Probability and Statistics

2 – Probability

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Intro

How to determine a probability?

- Eg.: What is the probability p_6 of getting a 6 when you roll a die?



- https://www.randomservices.org/random/apps/Dice.html

Intro

How to determine a probability?

- A tempting approach: roll the dice repeatedly, and if a_n gives the number of successes after n rolls, define:

$$p_6 \quad := \quad \lim_{n \to \infty} \frac{a_n}{n}$$

- But: This is NOT a meaningful definition!

Math.:

Given $\varepsilon > 0$ there exists some n_ε, s.t. $|p_6 - \frac{a_n}{n}| < \varepsilon$ for all $n \geq n_\varepsilon$

Intro

Axiomatic Approach

- A. N. Kolmogorow (1903 - 1987)

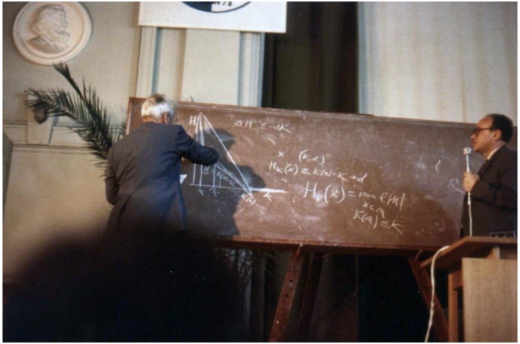
ERGEBNISSE DER MATHEMATIK
UND IHRER GRENZGEBIETE
HERAUSGEGEBEN VON DER SCHRIFTLEITUNG
DES
„ZENTRALBLATT FÜR MATHEMATIK"
ZWEITER BAND
3

GRUNDBEGRIFFE DER
WAHRSCHEINLICHKEITS-
RECHNUNG

VON

A. KOLMOGOROFF

BERLIN
VERLAG VON JULIUS SPRINGER
1933



Power Sets

Definition (2.1)

Let Ω be a set. The *power set* of Ω is defined by:

$$\mathcal{P}(\Omega) \;:=\; \{S \mid S \subseteq \Omega\}$$

Example

The power set of $\Omega = \{0, 1\}$ is given by:

$$\mathcal{P}(\Omega) \;=\; \{\{0,1\}, \{0\}, \{1\}, \emptyset\}$$

$$|\Omega| = n \Rightarrow |\mathcal{P}(\Omega)| = 2^n$$

σ-Algebras

Definition (2.3)

Let Ω be a set and $\mathcal{A}$ be a subset of the power set of Ω:

$$\mathcal{A} \subseteq \mathcal{P}(\Omega)$$

$\mathcal{A}$ is called a *σ-algebra* over Ω, if the following holds:

(i) $\Omega \in \mathcal{A}$

(ii) $A_i \in \mathcal{A}$ for $i \in \mathbb{N} \implies \bigcup_{i=1}^{\infty} A_i \in \mathcal{A}$

(iii) $A \in \mathcal{A} \implies \overline{A} := A^c := (\Omega \setminus A) \in \mathcal{A}$

σ-Algebras

Example (2.4)

(i) For every set Ω,

$$\mathcal{A} \;=\; \mathcal{P}(\Omega)$$

is a σ-algebra over Ω.

(ii) For every set Ω,

$$\mathcal{A} \;=\; \{\emptyset, \Omega\}$$

is a σ-algebra over Ω.

(iii) For $\Omega = \{1, 2, 3\}$,

$$\mathcal{A} = \{\emptyset, \{1\}, \{2, 3\}, \Omega\}$$

is a σ-algebra over Ω.

σ-Algebras

Lemma (2.5)

Let $\mathcal{A}$ be a σ-algebra over Ω. Then:

(i) $A_i \in \mathcal{A}$ for $i \in \mathbb{N} \implies \bigcap_{i=1}^{\infty} A_i \in \mathcal{A}$

$$\bigcap_{i=1}^{\infty} A_i = \Omega \setminus \left(\Omega \setminus \bigcap_{i=1}^{\infty} A_i\right) \underset{\text{De Morgan}}{=} \Omega \setminus \Big(\underbrace{\bigcup_{i=1}^{\infty} \underbrace{(\Omega \setminus A_i)}_{\overset{\cap}{\mathcal{A}}\ (*)}}_{\overset{\cap}{\mathcal{A}}\ (*)}\Big) \in \mathcal{A}$$

σ-Algebras

Lemma (2.5)

Let $\mathcal{A}$ be a σ-algebra over Ω. Then:

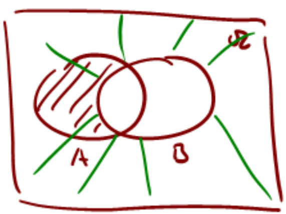
(ii) $A, B \in \mathcal{A} \quad \Longrightarrow \quad A \cup B,\ A \cap B,\ A \setminus B \in \mathcal{A}$

$$(A, B, B, \ldots) =: (A_1, A_2, \ldots)$$

$$\Rightarrow \quad A \cup B = \bigcup_{i=1}^{\infty} A_i \in \mathcal{A}$$

$$A \cap B = \bigcap_{i=1}^{\infty} A_i \in \mathcal{A}$$

$$A \setminus B = (\Omega \setminus B) \cap A \in \mathcal{A}$$



Sample Spaces and Events

Definition (2.6)

Let Ω be a set of possible outcomes of a random experiment and $\mathcal{A} \subseteq \mathcal{P}(\Omega)$ be a σ-algebra. Then Ω is called the *sample space* of the experiment and $\mathcal{A}$ is the *set of events* considered in the experiment. For some $x \in \Omega$ it is said that the event $E \in \mathcal{A}$ occurs if and only if $x \in E$.

Remark

If Ω is finite or countable infinite, all subsets of Ω are usually considered to be events, i.e.:

$$\mathcal{A} = \mathcal{P}(\Omega)$$

Sample Spaces and Events

Example (2.8)

Throwing a die can be considered to be a random experiment with sample space:

$$\Omega \;=\; \{1,2,3,4,5,6\}$$

In this example, the subset $A = \{2,4,6\}$ of Ω is the event, that an even number has been thrown.

Sample Spaces and Events

Example (2.9)

Throwing two dice can be considered to be a random experiment with sample space

$$\Omega = \{(i,j) \mid i,j \in \{1,2,3,4,5,6\}\} \qquad |\Omega| = 36$$

where i denotes the result from the first die and j the result from the second die. (Assuming that the dice are distinguishable.)

If dice are not distinguishable

$$\Omega = \{ \underset{\{1\}}{\{1,1\}}, \{1,2\}, \dots, \underset{\{6\}}{\{6,6\}} \} \qquad |\Omega| = \binom{6}{2} + 6 = 21$$

Sample Spaces and Events

Example (2.10)

Throwing a die, until a specific face, say six, shows up, can be considered to be a random experiment with sample space:

$$\Omega = \mathbb{N}$$

In this example, the subset $A = \{1, 2, 3, 4, 5, 6, 7\}$ of Ω is the event, that six shows up after the die has been thrown for at most seven times.

Sample Spaces and Events

Example (2.11)

The durations of cell-phone calls can be considered to be samples of a random experiment with sample space:

$$\Omega = [0, \infty)$$

Probability Measures

Definition (2.12)

A *probability measure* (or simply a *probability*) is a mapping

$$\mathrm{Pr} : \mathcal{A} \to \mathbb{R}$$

defined on a set of events $\mathcal{A}$ of a sample space Ω, such that:

(i) $\mathrm{Pr}(A) \geq 0 \quad$ for all $\; A \in \mathcal{A}$

(ii) $\mathrm{Pr}(\Omega) = 1$

(iii) For every countable sequence of pairwise disjoint events $A_i \in \mathcal{A} \;\; (i \in \mathbb{N})$:

$$\mathrm{Pr}\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \mathrm{Pr}(A_i)$$

$A_i \cap A_j = \emptyset$
if $i \neq j$

Probability Measures

Theorem (2.13)

Let $\Pr$ *be a probability measure defined on a set of events* $\mathcal{A}$ *of a sample space* Ω *and let* $A, B, A_i \in \mathcal{A}$ $(i = 1, \ldots, n)$, *then the following statements are true:*

(i) $\Pr(\emptyset) = 0$ $\quad \Pr(\emptyset) = \Pr\left(\bigcup_{i=1}^{\infty} \emptyset\right) = \sum_{i=1}^{\infty} \Pr(\emptyset) = \infty$ if $\Pr(\emptyset) \neq 0$ ✓ $\quad \Rightarrow = 0$

σ (ii) $\Pr\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} \Pr(A_i)$ *if* $A_i \cap A_j = \emptyset$ *for all* $i \neq j$

(iii) $\Pr(A^c) = 1 - \Pr(A)$ $\quad \Omega = A \,\dot\cup\, (\Omega \setminus A) \overset{(ii)}{\Rightarrow} 1 = \Pr(\Omega) = \Pr(A) + \Pr(A^c)$

(iv) $A \subseteq B \implies \Pr(A) \leq \Pr(B)$ $\quad B = A \,\dot\cup\, (B \setminus A) \Rightarrow \Pr(B) = \Pr(A) + \underbrace{\Pr(B \setminus A)}_{\geq 0}$

(v) $0 \leq \Pr(A) \leq 1$ $\quad \emptyset \subseteq A \subseteq \Omega$ Apply (iv)

(vi) $\Pr(A \setminus B) = \Pr(A \cap B^c) = \Pr(A) - \Pr(A \cap B)$ $\leftarrow$ $A = (A \setminus B) \,\dot\cup\, (A \cap B)$

(vii) $\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B)$

$A \cup B = (A \setminus B) \,\dot\cup\, B$

$\Pr(A \cup B) \underset{(vi)}{=} \Pr(A) - \Pr(A \cap B) + \Pr(B)$

Probability Measures

Theorem (2.13 (viii))

$$\Pr\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{\emptyset \neq I \subseteq \{1,2,\dots,n\}} (-1)^{|I|+1} \Pr\left(\bigcap_{i\in I} A_i\right)$$

$$= \sum_{i=1}^{n} \Pr(A_i) - \sum_{i_1<i_2} \Pr(A_{i_1} \cap A_{i_2}) + \sum_{i_1<i_2<i_3} \Pr(A_{i_1} \cap A_{i_2} \cap A_{i_3})$$

$$- \sum_{i_1<i_2<i_3<i_4} \Pr(A_{i_1} \cap A_{i_2} \cap A_{i_3} \cap A_{i_4})$$

$$+ \cdots + (-1)^{n+1} \Pr(A_1 \cap A_2 \cap \cdots \cap A_n)$$