_{t \to 0_+} x(t) = 1$ $\circ\!\!-\!\!\bullet$ $\lim_{s \to \infty} s \cdot X(s) = 1$

3) $x(t) = t$ $\circ\!\!-\!\!\bullet$ $X(s) = \frac{1}{s^2}$



$\lim_{t \to 0_+} x(t) = 0$ $\circ\!\!-\!\!\bullet$ $\lim_{s \to \infty} s \cdot X(s) = \lim_{s \to \infty} \frac{1}{s} = 0$

a) $$H(s) = \frac{1}{(s-p_1)(s-p_2)} = \frac{N_m(s)}{D_n(s)}$$ $h(t)$: ramp behavior

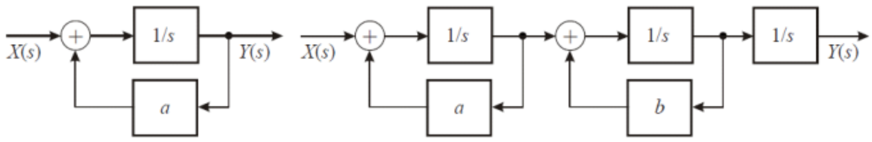
$$\tilde{H}(s) = \frac{(s-z_1)}{(s-p_1)(s-p_2)(s-p_3)}$$

$n - m = 2$

b) $H(s) = \frac{s-z_1}{(s-p_1)(s-p_2)}$ $h(t)$: step behavior : $n-m=1$

c) $n=m \longrightarrow h(t)$ with Dirac impulse

Problem 6 The block diagrams of two systems are given. a, b are real.



System 1 **System 2**

Determine for each of the systems:

a) system function, b) zero-pole plot, c) stability, d) impulse response, e) ARMA topology.

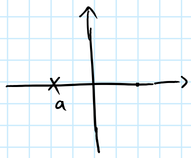
System 1

a) $Y(s) = X(s)\,\frac{1}{s} + Y(s)\,\frac{a}{s}$

$Y(s)\left(1-\frac{a}{s}\right) = X(s)\,\frac{1}{s}$

$H(s) = \frac{Y(s)}{X(s)} = \frac{1/s}{1-\frac{a}{s}} = \frac{1}{s-a}$

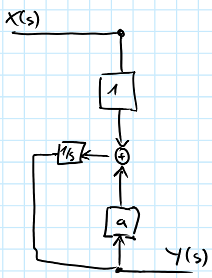
b)



c) stable for $a<0$

d) $h(t) = u(t)\,e^{at}$

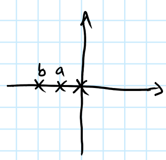
e) $b_0 = 1$ $a_0 = -a$ $a_1 = 1$



System 2

a) $H(s) = \frac{1}{s-a}\cdot\frac{1}{s-b}\cdot\frac{1}{s}$

b)



c) instable $\longrightarrow$ pole at 0

d) $\frac{1}{(s-a)(s-b)\,s} = \frac{A}{s-a} + \frac{B}{s-b}\;\frac{C}{s}$

$1 = A(s-b)\,s + B(s-a)\,s + C(s-a)(s-b)$

$s=0:$ $1 = a\cdot b\,C$ $C = \frac{1}{a\cdot b}$

$s=a:$ $1 = A(a-b)\,a$ $A = \frac{1}{(a-b)a}$

$s=b$ $1 = B(b-a)\,b$ $B = \frac{1}{(b-a)b}$

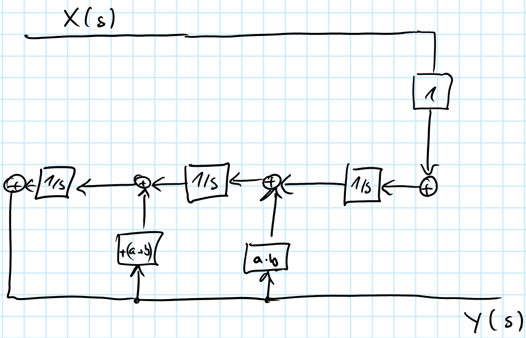
$H(s) = \frac{\frac{1}{(a-b)a}}{s-a} + \frac{1/(b-a)b}{s-b} + \frac{1/a\cdot b}{s} = \frac{1}{(s-a)(a-b)\cdot a} + \frac{1}{(s-b)(b-a)\cdot b} + \frac{1}{s\cdot a\cdot b}$

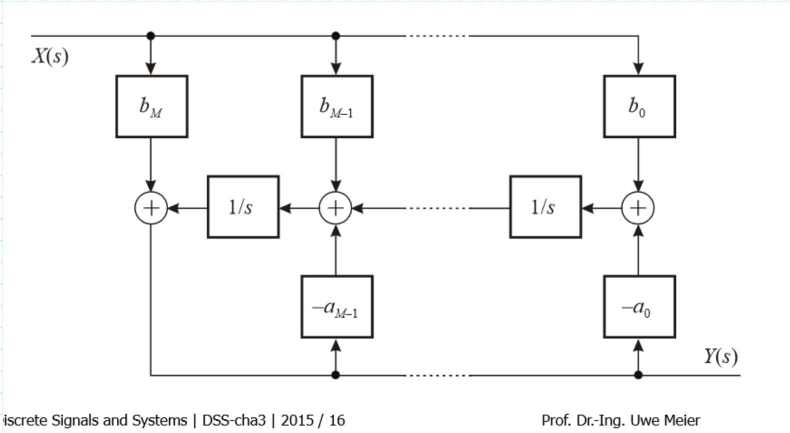
$h(t) = u(t)\left(\frac{1}{(a-b)a}\cdot e^{at} + \frac{1}{(b-a)b}\,e^{bt} + \frac{1}{ab}\right)$

e) $\frac{1}{(s-a)(s-b)\,s} = \frac{1}{\ldots} = \frac{1}{\ldots} = \frac{1}{\ldots}$

$$\frac{}{(s-a)(s-b)\,s} = \frac{''}{(s^2+ab-s\cdot b-s\cdot a)\cdot s} = \frac{''}{s^3+sab-s^2b-s^2a} = \frac{1}{s^3+s^2(-a-b)+s\cdot ab}$$

$b_0 = 1 \qquad a_0 = 0 \qquad a_1 = a\cdot b \qquad a_2 = -(a+b)$



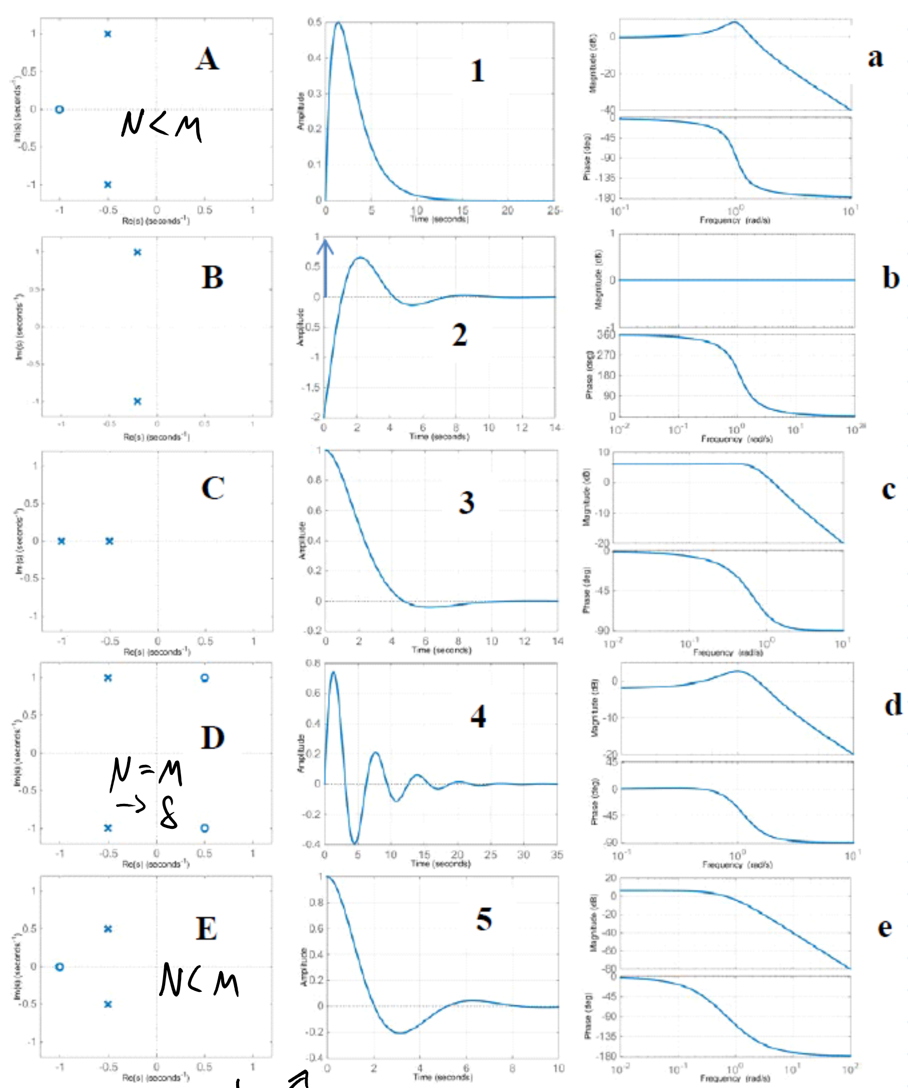


3-3

Freitag, 4. November 2016 08:44

Mapping task: Continuous-time systems

Different continuous-time systems shall be considered. Unfortunately, the given relation of the zero-pole plots to the impulse responses and to the BODE plots are erroneous. Find the correct relation and give the answer in the form (A, 2, c), if you think that this is a correct constellation.



A5d, B4a, C1e, D2b, E3c

initial value theorem

zero: left = +90° : Phase

right = −90°

pole : = −90°

Magnitude : |H|

$$H(s) = \frac{1}{s-p} \qquad p \text{ real}$$

$$H(j\omega) = \frac{1}{j\omega - p} \; ; \quad |H| = \frac{1}{\sqrt{\omega^2 + p^2}}$$

$$|H|_{dB} = 20\,dB \log_{..} (\omega^2+p^2)^{-1/2}$$

$$|H|_{dB} = 20\,dB \log_{10} (\omega^2+p^2)^{-1/2}$$

$$= -10\,dB \log_{10} (\omega^2+p^2)$$

---

asymdotic behavior : $\omega >> p$

$$|H|_{dB} = -20\,dB \cdot \log_{10} \omega$$

slope : $-20\,dB$ / decode

$-6\,dB$ / octave

$$H(s) = s - z$$

$|H|_{dB}$: $+20\,dB$ / decode

Phase contibution.

1 pole : $0 \ldots -90°$

1 zero: right : $\Delta\varphi = -90°$

left : $\Delta\varphi = +90°$

Problem 1 Is ideal impulse train sampling a linear operation? Is it a time-invariant operation?

$x_s(t) = x(t) \cdot p(t)$

$p(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT_s)$

$x_s(t) = \sum_{n=-\infty}^{\infty} x(nT_s) \cdot \delta(t - nT_s)$

linear?

new input $k \cdot x(t)$ ← old input

$x_s(t) = k\, x(t) \cdot p(t)$

$\stackrel{!}{=} \sum_{n=-\infty}^{\infty} k\, x(nT_s) \cdot \delta(t - nT_s)$

$= \underbrace{k \cdot \overbrace{\sum_{n=-\infty}^{\infty} x(nT_s) \cdot \delta(t - nT_s)}^{\text{old output}}}_{\text{new input}}$

linear ✓

time invariant?

$x(t-t_0) \to y(t-t_0)$

new input:

$x(t-t_0) \to x(t-t_0) \cdot p(t) =$

$x(t-t_0) \cdot \sum_{n=-\infty}^{\infty} \delta(t - nT_s) = \sum_{n=-\infty}^{\infty} x(nT_s - t_0) \cdot \delta(t - nT_s)$

$y(t-t_0) = x_s(t-t_0) = x(t-t_0) \cdot p(t-t_0)$

$= x(t-t_0) \cdot \sum \delta(t - t_0 - nT_s)$

$= \sum x(\cancel{t_0} + nT_s - \cancel{t_0}) \cdot \delta(t - t_0 - nT_s)$

$\neq$

not time-invariant!

Problem 2 Determine the sampling period T_s for a successful sampling and reconstruction of the signal

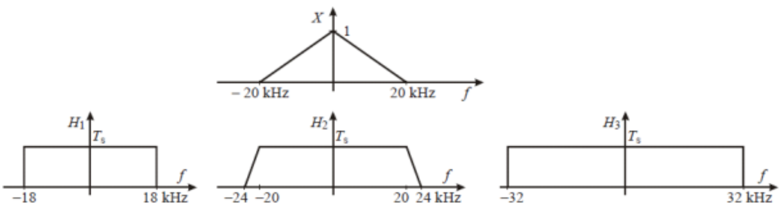
$$x(t) = 1 + \cos(2\,\text{Hz} \cdot \pi \cdot t) + 2 \cdot \sin(\underbrace{40\,\text{Hz} \cdot \pi \cdot t}_{2\pi f t})$$

$f = 20\,\text{Hz}$

$F_s > 2 \cdot F_{max}$ $\quad F_{max} = 20\,\text{Hz}$

$F_s > 40\,\text{Hz}$ $\quad T_s < 25\,\text{ms}$

Problem 3 A time-continuous signal $x(t)$ shall be sampled with one of the given sampling frequencies: 36 kHz, 44 kHz, 64 kHz. Subsequently the original signal shall be reconstructed with one of the given low-pass filters H_1, H_2, H_3.

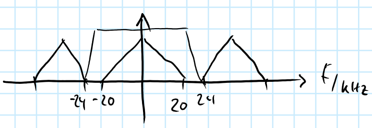


a) Select the minimal sampling frequency for a successful reconstruction.

b) Select one of the given low-pass filters: H_1, H_2, H_3

a) $f_s > 2 f_{max}$ $\quad f_s > 40\,\text{kHz} \to f_s = 44\,\text{kHz}$

b)



$\to H_2$.

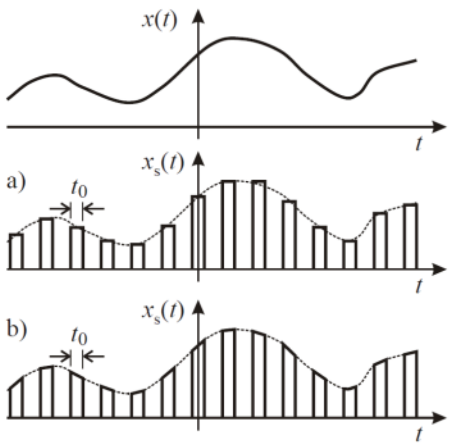
H_1 has too small T_s

H_3 will not filter repitition.

Problem 4 A real sampling system uses impulses of finite width t_0. Circuit a) is referred as sample-and-hold circuit. Circuit b) is referred as linear-gate circuit.

a) Determine and plot the spectra of the sampled signals $x_s(t)$.

b) Is a perfect reconstruction possible?



(a)

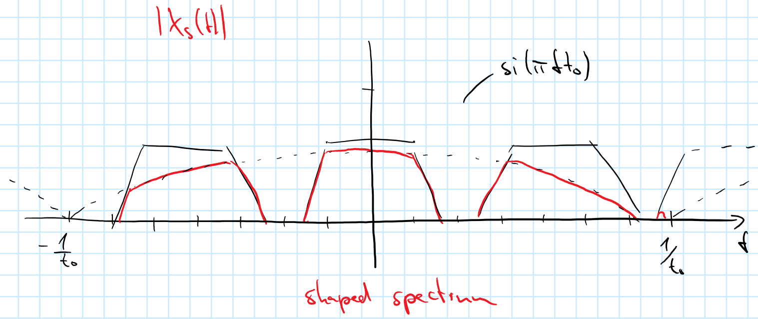
a) $x_s(t) = \sum_{n=-\infty}^{\infty} x(nT_s)\cdot \mathrm{rect}\frac{t-nT_s}{t_0}$

rectangular impulse train: $p(t) = \sum_{n=-\infty}^{\infty} \mathrm{rect}\frac{t-nT_s}{t_0} = \mathrm{rect}\frac{t}{t_0} * \sum_{n=-\infty}^{\infty} \delta(t-nT_s)$

$$x_s(t) = \underbrace{\mathrm{rect}\frac{t}{t_0}} * \underbrace{\sum_{n=-\infty}^{\infty} x(nT_s)\cdot\delta(t-nT_s)}$$

$$X_s(f) = \overbrace{si(\pi f t_0)}\cdot \overbrace{\frac{1}{T_s}\sum_{n=-\infty}^{\infty} X(f-n\cdot f_s)}$$

1st zero: $f = \frac{1}{t_0}$



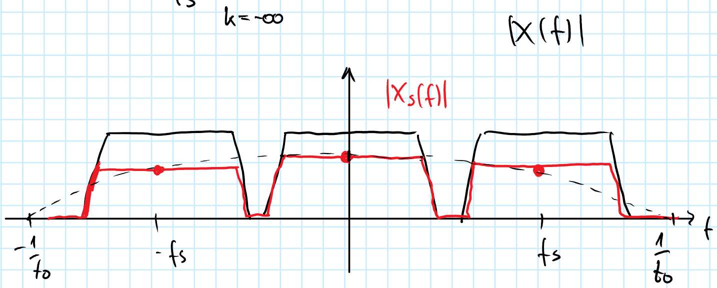
b) $x_s(t) = x(t)\cdot p(t)$ $\quad p(t) = \sum_{n=-\infty}^{+\infty} \mathrm{rect}\frac{t-nT_s}{t_0}$

$$x_s(t) = x(t)\cdot\left\{\mathrm{rect}\frac{t}{t_0} * \sum_{n=-\infty}^{+\infty}\delta(t-nT_s)\right\}$$

$$X_s(f) = X(f) * \left\{t_0\cdot si(\pi f t_0)\cdot\frac{1}{T_s}\sum_{k=-\infty}^{+\infty}\delta(f-k\cdot f_s)\right\} \qquad f_s = \frac{1}{T_s}$$

$$X_s(f) = X(f) * \frac{t_0}{T_s}\sum_{k=-\infty}^{+\infty} si(\pi\cdot k\cdot f_s\cdot t_0)\cdot\delta(f-kf_s)$$

$$= \frac{t_0}{T_s}\sum_{k=-\infty}^{+\infty} si(\pi k f_s t_0)\cdot X(f-k\cdot f_s)$$



4-2

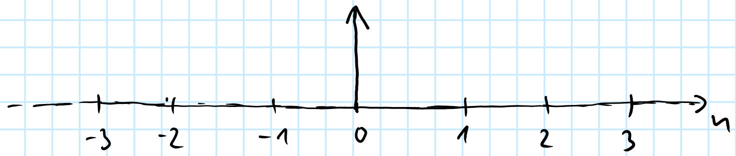
Freitag, 11. November 2016 08:29

Problem 1 Consider the finite-length sequence of length 7 defined for $-3 \leq n \leq 3$:

$$\{x[n]\} = \{0,\ 1+\mathrm{j}4,\ -2+\mathrm{j}3,\ 4-\mathrm{j}2,\ -5-\mathrm{j}6,\ -\mathrm{j}2,\ 3\}.$$

Determine the conjugate-symmetric and conjugate-antisymmetric sequences.

n:	-3	-2	-1	0	1	2	3
$x[n]$:	0	$1+j4$	$-2+j3$	$4-j2$	$-5-j6$	$-j2$	3
$x^*[-n]$:	3	$+j2$	$-5+j6$	$4+j2$	$-2-j3$	$1-j4$	0
$x_{cs}[n]$:	1.5	$0.5+j3$	$-3.5+j4.5$	4	$-3.5-j4.5$	$0.5-j3$	1.5



Problem 2 Consider the causal sequences defined by:

a) $x[n] = \begin{cases} 3\cdot(-1)^n, & n \geq 0 \\ 0, & \text{otherwise} \end{cases}$ b) $x[n] = \begin{cases} 1/n, & n \geq 1 \\ 0, & \text{otherwise} \end{cases}$

Are these sequences energy or power signals? If yes, compute the energy or power, respectively.

a) $$x[n] = \begin{cases} 3\cdot(-1)^n & n \geq 0 \\ 0 & n < 0 \end{cases}$$

$$\{x[n]\} = \{3, -3, 3, -3, \ldots\}$$

$$\{n\} = \{0, 1, 2, 3, \ldots\}$$

$$\{x^2[n]\} = \{9, 9, 9, 9, \ldots\}$$

$E_\infty \longrightarrow \infty$ no energy signal

$$P_\infty = \lim_{N\to\infty} \frac{1}{2N+1}\cdot 9(N+1) \longrightarrow \frac{9\cancel{N}}{2\cancel{N}} = \underline{\underline{4.5}}$$

b) $$\{x^2[n]\} = \{1, \tfrac{1}{4}, \tfrac{1}{9}, \ldots\}$$

$$\{n\} = \{1, 2, 3, \ldots\}$$

$$E_\infty = \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}$$

Problem 3 Compute the N-point DFT $X[k]$ of the length-N sequences with $0 \le k \le N-1$

a) $\{x[n]\} = \{2, 0, 1, 0, 0, 0, 1, 0\}$; b) $\{x[n]\} = \{1, 2, 3, 4\}$

Check for symmetry relations.

a) $\{x[n]\} = \{2, 0, 1, 0, 0, 0, 1, 0\}$

$\{n\} = \{0, 1, 2, 3, 4, 5, 6, 7\} \quad \Rightarrow \quad N = 8$

DFT $\quad X[k] = \sum_{n=0}^{N-1} x[n] \cdot e^{-j2\pi kn/N}$

$$X[k] = 2\cdot e^0 + 1\, e^{-j2\pi \frac{k\cdot 2}{8}} + 1\cdot e^{-j2\pi \frac{k\cdot 6}{8}}$$

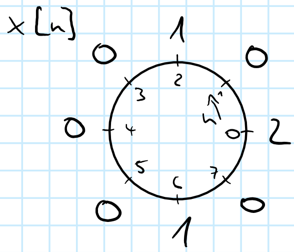
$$= 2 + e^{-j\frac{\pi}{2}\cdot k} + e^{-j\pi\frac{3}{2}\cdot k}$$

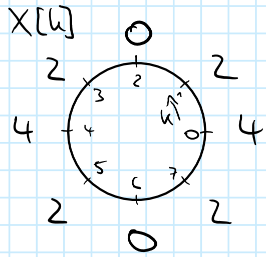
$$= 2 + e^{-j\pi k} \cdot \underbrace{\left\{ e^{j\frac{\pi}{2}k} + e^{-j\frac{\pi}{2}k} \right\}}_{2\cdot\cos\left(\frac{\pi}{2}\cdot k\right)}$$

$\{X[k]\} = \{4, 2, 0, 2, 4, 2, 0, 2\}$

$\{k\} = \{0, 1, 2, 3, 4, 5, 6, 7\}$

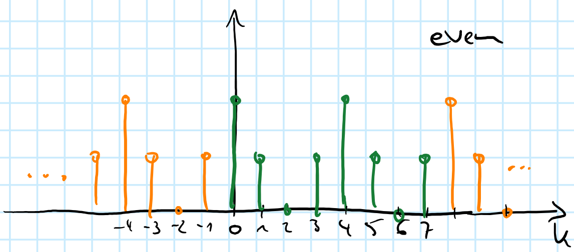
symmetry ?





circular even, real
(gleiche Werte im oder gegen den Uhrzeigersinn)

cicular even,
real



b) $\{x[n]\} = \{1, 2, 3, 4\}$

$\{n\} = \{0, 1, 2, 3\} \qquad X[k] = \sum_{n=0}^{N-1} x[n] \cdot e^{-j2\pi kn/N}$

$$X[k] = 1\cdot e^{0} + 2\cdot e^{-\frac{j8\pi k\cdot 1}{4\,2}} + 3\cdot e^{-\frac{j8\pi k 2}{4}} + 4\, e^{-\frac{j\pi 2\cdot 3\cdot k}{4\,2}}$$

$$X[k] = 1\, e^{0} + 2\, e^{-\frac{j\pi k}{2}} + 3\, e^{-j\pi k} + 4\, e^{-\frac{j\pi 3k}{2}}$$

$$\{X[k]\} = \{10, -2+2j, -2, -2-2j\}$$

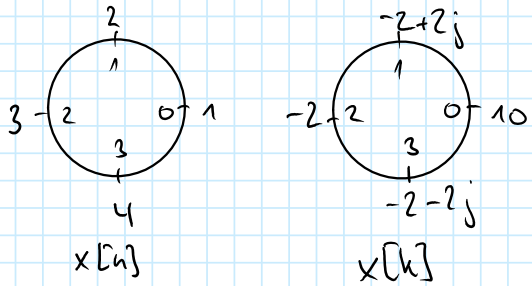
$$\{k\} = \{0, 1, 2, 3\} \qquad N-1 = k_{max} = 3$$

$$X[0] = 1+2+3+4 = 10$$

$$X[1] = 1 - 2j - 3 + 4j = -2+2j$$

$$X[2] = 1 - 2 + 3 - 4 = -2$$

$$X[3] = 1 + 2j - 3 - 4j = -2 - 2j$$



conjugate symmetric

Problem 4 Compute the N-point DFT $X[k]$ of the length-N sequences with $0 \leq k \leq N-1$

a) $x[n] = e^{j2\pi rn/N}$; $0 \leq n \leq N-1$,

b) $x[n] = \cos(2\pi rn/N)$; $0 \leq n \leq N-1$,

where r is an integer in the range $0 < r \leq N-1$.

$$0 \leq \left\{ \begin{matrix} n \\ k \\ r \end{matrix} \right\} \leq N-1$$

a) $$X[k] = \sum_{n=0}^{N-1} e^{j2\pi rn/N} \cdot e^{-j2\pi kn/N}$$

$$= \sum_{n=0}^{N-1} e^{j2\pi(r-k)\cdot\frac{n}{N}} = \sum_{n=0}^{N-1} \left(e^{j2\pi(r-k)/N}\right)^{n}$$

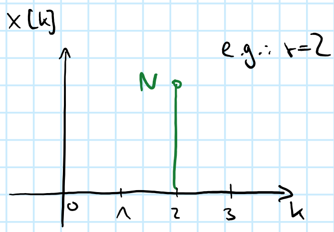
Geometric series

$$\sum_{i=0}^{I-1} x^i = 1 + x + x^2 + \ldots + x^{I-1} = \frac{x^I - 1}{x - 1}$$

$$= \frac{e^{j2\pi(r-k)} - 1}{e^{j2\pi(r-k)/N} - 1} = N\cdot\delta[k-r]$$

$$= \begin{cases} N & \text{for } k=r \\ 0 & \text{else} \end{cases}$$

e.g.: r=2



Discrete version of:

$x(t) = e^{j2\pi f_0 t} = e^{j\omega_0 t}$ ○—● $X(f) = \delta(f - f_0)$

$$f_0 = r\cdot\Delta f = r\cdot\frac{1}{T} = r\,\frac{1}{N\cdot T_s} = r\,\frac{f_s}{N}$$

b) $x[n]=\cos(2\pi rn/N)$; $0\le n\le N-1$,

$$X[k] = \begin{cases} N/2 & k=r \\ N/2 & k=N-r \\ 0, & \text{otherwise} \end{cases}$$

$\cos(2\pi nl/N),\ 0\le n\le N-1,\ 0<l\le N-1$ $\qquad \begin{cases} N/2, & k=l \\ N/2, & k=N-l \\ 0 & \text{, otherwise} \end{cases}$

App B. p.6

Problem 5 Building resonances between 0.8 Hz and 1.0 Hz must be avoided in skyscrapers, because they cause braking window glasses. Thus, civil engineers need to know exactly the resonance frequencies. The building is equipped with sensors and their signals are being processed digitally. After AD conversion an FFT algorithm is applied.

a) How many FFT points are necessary for the sampling frequency 5.4 Hz in order to achieve a frequency resolution of 0.001 Hz?

b) Determine the observation time.

a) FT: $5400 = \frac{5.4\,\text{Hz}}{0.001\,\text{Hz}}$ $\qquad$ FFT: $(2^{13}) \longrightarrow 8192 > 5400$ ✓

FFT muss mit Potenzen von 2 gemacht werden.

b) $\frac{N}{f_s} = T = 1517\ s$

F. 32 Kap 4.

Problem 6 Use the equation

$$\sum_{n=0}^{N-1} e^{-j2\pi kn/N} = \begin{cases} N\ , & k=l\cdot N\ ,\ l \text{ integer} \\ 0\ , & \text{otherwise} \end{cases}$$

for the computation of the length-N DFT.

a) Compute the continuous-time FOURIER transform $X(f)$ of $x(t)=\sin(2\pi f_0 t)$ and plot $|X(f)|$.

b) Compute the continuous-time FOURIER transform $X(f)$ of

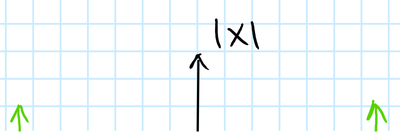
$$x(t)=\begin{cases} \sin(2\pi f_0 t)\ , & 0\le t\le 1/f_0 \\ 0 & ,\ \text{otherwise} \end{cases}$$

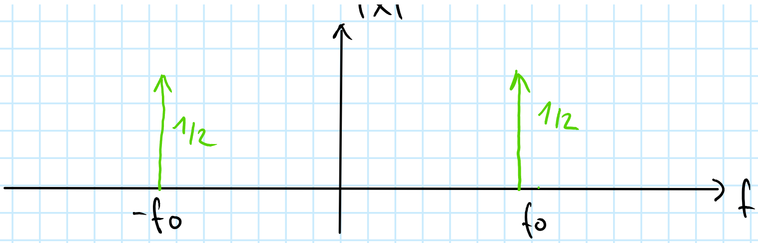
c) Compute $X[k]$ of $x[n]=\sin(2\pi f_0 n T_s)$ with $T_s = 1/(8f_0)$ and $N=8$. Plot $|X[k]|$.

d) Compute $X[k]$ of $x[n]=\sin(2\pi f_0 n T_s)$ with $T_s = 0.9/(8f_0)$ and $N=8$. Plot $|X[k]|$.

e) Perform suitable Matlab simulations and check the results.

a) $x(t) = \sin(2\pi f_0 t) \circ\!\!-\!\!\bullet\ X(f) = \frac{j}{2}\{\delta(f+f_0) - \delta(f-f_0)\}$





b) $$x(t) = \sin(2\pi f_0 t) \cdot \text{rect}\,\frac{t - T_0/2}{T_0}$$

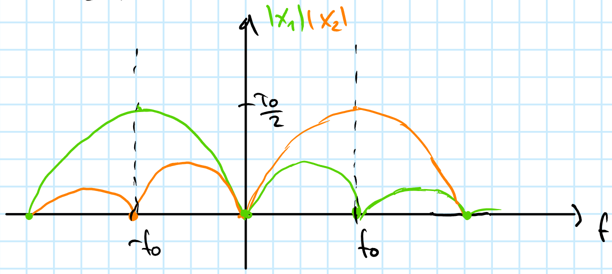
$$T_0 = 1/f_0 \qquad \text{rect}\,\frac{t}{T_0} \circ\!\!-\!\!\bullet\; T_0 \cdot si(\pi f T_0)$$

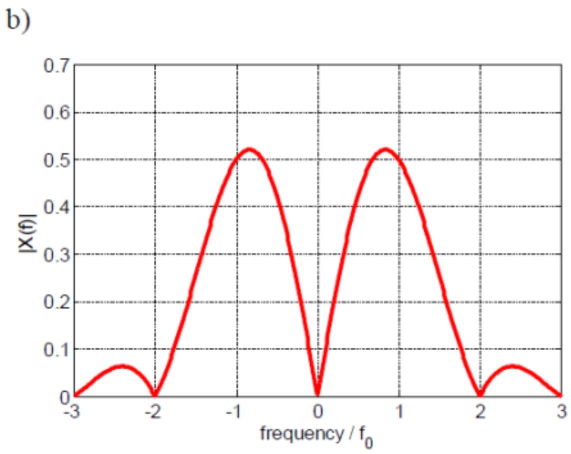
$$\text{rect}\,\frac{t - T_0/2}{T_0} \circ\!\!-\!\!\bullet\; T_0 \cdot si(\pi f T_0)\, e^{-j2\pi f T_0/2}$$

$$X(f) = \frac{j}{2}\left\{\delta(f+f_0) - \delta(f-f_0)\right\} * T_0 \cdot si(\pi f T_0) \cdot e^{-j2\pi f \frac{T_0}{2}}$$

$$X(f) = \frac{jT_0}{2} \cdot si\left[\pi(f+f_0)T_0\right] e^{-j2\pi(f+f_0)\frac{T_0}{2}} - \frac{jT_0}{2}\, si\left[\pi(f-f_0)\cdot T_0\right] e^{-j2\pi(f-f_0)\frac{T_0}{2}}$$

$$X(f) = X_1(f) + X_2(f)$$





c) $$x[n] = \sin(2\pi f_0 \cdot n \cdot T_s) \qquad N = 8$$

$$T = \underbrace{8 \cdot T_s}_{= T_0} = \frac{1}{f_0} \Leftarrow T_s = \frac{1}{8 \cdot f_0}$$

no leakage

$$x[n] = \frac{1}{2j}\left\{e^{j2\pi f_0 n T_s} - e^{-j2\pi f_0 n T_s}\right\} = \frac{1}{2j}\left\{e^{j2\pi \frac{n}{8}} - e^{-j2\pi \frac{n}{8}}\right\}$$

$$x[n] = \frac{1}{2j}\left\{e^{j2\pi f_0 n T_s} - e^{-j2\pi f_0 n T_s}\right\} = \frac{1}{2j}\left\{e^{j2\pi\frac{n}{8}} - e^{-j2\pi\frac{n}{8}}\right\}$$

$$2\pi \cancel{f_0} \cdot n \cdot \frac{1}{8 \cancel{f_0}} = 2\pi\,\frac{n}{8}$$

$$X[k] = \frac{1}{2j}\sum_{n=0}^{7} e^{-j2\pi\cdot n(k-1)/8} - \frac{1}{2j}\sum_{n=0}^{7} e^{-j2\pi n(k+1)/8}$$

$$X[k] = \underbrace{\frac{1}{2j}\cdot 8\cdot\delta\cdot[k-1]}_{} - \underbrace{\frac{1}{2j}\cdot 8\cdot\delta\cdot[k-7]}_{}$$

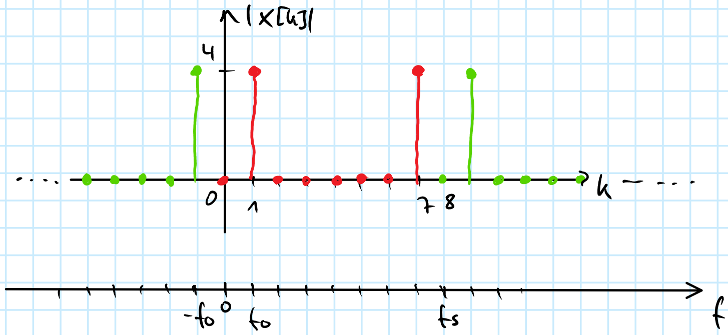
for $(k-1) = l\cdot 8$ for $(k+1) = l\cdot N$

$\Leftrightarrow$ for $k = 1 + \underset{0}{\underset{\uparrow}{l}}\cdot 8$ $\Leftrightarrow k = -1 + \underset{1}{\underset{\uparrow}{l}}\cdot N$

$k = 1$ $k = 7$

$0 \le k \le 7$

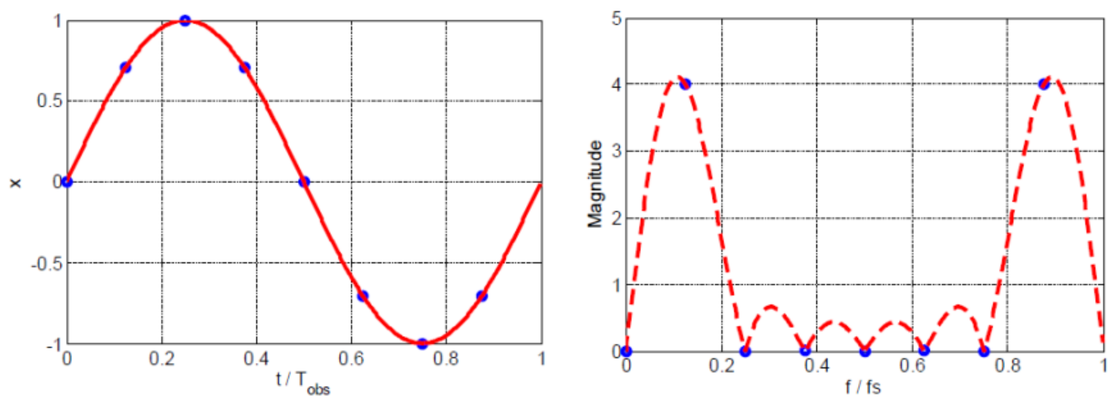
$$X[k] = \frac{4}{j}\,\delta[k-1] - \frac{4}{j}\,\delta[k-7]$$



$$\Delta f = \frac{1}{T} = \frac{1}{8\cdot T_s} = \frac{f_s}{8} = f_0$$

$$f_s = 8\cdot\Delta f$$

c) $T_s = 1/(8 f_0)$ and $N = 8$



d) Compute $X[k]$ of $x[n] = \sin(2\pi f_0 n T_s)$ with $T_s = 0.9/(8 f_0)$ and $N = 8$. Plot $|X[k]|$.

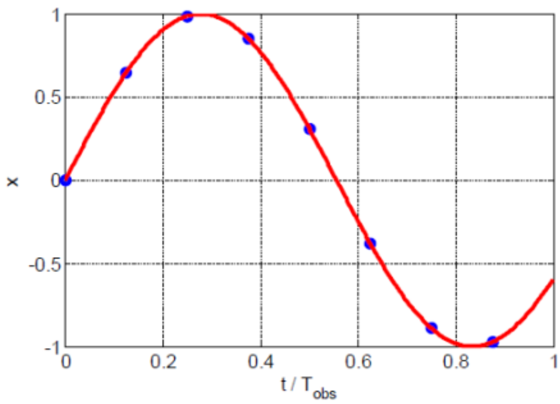
$$x[n] = \sin\left(\frac{2\pi n\, 0.9}{8}\right)$$

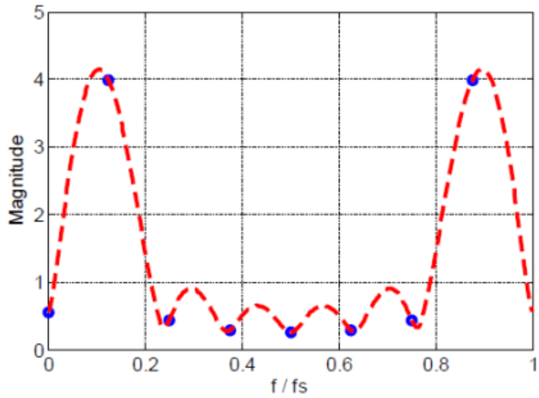
$$X[k] = \begin{cases} -j\cdot 8/2, & k = 0.9 \\ j\cdot 8/2, & k = 8 - 0.9 = 7{,}1 \\ 0, & \text{otherwise} \end{cases}$$

$\sin(2\pi nl / N),\ 0 \le n \le N-1,\ 0 < l \le N-1$

$$\begin{cases} -\mathrm{j}\cdot N/2, & k = l \\ \mathrm{j}\cdot N/2, & k = N-l \\ 0, & \text{otherwise} \end{cases}$$

d) $T_s = 0.9/(8 f_0)$ and $N = 8$





5-1

Mittwoch, 7. Dezember 2016 10:04

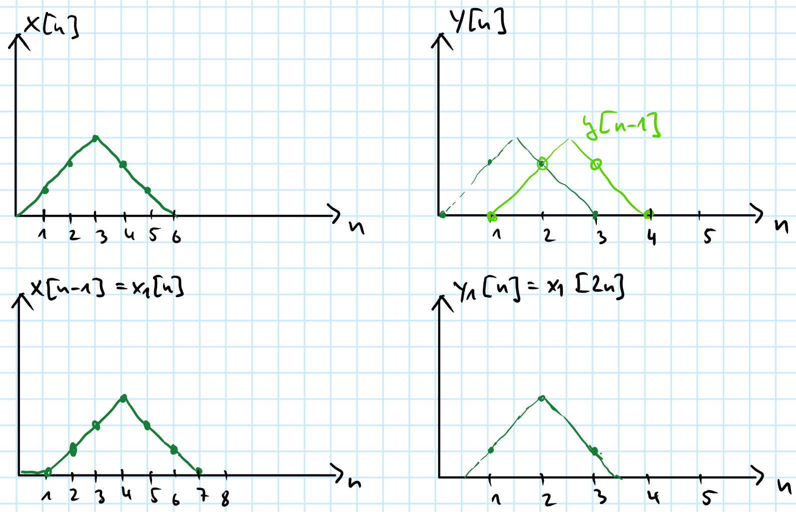
Problem 1 Is the down-sampler a linear and time-invariant system?

$$y[n] = x[n \cdot M] = x[2 \cdot n] \qquad M=2$$

linear? ✓

$$K \cdot x[n \cdot M] = K \cdot x[n \cdot M] = K \cdot y[n] \quad ✓$$

$$x_1[2n] + x_2[2n] = y_1[n] + y_2[n] \quad ✓$$



$$y[n-1] \neq y_1[n]$$

$\Rightarrow$ time variant

Problem 2 Is the M-point moving average filter a stable system? Its input-output relation is given by:

$$y[n] = \frac{1}{M}\sum_{i=0}^{M-1} x[n-i]$$

$$|x[n]| < B_x$$

$$|y[n]| < ?$$

$$|y[n]| = \left|\frac{1}{M}\sum x[n-i]\right|$$

$$|y[n]| \leq \frac{1}{M}\sum \underbrace{|x[n-i]|}_{\leq B_x} \leq B_x$$ Bounded by limits

$$|y[n]| \leq B_x \qquad \Rightarrow \text{Bibo stable}$$

Problem 3 Consider a causal LTI discrete-time system with the impulse response $h[n] = \beta^n \cdot u[n]$, where $|\beta| < 1$. Determine the output sequence $y[n]$ for the causal input sequence $x[n] = \alpha^n \cdot u[n]$ with $|\alpha| < 1$. $u[n]$ is the unit step sequence.

$h[n] = \beta^n \cdot u[n] \qquad |\beta| < 1$

FIR stable

$x[n] = \alpha^n \cdot u[n] \qquad |\alpha| < 1$

Does $y[n]$ exist?

$$y[n] = x[n] * h[n] = h[n] * x[n]$$

$$= \sum_{k=\cancel{-\infty}\,0}^{\cancel{\infty}\,n} \alpha^k \cdot \cancel{u[k]} \cdot \beta^{n-k} \cdot \cancel{u[n-k]}$$

$n-k \geq 0$

$\hookrightarrow k \leq n$

$$= \sum_{k=0}^{n} \alpha^k \cdot \beta^{n-k}$$

Problem 4 A discrete-time system is characterized by the following difference equation:

$$y[n] - 2^n \cdot y[n+1] + 3 \cdot y^2[n+2] = 4 \cdot x[n] - 2 \cdot x[n+1]$$

Check, whether the system is linear, time-invariant, causal, and memoryless.

linear: no because of $y^2[\dots]$

time invariant: no because of 2^n

causal: present output sample of $n+2$

$x[p] \quad y[k]$

$p \leq k$ sein, sonst nicht kausal

$n+2 = m \iff n = m-2$

$$\underbrace{y[m-2] - 2^{m-2} \cdot y[m-1] + 3y^2[n] = 4 \cdot x[m-2] - 2 \cdot x[m-1]}_{\text{memory}}$$

✓

Problem 5 A causal discrete-time LTI system is characterized by the following difference equation:

$$y[n] + y[n-1] - 6 \cdot y[n-2] = x[n]$$

Compute the impulse response. Is the system stable?

$$h[n] = y_h[n] = \alpha \cdot p^n$$

$$\cancel{\alpha} \cdot p^n + \cancel{\alpha} \cdot p^{n-1} - 6 \cdot \cancel{\alpha} \cdot p^{n-2} = 0$$

$$p^{n-2} \cdot \underbrace{(p^2 + p - 6)}_{=0} = 0$$

$$p_{1/2} = \frac{1}{2} \pm \sqrt{\frac{1}{4} + 6} = -\frac{1}{2} \pm \frac{5}{2}$$

$p_1 = 2; \quad p_2 = -3$

$$h[n] = \alpha_1\, 2^n + \alpha_2 \cdot (-3)^n$$

$h[n] + h[n-1] - 6 \cdot h[n-2] = \delta[n]$

$n=0: \quad h[0] = 1 = \alpha_1 + \alpha_2 \qquad \alpha_1 = 1 - \alpha_2$

$h[n] = 0$ for $n < 0$

$n=1: \quad h[1] + h[0] = 0 \qquad \alpha_1 \cdot 2 - 3 \cdot \alpha_2 + 1 = 0$

$\alpha_2 = \frac{3}{5} = 0.6 \qquad 3 = 5 \cdot \alpha_2$

$\alpha_1 = 1 - \frac{3}{5} = \frac{2}{5} = 0.4$

$h[n] = 0.4 \cdot 2^n + 0.6(-3)^n$ for $n \geq 0$

not stable bec. $|p_{1,2}| \geq 1$

Problem 5 A discrete-time LTI system is characterized by the following difference equation:

$$y[n] + y[n-1] - 6 \cdot y[n-2] = x[n]$$

Determine the solution for $n \geq 0$ for

a) the step input $x[n] = 8 \cdot u[n]$ and

b) for $x[n] = 2^n \cdot u[n]$

with initial conditions $y[0] = 0$ and $y[1] = 1$.

a) $x[n] = 8 \cdot u[n]$

$y[n] = y[n] + y_p[n]$

$y_h[n] = \alpha \cdot p^n$

$\cancel{\alpha} \cdot p^n + \cancel{\alpha} p^{n-1} - 6 \cdot \cancel{\alpha} \cdot p^{n-2} = 0$

$p^2 + p - 6 = 0$

$p_{1/2} = -\frac{1}{2} \pm \sqrt{\frac{1}{4} + 6} = -\frac{1}{2} \pm \sqrt{\frac{25}{4}} = -\frac{1}{2} \pm \frac{5}{2}$ — $p_1 = -3$, $p_2 = 2$

$y_h[n] = \alpha_1 (-3)^n + \alpha_2 \cdot (2)^n$

$x[n] = 8 \cdot u[n] \Rightarrow x[n] = 8$ for $n \geq 0$

$y_p[n] = \beta \cdot u[n]$

↳ check, if it is a solution.

weil größte Verschiebung 2 ist.

$n \geq 2: \quad y_p[2] + y_p[1] - 6 \cdot y_p[0] = 8 \qquad n = 2$

$\beta + \beta - 6 \cdot \beta = 8$

$-4\beta = 8 \Rightarrow \boxed{\beta = -2}$

$y[n] = \underbrace{\alpha_1 (-3)^n + \alpha_2 \cdot (2)^n}_{y_h} \underbrace{- 2 \cdot u[n]}_{y_p}$

$y[0] = +\alpha_1 + \alpha_2 - 2 = 0 \quad \Leftrightarrow \quad \alpha_2 = -\alpha_1 + 2$

$y[1] = -\alpha_1 3 + 2\alpha_2 - 2 = 1 \quad \Leftrightarrow \quad 1 = -\alpha_1 3 + 2(2-\alpha_1) - 2$

$\Leftrightarrow \quad 3 = -\alpha_1 3 + 4 - 2\alpha_1$

$\Leftrightarrow \quad -1 = -\alpha_1 3 - 2\alpha_1 \quad \Leftrightarrow \quad \alpha_1 = 1$

$\Leftrightarrow \quad \alpha_1 = \frac{1}{5}$

$\alpha_2 = \frac{9}{5}$

$y[n] = \frac{1}{5}(-3)^n + \frac{9}{5}2^n - 2 \cdot u[n]$

b)

$y_p[n] = \beta \cdot 2^n \cdot u[n]$ — geht nicht, da $\beta = 0$

$\downarrow$ modify

$y_p[n] = \beta \cdot n\, 2^n \cdot u[n] \qquad x[n] = 2^n \cdot u[n] = 2^n$ for $n > 0$

$n \geq 2 \qquad y_p[2] + y_p[1] + y_p[0] = 2^2 \qquad n = 2$

$\beta \cdot 8 + \beta \cdot 2 = 4 \qquad \frac{2}{5} = \beta$

$y[n] = \underbrace{\alpha_1(-3)^n + \alpha_2 \cdot (2)^n}_{y_h} + \underbrace{\frac{2}{5} n 2^n u[n]}_{y_p}$ for $n \geq 0$

$y[0] = +\alpha_1 + \alpha_2 = 0 \qquad \alpha_1 = -\alpha_2$

$y[1] = -\alpha_1 3 + 2\alpha_2 + \frac{4}{5} = 1 \quad \Leftrightarrow \quad \alpha_2 3 + 2\alpha_2 + \frac{4}{5} = 1$

$\Leftrightarrow \quad \alpha_2 = \frac{1}{25}$

$\Leftrightarrow \quad \alpha_1 = -\frac{1}{25}$

$y[n] = -\frac{1}{25}(-3)^n + \frac{1}{25}(2)^n + \frac{2}{5} n 2^n \cdot u[n]$ for $n \geq 0$

Problem 6 Compute the impulse response of a causal discrete-time system with the difference equation:

$x = \delta$ (Eingang)

$$y[n] - \frac{1}{2} \cdot y[n-1] = x[n]$$

$\circ\!\!-\!\!\bullet$ z

$Y(z) - \frac{1}{2}Y(z) \cdot z^{-1} = X(z) \quad \Leftrightarrow \quad Y(z)\left(1 - \frac{1}{2}z^{-1}\right) = X(z)$

$\Leftrightarrow \quad \frac{Y(z)}{X(z)} = \frac{1}{1 - \frac{1}{2}z^{-1}} \qquad X(z) = \delta(z) = 1$

$\Leftrightarrow \quad H(z) = \frac{1}{1 - \frac{1}{2}z^{-1}}$

$\bullet\!\!-\!\!\circ$

$h[n] = \left(\frac{1}{2}\right)^n \cdot u[n]$

- **Homogeneous** (complementary, natural) **solution** $y_h[n]$ solves
  $a_0 \cdot y[n] + a_1 \cdot y[n-1] + \ldots + a_N \cdot y[n-N] = 0$ with $y_h[n] = \alpha \cdot p^n$
  - **Characteristic polynomial:** $\sum_{k=0}^{N} a_k \cdot p^{N-k} = 0$
  - **Roots:** ...

Impulse Response Calculation

- The impulse response $h[n]$ of a **causal LTI** discrete-time system is the output observed with input $x[n] = \delta[n]$. Because $x[n] = 0$ for $n > 0$, the particular solution is zero: $y_p[n] = 0$.
- $h[n] = y_h[n]$ with coefficients which meet the **zero initial condition**, i.e. $y[n] = 0$ for $n < 0$.

Alternative:

$y_p = 0$

$\rightarrow h[n] - \frac{1}{2} h[n-1] = \delta[n]$

$\alpha \cdot p^n - \frac{1}{2} \alpha p^{n-1} = 0$

$\alpha \cdot p^{n-1}(p - \frac{1}{2}) = 0 \qquad p = \frac{1}{2}$

$h[n] = \alpha (\frac{1}{2})^n \longrightarrow h[0] = \alpha$

$h[0] - \frac{1}{2} \underbrace{h[-1]}_{0} = 1 \quad \delta[0] = 1 \qquad n \geq 0$

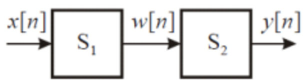
$h[n] = (\frac{1}{2})^n \cdot u[n]$

causal

Problem 7 Consider the cascade of the causal LTI systems S_1 and S_2.

$$S_1: w[n] = \frac{1}{2} \cdot w[n-1] + x[n] \quad ; \quad S_2: y[n] = \alpha \cdot y[n-1] + \beta \cdot w[n]$$

The difference equation relating $x[n]$ and $y[n]$ is:



$$y[n] = -\frac{1}{8} \cdot y[n-2] + \frac{3}{4} \cdot y[n-1] + x[n]$$

Determine α and β and the impulse response $h[n]$.

$S_2: \quad w[n] = \frac{1}{\beta} \cdot y[n] - \frac{\alpha}{\beta} \cdot y[n-1]$

$w[n-1] = \frac{1}{\beta} \cdot y[n-1] - \frac{\alpha}{\beta} \cdot y[n-2]$

$\rightarrow$ insert into S_1

$w[n] = \frac{1}{2} \frac{1}{\beta} y[n-1] - \frac{\alpha}{\beta 2} y[n-2] + x[n]$

$\frac{1}{\beta} y[n] - \frac{\alpha}{\beta} \cdot y[n-1] = \frac{1}{2} \frac{1}{\beta} y[n-1] - \frac{\alpha}{\beta 2} y[n-2] + x[n]$

$\cancel{\frac{1}{\beta}} y[n] = (\frac{1}{2\cancel{\beta}} + \frac{\alpha}{\cancel{\beta}}) y[n-1] - \frac{\alpha}{\cancel{\beta} 2} y[n-2] + x[n] \beta \qquad$ Koeff vgl:

$\frac{\alpha}{2} = \frac{1}{8} \Leftrightarrow \alpha = \frac{1}{4} \quad \beta = 1$

$$y[n] = -\frac{1}{8} \cdot y[n-2] + \frac{3}{4} \cdot y[n-1] + x[n]$$

$y_p = 0 \qquad y_h[n] = h[n]$

$h[n] = -\frac{1}{8} h[n-2] + \frac{3}{4} h[n-1] + \delta[n]$

$\delta[n] = h[n] + \frac{1}{8} h[n-2] - \frac{3}{4} h[n-1]$

$$\alpha \cdot p^n + \frac{1}{8}\alpha \cdot p^{n-2} - \frac{3}{4}\alpha p^{n-1} = 0$$

$$\alpha p^{n-2} \left(\frac{1}{8} - \frac{3}{4}p + p^2\right) = 0 \qquad p_1 = \frac{1}{2} \quad p_2 = \frac{1}{4}$$

$$h[n] = \alpha_1 \cdot p_1^n + \alpha_2 \cdot p_2^n = \left(\alpha_1 \cdot \left(\tfrac{1}{2}\right)^n + \alpha_2 \left(\tfrac{1}{4}\right)^n\right) \cdot u[n]$$

$$h[0] = \alpha_1 + \alpha_2$$

$$h[0] = -\frac{1}{8}h[-2] + \frac{3}{4}h[-1] + \delta[0] = 0 + 0 + 1 = 1 = \alpha_1 + \alpha_2$$

weil causal

$$h[1] = -\frac{1}{8}h[-1] + \frac{3}{4}h[0] + \delta[1] = 0 + \frac{3}{4} + 0 = \frac{3}{4} = \alpha_1\left(\tfrac{1}{2}\right) + \alpha_2\left(\tfrac{1}{4}\right)$$

$$\Rightarrow 1 - \alpha_2 = \alpha_1 \quad \Rightarrow \quad \frac{3}{4} = (1-\alpha_2)\,\frac{1}{2} + \alpha_2 \cdot \frac{1}{4} \quad \Leftrightarrow \quad -\frac{1}{4}\alpha_2 = \frac{1}{4} \quad \Leftrightarrow \quad \alpha_2 = -1$$

$$\Leftrightarrow \alpha_1 = +2$$

$$h[n] = \left(2\left(\tfrac{1}{2}\right)^n - \left(\tfrac{1}{4}\right)^n\right) \cdot u(n)$$

- **Homogeneous** (complementary, natural) **solution** $y_h[n]$ solves $a_0 \cdot y[n] + a_1 \cdot y[n-1] + \ldots + a_N \cdot y[n-N] = 0$ with $y_h[n] = \alpha \cdot p^n$
  - **Characteristic polynomial:** $\sum_{k=0}^{N} a_k \cdot p^{N-k} = 0$
  - **Roots:** $p_1 \ldots p_N$

Impulse Response Calculation

- The impulse response $h[n]$ of a **causal LTI** discrete-time system is the output observed with input $x[n] = \delta[n]$. Because $x[n] = 0$ for $n > 0$, the particular solution is zero: $y_p[n] = 0$.
- $h[n] = y_h[n]$ with coefficients which meet the **zero initial condition**, i.e. $y[n] = 0$ for $n < 0$.

Problem 8 a) Show that the sequence $x[n] = z^n$, where z is a complex constant, is an eigenfunction of an LTI discrete-time system. b) Is the sequence $x[n] = z^n \cdot u[n]$ with $u[n]$ being the unit step sequence also an eigenfunction of an LTI discrete-time system?

Problem 1 Compute the frequency response of a moving average filter. Its impulse response is given by

$$h[n] = \begin{cases} 1/M & , 0 \le n \le M-1 \\ 0 & , \text{otherwise} \end{cases}$$

$$H(e^{j\Omega}) = \, ? \quad \underset{\text{DTFT}}{\bullet\!\!-\!\!\circ} \quad h[n]$$

$$H(e^{j\Omega}) = \sum_{n=-\infty}^{+\infty} h[n] \cdot e^{-jn\Omega}$$

$$= \frac{1}{M} \sum_{n=0}^{M-1} 1 \cdot e^{-jn\Omega} = \frac{1}{M} \sum_{n=0}^{M-1} (e^{-j\Omega})^n$$

$$= \frac{1}{M} \cdot \frac{e^{-j\Omega M} - 1}{e^{-j\Omega} - 1} = \frac{1}{M} \frac{e^{-j\frac{\Omega}{2}M}}{e^{-j\frac{\Omega}{2}}} \cdot \frac{-e^{-j\frac{\Omega}{2}M} + e^{+j\frac{\Omega}{2}M}}{-e^{-j\frac{\Omega}{2}} + e^{+j\frac{\Omega}{2}}}$$

$$= \frac{1}{M} e^{j\frac{\Omega}{2}(1-M)} \cdot \frac{\cancel{2j} \sin(\frac{\Omega}{2}M)}{\cancel{2j} \cdot \sin(\frac{\Omega}{2})}$$

$$H(e^{j\Omega}) = \frac{1}{M} \frac{\sin(M\frac{\Omega}{2})}{\sin(\frac{\Omega}{2})} e^{-j\frac{\Omega}{2}(M-1)}$$

$$|H| = \frac{1}{M} \frac{|\sin(M\frac{\Omega}{2})|}{|\sin(\frac{\Omega}{2})|}$$

$$H(e^{j\Omega}) = H(\Omega) \cdot e^{j\varphi(\Omega)}$$

Problem 2 Consider a causal LTI discrete-time system with real impulse response $h[n]$. Compute the steady-state response for the sinusoidal input

$$x[n] = A \cdot \cos(\Omega_0 n + \phi)$$

where $\Omega_0 = \omega_0 \cdot T_s$ is the normalized radian frequency.

$$x[n] = \underbrace{\frac{A}{2} \cdot e^{j(\Omega_0 n + \phi)}}_{g[n]} + \underbrace{\frac{A}{2} e^{-j(\Omega_0 n + \phi)}}_{g^*[n]}$$

EF

$$y[n] = g[n] \, H(e^{j\Omega_0}) + g^*[n] \cdot H(e^{-j\Omega_0})$$

symmetry (App. B, DTFT)

$$\underset{\uparrow \text{ real}}{h[n]} \circ\!\!-\!\!\bullet \quad H(e^{j\Omega_0}) = H^*(e^{-j\Omega_0})$$

$$\Downarrow$$

$$H(e^{-j\Omega_0}) = H^*(e^{j\Omega_0})$$

$$y[n] = g[n]\cdot H(e^{j\Omega_0}) + g^*[n]\cdot H^*(e^{j\Omega_0})$$

$$= 2\cdot \mathrm{Re}\{\underbrace{g[n]}_{\frac{A}{2}\cdot e^{j(\Omega_0 n+\phi)}}\cdot \underbrace{H(e^{j\Omega_0})}_{|H(e^{j\Omega_0})|\cdot e^{j\varphi(\Omega_0)}}\}$$

$$= 2\cdot \mathrm{Re}\{\tfrac{A}{2}\,|H(e^{j\Omega_0})|\cdot e^{j(\Omega_0 n+\phi+\varphi(\Omega_0))}\}$$

$$\overset{!}{=} \underline{A\cdot |H(e^{j\Omega_0})|\cdot \cos(\Omega_0\cdot n+\phi+\varphi(\Omega_0))} \quad \text{in EF}$$

Problem 3 Consider a causal and stable LTI discrete-time system with real impulse response $h[n]$. Compute the response to the causal exponential sequence

$$x[n] = e^{jn\Omega}\cdot u[n] \quad \text{causal}$$

where $\Omega = \omega\cdot T_s$ is the normalized radian frequency and $u[n]$ is the unit step sequence. Identify and discuss the different parts of the solution.

$$y[n] = h[n] * x[n]$$

$$\overset{!}{=} \sum_{k=-\infty\,(0)}^{\infty\,(n)} \underbrace{h[k]}_{\text{causal}}\cdot e^{j(n-k)\Omega}\cdot \underbrace{u[n-k]}_{\rightarrow\, k\le n}$$

$$\overset{!}{=} \underbrace{\left\{\sum_{k=0}^{n} h[k]\cdot e^{-jk\Omega}\right\}}\, e^{jn\Omega}$$

$$\overset{!}{=} \underbrace{\sum_{k=0}^{\infty} \underbrace{h[k]}_{\text{causal}}\cdot e^{-jk\Omega}}_{H(e^{j\Omega})\ \text{DTFT}} - \sum_{k=n+1}^{\infty} h[k]\cdot e^{-jk\Omega}$$

$$y[n] = \underbrace{\underbrace{e^{jn\Omega}}_{x[n]}\cdot H(e^{j\Omega})}_{\substack{\text{steady state}\\ \text{behaviour}\\ \text{(particular solution)}}} - \underbrace{\underbrace{e^{jn\Omega}\sum_{k=n+1}^{\infty} h[k]\cdot e^{-jk\Omega}}_{\rightarrow\ \text{for } n\to\infty}}_{\text{transient}}$$

Problem 4 Determine the bilateral z-transform in a closed form and the region of convergence of the following series:

a) $x[n] = \begin{cases} \alpha^n & , M \le n \le N-1 \\ 0 & , \text{otherwise} \end{cases}$

b) $x[n] = \cosh(\alpha \cdot n) \cdot u[n]$

c) $x[n] = 7 \cdot \left(\frac{1}{3}\right)^n \cdot u[n] - 6 \cdot \left(\frac{1}{2}\right)^n \cdot u[n]$

Check the initial- and final-value theorems.

a) $$X(z) = \sum_{M}^{N-1} \alpha^n \cdot z^{-n} = \sum (\alpha \cdot z^{-1})^n = \alpha^M \cdot z^{-M} + \alpha^{M+1} \cdot z^{-(M+1)} + \alpha^{N-1} \cdot z^{-(N-1)}$$

$$= (\alpha \cdot z^{-1})^M \cdot \{1 + (\alpha + z^{-1})^1 + (\alpha \cdot z^{-1})^{N-1-M} = (\alpha \cdot z^{-1})^M \cdot \sum_{i=0}^{N-M-1} (\alpha \cdot z^{-1})^i$$

$$= (\alpha \cdot z^{-1})^M \cdot \frac{(\alpha \cdot z^{-1})^{N-M} - 1}{(\alpha - z^{-1} - 1)} = \frac{(\alpha \cdot z^{-1})^N - (\alpha \cdot z^{-1})^M}{\alpha \cdot z^{-1} - 1}$$

a) $M \le n \le N-1$

causal

$M \ge 0$

$\underbrace{\text{ROC: all } z \text{ except } z = 0}$

$$X(z) = \frac{\alpha^N \cdot z^{-N} - \alpha^M \cdot z^{-M}}{\alpha \cdot z^{-1} - 1}$$

$\alpha \cdot z^{-1} = 0 \iff z = \alpha = p$

$\Rightarrow$ $\cancel{z \ne \alpha}$?

$$\lim_{z \to \alpha} \frac{\overset{\text{numerator}}{dN/dz}}{\underset{\text{denom.}}{dD/dz}} = \lim_{z \to \alpha} = \frac{(-N)\,\alpha^N \cdot z^{-N-1} + \alpha^M \cdot M \cdot z^{-M-1}}{-\alpha \cdot z^{-2}}$$

$$\longrightarrow \frac{-N \cdot \cancel{\alpha^{-1}} + M \cdot \cancel{\alpha^{-1}}}{-\cancel{\alpha^{-1}}} = N - M \quad \checkmark$$

b) $M < 0$, $N > 0$

exclude $z = 0$

exclude $z \to \infty$

ROC: ring

c) $M < 0$, $N < 0$: exclude $z \to \infty$

Problem 5 Determine the inverse causal z-transforms:

a) $X(z)=\dfrac{z\cdot(z+2)}{(z-0.2)\cdot(z+0.6)}$

b) $X(z)=\dfrac{2+0.8\cdot z^{-1}+0.5\cdot z^{-2}+0.3\cdot z^{-3}}{1+0.8\cdot z^{-1}+0.2\cdot z^{-2}}$

c) $X(z)=\dfrac{3\cdot z^{2}+2\cdot z-10}{z^{3}-5\cdot z^{2}+8z-4}$ with $p_1=1$

Check the initial- and final-value theorems.

a)

$$X(z)=\frac{z(z+2)}{(z-0.2)(z+0.6)}=\underbrace{\frac{1+2\cdot z^{-1}}{(1-0.2\cdot z^{-1})(1+0.6\cdot z^{-1})}}_{\substack{M=1\\ N=2}}$$

poles: $p_1=0.2$; $p_2=-0.6$

$$X(z)=\frac{A_1}{(1-0.2\cdot z^{-1})}+\frac{A_2}{(1+0.6\cdot z^{-1})}$$

$$A_1=\left.\frac{1+2\cdot z^{-1}}{1+0.6\cdot z^{-1}}\right|_{z=0.2}=\frac{1+2\cdot 5}{1+0.6\cdot 5}=\frac{11}{4}=2.75$$

$$x[n]=2.75\cdot(0.2)^n\cdot u[n]-1.75\cdot(-0.6)^n\cdot u[n]$$

$A_2=-1.75$

$$\left.\begin{aligned}x[0]&=2.75-1.75=1\\ X(\infty)&=1\end{aligned}\right\}\checkmark$$

$$\left.\begin{aligned}x(\infty)&=0\\ \lim_{z\to 1}(z-1)\cdot X(z)&=0\end{aligned}\right\}\checkmark$$

b) $M=3$, $N=2$

polynomial division: must be reduced: $\tilde{M}<N$

$$(2+0.8\cdot z^{-1}+0.5\cdot z^{-2}+0.3\cdot z^{-3}):(1+0.8\cdot z^{-1}+0.2\cdot z^{-2})=1.5\cdot z^{-1}-3.5$$

$$\underline{1.5\, z^{-1}+1.2\cdot z^{-2}+0.3\cdot z^{-3}}$$

$$2-0.7\, z^{-1}-0.7\cdot z^{-2}$$

$$\underline{-3.5-2.8\cdot z^{-1}-0.7\cdot z^{-2}}$$

$$5.5+2.1\, z^{-1} \;-\; \tilde{M}$$

$$\underbrace{\frac{5.5+2.1\cdot z^{-1}}{1+0.8\cdot z^{-1}+0.2\, z^{-2}}}_{\text{proper fraction}}+1.5\cdot z^{-1}-3.5$$

residial polyn.

$$x[n]=-3.5\cdot\delta[n]+1.5\cdot\delta[n-1]+\text{Inv. ZT of }\{\text{prop fraction}\}$$

$$p_{1,2}=-0.4\pm j\cdot 0.2$$

$A_{1,2} = 2.75 \pm j \cdot 0.25$

c) $$X(z) = \frac{3z^{-1} + 2z^{-2} - 10 \cdot z^{-3}}{1 - 5z^{-1} + 8z^{-2} - 4z^{-3}}$$

$$-10z^{-3} + 2z^{-2} + 3z^{-1} \; : \; -4z^{-3} + 8z^{-2} - 5z^{-1} + 1 = 2.5$$
$$-10z^{-3} + 20z^{-2} - 12.5z^{-1} + 2.5$$
$$0 \quad -18z^{-2} + 15.5z^{-1} - 2.5$$

$$X(z) = \frac{-18z^{-2} + 15.5z^{-1} - 2.5}{4z^{-3} + 8z^{-2} - 5z^{-1} + 1} + 2.5 \qquad p = 1$$

$$z^{-1} \hookrightarrow (z^3 - 5z^2 + 8z + 4) : (z - 1) = z^2 - 4z + 4$$
$$z^3 - z^2$$
$$0 - 4z^2 + 8z + 4$$
$$-4z^2 + 4z$$
$$0 \quad +4z + 4$$
$$+4z - 4$$
$$0$$

$p_{2,3} = 2$

$$X(z) = 2.5 + \frac{A}{(1 - z^{-1})} + \frac{B}{(1 - 2z^{-1})^2} + \frac{C}{(1 - 2 \cdot z^{-1})}$$

$$\frac{-18z^{-2} + 15.5z^{-1} - 2.5}{\cancel{(1 - z^{-1})}\,\cancel{(1 - 2z^{-1})^2}} = A(1 - 2z^{-1})^2 + B(1 - z^{-1}) + C(1 - z^{-1})(1 - 2z^{-1})$$

$z = 1$: $\quad -18 + 15.5 - 2.5 = A \qquad A = -5$

$z = 2$: $\quad \frac{3}{4} = B\frac{1}{2} \qquad B = 1.5$

$z = 3$: $\quad \frac{2}{3} = A\frac{1}{9} + B\frac{3}{3} + C\frac{2}{9} \iff C = 1$

$$X(z) = 2.5 - \frac{5}{(1 - z^{-1})} + \frac{1.5}{(1 - 2z^{-1})^2} + \frac{1}{(1 - 2z^{-1})}$$

$$x[n] = 2.5\,\delta[n] + \left(-5 + 2^n \cdot 1.5(n+1) + 2^n\right) u[n]$$
$$= 2.5\,\delta[n] + \left(-5 + 2^n \cdot n \cdot 1.5 + 2^n \underset{\cdot 2.5}{\cancel{\cdot 1.5}} + \cancel{2^n}\right) u[n]$$

5-3

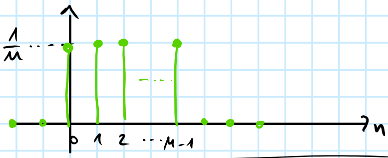
Freitag, 23. Dezember 2016 08:58

Problem 1 The impulse response of a moving average filter is given by

$$h[n] = \sum_{i=0}^{M-1} \frac{1}{M} \cdot \delta[n-i].$$

Compute the transfer function and the pole-zero location for $M > 1$. Is the system stable?

$h[n]$ M samples length $= M$



$$h[n] = \frac{1}{M} \sum_{i=0}^{M-1} \delta[n-i] \circ\!\!-\!\!\bullet \boxed{H(z) = \frac{1}{M} \sum_{i=0}^{M-1} z^{-i}} \quad (z^{-1})^i$$

$$H(z) = \frac{1}{M} \left(1 + z^{-1} + z^{-2} + \ldots + z^{-(M-1)} \right)$$

$(M-1)$ poles at $z = 0$

? zeros

$$H(z) = \frac{1}{M} \cdot \frac{z^{-M} - 1}{z^{-1} - 1} \cdot \frac{z^M}{z^M} \cdot \frac{1 - z^M}{z^M \cdot (z^{-1} - 1)}$$

$$= \frac{1}{M} \frac{1 - z^M}{z^{M-1}(1-z)}$$

poles: $p = 0$ with multiplicity $(M-1)$

$p = 1$

zeros: $z = 1$

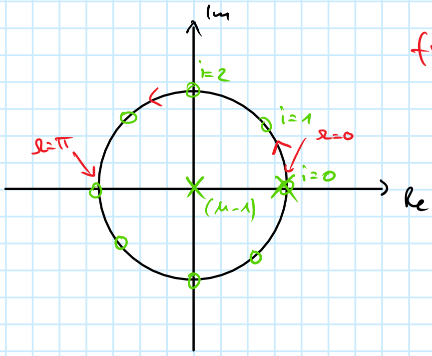
$(1 - z^M) = 0 \Leftrightarrow 1 = z^M$

M roots: $z(1)^{1/M}$

$$z = (1)^{1/M} = \left(1 \cdot e^{j2\pi}\right)^{1/M} = 1 \cdot e^{j\frac{2\pi}{M}}$$

$$z_i = e^{j\frac{2\pi}{M} \cdot i} \qquad 0 \le i \le M-1$$

$M = 8$



freq. resp.

$z = e^{j\Omega}$ $\Omega = 0 \ldots \pi$

$$H(z) = \frac{1}{M} \cdot \frac{\cancel{(z-1)}(z-z_1)(z-z_2)\ldots(z-z_{M\ldots})}{z^{M-1} \cdot \cancel{(z-1)}}$$

$M-1$ zeros

$M-1$ poles //

Problem 2 A causal, finite-dimensional, linear, time-invariant IIR filter is characterized by the constant coefficient difference equation

$$y[n] = x[n-1] - 1.2 \cdot x[n-2] + x[n-3] + 1.3 \cdot y[n-1] - 1.04 \cdot y[n-2] + 0.222 \cdot y[n-3].$$

Determine the transfer function and the pole-zero plot. Is the system stable?

Time shifting	$x[n-n_0]$	$z^{-n_0} \cdot X(z)$	R_x, except possibly the point $z = 0$ or $z = \infty$

$$Y(z) = X(z) \cdot \left(z^{-1} - 1.2 \cdot z^{-2} + z^{-3} \right) + Y(z) \cdot \left(1.3 z^{-1} - 1.04 \cdot z^{-2} + 0.222 \cdot z^{-3} \right)$$

$$\frac{Y(z)}{} = \frac{z^{-1} - 1.2 \cdot z^{-2} + z^{-3}}{} = \frac{z^2 - 1.2 \cdot z + 1}{}$$

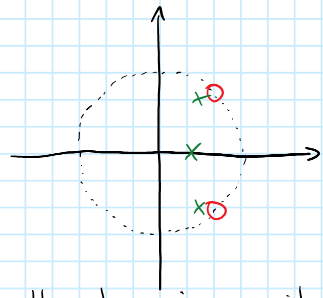
$$\frac{Y(z)}{X(z)} = \frac{z^{-1} - 1.2 \cdot z^{-2} + z^{-3}}{-1.3z^{-1} + 1.04\, z^{-2} - 0.222 \cdot z^{-3} + 1} = \frac{z^2 - 1.2 \cdot z + 1}{z^3 - 1.3 z^2 + 1.04 z - 0.222}$$

zeros: $z_{1,2} = 0.6 \pm \sqrt{0.6^2 - 1}$

$0.6 \pm 0.8j$

poles: $p_1 = \frac{3}{10}$

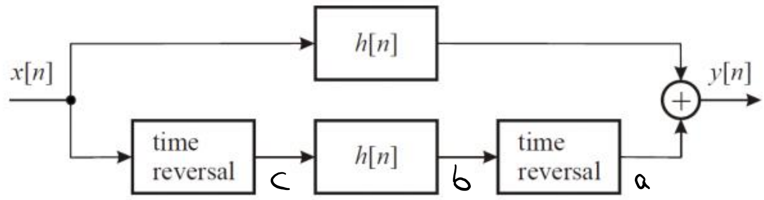
$p_{2/3} = 0.5 - 0.7j$



all poles in unit circle

$\rightarrow$ stable

Problem 3 Let a causal LTI discrete-time system be characterized by a real impulse response $h[n]$ with the DTFT $H(e^{j\Omega})$. Consider the given system with a complex finite-length sequence $x[n]$. Determine the frequency response $G(e^{j\Omega})$ of the overall system and characterize the system.



$C = X(e^{-j\Omega})$

$B = X(e^{-j\Omega}) \cdot H(e^{j\Omega})$

$A = X(e^{j\Omega}) \cdot H(e^{-j\Omega})$

$Y = X(e^{j\Omega}) \cdot H(e^{-j\Omega}) + X(e^{j\Omega}) \cdot H(e^{j\Omega})$

$= X(e^{j\Omega}) \left(H(e^{j\Omega}) + H^*(e^{j\Omega}) \right)$

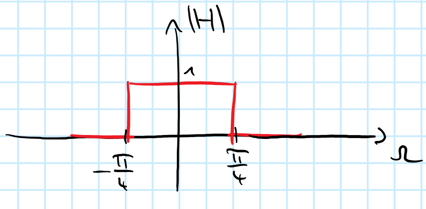
$= X(e^{j\Omega}) \cdot 2 \cdot \mathrm{Re}\{H(e^{j\Omega})\}$

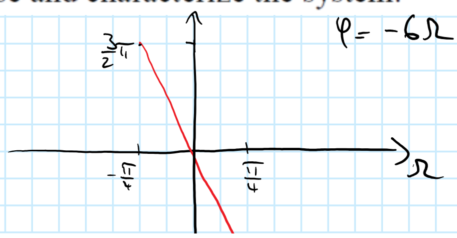
$G = \frac{Y}{X} = \underline{2 \cdot \mathrm{Re}\{H(e^{j\Omega})\}}$ real, zero-phase

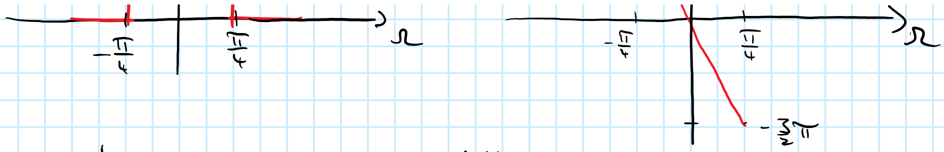
Problem 4 The frequency response of an LTI discrete-time system is given:

$$H(e^{j\Omega}) = \begin{cases} e^{-j6\Omega}, & 0 \le |\Omega| \le \frac{\pi}{4} \\ 0, & \frac{\pi}{4} < |\Omega| \le \pi \end{cases}$$

a) Plot the magnitude and phase response and characterize the system.







rectangular low-pass filter
„true" linear phase - system

b) Determine the impulse response.

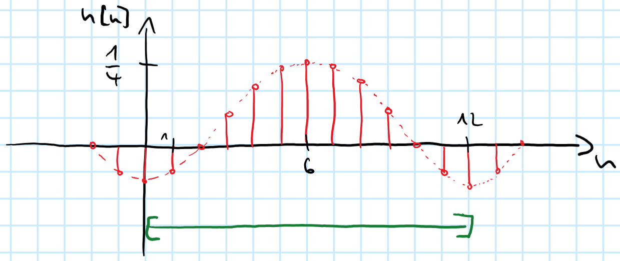
$h[n] = ?$

$e^{jn_0\Omega} = e^{-j6\Omega} \Rightarrow n_0 = 6$

$\text{rect}\,\frac{\Omega}{2\cdot\Omega_c} \bullet\!\!-\!\!\circ \; \frac{\Omega_c}{\pi} \cdot \frac{\sin(n\Omega_c)}{n\Omega_c}$!

$\Omega_c = \frac{\pi}{4}$

$h[n] = \frac{1}{4} \; \frac{\sin[(n-6)\cdot\Omega_c]}{(n-6)\cdot\Omega_c}$



$(n-6)\cdot\Omega_c = \pm\pi$

$n-6 = \pm\cancel{\pi}\,\frac{4}{\cancel{\pi}}$

$n = 6 \pm 4$

$\Rightarrow n = 10, \; n = 2$

c) Is the system realizable?

$h[n]$ infinite duration or length $\rightarrow$ not realizable

d) Truncate the impulse response to $0 \leq n \leq 12$ and discuss the realizability and the truncation effect.

13 samples $\Longleftrightarrow$ length = 13

$h_T[n] = h[n] \cdot \underbrace{\text{rect}\left[\frac{n-6}{13}\right]}_{\text{rect}_{13}[n-6]}$ ch. 4, F7

$H_T(e^{j\Omega}) = \underset{\text{previous response}}{H(e^{j\Omega})} * (\text{si function})$

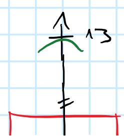
---

without time shifting

$\tilde{h}_T[n] = \tilde{h}[n] \cdot \text{rect}_{13}[n]$

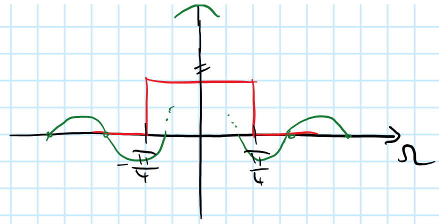
$= \frac{1}{4} \, \frac{\sin(n\cdot\Omega_c)}{n\cdot\Omega_c} \cdot \text{rect}_{13}[n]$

$H_T(e^{j\Omega}) = \tilde{H}_T(e^{j\Omega}) \cdot e^{-j6\Omega}$



$H_T(e^{j\Omega}) = \tilde{H}_T(e^{j\Omega}) \cdot e^{-j6\Omega}$

$\tilde{H}_T(e^{j\Omega}) = \mathrm{rect}\left(\frac{\Omega}{2\cdot\Omega_c}\right) * \frac{\sin(6.5\cdot\Omega)}{\sin\left(\frac{\Omega}{2}\right)}$



$\sin(x) = x + \frac{1}{3}x^3 + \dots$

max of — at $\Omega = 0$ $\left(\frac{6.5}{0.5} = 13\right)$

$6.5\,\Omega = \pi \qquad \Omega = \frac{\pi}{6.5} = \frac{2\pi}{13}$

Problem 5 Consider the following causal IIR transfer function:

$$H(z) = \frac{3\cdot z^3 + 2\cdot z^2 + 5}{(0.5\cdot z + 1)\cdot(z^2 + z + 0.6)}$$

a) Is $H(z)$ a stable transfer function?

poles: $p_1 = -2 \qquad p_{2,3} = -0.5 \pm j\,0.59$

$|p_1| > 1 \rightarrow$ not stable

b) If it is not stable, find a stable transfer function $G(z)$ such that $\left|G\left(e^{j\Omega}\right)\right| = \left|H\left(e^{j\Omega}\right)\right|$.

$G(z) = H(z) \cdot H_A(z)$

$H_A(z) = \frac{0.5\cdot z + 1}{D(z)}$

$|G(e^{j\Omega})| \overset{!}{=} |H(e^{j\Omega})| \overset{!}{=} |H(e^{j\Omega})| \cdot |H_A(e^{j\Omega})|$

$\Rightarrow |H_A(e^{j\Omega})| \overset{!}{=} \left|\frac{0.5\,e^{j\Omega}+1}{D(e^{j\Omega})}\right| \overset{!}{=} |\pm 1|$

$|\pm D(z)| \overset{!}{=} |0.5z + 1|$

$|e^{j\Omega} - p_4| = |0.5 e^{j\Omega} + 1|$

$|\pm(\cos(\Omega) + j\cdot\sin(\Omega) - p_4)| = |0.5\cos(\Omega) + 0.5j\sin(\Omega) + 1|$

① $\sqrt{(\pm\cos\Omega + p_4)^2 - \sin^2\Omega} = \sqrt{(0.5\cdot\cos\Omega + 1)^2 + 0.25\sin^2\Omega}$

$\Omega = \pi \qquad (1 + p_4) = -0.5$

$p_4 = -1.5$ ↯ unstable

② $\sqrt{(\cos\Omega - p_4)^2 + \sin^2\Omega} = -0.5$

$\Omega = \pi \qquad -1 - p_4 = -0.5 \Rightarrow \underline{p_4 = -0.5}$

$\Rightarrow H_A(z) = \frac{0.5z + 1}{z + 0.5} \Rightarrow G(z) = H(z)\cdot H_A(z)$

$$\Rightarrow \quad H_A(z) = \frac{\quad}{z+0.5} \quad \Rightarrow \quad G(z) = H(z) \cdot H_A(z)$$

c) Is there any other transfer function having the same magnitude response as that of *H*(*z*)?

infinitely many → $H(z)$ can have more poles and zeros

6-1

Dienstag, 10. Januar 2017 12:24

Problem 1 Determine the location of the notch frequency of the FIR notch filter given by

$H(z) = 1 + \sqrt{2} \cdot z^{-1} + z^{-2}$. $\Omega_0 = ?$

Compute the response to the input signal $x[n] = \cos(n \cdot \pi / 4)$.

Ch. 6, F 16 → Notch filter → compare coefficients

$-2 \cdot \cos(\Omega_0) = \sqrt{2}$

$\cos \Omega_0 = -\frac{1}{\sqrt{2}} \Rightarrow \underline{\Omega_0 = 135° = \frac{3}{4}\pi}$

1) $x[n] \circ\!\!-\!\!\bullet X(z) \Rightarrow Y(z) = H(z) \cdot X(z) \bullet\!\!-\!\!\circ y[n]$

2) $x[n] \circ\!\!-\!\!\circ X(e^{j\Omega}) \Rightarrow \ldots$

3) $x[n]$ as eigenfunction 5.3 F.22

$x[n] = \cos\left(\frac{n\pi}{4}\right) = \frac{1}{2} e^{j\frac{n\pi}{4}} + \frac{1}{2} e^{-j\frac{n\pi}{4}}$

$H(e^{j\Omega}) = 1 + \sqrt{2} \cdot e^{-j\Omega} + e^{-j2\Omega}$

$x_1[n] = \frac{1}{2} e^{jn\frac{\pi}{4}} \Rightarrow \Omega = \frac{\pi}{4}$

$y_1[n] = H(e^{j\Omega}) \cdot x_1[n]$

$H(e^{j\frac{\pi}{4}}) = 1 + \underbrace{\sqrt{2}\, e^{-j\frac{\pi}{4}}}_{(1-j)} + \underbrace{e^{-j\frac{\pi}{2}}}_{-j}$

$= 2 - 2j = 2 \cdot \sqrt{2} \cdot e^{-j45°}$ $\frac{\pi}{4}$

$y_1[n] = \sqrt{2} \cdot e^{j(n-1)\frac{\pi}{4}}$

$y_2[n] = \underbrace{H(e^{-j\frac{\pi}{4}})}_{H^*(e^{j\frac{\pi}{4}})} \cdot \overbrace{\frac{1}{2} e^{-j\frac{n\pi}{4}}}^{x_2[n]}$

$H^*(e^{j\frac{\pi}{4}}) = 2 \cdot \sqrt{2} \cdot e^{j\frac{\pi}{4}} = y_1^*[n]$

$y[n] = \underline{2 \cdot \mathrm{Re}\{y_1[n]\} = 2 \cdot \sqrt{2} \cdot \cos\left([n-1]\frac{\pi}{4}\right)}$

Problem 2 Design a first-order highpass filter with a normalized 3-dB cutoff frequency at 0.25 radian/sample.

Ω

$H_{HP,1}(e^{j\Omega}) = \frac{1+\alpha}{2} \cdot \frac{e^{j\Omega}-1}{e^{j\Omega}-\alpha}$ Ch. 6 p.20 $\Omega = 2\pi f \frac{1}{f_s} = \frac{\omega}{f_s}$

$\|$ 3dB cutoff

$|H(e^{j\Omega})| = \frac{1}{\sqrt{2}}$

$H(\Omega_c) = \frac{1+\alpha}{2} \cdot \frac{(\cos(\Omega_c) - 1 + j\sin(\Omega_c))}{(\cos(\Omega_c) - \alpha + j\sin(\Omega_c))}$

$|H(\Omega_c)| = \frac{(1+\alpha)}{2} \cdot \frac{\sqrt{(\cos(\Omega_c)-1)^2 + \sin^2(\Omega_c)}}{\ldots} = \frac{1}{\sqrt{2}}$

$$|H(\Omega_c)| = \frac{(1+\alpha)}{2} \cdot \frac{\sqrt{(\cos(\Omega_c)-1)^2 + \sin^2(\Omega_c)}}{\sqrt{(\cos(\Omega_c)-\alpha)^2 + \sin^2(\Omega_c)}} = \frac{1}{\sqrt{2}}$$

$$= \frac{(1+\alpha)^2}{2\cancel{4}} \cdot \frac{(\cos(\Omega_c)-1)^2 + \sin^2(\Omega_c)}{(\cos(\Omega_c)-\alpha)^2 + \sin^2(\Omega_c)} = \cancel{\frac{1}{2}}$$

$$\Leftrightarrow \quad \alpha = 0.7767$$

$$H(z) = \frac{1+\alpha}{2} \, \frac{z-1}{z-\alpha} = \frac{0.8884\,(1-z^{-1})}{1-0.7767\cdot z^{-1}}$$

Problem 3 Show that the causal FIR transfer function

$$H(z) = \frac{1}{1+\alpha}\cdot\left(1+\alpha\cdot z^{-1}\right);\ \alpha > 0$$

is a bounded-real (BR) function. $\xrightarrow{!}$

- **Bounded-real (BR) transfer function**

$\left|H\left(e^{j\Omega}\right)\right| \le 1$ with $H(z)$ causal, stable and real coefficients

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$$|H(z)| = \frac{1}{1+\alpha} \cdot \sqrt{(1+\alpha\cdot z^{-1})^2} \qquad \alpha > 0$$

$$(|H(z)|)^2 \le 1 \quad \Rightarrow \quad \frac{1}{(1+\alpha)^2}\cdot(1+\alpha z^{-1})^2 \le 1$$

$$(|H(e^{j\Omega})|)^2 = \frac{1}{(1+\alpha)^2}\cdot(1+\alpha\cdot e^{-j\Omega})^2 \le 1$$

$$\Leftrightarrow \quad \frac{1}{(1+\alpha)^2}\cdot(1+\alpha\cos(\Omega))^2 + (\alpha^2\sin^2(\Omega)) \le 1$$

$$\Leftrightarrow \quad \frac{1+\alpha^2\underbrace{\cancel{\cos^2(\Omega)}+\cancel{\sin^2(\Omega)}}_{1}+2\alpha\cos(\Omega)}{(1+\alpha)^2} \le 1$$

$$\Leftrightarrow \quad \frac{1+\alpha^2+2\alpha\cos(\Omega)}{(1+\alpha)^2} \le 1$$

$$\Leftrightarrow \quad \cancel{1}+\cancel{\alpha^2}+\cancel{2\alpha}\cos(\Omega) \le \cancel{1}+\cancel{\alpha^2}+\cancel{2\alpha}\ 1$$

$$\Leftrightarrow \quad \cos(\Omega) \le 1 \quad \checkmark$$

Problem 4 The frequency response of a causal digital filter is given

$$H(z) = 1-2\cdot z^{-1}+3\cdot z^{-2}-3\cdot z^{-4}+2\cdot z^{-5}-z^{-6}.$$

Determine the magnitude and phase responses.

$$H(e^{j\Omega}) = 1-2\cdot e^{-j\Omega}+3e^{-2j\Omega}-3e^{-4j\Omega}+2e^{-j\Omega 5}-e^{-j\Omega 6}$$

$$= e^{-j\Omega 3}\left(\underline{e^{j\Omega 3}}-\underline{2e^{j\Omega 2}}+\underline{3e^{j\Omega}}-\underline{3e^{-j\Omega}}+\underline{2e^{-j\Omega 2}}-\underline{e^{-j\Omega 3}}\right)$$

$$= e^{-j\Omega 3}\left(2j\sin(3\Omega)+6j\sin(\Omega)-4j\sin(2\Omega)\right)$$

$$= e^{-j3\Omega}\cdot \underbrace{j}_{e^{j\frac{\pi}{2}}}\underbrace{(2\sin(3\Omega)+6\sin(\Omega)-4\sin(2\Omega))}_{\tilde{H}(\Omega)\ \longrightarrow\ \text{magnitude response}}$$

$$= e^{-j3\Omega+j\frac{\pi}{2}}\cdot\tilde{H}(\Omega)$$

$$\Rightarrow \varphi(\Omega) = 3\Omega+\frac{\pi}{2} \qquad \text{phase response}$$

Problem 5 A causal LTI FIR discrete-time system is characterized by an impulse response

$$h[n] = a_1 \cdot \delta[n] + a_2 \cdot \delta[n-1] + a_3 \cdot \delta[n-2] + a_4 \cdot \delta[n-3] + a_5 \cdot \delta[n-4] + a_6 \cdot \delta[n-5].$$

For what values of the impulse response samples will its frequency response have a constant group delay?

length = 6 → even

$N = 5$ → odd   Type II or Type IV

$$H(z) = a_1 + a_2 z^{-1} + a_3 \cdot z^{-2} + a_4 z^{-3} + a_5 z^{-4} + a_6 \cdot z^{-5}$$

$$H(e^{j\Omega}) = a_1 + a_2 e^{-j\Omega} + a_3 e^{-2j\Omega} + a_4 e^{-3j\Omega} + a_5 e^{-j4\Omega} + a_6 e^{-j\Omega 5}$$

$$= e^{-j\frac{5}{2}\Omega}\underbrace{\left(a_1 e^{j\frac{5}{2}\Omega} + a_2 e^{j\frac{3}{2}\Omega} + a_3 e^{j\frac{1}{2}\Omega} + a_4 e^{-j\frac{1}{2}\Omega} + a_5 e^{-j\frac{3}{2}\Omega} + a_6 e^{-j\frac{5}{2}\Omega}\right)}_{\text{real and constant group delay if:}}$$

Type II (cos): $a_1 = a_6$, $a_2 = a_5$, $a_3 = a_4$ ⇒ $\varphi(\Omega) = -\frac{5}{2}\Omega$

Type IV (sin): $a_1 = -a_6$, $a_2 = -a_5$, $a_3 = -a_4$ ⇒ $\varphi(\Omega) = -\frac{5}{2}\Omega + \frac{\pi}{2}$

$$\tau_g = -\frac{\delta \varphi}{\delta \Omega} = \frac{5}{2} = \frac{N}{2} \quad \text{const.}$$

Problem 6 A type-II real-coefficient FIR filter (symmetric impulse response, N odd) with the transfer function $H(z)$ has the following zeros: $z_1 = 1$, $z_2 = -1$, $z_3 = 0.5$, $z_4 = 0.8 + \mathrm{j}$.

a) Determine the locations of the remaining zeros of $H(z)$ having the lowest order.

$z_1 = 1$, $z_2 = -1$, $z_3 = 0.5$, $z_4 = 0.8 + j$

| odd | Real zero

$z_6 = 1$ $\frac{1}{z_3} = 2 = z_5$

$z_7 = \frac{1}{z_4} = 0.488 - 0.61j$ $z_9 = z_4^* = 0.8 - j$

$z_8 = \frac{1}{z_4^*} = 0.488 + 0.61j$

- **Type II:** Symmetric impulse response, N odd

$$\tilde{H}(\Omega) = 2 \cdot \sum_{n=1}^{(N+1)/2} h\left[\frac{N+1}{2} - n\right] \cdot \cos\left(\left(n - \frac{1}{2}\right) \cdot \Omega\right)$$

- **Requirements for zeros**
  - Real zero ($z \neq \pm 1$): $z = r$ and $z = 1/r$ (2 zeros)
  - Zero on unit circle ($z \neq \pm 1$): $z = e^{j\phi}$ and $z = e^{-j\phi}$ (2 zeros)
  - Complex zero ($|z| \neq 1$, not real): $z = r \cdot e^{\pm j\phi}$ and $z = \frac{1}{r} \cdot e^{\pm j\phi}$ (4 zeros)
- **Additional zero requirements**
  - Filter type I: None or even number at $z = \pm 1$.
  - Filter type II: Odd number at $z = -1$. None or even number at $z = 1$. → no highpass
  - Filter type III: Odd number at $z = 1$ and at $z = -1$. → only bandpass
  - Filter type IV: Odd number at $z = 1$. None or even number at $z = -1$. → no lowpass

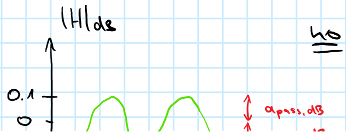
b) Determine the transfer function of the filter.

$$H(z) = \prod_{i=1}^{9} (1 - z_i \cdot z^{-1})$$

Problem 7 Estimate the order and the group delay of a linear-phase lowpass FIR filter with the following specifications: passband edge $f_{pass} = 1.8$ kHz, stopband edge $f_{stop} = 2$ kHz, peak passband ripple $a_{pass} = 0.1$ dB, minimum stopband attenuation $a_{stop} = 35$ dB, and sampling rate $f_s = 12$ kHz.

$f_{pass} = 1.8\,kHz$ $f_{stop} = 2\,kHz$ $a_{pass} = 0.1\,dB$ $a_{stop} = 35\,dB$

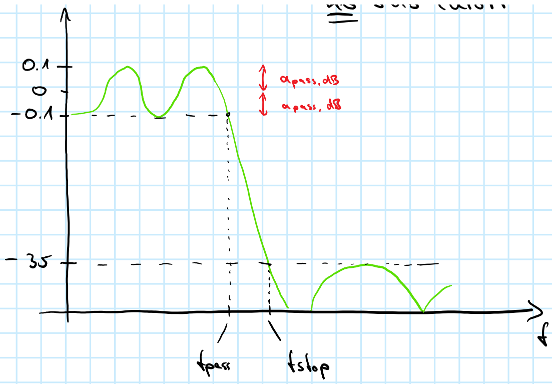
$f_s = 12\,kHz$



no 3dB cutoff

$$|H|_{dB} = -0.1\,dB = 20\,dB \cdot \log_{10}(1 - \delta_1)$$

$$1 - \delta_1 = 10^{-(0.1/20)} \Rightarrow \delta_1 = 1 - 10^{-(0.1/20)}$$



$|H|_{dB} = -0.1\,dB = 20\,dB \cdot \log_{10}(1-\delta_1)$

$1-\delta_1 = 10^{-(0.1/20)} \Rightarrow \delta_1 = 1-10^{-(0.1/20)}$

$= 0.01145$

$|H|_{dB,max} = 20\,dB \cdot \log_{10}(1+\delta_1) \approx 0.1\,dB$

← take minimum value for exact values

$|H|_{dB} = -35\,dB = 20\,dB \cdot \log_{10}(\delta_2)$

$\delta_2 = 0.01778$

$$\frac{\Omega_2-\Omega_1}{2\pi} \overset{\Omega = 2\pi \frac{f}{f_s}}{=} \frac{f_{stop}}{f_s} - \frac{f_{pass}}{f_s} = \frac{2}{12} - \frac{1.8}{12} \rightarrow \text{insert in Kaiser / Bellanger formula}$$

KAISER: $N = 98.3 \rightarrow N = 99$

BELLANGER: $N = 107.7 \rightarrow N = 108$

~ approx. same order

FIR-Filter: $\tau_g = \frac{N}{2} \approx 50$

$t_g = \frac{N}{2} \cdot T_s = \frac{50}{f_s} \approx 4.2\,ms$

IIR-Filter: $N \approx 14$ (order much lower)

Problem 8 Estimate the order of a linear-phase bandpass FIR filter with the following specifications: passband edges $f_{pass,1} = 0.35$ kHz and $f_{pass,2} = 1$ kHz, stopband edges $f_{stop,1} = 0.3$ kHz and $f_{stop,2} = 1.1$ kHz, passband ripple $\delta_{pass} = 0.002$, stopband ripple $\delta_{stop} = 0.01$, and sampling rate $f_s = 10$ kHz.

$f_{pass_{1/2}} = 0.35\,kHz / 1\,kHz$ $f_{stop_{1,2}} = 0.3\,kHz / 1.1\,kHz$ $\delta_{pass} = 0.002$ $\delta_{stop} = 0.01$

$f_s = 10\,kHz$

- KAISER's formula

$$N \approx \frac{-20 \cdot \log_{10}\left(\sqrt{\delta_1 \cdot \delta_2}\right) - 13}{14.6 \cdot (\Omega_2 - \Omega_1)/(2\pi)}$$

- BELLANGER's formula

$$N \approx \frac{-2 \cdot \log_{10}(10 \cdot \delta_1 \cdot \delta_2)}{3 \cdot (\Omega_2 - \Omega_1)/(2\pi)} - 1$$

- The formulas can also be used for highpass, bandpass, and bandstop filters. Dominant is the smallest transition band.

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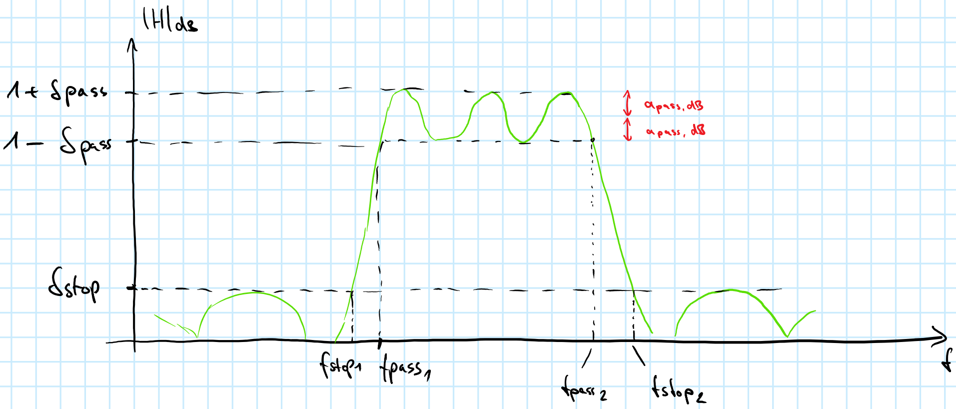


$$\frac{\Omega_2-\Omega_1}{2\pi} \overset{\Omega = 2\pi \frac{f}{f_s}}{=} \frac{f_{pass1}}{f_s} - \frac{f_{stop1}}{f_s} = \frac{0.35}{10} - \frac{0.3}{10} = \frac{1}{200} = \Omega \rightarrow \text{insert in Kaiser / Bellanger formula}$$

N Kaiser ≈ 466

N Bellanger ≈ 492

Problem 1 Determine the root-locus curve for the continuous-time feedback system with the system functions

$$H(s)=\frac{1}{s+1} \quad \text{and} \quad G(s)=\frac{1}{s+2}$$

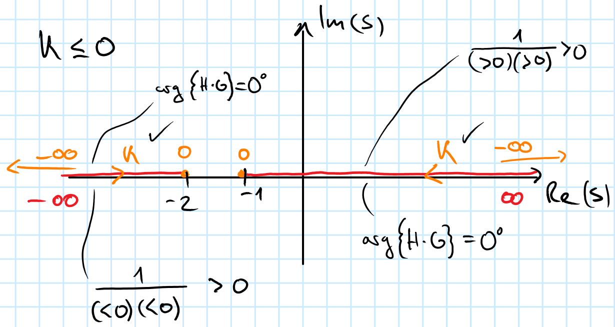
for the forward and feedback path, respectively. Use the real adjustable gain parameter K. Determine K for a stable feedback system.

$$Q(s)=\frac{\ldots}{1+K\cdot G(s)\cdot H(s)} \qquad 1+K\cdot G(s)\cdot H(s)=0$$

$$G(s)\cdot H(s)=-\frac{1}{K}$$

$$K\to\pm\infty \qquad G(s)\cdot H(s)=\frac{1}{(s+1)(s+2)}\overset{!}{=}0 \qquad |s|\to\infty,\ s\in\mathbb{C}$$

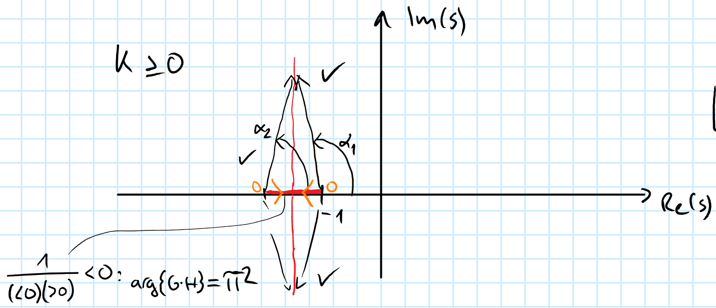
$$K\to 0 \qquad |G(s)\cdot H(s)|\to\infty \qquad p_1=-1;\ p_2=-2$$



Stability:

$$G(0)\cdot H(0)=\frac{1}{2}=-\frac{1}{K} \qquad K=-2$$

$$\underline{K>-2}$$



$$G(s)\cdot H(s)=\frac{1}{D_1\angle\alpha_1\cdot D_2\angle\alpha_2}$$

$$\mathrm{Im}(s)>0:\quad \arg\{G\cdot H\}=-\alpha_1-\alpha_2=-(\alpha_1+\alpha_2)=-\pi$$

Problem 2 Determine the root-locus curve for the discrete-time feedback system with the system functions

$$H(z)=\frac{1}{1-0.5\cdot z^{-1}} \quad \text{and} \quad G(z)=\frac{z^{-1}}{1-0.25\cdot z^{-1}}$$

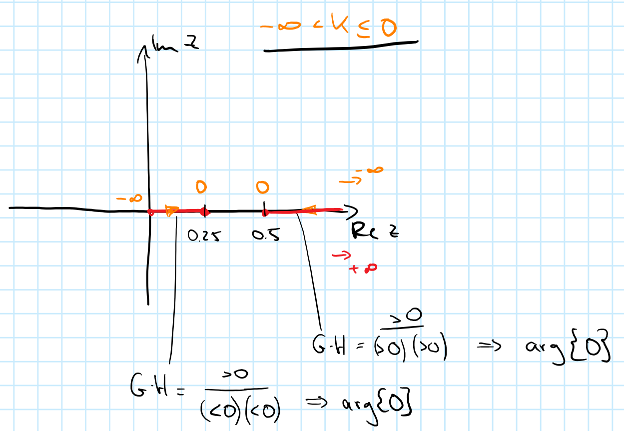
for the forward and feedback path, respectively. Use the real adjustable gain parameter K. Determine K for a stable feedback system.

$$H(z)=\frac{z}{\ldots} \qquad G(z)=\frac{1}{\ldots}$$

Determine K for a stable feedback system.

$$H(z) = \frac{z}{z-\frac{1}{2}} \qquad G(z) = \frac{1}{z-\frac{1}{4}}$$

$$H(z)\cdot G(z) = \frac{z}{(z-\frac{1}{2})(z-\frac{1}{4})}$$



$\underline{K \to 0}$: poles of $G \cdot H$

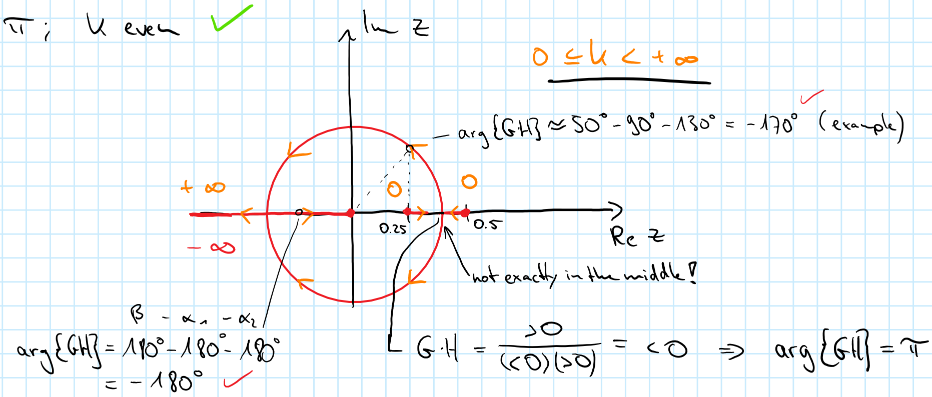
$p_1 = \frac{1}{2} \qquad p_2 = \frac{1}{4}$

$\underline{K \to \pm\infty}$: zeros of $G \cdot H$

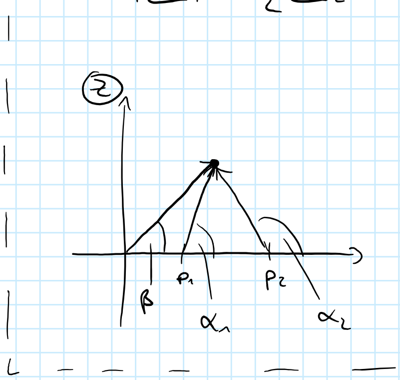
$z = 0 \; ; \quad z \to \pm\infty$

$k > 0$: $\arg\{G \cdot H\} = k \cdot \pi$; k odd

$k < 0$: $\arg\{G \cdot H\} = k \cdot \pi$; k even



$$G \cdot H = \frac{(z-0)}{(z-\frac{1}{2})(z-\frac{1}{4})} = \frac{N \angle \beta}{D_1 \angle \alpha_1 \cdot D_2 \angle \alpha_2}$$



Closed-loop poles

$G(...) \cdot H(...) = -\frac{1}{K}$ with K real, '...' = s or z

- $K \to 0 \Longrightarrow G(...) \cdot H(...) \to -\infty$ ← poles of $G(...) \cdot H(...)$
- $K \to \pm\infty \Longrightarrow G(...) \cdot H(...) \to 0$ ← zeros of $G(...) \cdot H(...)$
- $\arg\{G(...) \cdot H(...)\} = n \cdot \pi$; n: integer
  - $\arg\{G(...) \cdot H(...)\} = k \cdot \pi$; k: odd $\qquad K = \frac{1}{|G(...) \cdot H(...)|} > 0$
  - $\arg\{G(...) \cdot H(...)\} = k \cdot \pi$; k: even $\qquad K = \frac{-1}{|G(...) \cdot H(...)|} < 0$

$z = ?$ stability : $|p_i| < 1$

$$G(1) \cdot H(1) = \frac{1}{(1-\frac{1}{2})(1-\frac{1}{4})} = \frac{1}{\frac{1}{2} \cdot \frac{3}{4}} = \frac{8}{3} \overset{!}{=} -\frac{1}{K}$$

$$K = -\frac{3}{8} \Rightarrow \underline{K > -\frac{3}{8}}$$

$$G(-1) \cdot H(-1) = \frac{-1}{(-\frac{3}{2}) \cdot (-\frac{5}{4})} = -\frac{8}{15} \overset{!}{=} -\frac{1}{K}$$

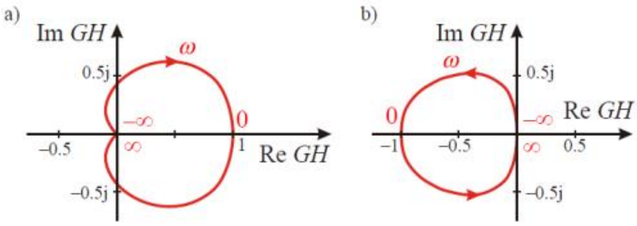
$$K = \frac{15}{8} \Rightarrow \underline{K < \frac{15}{8}}$$

Problem 3 Two continuous-time feedback systems

a) $G(s)\cdot H(s)=\dfrac{2}{(s+1)\cdot(s+2)}$ b) $G(s)\cdot H(s)=\dfrac{2(s+1)}{(s-1)\cdot(s+2)}$

shall be considered. Their NYQUIST plots $G(j\omega)\cdot H(j\omega)$ are given below. Derive stability requirements for the real adjustable gain parameter K.

NYQUIST plot



real $h(t)$ ⊶ $H(j\omega) = H(-j\omega)^*$

$H(-j\omega) = H(j\omega)^*$

Measured: $0 \le \omega \le \infty$

compute for $\omega < 0$

a) $p_1 = -1$; $p_2 = -2$ $\rightarrow$ $N=0$, $M=0$

- **Requirement for stability:** The number of counterclockwise encirclements of the point $-1/K$ by the NYQUIST plot of $H(j\omega)\cdot G(j\omega)$ must equal the number of right-half-plane poles of $H(s)\cdot G(s)$.

$\Rightarrow$ no ccw encirclements !

$-\frac{1}{K} < 0$ or $-\frac{1}{K} > 1$ (outside the NYQUIST plot)

$\downarrow$

$K > 0$ or $-1 < K < 0$

$\boxed{K > -1}$ different way to get K (alternative to root-locus curve)

b) $p_1 = 1$ $p_2 = -2$ $z = -1$ $\Rightarrow$ $N = 0$, $M = 1$

$\Rightarrow$ 1 cw encirclement

$-\frac{1}{K} > -1$ $-\frac{1}{K} < 0$

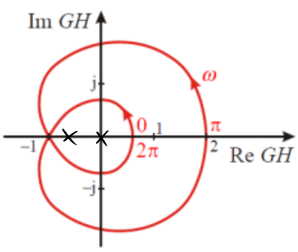
$\downarrow$ $\downarrow$

$K > 1$ $\frac{1}{K} > 0 \Rightarrow K > 0$ $K > 1$

Problem 4 The discrete-time feedback system

$$G(z)\cdot H(z)=\frac{z^{-2}}{1+0.5\cdot z^{-1}}$$

shall be considered. The NYQUIST plot $G\left(e^{j\Omega}\right)\cdot H\left(e^{j\Omega}\right)$ is shown on the right. Derive stability requirements for the real adjustable gain parameter K.



Requirement for stability: The number of counterclockwise encirclements of the point $-1/K$ by the NYQUIST plot of $H\left(e^{j\Omega}\right)\cdot G\left(e^{j\Omega}\right)$ as Ω varies from 0 to 2π must equal the number of poles of $H(z)\cdot G(z)$ outside the unit circle.

$$G(z)\cdot H(z) = \frac{1}{z^2 + 0.5z} \qquad z = 0 \qquad p_1 = 0$$

$$p_2 = -\frac{1}{2}$$

$M = 0 \quad N = 2 \qquad M - N = -2$ ccw

$$-\frac{1}{k} < -1 \Rightarrow k < 1$$

$$-\frac{1}{k} > 2 \Rightarrow k > -\frac{1}{2}$$

$$-\frac{1}{2} < k < 1$$