6 Hypothesis Testing

The objective of statistical hypothesis tests, is to check whether a hypothesis is consistent with data from a random sample.

(6.1) Notation. A hypothesis about a parameter θ of a random distribution is called *null hypothesis* H_0.

If H_0 completely determines the distribution (e.g. $H_0 : \theta = 0$), H_0 is called a *simple hypothesis*, otherwise (e.g. $H_0 : \theta > 0$), H_0 is called a *composite hypothesis*.

A *test of a null hypothesis* is always based on a random sample $X_1, \ldots, X_n$ and the (a priori) definition of a so-called *critical region* $C \subseteq \mathbb{R}^n$. A particular test consists of the observation of n values $x_1, \ldots, x_n$ from the random variables $X_1, \ldots, X_n$ and:

- H_0 is *accepted* if $(x_1, \ldots, x_n) \notin C$.
- H_0 is *rejected* if $(x_1, \ldots, x_n) \in C$.

In a test, two different types of errors may occur:

- Errors of *type I*: H_0 is rejected (i. e. $(x_1, \ldots, x_n) \in C$), although H_0 is correct.
- Errors of *type II*: H_0 is accepted (i. e. $(x_1, \ldots, x_n) \notin C$), although H_0 is false.

A test has *significance level* α if the probability of false rejection of H_0 (i.e. of a type I error) is at most α.

6.1 Tests concerning the mean of a normal distribution

Let $X_1, \ldots, X_n$ be a random sample with:

$$X_i \sim \mathcal{N}(\mu, \sigma)$$

6.1.1 Two-sided tests concerning μ if σ is known

For a given constant μ_0, let H_0 denote the null hypothesis:

$$H_0 : \mu = \mu_0$$

As the sample mean $\overline{X}$ is a suitable estimator for μ, the hypothesis should be rejected if the mean value $\overline{x}$ of a set of sampled data $x_1, \ldots, x_n$ differs significantly from μ_0. A critical region is therefore given by

$$C := \{(x_1, \ldots, x_n) \mid |\overline{x} - \mu_0| > c\}$$

for a suitable constant c.

Given a significance level α, c has to be determined, such that:

$$\Pr_{H_0}\big(|\overline{X} - \mu_0| > c\big) := \Pr\big(|\overline{X} - \mu_0| > c \mid \mu = \mu_0\big) = \alpha$$

From the determination of confidence intervals in (5.14), this is equivalent to

$$c \;=\; \frac{\sigma \cdot \Phi^{-1}\big(1-\frac{\alpha}{2}\big)}{\sqrt{n}}$$

and hence:

- H_0 is rejected if: $\quad \frac{\sqrt{n}}{\sigma} \cdot |\overline{x} - \mu_0| \;>\; \Phi^{-1}\big(1-\frac{\alpha}{2}\big)$
- H_0 is accepted if: $\quad \frac{\sqrt{n}}{\sigma} \cdot |\overline{x} - \mu_0| \;\leq\; \Phi^{-1}\big(1-\frac{\alpha}{2}\big)$

Starting with a set of sampled data,

$$v \;:=\; \frac{\sqrt{n}}{\sigma} \cdot |\overline{x} - \mu_0|$$

(left hand side of the above inequalities) can be calculated and the maximal value for α can be determined, such that H_0 will be accepted on the basis of the data sampled. This value

$$\alpha_{\overline{x}} \;:=\; 2(1 - \Phi(v))$$

is called the *p-value* of the sample.

While the probability of the occurrence of a type I error is set to α, the probability of the occurrence of a type II error depends on the true value of μ.

(6.2) Definition. The value of this probability (that H_0 is accepted when the true value of the mean is $\mu \neq \mu_0$) is given by the so-called *operating characteristic*:

$$\begin{aligned}
\beta(\mu) &= \Pr(H_0 \text{ is accepted} \mid \text{mean is } \mu) \\
&= \Pr\left(-\Phi^{-1}\Big(1-\frac{\alpha}{2}\Big) \;\leq\; \frac{\overline{X}-\mu_0}{\sigma/\sqrt{n}} \;\leq\; \Phi^{-1}\Big(1-\frac{\alpha}{2}\Big) \;\Big|\; \text{mean is } \mu \right) \\
&= \Pr\left(\frac{\mu_0-\mu}{\sigma/\sqrt{n}} - \Phi^{-1}\Big(1-\frac{\alpha}{2}\Big) \;\leq\; \frac{\overline{X}-\mu}{\sigma/\sqrt{n}} \;\leq\; \frac{\mu_0-\mu}{\sigma/\sqrt{n}} + \Phi^{-1}\Big(1-\frac{\alpha}{2}\Big) \;\Big|\; \text{mean is } \mu \right) \\
&= \Phi\left(\frac{\mu_0-\mu}{\sigma/\sqrt{n}} + \Phi^{-1}\Big(1-\frac{\alpha}{2}\Big) \right) \;-\; \Phi\left(\frac{\mu_0-\mu}{\sigma/\sqrt{n}} - \Phi^{-1}\Big(1-\frac{\alpha}{2}\Big) \right)
\end{aligned}$$

(6.3) Lemma.

(i)

$$\beta(\mu) \;=\; \Phi\left(\frac{\mu-\mu_0}{\sigma/\sqrt{n}} + \Phi^{-1}\Big(1-\frac{\alpha}{2}\Big) \right) \;-\; \Phi\left(\frac{\mu-\mu_0}{\sigma/\sqrt{n}} - \Phi^{-1}\Big(1-\frac{\alpha}{2}\Big) \right)$$

(ii) For small values of α:

$$\beta(\mu) \;\lessapprox\; \Phi\left(-\frac{|\mu_0-\mu|}{\sigma/\sqrt{n}} + \Phi^{-1}\Big(1-\frac{\alpha}{2}\Big) \right)$$

(iii) Assume $\mu \neq \mu_0$ and $\beta \in (0,1)$. Then H_0 will be accepted with a probability $\beta(\mu) < \beta$ if:

$$n \;:=\; \left\lceil \frac{\big(\Phi^{-1}(1-\frac{\alpha}{2}) + \Phi^{-1}(1-\beta)\big)^2\, \sigma^2}{(\mu_0-\mu)^2} \right\rceil$$

Figure 25 shows the operating characteristics for $\alpha = 0.1$, 0.05 and 0.01. The plots show the values of $\beta(\mu)$ with respect to the values of $\frac{|\mu_0 - \mu|}{\sigma/\sqrt{n}}$. The dashed lines correspond to the estimation from Lemma (6.3)(ii).

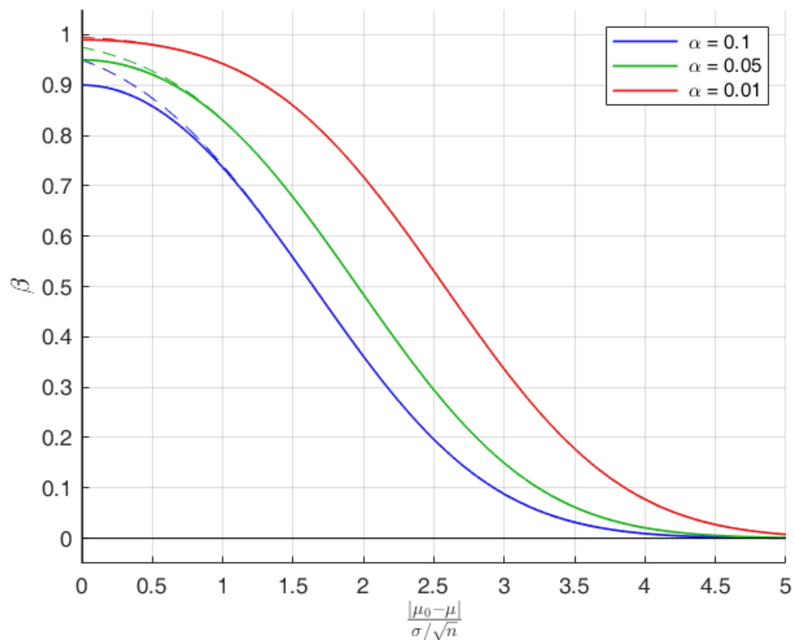

Figure 25: Operating characteristics for $\alpha = 0.1$, 0.05 and 0.01

6.1.2 One-sided tests concerning μ if σ is known

For a given constant μ_0, let H_0 denote the null hypothesis:

$$H_0: \ \mu \leq \mu_0$$

Given a significance level α, c has to be determined, such that:

$$\Pr_{H_0}\big(\overline{X} - \mu_0 > c\big) \ := \ \Pr\big(\overline{X} - \mu_0 > c \mid \mu \leq \mu_0\big) \ \leq \ \Pr\big(\overline{X} - \mu_0 > c \mid \mu = \mu_0\big) \ = \ \alpha$$

From the determination of confidence intervals in (5.14), the last equality is equivalent to

$$c \ = \ \frac{\sigma \cdot \Phi^{-1}(1-\alpha)}{\sqrt{n}}$$

and hence:

- H_0 is rejected if: $\quad \frac{\sqrt{n}}{\sigma} \cdot (\overline{x} - \mu_0) \ > \ \Phi^{-1}(1-\alpha)$
- H_0 is accepted if: $\quad \frac{\sqrt{n}}{\sigma} \cdot (\overline{x} - \mu_0) \ \leq \ \Phi^{-1}(1-\alpha)$

For a set of sampled data define the statistic

$$v \ := \ \frac{\sqrt{n}}{\sigma} \cdot (\overline{x} - \mu_0)$$

(left hand side of the above inequalities) in order to calculate its p-value:

$$\alpha_{\overline{x}} \;:=\; 1 - \Phi(v)$$

The operating characteristic is given by:

$$\begin{aligned}
\beta(\mu) &= \Pr(H_0 \text{ is accepted} \mid \text{mean is } \mu) \\
&= \Pr\left(\frac{\overline{X} - \mu_0}{\sigma/\sqrt{n}} \;\le\; \Phi^{-1}(1-\alpha) \;\middle|\; \text{mean is } \mu \right) \\
&= \Pr\left(\frac{\overline{X} - \mu}{\sigma/\sqrt{n}} \;\le\; \frac{\mu_0 - \mu}{\sigma/\sqrt{n}} \;+\; \Phi^{-1}(1-\alpha) \;\middle|\; \text{mean is } \mu \right) \\
&= \Phi\left(\frac{\mu_0 - \mu}{\sigma/\sqrt{n}} \;+\; \Phi^{-1}(1-\alpha) \right)
\end{aligned}$$

Similarly, for the null hypothesis

$$H_0 : \; \mu \ge \mu_0$$

and a significance level α:

- H_0 is rejected if $\quad \frac{\sqrt{n}}{\sigma} \cdot (\overline{x} - \mu_0) \;<\; \Phi^{-1}(\alpha) \;=\; -\Phi^{-1}(1-\alpha)$
- H_0 is accepted if $\quad \frac{\sqrt{n}}{\sigma} \cdot (\overline{x} - \mu_0) \;\ge\; \Phi^{-1}(\alpha) \;=\; -\Phi^{-1}(1-\alpha)$

For a set of sampled data define the statistic

$$v \;:=\; \frac{\sqrt{n}}{\sigma} \cdot (\overline{x} - \mu_0)$$

(left hand side of the above inequalities) in order to calculate its p-value:

$$\alpha_{\overline{x}} \;:=\; \Phi(v)$$

The operating characteristic is given by

$$\begin{aligned}
\beta(\mu) &= \Pr(H_0 \text{ is accepted} \mid \text{mean is } \mu) \\
&= \Pr\left(\frac{\overline{X} - \mu_0}{\sigma/\sqrt{n}} \;\ge\; \Phi^{-1}(\alpha) \;\middle|\; \text{mean is } \mu \right) \\
&= \Pr\left(\frac{\overline{X} - \mu}{\sigma/\sqrt{n}} \;\ge\; \frac{\mu_0 - \mu}{\sigma/\sqrt{n}} \;+\; \Phi^{-1}(\alpha) \;\middle|\; \text{mean is } \mu \right) \\
&= 1 \;-\; \Phi\left(\frac{\mu_0 - \mu}{\sigma/\sqrt{n}} \;+\; \Phi^{-1}(\alpha) \right)
\end{aligned}$$

6.1.3 Two-sided tests concerning μ if σ is unknown (*two-sided t-tests*)

For a given constant μ_0, let H_0 denote the null hypothesis

$$H_0: \ \mu = \mu_0$$

Given a significance level α, we have:

$$\Pr_{H_0}\left(\frac{|\overline{X}-\mu_0|}{S/\sqrt{n}} > c\right) \ := \ \Pr\left(\frac{|\overline{X}-\mu_0|}{S/\sqrt{n}} > c \,\middle|\, \mu = \mu_0\right) \ = \ \alpha$$

$$\iff \qquad c \ = \ F^{-1}_{t_{n-1}}\Big(1-\frac{\alpha}{2}\Big)$$

where $F_{t_{n-1}}$ denotes the cdf of a t-distribution with $n-1$ degrees of freedom. Hence:

- H_0 is rejected if $\quad \frac{\sqrt{n}}{s} \cdot |\overline{x}-\mu_0| \ > \ F^{-1}_{t_{n-1}}(1-\frac{\alpha}{2})$
- H_0 is accepted if $\quad \frac{\sqrt{n}}{s} \cdot |\overline{x}-\mu_0| \ \leq \ F^{-1}_{t_{n-1}}(1-\frac{\alpha}{2})$

For a set of sampled data define the statistic

$$v \ := \ \frac{\sqrt{n}}{s} \cdot |\overline{x}-\mu_0|$$

(left hand side of the above inequalities) in order to calculate its p-value:

$$\alpha_{\overline{x}} \ := \ 2(1-F_{t_{n-1}}(v))$$

6.1.4 One-sided tests concerning μ if σ is unknown (*one-sided t-tests*)

For a given constant μ_0, let H_0 denote the null hypothesis:

$$H_0: \ \mu \leq \mu_0$$

Given a significance level α, we have

$$\Pr_{H_0}\left(\frac{\overline{X}-\mu_0}{S/\sqrt{n}} > c\right) \ \leq \ \Pr_{H_0}\left(\frac{\overline{X}-\mu}{S/\sqrt{n}} > c\right) \ \overset{(*)}{=} \ \alpha$$

$$(*) \qquad \iff \qquad c \ = \ F^{-1}_{t_{n-1}}(1-\alpha)$$

and therefore:

- H_0 is rejected if $\quad \frac{\sqrt{n}}{s} \cdot (\overline{x}-\mu_0) \ > \ F^{-1}_{t_{n-1}}(1-\alpha)$
- H_0 is accepted if $\quad \frac{\sqrt{n}}{s} \cdot (\overline{x}-\mu_0) \ \leq \ F^{-1}_{t_{n-1}}(1-\alpha)$

For a set of sampled data define the statistic

$$v \ := \ \frac{\sqrt{n}}{s} \cdot (\overline{x}-\mu_0)$$

(left hand side of the above inequalities) in order to calculate its p-value:

$$\alpha_{\overline{x}} \ := \ 1-F_{t_{n-1}}(v)$$

Similarly, for the null hypothesis

$$H_0: \ \mu \geq \mu_0$$

and a significance level α:

- H_0 is rejected if $\frac{\sqrt{n}}{s} \cdot (\overline{x} - \mu_0) \ < \ F^{-1}_{t_{n-1}}(\alpha) \ = \ -F^{-1}_{t_{n-1}}(1-\alpha)$
- H_0 is accepted if $\frac{\sqrt{n}}{s} \cdot (\overline{x} - \mu_0) \ \geq \ F^{-1}_{t_{n-1}}(\alpha) \ = \ -F^{-1}_{t_{n-1}}(1-\alpha)$

For a set of sampled data define the statistic

$$v \ := \ \frac{\sqrt{n}}{s} \cdot (\overline{x} - \mu_0)$$

(left hand side of the above inequalities) in order to calculate its p-value:

$$\alpha_{\overline{x}} \ := \ F_{t_{n-1}}(v)$$

6.2 Analysis of variance (ANOVA)

The objective of ANOVA is to test the null hypothesis

$$H_0 : \; \mu_1 = \mu_2 = \cdots = \mu_m$$

for mean values of m different groups of normally distributed random samples of the same size n and same variance σ^2.

(6.4) Notation. Given independent random variables

$$X_{ij} \; \sim \; \mathcal{N}(\mu_i, \sigma) \qquad (i = 1, \ldots, m, \; j = 1, \ldots, n)$$

with unknown values for $\mu_i (i = 1, \ldots, m)$ and $\sigma > 0$, the sample mean for the i'th group of random variables $X_{i1}, \ldots, X_{in}$ is denoted by

$$\overline{X_i} \; := \; \frac{1}{n} \sum_{j=1}^{n} X_{ij}$$

and the sample mean off all involved random variables $X_{ij} (i = 1, \ldots, m, \; j = 1, \ldots, n)$ is denoted by:

$$\overline{X} \; := \; \frac{1}{m\,n} \sum_{i=1}^{m} \sum_{j=1}^{n} X_{ij}$$

(6.5) Definition. With the notation given in (6.4) the following random variables are defined.

(i) *Within samples sum of squares*:

$$S_w \; := \; \sum_{i=1}^{m} \sum_{j=1}^{n} \left(X_{ij} - \overline{X_i} \right)^2$$

(ii) *Between samples sum of squares*:

$$S_b \; := \; n \sum_{i=1}^{m} \left(\overline{X_i} - \overline{X} \right)^2$$

(6.6) Theorem. With the notation given in (6.4) and definition (6.5) the following holds:

(i)

$$\frac{1}{\sigma^2} S_w \; \sim \; \chi^2_{m(n-1)}$$

(ii) If

$$H_0 : \; \mu_1 = \mu_2 = \cdots = \mu_m$$

is true, then

$$\frac{1}{\sigma^2} S_b \; \sim \; \chi^2_{m-1}$$

Proof:

(i) Looking at the sample variance of the random sample $X_{i1}, \ldots, X_{in}$, we may conclude from Theorem (5.11)

$$\frac{1}{\sigma^2} \sum_{j=1}^{n} \left(X_{ij} - \overline{X_i}\right)^2 \;\sim\; \chi^2_{n-1}$$

and an application of Theorem (4.64) gives:

$$\frac{1}{\sigma^2} S_w \;=\; \sum_{i=1}^{m} \left(\frac{1}{\sigma^2} \sum_{j=1}^{n} \left(X_{ij} - \overline{X_i}\right)^2 \right) \;\sim\; \chi^2_{m(n-1)}$$

(ii) If $\mu := \mu_1 = \mu_2 = \cdots = \mu_m$, then

$$\overline{X_i} \;\sim\; \mathcal{N}(\mu, \sigma/\sqrt{n})$$

for $i = 1, \ldots, m$ and $\overline{X}$ is the sample mean of $\overline{X_1}, \ldots, \overline{X_m}$. Therefore, we may apply again Theorem (5.11):

$$\frac{1}{\sigma^2} S_b \;=\; \frac{n}{\sigma^2} \sum_{i=1}^{m} \left(\overline{X_i} - \overline{X}\right)^2 \;=\; \frac{1}{(\sigma/\sqrt{n})^2} \sum_{i=1}^{m} \left(\overline{X_i} - \overline{X}\right)^2 \;\sim\; \chi^2_{m-1}$$

$\square$

(6.7) Definition. If X_1 and X_2 are independent random variables with

$$X_1 \;\sim\; \chi^2_{n_1} \qquad \text{and} \qquad X_2 \;\sim\; \chi^2_{n_2}$$

then the distribution of

$$X \;:=\; \frac{\frac{1}{n_1} X_1}{\frac{1}{n_2} X_2}$$

is defined to be a F-distribution with parameters n_1 and n_2. Its cdf is denoted by:

$$F_{n_1,n_2}(x) \;=\; F_X(x) \qquad \text{for all } x \geq 0$$

As a direct consequence of Theorem (6.6) above and Theorem (4.62)(i), we have:

(6.8) Corollary.

(i) An unbiased estimator for σ^2 is always given by:

$$\frac{1}{m(n-1)} S_w$$

(ii) If

$$H_0: \; \mu_1 = \mu_2 = \cdots = \mu_m$$

is true, then an unbiased estimator for σ^2 is given by:

$$\frac{1}{m-1} S_b$$

(iii) Define the random variable:

$$T \;:=\; \frac{\frac{1}{m-1}\, S_b}{\frac{1}{m(n-1)}\, S_w}$$

If H_0 is true, then T has a F-distribution with parameters $m-1$ and $m(n-1)$.

(6.9) Lemma. With the notation from Corollary (6.8) the following holds:

$$E(T) > 1 \qquad \text{if } H_0 \text{ is false}$$

<u>*Proof:*</u> Let

$$\overline{\mu} \;:=\; \frac{1}{m}\sum_{i=1}^{m} \mu_i$$

and:

$$Y_i \;:=\; \overline{X_i} \,-\, \mu_i \,+\, \overline{\mu} \;\sim\; \mathcal{N}(\overline{\mu}, \sigma/\sqrt{n}) \qquad (i = 1, \ldots, m)$$

The sample mean $\overline{Y}$ of $Y_1, \ldots, Y_m$ equals $\overline{X}$:

$$\overline{Y} \;=\; \frac{1}{m}\sum_{i=1}^{m} Y_i \;=\; \frac{1}{m}\sum_{i=1}^{m} \left(\overline{X_i} \,-\, \mu_i \,+\, \overline{\mu}\right) \;=\; \frac{1}{m}\sum_{i=1}^{m} \overline{X_i} \,-\, \frac{1}{m}\sum_{i=1}^{m} \mu_i \,+\, \overline{\mu} \;=\; \overline{X}$$

and we have:

$$\begin{aligned}
E\left(\sum_{i=1}^{m} \left(\overline{X_i} - \overline{X}\right)^2\right) \;&=\; E\left(\sum_{i=1}^{m} \left(Y_i \,-\, \overline{Y} \,+\, \mu_i \,-\, \overline{\mu}\right)^2\right) \\
&=\; E\left(\sum_{i=1}^{m} \left(Y_i \,-\, \overline{Y}\right)^2\right) \,+\, \sum_{i=1}^{m} (\mu_i \,-\, \overline{\mu})^2 \,+\, 2\sum_{i=1}^{m} (\mu_i \,-\, \overline{\mu})\, E\left(Y_i \,-\, \overline{Y}\right) \\
&=\; \frac{(m-1)\sigma^2}{n} \,+\, \sum_{i=1}^{m} (\mu_i \,-\, \overline{\mu})^2 \qquad \left(\text{Note: } E\left(Y_i\right) = E\left(\overline{Y}\right)\right)
\end{aligned}$$

$$\Longrightarrow \qquad E(T) \;=\; \frac{E\left(\frac{n}{m-1}\sum_{i=1}^{m} \left(\overline{X_i} - \overline{X}\right)^2\right)}{\sigma^2} \;=\; \frac{\sigma^2 \,+\, \frac{n}{m-1}\sum_{i=1}^{m} (\mu_i \,-\, \overline{\mu})^2}{\sigma^2}$$

Therefore, $E(T) \geq 1$ and $E(T) > 1$ if $\mu_i \neq \overline{\mu}$ for some i, i. e. if H_0 is false. □

(6.10) ANOVA. The random variable T as defined in Corollary (6.8) can be used to test the null hypothesis:

$$H_0 : \; \mu_1 = \mu_2 = \cdots = \mu_m$$

The outcome of a random experiment described be random variables X_{ij} as introduced in (6.4) may be summarized in a matrix:

$$\mathbf{x} \;:=\; (x_{ij})_{i=1,\ldots,m,\; j=1,\ldots,n}$$

Then

$$T(\mathbf{x}) \;:=\; \frac{\frac{1}{m-1}\, S_b(\mathbf{x})}{\frac{1}{m(n-1)}\, S_w(\mathbf{x})} \;=\; \frac{\dfrac{n}{m-1}\displaystyle\sum_{i=1}^{m} (\overline{x_i} - \overline{x})^2}{\dfrac{1}{m(n-1)}\displaystyle\sum_{i=1}^{m}\sum_{j=1}^{n} (x_{ij} - \overline{x_i})^2}$$

(where $\overline{x}$ is the mean value of all values in $\mathbf{x}$ and $\overline{x_i}$ is the mean value of $x_{i1}, \ldots, x_{in}$) will be expected to be close to 1 if H_0 is true. On the other hand, if $T(\mathbf{x})$ exceeds 1 significantly, H_0 is presumably false and should be rejected.

Because T has a F-distribution with parameters $m-1$ and $m(n-1)$ if H_0 is true, we have

$$\mathrm{Pr}_{H_0}(T > c) \;=\; \alpha$$

$$\iff \qquad c \;=\; F^{-1}_{m-1,m(n-1)}(1-\alpha)$$

and therefore, for any given significance level α:

- H_0 is rejected if $\quad T(\mathbf{x}) \;>\; F^{-1}_{m-1,m(n-1)}(1-\alpha)$
- H_0 is accepted if $\quad T(\mathbf{x}) \;\leq\; F^{-1}_{m-1,m(n-1)}(1-\alpha)$

Finally, the p-value of a collection of sampled data (arranged in the matrix $\mathbf{x}$) is given by:

$$\alpha_{\mathbf{x}} \;=\; 1 \;-\; F_{m-1,m(n-1)}(T(\mathbf{x}))$$

(6.11) Example. Sampled data for three groups with seven random variables per group are displayed in the following matrix:

$$\mathbf{x} \;=\; \begin{pmatrix} 13.6 & 14.6 & 10.7 & 20.3 & 14.2 & 11.4 & 16.2 \\ 11.7 & 7.2 & 7.2 & 2.2 & 6.0 & 5.8 & 10.5 \\ 2.5 & 5.4 & 20.0 & 18.4 & 11.4 & 13.8 & 11.3 \end{pmatrix}$$

The values for the sample means $\overline{X_1}$, $\overline{X_2}$, $\overline{X_3}$ and $\overline{X}$ are:

$$(\overline{x_1}, \overline{x_2}, \overline{x_3}) \;=\; (14.4, 7.2, 11.8), \qquad \overline{x} \;=\; 11.2$$

The estimation for the variance σ^2 given by $\frac{1}{m(n-1)} S_w$ and the value of $\frac{1}{m-1} S_b$ are:

$$\frac{1}{m(n-1)} S_w(\mathbf{x}) \;=\; 20.2, \qquad \frac{1}{m-1} S_b(\mathbf{x}) \;=\; 93.1$$

Therefore $T(\mathbf{x}) = 93.1/20.2 = 4.6$ and the p-value of the sampled data is:

$$\alpha_{\mathbf{x}} \;=\; 1 \;-\; F_{m-1,m(n-1)}(T(\mathbf{x})) \;=\; 0.024$$

For example, on the basis of the data sampled, the null hypothesis $H_0 : \mu_1 = \mu_2 = \mu_3$ would be rejected if the significance level was chosen to be $\alpha = 5\%$ and would be accepted if α was chosen to be $\alpha = 1\%$.

(6.12) MatLab. MatLab provides an implementation of the ANOVA test with the function `anova1()`. Please note that sampled data values belonging to the same group of random variables have to be composed in columns instead of rows for an input matrix to `anova1()`.

The following listing shows a call of `anova1()` with the transpose of $\mathbf{x}$ from the above Example (6.11). Furthermore, the calculations of all values mentioned in Example (6.11) are performed explicitly.

```
x = [ 13.6  14.6  10.7  20.3  14.2  11.4  16.2; ...
      11.7   7.2   7.2   2.2   6.0   5.8  10.5; ...
       2.5   5.4  20.0  18.4  11.4  13.8  11.3]'

anova1(x)

[n m] = size(x);
x_i_mean = mean(x)
x_mean = mean(x_i_mean)
sb = x_i_mean - x_mean;
Sb = n*sb*sb'
var_b = Sb/(m-1)

sw = x - x_i_mean;
Sw = sum(sum(sw.*sw))
var_w = Sw/(m*(n-1))

T = var_b/var_w
p_Value = 1 - fcdf(T,m-1,m*(n-1))
```