Fascicle 6 Satisfiability

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Chapter 1

4.2.2.2 Satisfiability

1.1 MPR-105

(Random walk on an n-cycle.) Given integers a and n, with $0 \leq a \leq n$, let N be minimum such that $(a + (-1)^{X_1} + (-1)^{X_2} + \cdots + (-1)^{X_N}) \mod n = 0$, where $X_1, X_2, \ldots$ is a sequence of independent random bits. Find the generating function $g_a = \sum_{k=0}^{\infty} \Pr(N = k) z^k$. What are the mean and variance of N?

Proof. If $a = 0$ or $a = n$, then the sum is already zero and we need no bits at all. In those cases, $N = 0$ and therefore $g_0 = 1$ and $g_n = 1$. If a is between 0 and n, then one extra random bit added will reduce the problem to either $a + 1$ or $a - 1$ with equal probability. Therefore, $g_a = z\,(g_{a-1}/2 + g_{a+1}/2)$. Also, if N is a solution for a, then N is also a solution for $n - a$ (just complement the X_i bits). Therefore, $g_a = g_{n-a}$.

Let's break the problem in two cases. First suppose $N = 2m$. In this case $g_{m-1} = g_{2m-(m-1)} = g_{m+1}$. We also have $g_m = z/2(g_{m+1} + g_{m-1}) = zg_{m-1}$. Now let's define $G_a = g_{m-a}/g_m$. From this definition:

$$
\begin{aligned}
G_0 &= \frac{g_m}{g_m} = 1 \\
G_1 &= \frac{g_{m-1}}{g_m} = \frac{g_m}{z} \times \frac{1}{g_m} = \frac{1}{z} \\
G_a &= \frac{g_{m-a}}{g_m} \\
&= \frac{1}{g_m} \times \left(\frac{2g_{m-(a-1)}}{z} - g_{m-(a-2)} \right) \\
&= \frac{2}{z} G_{a-1} - G_{a-2}
\end{aligned}
$$

If we set $x = 1/z$, then it's clear that $G_a(x)$ has exactly the same definition of $T_a(x)$, the Chebyshev polynomials of the first kind. We can find out a closed form for g_a if we set $a = m$ and remember that $g_0 = 1$:

$$
\begin{aligned}
G_a(1/z) &= T_a(1/z) \\
\frac{g_{m-a}}{g_m} &= T_a(1/z) \\
\frac{g_{m-m}}{g_m} &= T_m(1/z) \\
g_m &= \frac{1}{T_m(1/z)} \\
g_{m-a} &= \frac{T_a(1/z)}{T_m(1/z)} \\
g_a &= \frac{T_{m-a}(1/z)}{T_m(1/z)}
\end{aligned}
$$

The mean of N is given by $g'_a(1)$. We start with the generating function for Chebyshev polynomials. We also define $TZ(z) = T(1/z)$ for simplicity.

$$
\begin{aligned}
T(w) &= \sum_{n\geq 0} T_n(z)w^n = \frac{1-wz}{1-2wz+w^2} \\
TZ(z) &= T(1/z) = \frac{z-w}{z-2w+zw^2} \\
TZ'(z) &= \frac{w(w^2-1)}{(z+w(wz-2))^2} \\
TZ'(1) &= \frac{w(w^2-1)}{(1+w(w-2))^2} = \frac{w(1+w)}{(w-1)^3} \\
E[T_n(1/z)] &= [w^n]TZ'(1) = -n^2
\end{aligned}
$$

Probability generating functions have the property that $E[F/G] = E[F] - E[G]$, therefore:

$$
\begin{aligned}
E[g_a] &= E[T_{m-a}(1/z)] - E[T_m(1/z)] = \\
&= (m-a)^2 - m^2 \\
&= 2ma - a^2 \\
&= a(n-a)
\end{aligned}
$$

The variance of N is given by $g''_a(1) + g'_a(1) - g'a(1)^2$:

$$\begin{aligned}
TZ''(z) &= \frac{-2w(w^4-1)}{(z+w(wz-2))^3} \\
TZ''(1) &= \frac{-2w(w^4-1)}{(1+w(w-2))^3} \\
T_n''(1) &= [w^n]TZ''(1) = \frac{n^2(5+n^2)}{3} \\
V[T_n(1/z)] &= \frac{n^2(5+n^2)}{3} - n^2 - n^4 = \frac{2(n^2-n^4)}{3} \\
V[g_a] &= V[T_{m-a}(1/z)] - V[T_m(1/z)] \\
&= \frac{2((m-a)^2-(m-a)^4-m^2+m^4)}{3} \\
&= \frac{a(n-a)(n^2-2a(n-a)-2)}{3}
\end{aligned}$$

Now the second case. Suppose $N = 2m-1$, in this case $g_{2m-1-m} = g_{m-1} = g_m$. Defining $G_a = g_{m-a}/g_m$ we have $G_0 = G_1 = 1$, which is not readily a Chebyshev polynomial. However, since the recursion is the same, we can rewrite it as a linear combination of Chebyshev polynomials:

$$\begin{aligned}
\begin{pmatrix} 1 & 1 \\ z & 2z \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} &= \begin{pmatrix} 1 \\ 1 \end{pmatrix} \\
\begin{pmatrix} a \\ b \end{pmatrix} &= \begin{pmatrix} 2-\frac{1}{z} \\ \frac{1}{z}-1 \end{pmatrix}
\end{aligned}$$

We conclude that $G_a = (2-1/z)T_a(z) + (1/z-1)U_a(z)$. This linear combination can be applied to the generating functions as well, since z is just a constant in relation to w:

$$
\begin{aligned}
G(w) &= \left(2 - \frac{1}{z}\right) \sum_{n\geq 0} T_n w^n + \left(\frac{1}{z} - 1\right) \sum_{n\geq 0} U_n w^n \\
&= \left(2 - \frac{1}{z}\right) \left(\frac{1 - wz}{1 - 2wz + w^2}\right) + \left(\frac{1}{z} - 1\right) \left(\frac{1}{1 - 2wz + w^2}\right) \\
&= \frac{1 + w - 2wz}{1 + w^2 - 2wz} \\
GZ(z) &= G(1/z) = \frac{w(z-2) + z}{z + w(wz-2)} \\
GZ'(z) &= \frac{2(w-1)w^2}{(z + w(wz-2))^2} \\
GZ'(1) &= \frac{2w^2}{(w-1)^3} \\
E[G_n] &= [w^n]GZ'(1) = n(1-n) \\
E[g_a] &= E[g_{m-a}] - E[g_m] \\
&= (m-a)(1-m+a) - m(1-m) \\
&= a(2m-1-a) \\
&= a(n-a)
\end{aligned}
$$

The mean of N turns out to have the same value as in the previous case. Now for the variance:

$$
\begin{aligned}
GZ''(z) &= \frac{4(w-1)w^2(1+w^2)}{(z + w(wz-2))^3} \\
GZ''(1) &= -\frac{4(w^2 + w^4)}{(w-1)^5} \\
G_n''(1) &= [w^n]GZ''(1) = \frac{n(n^3 - 2n^2 + 5n - 4)}{3} \\
V[G_n] &= \frac{n(n^3 - 2n^2 + 5n - 4)}{3} + n(1-n) - (n(1-n))^2 \\
&= -\frac{n(1 + n(1 + 2n(n-2)))}{3} \\
V[g_a] &= V[g_{m-a}] - V[g_a] \\
&= \frac{a(2m-1-a)(2a^2 - 1 + a(2-4m) + 4(m-1)m)}{3} \\
&= \frac{a(n-a)(n^2 - 2a(n-a) - 2)}{3}
\end{aligned}
$$

The variance also has the same value as the previous case.

□

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Consider the 2SAT problem with $n(n-1)$ clauses $(\bar{x}_j \vee x_k)$ for all $j \neq k$. Find the generating function for the number of flips taken by Algorithms P and W.

Proof. Suppose the vector x_i has exactly a bits one. In this case, for each bit one, there are $n-a$ clauses violated, one for each bit zero. The total of violated clauses is $a(n-a)$ and this value is zero only for $a=0$ and $a=n$, which are the two solutions for the problem.

Algorithm P will pick the first violated clause and flip a random literal on it. The violated clauses are always of the form $(\bar{x}_i \vee x_j)$ so it will change the state to $a+1$ or $a-1$ with equal probabilities. The generating function for the number of flips, thus, is exactly the same as studied in exercise MPR-105, whose mean is $g'_a(1) = a(n-a)$.

However, the a in this algorithm are not uniform, in fact there are $\binom{n}{a}$ vectors whose sum is a. The generating function for the entire algorithm is $g(z) = 2^{-n} \sum_{0 \le a \le n} \binom{n}{a} g_a$ and the mean is:

$$
\begin{aligned}
g'(1) &= 2^{-n} \sum_{0 \le a \le n} a(n-a)\binom{n}{a} \\
&= 2^{-n} \sum_{0 \le a \le n} an\binom{n-1}{a} \\
&= n2^{-n} \sum_{0 \le a \le n} (n-1)\binom{n-2}{a-1} \\
&= n(n-1)2^{-n} \sum_{0 \le a \le n-2} \binom{n-2}{a} \\
&= n(n-1)2^{-n}(2^{n-2}) \\
&= \frac{n(n-1)}{4} = \frac{1}{2}\binom{n}{2}
\end{aligned}
$$

For Algorithm W, notice that the cost for all bits one is the same, $a-1$, and the cost for every bit zero is $n-a-1$. For the analysis we can assume $a \le n/2$, since the problem is symmetric (just flip all bits). In this case, the greedy choice is always to flip a bit one to zero.

The greedy choice is only taken with certainty when $a=1$, since the cost is zero, in this case the generating function is $g_1 = z$. For all other cases, let p' be the greedy parameter of Algorithm W, we'll have the greedy choice taken with probability $1-p'$, and the random choice with probability p', in which case it goes to zero or to one with equal probability. Let $p = p'/2$ and $q = 1-p$, the generating function in those cases will be $g_a = z(qg_{a-1} + pg_{a+1})$.

Let's break into some cases now. When $p' = 0$, all choices are greedy, and we have $g_a = z^a$ for $a \le n/2$, and $g_a = z^{n-a}$ otherwise. We can remove the conditional by writing $g_a = z^{n/2-|a-n/2|}$. The overall generating function is $g = \sum_a 2^{-n}\binom{n}{a} z^{n/2-|a-n/2|}$ and the mean is:

$$\begin{aligned}
g'(1) &= \sum_a 2^{-n}\binom{n}{a}(n/2-|a-n/2|) \\
&= n2^{-1-n}\sum_a \binom{n}{a} - 2^{-n}\sum_a\binom{n}{a}|a-n/2| \\
&= n2^{-1-n}2^n - 2^{-n}\sum_a\binom{n}{a}|a-n/2| \\
&= \frac{n}{2} - 2^{-n}\sum_a\binom{n}{a}|a-n/2|
\end{aligned}$$

The remaining summation can be solved with the following identity from 1.2.7-68, using its $p = 1/2$:

$$\begin{aligned}
\sum_k \binom{n}{k} p^k (1-p)^{n-k}|k-np| &= 2\lceil np\rceil \binom{n}{\lceil np\rceil} p^{\lceil np\rceil}(1-p)^{n+1-\lceil np\rceil} \\
\sum_a \binom{n}{a} 2^{-n}\left|a-\frac{n}{2}\right| &= 2\left\lceil\frac{n}{2}\right\rceil\binom{n}{\lceil\frac{n}{2}\rceil}2^{-n-1} \\
2^{-n}\sum_a\binom{n}{a}\left|a-\frac{n}{2}\right| &= 2^{-n}\left\lceil\frac{n}{2}\right\rceil\binom{n}{\lceil\frac{n}{2}\rceil}
\end{aligned}$$

Substituting back in $g'(1)$:

$$g'(1) = \frac{n}{2} - 2^{-n}\left\lceil\frac{n}{2}\right\rceil\binom{n}{\lceil\frac{n}{2}\rceil}$$

The binomial is either $\binom{n}{n/2}$ or $\binom{n}{n/2+1}$. Since $\binom{n}{n/2+1} = \binom{n}{n/2}(1+2/n)$, then $\binom{n}{\lceil n/2\rceil} = \binom{n}{n/2}(1+O(1/n))$. We can expand an asymptotic now:

$$\begin{aligned}
g'(1) &= \frac{n}{2} - 2^{-n}\left\lceil\frac{n}{2}\right\rceil\binom{n}{\lceil\frac{n}{2}\rceil}\\
&= \frac{n}{2} - 2^{-n}\left(\frac{n}{2}+O(1)\right)\binom{n}{n/2}(1+O(1/n))\\
&= \frac{n}{2} - 2^{-n}\left(\frac{n}{2}+O(1)\right)\frac{2^n}{\sqrt{n\pi/2}}(1+O(1/n))(1+O(1/n))\\
&= \frac{n}{2} - \sqrt{\frac{2}{n\pi}}\left(\frac{n}{2}+O(1)\right)(1+O(1/n))\\
&= \frac{n}{2} - \sqrt{\frac{2}{n\pi}}\left(\frac{n}{2}+O(1)\right)\\
&= \frac{n}{2} - \sqrt{\frac{n}{2\pi}} + O\left(\frac{1}{\sqrt{n}}\right)\\
&= \frac{n}{2} - \sqrt{\frac{n}{2\pi}} + O(1)
\end{aligned}$$

When $p' = 1$, then $p = q = 1/2$, and we have almost the same case as Algorithm P, except that for $a = 1$ and $a = n - 1$ the choice is always greedy. We can use the same formulas if we use $n - 2$ as the length of the vector. The mean becomes:

$$\begin{aligned}
g'(1) &= 2^{-n}\sum_{1\le a\le n-1}\binom{n}{a}(1+(a-1)((n-2)-(a-1)))\\
&= 2^{-n}\sum_{1\le a\le n-1}\binom{n}{a}(2-a^2+n(a-1))\\
&= \frac{2n-4}{2^n} + 2^{-n}\sum_{0\le a\le n}\binom{n}{a}(2-a^2+n(a-1))\\
&= \frac{2n-4}{2^n} + 2 + 2^{-n}\sum_{0\le a\le n}\binom{n}{a}(-a^2+n(a-1))\\
&= \frac{2n-4}{2^n} + 2 - n + 2^{-n}\sum_{0\le a\le n}\binom{n}{a}(-a^2+na)
\end{aligned}$$

We now need two simple summations:

$$
\begin{aligned}
\sum_{0\le a\le n} a\binom{n}{a} &= \sum_a n\binom{n-1}{a-1} = n\sum_a \binom{n-1}{a} = n2^{n-1} \\
\sum_{0\le a\le n} a^2\binom{n}{a} &= \sum_a an\binom{n-1}{a-1} \\
&= \sum_a (a+1)n\binom{n-1}{a} \\
&= \sum_a an\binom{n-1}{a} + \sum_a n\binom{n-1}{a} \\
&= \sum_a n(n-1)\binom{n-2}{a-1} + (n2^{n-1}) \\
&= n(n-1)2^{n-2} + n2^{n-1} \\
&= n(n+1)2^{n-2}
\end{aligned}
$$

The total mean is:

$$
\begin{aligned}
g'(1) &= \frac{2n-4}{2^n} + 2 - n + 2^{-n}\left(-n(n+1)2^{n-2} + n^2 2^{n-1}\right) \\
&= \frac{2n-4}{2^n} + 2 - n + \frac{n(n-1)}{4} \\
&= \frac{2n-4}{2^n} + 2 + \frac{n(n-5)}{4}
\end{aligned}
$$

Let's consider now the case $0 \le p' \le 1$. Again we can consider the size to be $n-2$ to account for the forcibly greedy choice when $a = 0$ or $a = n$. The same reasoning used in MPR-105 also applies here. We have $g_0 = 1$, $g_{n-a} = g_a$, $g_a = z(qg_{a-1} + pg_{a+1})$.

We need to break into two subcases. When $n = 2m$, this implies $g_{m-1} = g_{m+1}$ and $zg_{m-1} = g_m$. Again we define $G_a = g_{m-a}/g_m$, which implies $G_0 = 1$ and $G_1 = 1/z$, and apply this definition to the recursion:

$$
\begin{aligned}
g_a &= z(qg_{a-1} + pg_{a+1}) \\
g_{m-a} &= z(qg_{m-a-1} + pg_{m-a+1}) \\
g_{m-a} &= z(qg_{m-(a+1)} + pg_{m-(a-1)}) \\
G_a &= z(qG_{a+1} + pG_{a-1}) \\
G_{a-1} &= z(qG_a + pG_{a-2}) \\
zqG_a &= G_{a-1} - zpG_{a-2} \\
G_a &= \frac{1}{zq}G_{a-1} - \frac{p}{q}G_{a-2}
\end{aligned}
$$

We can add the initial conditions to this recursion and find the generating function. For $a = 0$:

$$\begin{aligned} G_0 &= 1 \\ 1 &= \frac{1}{zq} \times 0 - \frac{p}{q} \times 0 + k_0[a = 0] \\ k_0 &= 1 \end{aligned}$$

For $a = 1$:

$$\begin{aligned} G_1 &= \frac{1}{z} \\ \frac{1}{z} &= \frac{1}{zq} \times 1 - \frac{p}{q} \times 0 + k_1[a = 1] \\ k_1 &= \frac{1}{z} - \frac{1}{zq} = \frac{1-q}{zq} = -\frac{p}{zq} \end{aligned}$$

In the general case:

$$\begin{aligned} G_a &= \frac{1}{zq} G_{a-1} - \frac{p}{q} G_{a-2} - \frac{p}{zq}[a = 1] + [a = 0] \\ G &= \frac{w}{zq} G - \frac{pw^2}{q} G - \frac{pw}{zq} + 1 \\ G\left(\frac{zq - w + pzw^2}{zq}\right) &= \frac{zq - pw}{zq} \\ G &= \frac{zq - pw}{zq - w + pzw^2} \end{aligned}$$

We can find the mean for G now:

$$\begin{aligned} \left.\frac{dG}{dz}\right|_{z=1} &= \frac{w(p^2w^2 - q^2)}{(q + pw^2 - w)^2} \\ &= \frac{w(pw - q)(pw + q)}{(w-1)^2(pw - q)^2} \\ &= \frac{w(pw + q)}{(w-1)^2(pw - q)} \end{aligned}$$

This generating function can be opened in parts:

$$
\begin{aligned}
\frac{w(pw+q)}{pw-q} &= \frac{\frac{p}{q}w^2+w}{\frac{p}{q}w-1} \\
&= -\frac{p}{q}w^2\frac{1}{1-\frac{p}{q}w} - w\frac{1}{1-\frac{p}{q}w} \\
&= -\frac{p}{q}w^2\left(\sum_{n\geq 0}\left(\frac{p}{q}\right)^n w^n\right) - w\left(\sum_{n\geq 0}\left(\frac{p}{q}\right)^n w^n\right) \\
&= -\sum_{n\geq 0}\left(\frac{p}{q}\right)^{n+1} w^{n+2} - \sum_{n\geq 0}\left(\frac{p}{q}\right)^{n} w^{n+1} \\
&= -\sum_{n\geq 2}\left(\frac{p}{q}\right)^{n-1} w^{n} - \sum_{n\geq 1}\left(\frac{p}{q}\right)^{n-1} w^{n} \\
&= -\sum_{n\geq 2}\left(\frac{p}{q}\right)^{n-1} w^{n} - w - \sum_{n\geq 2}\left(\frac{p}{q}\right)^{n-1} w^{n} \\
&= -w - \sum_{n\geq 2} 2\left(\frac{p}{q}\right)^{n-1} w^{n} \\
[w^n]\left(\frac{w(pw+q)}{pw-q}\right) &= -2\left(\frac{p}{q}\right)^{n-1}[n\geq 2] - [n=1]
\end{aligned}
$$

At this point we can use the accumulation property:

$$
\begin{aligned}
[w^a]\left(\frac{w(pw+q)}{(w-1)^2(pw-q)}\right) &= \sum_{0\leq b\leq a}\sum_{0\leq n\leq b} -2\left(\frac{p}{q}\right)^{n-1}[n\geq 2] - [n=1] \\
&= \sum_{0\leq b\leq a} -1 - 2\sum_{2\leq n\leq b}\left(\frac{p}{q}\right)^{n-1} \\
&= \sum_{0\leq b\leq a} -1 - 2\sum_{1\leq n\leq b-1}\left(\frac{p}{q}\right)^{n} \\
&= \sum_{0\leq b\leq a} -1 - 2\frac{\left(\frac{p}{q}\right)^{b-1+1} - \frac{p}{q}}{\frac{p}{q}-1} \\
&= \sum_{0\leq b\leq a} -1 - 2\frac{q\left(\frac{p}{q}\right)^{b} - p}{p-q} \\
&= \sum_{0\leq b\leq a} \frac{p+q}{p-q} - \frac{2q}{p-q}\left(\frac{p}{q}\right)^{b}
\end{aligned}
$$

Continuing:

$$
\begin{aligned}
[w^a]\left(\frac{w(pw+q)}{(w-1)^2(pw-q)}\right) &= \sum_{0\le b\le a} \frac{p+q}{p-q} - \frac{2q}{p-q}\left(\frac{p}{q}\right)^b \\
&= (a+1)\frac{p+q}{p-q} - \frac{2q}{p-q}\left(\frac{\left(\frac{p}{q}\right)^{a+1}-1}{\frac{p}{q}-1}\right) \\
&= \frac{a+1}{p-q} - \frac{2q}{p-q}\left(\frac{p\left(\frac{p}{q}\right)^a - q}{p-q}\right) \\
&= \frac{a}{p-q} + \frac{p-q+2pq\left(\frac{p}{q}\right)^a + 2q^2}{(p-q)^2} \\
&= \frac{a}{p-q} + \frac{p-q+2q^2}{(p-q)^2} - \frac{2pq\left(\frac{p}{q}\right)^a}{(p-q)^2}
\end{aligned}
$$

Since $G_a(1) = 1$, the mean of the original recurrence is given by the formula below, when $0 \le a \le m$:

$$
\begin{aligned}
g'_a(1) &= G'_{m-a}(1) - G'_m(1) \\
&= \frac{m-a}{p-q} - \frac{2pq\left(\frac{p}{q}\right)^{m-a}}{(p-q)^2} - \frac{m}{p-q} + \frac{2pq\left(\frac{p}{q}\right)^m}{(p-q)^2} \\
&= \frac{a}{q-p} - \frac{2pq\left(\left(\frac{p}{q}\right)^{m-a} - \left(\frac{p}{q}\right)^m\right)}{(q-p)^2} \\
&= \frac{a}{q-p} - \frac{2pq\left(\frac{p}{q}\right)^m\left(\left(\frac{q}{p}\right)^a - 1\right)}{(q-p)^2}
\end{aligned}
$$

The final mean over all a must take into account the forced greedy move when $a = 1$ and $a = n$, and also the binomial distribution of the a:

$$
\begin{aligned}
g'(1) &= \frac{1}{2^{2m+2}} \sum_{0\le a\le 2m} \binom{2m+2}{a+1}(1+g'_a(1)) \\
&= \frac{1}{2^{2m+2}} \left(\sum_{0\le a\le m} \binom{2m+2}{a+1}(1+g'_a(1)) + \sum_{0\le a\le m-1} \binom{2m+2}{a+1}(1+g'_a(1)) \right) \\
&= \frac{1}{2^{2m+2}} \left(\binom{2m+2}{m+1}(1+g'_m(1)) + \sum_{0\le a\le m-1} 2\binom{2m+2}{a+1}(1+g'_a(1)) \right)
\end{aligned}
$$

There is no evident closed form for this expression; we can calculate it numerically though. For p' in $(.001, .01, .1, .5, .9, .99, .999)$, the means are:

$$
g'(1) \sim (487.9, 492.3, 541.4, 973.7, 4853.4, 44688.2, 183063.4)
$$

The last subcase is when $n = 2m - 1$, which implies $g_{m-1} = g_m$. Defining once again $G_a = g_{m-a}/g_m$, we have $G_0 = 1$ and $G_1 = 1$. The recursion is the same as the previous case, $G_a = (1/zq)G_{a-1} - (p/q)G_{a-2}$, from which we can calculate the mean $g'(1)$:

$$
\begin{aligned}
G &= \frac{-w + zq(1+w)}{qz + w(pwz - 1)} \\
\left.\frac{dG}{dz}\right|_{z=1} &= \frac{w^2(pw-q)}{(q + w(pw-1))^2} \\
[w^a]\left.\frac{dG}{dz}\right|_{z=1} &= \frac{a}{p-q} + \frac{q\left(1 - \left(\frac{p}{q}\right)^a\right)}{(p-q)^2} \\
g'_a(1) &= \frac{a}{q-p} - \frac{q\left(\frac{p}{q}\right)^m\left(\left(\frac{q}{p}\right)^a - 1\right)}{(q-p)^2} \\
g'(1) &= \frac{1}{2^{2m+1}} \sum_{0\le a\le m-1} 2\binom{2m+1}{a+1}(1+g'_a(1))
\end{aligned}
$$

□