Chapter 2

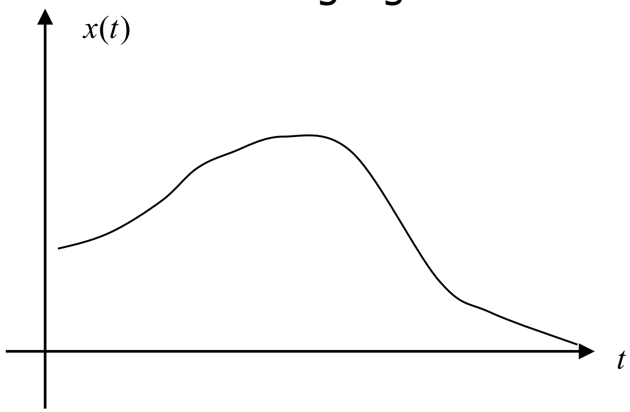
Continuous-Time Signals

- 2.1 Overview
- 2.2 Signal Properties and Classification
- 2.3 Test Signals
- 2.4 Basic Signal Operations
- 2.5 Continuous-Time FOURIER Transform (CTFT)
- 2.6 HILBERT Transform

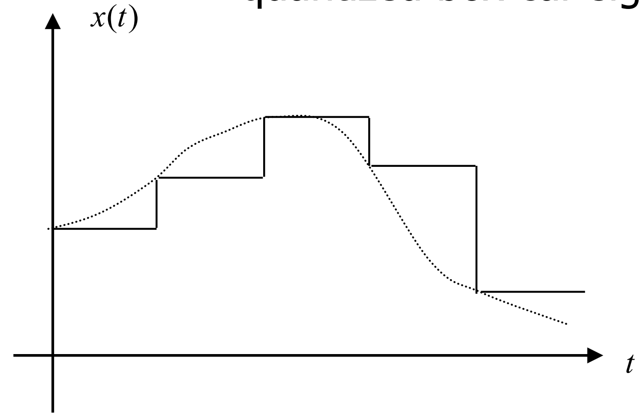
2.1 Overview

Continuous-Time Signals

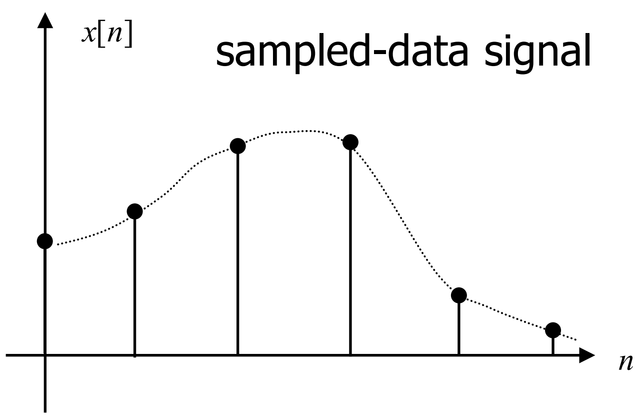
analog signal



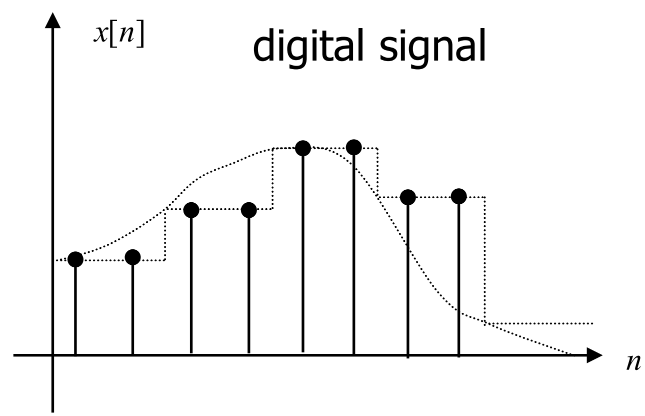
quantized box-car signal



sampled-data signal



digital signal



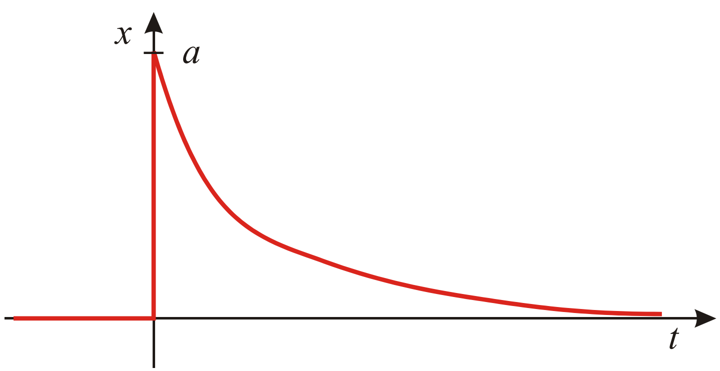
2.2 Signal Properties and Classification

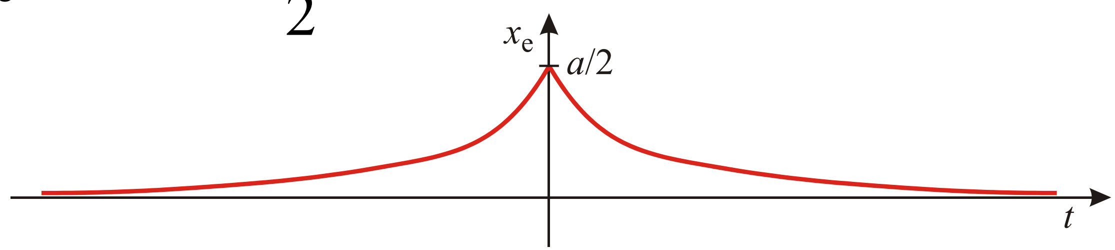
- $x = x(t)$ with **real** or **complex** value x
- Without loss of generality we use the independent variable t, which can be the normalized time.
- **Periodic**: $x(t) = x(t \pm nT)$ with period T , $n = 1, 2, 3, \ldots$
- **Deterministic**: The value of x for a given time t is known.
- **Causal:** $x(t) = 0$ for $t < 0$ (no physical meaning)
- **Random**: Only statistical information (e.g. mean value, median value, probabilibty density) about the value of x is available for a given time t. → not addressed in this course

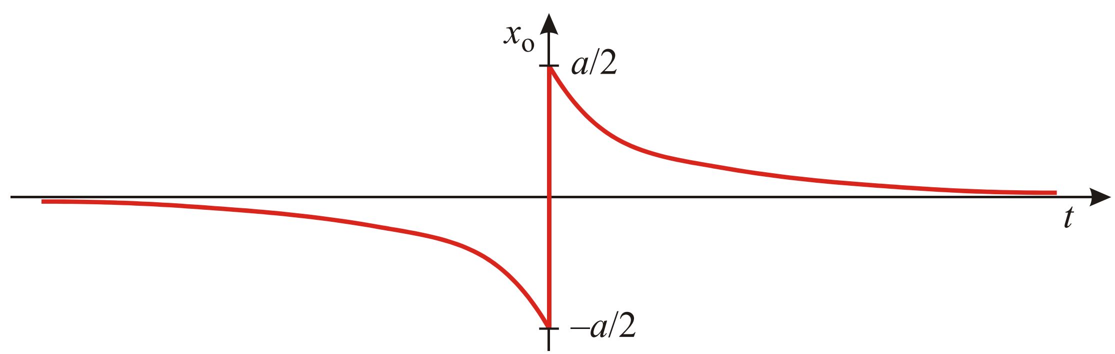
Even and Odd Real Signals

- Real **even** signal: $x_e(-t) = x_e(t)$
- Real **odd** signal: $x_o(-t) = -x_o(t)$
- Each real signal $x(t)$ can be written as: $x(t) = x_e(t) + x_o(t)$

$$x_e(t) = \frac{x(t) + x(-t)}{2}, \quad x_o(t) = \frac{x(t) - x(-t)}{2}$$







Symmetry Relations for Complex Signals

- **Conjugate symmetric** signal: $x_{\mathrm{cs}}(t) = x_{\mathrm{cs}}{}^*(-t)$
- **Conjugate antisymmetric** signal: $x_{\mathrm{ca}}(t) = -x_{\mathrm{ca}}{}^*(-t)$

- Any complex signal $x(t)$ can be composed as:

$$x(t) = x_{\mathrm{cs}}(t) + x_{\mathrm{ca}}(t)$$

$$x_{\mathrm{cs}}(t) = \frac{x(t) + x^*(-t)}{2} \quad , \quad x_{\mathrm{ca}}(t) = \frac{x(t) - x^*(-t)}{2}$$

$$x_{\mathrm{cs}}(0) = \text{real} \quad ; \quad x_{\mathrm{ca}}(0) = \text{imaginary}$$

Power and Energy

- **Power**: $P(t) = x(t) \cdot x(t) = x^2(t) \quad , \quad x \text{ real}$

  $P(t) = x(t) \cdot x^*(t) = |x(t)|^2 \quad , \quad x \text{ complex}$

- **Energy**: $E_{12} = \int_{t_1}^{t_2} P(t)\,\mathrm{d}\,t$

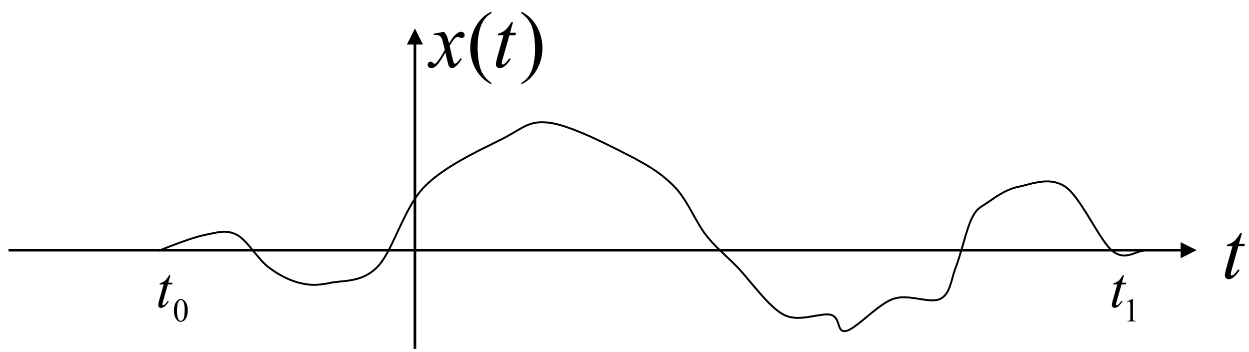
- **Average power**: $P_{12} = \dfrac{1}{t_2 - t_1} \cdot \int_{t_1}^{t_2} P(t)\,\mathrm{d}\,t$

- **Energy signal**: $E_\infty = \int_{-\infty}^{\infty} P(t)\,\mathrm{d}\,t < \infty \quad , \quad P_\infty = 0$

- **Power Signal**: $P_\infty = \lim\limits_{T \to \infty} \dfrac{1}{T} \cdot \int_{-T/2}^{T/2} P(t)\,\mathrm{d}\,t < \infty \quad , \quad E_\infty \to \infty$

Properties of Many Real-World Signals

- **Irregular**
- **Aperiodic**
- **Finite duration**: $x(t) = 0$ for $t < t_0$ and $t > t_1$
- **Steady**, i.e. no jumps
- **Differentiation** is always several times possible

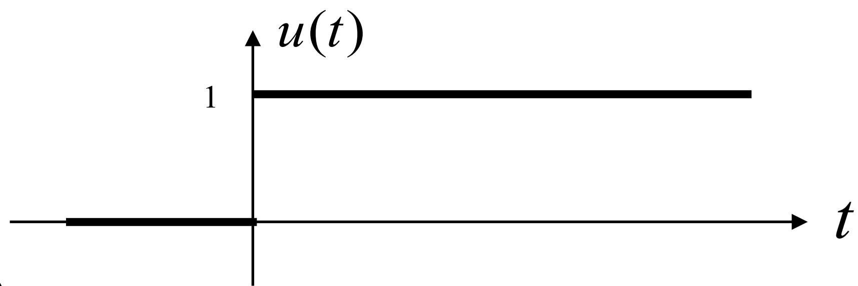


2.3 Test Signals

- We introduce some **basic continuous-time signals**.
- They
  - Occur frequently
  - Can be used as building blocks to construct other signals
  - Can be used to stimulate systems for means of characterization

Unit Step Function

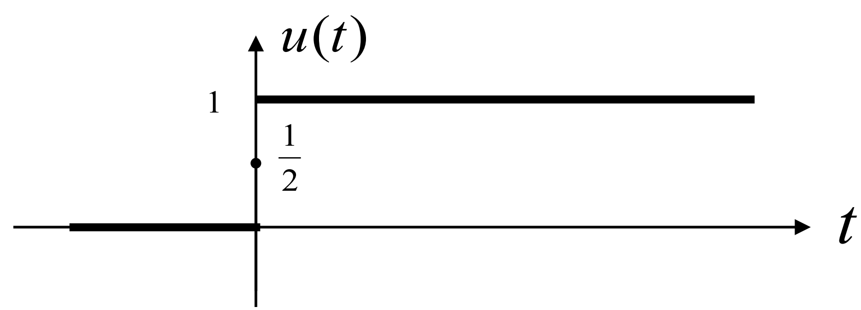
$$u(t) = \begin{cases} 0 \ , & t < 0 \\ 1 \ , & t > 0 \end{cases} = \frac{1}{2} \cdot \left(1 + \mathrm{sgn}(t)\right)$$



$$\mathrm{sgn}(t) = \begin{cases} 1 \ , & t > 0 \\ -1 \ , & t < 0 \end{cases}$$

Also used by some authors

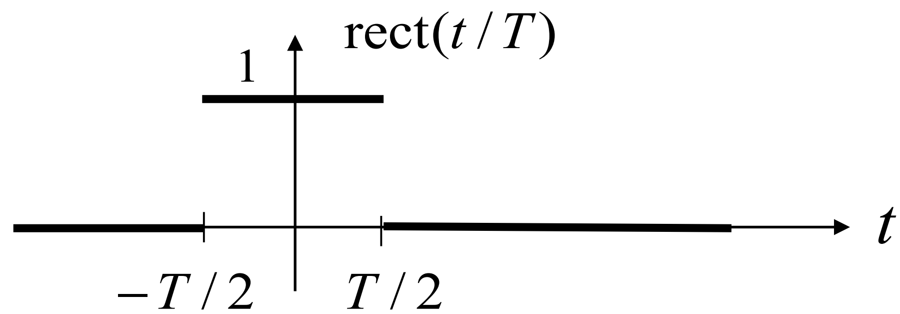
$$u(t) = \begin{cases} 0 \ , & t < 0 \\ 0.5 \ , & t = 0 \\ 1 \ , & t > 0 \end{cases}$$



→ Will be discussed in section 2.5 CTFT

Rectangular Function

$$\mathrm{rect}\left(\frac{t}{T}\right) = \begin{cases} 1\ , & |t| < T/2 \\ 0\ , & |t| > T/2 \end{cases}$$



Useful relation

$$\mathrm{rect}\left(\frac{t}{T}\right) = u\left(t + \frac{T}{2}\right) - u\left(t - \frac{T}{2}\right)$$

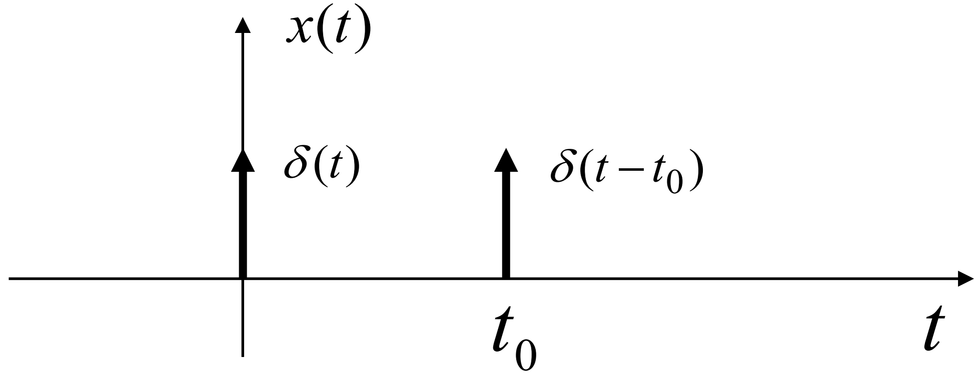
$$\int_{-\infty}^{\infty} \mathrm{rect}\left(\frac{t}{T}\right) \cdot x(t)\,\mathrm{d}t = \int_{-T/2}^{T/2} x(t)\,\mathrm{d}t$$

→ rect used as **window function**

Unit Impulse Function - DIRAC Impulse

$$\delta(t)=\begin{cases}\infty\ , & t=0\\ 0\ , & t\neq 0\end{cases}$$

$$\int_{-\infty}^{\infty}\delta(t)\,\mathrm{d}t=1$$



PAUL DIRAC
1902 - 1984

Important relations

$$\delta(t)=\frac{\mathrm{d}u(t)}{\mathrm{d}t}=\frac{1}{2}\cdot\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{sgn}(t)\ ,\quad u(t)=\int_{-\infty}^{t}\delta(\tau)\,\mathrm{d}\tau$$

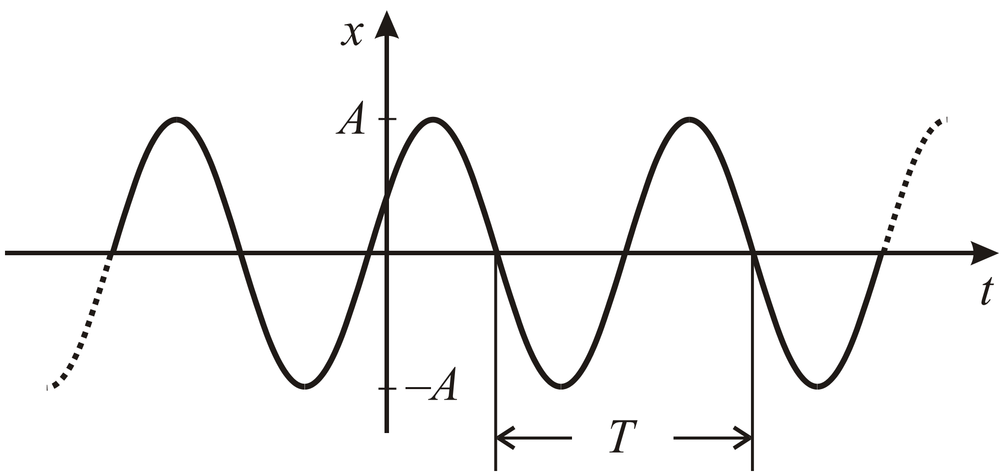
$$\int_{-\infty}^{\infty}x(t)\cdot\delta(t-t_0)\,\mathrm{d}t=x(t_0)$$

→ **sifting property** (fundamental for discrete signal processing)

Sinusoidal Signals

$$x(t) = A \cdot \cos\left(\omega t + \varphi\right) \quad \text{with} \quad \omega = 2\pi f$$

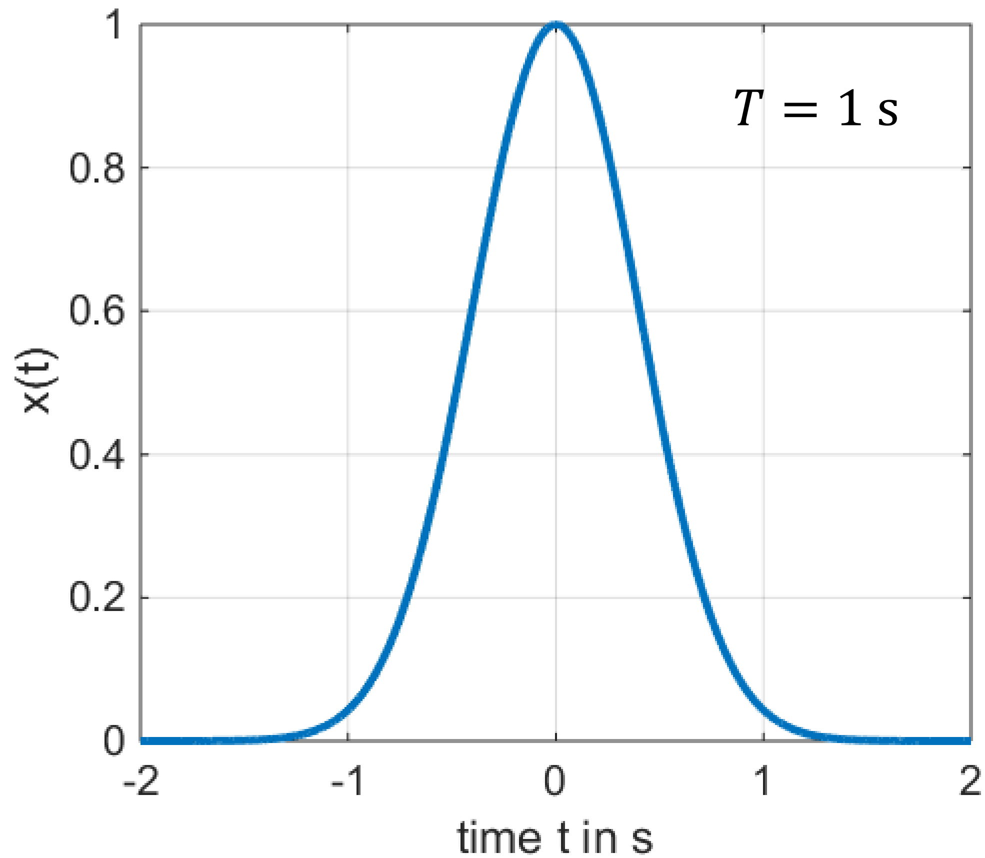
A : amplitude
ω : radian frequency
f : frequency
φ : zero phase angle
B : complex amplitude



$$x(t) = \frac{1}{2}\left(B \cdot e^{j\omega t} + B^* \cdot e^{-j\omega t}\right) = \mathrm{Re}\left\{B \cdot e^{j\omega t}\right\} \quad \text{with} \quad B = A \cdot e^{j\varphi}$$

GAUSS signal

$$x(t) = \mathrm{e}^{-\pi\cdot(t/T)^2}$$



2.4 Basic Signal Operations

- Addition, subtraction: $y(t) = x_1(t) \pm x_2(t)$
- Scaling
  - Vertical: $y(t) = a \cdot x(t)$
  - Horizontal: $y(t) = x(a \cdot t)$
- Shifting
  - Horizontal: $y(t) = x(t \pm t_0)$
  - Vertical: $y(t) = x(t) \pm a$
- Flipping, mirroring
  - Vertical: $y(t) = -x(t)$
  - Horizontal: $y(t) = x(-t)$ $\rightarrow$ time reversal

- Differentiation: $y(t) = \frac{\mathrm{d}\,x(t)}{\mathrm{d}\,t}$
- Integration: $y(t) = \int_{-\infty}^{t} x(\tau)\,\mathrm{d}\,\tau$
- Modulation, demodulation: → not addressed in this course

Scalar Product

- $x(t), y(t)$ are **energy signals**

$$\langle x(t), y(t) \rangle = \int_{-\infty}^{\infty} x(t) \cdot y^*(t)\,\mathrm{d}t$$

- $x(t), y(t)$ are **power signals**

$$\langle x(t), y(t) \rangle = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} x(t) \cdot y^*(t)\,\mathrm{d}t$$

- $x(t), y(t)$ are **periodic signals**, with common period T

$$\langle x(t), y(t) \rangle = \frac{1}{T} \int_{T} x(t) \cdot y^*(t)\,\mathrm{d}t$$

Scalar Product

- $x(t), y(t)$ are **orthogonal signals** if $\langle x(t), y(t)\rangle = 0$
- Properties

$$\langle x(t), y(t)\rangle = \langle y(t), x(t)\rangle^*$$

$$\langle a \cdot x(t), y(t)\rangle = a \cdot \langle x(t), y(t)\rangle$$

$$\langle x(t), a \cdot y(t)\rangle = a^* \cdot \langle x(t), y(t)\rangle$$

$$\langle x_1(t) + x_2(t), y(t)\rangle = \langle x_1(t), y(t)\rangle + \langle x_2(t), y(t)\rangle$$

$$\langle x(t), x(t)\rangle = \int_{-\infty}^{\infty} x(t) \cdot x^*(t)\,\mathrm{d}\,t = \int_{-\infty}^{\infty} |x(t)|^2\,\mathrm{d}\,t = E_\infty$$

Convolution

$$y(t) = \int_{-\infty}^{\infty} x_1(\tau) \cdot x_2(t-\tau)\,\mathrm{d}\tau = x_1(t) * x_2(t)$$

- Properties

$$x_1(t) * x_2(t) = x_2(t) * x_1(t) \qquad \rightarrow \text{commutative property}$$

$$x_1(t) * [x_2(t) * x_3(t)] = [x_1(t) * x_2(t)] * x_3(t) \qquad \rightarrow \text{associative property}$$

$$x_1(t) * [x_2(t) + x_3(t)] = [x_1(t) * x_2(t)] + [x_1(t) * x_3(t)] \qquad \rightarrow \text{distributive property}$$

- **Sifting equation**

$$x(t) * \delta(t) = \int_{-\infty}^{\infty} x(\tau) \cdot \delta(t-\tau)\,\mathrm{d}\tau = x(t)$$

$$x(t) * \delta(t-t_0) = \delta(t-t_0) * x(t) = x(t-t_0)$$

Correlation for Real Energy Signals

- Cross-correlation function

$$\phi_{xy}^{\mathrm{E}}(t) = \int_{-\infty}^{\infty} x(\tau) \cdot y(\tau + t)\,\mathrm{d}\,\tau = x(t) * y(-t) = \phi_{yx}^{\mathrm{E}}(-t)$$

- Auto-correlation function

$$\phi_{xx}^{\mathrm{E}}(t) = \int_{-\infty}^{\infty} x(\tau) \cdot x(\tau + t)\,\mathrm{d}\,\tau = x(t) * x(-t) = \phi_{xx}^{\mathrm{E}}(-t)$$

$$\phi_{xx}^{\mathrm{E}}(0) = \int_{-\infty}^{\infty} x(\tau) \cdot x(\tau)\,\mathrm{d}\,\tau = E_{\infty}$$

Correlation for Complex Energy Signals

- Cross-correlation function

$$\phi_{xy}^{\mathrm{E}}(t) = \int_{-\infty}^{\infty} x(\tau) \cdot y^*(\tau + t)\,\mathrm{d}\tau = x(t) * y^*(-t) = \left[\phi_{yx}^{\mathrm{E}}(-t)\right]^*$$

- Auto-correlation function

$$\phi_{xx}^{\mathrm{E}}(t) = \int_{-\infty}^{\infty} x(\tau) \cdot x^*(\tau + t)\,\mathrm{d}\tau = x(t) * x^*(-t) = \left[\phi_{xx}^{\mathrm{E}}(-t)\right]^*$$

$$\phi_{xx}^{\mathrm{E}}(0) = \int_{-\infty}^{\infty} x(\tau) \cdot x^*(\tau)\,\mathrm{d}\tau = E_\infty$$

Correlation for Complex Power Signals

- Cross-correlation function

$$\phi_{xy}^{\mathrm{P}}(t) = \lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} x(\tau)\cdot y^*(\tau+t)\,\mathrm{d}\tau = \left[\phi_{yx}^{\mathrm{P}}(-t)\right]^*$$

- Auto-correlation function

$$\phi_{xx}^{\mathrm{P}}(t) = \lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} x(\tau)\cdot x^*(\tau+t)\,\mathrm{d}\tau = \left[\phi_{xx}^{\mathrm{P}}(-t)\right]^*$$

$$\phi_{xx}^{\mathrm{P}}(0) = \lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} x(\tau)\cdot x^*(\tau)\,\mathrm{d}\tau = P_\infty$$

2.5 Continuous-Time FOURIER Transform - CTFT

- The CTFT is based on **complex exponential functions** from which **sinusoidal oscillations** can be derived easily.
  - Useful for characterizing systems
  - Don't change the form when linear signal operations are applied
  - **Eigenfunctions** of lossless LTI systems
- A periodic signal $x(t)$ with period T can be represented as a linear combination of complex exponentials → **FOURIER series.** f_1 is the frequency of the fundamental oscillation. c_k are the complex FOURIER coefficients.

$$x(t) = \sum_{k=-\infty}^{+\infty} c_k \cdot \mathrm{e}^{\mathrm{j}k\omega_1 t} \quad \text{with} \quad \omega_1 = \frac{2\pi}{T} = 2\pi \cdot f_1$$

JOSEPH FOURIER
1768 - 1830

- **Fourier transform** or **Fourier integral** of $x(t)$

$$X(\mathrm{j}\omega) = \int_{-\infty}^{\infty} x(t) \cdot \mathrm{e}^{-\mathrm{j}\omega t} \,\mathrm{d}\, t \qquad x(t) \circ\!\!-\!\!\bullet \; X(\mathrm{j}\omega) = \mathrm{F}\{x(t)\}$$

$$X(f) = \int_{-\infty}^{\infty} x(t) \cdot \mathrm{e}^{-\mathrm{j}2\pi f t} \,\mathrm{d}\, t \qquad x(t) \circ\!\!-\!\!\bullet \; X(f) = \mathrm{F}\{x(t)\}$$

- **Inverse Fourier transform** of $X(\mathrm{j}\omega)$, $X(f)$

$$x(t) = \frac{1}{2\pi} \cdot \int_{-\infty}^{\infty} X(\mathrm{j}\omega) \cdot \mathrm{e}^{\mathrm{j}\omega t} \,\mathrm{d}\, \omega \qquad X(\mathrm{j}\omega) \bullet\!\!-\!\!\circ \; x(t) = \mathrm{F}^{-1}\{X(\mathrm{j}\omega)\}$$

$$x(t) = \int_{-\infty}^{\infty} X(f) \cdot \mathrm{e}^{\mathrm{j}2\pi f t} \,\mathrm{d}\, f \qquad X(f) \bullet\!\!-\!\!\circ \; x(t) = \mathrm{F}^{-1}\{X(f)\}$$

- $X(\mathrm{j}\omega)$, $X(f)$ is the **spectrum** of $x(t)$.
- The signal representation by either $x(t)$ or $X(\mathrm{j}\omega)$, $X(f)$ is equivalent. All information are incorporated in both representations.

Convergence of FOURIER transforms

- The inverse FOURIER transform constitutes $x(t)$ successfully if:
  - $x(t)$ and its derivative are steady functions within intervals
  - $x(t)$ is of finite duration: $x(\infty) = x(-\infty) = 0$
  - at discontinuities: $x(t_0) = \frac{1}{2} \cdot \{x(t_{0+}) + x(t_{0-})\}$
  - $x(t)$ is absolutely integrable: $\int_{-\infty}^{\infty} |x(t)| \, \mathrm{d}t < \infty$
- This is a **sufficient** but not a necessary **condition**.

→ Absolutely integrable signals that are continuous or that have a finite number of discontinuities have a FOURIER transform.

FOURIER transform of periodic signals

- $$X(\mathrm{j}\omega) = 2\pi \cdot \delta(\omega - \omega_1) \quad \bullet\!-\!\circ \quad x(t) = \mathrm{e}^{\mathrm{j}\omega_1 t}$$

  $$X(f) = \delta(f - f_1) \quad \bullet\!-\!\circ \quad x(t) = \mathrm{e}^{\mathrm{j}2\pi f_1 t}$$

- $$X(\mathrm{j}\omega) = \sum_{k=-\infty}^{\infty} 2\pi \cdot c_k \cdot \delta(\omega - k\omega_1) \quad \bullet\!-\!\circ \quad x(t) = \sum_{k=-\infty}^{\infty} c_k \cdot \mathrm{e}^{\mathrm{j}k\omega_1 t}$$

  $$X(f) = \sum_{k=-\infty}^{\infty} c_k \cdot \delta(f - kf_1) \quad \bullet\!-\!\circ \quad x(t) = \sum_{k=-\infty}^{\infty} c_k \cdot \mathrm{e}^{\mathrm{j}k2\pi f_1 t}$$

- The FOURIER transform of a periodic signal $x(t)$ with FOURIER series coefficients c_k can be interpreted as a train of DIRAC impulses of area $2\pi c_k$ or c_k occurring at harmonically related frequencies $k\omega_1$ or kf_1.

Properties of the FOURIER transform

- Linearity

$$a \cdot x(t) + b \cdot y(t) \quad \circ\!\!-\!\!\!-\!\!\bullet \quad a \cdot X(\mathrm{j}\omega) + b \cdot Y(\mathrm{j}\omega)$$

- Conjugation and conjugate symmetry

$$x^*(t) \quad \circ\!\!-\!\!\!-\!\!\bullet \quad X^*(-\mathrm{j}\omega)$$

$$x(t) = x^*(t) \quad \circ\!\!-\!\!\!-\!\!\bullet \quad X(-\mathrm{j}\omega) = X^*(\mathrm{j}\omega)$$

real signal $\circ\!\!-\!\!\!-\!\!\bullet$ conjugate complex spectrum

Real part of $X(\mathrm{j}\omega)$ is even

Imaginary part of $X(\mathrm{j}\omega)$ is odd

- Even and odd functions

$$x(t) = x_\mathrm{e}(t) + x_\mathrm{o}(t) \;\circ\!\!-\!\!\bullet\; X(\mathrm{j}\omega) = X_\mathrm{e}(\mathrm{j}\omega) + X_\mathrm{o}(\mathrm{j}\omega)$$

$$x_\mathrm{e}(t) \;\circ\!\!-\!\!\bullet\; X_\mathrm{e}(\mathrm{j}\omega) = \mathrm{Re}\{X(\mathrm{j}\omega)\}$$

$$x_\mathrm{o}(t) \;\circ\!\!-\!\!\bullet\; X_\mathrm{o}(\mathrm{j}\omega) = \mathrm{j}\cdot \mathrm{Im}\{X(\mathrm{j}\omega)\}$$

- $x(t)$ real and even $\circ\!\!-\!\!\bullet$ $X(\mathrm{j}\omega)$ real and even
- $x(t)$ real and odd $\circ\!\!-\!\!\bullet$ $X(\mathrm{j}\omega)$ imaginary and odd

- Time shifting

$$x(t-t_0) \quad \circ\!\!-\!\!\bullet \quad e^{-j\omega t_0}\cdot X(j\omega)$$

- Frequency shifting

$$e^{j\omega_0 t}\cdot x(t) \quad \circ\!\!-\!\!\bullet \quad X\big(j(\omega-\omega_0)\big)$$

- Time and frequency scaling

$$x(at) \quad \circ\!\!-\!\!\bullet \quad \frac{1}{|a|}\cdot X\left(\frac{j\omega}{a}\right)$$

$$x(-t) \quad \circ\!\!-\!\!\bullet \quad X(-j\omega)$$ → time reversal

- Duality

$$X(t) \quad \circ\!\!-\!\!\bullet \quad x(-j\omega) \qquad\qquad X(t) \quad \circ\!\!-\!\!\bullet \quad x(-f)$$

- Differentiation

$$\frac{\mathrm{d}\,x(t)}{\mathrm{d}\,t} \quad \circ\!\!-\!\!\!-\!\!\bullet \quad \mathrm{j}\omega \cdot X(\mathrm{j}\omega)$$

- Integration

$$\int_{-\infty}^{t} x(\tau)\,\mathrm{d}\,\tau \quad \circ\!\!-\!\!\!-\!\!\bullet \quad \frac{1}{\mathrm{j}\omega} \cdot X(\mathrm{j}\omega) + \pi \cdot X(0) \cdot \delta(\omega)$$

- Convolution

$$x(t) * y(t) \quad \circ\!\!-\!\!\!-\!\!\bullet \quad X(\mathrm{j}\omega) \cdot Y(\mathrm{j}\omega)$$

→ transmission, frequency-selective filtering

- Multiplication

$$x(t) \cdot y(t) \quad \circ\!\!-\!\!\!-\!\!\bullet \quad \frac{1}{2\pi}\big(X(\mathrm{j}\omega) * Y(\mathrm{j}\omega)\big)$$

→ modulation, time-selective filtering

- **PARSEVAL's relation for aperiodic signals**

MARC-ANTOINE PARSEVAL
1755 - 1836

$$E_\infty = \int_{-\infty}^{\infty} |x(t)|^2 \,\mathrm{d}t = \frac{1}{2\pi}\int_{-\infty}^{\infty} |X(\mathrm{j}\omega)|^2 \,\mathrm{d}\omega$$

- $|X(\mathrm{j}\omega|^2$ is the **energy-density spectrum** of the signal $x(t)$.

- **PARSEVAL's relation for periodic signals**

$$P = \frac{1}{T}\int_T |x(t)|^2 \,\mathrm{d}t = \sum_{k=-\infty}^{+\infty} |c_k|^2 = \sum_{k=0}^{\infty} P_k$$

- P_k is the power of the kth sinusoidal oscillation (= kth harmonic).

Basic FOURIER transform pairs

- **DC signal**

$$x(t)=1 \quad \circ\!-\!\bullet \quad X(\mathrm{j}\omega)=2\pi\cdot\delta(\omega)\ ;\ X(f)=\delta(f)$$

- **Unit impulse function - DIRAC impulse**

$$x(t)=\delta(t) \quad \circ\!-\!\bullet \quad X(\mathrm{j}\omega)=1;\ X(f)=1$$

- **Unit step function**

$$x(t)=u(t) \quad \circ\!-\!\bullet \quad X(\mathrm{j}\omega)=\frac{1}{\mathrm{j}\omega}+\pi\cdot\delta(\omega)$$

$$x(t)=\mathrm{sgn}(t) \quad \circ\!-\!\bullet \quad X(\mathrm{j}\omega)=\frac{2}{\mathrm{j}\omega}$$

- **Complex exponential signal**

$$x(t) = \mathrm{e}^{\mathrm{j}\omega_0 t} \quad \circ\!-\!\bullet \quad X(\mathrm{j}\omega) = 2\pi \cdot \delta(\omega - \omega_0)$$

- **Cosine signal**

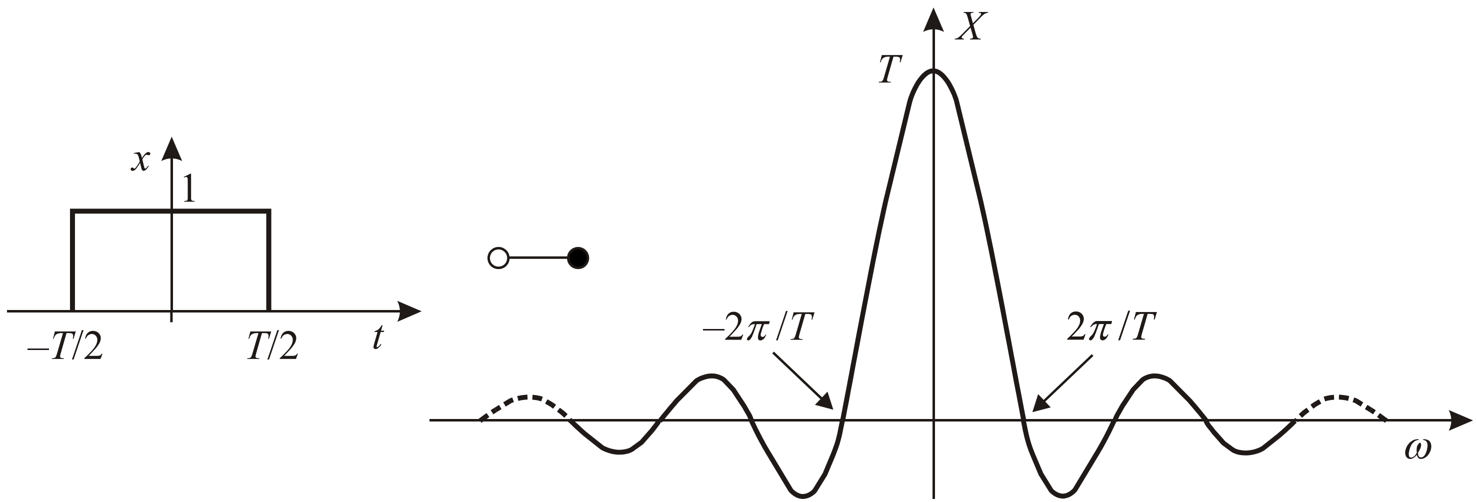
$$x(t) = \cos\left(\omega_0 t\right) \quad \circ\!-\!\bullet \quad X(\mathrm{j}\omega) = \pi \cdot \left[\delta(\omega + \omega_0) + \delta(\omega - \omega_0)\right]$$

- **Sine signal**

$$x(t) = \sin\left(\omega_0 t\right) \quad \circ\!-\!\bullet \quad X(\mathrm{j}\omega) = \mathrm{j}\pi \cdot \left[\delta(\omega + \omega_0) - \delta(\omega - \omega_0)\right]$$

- **Rectangular function**

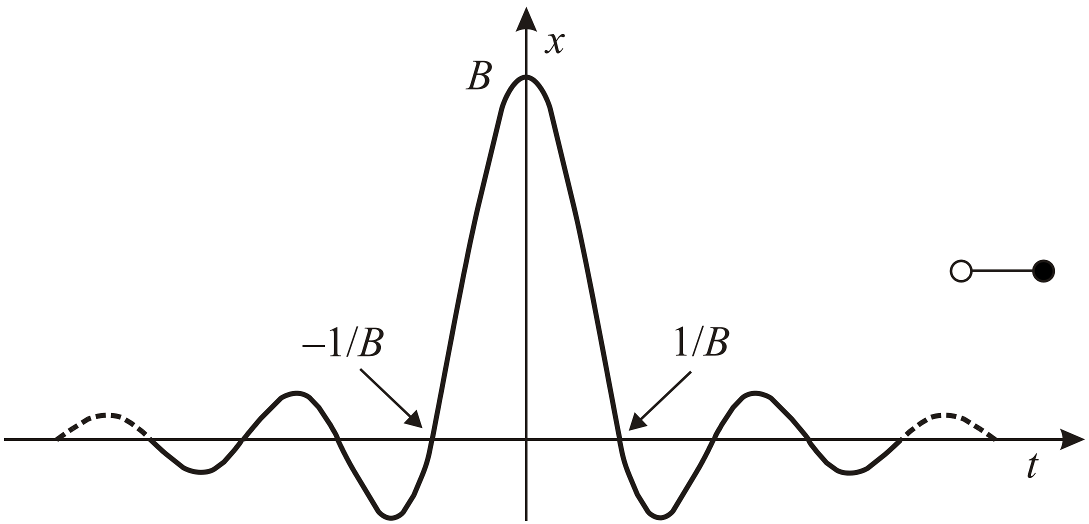
$$x(t) = \mathrm{rect}\left(\frac{t}{T}\right) \quad \circ\!-\!\bullet \quad X(\mathrm{j}\omega) = T \cdot \frac{\sin(\omega T/2)}{\omega T/2} = T \cdot \mathrm{si}\left(\frac{\omega T}{2}\right) = T \cdot \mathrm{sinc}\left(\frac{\omega T}{2\pi}\right)$$

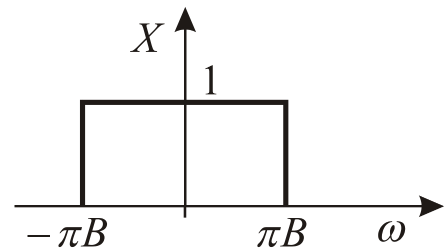


$$\text{DE:}\;\; \mathrm{si}\, x = \frac{\sin x}{x} \quad ; \quad \text{EN:}\;\; \mathrm{sinc}\, x = \frac{\sin(\pi x)}{\pi x}$$

Sinc function

$$x(t) = B \cdot \mathrm{si}\left(\pi B t\right) = B \cdot \mathrm{sinc}\left(B t\right) \quad \circ\!-\!\bullet \quad X(\mathrm{j}\omega) = \mathrm{rect}\left(\frac{\omega}{2\pi B}\right)$$





Impulse train

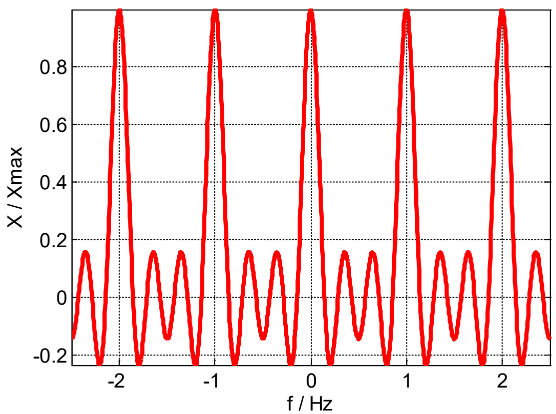
$$X(f) = 1 + 2\sum_{n=1}^{\infty} \cos\left(2\pi f n T\right)$$

$$x(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT) \quad \circ\!-\!\bullet$$

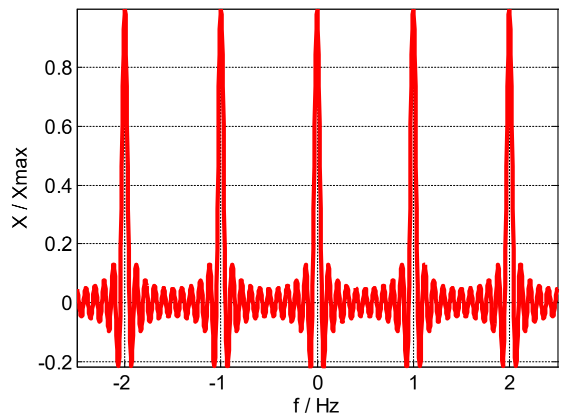
$$X(f) = \frac{1}{T} \cdot \sum_{k=-\infty}^{\infty} \delta\left(f - k\frac{1}{T}\right)$$

$$X(f) = 1 + 2\sum_{n=1}^{n_{\max}} \cos\left(2\pi f n T\right) \quad \text{with} \quad T = 1\ \text{s}$$

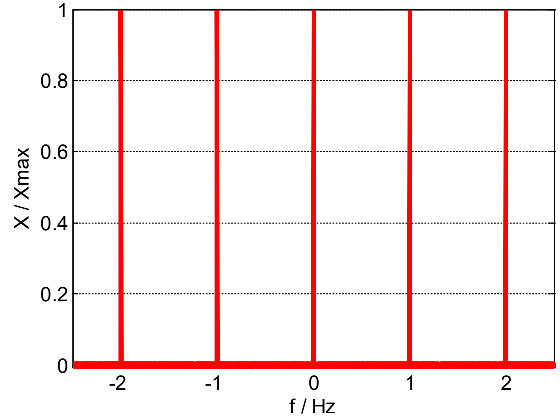
$n_{\max} = 3$



$n_{\max} = 10$



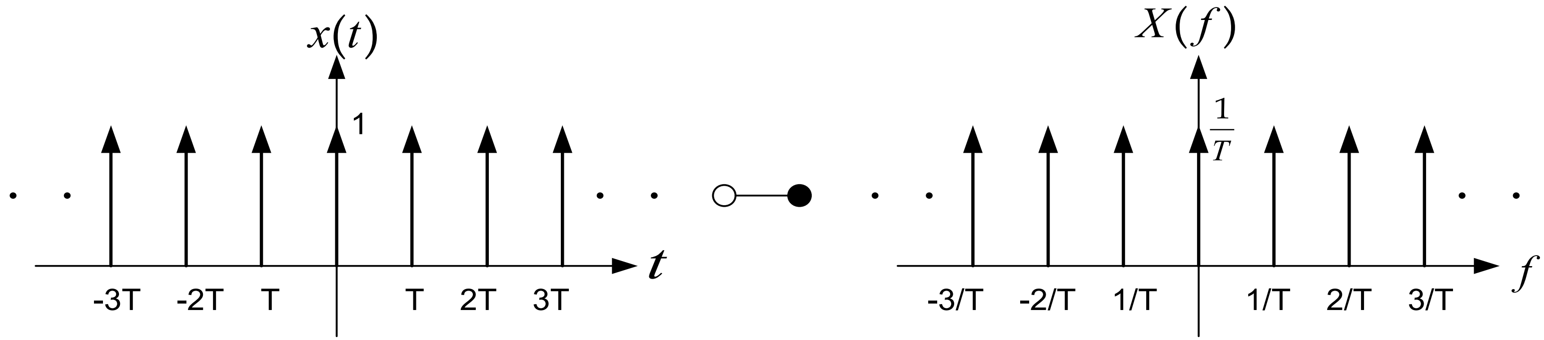
$n_{\max} = 100$



- **Impulse train**

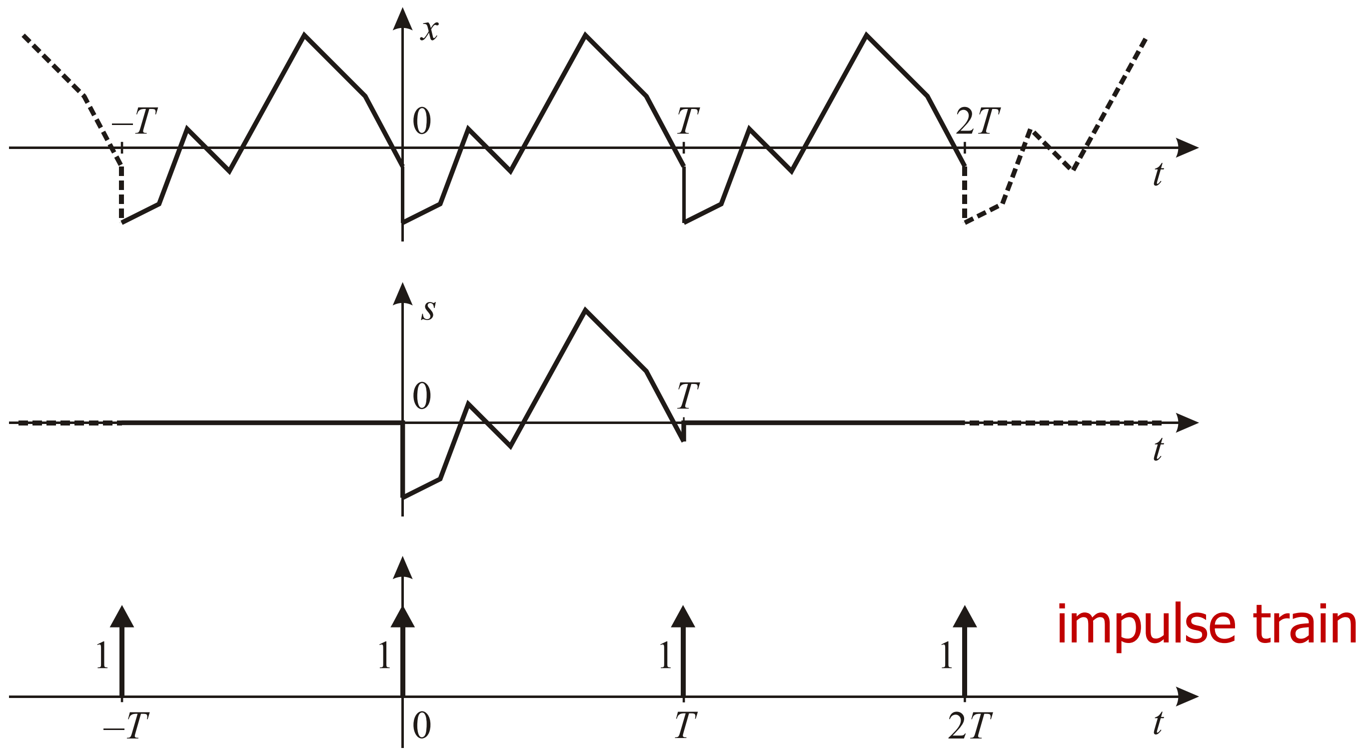
$$x(t)=\sum_{n=-\infty}^{\infty}\delta(t-nT) \quad \circ\!-\!\bullet \quad X(\mathrm{j}\omega)=\frac{2\pi}{T}\cdot\sum_{k=-\infty}^{\infty}\delta\left(\omega-k\frac{2\pi}{T}\right)$$

$$X(f)=\frac{1}{T}\cdot\sum_{k=-\infty}^{\infty}\delta\left(f-k\frac{1}{T}\right)$$



- Period T ○—● spacing $\Delta\omega = 2\pi/T$
- Spacing T ○—● period $\Delta\omega = 2\pi/T$

Periodic signals



$$x(t) = \sum_{n=-\infty}^{\infty} s(t - nT) = s(t) * \sum_{n=-\infty}^{\infty} \delta(t - nT);\ n \text{ integer}$$

- **Periodic signals**

$$x(t)=\sum_{n=-\infty}^{\infty} s(t-nT)=s(t)*\sum_{n=-\infty}^{\infty}\delta(t-nT);\ n\ \text{integer}$$

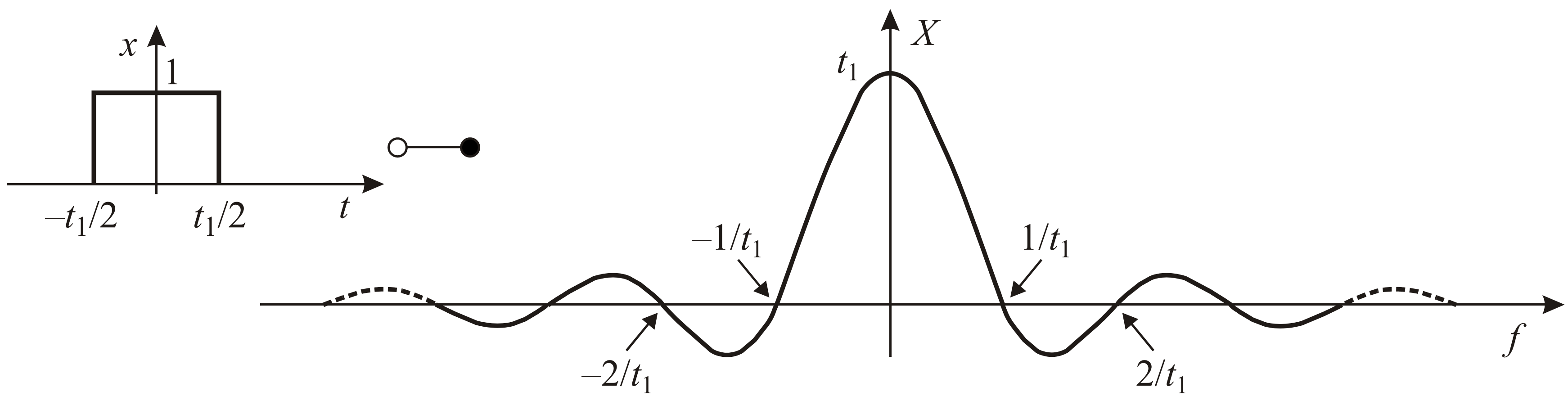
$$\circ\!\!-\!\!\!-\!\!\bullet$$

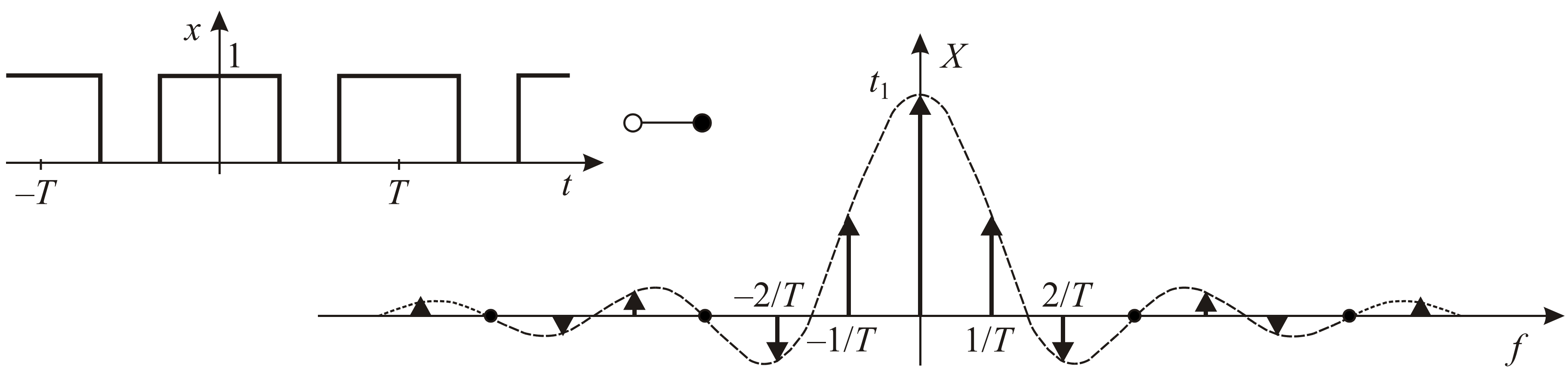
$$X(\mathrm{j}\omega)=S(\mathrm{j}\omega)\cdot\frac{2\pi}{T}\cdot\sum_{k=-\infty}^{\infty}\delta\left(\omega-k\frac{2\pi}{T}\right);\ k\ \text{integer}$$

$$X(f)=S(f)\cdot\frac{1}{T}\cdot\sum_{k=-\infty}^{\infty}\delta\left(f-k\frac{1}{T}\right);\ k\ \text{integer}$$

Periodic signals with period T have a **discrete spectrum** with spacing $\Delta\omega=2\pi/T$ and $\Delta f=1/T$, respectively.

Aperiodic and periodic signals

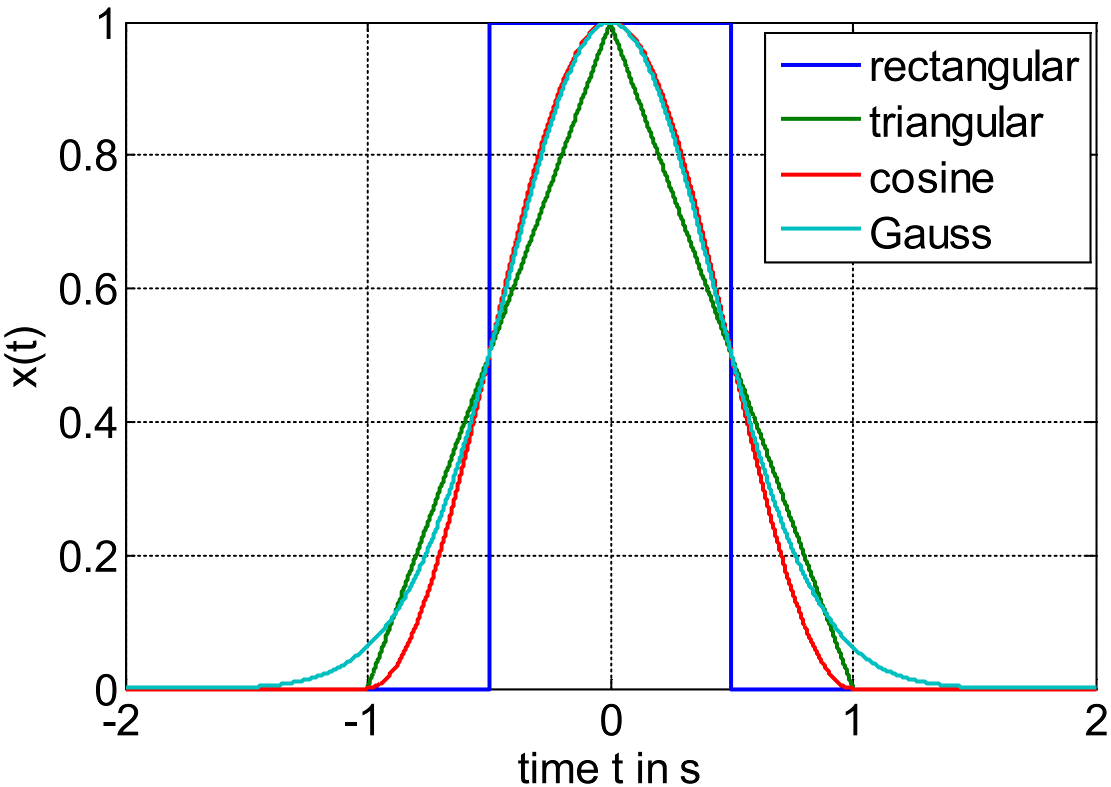




Duality

- One domain ↔ the other domain
  - shifting ↔ multiplication with an exponential function
  - differentiation ↔ multiplication with $\mathrm{j}\omega$ or $\mathrm{j}t$, respectively
  - convolution ↔ multiplication
  - rect function ↔ sinc function
  - ...
- A signal cannot be limited in the time **and** frequency domain
  - time limitation ○—● no frequency limitation
  - frequency limitation ●—○ no time limitation
- **Uncertainty relation**: 'duration' times 'bandwidth' = const.

Window Signals ($T = 1s$)



Rectangular signal

$$x(t) = \mathrm{rect}\left(\frac{t}{T}\right)$$

Triangular signal (BARLETT)

$$x(t) = \begin{cases} 1 - t/T \ , & 0 \le t \le T \\ 1 + t/T \ , & -T \le t \le 0 \\ 0 \ , & \text{otherwise} \end{cases}$$

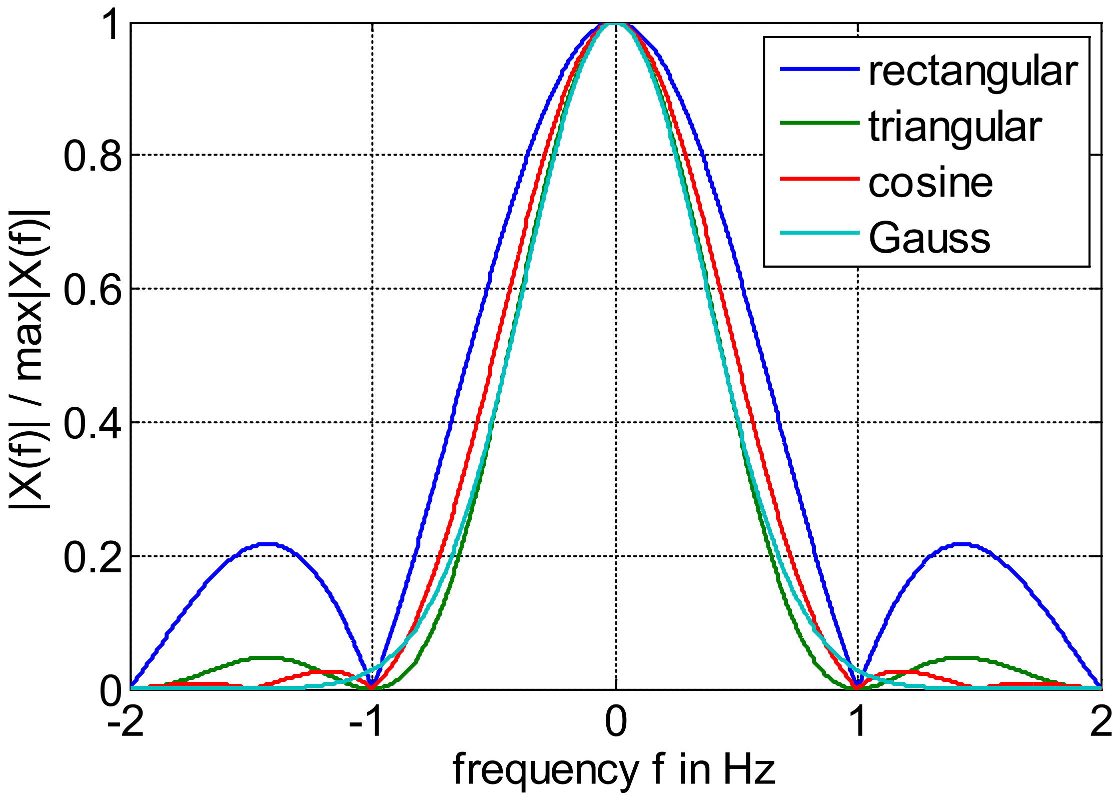
Cosine signal (HANNING)

$$x(t) = \frac{1}{2} \cdot \left[1 + \cos(\pi \cdot t / T)\right] \text{ with } \left|t\right| \le T$$

GAUSS signal

$$x(t) = \mathrm{e}^{-\pi \cdot (t/T)^2}$$

Window Signals



Duration Bandwidth Product (DB)

rectangular: DB= 1.2067
triangular: DB= 0.8859
cosine: DB= 1.0000
Gauss: DB= 0.8826

Duration and bandwidth related to 50 % level.

2.6 HILBERT Transform

- $x(t)$ is a real and causal signal with $\delta(t) = 0 \rightarrow$ even and odd parts are related

$$x_{\mathrm{e}}(t) = x_{\mathrm{o}}(t) \cdot \left[2 \cdot u(t) - 1\right], \quad x_{\mathrm{o}}(t) = x_{\mathrm{e}}(t) \cdot \left[2 \cdot u(t) - 1\right]$$

- Apply FOURIER transformation $x(t) \circ\!\!-\!\!\bullet\, X(\mathrm{j}\omega)$

$$X_{\mathrm{e}}(\mathrm{j}\omega) = -\mathrm{j} \cdot \left\{ \frac{1}{\pi} \cdot \int_{-\infty}^{\infty} \frac{X_{\mathrm{o}}(\mathrm{j}\Omega)}{(\omega - \Omega)} \mathrm{d}\Omega \right\} = -\mathrm{j} \cdot \mathrm{H}\left\{ X_{\mathrm{o}}(\mathrm{j}\omega) \right\}$$

$$X_{\mathrm{Re}}(\mathrm{j}\omega) = \mathrm{H}\left\{ X_{\mathrm{Im}}(\mathrm{j}\omega) \right\}$$

- $\mathrm{H}\{\ldots\}$ denotes the **HILBERT transform**

DAVID HILBERT
1862 - 1943

- The imaginary part can be derived from the real part of the FOURIER transform

$$X_{\mathrm{o}}(\mathrm{j}\omega) = -\mathrm{j} \cdot \left\{ \frac{1}{\pi} \cdot \int_{-\infty}^{\infty} \frac{X_{\mathrm{e}}(\mathrm{j}\Omega)}{(\omega - \Omega)} \mathrm{d}\,\Omega \right\} = -\mathrm{j} \cdot \mathrm{H}\left\{ X_{\mathrm{e}}(\mathrm{j}\omega) \right\}$$

$$X_{\mathrm{Im}}(\mathrm{j}\omega) = -\mathrm{H}\left\{ X_{\mathrm{Re}}(\mathrm{j}\omega) \right\}$$

- **HILBERT transform**

$$\mathrm{H}\left\{ x(t) \right\} = \frac{1}{\pi} \cdot \int_{-\infty}^{\infty} \frac{x(\tau)}{t - \tau} \mathrm{d}\,\tau = x(t) * \frac{1}{\pi \cdot t}$$