Probability and Statistics

2 – Probability

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Probability Measures

① Sample space Ω

σ - algebra

② events $\mathcal{A} \subseteq \mathcal{P}(\Omega)$

③ prob. measure

$\Pr : \mathcal{A} \to \mathbb{R}$

Definition (2.12)

A *probability measure* (or simply a *probability*) is a mapping

$$\Pr : \mathcal{A} \to \mathbb{R}$$

defined on a set of events $\mathcal{A}$ of a sample space Ω, such that:

(i) $\Pr(A) \geq 0$ for all $A \in \mathcal{A}$

(ii) $\Pr(\Omega) = 1$

(iii) For every countable sequence of pairwise disjoint events $A_i \in \mathcal{A}$ $(i \in \mathbb{N})$:

$$\Pr\left(\bigcup_{i=1}^{\infty} A_i\right) = \sum_{i=1}^{\infty} \Pr(A_i)$$

Probability Measures

Theorem (2.13)

Let $\Pr$ *be a probability measure defined on a set of events* $\mathcal{A}$ *of a sample space* Ω *and let* $A, B, A_i \in \mathcal{A}$ $(i = 1, \ldots, n)$*, then the following statements are true:*

(i) $\Pr(\emptyset) = 0$

(ii) $\Pr\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} \Pr(A_i) \qquad$ *if* $A_i \cap A_j = \emptyset$ *for all* $i \neq j$

(iii) $\Pr(A^c) = 1 - \Pr(A)$

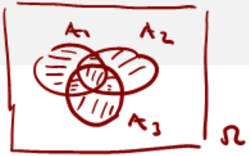
(iv) $A \subseteq B \implies \Pr(A) \leq \Pr(B)$

(v) $0 \leq \Pr(A) \leq 1$

(vi) $\Pr(A \setminus B) = \Pr(A \cap B^c) = \Pr(A) - \Pr(A \cap B)$

(vii) $\Pr(A \cup B) = \Pr(A) + \Pr(B) - \Pr(A \cap B)$

Probability Measures



Theorem (2.13 (viii))

$$\Pr\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{\emptyset \neq I \subseteq \{1,2,\ldots,n\}} (-1)^{|I|+1} \Pr\left(\bigcap_{i \in I} A_i\right)$$

2^n-1 summands

$$= \overset{|I|=1}{\sum_{i=1}^{n}} \Pr(A_i) - \overset{|I|=2}{\sum_{i_1<i_2}} \Pr(A_{i_1} \cap A_{i_2}) + \overset{|I|=3}{\sum_{i_1<i_2<i_3}} \Pr(A_{i_1} \cap A_{i_2} \cap A_{i_3})$$

$$- \overset{|I|=4}{\sum_{i_1<i_2<i_3<i_4}} \Pr(A_{i_1} \cap A_{i_2} \cap A_{i_3} \cap A_{i_4})$$

$$+ \cdots + \overset{|I|=n}{(-1)^{n+1} \Pr(A_1 \cap A_2 \cap \cdots \cap A_n)}$$

Proof of Theorem (2.13)

(viii)

$$\Pr\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{\emptyset \neq I \subseteq \{1,2,\ldots,n\}} (-1)^{|I|+1} \Pr\left(\bigcap_{i \in I} A_i\right)$$

Proof by induction.

$n = 1$: $\Pr\left(\bigcup_{i=1}^{1} A_i\right) = \Pr(A_1)$

$\{A_1\}$

$\sum_{\phi \neq I = \{1\}} (-1)^{|I|+1} \Pr\left(\bigcap_{i \in I} A_i\right) = -1^{|\{1\}|+1} \Pr\left(\bigcap_{i=1} A_i\right) =$ ✓

$= +1 \cdot \Pr(A_1)$

Proof of Theorem (2.13)

(viii) $n \to (n+1)$: Set $\underline{A_i^* \ := \ A_i \cap A_{n+1}}$ for $i = 1, 2, \ldots, n$. Then:

$$\Pr\left(\bigcup_{i=1}^{n+1} A_i\right) = \Pr\left(\left(\bigcup_{i=1}^{n} A_i\right) \cup A_{n+1}\right)$$

$$\overset{(vii)}{=} \Pr\left(\bigcup_{i=1}^{n} A_i\right) + \Pr(A_{n+1}) - \Pr\left(\left(\bigcup_{i=1}^{n} A_i\right) \cap A_{n+1}\right)$$

$$= \Pr\left(\bigcup_{i=1}^{n} A_i\right) + \Pr(A_{n+1}) - \Pr\left(\bigcup_{i=1}^{n} A_i^*\right)$$

$$\left(\bigcup_{i=1}^{n} A_i\right) \cap A_{n+1} = \bigcup_{i=1}^{n} (A_i \cap A_{n+1})$$

Proof of Theorem (2.13)

$I = \{1,3\}$

$A_1^* \cap A_3^* = (A_1 \cap A_8) \cap (A_3 \cap A_8)$
$= A_1 \cap A_3 \cap A_8$

Example

Ind. hypoth.

(viii) $\ldots \overset{\text{Ind. hypoth.}}{=} \displaystyle\sum_{\emptyset \neq I \subseteq \{1,2,\ldots,n\}} (-1)^{|I|+1} \Pr\left(\bigcap_{i\in I} A_i\right) + \Pr(A_{n+1}) - \sum_{\emptyset \neq I \subseteq \{1,2,\ldots,n\}} (-1)^{|I|+1} \Pr\left(\bigcap_{i\in I} A_i^*\right)$

$$= \sum_{\emptyset \neq I \subseteq \{1,2,\ldots,n\}} (-1)^{|I|+1} \Pr\left(\bigcap_{i\in I} A_i\right) + \Pr(A_{n+1}) - \underbrace{\sum_{\emptyset \neq I \subseteq \{1,2,\ldots,n\}}}_{I} (-1)^{|I|+1} \Pr\left(\underbrace{\bigcap_{i\in I} A_i \cap A_{n+1}}\right)$$

$J = I \cup \{n+1\}$

$$= \sum_{\emptyset \neq I \subseteq \{1,2,\ldots,n\}} (-1)^{|I|+1} \Pr\left(\bigcap_{i\in I} A_i\right) + \Pr(A_{n+1}) - \sum_{\substack{J \subseteq \{1,2,\ldots,n,n+1\} \\ (n+1)\in J \neq \{n+1\}}} (-1)^{|J|} \Pr\left(\bigcap_{i\in J} A_i\right)$$

Proof of Theorem (2.13)

(viii) ...
$$= \sum_{\emptyset \neq I \subseteq \{1,2,\dots,n\}} (-1)^{|I|+1} \Pr\left(\bigcap_{i\in I} A_i\right) + \Pr(A_{n+1}) - \sum_{\substack{I \subseteq \{1,2,\dots,n,n+1\} \\ (n+1) \in I \neq \{n+1\}}} (-1)^{|I|} \Pr\left(\bigcap_{i\in I} A_i\right)$$

$$= \sum_{\substack{\emptyset \neq I \subseteq \{1,2,\dots,n,n+1\} \\ (n+1) \notin I}} (-1)^{|I|+1} \Pr\left(\bigcap_{i\in I} A_i\right) + \sum_{\substack{I \subseteq \{1,2,\dots,n,n+1\} \\ (n+1) \in I}} (-1)^{|I|+1} \Pr\left(\bigcap_{i\in I} A_i\right)$$

$$= \sum_{\emptyset \neq I \subseteq \{1,2,\dots,n,n+1\}} (-1)^{|I|+1} \Pr\left(\bigcap_{i\in I} A_i\right)$$

□

Finite Sample Spaces

Notation (2.14)

Let Ω be a finite sample space

$$\Omega \;=\; \{\omega_1, \omega_2, \ldots, \omega_N\}$$

with probability measure Pr defined on $\mathcal{A} = \mathcal{P}(\Omega)$. The probabilities of the elementary events (defined by the elements of Ω) are then given as follows:

$$\boxed{p_i \;:=\; \Pr(\{\omega_i\})} \qquad (i = 1, 2, \ldots, N)$$

Simplification of notation $\quad p_i = \Pr(\omega_i)$

Ω	ω_1	ω_2	---	ω_N
Pr	p_1	p_2		p_N

(1) $0 \le p_i\ [\le 1]$

(2) $\sum_{i=1}^{N} p_i = 1$

$\Pr(\{\omega_1, \omega_2\}) = \Pr(\{\omega_1\} \,\dot{\cup}\, \{\omega_2\}) = \Pr(\{\omega_1\}) + \Pr(\{\omega_2\}) = p_1 + p_2$

Finite Sample Spaces

Lemma (2.15)

With the notation from (2.14), we have

$$\boxed{\Pr(A) = \sum_{\{i|\omega_i \in A\}} p_i \qquad \textit{for all } A \in \mathcal{A}} \qquad (*)$$

In particular:

$$\sum_{i=1}^{N} p_i = 1$$

see last slide

Finite Sample Spaces

Lemma (2.15)

On the other hand, given any sequence of non-negative numbers

$$p_1, p_2, \dots, p_N$$

such that $\sum_{i=1}^{N} p_i = 1$, *a probability measure* Pr *is defined on* $\mathcal{A} = \mathcal{P}(\Omega)$ *from the formula* (*) *above.*

Simple Sample Spaces

Definition (2.16)

A sample space with equally likely outcomes is called a *simple sample space*. With the notation from (2.14), Ω is a simple sample space, iff:

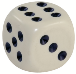
$$p_i = \frac{1}{|\Omega|} = \frac{1}{N} \qquad \text{for all } \; i \in \{1, 2, \ldots, N\}$$

For a simple sample space Ω:

$$\Pr(A) = \frac{|A|}{|\Omega|} \qquad \text{for all } \; A \in \mathcal{A}$$

Simple Sample Spaces

Example (2.17)



A fair die, used in the random experiment described in (2.8), is expressed by a uniform probability measure:

$$p_i \quad = \quad \Pr(i) \quad := \quad \frac{1}{6} \qquad \text{for} \;\; i = 1, 2, \ldots, 6$$

Countable Infinite Sample Spaces

Notation (2.18)

The notation and results introduced in (2.14) and (2.15) can also be used for countable infinite sample spaces

$$\Omega \;=\; \{\omega_i \mid i \in \mathbb{N}\}$$

to characterise the probability measures on $\mathcal{A} = \mathcal{P}(\Omega)$. These probability measures correspond to all sequences $(p_i)_{i\in\mathbb{N}}$ of non-negative numbers, such that

$$1 = \Pr(\Omega) = \Pr\Big(\bigcup_{i=1}^{\infty}\{\omega_i\}\Big) = \sum_{i=1}^{\infty}\Pr(\{\omega_i\}) \Rightarrow \sum_{i=1}^{\infty} p_i \;=\; 1$$

again by simply setting:

$$p_i \;:=\; \Pr(\{\omega_i\}) \qquad \text{for all } \; i \in \mathbb{N}$$

Countable Infinite Sample Spaces

Example (2.19)

If the random experiment described in (2.10) is based on a fair die, the probabilities for the elementary events are given by:

$$p_i \quad := \quad \left(\frac{5}{6}\right)^{i-1} \cdot \frac{1}{6} \qquad \text{for all} \quad i \in \mathbb{N}$$

$p_1 = \frac{1}{6}$, $p_2 = \frac{5}{36}$, $p_3 = \frac{5^2}{6^3}$

$p_i = \frac{5^{i-1}}{6^i}$

$\sum_{i=1}^{\infty} p_i = \frac{1}{6} \sum_{i=1}^{\infty} \left(\frac{5}{6}\right)^{i-1} = \frac{1}{6} \sum_{i=0}^{\infty} \left(\frac{5}{6}\right)^{i} = \frac{1}{6} \cdot \frac{1}{1-5/6} = 1$ ✓

$\Omega = \{ (x_1, x_2, \ldots, x_r) \mid x_i \in \{1, \ldots, 5\}$ for $i < r$, $x_r = 6 \}$

$\Omega_2 = \{ (x_1, x_2) \mid x_1 < 6, x_2 = 6 \}$

$|\Omega_2| = 5$

$\Omega_2^* = \{1, \ldots, 6\}^2$

Countable Infinite Sample Spaces

Example (2.20)

Generalizing (2.10) by considering the repetition of a simple random experiment with probability p for success, gives rise to a probability measure for $\mathbb{N}$ with:

$$p_i \quad = \quad p \cdot (1-p)^{i-1} \qquad \text{for all} \quad i \in \mathbb{N}$$

The probability for the event, that success occurs after at most n repetitions, is given by:

$$s_n = \sum_{i=1}^{n} p_i = p \cdot \sum_{i=1}^{n} (1-p)^{i-1} = p \cdot \sum_{i=0}^{n-1} (1-p)^{i} = \cancel{p} \cdot \frac{1-(1-p)^n}{1-\cancel{(1-p)}} = 1-(1-p)^n$$

Question: needed n to get $s_n \geq 1/2$

$$s_n = 1-(1-p)^n \geq 1/2 \iff (1-p)^n \leq 1/2 \iff n \cdot \ln(1-p) \leq -\ln(2)$$

$$\iff n \geq -\frac{\ln(2)}{\ln(1-p)}$$

Ex. $p = \frac{1}{6}$

$p = 10^{-10}, \dots$

The Matching Problem

Example (2.21)

Let p_n denote the probability that a random permutation of $n \in \mathbb{N}$ elements has at least one fixed point. Then:

$$\begin{array}{c|c|c|c|c|c} 1 & 2 & 3 & \ldots i & & n \\ \hline 3 & 12 & n & i & & 1 \end{array}$$

$$f : \{1, \ldots, n\} \to \{1, \ldots, n\} \in \Sigma_n$$

$$|\Omega| = |\Sigma_n| = n!$$

The Matching Problem

Example (2.21)

Let p_n denote the probability that a random permutation of $n \in \mathbb{N}$ elements has at least one fixed point. Then:

$$p_n \;=\; 1 - \sum_{i=0}^{n} \frac{(-1)^i}{i!} \quad \xrightarrow[n\to\infty]{} \quad 1 - \frac{1}{e} \;\approx\; 0,6321$$

The Matching Problem

$A_i \subseteq \Sigma_n$, $A_i = \{f \in \Sigma_n \mid f(i) = i\}$

$|A_1| = (n-1)!$

$|A_{i_1} \cap A_{i_2}| = (n-2)!$

Remember Theorem (2.13)(viii):

$$\Pr\left(\bigcup_{i=1}^{n} A_i\right) = \sum_{\emptyset \neq I \subseteq \{1,2,\dots,n\}} (-1)^{|I|+1} \Pr\left(\bigcap_{i \in I} A_i\right)$$

$$= \sum_{i=1}^{n} \Pr(A_i) - \overset{\binom{n}{2}}{\sum_{i_1 < i_2}} \underset{\frac{(n-2)!}{n!}}{\Pr(A_{i_1} \cap A_{i_2})} + \overset{\binom{n}{3}}{\sum_{i_1 < i_2 < i_3}} \underset{\frac{(n-3)!}{n!}}{\Pr(A_{i_1} \cap A_{i_2} \cap A_{i_3})}$$

$$- \sum_{i_1 < i_2 < i_3 < i_4} \Pr(A_{i_1} \cap A_{i_2} \cap A_{i_3} \cap A_{i_4})$$

$$+ \cdots + (-1)^{n+1} \Pr(A_1 \cap A_2 \cap \cdots \cap A_n)$$

The Matching Problem

Let A_i denote the event, that the i'th element is fixed.

$$
\begin{aligned}
p_n &= \Pr\left(\bigcup_{i=1}^{n} A_i\right) = n \cdot \frac{1}{n} - \binom{n}{2} \cdot \frac{1}{n\cdot(n-1)} + \binom{n}{3} \cdot \frac{1}{n\cdot(n-1)\cdot(n-2)} \mp \dots \\
&= 1 - \frac{1}{2!} + \frac{1}{3!} - \frac{1}{4!} + \dots + (-1)^{n+1}\frac{1}{n!} \\
&= -\sum_{i=1}^{n} \frac{(-1)^i}{i!} = 1 - \sum_{i=0}^{n} \frac{(-1)^i}{i!} \\
&\underset{n\to\infty}{\longrightarrow} 1 - \sum_{i=0}^{\infty} \frac{(-1)^i}{i!} = 1 - e^{-1}
\end{aligned}
$$