Probability and Statistics

5 – Statistics

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$X_i \sim$ Bernoulli(p): Approximate Confidence Intervals for p

(5.21) Approximation of n for confidence intervals of given width

Given α, δ and an estimation $\overline{x} \approx p$ (e.g. from a small preliminary sample), an estimation for the sample size n, such that a two-sided confidence interval for p of confidence level $1-\alpha$ has width 2δ, is given by:

$$n \approx \frac{p(1-p)\cdot\left(\Phi^{-1}\left(1-\frac{\alpha}{2}\right)\right)^2}{\delta^2} \approx \frac{\overline{x}(1-\overline{x})\cdot\left(\Phi^{-1}\left(1-\frac{\alpha}{2}\right)\right)^2}{\delta^2}$$

(meaningful only an estimate for $\bar{x}$)

As $p(1-p) \leq \frac{1}{4}$ for all $p \in \mathbb{R}$, the estimation

$$n \lessapprox \frac{\left(\Phi^{-1}\left(1-\frac{\alpha}{2}\right)\right)^2}{4\delta^2}$$

always true

holds true for all values of p.

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Example (5.22)

Obtain an estimation for the number n of times a fair coin must be thrown in order to see head in 49%–51% of all cases with a probability of at least 98%.

Exercise

$X_i \sim \exp(\lambda)$

Lemma (5.23)

Let $X_1, \dots, X_n$ be a random sample with $X_i \sim \exp(\lambda)$. Then:

$$Y_{2n} := 2\lambda \cdot \sum_{i=1}^{n} X_i = \chi^2_{2n}$$

$\wr$

Proof:

$$2\lambda \cdot \Gamma(n, \lambda) \underset{(4.58)(iv)}{=} \Gamma\left(n, \tfrac{1}{2}\right) = \Gamma\left(\tfrac{2n}{2}, \tfrac{1}{2}\right) = \chi^2_{2n}$$

$X_i \sim \exp(\lambda)$: Confidence intervals for λ and $1/\lambda = E(X_i)$

Notation: $Y_n \sim \chi^2_n$

(5.24)(i) Two-sided confidence intervals

Given $\alpha \in (0,1)$, put: $\quad b_1 := F^{-1}_{Y_{2n}}\left(\frac{\alpha}{2}\right) \quad$ and $\quad b_2 := F^{-1}_{Y_{2n}}\left(1 - \frac{\alpha}{2}\right)$

Then:

$$\Pr\left(\frac{2n\overline{X}}{b_2} \leq 1/\lambda \leq \frac{2n\overline{X}}{b_1}\right) = \Pr\left(\frac{b_1}{2n\overline{X}} \leq \lambda \leq \frac{b_2}{2n\overline{X}}\right)$$

$$= \Pr\left(b_1 \leq \underbrace{2\lambda n\overline{X}}_{\sim \chi^2_{2n} \ (5.23)} \leq b_2\right) = 1 - \alpha$$

and therefore:

$$1/\lambda \in \left[\frac{2n}{b_2}\cdot\overline{X}, \frac{2n}{b_1}\cdot\overline{X}\right] \qquad \text{with confidence level } 1-\alpha$$

$$\lambda \in \left[\frac{b_1}{2n}\cdot\frac{1}{\overline{X}}, \frac{b_2}{2n}\cdot\frac{1}{\overline{X}}\right] \qquad \text{with confidence level } 1-\alpha$$

$X_i \sim \exp(\lambda)$: Confidence intervals for λ and $1/\lambda$

(5.24)(ii) One-sided lower confidence intervals

Given $\alpha \in (0,1)$, put:

$$b \; := F^{-1}_{Y_{2n}}(\alpha)$$

Then:

$$\Pr\left(1/\lambda \le \frac{2n\overline{X}}{b}\right) = \Pr\left(\lambda \ge \frac{b}{2n\overline{X}}\right) = \Pr\left(2\lambda n\overline{X} \ge b\right) = 1-\alpha$$

2

χ^2_{2n}

and:

$$1/\lambda \le \frac{2n}{b}\cdot\overline{X} \qquad \text{with confidence level } 1-\alpha$$

$$\lambda \ge \frac{b}{2n}\cdot\frac{1}{\overline{X}} \qquad \text{with confidence level } 1-\alpha$$

$X_i \sim \exp(\lambda)$: Confidence intervals for λ and $1/\lambda$

(5.24)(iii) One-sided upper confidence intervals

Given $\alpha \in (0,1)$, put:

$$b \ := F_{Y_{2n}}^{-1}(1-\alpha)$$

Then:

$$\Pr\left(1/\lambda \ \geq \ \frac{2n\overline{X}}{b}\right) \ = \ \Pr\left(\lambda \ \leq \ \frac{b}{2n\overline{X}}\right) \ = \ \Pr\left(2\lambda n\overline{X} \ \leq \ b\right) \ = \ 1-\alpha$$

and:

$$1/\lambda \ \geq \ \frac{2n}{b}\cdot\overline{X} \qquad \text{with confidence level } 1-\alpha$$

$$\lambda \ \leq \ \frac{b}{2n}\cdot\frac{1}{\overline{X}} \qquad \text{with confidence level } 1-\alpha$$