1.) Let F_X be a cumulative distribution function. Prove Lemma 4.3(ii) and the first assertion of Lemma 4.4(iii):

$$\lim_{x \to -\infty} F_X(x) \;=\; 0$$

2.) If $X \sim$ uniform[a, b], show that $\mathrm{Var}(X) = \frac{(b-a)^2}{12}$.

3.) Let $X \sim \exp(\lambda)$ with $\lambda \in \mathbb{R}^+$. Show that:

(i) $E(X^n) = \dfrac{n!}{\lambda^n}$ for $n \in \mathbb{N}$

(ii) $\phi_X(t) = \dfrac{\lambda}{\lambda - t}$ for $t < \lambda$

(iii) $F_X^{-1}(p) = -\dfrac{\ln(1-p)}{\lambda}$ for $p \in (0,1)$

4.) The time (in hours) required to repair a machine is an exponentially distributed random variable with parameter $\lambda = 1$.

(i) What is the probability that a repair time exceeds 2 hours?

(ii) What is the conditional probability that a repair takes at least 3 hours, given that its duration exceeds 2 hours?

5.) The number of years a radio functions is exponentially distributed with parameter $\lambda = \frac{1}{8}$. If Jones buys a used radio, what is the probability that it will be working after an additional 10 years?

6.) Jones figures that the total number of thousands of miles that a used auto can be driven before it would need to be junked is an exponential random variable with parameter $\frac{1}{20}$. Smith has a used car that he claims has been driven only 10,000 miles. If Jones purchases the car, what is the probability that she would get at least 20,000 additional miles out of it? Repeat under the assumption that the lifetime mileage of the car is not exponentially distributed but rather is (in thousands of miles) uniformly distributed over (0, 40).

7.) Let $X : \Omega \to \mathbb{R}$ be a random variable with $X(\omega) > 0$ for all $\omega \in \Omega$ and $X \sim$ uniform$[0,1]$. Find the density of:

$$Y \;:=\; \ln(1/X)$$

8.) Given $\lambda > 0$, show that

$$f(x) \;=\; \frac{\lambda}{2} e^{-\lambda|x|}$$

is a probability density function.

If X is a random variable defined by the pdf f given above, then X is said to be a *Laplace distribution* (*double-sided exponential distribution*) denoted by $X \sim \mathrm{Laplace}(\lambda)$.

Determine $F_X(x)$, $E(X)$, $\mathrm{Var}(X)$ and $\phi_X(t)$ if $X \sim \mathrm{Laplace}(\lambda)$.

9.) Let $\lambda > 0$. Show that

$$f(x) = \frac{\lambda/\pi}{\lambda^2 + x^2}$$

is a probability density function.

If X is a random variable defined by the pdf f given above, then X is said to be a *Cauchy distribution* denoted by $X \sim \text{Cauchy}(\lambda)$.

Determine (if possible) $F_X(x)$, $E(X)$, $\text{Var}(X)$ and $\phi_X(t)$ if $X \sim \text{Cauchy}(\lambda)$.

10.) Let X be a random variable with $X \sim \text{Cauchy}(\lambda)$. Determine the characteristic function $\varphi_X(v)$.

11.) Show that

$$f(x) \;=\; \begin{cases} \frac{x}{\lambda^2}\, e^{\frac{-(x/\lambda)^2}{2}} & \text{for } \; x \geq 0 \\ 0 & \text{for } \; x < 0 \end{cases}$$

is a probability density function.

If X is a random variable defined by the pdf f given above, then X is said to be a *Rayleigh distribution* denoted by $X \sim \text{Rayleigh}(\lambda)$.

Determine $F_X(x)$, $E(X)$ and $\text{Var}(X)$ if $X \sim \text{Rayleigh}(\lambda)$.

$$\left(\text{Hint: } \int_0^\infty e^{\frac{-x^2}{2}}\, dx \;=\; \sqrt{\pi/2}\right)$$

12.) If $X \sim \exp(1)$, show that $\sqrt{X} \sim \text{Rayleigh}(1/\sqrt{2})$.

13.) Find $c \in \mathbb{R}$, such that

$$f_{XY}(x,y) \;:=\; \begin{cases} \frac{c}{(x+y+1)^3} & \text{if } \; x, y \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

is a joint probability density function for two random variables X and Y.

Determine $f_X(x), f_Y(y), F_X(x), F_Y(y), F_{XY}(x,y)$ and show that X and Y are not independent. (Hint: Consider the events $X \leq 1$ and $Y \leq 1$.)

14.) Show that

$$F_{XY}(x,y) \;=\; \begin{cases} 1 - e^{-xy} & \text{if } \; x, y \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

is not a joint cdf of two random variables.

15.) Let X_i be independent random variables with $X_i \sim \text{uniform}[-0.5, 0.5]$ for $i = 1, 2, 3$. Determine the pdf and cdf of $X_1 + X_2$ and $X_1 + X_2 + X_3$.