Probability and Statistics

3 – Discrete Random Variables

Stefan Heiss

Technische Hochschule Ostwestfalen-Lippe
Dep. of Electrical Engineering and Computer Science

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$Var(X_1 + \cdots + X_n)$

Corollary (3.38)

Let $X_1, \ldots, X_n$ be random variables, such that X_i and X_j are uncorrelated if $i \neq j$. Then:

$$Var\left(\sum_{i=1}^{n} X_i\right) = \sum_{i=1}^{n} Var(X_i) \qquad (*)$$

In particular: $(*)$ holds if $X_1, \ldots, X_n$ are independent!

Selected Discrete Probability Distributions

- Uniform Distributions
- Bernoulli Distributions
- Binomial Distributions
- Geometric Distributions
- Negative Binomial Distributions
- Poisson Distributions
- Hypergeometric Distributions

Binomial Random Variables

$X = X_1 + \dots + X_n \quad , \quad X_i \sim \text{Bernoulli}(p)$

X_i are independent

Lemma (3.50)

If $X \sim \text{binomial}(n, p)$*, then:*

(i) $E(X) = \sum_{i=1}^{n} E(X_i) = n \cdot p$ (3.25)

(ii) $Var(X) = \sum_{i=1}^{n} Var(X_i) = n \cdot p(1-p)$ (3.38)

(iii) $\phi_X(t) = \phi_{X_1 + \dots + X_n}(t) = (\phi_{X_1}(t))^n = ((1-p) + pe^t)^n$ (3.23) (iii)

Geometric Random Variables

Definition (3.51)

Let $p \in (0,1)$. A random variable X with a distribution given by

$$p_X(i) \;:=\; (1-p)^{i-1}p \qquad \text{for} \;\; i \in \mathbb{N}$$

is called a *geometric random variable*. This may be denoted by $X \sim \mathrm{geometric}(p)$.

Random Experiments with Geometric Distributions

Random Experiment: A Bernoulli experiment is repeated until the first success occurs.

- Parameters: $p \in (0,1)$
- $\Omega = \{\omega = (\omega_1, \omega_2, \ldots, \omega_i) \mid \omega_i \in \{0,1\}, \omega_1 = \ldots \omega_{i-1} = 0, \omega_i = 1\}$

$Pr(\{(1)\}) = p$ $\qquad X : \Omega \to \mathbb{R}$

$X((1)) = 1, \quad X^{-1}(1) = \{(1)\}, \quad P_X(1) = p$

$Pr(\{(0,1)\}) = (1-p)\cdot p$ $\qquad P_X(2) = (1-p)\cdot p$

$\vdots$ $\qquad \vdots$

$Pr(\{(\underbrace{0,\ldots,0}_{i-1},1)\}) = (1-p)^{i-1}\cdot p$ $\qquad P_X(i) = (1-p)^{i-1}\cdot p$

Geometric Random Variables

Lemma (3.52)

If $X \sim \text{geometric}(p)$, *then:*

(i) $E(X) = \frac{1}{p}$

(ii) $Var(X) = \frac{1}{p^2} - \frac{1}{p} = \frac{1}{p}\left(\frac{1}{p} - 1\right) = \frac{1-p}{p^2}$

(iii) $\phi_X(t) = \frac{pe^t}{1-(1-p)\cdot e^t}$

Pf.: see script

Negative Binomial Random Variables

Definition (3.54)

Let $p \in (0,1)$ and $n \in \mathbb{N}$. A *negative binomial random variable* (*Pascal random variable*) with parameters p and n is a random variable X having a distribution given by:

$$p_X(i) := \binom{i-1}{n-1} p^n (1-p)^{i-n} \qquad \text{for all } i \in \{\underline{n}, n+1, \dots\}$$

This may be denoted by $X \sim \mathrm{nbino}(n, p)$.

Random Experiments with Negative Binomial Distributions

Random Experiment: For a fixed $n \in \mathbb{N}$ a Bernoulli experiment is repeated until the n'th success occurs.

- Parameters: $p \in (0,1)$, $n \in \mathbb{N}$ ← Bernoulli (p)
- $\Omega = \{\omega = (\omega_1, \omega_2, \ldots, \omega_i) \mid \omega_i \in \{0,1\}, \underline{wt(\omega) = n}, \omega_i = 1\}$
- $n = 1$: geometric distribution

$P_X(i) = 0$ for $i < n$

$\omega = (0\,0\,\overset{1}{1}\,0\,0\,0\,\overset{2}{1}\,0\,\text{-------}\,\overset{n}{1})$ $\quad wt = n$

$(1-p)(1-p)\cdot p \;\cdots$ $\qquad i \geq n$

$Pr(\{\omega\}) = p^n (1-p)^{i-n}$

$\binom{i-1}{n-1}$

Negative Binomial Random Variables

$X = X_1 + \dots + X_n$, $X_i \sim$ geometric(p)

X_i's are independent

Lemma (3.55)

If $X \sim \mathrm{nbino}(n, p)$, *then:*

(i) $E(X) = \sum_{i=1}^{n} E(X_i) = n \cdot \frac{1}{p}$

(ii) $Var(X) = \sum_{i=1}^{n} Var(X_i) = n \cdot \frac{1-p}{p^2}$

(iii) $\phi_X(t) = \phi_{X_1+\dots+X_n}(t) = (\phi_{X_1}(t))^n = \left(\frac{pe^t}{1-(1-p)\cdot e^t}\right)^n$

Poisson Random Variables

Definition (3.56)

A *Poisson random variable* is a random variable X having a distribution given by

$$p_X(i) := e^{-\lambda}\frac{\lambda^i}{i!} \qquad \text{for all } \underline{i \in \mathbb{N}_0},$$

where $\underline{\lambda > 0}$ is some fixed parameter. This may be denoted by $X \sim \mathrm{Poisson}(\lambda)$.

$$\sum_{i=0}^{\infty} p_X(i) = \sum_{i=0}^{\infty} e^{-\lambda}\frac{\lambda^i}{i!} = e^{-\lambda}\cdot\sum_{i=0}^{\infty}\frac{\lambda^i}{i!} = e^{-\lambda}e^{\lambda} = 1$$

Random Experiments with Poisson Distributions

Random Experiment: Number of “arrivals” in a fixed time interval in a Poisson Process.

Examples of such “arrivals”:

- emited particles from a radioactive source,
- customers at a service station,
- requests at a web server,
- devices that fail and are replaced by new devices in some system.

- https://www.randomservices.org/random/apps/PoissonExperiment.html

Poisson Random Variables

Lemma (3.57)

If $X \sim \mathrm{Poisson}(\lambda)$, *then:*

(i) $E(X) = \sum_{i=\cancel{0}\,1}^{\infty} \cancel{i} \cdot e^{-\lambda} \cdot \frac{\lambda^{i}}{\cancel{i}\,(i-1)!} = e^{-\lambda} \lambda \sum_{i=1}^{\infty} \frac{\lambda^{i-1}}{(i-1)!} = e^{-\lambda} \cdot \lambda \cdot e^{\lambda} = \lambda$

(ii) $Var(X) = \lambda$ (Exercise)

(iii) $\phi_X(t) = e^{\lambda(e^{t}-1)}$ (Exercise)