The Art of Computer Programming

Chapter 1

Mathematical Preliminaries Redux

1.1 Exercise 1.a

I've written a python script to check the probabilities of the dice (file redux.1a.py). They are, indeed, all equal to $5/9$.

1.2 Exercise 1.b

I've written a MIP model to find dice with the given probabilities (file redux.1b.cc). The MIP model is:

$$\begin{gathered}\text{Integer variable } dice[3][6] \\ \text{Integer variable } count[3] \\ \text{Binary variable } greater[3][6]\end{gathered}$$

$$\begin{aligned}0 &\le dice[i+1][k] - dice[i][j] + greater[i][j][k] \le 5 \\ 0 &\le count[i] - \sum_{j,k} greater[i][j][k] \le 0 \\ 21 &\le count[i] \le 36 \\ -6 &\le dice[i][j] - dice[i][k] \le 0\end{aligned}$$

This model has a solution of $A = [3, 3, 3, 3, 3, 6]$, $B = [2, 2, 2, 5, 5, 5]$, $C = [1, 4, 4, 4, 4, 4]$.

1.3 Exercise 1.c

I've modified the MIP model from 1.b to check solution for dices with F_m sides (file redux.1c.cc). For $n = 8$ the solution found is $A = [1, 1, 1, 5, 5, 5, 5, 5]$, $B = [3, 3, 3, 3, 3, 3, 3, 3]$, $C = [2, 2, 2, 2, 2, 8, 8, 8]$. This suggest a pattern of:

$$\begin{aligned} A &= [F_{m-2} \times 1, F_{m-1} \times 5] \\ B &= [F_m \times 3] \\ C &= [F_{m-1} \times 2, F_{m-2} \times 8] \end{aligned}$$

This will provide probabilities given by:

$$\begin{aligned} P(A > B) &= \frac{F_{m-1}F_m}{F_m^2} \\ P(B > C) &= \frac{F_{m-1}F_m}{F_m^2} \\ P(C > A) &= \frac{F_{m-2}F_m + F_{m-1}F_{m-2}}{F_m^2} \end{aligned}$$

$P(A > B)$ and $P(B > C)$ will readily simplify to the desired probabilities, let's prove $P(C > A)$ does as well. We'll start with Cassini's identity:

$$F_m F_{m-2} = F_{m-1}^2 + (-1)^n$$

Substituting it into the original expression:

$$\begin{aligned} F_{m-2}F_m + F_{m-1}F_{m-2} &= F_{m-1}^2 + (-1)^n + F_{m-1}F_{m-2} \\ &= F_{m-1}(F_{m-1} + F_{m-2}) + (-1)^n \\ &= F_{m-1}F_m + (-1)^n \\ &= F_{m-1}F_m \pm 1 \end{aligned}$$

QED