Probability and Statistics

4 – Continuous Random Variables

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Jointly Continuous Random Variables

Definition (4.26)

Two random variables X and Y are *jointly continuous random variables*, if there exists a function $f_{XY} : \mathbb{R}^2 \to \mathbb{R}_0^+$, such that

$$F_{XY}(x, y) = \int_{-\infty}^{y} \int_{-\infty}^{x} f_{XY}(s, t)\, ds\, dt$$

for all $x, y \in \mathbb{R}$. The function f_{XY} is called a *joint probability density function* of X and Y.

Jointly Continuous Random Variables

Lemma (4.27)

Let f_{XY} be a joint probability density function of two random variables X and Y. The (marginal) probability density functions of X and Y are determined by f_{XY} as follows:

$$\begin{aligned} f_X(x) &= \int_{-\infty}^{\infty} f_{XY}(x,t)\,dt \\ f_Y(y) &= \int_{-\infty}^{\infty} f_{XY}(s,y)\,ds \end{aligned}$$

Jointly Continuous Random Variables

Lemma (4.28)

Let f_{XY} be a joint probability density function of two random variables X and Y. If f_{XY} is continuous, then for all $x, y \in \mathbb{R}$:

$$\frac{\partial^2}{\partial y \partial x} F_{XY}(x,y) = \frac{\partial^2}{\partial x \partial y} F_{XY}(x,y) = f_{XY}(x,y)$$

Independent Jointly Continuous Random Variables

Lemma (4.29)

Two continuous random variables X and Y with probability density functions f_X and f_Y are independent, if and only if

$$f(s,t) = f_X(s) \cdot f_Y(t)$$

$$f(x,y) := f_X(x) \cdot f_Y(y) \qquad \text{for all } x, y \in \mathbb{R}$$

defines a joint density function of X and Y.

1) Assume $f(x,y)$ def. j.d.f. of X and Y

$$\Rightarrow F_{XY}(x,y) = \int_{-\infty}^{y} \int_{-\infty}^{x} f_X(s) \cdot f_Y(t) \, ds \, dt = \int_{-\infty}^{y} f_Y(t) \Big(\underbrace{\int_{-\infty}^{x} f_X(s) \, ds}_{F_X(x)} \Big) dt$$

$$F_X(x) \cdot F_Y(y)$$

Proof of Lemma (4.29)

2) X, Y are independent

assumption

$$\underline{f_{XY}(x,y)} \overset{\downarrow}{=} \frac{\partial^2}{\partial x\,\partial y} F_{XY}(x,y) = \frac{\partial}{\partial x}\left(\frac{\partial}{\partial y} F_X(x)\cdot F_Y(y)\right)$$

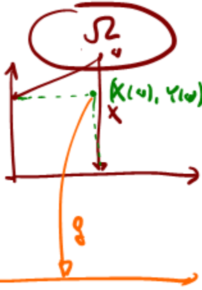
$$= \frac{\partial}{\partial x}\left(F_X(x)\cdot\frac{\partial}{\partial y} F_Y(y)\right) = \left(\frac{\partial}{\partial x} F_X(x)\right)\cdot\left(\frac{\partial}{\partial y} F_Y(y)\right)$$

$$= \underline{f_X(x)\cdot f_Y(y)}$$

Lemma (4.30)

$g(X,Y): \Omega \to \mathbb{R}, \quad \omega \mapsto g(X(\omega), Y(\omega))$

Let f_{XY} be a joint probability density function of two random variables X and Y.



(i) For any (essential) continuous mapping $g : \mathbb{R}^2 \to \mathbb{R}$ the following holds:

$$E(g(X,Y)) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y)\, f_{XY}(x,y)\, dx\, dy$$

(ii)

$$E(X+Y) = E(X) + E(Y)$$

pf of (ii): $g(x,y) = x+y$

$$E(X+Y) \overset{(i)}{=} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} (x+y)\, f_{XY}(x,y)\, dx\, dy = \int_{-\infty}^{\infty}\left(\int_{-\infty}^{\infty} x\cdot f_{XY}(x,y)\,dx + y\cdot\underbrace{\int_{-\infty}^{\infty} f_{XY}(x,y)\,dx}_{f_Y(y)}\right) dy$$

(4.27)

$$= \int_{-\infty}^{\infty} x \underbrace{\int_{-\infty}^{\infty} \cancel{y}\cdot f_{XY}(x,y)\,\cancel{dx\,dy}}_{f_X(x)}\; dy\, dx + \int_{-\infty}^{\infty} y\cdot f_Y(y)\, dy = E(X) + E(Y)$$

idea of proof of (i)

the quantizes

and (3.24)

Lemma (4.30)

Let f_{XY} be a joint probability density function of two random variables X and Y.

(iii) If X and Y are independent and if $h, k : \mathbb{R} \to \mathbb{R}$ are piecewise continuous functions, then:

$g : (x,y) \mapsto h(x) \cdot k(y)$

$$E(h(X) \cdot k(Y)) \;=\; E(h(X)) \cdot E(k(Y))$$

$$E(h(X) \cdot k(Y)) = E(g(X,Y)) \overset{(i)}{\underset{(4.29)}{=}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(x) \cdot k(y) \cdot f_X(x) \cdot f_Y(y) \, dx \, dy$$

$$= \left(\int_{-\infty}^{\infty} k(y) \cdot f_Y(y) \, dy \right) \left(\int_{-\infty}^{\infty} h(x) \, f_X(x) \, dx \right) \cancel{dy}$$

$$= E(k(Y)) \cdot E(h(X))$$

$\square$

Remark (4.31)

The definitions and results from section 3 concerning a pair of random variables ((3.29)–(3.40)) hold in full generality or their proofs make use of the properties given in (4.25), (4.30)(ii) and (iii). Therefore, all of these definitions and results also apply to continuous random variables.

Sums of Random Variables

Remark (4.31)

Let X be the sum of n random variables:

$$X = X_1 + X_2 + \ldots + X_n$$

Then:

(i)

$$E(X) = E(X_1) + E(X_2) + \ldots + E(X_n)$$

Sums of Independent Random Variables

Remark (4.31)

(ii) If $X_1, \ldots, X_n$ are independent, then:

$$Var(X) = Var(X_1) + Var(X_2) + \ldots + Var(X_n)$$

(iii) If $X_1, \ldots, X_n$ are independent, then:

$$\varphi_X(v) = \varphi_{X_1}(v) \cdot \varphi_{X_2}(v) \cdots \varphi_{X_n}(v)$$

(iv) If $X_1, \ldots, X_n$ are independent and its moment generating functions exist, then:

$$\phi_X(t) = \phi_{X_1}(t) \cdot \phi_{X_2}(t) \cdots \phi_{X_n}(t)$$

PDF's for Sums of Random Variables

Theorem (4.32)

If X and Y are independent random variables with pdf's f_X and f_Y, respectively, then the convolution of f_X and f_Y is a pdf of $X + Y$:

$$f_{X+Y}(z) = (f_X * f_Y)(z) = \int_{-\infty}^{\infty} f_X(x) \cdot f_Y(z - x)\, dx$$

PDF's for Products of Random Variables

Theorem (4.33)

Let X and Y be independent continuous random variables with pdf's f_X and f_Y, respectively. A pdf of $X \cdot Y$ is given by:

$$f_{X \cdot Y}(z) = \int_{-\infty}^{\infty} f_X(x) \cdot f_Y(z/x) \cdot \left|\frac{1}{x}\right| \, dx = \int_{-\infty}^{\infty} f_X(z/y) \cdot f_Y(y) \cdot \left|\frac{1}{y}\right| \, dy$$

PDF's for Quotients of Random Variables

Theorem (4.34)

Let X and Y be independent continuous random variables with pdf's f_X and f_Y, respectively. A pdf of $\frac{X}{Y}$ is given by:

$$f_{\frac{X}{Y}}(z) = \int_{-\infty}^{\infty} f_X(zy)\, f_Y(y)\, |y|\, dy$$