# Probability and Statistics

## 4 – Continuous Random Variables

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November 21, 2023

# Continuous Random Variables

## Lemma (4.12)

*Let $X$ be a continuous random variable with pdf $f_X(x)$.*

(i) *If $g : \mathbb{R} \to \mathbb{R}$ is piecewise continous and $Y = g(X) := g \circ X$, then:*

$$E(Y) \;=\; \int_{-\infty}^{\infty} g(x) \cdot f_X(x)\, dx$$

← pf with quantion is possible

(ii) *For any $a, b \in \mathbb{R}$:*

$$E(aX + b) \;=\; a\,E(X) + b$$

$g : \mathbb{R} \to \mathbb{R}$
$x \mapsto ax + b$

# Continuous Random Variables

## Definition (4.13)

Let $X$ be a random variable. Then the following numbers and functions are defined:

(i) *n–th moment* of $X$:

$$E(X^n)$$

(ii) *variance* of $X$:

$$\sigma^2 := Var(X) := E((X - E(X))^2) = E(X^2) - (E(X))^2$$

(iii) *standard deviation* of $X$:

$$\sigma = \sqrt{Var(X)}$$

# Continuous Random Variables

## Definition (4.13)

(iv) *moment generating function* of $X$:

$$\phi_X(t) \;=\; E\left(e^{tX}\right) \;=\; \int_{-\infty}^{\infty} e^{tx} \cdot f_X(x)\,dx \qquad (t \in \mathbb{R})$$

(v) *complex version of moment generating function* of $X$:

$$\phi_X(s) \;=\; E\left(e^{sX}\right) \;=\; \int_{-\infty}^{\infty} e^{sx} \cdot f_X(x)\,dx \qquad (s \in \mathbb{C})$$

(vi) *characteristic function* of $X$:

$$\varphi_X(v) \;=\; \phi_X(j \cdot v) \;=\; E\left(e^{(j\cdot v)X}\right) \qquad (v \in \mathbb{R})$$

# Moment Generating Functions

## Theorem (4.14)

*If there is some $r > 0$, such that $\phi_X(t)$ is defined for all $t \in [-r, r]$, then $\phi_X(s)$ is defined for all complex $s$ with $|s| < r$, all moments $E(X^i)$ are defined and:*

$$\phi_X(s) = \sum_{i=0}^{\infty} \frac{E(X^i)}{i!} s^i \qquad \text{for } |s| < r$$

$\Rightarrow E(X^i) = \phi_X^{(i)}(0)$

$$\phi_X(s) = E(e^{s \cdot X}) = \int_{-\infty}^{\infty} e^{sx} \cdot f_X(x)\, dx = \int_{-\infty}^{\infty} \left( \sum_{i=0}^{\infty} \frac{(sx)^i}{i!} \right) \cdot f_X(x)\, dx$$

$$= \sum_{i=0}^{\infty} \int_{-\infty}^{\infty} \frac{s^i x^i}{i!} \cdot f_X(x)\, dx = \sum_{i=0}^{\infty} \frac{s^i}{i!} \underbrace{\int_{-\infty}^{\infty} x^i \cdot f_X(x)\, dx}_{E(X^i)} \quad ✓$$

# Pdf's for discrete random variables using $\delta$-functions

## Remark (4.15)

Let $X$ be a discrete random variable defined with respect to a pmf $p_X(x)$, where $\{x \mid x \in \mathbb{R}, p_X(x) \neq 0\} = \{x_i \mid i \in I\}$ is countable. Using the $\delta$-function $\delta(x)$, the cdf of $X$ can be expressed as

$$F_X(x) \;=\; \int_{-\infty}^{x} \left( \sum_{i \in I} p_X(x_i) \cdot \delta(t - x_i) \right) dt$$

and

$$f_X(x) \;=\; \sum_{i \in I} p_X(x_i) \cdot \delta(x - x_i)$$

is a pdf of $X$.

# Moment Generating Functions $\leftrightarrow$ Laplace Transforms of PDF's

**Remark (4.16)**

$\phi_X(s)$ is the Laplace transform of $f_X$ evaluated at $-s$:

$$\begin{aligned} \phi_X(s) &= E(e^{sX}) = \int_{-\infty}^{\infty} e^{sx} \cdot f_X(x)\,dx \\ &= \int_{-\infty}^{\infty} e^{-(-s)x} \cdot f_X(x)\,dx = (\mathcal{L}(f_X))(-s) \end{aligned}$$

# Characteristic Functions $\leftrightarrow$ Fourier Transforms of PDF's

**Remark (4.17)**

$\varphi_X(v)$ is the Fourier transform of $f_X$ evaluated at $-\frac{v}{2\pi}$:

$$\varphi_X(v) = E(e^{(jv)X}) = \int_{-\infty}^{\infty} e^{jvx} \cdot f_X(x)\, dx$$

$$= \int_{-\infty}^{\infty} e^{-j\cdot 2\pi(-\frac{v}{2\pi})x} \cdot f_X(x)\, dx = (\mathcal{F}(f_X))\left(-\frac{v}{2\pi}\right)$$

# Characteristic Functions $\leftrightarrow$ Fourier Transforms of PDF's

**Theorem (4.18)**

*The pdf of a random variable $X$ is determined by its characteristic function:*

$$f_X(x) = \frac{1}{2\pi} \cdot \left(\mathcal{F}^{-1}(\varphi_X)\right)\left(-\frac{x}{2\pi}\right)$$

$$\varphi_X(x) = \mathcal{F}(f_X)\left(-\frac{x}{2\pi}\right)$$

$$\frac{1}{2\pi} \cdot \varphi_X(x) = \frac{1}{|-2\pi|}\mathcal{F}(f_X)\left(-\frac{x}{2\pi}\right)$$

$$\Big\downarrow \mathcal{F}^{-1} \quad \text{(scaling theorem)}$$

$$\frac{1}{2\pi} \cdot \mathcal{F}^{-1}(\varphi_X)(x) = f_X(-2\pi \cdot x) \quad \Rightarrow \quad f_X(\tilde{x}) = \frac{1}{2\pi}\mathcal{F}^{-1}(\varphi_X)\left(-\frac{\tilde{x}}{2\pi}\right) \quad \checkmark$$

$$x = -\tilde{x}/2\pi \qquad -2\pi \cdot x = \tilde{x}$$

# Convergence of random variables

**Theorem (4.19)**

*If $(X_i)_{i\in\mathbb{N}}$ is a sequence of random variables, such that the sequence of characteristic functions $(\varphi_{X_i})_{i\in\mathbb{N}}$ converges to some characteristic function $\varphi_X$ for some random variable $X$, then $(X_i)_{i\in\mathbb{N}}$ converges to $X$.*

$$X_1, X_2, X_3, \ldots \longrightarrow X$$

$$\varphi_{X_1}, \varphi_{X_2}, \varphi_{X_3}, \ldots \longrightarrow \varphi_X$$

# Selected Continuous Probability Distributions

- Uniform Distributions
- Exponential Distributions
- Normal Distributions
- Gamma Distributions
- Chi-Square Distributions
- Erlang Distributions
- Beta Distributions
- t-Distributions

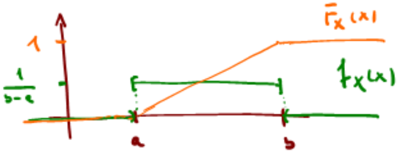
# Uniform Distributions

### Definition (4.35)

A random variable $X$ with a pdf given by

$$f_X(x) \;=\; \begin{cases} \frac{1}{b-a} & \text{for } \; a \leq x \leq b \\ 0 & \text{otherwise} \end{cases}$$

is said to have a *uniform distribution*, $X \sim \mathrm{uniform[a, b]}$.



# Uniform Distributions

**Lemma (4.36)**

*If* $X \sim \mathrm{uniform[a,b]}$, *then:*

(i) $$F_X(x) = \begin{cases} 0 & \text{for } x \leq a \\ \frac{x-a}{b-a} & \text{for } a < x \leq b \\ 1 & \text{for } x > b \end{cases}$$

(ii) $$E(X) = \frac{a+b}{2}$$

(iii) $$Var(X) = \frac{(b-a)^2}{12}$$

(iv) $$\phi_X(t) = \frac{e^{bt} - e^{at}}{(b-a)t}$$

(ii) $$E(X) = \int_{-\infty}^{\infty} x \cdot f_X(x)\, dx = \int_a^b x \cdot \frac{1}{b-a}\, dx = \frac{b+a}{2}$$

(iii) $$E(X^2) = \int_{-\infty}^{\infty} x^2 \cdot f_X(x)\, dx = \int_a^b x^2 \cdot \frac{1}{b-a}\, dx$$

$$= \frac{b^3-a^3}{3(b-a)} = \frac{b^2+ab+a^2}{3}$$

$$Var(X) = \frac{b^2+ab+a^2}{3} - \left(\frac{b+a}{2}\right)^2 \underset{HW}{=} \frac{(b-a)^2}{12}$$

(iv) $$\phi_X(t) = \int_a^b e^{tx} \cdot \frac{1}{b-a}\, dx$$

$$= \frac{1}{b-a} \, \frac{1}{t} \cdot e^{tx} \Big|_a^b = \frac{e^{tb} - e^{ta}}{(b-a) \cdot t} \checkmark$$

### Remark (4.38)

If a random number generator `rand` is available, that generates uniformly distributed numbers from the interval $(0, 1)$, then (4.37)(ii) may be applied to construct a random number generator with a distribution given by any cdf $F_X$: Simply apply $F_X^{-1}$ to the output sequence of `rand`.