Probability and Statistics

4 – Continuous Random Variables

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1.) Let X be a random variable with cdf $F_X(x)$. For $a, b \in \mathbb{R}, a \neq 0$ consider the random variable

$$Y = aX + b$$

and show that its cdf is given by:

$$F_Y(x) = \begin{cases} F_X\left(\frac{x-b}{a}\right) & \text{if } a > 0 \\ 1 - F_X\left(\frac{x-b}{a}\right) + \Pr\left(X = \frac{x-b}{a}\right) & \text{if } a < 0 \end{cases}$$ ✓

Y r.v.

$Y: \Omega \to \mathbb{R}$

$(\Omega, \mathcal{A}, \Pr)$

Proof: $F_Y(x) = \Pr(Y \leq x) = \Pr(Y^{-1}((-\infty, x])) = \Pr(\{\omega \in \Omega \mid Y(\omega) \leq x\})$

$= \Pr(a \cdot X + b \leq x) = \Pr(a \cdot X \leq x - b)$

$$= \begin{cases} \Pr\left(X \leq \frac{x-b}{a}\right) = F_X\left(\frac{x-b}{a}\right) & a > 0 \\ \Pr\left(X \geq \frac{x-b}{a}\right) = \underbrace{1 - \Pr\left(X \leq \frac{x-b}{a}\right)}_{1 - F_X\left(\frac{x-b}{a}\right)} + \Pr\left(X = \frac{x-b}{a}\right) & a < 0 \end{cases}$$

2.) Let X and Y be as in exercise 1. Furthermore assume that X is continuous with pdf $f_X(x)$.
Show that Y is continuous with pdf:

$$f_Y(x) = \frac{1}{|a|} \cdot f_X\left(\frac{x-b}{a}\right)$$

$$F_Y(x) = \int_{-\infty}^{x} f_Y(t)\,dt = \int_{-\infty}^{x} \frac{1}{|a|} \cdot f_X\left(\frac{t-b}{a}\right) dt$$

$$u = \frac{t-b}{a}$$

$$\frac{du}{dt} = \frac{1}{a}$$

$$= \frac{1}{|a|} \cdot \int_{-\infty}^{\frac{x-b}{a}} f_X(u) \cdot a\, du$$

$a > 0$

$+\infty$

$a < 0$

$$= \begin{cases} \dfrac{a}{|a|} F_X\left(\dfrac{x-b}{a}\right) & a > 0 \\ \underbrace{\dfrac{a}{|a|}}_{=-1} \cdot \left(-\displaystyle\int_{\frac{x-b}{a}}^{\infty} f_X(u)\,du\right) = \cancel{(-1)}\left(\cancel{-}\left(1 - F_X\left(\frac{x-b}{a}\right)\right)\right) \end{cases}$$

3.) Let X be a continuous random variable with cdf $F_X(x)$ and pdf $f_X(x)$. For $a \in \mathbb{R}^+$ and $b \in \mathbb{R}$ consider the random variable

$$Y = \frac{X - b}{a} \qquad = \frac{1}{a} \cdot X - \frac{b}{a}$$

and show that its cdf and pdf are given by:

$$F_Y(x) = F_X(a\,x + b)$$

$$f_Y(x) = a \cdot f_X(a\,x + b) = a \cdot f_X\left(a\left(x + \frac{b}{a}\right)\right)$$

In particular, if X has finite expectation $\mu = E(X)$ and variance $\sigma^2 = \mathrm{Var}(X)$ then the normalized version of X given by $\underline{X^\bullet = \frac{X - \mu}{\sigma}}$ possesses the pdf:

$$f_{X^\bullet}(x) = \sigma \cdot f_X(\sigma\,x + \mu)$$

Normal Distributions

Theorem (4.48)

(i) $$\int_{-\infty}^{\infty} e^{-x^2}\,dx \;=\; \sqrt{\pi}$$

(ii) $$\frac{1}{\sqrt{2\pi}\cdot\sigma}\int_{-\infty}^{\infty} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}\,dx \;=\; 1 \qquad \textit{for all } \mu,\sigma\in\mathbb{R} \textit{ with } \sigma>0$$

Definition (4.49)

A random variable has a *normal distribution* $\mathcal{N}(\mu,\sigma)$ for some parameters $\mu,\sigma\in\mathbb{R}$ and $\sigma>0$ if it has a pdf defined by:

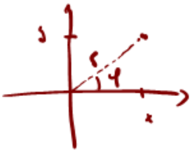
$$f_X(x) \;=\; \frac{1}{\sqrt{2\pi}\cdot\sigma}\cdot e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$$

Proof of Theorem (4.48)

(i) A substitution with polar coordinates, $0 < r\,,\ \varphi \in (-\pi, \pi]$

$$\underline{(x,y) = (r\cos\varphi, r\sin\varphi)}, \qquad \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \varphi} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \varphi} \end{vmatrix} = \begin{vmatrix} \cos(\varphi) & -r\sin(\varphi) \\ \sin\varphi & r\cos\varphi \end{vmatrix} = r$$

gives:
$$\left(\int_{-\infty}^{\infty} e^{-x^2}\,dx\right)^2 = \int_{-\infty}^{\infty} e^{-x^2}\,dx \cdot \int_{-\infty}^{\infty} e^{-y^2}\,dy$$



$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)}\,dx\,dy = \int_{0}^{\infty}\int_{-\pi}^{\pi} e^{-r^2}\, r\,d\varphi\,dr$$

$$= \int_0^{\infty} 2\pi r\, e^{-r^2}\,dr = -\pi \int_0^{\infty} -2r\cdot e^{-r^2}\,dr = -\pi\cdot e^{-r^2}\Big|_0^{\infty} = 0 + \pi$$

Proof of Theorem (4.48)

$u = \frac{x-\mu}{\sqrt{2}\cdot\sigma}$, $\frac{du}{dx} = \frac{1}{\sqrt{2}\cdot\sigma}$

(ii) $$\frac{1}{\sqrt{2\pi}\cdot\sigma}\int_{-\infty}^{\infty} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}\,dx = \frac{1}{\sqrt{2\pi}\cdot\sigma}\int_{-\infty}^{\infty}\sqrt{2}\cdot\sigma\, e^{-u^2}\,du = \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} e^{-u^2}\,du = 1$$

Standard Normal Distribution

Definition (4.50)

$\mu = 0, \sigma = 1$

The normal distribution $\mathcal{N}(0,1)$ is called the *standard normal distribution*. Its pdf and cdf are denoted by

$$\varphi(x) = \frac{1}{\sqrt{2\pi}} \cdot e^{-\frac{1}{2}x^2}$$

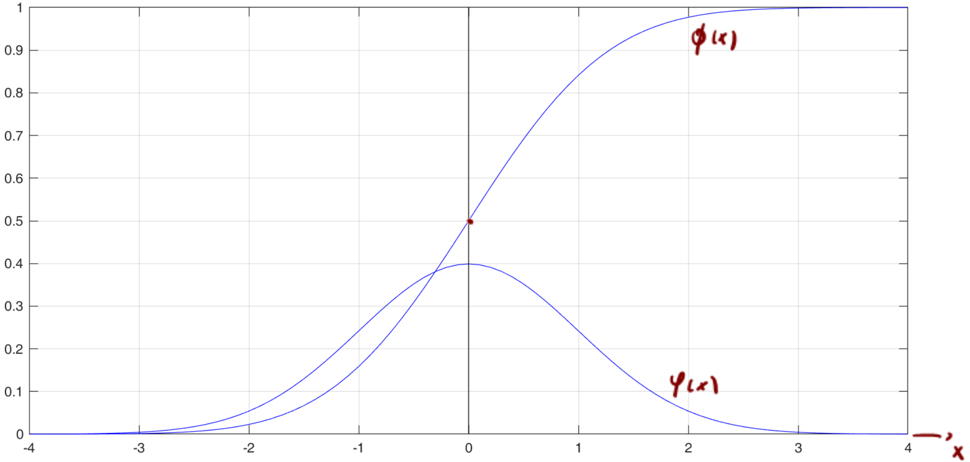
and

$$\Phi(x) = \frac{1}{\sqrt{2\pi}} \cdot \int_{-\infty}^{x} e^{-\frac{1}{2}u^2}\, du$$

no closed form to calculate values

respectively.

Standard Normal Distribution



Normal Distributions

Lemma (4.51)

If $X \sim \mathcal{N}(\mu, \sigma)$, then:

(i) $E(X) = \mu$

(ii) $Var(X) = \sigma^2$

(iii) $\phi_X(t) = e^{\mu t + \frac{1}{2}\sigma^2 t^2}$ (*for all* $t \in \mathbb{R}$)

(iv) $aX + b \sim \mathcal{N}(a\mu + b, |a| \cdot \sigma)$ *if* $a \neq 0$

(v) $F_X(x) = \Phi\left(\frac{x-\mu}{\sigma}\right)$

(vi) $\Phi(-x) = 1 - \Phi(x)$ *for all* $x \in \mathbb{R}$

(vii) $\Phi^{-1}(p) = -\Phi^{-1}(1-p)$ *for all* $p \in (0, 1)$

Proof of (4.51)(i): $E(X) = \mu$

$$
\begin{aligned}
E(X) &= \frac{1}{\sqrt{2\pi}\cdot\sigma}\int_{-\infty}^{\infty} x\, e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}\,dx \\
&= \frac{1}{\sqrt{2\pi}\cdot\sigma}\int_{-\infty}^{\infty} (x-\mu+\mu)\, e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}\,dx \\
&= \frac{\sigma}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \frac{x-\mu}{\sigma^2}\, e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}\,dx + \frac{1}{\sqrt{2\pi}\cdot\sigma}\int_{-\infty}^{\infty} \mu\, e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}\,dx \\
&= -\frac{\sigma}{\sqrt{2\pi}}\, e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}\Bigg|_{-\infty}^{\infty} + \mu = \mu
\end{aligned}
$$

Proof of (4.51)(ii): $Var(X) = \sigma^2$

$E((X-\mu)^2)$

$$Var(X) = \frac{1}{\sqrt{2\pi}\cdot\sigma}\int_{-\infty}^{\infty}(x-\mu)^2\, e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}\,dx$$

$$= -\frac{\sigma}{\sqrt{2\pi}}\int_{-\infty}^{\infty}(x-\mu)\cdot\left(-\frac{x-\mu}{\sigma^2}\right)\, e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}\,dx$$

Integration by parts

$$= -\frac{\sigma}{\sqrt{2\pi}}\left(\left.(x-\mu)\cdot e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}\right|_{-\infty}^{\infty} - \int_{-\infty}^{\infty} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}\,dx\right)$$

$$= -\frac{\sigma}{\sqrt{2\pi}}\left(0 - \sqrt{2\pi}\cdot\sigma\right) = \sigma^2$$

Proof of (4.51)(iii): $\phi_X(t) = e^{\mu t + \frac{1}{2}\sigma^2 t^2}$ (for all $t \in \mathbb{R}$)

$$\begin{aligned}
\phi_X(t) &= \frac{1}{\sqrt{2\pi}\cdot\sigma}\int_{-\infty}^{\infty} e^{tx}\, e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}\, dx \\
&= \frac{1}{\sqrt{2\pi}\cdot\sigma}\int_{-\infty}^{\infty} e^{-\frac{1}{2}\frac{x^2-2\mu x+\mu^2-2\sigma^2 tx}{\sigma^2}}\, dx \\
&= \frac{1}{\sqrt{2\pi}\cdot\sigma}\int_{-\infty}^{\infty} e^{-\frac{1}{2}\frac{x^2-2(\mu+\sigma^2 t)x+(\mu+\sigma^2 t)^2-2\mu\sigma^2 t-\sigma^4 t^2}{\sigma^2}}\, dx \\
&= \frac{1}{\sqrt{2\pi}\cdot\sigma}\int_{-\infty}^{\infty} e^{\mu t+\frac{1}{2}\sigma^2 t^2}\, e^{-\frac{1}{2}\left(\frac{x-(\mu+\sigma^2 t)}{\sigma}\right)^2}\, dx \\
&= e^{\mu t+\frac{1}{2}\sigma^2 t^2}
\end{aligned}$$

pdf of $\mathcal{N}(\mu+\sigma^2 t, \sigma)$