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Excercise 6: Possibility Theory

Possibility Theory is a mathematical theory which is closely related to Fuzzy Set Theory but also to Probability Theory. It is especially well suited to cope with imprecision and absence of information.

The aims of this exercise are (i) to get accustomed to possibility and necessity measures, (ii) to construct possibility distributions from evidential masses, and (iii) to get an intuition of the meaning of possibility distributions. This exercise require you to have read and understood the lecture L_IFU_34.

6.1 Possibility and Necessity Measures

Possibility Theory provides a pair of dual set-based measures: *Possibility* $\Pi(A)$ and *Necessity* $N(A)$ (analogous to the Probability $P(A)$ in Probability Theory). They are calculated as follows:

$$\Pi(A) = \max_{\theta \in A}(\pi_A(x) \mid x = \theta), \qquad N(A) = \min_{\theta \notin A}(1 - \pi_A(x) \mid x = \theta).$$

Take a look at the example 3.4.5 „distance measurement“ from lecture L_IFU_34 (slides 36-39). Four sensors measure the distance between two objects. Each sensor i outputs an interval A_i together with an evidential mass m_i. The A_i are nested subsets. In the lecture, the evidential masses are used to construct a possibility distribution $\pi(x)$ which is illustrated in Fig. 1.

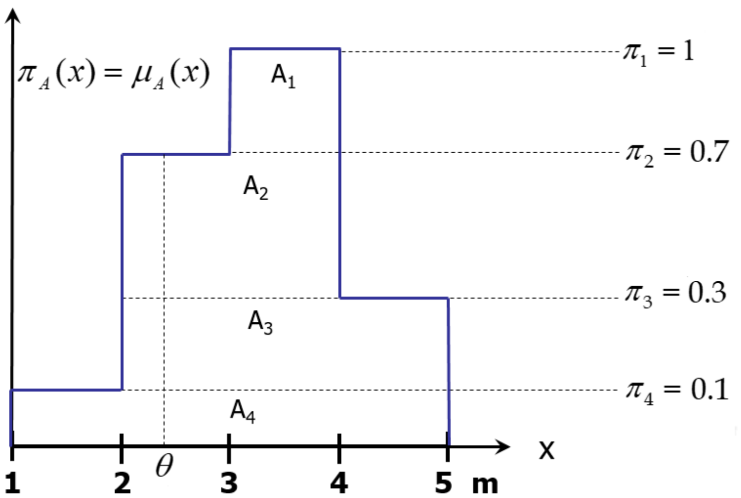

Figure 1: Example 3.4.5 from lecture L_IFU_34. The example shows a possibility distribution based on four intervals each obtained by a sensor.

1. Determine the possibility measures $\Pi(A_i)$ and necessity measures $N(A_i)$ for all $i \in \{1, 2, 3, 4\}$.
2. Determine $\Pi(A_i{}^{\mathrm{c}})$ and $N(A_i{}^{\mathrm{c}})$.

6.2 From Belief Assignments to Possibility Distributions

If propositions form nested sets, then a possibility distribution can be constructed based on the basic beliefs (evidential masses). Assume for this task the following sets of measurement ranges obtained from five sensors:

- $A_1 = [370, 375]$mm,
- $A_2 = [370, 380]$mm,
- $A_3 = [345, 390]$mm,
- $A_4 = [340, 400]$mm, and
- $A_5 = [320, 400]$mm.

Several experts (a, b, c) assign basic belief distributions. The experts assign for each measurement interval an evidential mass as follows:

	A_1	A_2	A_3	A_4	A_5
a	0.2	0.2	0.4	0.1	0.1
b	0	0	0	0.5	0.5
c	0.6	0.3	0.1	0	0

1. Determine (and draw) the possibility distributions $\pi_A^{(a)}, \pi_A^{(b)}, \pi_A^{(c)}$ from the given basic belief distributions.
2. Which of the possibility distributions is the most specific one? Which is the least specific one?
3. The concept of specificity is closely related to ignorance. Describe in your own words or illustrate in a drawing the concepts of *partial ignorance*, *total ignorance*, and *complete knowledge* within the Possibility Theory.

6.3 Interpretation - Possibility Theory vs. Fuzzy Set Theory

The Possibility Theory (PosT) was introduced by Zadeh [3] in 1978 motivated by the observation that Probability Theory (ProbT) handles epistemic uncertainty only insufficiently. Zadeh defines PosT as an extension of fuzzy sets [1].

Because PosT is directly based on Fuzzy Set Theory, fuzzy membership functions $\mu(x)$ can directly be used as possibility distributions $\pi(x)$. Consequently, mathematical operations defined on fuzzy memberships functions can be directly applied to possibility distributions. But, as hinted at in the last exercise, the interpretational meaning of $\mu(x)$ and $\pi(x)$ are different.

1. To stress the difference between the meaning of fuzzy membership functions and possibility distributions, Dubois and Prade introduced the notation

$$\pi_{\mathrm{A}}(x = \theta \mid \theta \in A) = \mu_{\mathrm{A}}(\theta \in A \mid x = \theta).$$

   The difference in meaning is quite subtle but distinctive. Are you able to describe this difference with the equation at hand in your own words?

2. In [2, p. 116 f.] Solaiman and Bossé discuss whether it makes sense semantically to apply fuzzy similarity measures to possibility distributions[1]. They provide an example with three experts estimating the distance between two objects. Their estimation is modelled fuzzily, e. g., the first expert says the distance is „close to 10 metres“. These fuzzy sets are used as possibility distributions; this can be seen in Fig. 2.

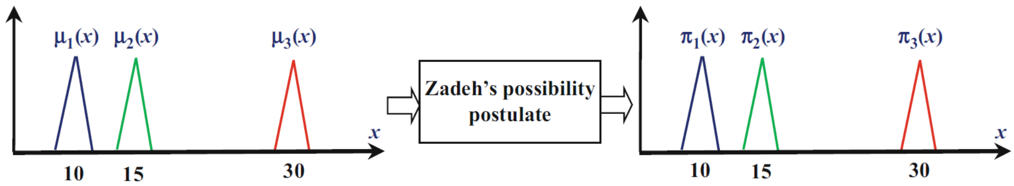

Figure 2: Example given in [2, p. 117]: Three experts estimate the distance of two objects.

Read the example in the provided book ([2, p. 116 f.]). Explain why it may not always be a good idea to apply fuzzy operations to possibility distribution.

6.4 Interpretation - Possibility Theory vs. Probability Theory

Possibility distributions can also be directly compared to probability density functions.

1. Zadeh [3] describes Possibility Theory as analogous yet different to Probability Theory. Come up with a real-world example for which it is reasonable to model it with a possibility distribution (Zadeh's paper [3] may be a good place to look for ideas). Draw a possibility distribution $\pi(x)$ for your example and label it correctly (e. g., abscissa, ordinate, ...). Model the same example with a probability density function (again: draw $p(x)$).
2. What are – in your opinion – the strengths and weaknesses of possibility distributions in comparison to probability density functions?

References

[1] Thierry Denœux, Didier Dubois, and Henri Prade. Representations of uncertainty in artificial intelligence: Probability and possibility. In Pierre Marquis, Odile Papini, and Henri Prade, editors, *A Guided Tour of Artificial Intelligence Research: Volume I: Knowledge Representation, Reasoning and Learning*, pages 69–117. Springer International Publishing, 2020.

[2] B. Solaiman and É. Bossé. *Possibility Theory for the Design of Information Fusion Systems*. Information Fusion and Data Science. Springer International Publishing, 2019.

[3] Lotfi Aliasker Zadeh. Fuzzy sets as a basis for a theory of possibility. *Fuzzy Sets and Systems*, 1:3–28, 1978.

[1] In this exercise we are not interested in similarity measurements per se, but this discussion shows that the semantic difference in meaning of $\mu(x)$ and $\pi(x)$ is actually important.