Probability and Statistics

3 – Discrete Random Variables

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Lemma (3.12)

Let X be a discrete random variable. Then, every function $g : \mathbb{R} \to \mathbb{R}$ defines a random variable:

$$Y \; = \; g(X) \; := \; g \circ X$$

(i) *The pmf of Y is given by:*

$$p_Y(y) \; = \sum_{x \in g^{-1}(y)} p_X(x) \qquad \text{for all } y \in \mathbb{R}$$

(ii) *If $E(Y)$ exists, then:*

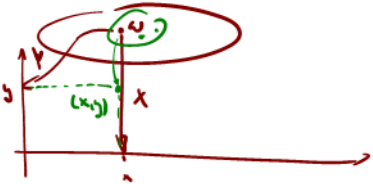
$$\boxed{E(Y) \; = \; \sum_{x \in \mathbb{R}} g(x) \cdot p_X(x)}$$

Joint Distributions (Joint Probability Mass Functions)

Definition (3.21)

Let X, Y be discrete random variables with respect to the same probability measure Pr, i.e. with respect to the same triple $(\Omega, \mathcal{A}, \mathrm{Pr})$. The *joint probability mass function* of X and Y, $p_{XY} : \mathbb{R}^2 \to [0, 1]$, is defined by:

$$p_{XY}(x, y) \;:=\; \mathrm{Pr}(X = x, Y = y) \;=\; \mathrm{Pr}\left(X^{-1}(x) \cap Y^{-1}(y)\right) \qquad \text{for all } \; x, y \in \mathbb{R}$$



Marginal Distributions

Lemma (3.22)

Let p_{XY} be the joint probability mass function of two random variables X and Y. The probability mass functions of X and Y are determined from the marginal probability mass functions *of p_{XY} as follows:*

$$(i) \qquad p_X(x) = \sum_{y\in\mathbb{R}} p_{XY}(x,y)$$

$$(ii) \qquad p_Y(y) = \sum_{x\in\mathbb{R}} p_{XY}(x,y)$$

ad (i) $\quad p_X(x) = \Pr(\{\omega \mid X(\omega)=x\}) = \Pr(\{\omega \mid X(\omega)=x\}) = \Pr\Big(\bigcup_{y\in\mathbb{R}} \{\omega \mid X(\omega)=x, Y(\omega)=y\}\Big)$

$= \sum_{y\in\mathbb{R}} \Pr(\{\omega \mid X(\omega)=x, Y(\omega)=y\})$

$p_Y(y) \neq 0$

Marginal Distributions

Remark (3.23)

Given two random variables whose probability mass functions are $\neq 0$ for only finitely many numbers, i.e. there are $m, n \in \mathbb{N}$ such that

$$\mathcal{X} \;:=\; \{x \in \mathbb{R} \mid p_X(x) \neq 0\} \;=\; \{x_1, \ldots, x_m\}$$

and:

$$\mathcal{Y} \;:=\; \{y \in \mathbb{R} \mid p_Y(y) \neq 0\} \;=\; \{y_1, \ldots, y_n\}$$

If $p_{XY}(x, y) \neq 0$ for some $(x, y) \in \mathbb{R}^2$, then $p_X(x) \neq 0$ and $p_Y(y) \neq 0$ by ($\overset{\text{def}}{\cancel{3.18}}$), i.e. $x \in \mathcal{X}$ and $y \in \mathcal{Y}$. Hence $p_{XY}(x, y) \neq 0$ may only hold, if $(x, y) \in \mathcal{X} \times \mathcal{Y}$, and p_{XY} is completely determined by the $m \cdot n$ numbers:

$$p_{ij} \;:=\; p_{XY}(x_i, y_j) \qquad (i = 1, \ldots, m,\; j = 1, \ldots, n)$$

Marginal Distributions

Putting these values in a tabular scheme, the values for the marginal probability mass functions can be calculated by summing up all entries from a row or column of the table, respectively:

	y_1	y_2	$\dots$	y_n	
x_1	p_{11}	p_{12}	$\dots$	p_{1n}	$\sum_{j=1}^{n} p_{1j} = p_X(x_1)$
x_2	p_{21}	p_{22}	$\dots$	p_{2n}	$\sum_{j=1}^{n} p_{2j} = p_X(x_2)$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
x_m	p_{m1}	p_{m2}	$\dots$	p_{mn}	$\sum_{j=1}^{n} p_{mj} = p_X(x_m)$
	$\sum_{i=1}^{m} p_{i1}$ $= p_Y(y_1)$	$\sum_{i=1}^{m} p_{i2}$ $= p_Y(y_2)$	$\dots$ $\dots$	$\sum_{i=1}^{m} p_{in}$ $= p_Y(y_n)$	

$g : \mathbb{R}^2 \to \mathbb{R}, \quad Z = g(X, Y) : \Omega \to \mathbb{R}$

Lemma (3.24)

Let p_{XY} be the joint probability mass function of two random variables X, Y. Given any function $g : \mathbb{R}^2 \to \mathbb{R}$, a random variable

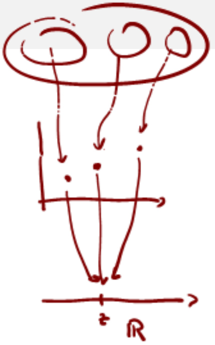
$$Z = g(X, Y) : \Omega \to \mathbb{R}$$

can be defined by

$$Z(\omega) = g(X, Y)(\omega) := g(X(\omega), Y(\omega)) \qquad \text{for all } \omega \in \Omega$$

and the following holds:

$g : \mathbb{R}^2 \to \mathbb{R}, \quad Z = g(X,Y) : \Omega \to \mathbb{R}$



(i)

$$p_Z(z) = \sum_{(x,y)\in g^{-1}(z)} p_{XY}(x,y) \qquad \text{for all } z \in \mathbb{R}$$

(ii)

$$E(Z) = \sum_{(x,y)\in\mathbb{R}^2} g(x,y)\cdot p_{XY}(x,y)$$

(i) $P_Z(z) = \Pr(Z^{-1}(z)) = \Pr(\{\omega \mid (X(\omega),Y(\omega)) \in g^{-1}(z)\}) = \bigcup_{(x,y)\in g^{-1}(z)} \Pr(\{\omega \mid (X(\omega),Y(\omega)) = (x,y)\})$

$= \sum_{(x,y)\in g^{-1}(z)} P_{XY}(x,y)$

(ii) Similar to the proof of (3.12).

$Z = X + Y$

Application: $X_i \sim \text{Bernoulli}(p)$

$E(X_i) = p$

$E(X_1 + \dots + X_n) = E(X_1) + \dots + E(X_n)$
$= n \cdot p$

Lemma (3.25)

Let X, Y be discrete random variables with respect to the same probability measure Pr*, i.e. with respect to the same triple* $(\Omega, \mathcal{A}, \Pr)$*. Then:*

$$E(X+Y) = E(X) + E(Y)$$

$g : \mathbb{R}^2 \to \mathbb{R}, \quad (x,y) \mapsto x+y$

$$E(X+Y) \overset{(3.24)(ii)}{=} \sum_{(x,y)\in\mathbb{R}^2} g(x,y)\cdot p_{XY}(x,y) = \sum_{(x,y)\in\mathbb{R}^2} (x+y)\cdot p_{XY}(x,y)$$

$$= \sum_{(x,y)} x \cdot p_{XY}(x,y) + \sum_{(x,y)} y \cdot p_{XY}(x,y) = \sum_{x\in\mathbb{R}} x \cdot \underbrace{\sum_{y\in\mathbb{R}} p_{XY}(x,y)}_{p_X(x)} + \sum_y y \cdot p_Y(y) = E(X) + E(Y)$$

Selected Discrete Probability Distributions

- Uniform Distributions
- Bernoulli Distributions
- Binomial Distributions
- Geometric Distributions
- Negative Binomial Distributions
- Poisson Distributions
- Hypergeometric Distributions

Binomial Random Variables

Ex.: $(X_1 = 1) = \{\omega \in \Omega \mid X_1(\omega) = 1\}$
$= \{(1,0,0), (1,0,1), (1,1,0), (1,1,1)\}$

$- \; p_X(2) = 3 \cdot p^2 (1-p)^1 = \binom{3}{2} p^2 (1-p)^{3-2}$

Definition (3.49)

A *binomial random variable* is a random variable X having a distribution given by:

$$p_X(i) \; := \; \binom{n}{i} p^i (1-p)^{n-i} \qquad \text{for} \;\; i = 0, 1, \ldots, n$$

where $n \in \mathbb{N}$ and $p \in (0,1)$ are fixed parameters. This may be denoted by $X \sim \mathrm{binomial}(n,p)$.

Ex.: $n = 3$, $\Omega = \{(x_1, x_2, x_3) \mid x_i \in \{0,1\}\}$

$p_X(2) = \Pr(\{(1,1,0), (1,0,1), (0,1,1)\})$

$\Pr((1,1,0)) = \Pr((X_1 = 1) \cap (X_2 = 1) \cap (X_3 = 0))$
$= \Pr(X_1 = 1) \cdot \Pr(X_2 = 1) \cdot \Pr(X_3 = 0)$
$= \quad p \quad \cdot \quad p \quad \cdot \; (1-p)$

Binomial Theorem

$1 \overset{!}{=} \sum_{i=0}^{n} p_X(i) = \sum_{i=0}^{n} \binom{n}{i} p^i (1-p)^{n-i} \overset{\text{Binomial Theorem}}{=} (p + (1-p))^n = 1^n = 1$

n independent Bernoulli experiments $X_1, X_2, \ldots, X_n \sim \mathrm{Bernoulli}(p)$

$(X_1, X_2, \ldots, X_n) \to (1, 0, 0, 1, \ldots, 1)$

$\Pr\left(\sum_{j=1}^{n} X_j = i\right) = p_X(i)$

Random Experiments with Binomial Distributions

Random Experiment: Perform n similar Bernoulli experiments and count the total number of "successes".

- Parameters: $p \in (0,1)$, $n \in \mathbb{N}$
- $\Omega = \{1,0\}^n = \{(\omega_1, \omega_2, \ldots, \omega_n) \mid \omega_i \in \{0,1\}\}$
- $$\Pr(\omega) = p^{wt(\omega)}(1-p)^{n-wt(\omega)}$$

  Here, $wt(\omega)$ denotes the *weight* of $\omega = (\omega_1, \omega_2, \ldots, \omega_n)$:

  $$wt((\omega_1, \omega_2, \ldots, \omega_n)) \quad := \quad \sum_{i=1}^{n} \omega_i$$
- $wt : \Omega \to \mathbb{R}$ is a random variable with $wt \sim \mathrm{binomial}(n,p)$.
- https://www.randomservices.org/random/apps/BinomialCoinExperiment.html

Binomial Random Variables

Lemma (3.50)

If $X \sim \mathrm{binomial}(n, p)$, then:

(i) $E(X) = \sum_{i=0}^{n} i \cdot p_X(i) = \sum_{i=0}^{n} i \cdot \binom{n}{i} p^i (1-p)^{n-i} = \overset{\text{Exercise}}{\cdots} = n \cdot p$

$\phi_X'(0)$

$E(X^2) = \phi_X''(0)$

(ii) $Var(X) = \underline{\sum_{i=0}^{n} i^2 p_X(i)} - (n \cdot p)^2 = \cdots = \, ?$

(iii) $\phi_X(t) = \sum_{i=0}^{n} e^{ti} \binom{n}{i} p^i (1-p)^{n-i} = \sum_{i=0}^{n} \binom{n}{i} \cdot (p \cdot e^t)^i (1-p)^{n-i} = (p \cdot e^t + 1 - p)^n$