Probability and Statistics

2 – Probability

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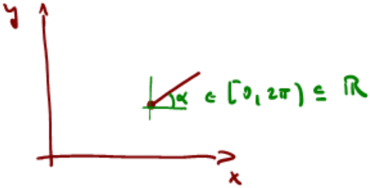
Non-countable Infinite Sample Spaces

Example (2.23)

A needle is thrown randomly and its direction $\alpha \in [0, 2\pi)$ (with respect to some fixed orientation in the plane) is measured. The result can be considered to be a sample of the sample space $\Omega = [0, 2\pi)$.



$\mathcal{P}([0, 2\pi))$ has no probability measure

$\mathcal{A} \subseteq \mathcal{P}([0, 2\pi))$ σ-algebra generated by all subintervals of $[0, 2\pi)$

$$\langle J \subseteq [0, 2\pi) \mid J \text{ is an interval} \rangle$$

$$I = [a, b) \subseteq [0, 2\pi), \quad \Pr([a, b)) = \frac{b-a}{2\pi}$$

Conditional probabilities

Definition (2.24)

Let Pr be a probability measure defined on a set of events $\mathcal{A}$ of a sample space Ω and let $A, B \in \mathcal{A}$. If $\Pr(B) \neq 0$, then the *conditional probability* of A given the event B is defined to be:

$$\Pr(A|B) \quad := \quad \frac{\Pr(A \cap B)}{\Pr(B)}$$

Ex. $\Omega = \{1, 2, 3, 4, 5, 6\}$

$A = \{1, 2, 3\}$ $\quad \Pr(A) = 1/2$

$B = \{2, 4, 6\}$ $\quad \Pr(B) = 1/2$

$\Pr(A|B) = \frac{1}{3}$

Conditional probabilities

Lemma (2.25)

Let Pr *be a probability measure defined on a set of events* $\mathcal{A}$ *of a sample space* Ω *and let* $A, B \in \mathcal{A}$.

(i) *If* $\Pr(B) \neq 0$, *then:*

$$\Pr(A \cap B) = \Pr(A|B)\Pr(B)$$

$$= \Pr(B|A) \cdot \Pr(A) \qquad \Pr(A) \neq 0$$

(ii) *If* $\Pr(A) \neq 0$ *and* $\Pr(B) \neq 0$, *then:*

$$\Pr(B|A) = \frac{\Pr(A|B)\Pr(B)}{\Pr(A)}$$

Conditional probabilities

$\Omega = B \mathbin{\dot\cup} B^c$

Lemma (2.25)

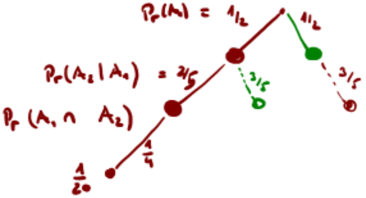
(iii) *If* $\Pr(B) \neq 0$ *and* $\Pr(B^c) \neq 0$, *then:*

$$\Pr(A) \;=\; \Pr(A|B)\Pr(B) + \Pr(A|B^c)\Pr(B^c)$$

$$\begin{aligned}
\Pr(A) &= \Pr(A \cap \Omega) = \Pr\big((A\cap B) \mathbin{\dot\cup} (A \cap B^c)\big)\\
&= \Pr(A\cap B) + \Pr(A \cap B^c)\\
&= \Pr(A|B)\cdot\Pr(B) + \Pr(A|B^c)\cdot\Pr(B^c)
\end{aligned}$$

Conditional probabilities

Q: Prob. to get 3 red balls after 3 drawings (without replacement)



A_i : Event that i'th ball is red

$Pr(A_1) = 1/2$

$Pr(A_2) = 1/2$

$Pr(A_1 \cap A_2) = Pr(A_2|A_1) \cdot Pr(A_1) = 1/5$

$$Pr((A_1 \cap A_2) \cap A_3) = \underbrace{Pr(A_3|(A_1 \cap A_2))}_{1/4} \cdot \underbrace{Pr(A_1 \cap A_2)}_{1/5}$$

$$= 1/20$$

Law of total probability

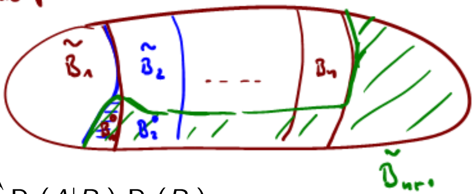
Theorem (2.26)

Let Pr *be a probability measure defined on a set of events* $\mathcal{A}$ *of a sample space* Ω*. Furthermore, let* $B_i \in \mathcal{A}$ $(i = 1, \ldots, n)$ *with* $\Pr(B_i) \neq 0$ *for* $i = 1, \ldots, n$ *and:*

- $\Pr(B_i \cap B_j) = 0$ *for* $i \neq j$
- $\displaystyle\sum_{i=1}^{n} \Pr(B_i) \;=\; 1$

typical example: $\Omega = \dot{\bigcup}_{i=1}^{n} B_i$

more general



Then for any $A \in \mathcal{A}$ *the following holds:*

(i) Law of total probability*:*

$$\Pr(A) \;=\; \sum_{i=1}^{n} \Pr(A|B_i)\,\Pr(B_i)$$

Law of total probability

- $\Pr(B_i \cap B_j) = 0$ for $i \neq j$
- $\displaystyle\sum_{i=1}^{n} \Pr(B_i) \; = \; 1$

Assume $\Omega = \dot{\bigcup}_{i=1}^{n} B_i$

$$A = A \cap \Omega = A \cap \dot{\bigcup_{i=1}^{n}} B_i = \dot{\bigcup_{i=1}^{n}} (A \cap B_i)$$

$$\Pr(A) = \sum_{i=1}^{n} \Pr(A \cap B_i) = \sum_{i=1}^{n} \Pr(A \mid B_i) \cdot \Pr(B_i)$$

Bayes' rule

Theorem (2.26)

Let Pr *be a probability measure defined on a set of events* $\mathcal{A}$ *of a sample space* Ω*. Furthermore, let* $B_i \in \mathcal{A}$ $(i = 1, \dots, n)$ *with* $\Pr(B_i) \neq 0$ *for* $i = 1, \dots, n$ *and:*

- $\Pr(B_i \cap B_j) = 0$ *for* $i \neq j$
- $\sum_{i=1}^{n} \Pr(B_i) = 1$

Then for any $A \in \mathcal{A}$ *the following holds:*

(ii) Bayes' rule*:*

$$\Pr(B_j|A) = \frac{\Pr(A|B_j)\,\Pr(B_j)}{\sum_{i=1}^{n} \Pr(A|B_i)\,\Pr(B_i)} \qquad \text{if} \quad \Pr(A) \neq 0$$

Proof of Bayes' rule

$$\Pr(B_j|A) = \frac{\Pr(A|B_j)\Pr(B_j)}{\sum_{i=1}^{n}\Pr(A|B_i)\Pr(B_i)} \underset{(i)}{=} \frac{\Pr(A \cap B_j)}{\Pr(A)} \quad \checkmark$$

Independent Events

Definition (2.27)

Let Pr be a probability measure defined on a set of events $\mathcal{A}$ of a sample space Ω. $A, B \in \mathcal{A}$ are called *(stochastically) independent* if:

$$\boxed{\Pr(A \cap B) \quad = \quad \Pr(A)\Pr(B)}$$

$\Pr(B) \neq 0 \Rightarrow \Pr(A|B) = \frac{\Pr(A \cap B)}{\Pr(B)} = \frac{\Pr(A) \cdot \cancel{\Pr(B)}}{\cancel{\Pr(B)}} = \Pr(A)$

$\Pr(A) \neq 0 \Rightarrow \Pr(B|A) = \Pr(B)$

Independent Events

Remark (2.28)

If $A, B \in \mathcal{A}$ are events with $\Pr(A) \neq 0$ and $\Pr(B) \neq 0$. Then the following statements are equivalent:

- A and B are stochastically independent.
- $\Pr(A|B) = \Pr(A)$
- $\Pr(B|A) = \Pr(B)$

Proof: last slide

Independent Events

Definition (2.29)

Let Pr be a probability measure defined on a set of events $\mathcal{A}$ of a sample space Ω. Then, $A_i \in \mathcal{A}$ $(i \in I)$ are called *independent* if

$$\Pr\left(\bigcap_{j\in J} A_j\right) = \prod_{j\in J} Pr(A_j)$$

for every finite subset $J \subseteq I$.