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Information Fusion – Basics on ET

Combination Techniques for Uncertain Information in Measurement and Signal Processing

3.4 Possibility Theory

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Lectures – Contents

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3.4 Possibility Theory – Introduction

- The main researchers in this area:
  Didier Dubois and Henri Prade
  Université Paul Sabatier 31062 TOULOUSE, France
- Possibility theory is an uncertainty theory devoted to the handling of incomplete information.
- Similar to Probability Theory because it is based on set-functions.
- Differs by the use of a pair of dual set functions (possibility and necessity measures) instead of only one.
- It is not additive and makes sense on ordinal structures.

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3.4 Possibility Theory – Introduction

- In 1978 **Lotfi Zadeh**, coined the concept of Possibility Theory.
- Zadeh proposed an interpretation of membership functions of fuzzy sets as possibility distributions encoding flexible constraints induced by natural language statements.
- Relationship between possibility and probability:
  - **What is probable must preliminarily be possible.**
- Refers to the idea of graded feasibility ("degrees of ease") rather than to the epistemic notion of plausibility.
  (after Dubois & Prade)

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3.4 Possibility Theory – Introduction

- General statements
  - What is **possible may not** be **probable!**
  - What is **impossible is also improbable!**

- Interpretations of **Possibility**
  - **Feasibility:** *It is possible to do something* (physical)
  - **Plausibility**: *It is possible that something occurs* (epistemic)
  - **Consistency :** *Compatible with what is known* (logical)
  - **Permission:** *It is allowed to do something* (deontic, social rules)

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3.4 Possibility Theory – Introduction

- As imprecision and vagueness are rather possibilistic than probabilistic (frequentist's view) in its description (... *It is possible that we will have snow in summer, however, it is highly improbable* ...), it is reasonable to describe the meaning of information, especially the meaning of incomplete information within a possibilistic framework.

- We will concentrate on numerical Possibility Theory with its special branches based on consonant (nested) sets (Shafer, 1976) and on membership functions (Zadeh, 1978).

- Possibility Theory can be interpreted as a framework for handling incomplete information and aggregate information coming from multiple sources (sensors, experts, databases, etc.).

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3.4 Possibility Theory – Axioms

- Finite frame of discernment $\Theta = \{\theta_1, \theta_2, ..., \theta_n\}$
- Possibility measure is a mapping $\Pi : 2^{\Theta} \to [0,1]$
  - Axiom 1 (empty set)
    $$\Pi(\varnothing) = 0.$$
  - Axiom 2 (universal set)
    $$\Pi(\Theta) = 1.$$
  - Axiom 3 (maxitivity)
    $$\Pi(A \cup B) = \max[\Pi(A), \Pi(B)]$$

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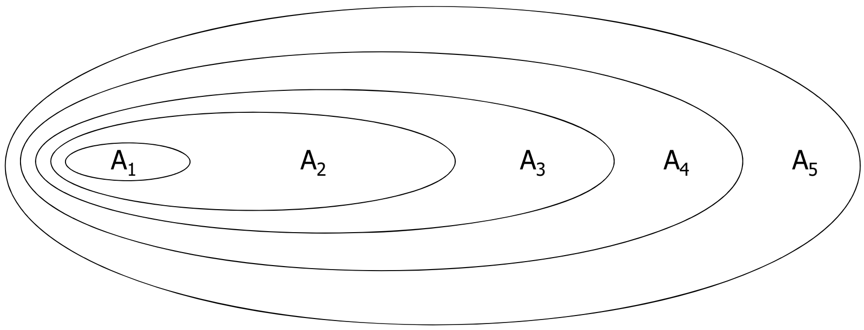
3.4 Possibility Theory – Consonant Sets

- Definition: **Consonant sets** aka **Nested sets**



Nested Sets: $A_1 \subset A_2 \subset A_3 \subset A_4 \subset A_5$ (Consonant Evidence)

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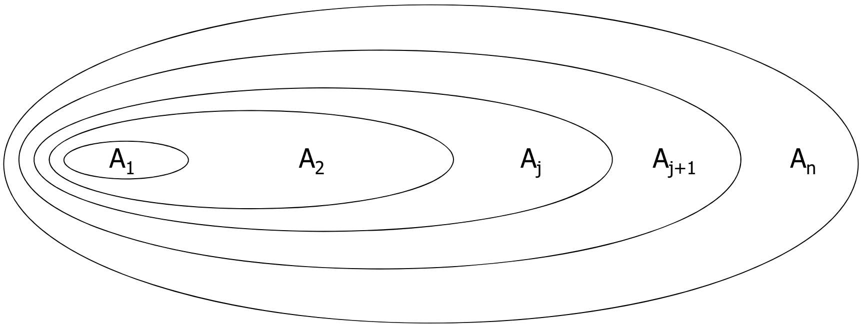
3.4 Possibility Theory – Consonant Sets

- Frame of discernment or universal set $\Theta = \{\theta_1, \theta_2, ..., \theta_n\}, n \in \mathbb{N}$
- Consonant Subset $A_i = \{\theta_1, \theta_2, ..., \theta_i\}, i \in \mathbb{N}, i \leq n$



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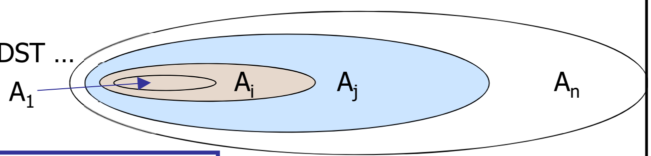
3.4 Possibility Theory – Consonant Sets

- Now let us go back to DST …



$$\boxed{\mathrm{Bel}(A_i \cap A_j) = \min\left[\mathrm{Bel}(A_i), \mathrm{Bel}(A_j)\right]}$$

- Proof (Minitivity-Lemma)

$$\forall i < j : A_i \cap A_j = A_i \Rightarrow \mathrm{Bel}(A_i \cap A_j) = \mathrm{Bel}(A_i),$$

$$\mathrm{Bel}(A_i) = \sum_{k=1}^{i} m(A_k) = \sum_{k=1}^{\min(i,j)} m(A_k) = \min\left[\sum_{k=1}^{i} m(A_k), \sum_{k=1}^{j} m(A_k)\right] = \cdots$$

$$\cdots = \min\left[\mathrm{Bel}(A_i), \mathrm{Bel}(A_j)\right],$$

$$\mathrm{Bel}(A_i \cap A_j) = \min\left[\mathrm{Bel}(A_i), \mathrm{Bel}(A_j)\right]. \qquad \blacksquare$$

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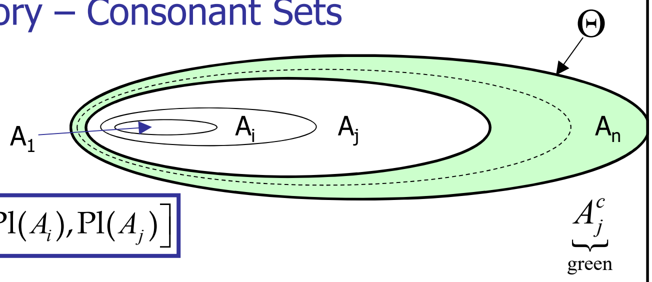
3.4 Possibility Theory – Consonant Sets

- Axiom 3 (maxitivity)



$$\boxed{\mathrm{Pl}(A_i \cup A_j) = \max\left[\mathrm{Pl}(A_i), \mathrm{Pl}(A_j)\right]}$$

- Proof

$$\forall i < j : A_i \cup A_j = A_j \Rightarrow \mathrm{Pl}(A_i \cup A_j) = \mathrm{Pl}(A_j),$$

$$\mathrm{Pl}(A_j) = 1 - \mathrm{Bel}(A_j^c) = 1 - \mathrm{Bel}\left[\left(A_i \cup A_j\right)^c\right] = 1 - \mathrm{Bel}(A_i^c \cap A_j^c) = \cdots$$

$$\cdots = 1 - \min\left[\mathrm{Bel}(A_i^c), \mathrm{Bel}(A_j^c)\right] = \max\left[1 - \mathrm{Bel}(A_i^c), 1 - \mathrm{Bel}(A_j^c)\right] = \cdots$$

$$\cdots = \max\left[\mathrm{Pl}(A_i), \mathrm{Pl}(A_j)\right] \rightarrow \mathrm{Pl}(A_i \cup A_j) = \max\left[\mathrm{Pl}(A_i), \mathrm{Pl}(A_j)\right]. \quad \blacksquare$$

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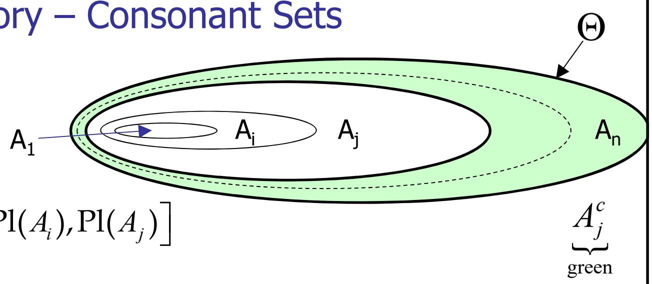
3.4 Possibility Theory – Consonant Sets

- Axiom 3 (maxitivity)



$$\mathrm{Pl}(A_i \cup A_j) = \max\left[\mathrm{Pl}(A_i), \mathrm{Pl}(A_j)\right]$$

- Proof

$$\forall i < j : A_i \cup A_j = A_j \Rightarrow \mathrm{Pl}(A_i \cup A_j) = \mathrm{Pl}(A_j),$$

$$\mathrm{Pl}(A_j) = 1 - \mathrm{Bel}(A_j^c) = 1 - \sum_{k=j+1}^{n} m(A_k) = 1 - \left[\sum_{k=1}^{n} m(A_k) - \sum_{k=1}^{j} m(A_k)\right] = \cdots$$

$$\cdots = 1 - \sum_{k=1}^{n} m(A_k) + \sum_{k=1}^{j} m(A_k) = \sum_{k=1}^{j} m(A_k) = \sum_{k=1}^{\max(i,j)} m(A_k) = \max\left[\sum_{k=1}^{i} m(A_k), \sum_{k=1}^{j} m(A_k)\right]$$

$$\mathrm{Pl}(A_j) = \mathrm{Pl}(A_i \cup A_j) = \max\left[\mathrm{Pl}(A_i), \mathrm{Pl}(A_j)\right]. \quad \blacksquare$$

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3.4 Possibility Theory – Consonant Sets

- Necessity and Possibility functions
  - Iff **Belief** and **Plausibility** are based on nested sets, then intersection and union of two nested sets are called **Necessity*** (credibility, confidence) and **Possibility** (functions).

$$\boxed{\begin{aligned}&\mathrm{Pl}(A_i \cup A_j) = \max\left[\mathrm{Pl}(A_i), \mathrm{Pl}(A_j)\right] \Leftrightarrow \Pi(A_i \cup A_j) = \max\left[\Pi(A_i), \Pi(A_j)\right]\\ &\mathrm{Bel}(A_i \cap A_j) = \min\left[\mathrm{Bel}(A_i), \mathrm{Bel}(A_j)\right] \Leftrightarrow \mathrm{N}(A_i \cap A_j) = \min\left[\mathrm{N}(A_i), \mathrm{N}(A_j)\right]\end{aligned}}$$

*Das Vertrauen darauf …
necessity: Notwendigkeit, Bedarf, Unumgänglichkeit
credibility: Glaubwürdigkeit
confidence: Vertrauen, Sicherheit

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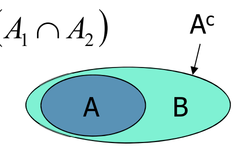
3.4 Possibility Theory – Consonant Sets

- Some more corollaries
  - From DST: $\mathrm{Bel}(A_1 \cup A_2) \geq \mathrm{Bel}(A_1) + \mathrm{Bel}(A_2) - \mathrm{Bel}(A_1 \cap A_2)$
  - Let: $A^c = B / A, \Theta = B = A \cup A^c$
    $\mathrm{Bel}(A \cup A^c) \geq \mathrm{Bel}(A) + \mathrm{Bel}(A^c) - \mathrm{Bel}(A \cap A^c)$
  - then $1 \geq \mathrm{Bel}(A) + \mathrm{Bel}(A^c)$.
  - It follows: $\boxed{\mathrm{N}(A) + \mathrm{N}(A^c) \leq 1}$



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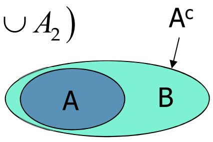
3.4 Possibility Theory – Consonant Sets

- Some more corollaries
  - From DST: $\mathrm{Pl}(A_1 \cap A_2) \le \mathrm{Pl}(A_1) + \mathrm{Pl}(A_2) - \mathrm{Pl}(A_1 \cup A_2)$
  - Let: $A^c = B / A, \Theta = B = A \cup A^c$
    $\mathrm{Pl}(A \cap A^c) \le \mathrm{Pl}(A) + \mathrm{Pl}(A^c) - \mathrm{Pl}(A \cup A^c)$



  - then $0 \le \mathrm{Pl}(A) + \mathrm{Pl}(A^c) - 1 \to 1 \le \mathrm{Pl}(A) + \mathrm{Pl}(A^c).$
  - It follows: $\boxed{\Pi(A) + \Pi(A^c) \ge 1}$
  - Furthermore: $\boxed{\Pi(A) = 1 - \mathrm{N}(A^c), \quad \mathrm{N}(A) = 1 - \Pi(A^c).}$

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3.4 Possibility Theory – Consonant Sets

- Some more corollaries
  - Let $A_i = A, A_j / A_i = A^c : \Theta = A \cup A^c, \varnothing = A \cap A^c; i < j$

$$\Pi(\Theta) = \max\left[\Pi(A), \Pi(A^c)\right] = 1 \qquad \text{(Axiom 1)}$$

$$\mathrm{N}(\varnothing) = 1 - \Pi(\varnothing^c) = 1 - \Pi(\Theta) = \min\left[\mathrm{N}(A), \mathrm{N}(A^c)\right] = 0$$

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3.4 Possibility Theory – Consonant Sets

- Possibility and Necessity implications based on
  $\max\left[\Pi(A),\Pi(A^c)\right]=1, \min\left[\mathrm{N}(A),\mathrm{N}(A^c)\right]=0$
  - Let $\boxed{\begin{array}{ll}\text{i)} & \mathrm{N}(A)>0 \Rightarrow \Pi(A)=1, \\ \text{ii)} & \Pi(A)<1 \Rightarrow \mathrm{N}(A)=0.\end{array}}$
  - Proof

$$\text{i)}\quad \mathrm{N}(A)>0 \Rightarrow \Pi(A)=1 \;\rightarrow\; \min\left[\mathrm{N}(A),\mathrm{N}(A^c)\right]=0 \rightarrow \mathrm{N}(A^c)=0$$

$$\Pi(A)=1-\mathrm{N}(A^c)=1.$$

$$\text{ii)}\quad \Pi(A)<1 \Rightarrow \mathrm{N}(A)=0 \;\rightarrow\; \max\left[\Pi(A),\Pi(A^c)\right]=1 \rightarrow \Pi(A^c)=1$$

$$\mathrm{N}(A)=1-\Pi(A^c)=0. \qquad \blacksquare$$

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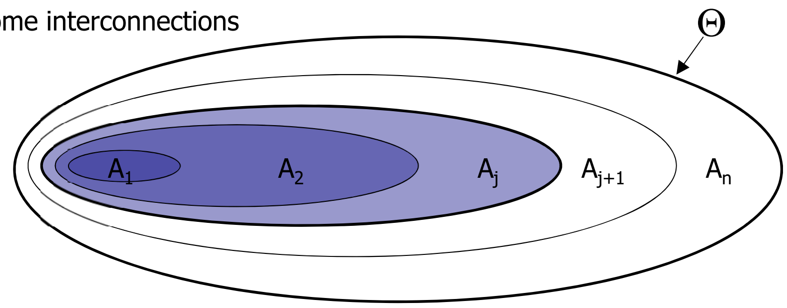
3.4 Possibility Theory – Consonant Sets

- Some interconnections



$$\mathrm{N}(A_j)=\sum_{k=1}^{j} m(A_k) \qquad \Pi(A_j)=\sum_{k=1}^{n} m(A_k)=1$$

$$\mathrm{N}(A_j^c)=0 \qquad \Pi(A_j^c)=\sum_{k=j+1}^{n} m(A_k)$$

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3.4 Possibility Theory – Possibility Distribution Function

- A **possibility distribution** is the representation of a state of knowledge:
  A description of how we think what the status of an event is.
- Let Θ be a frame of discernment and *A* be a subset $A \subset \Theta$.
- The variable θ is an ill-known description of an event taking its value on *A*.
- Let $L \in [0,1]$ be a possibility level for which $\theta \in A$ represents an existing, but unknown element (incomplete information) within in the crisp set *A*.
- A possibility distribution $\pi_x(\theta)$ attached to *x* is a mapping from Θ to L :

$$\forall \theta, \pi_x(\theta) \in L : \exists \theta_i \ \pi_x(\theta_i) = 1$$

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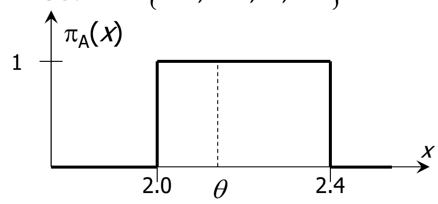
3.4 Possibility Theory – Possibility Distribution Function

- Conventions:
  $\pi_x(\theta)$ = 0 iff $x = \theta$ is impossible, totally excluded
  $\pi_x(\theta)$ = 1 iff $x = \theta$ is normal, fully plausible, unsurprizing

- Example 3.4-1
  - **Partial Ignorance**: I assume the pressure of a tyre is between 2.0 bar and 2.4 bar.
  - Set: $A = \{2.0, 2.1, \ldots, 2.4\}$



$$\pi_A(x) = 1 \, \middle| \, x = \theta, \theta \in A$$

$$\pi_A(x) = 0 \, \middle| \, x = \theta, \theta \notin A$$

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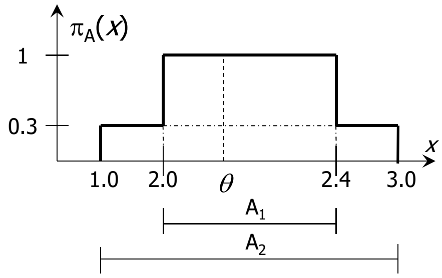
3.4 Possibility Theory – Possibility Distribution Function

- Example 3.4-2
  - **Partial Ignorance**: I assume the pressure of a tyre is between 2.0 bar and 2.4 bar. Maybe it is possible to a level L = 0.3 that the pressure is in the range of 1 to 3 bar.
  - Set: $A_2 = \{1.0, 1.1, ..., 3.0\}$, $A_1 = \{2.0, 2.1, ..., 2.4\}$



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3.4 Possibility Theory – Possibility Distribution Function

- Specitivity

Iff $\pi^` \leq \pi$ then $\pi^`$ is more specific than π in the wide sense.

  - Any value possible for $\pi^`$ should be at least as possible for $\pi$, that is, $\pi^`$ is more informative than π.

- Definitions
  - Complete knowledge: $\forall\, \theta_1 \in A : \pi_A(\theta_1) = 1$
    $\forall i \in \mathbb{N}, i \neq 1, \theta_i \in A : \pi_A(\theta_i) = 0$
  - Total ignorance: $\forall i \in \mathbb{N}, \theta_i \in A : \pi_A(\theta_i) = 1$
    $\forall i \in \mathbb{N}, \theta_i \notin A : \pi_A(\theta_i) = 0$

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3.4 Possibility Theory – Possibility Distribution Function

- Possibility measure
  - Theorem 3.4-1

    Every **possibility measure** $\Pi(\theta)$ on a finite power set 2^Θ is uniquely determined by a **possibility distribution function**

$$\pi : \Theta \to [0,1]$$

by the formulae

$$\pi_i = \pi_i(\theta) = \Pi(\{\theta_i\}),$$

$$\Pi(\theta) = \max_{\theta \in A} \pi(\theta).$$

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3.4 Possibility Theory – Possibility Distribution Function

- Possibility measure
  - Proof:

    Let $A_i \subset 2^\Theta$ and $i = 1 \Rightarrow A_1 = \{\theta_1\}, |A_1| = 1$

    It follows that $\Pi(\theta_1) = \max_{\theta_1 \in A_1} \pi(\theta_1) = \pi(\theta_1).$

    now let $A_{n-1} = \{\theta_1, \theta_2, \ldots, \theta_{n-1}\}, |A_{n-1}| = n-1$ and

$$A_n = \{\theta_1, \theta_2, \ldots, \theta_n\}, |A_n| = n.$$

then $\Pi(A_n) = \max_{\theta \in A_n}\left[\Pi(A_{n-1}), \Pi(\theta_n)\right] = \max_{\theta \in A_n}\left[\Pi(\{\theta_1, \theta_2, \ldots, \theta_{n-1}\}), \Pi(\theta_n)\right] = \cdots$

$$\cdots = \max_{\theta \in A_n}\left[\max_{\theta \in A_{n-1}}\left[\Pi(\{\theta_1, \theta_2, \ldots, \theta_{n-2}\}), \Pi(\theta_{n-1})\right], \Pi(\theta_n)\right] = \cdots$$

$$\cdots = \max_{\theta \in A_n}\left[\Pi(\theta_1), \ldots, \Pi(\theta_{n-1}), \Pi(\theta_n)\right] = \max_{\theta \in A_n} \pi(\theta).$$

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3.4 Possibility Theory – Possibility Distribution Function

- Degree of Possibility $\theta \in A \subset \Theta$

$$\Pi(A) = \max_{\theta \in A}\left(\pi_A(x) \mid x = \theta \in A\right)$$

- Degree of Necessity (confidence) $\theta \notin A \subset \Theta$

$$\mathrm{N}(A) = \min_{\theta \notin A}\left(1 - \pi_A(x) \mid x = \theta \notin A\right)$$

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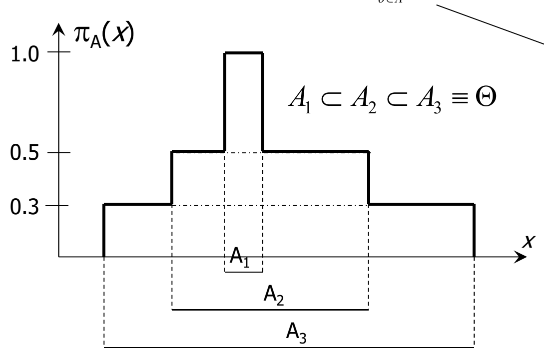
3.4 Possibility Theory – Possibility Distribution Function

- Example 3.4-3
  - Possibility distribution $\Pi(A) = \max_{\theta \in A}\left(\pi_A(x) \mid x = \theta \in A\right)$ $\mathrm{N}(A) = \min_{\theta \notin A}\left(1 - \pi_A(x) \mid x = \theta \notin A\right)$



$$\Pi(A_1) = 1.0, \quad \mathrm{N}(A_1) = 0.5$$
$$\Pi(A_2) = 1.0, \quad \mathrm{N}(A_2) = 0.7$$
$$\Pi(A_3) = 1.0, \quad \mathrm{N}(A_3) = 1.0$$

$$\Pi(A_1^c) = 0.5, \quad \mathrm{N}(A_1^c) = 0.0$$
$$\Pi(A_2^c) = 0.3, \quad \mathrm{N}(A_2^c) = 0.0$$
$$\Pi(A_3^c) = 0.0, \quad \mathrm{N}(A_3^c) = 0.0$$

$\Pi(A^c) = 1 - \mathrm{N}(A)$ $\mathrm{N}(A^c) = 1 - \Pi(A)$

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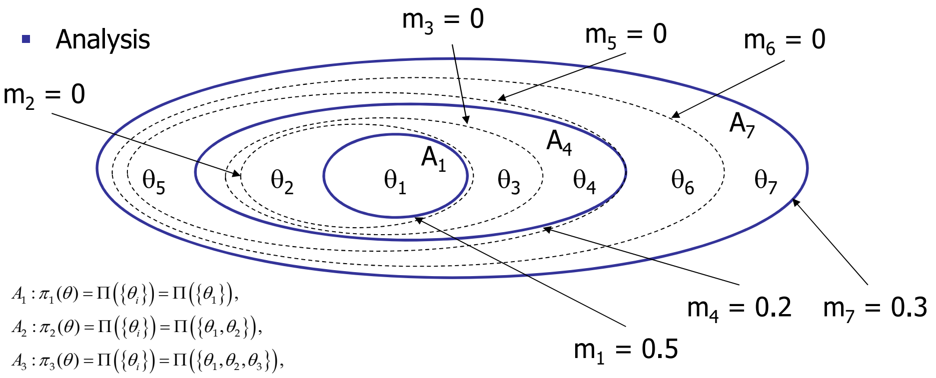
3.4 Possibility Theory – Possibility Distribution Function

- Analysis



$$
\begin{aligned}
A_1 &: \pi_1(\theta) = \Pi\left(\{\theta_i\}\right) = \Pi\left(\{\theta_1\}\right),\\
A_2 &: \pi_2(\theta) = \Pi\left(\{\theta_i\}\right) = \Pi\left(\{\theta_1,\theta_2\}\right),\\
A_3 &: \pi_3(\theta) = \Pi\left(\{\theta_i\}\right) = \Pi\left(\{\theta_1,\theta_2,\theta_3\}\right),\\
A_4 &: \pi_4(\theta) = \Pi\left(\{\theta_i\}\right) = \Pi\left(\{\theta_1,\theta_2,\theta_3,\theta_4\}\right),\\
A_5 &: \pi_5(\theta) = \Pi\left(\{\theta_i\}\right) = \Pi\left(\{\theta_1,\theta_2,\theta_3,\theta_4,\theta_5\}\right),\\
A_6 &: \pi_6(\theta) = \Pi\left(\{\theta_i\}\right) = \Pi\left(\{\theta_1,\theta_2,\theta_3,\theta_4,\theta_5,\theta_6\}\right),\\
A_7 &: \pi_7(\theta) = \Pi\left(\{\theta_i\}\right) = \Pi\left(\{\theta_1,\theta_2,\theta_3,\theta_4,\theta_5,\theta_6,\theta_7\}\right).
\end{aligned}
$$

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3.4 Possibility Theory – Possibility Distribution Function

- Analysis

$$\mathrm{Pl}(X) = \sum_{A \cap X \neq \varnothing} m(A) \rightarrow \mathrm{Pl}(A_j) = \Pi\left(A_j\right) = \sum_{\substack{A_i \cap A_j \neq \varnothing \\ A_i \subset A_j}} m(A_i)$$

$$\Pi\left(\{\theta_i\}\right) = \sum_{\theta_i \cap A_i \neq \varnothing} m(A_i) \rightarrow \pi_i(\theta) = \sum_{\theta_i \cap A_i \neq \varnothing} m_i = \sum_{k=i}^{n} m_k$$

$$
\begin{aligned}
\pi_1 &= m_1 + m_2 + \cdots + m_n\\
\pi_2 &= \phantom{m_1 +{}} m_2 + \cdots + m_n\\
&\vdots\\
\pi_n &= \phantom{m_1 + m_2 + \cdots +{}} m_n
\end{aligned}
\qquad
\begin{aligned}
&\pi_{n-1} = m_{n-1} + m_n = m_{n-1} + \pi_n \rightarrow m_{n-1} = \pi_{n-1} - \pi_n\\
&\pi_{n-2} = m_{n-2} + m_{n-1} + m_n = m_{n-2} + \pi_{n-1} - \pi_n + \pi_n \rightarrow m_{n-2} = \pi_{n-2} - \pi_{n-1}\\
&\vdots\\
&\cdots \rightarrow m_1 = \pi_1 - \pi_2
\end{aligned}
$$

$$\boxed{\pi_{n+1} = 0}$$

$$\boxed{\pi_i(\theta) = \sum_{k=i}^{n} m_k} \Rightarrow \boxed{m_i = \pi_i - \pi_{i+1}}$$

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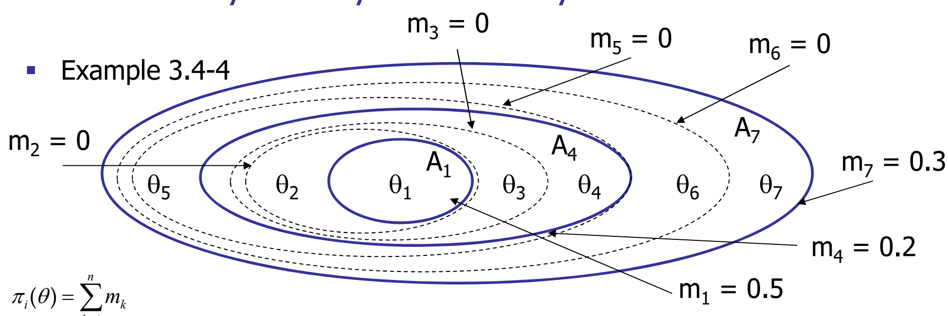
3.4 Possibility Theory – Possibility Distribution Function

- Example 3.4-4



$$\pi_i(\theta) = \sum_{k=i}^{n} m_k$$

$$\pi_1 = 0.5+0+0+0.2+0+0+0.3 = 1.0 \qquad A_1 : \pi_1(\theta) = \Pi(\{\theta_i\}) = \Pi(\{\theta_1\}),$$

$$\pi_2 = 0+0+0.2+0+0+0.3 = 0.5 \qquad A_2 : \pi_2(\theta) = \Pi(\{\theta_i\}) = \Pi(\{\theta_1,\theta_2\}),$$

$$\pi_3 = 0+0.2+0+0+0.3 = 0.5 \qquad A_3 : \pi_3(\theta) = \Pi(\{\theta_i\}) = \Pi(\{\theta_1,\theta_2,\theta_3\}),$$

$$\pi_4 = 0.2+0+0+0.3 = 0.5 \qquad A_4 : \pi_4(\theta) = \Pi(\{\theta_i\}) = \Pi(\{\theta_1,\theta_2,\theta_3,\theta_4\}),$$

$$\pi_5 = 0+0+0.3 = 0.3 \qquad A_5 : \pi_5(\theta) = \Pi(\{\theta_i\}) = \Pi(\{\theta_1,\theta_2,\theta_3,\theta_4,\theta_5\}),$$

$$\pi_6 = 0+0.3 = 0.3 \qquad A_6 : \pi_6(\theta) = \Pi(\{\theta_i\}) = \Pi(\{\theta_1,\theta_2,\theta_3,\theta_4,\theta_5,\theta_6\}),$$

$$\pi_7 = 0.3 = 0.3 \qquad A_7 : \pi_7(\theta) = \Pi(\{\theta_i\}) = \Pi(\{\theta_1,\theta_2,\theta_3,\theta_4,\theta_5,\theta_6,\theta_7\}).$$

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3.4 Possibility Theory – Possibility Distribution Function

- Possibility Theory is defined axiomatically as an independent theory, though, as a second branch it can be defined by the possibility distribution as a special form of the membership function (Zadeh, 1978).
- However, it has to be pointed out that the membership function may not be readily interpreted as a possibility distribution function, because the meaning of both functions is completely different.
- Whereas the membership function describes a fuzzy variable $\theta \in$ A, the possibility distribution function characterizes a representation of what is known (expert, sensor measurement, etc.) about the value of some quantity θ ranging on A $\subseteq \Theta$ (not necessarily a random quantity).

Zadeh, L. A. (1978). Fuzzy Sets as a Basis for a Theory of Possibility, Fuzzy Sets and Systems 1: pp. 3–28.

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3.4 Possibility Theory – Possibility Distribution Function

- The function $\pi_A(x)$ reflects the more or less plausible values of the unknown quantity roughly. We tend to another notation proposed by Dubois and Prade to circumvent any ambiguity:

$$\boxed{\pi_A(\theta \mid \theta \in A) = \mu_A(\theta \in A \mid \theta).}$$

> The possibility that $x = \theta$, knowing that $\theta \in$ A, is $\pi(\theta \mid$ A). The level of possibility is described, knowing that θ is element of a crisp set A, $\theta \in$ A, with $\mu(\mathrm{A} \mid \theta)$.

- An interesting interrelationship exists between α-cuts, basic belief assignments m, possibility distributions, and membership functions ***iff the sets are nested***.

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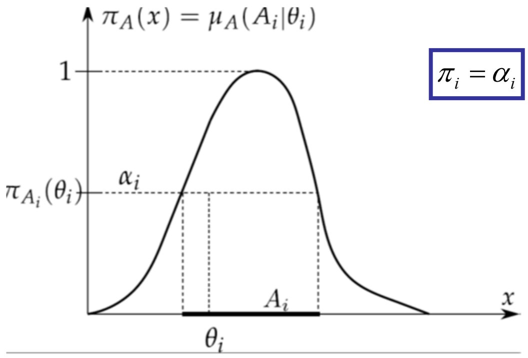
3.4 Possibility Theory – Possibility Distribution Function

- Membership Function as graded possibility
  - Let α_i be a graded value of $\mu_A(A_{\alpha_i} \mid \theta_i)$ then A_{α_i} denotes the corresponding α-cut.



$$\boxed{\pi_i = \alpha_i}$$

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3.4 Possibility Theory – Possibility Distribution Function

- Membership Function as graded possibility
  - It follows that $\alpha_i > \alpha_{i+1}$ and $A_{\alpha_i} \subset A_{\alpha_{i+1}}$.
  - Therefore, necessity and possibility can be rewritten in terms of alpha-cuts:

$$\mathrm{N}(A_{\alpha_i}) = \sum_{k=1}^{i} m(A_{\alpha_k}) \qquad \Pi(A_{\alpha_i}) = \sum_{k=1}^{n} m(A_{\alpha_k}) = 1$$

$$\mathrm{N}(A_{\alpha_i}^c) = 0 \qquad \Pi(A_{\alpha_i}^c) = \sum_{k=i+1}^{n} m(A_{\alpha_k})$$

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3.4 Possibility Theory – Possibility Distribution Function

- Membership Function as graded possibility
  - With

$$\begin{array}{l|l}
\alpha_1 = m_1 + m_2 + \cdots + m_n & \alpha_{n-1} = m_{n-1} + m_n = m_{n-1} + \alpha_n \to m_{n-1} = \alpha_{n-1} - \alpha_n \\
\alpha_2 = \quad m_2 + \cdots + m_n & \alpha_{n-2} = m_{n-2} + m_{n-1} + m_n = m_{n-2} + \alpha_{n-1} - \alpha_n + \alpha_n \to m_{n-2} = \alpha_{n-2} - \alpha_{n-1} \\
\vdots & \vdots \\
\alpha_n = \qquad\quad m_n & \cdots \to m_1 = \alpha_1 - \alpha_2
\end{array}$$

$$\boxed{\alpha_{n+1} = 0}$$

$$\boxed{\alpha_i = \sum_{k=i}^{n} m_k} \Rightarrow \boxed{m_i = m(A_{\alpha_i}) = \alpha_i - \alpha_{i+1}}$$

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3.4 Possibility Theory – Possibility Distribution Function

- Membership Function as graded possibility
  - With:

$$\boxed{\Pi(A_{\alpha_i}^c) = \sum_{k=i+1}^{n} m(A_{\alpha_k}) = m(A_{\alpha_{i+1}}) + m(A_{\alpha_{i+2}}) + \cdots + m(A_{\alpha_n}) = \alpha_{i+1} \,,}$$

$$\boxed{\mathrm{N}(A_{\alpha_i}) = 1 - \Pi(A_{\alpha_i}^c) = 1 - \alpha_{i+1} \,.}$$

  - In the case of $\lim_{n\to\infty}(\alpha_i - \alpha_{i+1}) \to 0$ it follows that

$\boxed{\Pi(A_\alpha^c) = \alpha}$ and $\boxed{\mathrm{N}(A_\alpha) = 1 - \alpha.}$

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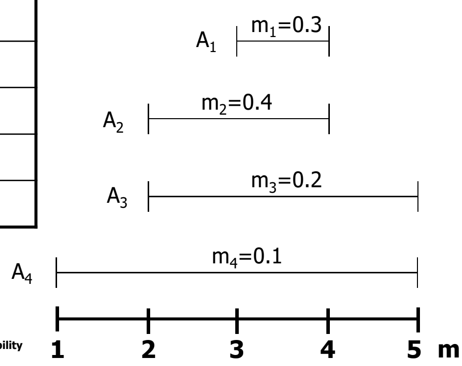
3.4 Possibility Theory – Possibility Distribution Function

- Example 3.4-5 (Distance measurement)

Sensor	**Distance range (m)**	**m_i**
1	[3, 4]	0.3
2	[2, 4]	0.4
3	[2, 5]	0.2
4	[1, 5]	0.1



Inspired by: Zissimos P. Mourelatos, DETC06: Uncertainty Workshop; Evidence & Possibility Theories, Oakland University, 2006

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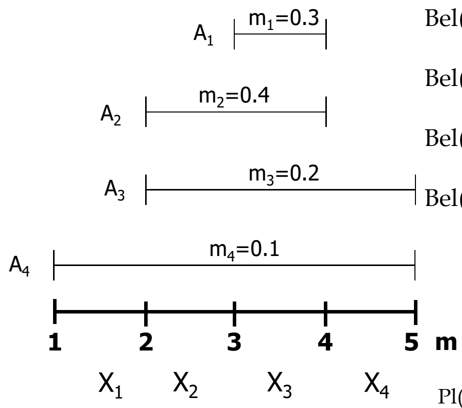
3.4 Possibility Theory – Possibility Distribution Function

- Example 3.4-5 (Distance measurement)



$$\mathrm{Bel}(X_1)=\sum_{A\subset X_1} m(A)=0 \qquad \mathrm{Pl}(X_1)=\sum_{A\cap X_1\neq\varnothing} m(A)=0.1$$

$$\mathrm{Bel}(X_2)=\sum_{A\subset X_2} m(A)=0 \qquad \mathrm{Pl}(X_2)=\sum_{A\cap X_2\neq\varnothing} m(A)=0.7$$

$$\mathrm{Bel}(X_3)=\sum_{A\subset X_3} m(A)=0.3 \qquad \mathrm{Pl}(X_3)=\sum_{A\cap X_3\neq\varnothing} m(A)=1$$

$$\mathrm{Bel}(X_4)=\sum_{A\subset X_4} m(A)=0 \qquad \mathrm{Pl}(X_4)=\sum_{A\cap X_4\neq\varnothing} m(A)=0.3$$

$$\mathrm{Pl}(X_2)=1-\mathrm{Bel}(X_2^c)=1-\left[\underbrace{\mathrm{Bel}(X_1)}_{0}+\underbrace{\mathrm{Bel}(X_3)}_{0.3}+\underbrace{\mathrm{Bel}(X_4)}_{0}\right]=0.7$$

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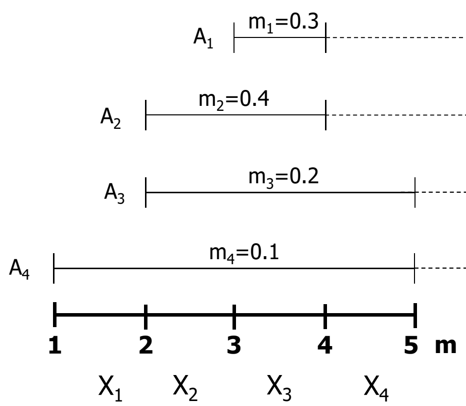
3.4 Possibility Theory – Possibility Distribution Function

- Example 3.4-5 (Distance measurement)



$$\pi_1=\sum_{k=1}^{4} m_k=1$$

$$\pi_2=\sum_{k=2}^{4} m_k=0.7$$

$$\pi_3=\sum_{k=3}^{4} m_k=0.3$$

$$\pi_4=\sum_{k=4}^{4} m_k=0.1$$

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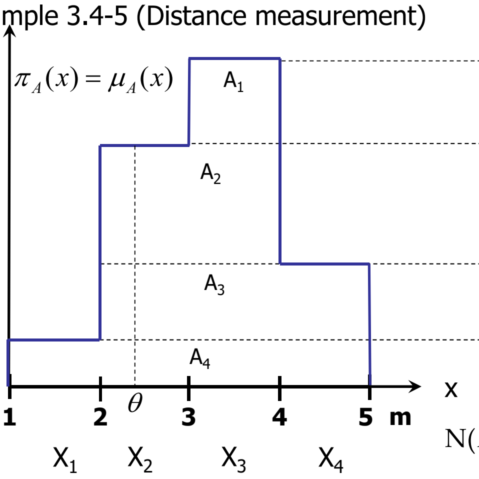
3.4 Possibility Theory – Possibility Distribution Function

- Example 3.4-5 (Distance measurement)



$$\pi_1 = \sum_{k=1}^{4} m_k = 1$$

$$\pi_2 = \sum_{k=2}^{4} m_k = 0.7$$

$$\pi_3 = \sum_{k=3}^{4} m_k = 0.3$$

$$\pi_4 = \sum_{k=4}^{4} m_k = 0.1$$

$$\mathrm{N}(A) = \min_{\theta \notin A}\left(1 - \pi_A(x) \mid x = \theta \notin A\right) = ?$$

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3.4 Possibility Theory – Summary

- Possibility Theory provides a promising framework for sensor and information fusion based on incomplete or party unknown data.
- It is able to handle uncertainty, ignorance, and vagueness based on crisp sets instead of fuzzy sets.
- However, being a distinct theory, its branches interconnect Dempster-Shafer Theory, Fuzzy Set Theory, and Probability Theory.
- Insofar PosT can be interpreted as an unifying Theory.
- PosT is, although 40 years old, in its childhood and is still under research.
- PosT is not 100% understood in all of its branches.

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