Fascicle 6 Satisfiability, exercises 297-299

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Chapter 1

4.2.2.2 Satisfiability

1.1 Exercise 297 item a

Find a closed form for $G_q(z) = \sum_p C_{p,p+q-1}(z/3)^{p+q}(2z/3)^p$, using formulas from Section 7.2.1.6.

Proof. Let's take $(z/3)^q$ outside:

$$\begin{aligned}
\sum_p C_{p,p+q-1}\left(\frac{z}{3}\right)^{p+q}\left(\frac{2z}{3}\right)^p &= \left(\frac{z}{3}\right)^q \sum_p C_{p,p+q-1}\left(\frac{z}{3}\right)^p\left(\frac{2z}{3}\right)^p \\
&= \left(\frac{z}{3}\right)^q \sum_p C_{p,p+q-1}\left(\frac{2z^2}{9}\right)^p
\end{aligned}$$

If we make $y = 2z^2/9$, then we can use eq. 24 from the book:

$$C_{pq} = [z^p]C(z)^{q-p+1} \implies \sum_p C_{pq}z^p = C(z)^{q-p+1}$$

Which in our case is:

$$\begin{aligned}
\left(\frac{z}{3}\right)^q \sum_p C_{p,p+q-1}\left(\frac{2z^2}{9}\right)^p &= \left(\frac{z}{3}\right)^q C\left(\frac{2z^2}{9}\right)^{(p+q-1)-p+1} \\
&= \left(\frac{z}{3}\right)^q C\left(\frac{2z^2}{9}\right)^q
\end{aligned}$$

Let's check eq. 18 from the book:

$$C(z) = \frac{1-\sqrt{1-4z}}{2z}$$

In our case:

$$
\begin{aligned}
\left(\frac{z}{3}\right)^q C\left(\frac{2z^2}{9}\right)^q &= \left(\frac{z}{3}\right)^q \left(\frac{1-\sqrt{1-4(2z^2/9)}}{2(2z^2/9)}\right)^q \\
&= \left(\frac{3}{4z}\right)^q \left(1-\sqrt{1-\frac{8z^2}{9}}\right)^q \\
&= \left(\frac{3-\sqrt{9-8z^2}}{4z}\right)^q
\end{aligned}
$$

□

1.2 Exercise 297 item b

Explain why $G_q(1)$ is less than 1.

Proof.

$$G_q(1) = \left(\frac{3-\sqrt{9-8}}{4}\right)^q = 2^{-q}$$

This is indeed less than 1 for all $q > 0$. Let's interpret this result. q is a measure of how "distant" the initial configuration is from the solution. $t = 2p + q$ is the time taken to reach a solution when the path rises by p.

$C_{p,p+q-1}$ is the number of possible paths from q to the solution of size t. Each of these paths has a chance of $(1/3)^{p+q}(2/3)^q$ to happen. Therefore, if you sum over all possible p, then it's the same as summing over all possible t, and the meaning of $G_q(1)$ is the chance of reaching the solution when starting from q, with any possible time. Since it is a chance, it must be smaller than 1.

□

1.3 Exercise 297 item c

Evaluate and interpret the quantity $G'_q(1)/G_q(1)$.

Proof. Let's use Bayes' Rule:

$$P(\text{finishes at}\, t \mid \text{finishes at all}) = \frac{P(\text{finishes at all} \mid \text{finishes at}\, t)P(\text{finishes at}\, t)}{P(\text{finishes at all})}$$

P(finishes at all | finishes at t) is clearly 1. P(finishes at t) is $[z^t]G_q(z)$. P(finishes at all) is $G_q(1)$.

Therefore:

$$\frac{G_q(z)}{G_q(1)} = \sum P(\text{finishes at } t \mid \text{finishes at all}) z^t$$
$$\frac{G'_q(z)}{G_q(1)} = \sum tP(\text{finishes at } t \mid \text{finishes at all}) z^{t-1}$$
$$\frac{G'_q(1)}{G_q(1)} = E(t) \mid \text{finishes at all}$$

The interpretation for $G'_q(1)/G_q(1)$ is the expected time to reach the solution, given it has finished at all.

After long calculation. $G'_q(z)$ turns out to be:

$$G'_q(z) = \frac{3q}{z\sqrt{9-8z^2}} \left(\frac{3-\sqrt{9-8z^2}}{4z} \right)^q$$

And therefore $G'_q(1)/G_q(1) = 3q$.

We can also calculate the variance as:

$$\left(\frac{G''_q(1) + G'_q(1)}{G_q(1)} \right) - \left(\frac{G'_q(1)}{G_q(1)} \right)^2 = 24q$$

(This last calculation made by Mathematica).

□

1.4 Exercise 297 item d

Use Markov's inequality to bound the probability that $Y_t = 0$ for some $t \leq N$.

Proof. The probability that a given state is distant q bits from the solution is the number of n-sized binary words with exactly q bits set (this is $\binom{n}{q}$), divided by the total number of n-sized binary words, which is 2^n. The generating function for the probability of the algorithm ending when running with n variables is:

$$T(z) = \sum_q \binom{n}{q} 2^{-n} G_q(z)$$

Using the same argument as the last exercises, the expected path length to the solution, given that it terminates, is $E[t] = T'(1)/T(1)$:

$$
\begin{aligned}
T(1) &= \sum_q \binom{n}{q} 2^{-n} G_q(1) \\
&= 2^{-n} \sum_q \binom{n}{q} 2^{-q} \\
&= 2^{-n} \sum_q \binom{n}{q} \left(\frac{1}{2}\right)^q (1)^{n-q} \\
&= 2^{-n} \left(\frac{1}{2} + 1\right)^n \\
&= \left(\frac{3}{4}\right)^n
\end{aligned}
$$

$$
\begin{aligned}
T'(1) &= \sum_q \binom{n}{q} 2^{-n} G'_q(1) \\
&= 2^{-n} \sum_q \binom{n}{q} 3q 2^{-q} \\
&= 3 \times 2^{-n} \sum_q q \binom{n}{q} \left(\frac{1}{2}\right)^q \\
&= 3 \times 2^{-n} \sum_q n \binom{n-1}{q-1} \left(\frac{1}{2}\right)^q \\
&= 3n2^{-n} \sum_{q \geq 0} \binom{n-1}{q-1} \left(\frac{1}{2}\right)^q \\
&= \frac{3n2^{-n}}{2} \sum_{q \geq 0} \binom{n-1}{q-1} \left(\frac{1}{2}\right)^{q-1} \\
&= \frac{3n2^{-n}}{2} \left(\binom{n-1}{-1} \left(\frac{1}{2}\right)^{-1} + \sum_{q-1 \geq 0} \binom{n-1}{q-1} \left(\frac{1}{2}\right)^{q-1} \right) \\
&= \frac{3}{2} n 2^{-n} \left(\frac{3}{2}\right)^{n-1} \\
&= n2^{-n} \left(\frac{3}{2}\right)^n \\
&= n \left(\frac{3}{4}\right)^n
\end{aligned}
$$

$$\frac{T'(1)}{T(1)} = \frac{n\left(\frac{3}{4}\right)^n}{\left(\frac{3}{4}\right)^n} = n$$

Now we can use Markov's inequality:

$$P(X \geq a) \leq \frac{E[X]}{a}$$

Which in our case is:

$$P(t \geq N|\text{given the algorith terminates}) \leq \frac{n}{N}$$

□

1.5 Exercise 297 item e

Show that Corollary W follows from this analysis.

Proof. Let's define p as the probability that the algorithm succeeds within N steps. Then:

$$\begin{aligned}
P(\text{succeeds within N}) &= P(\text{succeeds within N } | \text{ succeeds at all})P(\text{succeeds at all}) \\
&= (1 - P(X \geq N|\text{succeeds at all}))\left(\frac{3}{4}\right)^n \\
&\geq \left(1 - \frac{n}{N}\right)\left(\frac{3}{4}\right)^n
\end{aligned}$$

The probability it succeeds in Q trials is:

$$\begin{aligned}
P(\text{succeeds in Q trials}) &= p + (1-p)p + (1-p)^2 p + \cdots + (1-p)^{Q-1}p \\
&= p\frac{1-(1-p)^Q}{1-(1-p)} \\
&= 1 - (1-p)^Q \\
&= 1 - \exp\left(Q \log\left(1-p\right)\right) \\
&= 1 - \exp\left(-Q \sum p + \frac{p^2}{2} + \frac{p^3}{3} + \cdots\right) \\
&\geq 1 - \exp\left(-Qp\right)
\end{aligned}$$

For $Q = K(4/3)^n$ and p given above, we have:

$$\begin{aligned}P(\text{succeeds in Q trials}) &\geq 1 - \exp\left(-K\left(\frac{4}{3}\right)^n\left(1-\frac{n}{N}\right)\left(\frac{3}{4}\right)^n\right)\\ &\geq 1-\exp\left(-K\left(1-\frac{n}{N}\right)\right)\end{aligned}$$

Now we choose $N = 2n$ to conclude:

$$\begin{aligned}P(\text{succeeds in Q trials}) &\geq 1 - \exp\left(-K\left(1-\frac{n}{2n}\right)\right)\\ &\geq 1-\exp\left(-\frac{K}{2}\right)\end{aligned}$$

□

1.6 Exercise 298

Generalize Theorem U and Corollary W to the case where each clause has at most k clauses, where $k \leq 3$.

Proof. We'll follow the steps of the previous exercise, but swapping the probabilities $1/3$ and $2/3$ to $1/k$ and $(k-1)/k$. First, $G_q(z)$ becomes:

$$\begin{aligned}G_q(z) &= \sum_p C_{p,p+q-1}\left(\frac{z}{k}\right)^{p+q}\left(\frac{z(k-1)}{k}\right)^p\\ &= \left(\frac{z}{k}\right)^q\sum_p C_{p,p+q-1}\left(\frac{z^2(k-1)}{k^2}\right)^p\\ &= \left(\frac{z}{k}\right)^q C\left(\frac{z^2(k-1)}{k^2}\right)^q\\ &= \left(\frac{z}{k}\left(\frac{1-\sqrt{1-4\frac{z^2(k-1)}{k^2}}}{2\frac{z^2(k-1)}{k^2}}\right)\right)^q\\ &= \left(\frac{1}{2z(k-1)}\left(k-\sqrt{k^2-4z^2(k-1)}\right)\right)^q\end{aligned}$$

$$\begin{aligned}G_q(1) &= \left(\frac{1}{2(k-1)}\left(k-\sqrt{k^2-4k+4}\right)\right)^q\\ &= \left(\frac{1}{2(k-1)}\left(k-(k-2)\right)\right)^q\\ &= \left(\frac{1}{k-1}\right)^q\end{aligned}$$

For the derivative, let's write $G_q(z)$ as $(G(z))^q$:

$$
\begin{aligned}
G(z) &= \frac{1}{2z(k-1)}\left(k-\sqrt{k^2-4z^2(k-1)}\right)\\
G'(z) &= \frac{k}{2(k-1)z^2}\left(-1+\frac{k}{\sqrt{k^2-4(k-1)z^2}}\right)\\
G'(1) &= \frac{k}{2(k-1)}\left(-1+\frac{k}{k-2}\right)\\
&= \frac{k}{(k-1)(k-2)}\\
G_q'(z) &= qG(z)^{q-1}G'(z)\\
G_q'(1) &= q\left(\frac{1}{k-1}\right)^{q-1}\frac{k}{(k-1)(k-2)}\\
&= \left(\frac{1}{k-1}\right)^{q}\frac{qk}{k-2}
\end{aligned}
$$

Let's write $T(z)$ in terms of $G(z)$:

$$
\begin{aligned}
T(z) &= \sum_q \binom{n}{q} 2^{-n}\left(G(z)\right)^q\\
&= 2^{-n}\left(1+G(z)\right)^n\\
T'(z) &= n2^{-n}\left(1+G(z)\right)^{n-1}G'(z)\\
\frac{T'(z)}{T(z)} &= \frac{n2^{-n}\left(1+G(z)\right)^{n-1}G'(z)}{2^{-n}\left(1+G(x)\right)^n}\\
&= \frac{nG'(z)}{1+G(z)}\\
\frac{T'(1)}{T(1)} &= \frac{\frac{nk}{(k-1)(k-2)}}{1+\frac{1}{k-1}} = \frac{nk}{(k-1)(k-2)}\frac{k-1}{k} = \frac{n}{k-2}
\end{aligned}
$$

The probability of success within N steps is:

$$
\begin{aligned}
p &\geq \left(1-\frac{T'(1)/T(1)}{N}\right)T(1)\\
&\geq \left(1-\frac{n}{N(k-2)}\right)\left(2^{-n}\left(1+\frac{1}{k-1}\right)^n\right)\\
&\geq \left(1-\frac{n}{N(k-2)}\right)\left(\frac{k}{2(k-1)}\right)^n
\end{aligned}
$$

We know that the probability of success with Q trials is $P \geq 1-exp(-Qp)$, so we want to choose Q and N such that $Qp = K/2$. For example, $Q = K(2-2/k)^n$ and $N = 2n/(k-2)$:

$$\begin{aligned} Qp &= K\left(2-\frac{2}{k}\right)^n\left(1-\frac{1}{2}\right)\left(\frac{k}{2(k-1)}\right)^n \\ &= \frac{K}{2} \end{aligned}$$

Both Q and N need to be integers. For Q, you can always choose a suitable K to make the quantity an integer, but that's not possible for N since n and k are given. Therefore, you must round down N to:

$$N = \left\lfloor \frac{2n}{k-2} \right\rfloor$$

□

1.7 Exercise 299

Continuing the previous exercise, investigate the case $k = 2$.

Proof. The approach used in previous exercises doesn't work here, because $G'(1)$ will have a division by zero. We'll resort to the original probabilities. If we choose $N = n^2$, then the chance of failure is:

$$\begin{aligned} P(X > n^2) &= \sum_{p,q} f(p,q)[2p+q > n^2] \\ &= \sum_{p,q} \frac{1}{2^n}\binom{n}{q}\frac{q}{2p+q}\binom{2p+q}{p}\left(\frac{1}{2}\right)^{p+q}\left(\frac{1}{2}\right)^p[2p+q > n^2] \end{aligned}$$

Let's make $t = 2p + q$, then:

$$\begin{aligned} P(X > n^2) &= \sum_{p,t} \frac{1}{2^n}\binom{n}{t-2p}\frac{t-2p}{t}\binom{t}{p}\left(\frac{1}{2}\right)^t[t > n^2] \\ &= \frac{1}{2^n}\sum_{t>n^2}\frac{1}{t2^t}\sum_p (t-2p)\binom{n}{t-2p}\binom{t}{p} \end{aligned}$$

Now we can use the following bound, which is true because the middle element in any row of Pascal's triangle is always the greatest:

$$\binom{n}{p} \le \binom{n}{\lfloor n/2 \rfloor}$$

In our case:

$$
\begin{aligned}
P(X > n^2) &= \frac{1}{2^n} \sum_{t>n^2} \frac{1}{t2^t} \sum_p (t-2p) \binom{n}{t-2p} \binom{t}{p} \\
&\le \frac{1}{2^n} \sum_{t>n^2} \frac{1}{t2^t} \binom{t}{\lfloor t/2 \rfloor} \sum_p (t-2p) \binom{n}{t-2p} \\
&\le \frac{1}{2^n} \sum_{t>n^2} \frac{1}{t2^t} \binom{t}{\lfloor t/2 \rfloor} \sum_q (t-q) \binom{n}{t-q} [q \text{ even}] \\
&\le \frac{1}{2^n} \sum_{t>n^2} \frac{1}{t2^t} \binom{t}{\lfloor t/2 \rfloor} \sum_q (t-q) \binom{n}{t-q} \left(\frac{1}{2} \left(1 - (-1)^q\right) \right) \\
&\le \frac{1}{2^n} \sum_{t>n^2} \frac{1}{t2^t} \binom{t}{\lfloor t/2 \rfloor} \left(\sum_q \frac{t-q}{2} \binom{n}{t-q} + \sum_q \frac{t-q}{2} \binom{n}{t-q} (-1)^q \right)
\end{aligned}
$$

The second inner sum is zero, because it's a special case of eq. (5.42) from Concrete Mathematics. Namely, the sum of an alternating binomial times a polynomial is zero when the degree of the polynomial is less than n.

$$
\begin{aligned}
P(X > n^2) &\le \frac{1}{2^n} \sum_{t>n^2} \frac{1}{t2^t} \binom{t}{\lfloor t/2 \rfloor} \left(\frac{1}{2} \sum_q (t-q) \binom{n}{t-q} \right) \\
&\le \frac{1}{2^n} \sum_{t>n^2} \frac{1}{t2^t} \binom{t}{\lfloor t/2 \rfloor} \left(\frac{1}{2} \sum_k k \binom{n}{k} \right) \\
&\le \frac{1}{2^n} \sum_{t>n^2} \frac{1}{t2^t} \binom{t}{\lfloor t/2 \rfloor} \left(\frac{n2^{n-1}}{2} \right) \\
&\le \frac{n}{4} \sum_{t>n^2} \frac{1}{t2^t} \binom{t}{\lfloor t/2 \rfloor}
\end{aligned}
$$

From Stirling's approximation we have:

$$
\begin{aligned}
\binom{2n}{n} &\le \frac{4^n}{\sqrt{\pi n}} \\
\binom{t}{\lfloor t/2 \rfloor} &\le \frac{2^t}{\sqrt{\pi t/2}}
\end{aligned}
$$

Applied to our case:

$$
\begin{aligned}
P(X > n^2) &\leq \frac{n}{4} \sum_{t > n^2} \frac{1}{t2^t} \left(\frac{2^t \sqrt{2}}{\sqrt{\pi t}} \right) \\
&\leq \frac{n}{4} \sum_{t > n^2} \frac{1}{t2^t} \left(\frac{2^t \sqrt{2}}{\sqrt{\pi t}} \right) \\
&\leq \frac{n}{\sqrt{8\pi}} \sum_{t > n^2} \sqrt{\frac{1}{t^3}}
\end{aligned}
$$

Now we can use the following property:

$$
\sum_{x > a} f(x) = \int_a^\infty f(\lceil x \rceil) dx
$$

This is just a different way of writing the same thing. We are replacing a sum of values by a sum of rectangles with width 1 and height $f(x)$, both evaluate numerically to the same value.

$$
P(X > n^2) \leq \frac{n}{\sqrt{8\pi}} \int_{n^2}^\infty \frac{dx}{\lceil x \rceil^{3/2}}
$$

Since $x^{-3/2}$ is decreasing in this region, we can drop the ceiling:

$$
\int_{n^2}^\infty \frac{dx}{\lceil x \rceil^{3/2}} \leq \int_{n^2}^\infty \frac{dx}{x^{3/2}}
$$

Therefore:

$$
\begin{aligned}
P(X > n^2) &\leq \frac{n}{\sqrt{8\pi}} \int_{n^2}^\infty \frac{dx}{x^{3/2}} \\
&\leq \frac{n}{\sqrt{8\pi}} \left(-\frac{2}{\sqrt{x}} \bigg|_{n^2}^\infty \right) \\
&\leq \frac{1}{\sqrt{2\pi}}
\end{aligned}
$$

□