Probability and Statistics

6 – Hypothesis Testing

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$X_i \sim \mathcal{N}(\mu, \sigma)$: Two-sided tests concerning μ

$$\Pr\left(|\overline{X} - \mu_0| > c \mid \underline{\mu = \mu_0}\right) \;=\; \alpha$$

$$\Pr\left(-c \;<\; \overline{X} - \mu \;<\; c\right) \;=\; 1 - \alpha$$

$$\iff \quad V := \frac{\sqrt{n}}{\sigma}\,|\bar{X} - \mu_0| < \Phi^{-1}\left(1 - \frac{\alpha}{2}\right)$$

$$\Pr\left(-\frac{c}{\sigma/\sqrt{n}} \;<\; \underbrace{\overset{=V}{\frac{\overline{X} - \mu}{\sigma/\sqrt{n}}}}_{\sim \mathcal{N}(0,1)} \;<\; \frac{c}{\sigma/\sqrt{n}}\right) \;=\; 1 - \alpha \quad\iff\quad \frac{c}{\sigma/\sqrt{n}} \;=\; \Phi^{-1}\left(1 - \frac{\alpha}{2}\right)$$

$$\iff \quad c \;=\; \frac{\sigma}{\sqrt{n}} \cdot \Phi^{-1}\left(1 - \frac{\alpha}{2}\right)$$

$$\Pr\left(-\frac{c}{S/\sqrt{n}} \;<\; \frac{\overline{X} - \mu}{S/\sqrt{n}} \;<\; \frac{c}{S/\sqrt{n}}\right) \;=\; 1 - \alpha \quad\iff\quad c \;=\; \frac{S}{\sqrt{n}} \cdot F^{-1}_{t_{n-1}}\left(1 - \frac{\alpha}{2}\right)$$

$X_i \sim \mathcal{N}(\mu, \sigma)$: Two-sided tests concerning μ

- H_0 is rejected if:

$$|\overline{x} - \mu_0| > c \iff \frac{\sqrt{n}}{\sigma} \cdot |\overline{x} - \mu_0| > \Phi^{-1}\left(1 - \frac{\alpha}{2}\right)$$

← σ known

$$\frac{\sqrt{n}}{S} \cdot |\overline{x} - \mu_0| > F^{-1}_{t_{n-1}}\left(1 - \frac{\alpha}{2}\right)$$

← σ estimated

- H_0 is accepted if:

$$|\overline{x} - \mu_0| < c \iff \frac{\sqrt{n}}{\sigma} \cdot |\overline{x} - \mu_0| < \Phi^{-1}\left(1 - \frac{\alpha}{2}\right)$$

$$\frac{\sqrt{n}}{S} \cdot |\overline{x} - \mu_0| < F^{-1}_{t_{n-1}}\left(1 - \frac{\alpha}{2}\right)$$

$X_i \sim \mathcal{N}(\mu, \sigma)$: Two-sided tests concerning μ, σ is known

Starting with a set of sampled data,

$$v \quad := \quad \frac{\sqrt{n}}{\sigma} \cdot |\overline{x} - \mu_0| \quad = \quad \Phi^{-1}(1 - \alpha_{\overline{x}}/2)$$

can be calculated and the maximal value for α can be determined, such that H_0 will be accepted on the basis of the data sampled. This value

$$\alpha_{\overline{x}} \quad := \quad 2(1 - \Phi(v))$$

is called the *p-value* of the sample.

Example (Ross, Chapter 8, Exc. 4)

In a certain chemical process, it is very important that a particular solution that is to be used as a reactant have a pH of exactly 8.20. A method for determining pH that is available for solutions of this type is known to give measurements that are normally distributed with a mean equal to the actual pH and with a standard deviation of 0.02. Suppose 10 independent measurements yielded the following pH values:

$$8.18, \quad 8.17, \quad 8.16, \quad 8.15, \quad 8.17, \quad 8.21, \quad 8.22, \quad 8.16, \quad 8.19, \quad 8.18$$

$H_0 : \mu = 8.2$, $\sigma = 0.02$, $\bar{x} = 8.179$ $\qquad |\bar{x} - \mu| = 0.021 = d$

(a) Accept or reject H_0 at a level of significance $\alpha = 0.05$ or $\alpha = 0.10$.

(b) Calculate the p-value.

$\frac{\sqrt{10}}{0.02} \cdot d \approx 3.32 > 1.96 \approx \Phi^{-1}(0.975) \rightarrow$ reject

$> 1.65 \approx \Phi^{-1}(0.95) \rightarrow$ reject

$\alpha_* \approx 8.99 \cdot 10^{-4}$

Operating Characteristic

$X_i \sim \mathcal{N}(\mu, \sigma)$, σ is known

While the probability of the occurrence of a type I error is set to α, the probability of the occurrence of a type II error depends on the true value of μ.

$H_0: \mu = \mu_0$

Definition (6.2)

The value of this probability (that H_0 is accepted when the true value of the mean is $\mu \neq \mu_0$) is given by the so-called *operating characteristic*:

$$\beta(\mu) \quad = \quad \Pr(H_0 \text{ is accepted} \mid \text{mean is } \mu)$$

$$\lim_{\mu \to \mu_0} \beta(\mu) = 1 - \alpha$$

$X_i \sim \mathcal{N}(\mu, \sigma)$, known σ: Operating Characteristic

α fixed significance level

$$\beta(\mu) = \Pr(H_0 \text{ is accepted} \mid \text{mean is } \mu)$$

$$= \Pr\left(-\Phi^{-1}\left(1-\frac{\alpha}{2}\right) \le \frac{\overline{X}-\mu_0}{\sigma/\sqrt{n}} \le \Phi^{-1}\left(1-\frac{\alpha}{2}\right) \,\middle|\, \text{mean is } \mu \right)$$

$= \checkmark$

$\leftarrow + \frac{\mu_0-\mu}{\sigma/\sqrt{n}}$

$$= \Pr\left(\underline{\frac{\mu_0-\mu}{\sigma/\sqrt{n}}} - \Phi^{-1}\left(1-\frac{\alpha}{2}\right) \le \frac{\overline{X}-\mu}{\sigma/\sqrt{n}} \le \underline{\frac{\mu_0-\mu}{\sigma/\sqrt{n}}} + \Phi^{-1}\left(1-\frac{\alpha}{2}\right) \,\middle|\, \text{mean is } \mu \right)$$

$\sim \mathcal{N}(0,1)$

$$\beta(\mu) = \Phi\left(\frac{\mu_0-\mu}{\sigma/\sqrt{n}} + \Phi^{-1}\left(1-\frac{\alpha}{2}\right) \right) - \Phi\left(\frac{\mu_0-\mu}{\sigma/\sqrt{n}} - \Phi^{-1}\left(1-\frac{\alpha}{2}\right) \right)$$

$\mu_0 - \mu < 0$

$X_i \sim \mathcal{N}(\mu, \sigma)$, known σ: Operating Characteristic

Lemma (6.3)

(i)

$$\beta(\mu) = \Phi\left(\frac{\mu-\mu_0}{\sigma/\sqrt{n}} + \Phi^{-1}\left(1-\frac{\alpha}{2}\right)\right) - \cancel{\Phi\left(\frac{\mu-\mu_0}{\sigma/\sqrt{n}} - \Phi^{-1}\left(1-\frac{\alpha}{2}\right)\right)}$$

(ii) *For small values of* α:

$\mu - \mu_0 < 0$

$$\beta(\mu) \lessapprox \Phi\left(-\frac{|\mu_0-\mu|}{\sigma/\sqrt{n}} + \Phi^{-1}\left(1-\frac{\alpha}{2}\right)\right)$$

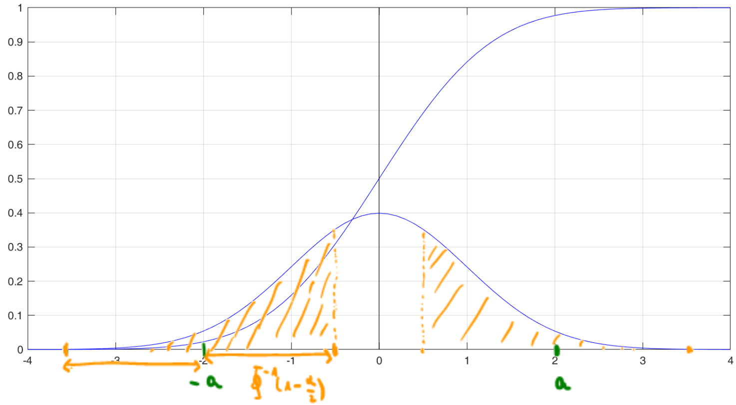
(iii) *Assume* $\mu \neq \mu_0$ *and* $\beta \in (0,1)$. *Then* H_0 *will be accepted with a probability* $\beta(\mu) < \beta$ *if:*

$$n := \left\lceil \frac{\left(\Phi^{-1}(1-\frac{\alpha}{2}) + \Phi^{-1}(1-\beta)\right)^2 \sigma^2}{(\mu_0-\mu)^2} \right\rceil$$

Lemma (6.3)(i)

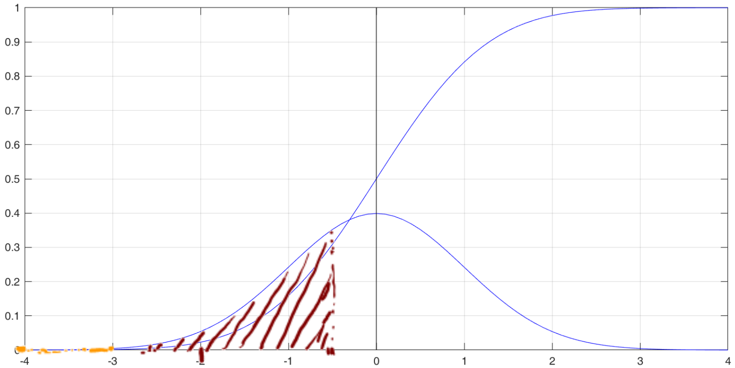
$\Phi(-a + \Phi^{-1}(1-\frac{\alpha}{2})) - \Phi(-a - \Phi^{-1}(1-\frac{\alpha}{2})) = \beta(\mu)$

$$\beta(\mu) = \Phi\Big(\underbrace{\frac{\mu-\mu_0}{\sigma/\sqrt{n}}}_{a} + \Phi^{-1}\Big(1-\frac{\alpha}{2}\Big)\Big) - \Phi\Big(\frac{\mu-\mu_0}{\sigma/\sqrt{n}} - \Phi^{-1}\Big(1-\frac{\alpha}{2}\Big)\Big)$$



Lemma (6.3)(ii)

$$\beta(\mu) \lessapprox \Phi\left(-\frac{|\mu_0-\mu|}{\sigma/\sqrt{n}} + \Phi^{-1}\left(1-\frac{\alpha}{2}\right)\right)$$



Lemma (6.3)(iii)

Assume $\mu \neq \mu_0$ and $\beta \in (0,1)$. Then H_0 will be accepted with a probability $\beta(\mu) < \beta$ if:

$$n \quad := \quad \left\lceil \frac{\left(\Phi^{-1}(1-\frac{\alpha}{2}) + \Phi^{-1}(1-\beta)\right)^2 \sigma^2}{(\mu_0-\mu)^2} \right\rceil \qquad (*)$$

(ii) : $\quad \beta(\mu) \lessapprox \Phi\left(-\frac{|\mu_0-\mu|}{\sigma/\sqrt{n}} + \Phi^{-1}\left(1-\frac{\alpha}{2}\right)\right) < \beta$

$\Leftrightarrow \quad \Phi^{-1}(\beta) > -\frac{|\mu_0-\mu|}{\sigma/\sqrt{n}} + \Phi^{-1}(1-\frac{\alpha}{2})$

$\Leftrightarrow \quad \frac{\sqrt{n}}{\sigma}|\mu_0-\mu| > \Phi^{-1}(1-\frac{\alpha}{2}) + \Phi^{-1}(1-\beta)$

$\Leftrightarrow \quad n > (*)$

$X_i \sim \mathcal{N}(\mu, \sigma)$, known σ: Operating Characteristic

doted lines show estimation from Lemma (6.3) (ii)

$1-\alpha$