Probability and Statistics

4 – Continuous Random Variables

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Quantile Functions

Definition (4.5)

Let X be a random variable with cdf F_X. The *quantile function* of X is defined for all $p \in (0,1)$ by:

$$F_X^{-1}(p) \quad := \quad \min\{x \mid F_X(x) \geq p\}$$

First quartile, *median* and *third quartile* are defined to be: $F_X^{-1}\left(\frac{1}{4}\right)$, $F_X^{-1}\left(\frac{1}{2}\right)$ and $F_X^{-1}\left(\frac{3}{4}\right)$

Note: If F_X is continuous and strictly increasing, then restricting the codomain of F_X to $F_X(\mathbb{R}) = (0,1)$ yields a bijective mapping with the quantile function as the inverse mapping. Moreover:

(i) $F_X(F_X^{-1}(p)) \geq p$ for all $p \in (0,1)$

(ii) $F_X(F_X^{-1}(p)) = p$ for all $p \in (0,1)$ if F_X is continuous

(iii) F_X^{-1} is strictly increasing if F_X is continuous

(iv) $F_X^{-1}(F_X(x)) \leq x$ for all $x \in \mathbb{R}$ with $F_X(x) \in (0,1)$

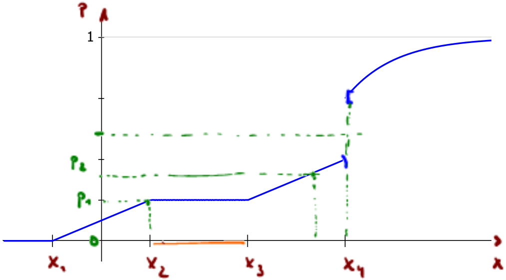
Quantile Functions

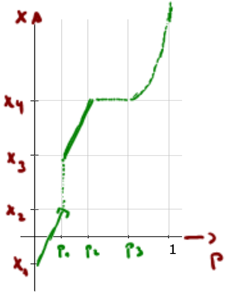
(i) $F_X(F_X^{-1}(p)) \geq p$ for all $p \in (0,1)$ "$>$" $\Leftrightarrow$ $p \notin F_X(\mathbb{R})$

(ii) $F_X(F_X^{-1}(p)) = p$ for all $p \in (0,1)$ if F_X is continuous

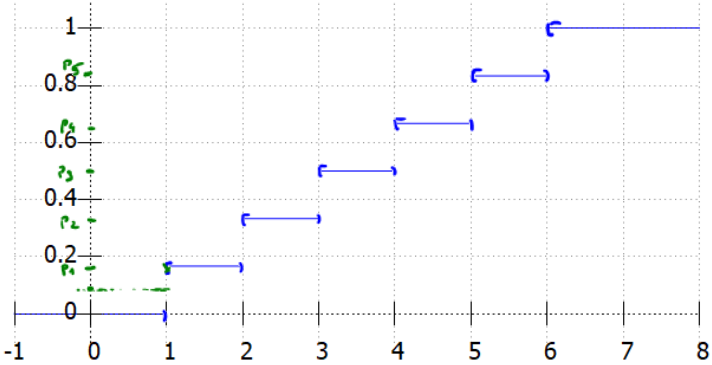
(iii) F_X^{-1} is strictly increasing if F_X is continuous

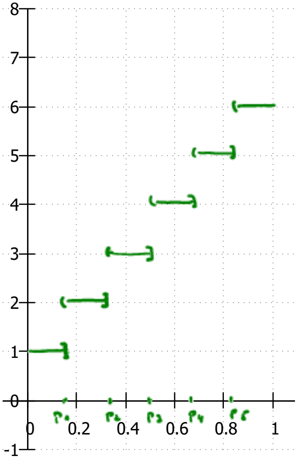
(iv) $F_X^{-1}(F_X(x)) \leq x$ for all $x \in \mathbb{R}$ with $F_X(x) \in (0,1)$ "$<$" $\Leftrightarrow$ There exists some $x_0 < x$ and $F(x_0) = F(x)$



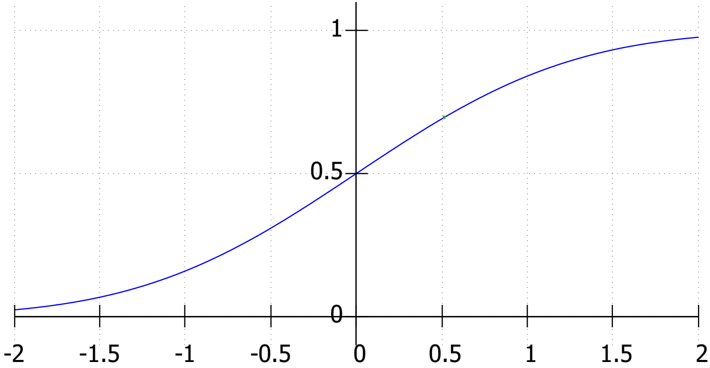


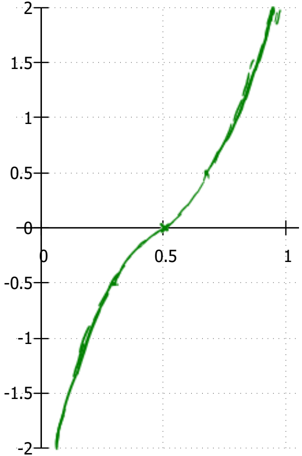
Quantile Functions



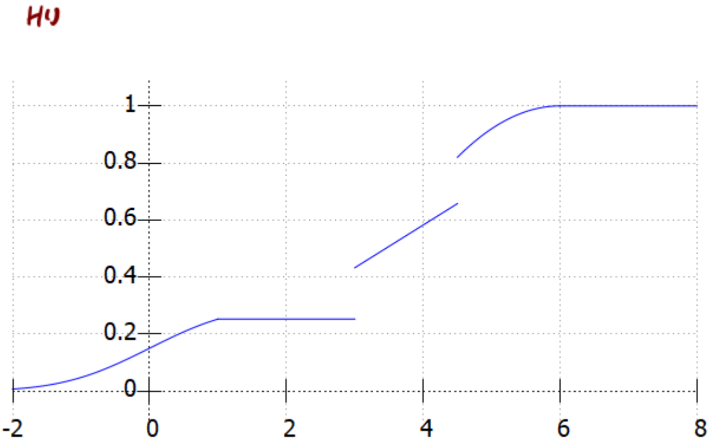


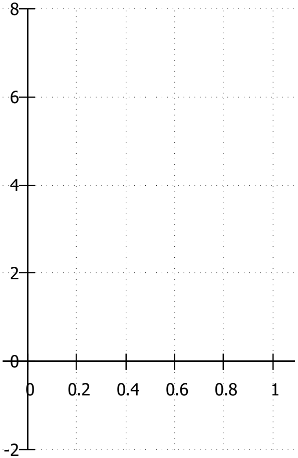
Quantile Functions





Quantile Functions





Quantizer

Definition (4.6)

Any random variable X can be approximated by discrete random variables defined by quantizing the possible values of X:

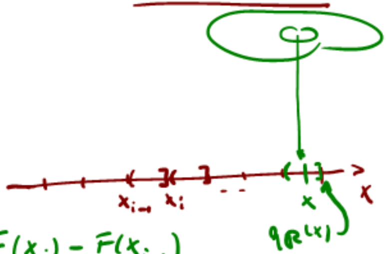
$$\mathcal{P} = \{x_i \mid i \in \mathbb{Z}\} \qquad \text{with } x_{i-1} < x_i \text{ for all } i \in \mathbb{Z} \text{ and } \lim_{i \to \pm\infty} x_i = \pm\infty$$

$$\Rightarrow \quad \mathbb{R} = \dot{\bigcup_{i \in \mathbb{Z}}} (x_{i-1}, x_i]$$

$$q_{\mathcal{P}} : \mathbb{R} \to \mathcal{P}, \qquad q_{\mathcal{P}}(x) = \min\{x_i \mid x_i \geq x\}$$

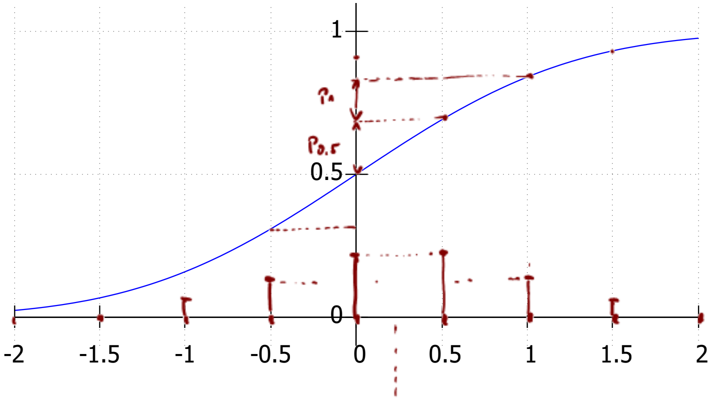
$$X_{\mathcal{P}} := q_{\mathcal{P}}(X)$$

$$p_i = \Pr(X_{\mathcal{P}} = x_i) = \Pr(x_{i-1} < X \leq x_i) = F(x_i) - F(x_{i-1})$$



Quantizer

$\mathcal{P} = \{ \frac{i}{2} \mid i \in \mathbb{Z} \}$



Continuous Random Variables

Definition (4.7)

A random variable X is called a *continuous random variable* if there exists a function $f_X : \mathbb{R} \to \mathbb{R}_0^+$ such that

$$F_X(x) = \int_{-\infty}^{x} f_X(\xi)\, d\xi$$

for all $x \in \mathbb{R}$. The function f_X is called a *probability density function* (pdf) of X.

Continuous Random Variables

Lemma (4.8)

If X is a continuous random variable with pdf f_X, then the following holds:

(i) $$\int_{-\infty}^{\infty} f_X(x)\,dx = \lim_{x\to\infty} \int_{-\infty}^{x} f_X(s)\,ds = \lim_{x\to\infty} F_X(x) = 1$$

(ii) $\Pr(X = x) = 0$ *for all* $x \in \mathbb{R}$ $\qquad \int_x^x f_X(s)\,ds = 0$

(iii) $$\int_a^b f_X(x)\,dx = \Pr(a \le X \le b) \quad \text{for all } a, b \in \mathbb{R},\ a < b$$

$= F_X(b) - F_X(a)$

(iv) *If f_X is continuous, then:*

$$F_X{}'(x) = f_X(x) \quad \text{for all } x \in \mathbb{R}$$

Expectation

Definition (4.9)

Let X be a continuous random variable with pdf $f_X(x)$. The *expectation* (*mean*) of X is defined to be

$$E(X) := \int_{-\infty}^{\infty} x \cdot f_X(x)\, dx$$

if the integral exists.

discrete case $\qquad \sum_{x \in \mathbb{R}} x \cdot p_X(x)$

Expectation

Remark (4.10)

Let $\mathcal{P}_n$ ($n \in \mathbb{N}$) define a sequence ~~series~~ of quantizers with

$$\lim_{n\to\infty} \Delta(\mathcal{P}_n) \;=\; 0$$

then:

$$\lim_{n\to\infty} E(X_{\mathcal{P}_n}) \;=\; E(X)$$

Expectation

Remark (4.11)

Virtually all definitions and theorems based on expectations of discrete random variables can be transferred to corresponding definitions and statements for continuous random variables.

This can be achieved using discrete approximations $q_{\mathcal{P}_n}(X)$ for a given continuous random variable X, where the quantizers are defined by sets $\mathcal{P}_n$ ($n \in \mathbb{N}$) with $\lim_{n \to \infty} \Delta(\mathcal{P}_n) = 0$.

Continuous Random Variables

Lemma (4.12)

Let X be a continuous random variable with pdf $f_X(x)$.

(i) *If $g : \mathbb{R} \to \mathbb{R}$ is piecewise continous and $Y = g(X) := g \circ X$, then:*

$$E(Y) = \int_{-\infty}^{\infty} g(x) \cdot f_X(x)\, dx$$

(ii) *For any $a, b \in \mathbb{R}$:*

$$E(aX + b) = a\,E(X) + b \quad \leftarrow \text{HW}$$