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Information Fusion – Basics on ET

Combination Techniques for Uncertain Information in Measurement and Signal Processing

3.3 Fuzzy Set Theory

Information Fusion Prof. Dr.-Ing. Volker Lohweg

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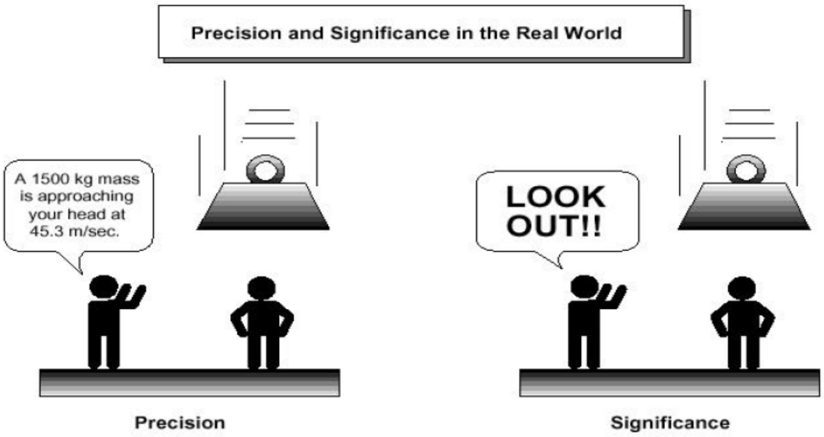
3.3 Fuzzy Set Theory – Introduction

- Example



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3.3 Fuzzy Set Theory – Introduction

- "Many decision-making and problem-solving tasks are **too complex** to be understood quantitatively, however, people succeed by using knowledge which is imprecise rather than precise."
- *Fuzzy Set Theory*, originally introduced by Lotfi Zadeh in the 1960's, resembles human reasoning in its use of approximate information and uncertainty to generate decisions.
- It was specifically designed to mathematically represent *uncertainty and vagueness* and provide formalized tools for dealing with the *imprecision* intrinsic to many problems. By contrast, traditional computing demands precision down to each bit.

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3.3 Fuzzy Set Theory – Introduction

- Fuzzy, or multi-valued logic was introduced in the 1930s by **Jan Lukasiewicz**, a Polish philosopher.
- While classical logic operates with only two values 1 (true) and 0 (false), Lukasiewicz introduced logic that extended the range of truth values to all real numbers in the interval between 0 and 1.
- He used a number in this interval to represent the possibility that a given statement was true or false. For example, the possibility that a man 181 cm tall is really tall might be set to a value of 0.86.
- It is *likely* that the man is tall. This work led to an inexact reasoning technique often called **Possibility Theory** coined by Zadeh and Dubois & Prade.

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3.3 Fuzzy Set Theory – Introduction

- Later, in 1937, **Max Black** published a paper "Vagueness: an exercise in logical analysis".
- In this paper, he argued that a continuum implies degrees. Imagine, he said, a line of countless "chairs". At one end is a Chippendale. Next to it is a near-Chippendale, in fact indistinguishable from the first item. Succeeding "chairs" are less and less chair-like, until the line ends with a log.
- When does a *chair* become a *log*? Max Black stated that if a continuum is discrete, a number can be allocated to each element.
- He accepted **vagueness as a matter of probability**.

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3.3 Fuzzy Set Theory – Introduction

- In 1965 **Lotfi Zadeh**, published his famous paper "Fuzzy Sets".
- Zadeh puts different concepts of multi-valued logic into a formal system of mathematical logic, and introduced a new concept for applying natural language terms.
- This new logic (not the whole theory → Fuzzy Set Theory (FST)) for representing and manipulating fuzzy terms was called **fuzzy logic**, and Zadeh became the Master of *fuzzy logic*.

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3.3 Fuzzy Set Theory – Introduction

- Experts rely on **common sense** when they solve problems.
- **How can we represent expert knowledge that uses vague and ambiguous terms in a computer?**
- **Fuzzy logic** (in comparison to **Boolean logic**) is not logic that is fuzzy, but logic that is used to describe fuzziness. Fuzzy logic is the theory of fuzzy sets, sets that calibrate vagueness.
- **Fuzzy Set Theory** (Fuzzy logic) is based on the idea that all things admit of degrees. Temperature, height, speed, distance, beauty – all come on a sliding scale.
  - The motor is running *really hot*.
  - Tom is a *very tall* man.

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3.3 Fuzzy Set Theory – Introduction

- **Fuzzy Set Theory is a set of mathematical principles for knowledge representation based on degrees of membership.**
  - Unlike two-valued Boolean logic, fuzzy logic is **multi-valued** →

    It deals with **degrees of membership** and **degrees of truth**.

  - Fuzzy logic uses the continuum of logical values between 0 (completely false) and 1 (completely true). Instead of just black and white, it employs the spectrum of colours, accepting that things can be partly true and partly false at the same time.

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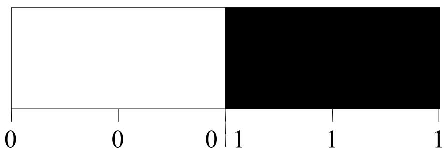
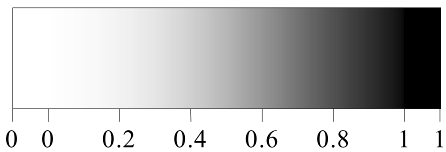
3.3 Fuzzy Set Theory – Introduction



(a) Boolean Logic.



(b) Multi-valued Logic.

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3.3 Fuzzy Set Theory – Fuzzy Sets

- Definitions – Sets

  1. Classical sets – either an element belongs to the set or it does not. For example, for the set of integers, either an integer is even or it is not (it is odd). However, either you are in Germany or you are not.

- Classical sets are also called *crisp* (sets).

  Lists: A = {apples, oranges, cherries, mangoes}

  $A = \{a_1, a_2, a_3\}$

  A = {2, 4, 6, 8, …}

  Formulas: A = {x | x is an even natural number}

  A = {x | x = 2n, n is a natural number}

Membership or characteristic *function* $\mu_A(x) = \begin{cases} 1 \text{ if } x \in A \\ 0 \text{ if } x \notin A \end{cases}$

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3.3 Fuzzy Set Theory – Fuzzy Sets

- Definitions – Sets

  2. Fuzzy sets – admits gradation such as all tones between black and white. A fuzzy set has a graphical description which expresses how the transition from one to another takes place.

- This graphical description is called a *membership function*.
  - When fuzzy variables and fuzzy sets are considered, the function describing a membership function of a variable to its set is allowed to take all values within the interval [0 … 1].
  - A fuzzy variable of an **universal set (frame of discernment)** *X* with a subset *A* is defined by its membership function $\mu_A(x)$, where $x \in \boldsymbol{R}$.
  - The membership function of a fuzzy variable satisfies the following properties:

$$0 \leq \mu_A(x) \leq 1$$
$$\mu_A(x) \text{ is convex.}$$
$$\mu_A(x) \text{ is normal. } \rightarrow \max(\mu_A(x)) = 1.$$

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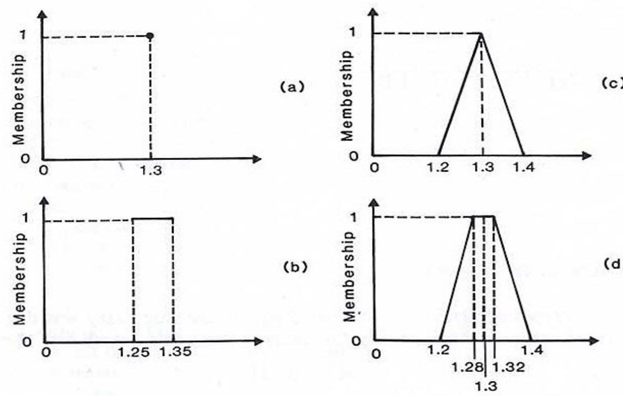
3.3 Fuzzy Set Theory – Fuzzy Sets

- Crisp numbers vs. Fuzzy numbers



A comparison of a real number and a crisp interval with a fuzzy number and a fuzzy interval, respectively.

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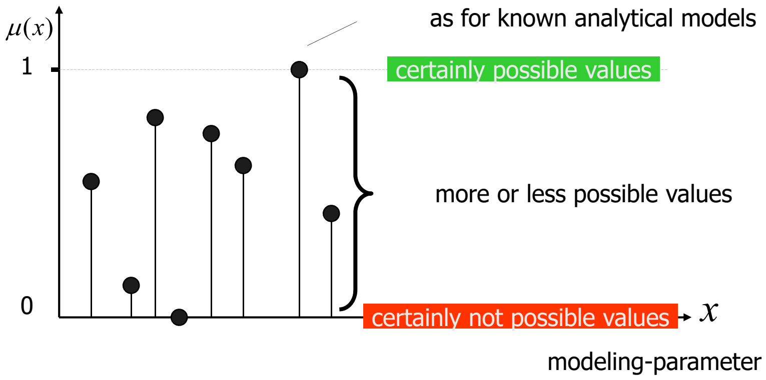
3.3 Fuzzy Set Theory – Fuzzy Sets

- Fuzzy Sets
  - Degree of membership to a fuzzy set



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3.3 Fuzzy Set Theory – Fuzzy Sets

- Fuzzy Sets
  - Definition of discrete memberships

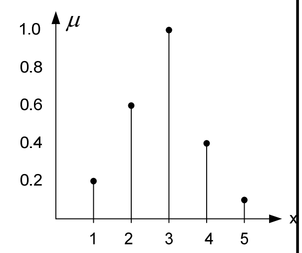
$$\mu_A(x) = \mu_0 / x_0 + \mu_1 / x_1 + \cdots + \mu_{N-1} / x_{N-1} = \sum_{i=0}^{N-1} (\mu_i / x_i)$$

$$A = \{x_0, \ldots, x_{N-1}\}$$

$$0 \le \mu_i \le 1$$

  - Example

$$\mu_A(x) = 0.2/1 + 0.6/2 + 1.0/3 + 0.4/4 + 0.1/5$$



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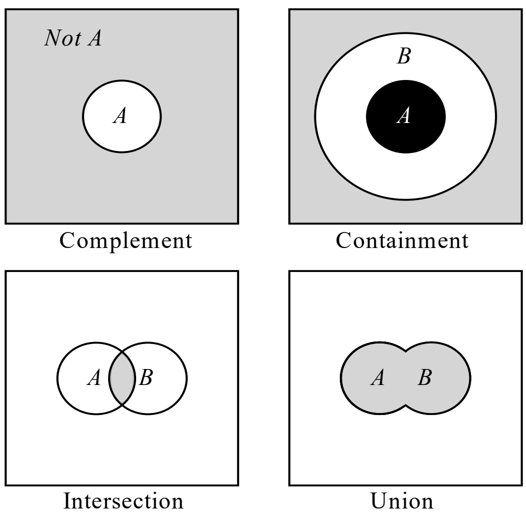
3.3 Fuzzy Set Theory – Fuzzy Operations



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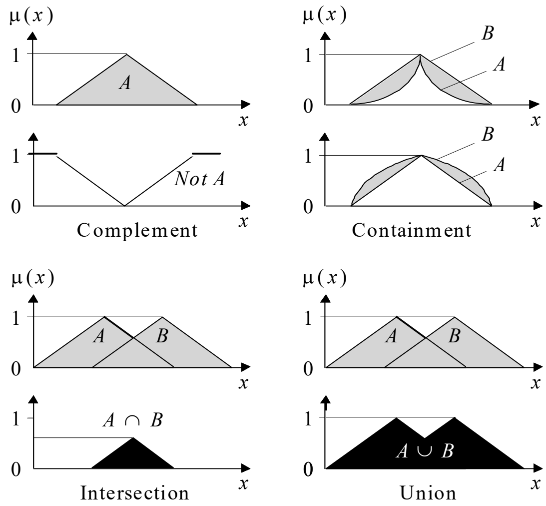
3.3 Fuzzy Set Theory – Fuzzy Operations



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3.3 Fuzzy Set Theory – Fuzzy Operations

- The set operators aim for combining information in order to generate a single fuzzy variable. The basic operations are the equivalents to the above mentioned operations in classical set theory.

- Combination by Maximum
  - For disjunction (union) of fuzzy propositions, elements are assigned the maximum membership grade given. The fuzzy union

$$\mu_A(x) \cup \mu_B(x)$$

based on a s-norm, s: [0, 1] x [0, 1] → [0, 1], is defined as:

$$\mu_{A \cup B}(x) = \max(\mu_A(x), \mu_B(x)).$$

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3.3 Fuzzy Set Theory – Fuzzy Operations

- Combination by Minimum
  - For conjunction (intersection) of fuzzy propositions, elements are assigned the minimum membership grade given. The fuzzy intersection

$$\mu_A(x) \cap \mu_B(x)$$

based on a t-norm, t: [0, 1] x [0, 1] → [0, 1], is defined as:

$$\mu_{A \cap B}(x) = \min(\mu_A(x), \mu_B(x)).$$

Again, contrary to the probability theory, fuzzy union and intersection is *non-additive*.

3.3 Fuzzy Set Theory – Fuzzy Operations

- Complement
  - The complement of a normalised membership value of the fuzzy set A is defined as:

$$\mu_{A^C}(x) = 1 - \mu_A(x).$$

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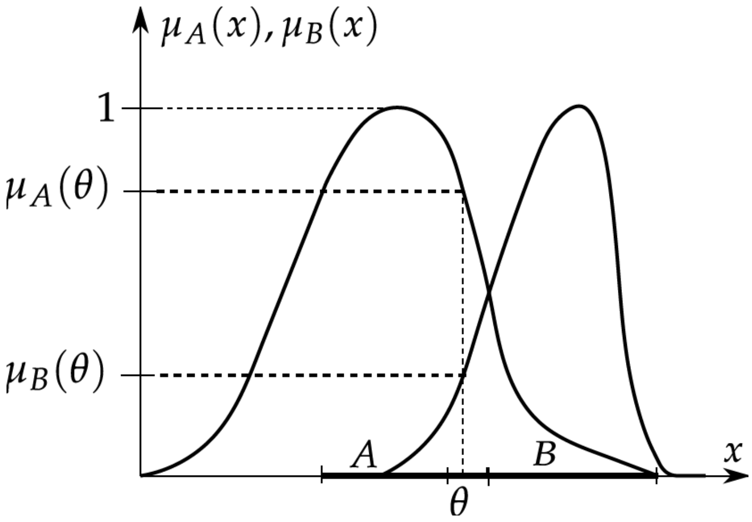
3.3 Fuzzy Set Theory – Fuzzy Operations



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3.3 Fuzzy Set Theory – Fuzzy Operations

- Operations – simple modifications

Nomination	Mathematical definition
Negation (NOT)	$\bar{\mu}_A(x) = \mu_{A^c}(x) = 1 - \mu_A(x)$
Disjunction (OR)	$\mu_{A \cup B}(x) = \max(\mu_A(x), \mu_B(x))$
Conjunction (AND)	$\mu_{A \cap B}(x) = \min(\mu_A(x), \mu_B(x))$
Concentration	$\mu_{kon}(x) = \mu_A^{\,2}(x)$
Reduction	$\mu_{ab}(x) = \sqrt{\mu_A(x)}$
Product	$\mu_{AB}(x) = \mu_A(x) \cdot \mu_B(x)$

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3.3 Fuzzy Set Theory – Fuzzy Operations

- The membership function (MSF) of a fuzzy number or variable can also be described in terms of so called *Alpha-Cuts* (α*-cuts*).
- Each α–cut, at a generic level α, is defined as:

$$A_\alpha = \left\{ x \in A \middle| \mu_A(x) \geq \alpha \right\}$$

- The importance of representing a fuzzy variable in terms of α-cuts is that the corresponding levels α can be considered as a **set of confidence intervals** and their associated levels of certitude.
- The **level of certitude** (certainty) contains information about how certain a human is about its knowledge.
- **"The higher the certainty the lower the confidence"** → human behaviour

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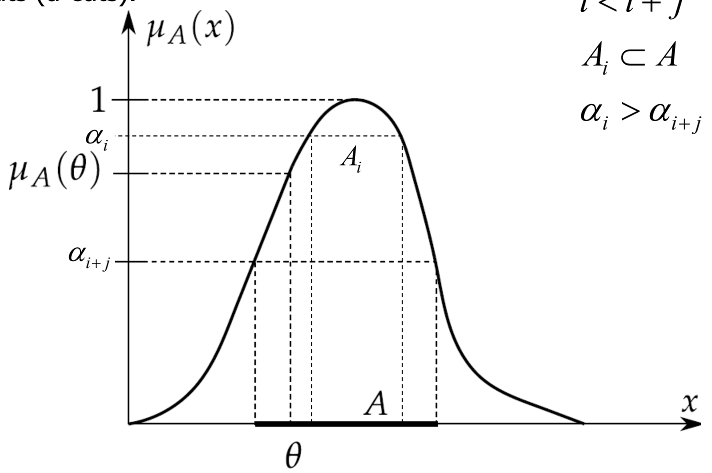
3.3 Fuzzy Set Theory – Fuzzy Operations

- Alpha-Cuts (α-cuts).



$$i < i + j$$
$$A_i \subset A$$
$$\alpha_i > \alpha_{i+j}$$

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3.3 Fuzzy Set Theory – Fuzzy Operations

- Fuzzy Mathematics based on cuts
  - Each fuzzy variable with finite support can fully and uniquely be represented by its α-cuts.
  - The α-cuts of each fuzzy variable are closed intervals of real numbers.
  - The operations are topic of *interval analysis*, a well-established area of classic mathematics.
- Let the symbol $\circ$ denote any of the four arithmetic operations (addition, subtraction, multiplication and division). Then, a general property of all arithmetic operations on closed intervals is given by

$$[a,b]\circ[c,d]=\{f\circ g \mid a\le f\le b,\ c\le g\le d\}$$

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3.3 Fuzzy Set Theory – Fuzzy Operations

- Fuzzy Mathematics based on cuts
  - Addition

$$[a,b]+[c,d]=[a+c,b+d]$$

  - Subtraction

$$[a,b]-[c,d]=[a-d,b-c]$$

  - Multiplication

$$[a,b]\cdot[c,d]=[\min(ac,ad,bc,bd),\max(ac,ad,bc,bd)]$$

  - Division

$$[a,b]/[c,d]=[a,b]\cdot\left[\frac{1}{c},\frac{1}{d}\right]=\cdots$$

$$\cdots=[\min(a/c,a/d,b/c,b/d),\max(a/c,a/d,b/c,b/d)]$$

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3.3 Fuzzy Set Theory – Fuzzy Operations

- Fuzzy Mathematics based on cuts – examples
  - Addition
    $[2,5]+[1,3]=[3,8]$ $\quad$ $[0,1]+[-6,5]=[-6,6]$
  - Subtraction
    $[0,1]-[-6,5]=[-5,7]$
  - Multiplication
    $[3,4]\cdot[2,2]=[6,8]$
  - Division
    $[4,10]/[1,2]=[2,10]$

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3.3 Fuzzy Set Theory – Fuzzy Operations

- Fuzzy Variables and Measurement
  - We are able to represent measurement results with uncertainty by means of membership functions and alpha-cuts.
  - When an indirect measurement procedure is considered, the mathematics of fuzzy variables allows us to directly obtain the final result in terms of a fuzzy number (*interval* ***and*** *certitude*).

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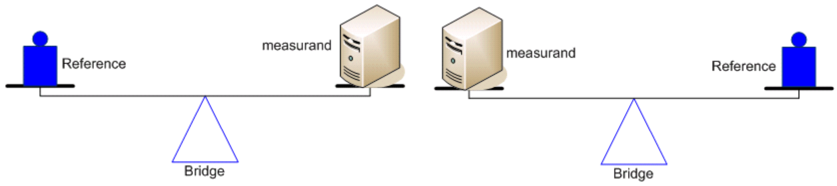
3.3 Fuzzy Set Theory – Fuzzy Operations

- Example 3.3-1 (cf. Example 1.2-4)
  - Again, we refer to double weighing. The variable x_1(q) is the value of the first measurement which is now 2 kg, the variable x_2(q) is the value of the second measurement which is now 2.4 kg. The measurement results are represented by two fuzzy variables M_1 and M_2. Let us suppose that a contribution of not known nature, but not random, is present and the interval of confidence is +/- 100 g.



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3.3 Fuzzy Set Theory – Fuzzy Operations

- Example 3.3-1, cont'd
  - The available information is that the measurement belongs to the given interval and each point of the interval is as plausible as all others. Therefore, we have to apply rectangular membership functions and calculate the result as follows:

$$R = \frac{M_1 + M_2}{2} \rightarrow [R] = \frac{1}{2}\left([1.9, 2.1] + [2.3, 2.5]\right)kg = \frac{1}{2}\left([4.2, 4.6]\right) = [2.1, 2.3]kg$$

  - As already known, we have more information available. The uncertain contribution behaves systematically due to the length difference of the weighing beams. We assume the systematic error is $b \in \mathbf{R}$ in the interval of +/-100g. The double weighing will compensate the effect. It is formally:

$$R = \frac{M_1 + M_2}{2} \rightarrow [R] = \frac{1}{2}\left([2+b, 2+b] + [2.4-b, 2.4-b]\right)kg = \frac{1}{2}\left([4.4, 4.4]\right) = [2.2, 2.2]kg$$

CRISP!

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3.3 Fuzzy Set Theory – Fuzzy Operations

- Fuzzy Sets – example 3.3-2 (Tall man)
  - The *x*-axis represents the **universe of discourse / frame of discerment** – the range of all possible values applicable to a chosen variable. In our case, the variable is the man height. According to this representation, the universe of men's heights consists of all tall men.
  - The *y*-axis represents the **membership value of the fuzzy set**. In our case, the fuzzy set of "*tall men*" maps height values into corresponding membership values.

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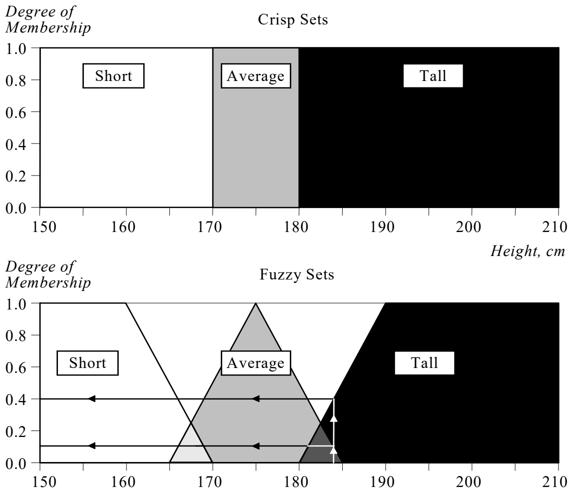
3.3 Fuzzy Set Theory – Fuzzy Operations



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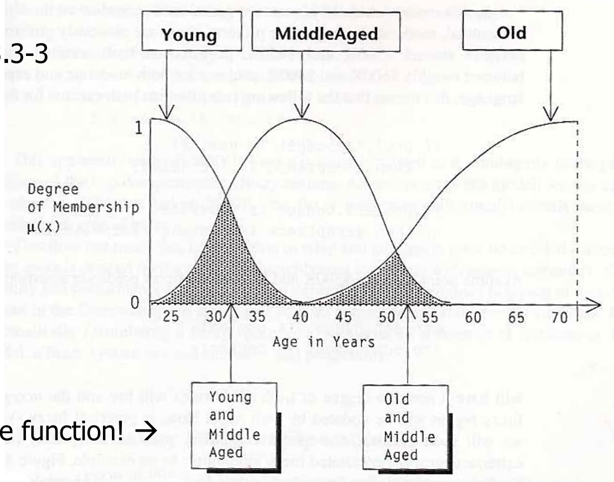
3.3 Fuzzy Set Theory – Fuzzy Operations

- Fuzzy Sets – example 3.3-3

Define the function! →



Ambiguous Fuzzy Regions in a Shared Domain

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3.3 Fuzzy Set Theory – Linguistic Variables and Modifiers

- At the root of Fuzzy Set Theory lies the idea of linguistic variables.
- **A linguistic variable is a fuzzy variable.**
  - For example, the statement "John is tall" implies that the linguistic variable *John* takes the linguistic value *tall*.

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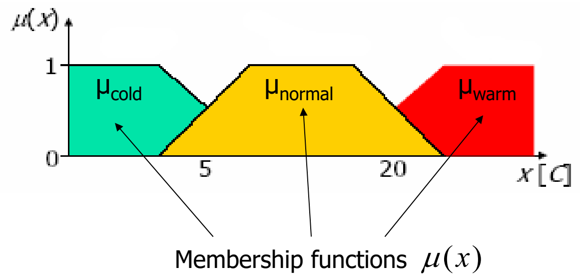
3.3 Fuzzy Set Theory – Linguistic Variables and Modifiers

- Linguistic Variable and Fuzzy Rules
  - A linguistic variable consits of more than one fuzzy quantities.
  - Example:
    linguistic variable: temperature
    linguistic term (fuzzy quantity): {cold, normal, warm}



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3.3 Fuzzy Set Theory – Linguistic Variables and Modifiers

- Linguistic Variable and Fuzzy Rules
  - Causally determined dependencies are verbalised in form of „IF ... AND ... THEN .... ELSE ... IS ...“ - Rules
  - General form:
    if < Premisse (assumption)> then <Conclusion>
  - linguistic variables:
    - temperature : {cold, normal, warm}
    - Price of oil : {cheap, normal, expensive}
    - Consumption of oil : {low, middle, high}
  - Fuzzy rule:
    if *temp* is *low* and *price of oil* is *cheap* then *consumption of oil* is high

„Mathematics with words“

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3.3 Fuzzy Set Theory – Linguistic Variables and Modifiers

A fuzzy rule can have multiple premisses, for example:

IF < project_duration is long >
AND < project_staffing is large >
AND < project_funding is inadequate >
THEN < risk is high >

IF < service is excellent >
OR < food is delicious >
THEN < tip is generous >

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3.3 Fuzzy Set Theory – Linguistic Variables and Modifiers

The conclusion of a fuzzy rule can also include multiple parts, for instance:

IF < temperature is hot >
THEN < hot_water is reduced > AND < cold_water is increased >

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3.3 Fuzzy Set Theory – Linguistic Variables and Modifiers

- The range of possible values of a linguistic variable represents the universe of discourse of that variable. For example, the universe of discourse of the linguistic variable *speed* might have the range between 0 and 220 km/h and may include such fuzzy subsets as *very slow*, *slow*, *medium*, *fast*, and *very fast*.
- A linguistic variable carries with it the concept of fuzzy set qualifiers, called linguistic modifiers.
- Linguistic modifiers are terms that modify the shape of fuzzy sets. They include adverbs such as *very*, *somewhat*, *quite*, *more or less* and *slightly*.

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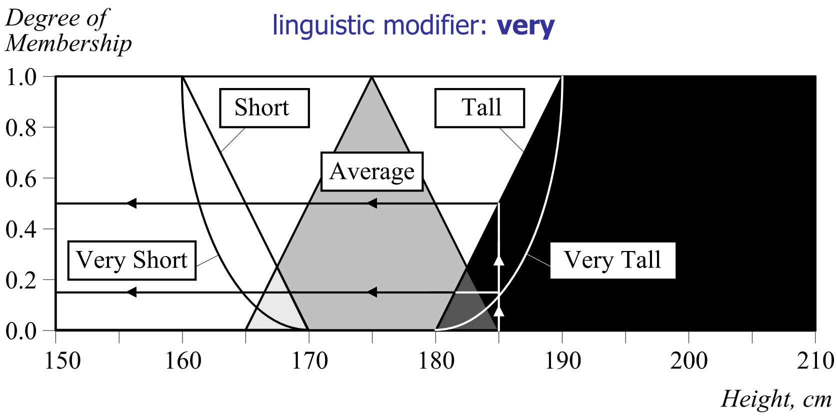
3.3 Fuzzy Set Theory – Linguistic Variables and Modifiers

linguistic modifier: **very**



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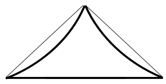
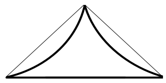
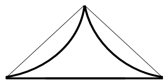
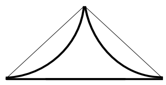
3.3 Fuzzy Set Theory – Linguistic Variables and Modifiers

Adverb	*Mathematical Expression*	*Graphical Representation*
A little	$[\mu_A(x)]^{1.3}$	
Slightly	$[\mu_A(x)]^{1.7}$	
Very	$[\mu_A(x)]^{2}$	
Extremely	$[\mu_A(x)]^{3}$	

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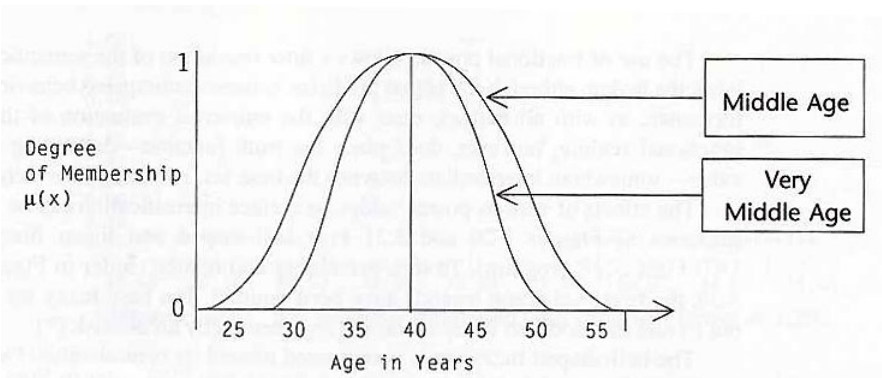
3.3 Fuzzy Set Theory – Linguistic Variables and Modifiers

- Fuzzy Sets – example 3.3-4



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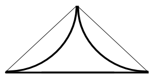
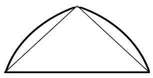
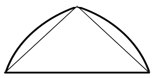
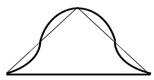
3.3 Fuzzy Set Theory – Linguistic Variables and Modifiers

Adverb	*Mathematical Expression*	*Graphical Representation*
Very very	$[\mu_A(x)]^4$	
More or less	$\sqrt{\mu_A(x)}$	
Somewhat	$\sqrt{\mu_A(x)}$	
Indeed	$2\,[\mu_A(x)]^2$ if $0 \le \mu_A \le 0.5$; $1 - 2\,[1 - \mu_A(x)]^2$ if $0.5 < \mu_A \le 1$	

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3.3 Fuzzy Set Theory – Limitations

- Choosing Membership Functions
  - Like choosing probabilities for Bayesian approaches, in the absence of solid evidence or data, the design of membership functions can be difficult.
  - However, … (☺) [**There is hope!**]
- This problem can be solved by using
  - **unimodal potential functions** based on
  - **measurement results**.

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3.3 Fuzzy Set Theory – Summary

- Fuzzy logic reflects how people think.
- It attempts to model our sense of words, our decision making and our common sense.
- As a result, it is leading to new, more human, intelligent systems.
- Like choosing probabilities for Bayesian approaches, in the absence of solid evidence or data, the design of membership functions can be difficult.

Information Fusion Only valid with verbal explanations Prof. Dr.-Ing. Volker Lohweg 45