Information Fusion (IFU), Summer Semester 2023

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Excercise 5: Fuzzy Set Theory

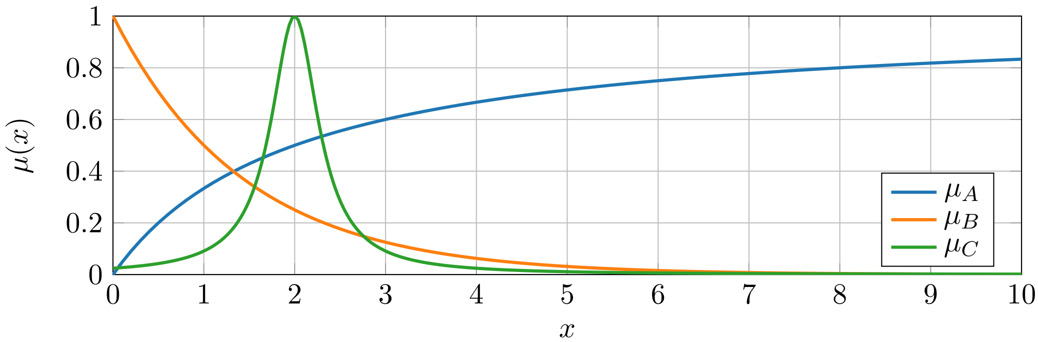
A fuzzy set allows objects to have gradual memberships. The degree of membership of a fuzzy set A is described by its fuzzy membership function $\mu_A(x)$ with $x \in \mathbb{R}$. Literature which focus on the theory of fuzzy sets but also on applications are [1, 2, 3, 4]. Klir's and Yuan's book [1], Ross' book [2], and Zadeh's paper [4] are freely available online.

In this exercise we focus on characteristics of fuzzy sets regarding convexity, α-cuts, and general fusion rules. You need to have worked through the lectures up to and including L_IFU_33 to be able to comprehend this exercise.

For this exercise take the fuzzy sets A, B, and C as examples, which are defined on the real-numbered interval $X = [0, 10]$ by their membership functions

$$\mu_A(x) = \frac{x}{x+2},$$
$$\mu_B(x) = 2^{-x}, \text{ and}$$
$$\mu_C(x) = \frac{1}{1 + 10 \cdot (x-2)^2}$$

with $x \in X$.



5.1 α-cuts

An important concept of fuzzy sets are α-cuts.

1. Describe the concept of α-cuts in your own words. Why are they important?
2. In the lecture it has been stated that „the higher the certainty the lower the confidence“. Why is that the case? How does this statement relate to α-cuts?
3. Given the membership functions μ_A, μ_B, and μ_C, calculate the α-cuts A_α, B_α, and C_α for $\alpha \in \{0.2, 0.6, 1\}$.

5.2 Normality and Convexity

If a fuzzy membership function satisfies specific requirements, the fuzzy set is called normal or convex.

1. Formulate these requirements and interpret them. You may illustrate your ideas with drawings if necessary.
2. Determine whether the fuzzy sets described by $\mu_A(x)$, $\mu_B(x)$, and $\mu_C(x)$ are normal or convex for $x \in X$.

5.3 Fusion in Fuzzy Set Theory

A fusion or aggregation operator $h : [0, 1]^n \to [0, 1]$ on n fuzzy sets needs to be *bounded*, *monotonic*, and *continuous* [1]. They are usually expected to be *symmetric* and *idempotent* as well.

1. What does it mean for a function to be symmetric and idempotent?
2. A popular fuzzy aggregator is the *ordered weighted averaging (OWA) operation.*

   Let $\mathbf{w} = \{w_1, w_2, \ldots, w_n\}$ be a weighting vector such that $w_i \in [0, 1]$ and $\sum_{i=1}^{n} w_i = 1$. Then the OWA operation is defined by the function

   $$h_{\mathbf{w}}(\mu_1, \mu_2, \ldots, \mu_n) = w_1 b_1 + w_2 b_2 + \ldots + w_n b_n$$

   in which b_i is the ith largest element in $\{\mu_1, \mu_2, \ldots, \mu_n\}$.

   Show that the OWA operation satisfies the axioms of symmetry and idempotency.
3. Another prominent aggregation operator is the *generalised means* operator:

   $$h_p(\mu_1, \mu_2, \ldots, \mu_n) = \left(\frac{\mu_1^p + \mu_2^p + \ldots + \mu_n^p}{n} \right)^{\frac{1}{p}}.$$

   Proof that for $p \to \infty$ the generalised means operator converts into the maximum operator, i. e., $h_\infty(\mu_1, \mu_2, \ldots, \mu_n) = \max(\mu_1, \mu_2, \ldots, \mu_n)$.

References

[1] George J. Klir and Bo Yuan. *Fuzzy Sets and Fuzzy Logic: Theory and Applications.* Prentice Hall PTR, 1995.

[2] Timothy J. Ross. *Fuzzy Logic with Engineering Applications.* John Wiley & Sons, 3rd edition, 2010.

[3] Simona Salicone. *Measurement Uncertainty: An Approach via the Mathematical Theory of Evidence.* Springer Series in Reliability Engineering. Springer, New York, NY and Berlin and Heidelberg, 2007.

[4] Lotfi A. Zadeh. Fuzzy sets. *Information and Control*, 8(3):338–353, 1965.