Probability and Statistics

4 – Continuous Random Variables

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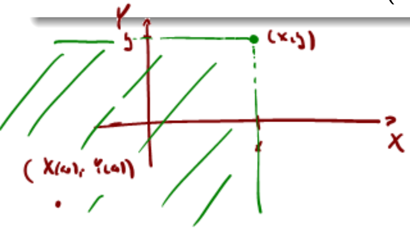
Joint Random Variables

Definition (4.20)

Let X, Y be random variables with respect to the same probability measure Pr, i.e. with respect to the same triple $(\Omega, \mathcal{A}, \Pr)$. The *joint cumulative distribution function* of X and Y, $F_{XY} : \mathbb{R}^2 \to [0,1]$, is defined by:

$$\begin{aligned} F_{XY}(x,y) &:= \Pr(X \le x, Y \le y) \\ &= \Pr(X^{-1}((-\infty,x]) \cap Y^{-1}((-\infty,y])) \qquad \text{for all } x,y \in \mathbb{R} \end{aligned}$$



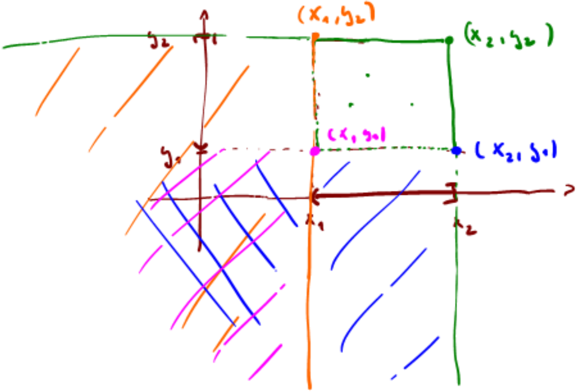
Joint Cumulative Distribution Functions

Lemma (4.21)

Let X, Y be random variables with respect to the same probability measure $\Pr$ and $(x_1, x_2] \times (y_1, y_2]$ be a rectangle. Then:

$$\begin{aligned}&\Pr(x_1 < X \leq x_2,\ y_1 < Y \leq y_2) \\ &\quad = F_{XY}(x_2, y_2) - F_{XY}(x_1, y_2) - F_{XY}(x_2, y_1) + F_{XY}(x_1, y_1)\end{aligned}$$

Proof of Lemma (4.21)



$$F_{XY}(x_2, y_2)$$
$$- F_{XY}(x_1, y_2)$$
$$- \left(F_{XY}(x_2, y_1) - F_{XY}(x_1, y_1)\right)$$

$\square$

Selected Continuous Probability Distributions

- Uniform Distributions
- Exponential Distributions
- Normal Distributions
- Gamma Distributions
- Chi-Square Distributions
- Erlang Distributions
- Beta Distributions
- t-Distributions

Uniform Distributions

Definition (4.35)

A random variable X with a pdf given by

$$f_X(x) = \begin{cases} \frac{1}{b-a} & \text{for } a \leq x \leq b \\ 0 & \text{otherwise} \end{cases}$$

is said to have a *uniform distribution*, $X \sim \mathrm{uniform[a, b]}$.

Uniform Distributions

Lemma (4.36)

If $X \sim \mathrm{uniform[a, b]}$, *then:*

(i) $F_X(x) = \begin{cases} 0 & \textit{for } x \leq a \\ \frac{x-a}{b-a} & \textit{for } a < x \leq b \\ 1 & \textit{for } x > b \end{cases}$

(ii) $E(X) = \dfrac{a+b}{2}$

(iii) $Var(X) = \dfrac{(b-a)^2}{12}$

(iv) $\phi_X(t) = \dfrac{e^{bt} - e^{at}}{(b-a)t}$

Probability Integral Transformations

Theorem (4.37)

Let X be a continuous random variable with cdf F_X.

(i) *The so-called probability integral transformation*

$$Y = F_X(X)$$

has a uniform distribution over $[0,1]$, i.e.:

$$Y \sim \mathrm{uniform}[0,1]$$

(ii) *let U be a uniformly distributed random variable $U \sim \mathrm{uniform}[0,1]$. Then:*

$$F_X = F_{F_X^{-1}(U)}$$

Proof of Theorem (4.37)

(i) $Y = F_X(X) \sim \mathrm{uniform}[0,1]$

$y \in (0,1) \qquad F_X^{-1}(y) = [x_0, x_1]$

$F_Y(y) = \Pr(Y \le y) = \Pr(F_X(X) \le y) = \Pr(X \le x_1) = F_X(x_1) = y$

Proof of Theorem (4.37)

(ii) $U \sim \mathrm{uniform}[0,1] \quad \Longrightarrow \quad F_X = F_{F_X^{-1}(U)}$

$U \sim \mathrm{uniform}[0,1]$

$$F_{F_X^{-1}(U)}(x) = P\left(F_X^{-1}(U) \le x\right) \underset{(*)}{=} P\left(U \le F_X(x)\right) \overset{\downarrow}{=} F_X(x)$$

(*) 1.) $F_X^{-1}(U(\omega)) \le x \Rightarrow \underbrace{F_X\left(F_X^{-1}(U(\omega))\right)}_{= U(\omega)} \le F_X(x) \Rightarrow U(\omega) \le F_X(x)$

2.) $U(\omega) \le F_X(x) \Rightarrow F_X^{-1}(U(\omega)) \le F_X^{-1}\left(F_X(x)\right) \le x$

$\Rightarrow F_X^{-1}(U(\omega)) \le x$

Remark (4.38)

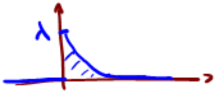
If a random number generator `rand` is available, that generates uniformly distributed numbers from the interval $(0, 1)$, then (4.37)(ii) may be applied to construct a random number generator with a distribution given by any cdf F_X: Simply apply F_X^{-1} to the output sequence of `rand`.

Exponential Distributions

Definition (4.39)

A random variable X has a *exponential distribution*, $X \sim \exp(\lambda)$, for some parameter $\lambda > 0$, if it has a pdf defined by:

$$f_X(x) \;=\; \begin{cases} \lambda e^{-\lambda x} & \text{for } \; x \geq 0 \\ 0 & \text{for } \; x < 0 \end{cases}$$



$$\left(\text{Note:} \quad \int_0^\infty \lambda e^{-\lambda x}\, dx \;=\; -e^{-\lambda x}\Big|_0^\infty \;=\; 1 \right)$$

Exponential Distributions

Lemma (4.40)

If $X \sim \exp(\lambda)$, *then:*

(i) $F_X(x) = \begin{cases} 1 - e^{-\lambda x} & \text{if } x \geq 0 \\ 0 & \text{if } x < 0 \end{cases}$ ✓ ✓

(ii) $E(X^n) = \dfrac{n!}{\lambda^n}$ for $n \in \mathbb{N}$

(iii) $E(X) = \dfrac{1}{\lambda}$, $Var(X) \overset{\checkmark}{=} \dfrac{1}{\lambda^2}$,

(iv) $\phi_X(t) = \dfrac{\lambda}{\lambda - t}$ for $t < \lambda$ HW (+ Reproof of (iii))

(v) $F_X^{-1}(p) \overset{\checkmark}{=} -\dfrac{\ln(1-p)}{\lambda}$ for $p \in (0,1)$

(vi) *first quartile, median and third quartile are:*

$\ln(4/3)/\lambda \approx 0.288/\lambda$, $\ln(2)/\lambda \approx 0.693/\lambda$, $\ln(4)/\lambda \approx 1.386/\lambda$

(i) $\int_0^x \lambda e^{-\lambda s}\, ds = -e^{-\lambda s}\Big|_0^x = -e^{-\lambda x} + 1$

(ii) $E(X^n) = \int_0^\infty x^n \lambda e^{-\lambda x}\, dx \overset{!}{=} \frac{n!}{\lambda^n}$

Induction for $n \in \mathbb{N}_0$ $n = 0$ ✓

$n \to (n+1)$ $\int_0^\infty x^{n+1} \lambda e^{-\lambda x}\, dx = x^{n+1} \cdot (-e^{-\lambda x})\Big|_0^\infty - \int_0^\infty (n+1) x^n (-e^{-\lambda x})\, dx$

$= 0 - 0 + (n+1) \cdot \frac{1}{\lambda} \int_0^\infty x^n \lambda e^{-\lambda x}\, dx \overset{I.H.}{=} \frac{n+1}{\lambda} \frac{n!}{\lambda^n} = \frac{(n+1)!}{\lambda^{n+1}}$

$E(X^2) = \frac{2}{\lambda^2}$

(v) $p = 1 - e^{-\lambda x} \Rightarrow 1 - p = e^{-\lambda x} \Rightarrow \ln(1-p) = -\lambda x$

Memorylessness of Exponential Distributions

Theorem (4.42)

X has an exponential distribution, if and only if X has the memoryless property:

$$\Pr(X > t + h \mid X > t) = \Pr(X > h) \qquad \textit{for all } t, h \geq 0$$

$X \sim \exp(\lambda)$

$$\frac{1 - F_X(t+h)}{1 - F_X(t)} = \frac{e^{-\lambda(t+h)}}{e^{-\lambda t}} = e^{-\lambda h} = 1 - F_X(h)$$

Proof of Theorem (4.42)

Assume X has the memoryless property

$a := \Pr(X > 1)$

$n \in \mathbb{N}$: $n > 1$ $\quad \Pr(X > n) = \Pr(X > 1) \dfrac{\Pr(X > n)}{\Pr(X > 1)} = \Pr(X > 1)\Pr(X > n \mid X > 1) \overset{\text{m.p.}}{=} a \cdot \Pr(X > n-1)$

$\Rightarrow \Pr(X > n) = a^n$

$a = \Pr(X > n \cdot \frac{1}{n}) \underset{\text{as before}}{=} (\Pr(X > \frac{1}{n}))^n \Rightarrow \Pr(X > \frac{1}{n}) = a^{1/n}$

$\Pr(X > \frac{m}{n}) = (\Pr(X > \frac{1}{n}))^m = a^{\frac{m}{n}}$ $\qquad 1 - F_X(\frac{m}{n}) = a^{\frac{m}{n}}$

$\Rightarrow F_X(\frac{m}{n}) = 1 - a^{\frac{m}{n}}$

Continuity $\Rightarrow F_X(x) = 1 - a^x = 1 - e^{\ln(a) \cdot x}$

$\Rightarrow X \sim \exp(\ln(a))$