Sol^n of Non-homogeneous Linear differential eq^n with constant coefficient

(RHS $\neq 0$)

$$\boxed{y = y_c + y_p}$$

• Case ①

$X = e^{ax} \Rightarrow P.I = \frac{1}{f(D)} e^{ax} = \frac{1}{f(a)} e^{ax}$ $\quad f(a) \neq 0$ [$D \to a$]

(RHS) $\longrightarrow$

eg. $y'' - 2y' - 3y = 3e^{2x}$

Sol^n. $D^2y - 2Dy - 3y = 0$

$\Rightarrow m^2 - 2m - 3 = 0$

$\Rightarrow m^2 - 3m + m - 3 = 0$

$\Rightarrow m(m-3) + 1(m-3) = 0$

$\Rightarrow (m-3)(m+1) = 0$

$\Rightarrow \boxed{m_1 = 3 \ \& \ m_2 = -1}$

$\therefore \boxed{y_c = c_1 e^{3x} + c_2 e^{-x}}$

Now.

$P.I = y_p = \frac{1}{f(D)} 3e^{2x}$

$\Rightarrow \frac{3 \cdot 1}{D^2 - 2D - 3} e^{2x}$

($D \to a$)
$\to 2$ $\Rightarrow \frac{3}{2^2 - 2 \cdot 2 - 3} e^{2x} = \frac{3}{4-4-3} e^{2x}$

$= \frac{3}{-3} e^{2x} = -e^{2x}$

$\therefore$ General sol^n

$y = y_c + y_p$

$\Rightarrow \boxed{y = c_1 e^{3x} + c_2 e^{-x} - e^{2x}}$

Ans.

CASE (2) When $x = \sin(ax+b) \,/\, \cos(ax+b)$

$(D^2 \to -a^2)$

$$\Rightarrow P.I = \frac{1}{f(D^2)} \sin(ax+b) \,/\, \cos(ax+b)$$

$$= \frac{1}{f(-a^2)} \sin(ax+b) \,/\, \cos(ax+b)$$

Eg $y''' - y'' + 4y' - 4y = \sin 3x$

Sol^n. $D^3y - D^2y + 4Dy - 4y = \sin 3x$

$\Rightarrow m^3 - m^2 + 4m - 4 = 0$

$m_1 = 1, \; m_2 = -2i, \; m_3 = 2i$

$$\boxed{y_c = c_1 e^x + c_2 \cos 2x + c_3 \sin 2x}$$

$$y_p = \frac{1}{f(D)} x = \frac{1}{D^3 - D^2 + 4D - 4} \sin 3x$$

$D^2 \to (-a^2) \to (-9)$

$$\Rightarrow \frac{1}{-9D + 9 + 4D - 4} \sin 3x$$

$$\Rightarrow \frac{1}{5 - 5D} \sin 3x$$

$$\frac{1}{5-5D} \sin 3x$$

$$\Rightarrow \frac{1}{5-5D} \times \frac{(5+5D)}{5+5D} \sin 3x$$

$$\Rightarrow \frac{5+5D}{25 - 25D^2} \sin 3x \qquad (D^2 \to -9)$$

$$\Rightarrow \frac{5+5D}{250} \sin 3x$$

$$\Rightarrow \frac{\sin 3x}{50} + \frac{3\cos 3x}{50}.$$

Given $y = y_c + y_p$

$$\Rightarrow \boxed{y = c_1 e^x + c_2 \cos 2x + c_3 \sin 2x + \frac{\sin 3x}{50} + \frac{3\cos 3x}{50}}$$

Ans.

More Questions Practise.

Ques $y''' - 2y'' - 5y' + 6y = 4e^{-x} - e^{2x}$

Soln. $D^3y - 2D^2y - 5Dy + 6 = 4e^{-x} - e^{2x}$

$\Rightarrow m^3 - 2m^2 - 5m + 6 = 0$

$\Rightarrow (m-1)(m+2)(m-3) = 0$ $(1, -2, 3)$

$\therefore \boxed{y_c = c_1e^{x} + c_2e^{-2x} + c_3e^{3x}}$

$y_p(x) = \frac{1}{f(D)} x = \frac{1}{D^3 - 2D^2 - 5D + 6} \cdot 4e^{-x} - e^{2x}$

$\Rightarrow \frac{1}{D^3 - 2D^2 - 5D + 6} 4e^{-x} - \frac{1}{D^3 - 2D^2 - 5D + 6} e^{2x}$

$\Rightarrow (D \to -1)$ $(D \to 2)$

$\Rightarrow \frac{1}{-1 - 2 + 5 + 6} 4e^{-x} - \frac{1}{8 - 8 - 10 + 6} e^{2x}$

$\Rightarrow \frac{4e^{-x}}{8} - \frac{1}{-4} e^{2x}$

$y_p \Rightarrow \frac{e^{-x}}{2} + \frac{e^{2x}}{4}$

G. soln. $y = y_c + y_p$

$\Rightarrow \boxed{y = c_1e^{x} + c_2e^{-2x} + c_3e^{3x} + \frac{e^{-x}}{2} + \frac{e^{2x}}{4}}$

Ans.

Que. $y''' - y'' - y' + y = e^x$

Sol. $y_c = ??$ (find yourself).

$y_p = \frac{1}{f(D)} X = \frac{1}{D^3 - D^2 - D + 1} e^x$

$(D \to 1)$

$\Rightarrow \frac{1}{1-1-1+1} e^x = \frac{e^x}{0}$

$\frac{x}{f'(D)} \cdot 1 \, e^{ax} \Rightarrow \frac{x}{3D^2 - 2D - 1} e^x$

$(D \to 1)$

$\Rightarrow \frac{x}{3-2-1} e^x \Rightarrow \frac{xe^x}{0}$

$\frac{x \cdot x \, e^{ax}}{6D - 2} \Rightarrow \frac{x^2 e^x}{6D - 2} \Rightarrow$ $(D \to 1)$ $\frac{x^2 e^x}{6-2} \Rightarrow \frac{x^2 e^x}{4}$

*** **Special case**

when Denominator $= 0$

G.S. $y = y_c + y_p$

$\Rightarrow \boxed{y = y_c + \frac{x^2 e^x}{4}}$ Ans

Ques $y'' + 4y = \cos 2x$

$D \to$ diff

$\frac{1}{D} \to$ int.

Sol^n. $D^2y + 4y = \cos 2x$

$\Rightarrow m^2 + 4 = 0$

$m^2 = -4$

$\Rightarrow \boxed{m = \pm 2i}$

$\boxed{y_c = c_1 \cos 2x + c_2 \sin 2x}$

$$\boxed{\begin{aligned} y &= y_c + y_p \\ &= c_1 \cos 2x + c_2 \sin 2x + \frac{x}{4}\sin 2x \end{aligned}}$$

Ans.

$y_p = \frac{1}{f(D)} x = \frac{1}{D^2+4} \cos 2x$ $\boxed{a=2}$

$(D^2 = -a^2 = -4)$

$\Rightarrow \frac{1}{-4+4} \cos 2x$

$\Rightarrow \frac{\cos 2x}{0}$

$\Rightarrow \frac{x \cos 2x}{2D} \Rightarrow \frac{x}{2} \cdot \frac{1}{D} \cos 2x \Rightarrow \frac{x}{2} \int \cos 2x = \frac{x}{2} \frac{\sin 2x}{2}$

$= \frac{x \sin 2x}{4}$

$$\boxed{\begin{aligned} &\frac{\cos 2x}{D} \\ &D \to \text{diff} \\ &1/D \to \text{int.} \end{aligned}}$$

Que: $y^{IV} + 5y'' + 4y = 16\sin x + 64\cos 2x$

Solⁿ. $D^4y + 5D^2y + 4 = 16\sin x + 64\cos 2x$

$\Rightarrow m^4 + 5m^2 + 4 = 0$

$m_1 = \pm i \quad , m_2 = \pm 2i$

$\boxed{y_c = c_1\cos x + c_2\sin x + c_3\cos 2x + c_4\sin 2x}$

$y_p = \frac{1}{f(D)}x = \frac{1}{f(D^2)}x = \frac{1}{f(-a^2)}x$

$\Rightarrow \frac{16\sin x}{D^4 + 5D^2 + 4} + \frac{64\cos 2x}{D^4 + 5D^2 + 4}$

$(D^2 = -1) \qquad (D^2 = -4)$

$\Rightarrow \frac{16\sin x}{1 - 5 + 4} + \frac{64\cos 2x}{16 - 20 + 4}$

$\Rightarrow \frac{16\sin x}{0} + \frac{64\cos 2x}{0}$

$\Rightarrow \frac{x \cdot 16\sin x}{4D^3 + 10D} + \frac{x \cdot 64\cos 2x}{4D^3 + 10D}$

Now,

$\frac{16x\sin x}{-4D + 10D} + \frac{64x\cos 2x}{-16D + 10D}$

$\left\{ \frac{1}{D} = \int \text{integration} \right.$

$\Rightarrow \frac{16x\sin x}{6D} + \frac{64x\cos 2x}{-6D}$

$\Rightarrow \frac{16x}{6}\int \sin x - \frac{64x}{6}\int \cos 2x$

$\Rightarrow \frac{-8x}{3}\cos x - \frac{32x}{3} \cdot \frac{\sin 2x}{2}$

$\Rightarrow \frac{-8x\cos x}{3} - \frac{16x}{3}\sin 2x$

∴ General solⁿ: $y = y_c + y_p$

$\Rightarrow y = c_1\cos x + c_2\sin x + c_3\cos 2x + c_4\sin 2x - \frac{8x\cos x}{3} - \frac{16x}{3}\sin 2x$

Case 3 When $X = x^n$

$$P.I = \frac{1}{f(D)} X = [1 + \phi D]^{-1}$$

Expand according to the type of expansion

- $(1-D)^{-1} \rightarrow 1 + D + D^2 + D^3 + \cdots$
- $(1+D)^{-1} \rightarrow 1 - D + D^2 - D^3 + \cdots$
- $(1-D)^{-2} \rightarrow 1 + 2D + 3D^2 + 4D^3 + \cdots$
- $(1+D)^{-2} \rightarrow 1 - 2D + 3D^2 - 4D^3 + \cdots$

Ques $y'' + 16y = 64x^2$

Solⁿ $\boxed{y_c = c_1 \cos 4x + c_2 \sin 4x}$

$$y_P = \frac{1}{f(D)} X = \frac{1}{[1+\phi D]} X$$

$$= \frac{64}{D^2 + 16} x^2$$

$$\frac{64 x^2}{16\left(\frac{D^2}{16} + 1\right)} \Rightarrow \frac{4x^2}{\left[1 + \left(\frac{D}{4}\right)^2\right]}$$

$$\Rightarrow 4\left[1 + \left(\frac{D^2}{16}\right)\right]^{-1} x^2$$

$$\Rightarrow 4\left[1 - \frac{D^2}{16} + \frac{D^4}{64} - \cdots\right] x^2$$

$$\Rightarrow 4\left[x^2 - \frac{1}{16} \cdot 2 + 0 - \cdots\right]$$

$$\Rightarrow 4x^2 - \frac{1}{2}$$

$\therefore$ General solⁿ $y = y_c + y_P$

$$\therefore \boxed{y = c_1 \cos 4x + c_2 \sin 4x + 4x^2 - \frac{1}{2}}$$

Que. $y'' + 25y = 9x^3 + 4x^2$

Sol^n $D^2y + 25y = 9x^3 + 4x^2$

$\Rightarrow m^2 + 25 = 0$

$\boxed{m = \pm 5i}$

$\boxed{y_c = c_1 \cos 5x + c_2 \sin 5x}$

$y_p = \frac{1}{f(D)}x = \frac{1}{[1+\phi D]}x = \frac{1}{D^2+25}(9x^3 + 4x^2)$

$\Rightarrow \frac{9x^3}{D^2+25} + \frac{4x^2}{D^2+25}$

$\Rightarrow \frac{9}{25}\left(\frac{1}{1+\left(\frac{D^2}{25}\right)}\right)x^3 + \frac{4}{25}\left(\frac{1}{1+\frac{D^2}{25}}\right)x^2$

$\Rightarrow \frac{9}{25}\left[1 + \frac{D^2}{25}\right]^{-1}x^3 + \frac{4}{25}\left[1 + \frac{D^2}{25}\right]^{-1}x^2$

$\Rightarrow \frac{9}{25}\left[1 - \frac{D^2}{25} + \frac{D^4}{25^2} \cdots\right]x^3 + \frac{4}{25}\left[1 - \frac{D^2}{25} + \frac{D^4}{25^2} \cdots\right]x^2$

$\Rightarrow \frac{9x^3}{25} - \frac{9}{625} \cdot 6x. + \frac{4x^2}{25} - \frac{4}{625} \cdot 2$

$\therefore y_p = \frac{9}{25}x^3 + \frac{4}{25}x^2 - \frac{54}{625}x - \frac{8}{625}$

$\therefore \boxed{y = y_c + y_p}$

$\Rightarrow \boxed{y = c_1 \cos 5x + c_2 \sin 5x + \frac{9}{25}x^3 + \frac{4}{25}x^2 - \frac{54}{625}x - \frac{8}{625}}$

Ans.

Ques $y'' + 6y' + 9y = 4x^2 - 1$

Sol. $D^2y + 6Dy + 9y = 4x^2 - 1$

$\Rightarrow m^2 + 6m + 9 = 0$

$\Rightarrow \boxed{m_1 = m_2 = -3}$

$\therefore y_c = (c_1 + c_2 x)e^{-3x}$

Now

$y_P = \frac{1}{f(D)}x = \frac{1}{[1+4D]}x = \frac{1}{D^2+6D+9}(4x^2-1)$

$\Rightarrow \frac{1}{9}\left[\frac{1}{1+\left(\frac{6D+D^2}{9}\right)}\right](4x^2-1)$

$\Rightarrow \frac{1}{9}\left[1 + \left(\frac{6D+D^2}{9}\right)\right]^{-1}(4x^2-1)$

$\Rightarrow \frac{1}{9}\left[1 - \left(\frac{6D}{9} + \frac{D^2}{9}\right) + \left(\frac{6D}{9} + \frac{D^2}{9}\right)^2 \dots\right](4x^2-1)$

$\Rightarrow \frac{1}{9}\left[1 - \frac{6D}{9} - \frac{D^2}{9} + \frac{36D^2}{81} + \frac{D^4}{81} + \frac{12}{81}D^3 \dots\right](4x^2-1)$

$\Rightarrow \frac{1}{9}\left[4x^2 - 1 - \frac{6}{9}[8x] - \frac{1}{9}[8] + \frac{36}{81}(8) + 0 + 0 + 0\right]$

$\Rightarrow \frac{1}{9}\left[4x^2 - 1 - \frac{48x}{9} - \frac{8}{9} + \frac{32}{9}\right]$

$\Rightarrow \frac{1}{9}\left[4x^2 - \frac{48x}{9} + \frac{32-8-9}{9}\right]$

$\Rightarrow \frac{1}{9}\left[4x^2 - \frac{48x}{9} + \frac{5}{3}\right]$

$\Rightarrow \frac{1}{9}\left[\frac{36x^2 - 48x + 15}{9}\right]$

$\Rightarrow y_P = \frac{12x^2 - 16x + 5}{27}$

$\therefore y(x) = (c_1 + c_2 x)e^{-3x} + \frac{12x^2 - 16x + 5}{27}$

G.S. $\Rightarrow y_c + y_p$

Ans.

Questions for Practice.

① $y'' - 2y' - 3y = 2x^2 + 6x$	① $c_1 e^{-x} + c_2 e^{3x} + \frac{18x^2 + 30x - 8}{27}$
② $y'' - 3y' + 2y = e^{3x}$	② $c_1 e^{x} + c_2 e^{2x} + \frac{1}{2} e^{3x}$
③ $(D^2 + a^2)y = \cos ax$	③ $c_1 \cos ax + c_2 \sin ax + \frac{x}{2a} \sin ax$
④ $4y'' + 12y' + 9y = 144 e^{-3x}$	④ $(c_1 + c_2 x) e^{-3x/2} + 16 e^{-3x}$
⑤ $(D^2 + 4D + 4)y = e^{2x} - e^{-2x}$	⑤ $(c_1 + c_2 x) e^{-2x} + \frac{1}{16} e^{2x} - \frac{x^2}{2} e^{-2x}$
⑥ $(D+2)(D-1)^3 y = e^{x}$	⑥ $c_1 e^{-2x} + (c_2 + c_3 x + c_4 x^2) e^{x} + \frac{1}{81} x^3 e^{x}$
⑦ $(D^3 - D)y = e^{x} + e^{-x}$	⑦ $c_1 + c_2 e^{x} + c_3 e^{-x} + \frac{x}{2}(e^{x} + e^{-x})$
⑧ $y''' - y = (e^{x} + 1)^2$	⑧ $c_1 e^{x} + e^{-x/2}\left[c_2 \cos \frac{\sqrt{3}x}{2} + c_3 \sin \frac{\sqrt{3}x}{2}\right] + \frac{1}{7} e^{2x} + \frac{2x}{3} e^{x} - 1$
⑨ $y'' + y = \cos 2x$	⑨ $c_1 \cos x + c_2 \sin x - \frac{1}{3} \cos 2x$
⑩ $y'' + 9y = \cos 4x$	⑩ $c_1 \cos 3x + c_2 \sin 3x - \frac{1}{7} \cos 4x$

(11) $(D^3 + 9D)y = \sin 3x$

(11) $c_1 + c_2 \cos 3x + c_3 \sin 3x - \frac{x}{18} \sin 3x$

(12) $y'' - 8y' + 9y = 40 \sin 5x$

(12) $e^{4x}(c_1 \cos \sqrt{7}x + c_2 \sin \sqrt{7}x) + \frac{5}{29}(5 \cos 5x - 2 \sin 5x]$

(13) $(D^2 + 9)y = \cos 2x + \sin 2x$

(13) $c_1 \cos 3x + c_2 \sin 3x + \frac{1}{5}(\cos 2x + \sin 2x)$

(14) $y'' + y = \cos x \sin 3x$

(14) $c_1 \cos x + c_2 \sin x - \frac{1}{30} \sin 4x - \frac{1}{16} \sin 2x$

(15) $(D^4 - D^2)y = 2$

(15) $c_1 + c_2 x + c_3 e^{x} + c_4 e^{-x} - x^2$

(16) $(D^4 - a^4)y = x^4$

(16) $c_1 e^{ax} + c_2 e^{-ax} + c_3 \cos ax + c_4 \sin ax - \frac{1}{a^4}[x^4 + \frac{24}{a^4}]$

(17) $y''' + 3y'' + 2y' = x$

(17) $c_1 + c_2 e^{-x} + c_3 e^{-2x} + \frac{x^3}{6} - \frac{3x^2}{4} + \frac{7x}{4}$

(18) $(D^3 + 8)y = x^4 + 2x + 1$

(18) $c_1 e^{-2x} + e^{x}(c_2 \cos \sqrt{3}x + c_3 \sin x\sqrt{3}] + \frac{1}{8}(x^4 - x + 1)$

(19) $(D^4 - 2D^3 + 5D^2 - 8D + 4)y = x^2$

(19) $(c_1 + c_2 x)e^{x} + c_3 \cos 2x + c_4 \sin 2x + \frac{1}{4}(x^2 + 4x + \frac{11}{12})$

(20) $y'' + 2y' + y = 2x + x^2$

(20) $(c_1 + c_2 x)e^{-x} + x^2 - 2x + 2$

Case IV

when $X = e^{ax} v(x)$

$\longrightarrow \sin(ax+b) / \cos(ax+b)$ or a^n

$$P.I \Rightarrow \frac{1}{f(D)} e^{ax} v(x) = \frac{e^{ax}}{f(D+a)} v(x)$$

$(D \rightarrow D+a)$

Ques. $16y'' + 8y' + y = 48e^{-x/4} . x$

Solⁿ. $y_c \Rightarrow 16D^2y + 8Dy + y = 48e^{-x/4} . x$

$\Rightarrow 16m^2 + 8m + 1 = 0$

$\Rightarrow \boxed{m_1 = m_2 = \frac{-1}{4}}$

$\therefore y_c = (c_1 + c_2 x) e^{-1/4 x}$

$$y_p \rightarrow \frac{1}{f(D)} X \Rightarrow \frac{e^{ax}}{f(D+a)} v(x)$$

$$\Rightarrow \frac{48e^{-x/4}}{16D^2 + 8D + 1} . x \qquad \left(D \rightarrow D+a\right) \Rightarrow \left(D - \frac{1}{4}\right)$$

$$\Rightarrow \frac{48e^{-x/4} . x}{16\left[\left(D - \frac{1}{4}\right)^2\right] + 8\left(D - \frac{1}{4}\right) + 1}$$

Now.
$$\frac{48e^{-x/4} . x}{16\left(D^2 + \frac{1}{16} - \frac{D}{2}\right) + 8D - 2 + 1}$$

$$\Rightarrow \frac{48e^{-x/4} . x}{16D^2 + 1 - 8D + 8D - 2 + 1}$$

$$\Rightarrow \frac{48e^{-x/4} . x}{16D^2}$$

$$\Rightarrow 3e^{-x/4} . \frac{x}{D^2}$$

$\frac{1}{D} \rightarrow$ int

$\frac{1}{D \cdot D} \rightarrow$ int $\rightarrow$ int

$$\Rightarrow 3e^{-x/4} . \frac{x^3}{6}$$

$$y_p \Rightarrow \frac{e^{-x/4} . x^3}{2}$$

$$\therefore y = (c_1 + c_2 x) e^{\frac{-1}{4} x} + \frac{x^3}{2} e^{-x/4}$$

Ans

Que. $y'' - 4y' + 13y = 18 e^{2x} \sin 3x$

Solⁿ. $D^2y - 4Dy + 13y = 18 e^{2x} \sin 3x$

$\Rightarrow m^2 - 4m + 13 = 0$

$\boxed{m = 2 \pm 3i}$

$y_c = e^{2x} [c_1 \cos 3x + c_2 \sin 3x]$

$y_p = \frac{1}{f(D)} x = \frac{e^{ax} \cdot v(x)}{f(D+a)}$

$\Rightarrow \frac{18 e^{2x} \sin 3x}{D^2 - 4D + 13}$

$D \to D+a \to D+2$

$\Rightarrow \frac{18 \cdot e^{2x} \cdot \sin 3x}{(D+2)^2 - 4(D+2) + 13}$

$\Rightarrow \frac{18 \cdot e^{2x} \cdot \sin 3x}{D^2 + 4 + 4D - 4D - 8 + 13}$

$\Rightarrow \frac{18 \cdot e^{2x} \cdot \sin 3x}{D^2 + 9}$

$\Rightarrow \frac{18 e^{2x}}{D^2 + 9} \sin 3x$

$D^2 = -a^2 = -3^2 = -9$

$\Rightarrow \frac{18 e^{2x}}{0} \sin 3x$

$\frac{18 e^{2x} \cdot x \cdot \sin 3x}{2D}$

$9 e^{2x} \cdot x \int \sin 3x$

$\Rightarrow 9 e^{2x} \cdot x \left(\frac{-\cos 3x}{3}\right)$

$\boxed{y_p \Rightarrow -3 e^{2x} \cdot x \cdot \cos 3x}$

$\therefore y = e^{2x} [c_1 \cos 3x + c_2 \sin 3x] - 3 e^{2x} \cdot x \cdot \cos 3x$

Questions Practise

① $y'' - 2y' + y = e^{3x} x^2$

② $(D-1)^2 y = e^{x} \sec^2 x \tan x$

③ $y''' - 3y' - 2y = 540 e^{-x} x^3$

④ $y''' - y'' + 3y' + 5y = e^{x} \cos x$

⑤ $y^{iv} + y'' + y = e^{-x/2} \cos\left(x\frac{\sqrt{3}}{2}\right)$

⑥ $y'' - 4y' + 4y = e^{2x} \sin 2x$

⑦ $y''' + y = e^{2x} \sin x + e^{x/2} \sin \frac{\sqrt{3}x}{2}$

① $(c_1 + c_2 x) e^{x} + \frac{1}{8} e^{3x} (2x^2 - 4x + 3)$

② $y_p = \frac{e^{x}}{2} (\tan x - x)$

③ $c_1 e^{2x} + (c_2 + c_3 x) e^{-x} - e^{-x} [9x^5 + 15x^4 + 20x^3 + 20x^2]$

④ $c_1 e^{-x} + e^{x} (c_2 \cos 2x + c_3 \sin 2x) + \frac{1}{34} e^{x} (3 \sin x + 5 \cos x)$

⑤ $e^{-x/2} [c_1 \cos \frac{\sqrt{3}}{2} x + c_2 \sin \frac{\sqrt{3}}{2} x] + e^{x/2} [c_3 \cos \frac{\sqrt{3}}{2} x + c_4 \sin \frac{\sqrt{3}}{2} x] + \frac{x}{4\sqrt{3}} e^{-x/2} [\sin \frac{\sqrt{3}}{2} x + \sqrt{3} \cos \frac{\sqrt{3}}{2} x]$

⑥ $(c_1 + c_2 x) e^{2x} - e^{2x} \sin x$

⑦ $c_1 e^{-x} + e^{x/2} [c_2 \cos \frac{\sqrt{3}}{2} x + c_3 \sin \frac{\sqrt{3}}{2} x] - \frac{x}{6} e^{x/2} [\sin \frac{\sqrt{3}}{2} x + \sqrt{3} \cos \frac{\sqrt{3}}{2} x]$

case ⑤

$X = \sin(ax+b) / \cos(ax+b) \cdot v(x)$ | $X = x\, v(x)$

P.I $\rightarrow \frac{1}{f(D)} \sin(ax+b)/\cos(ax+b) \cdot v(x)$

$\rightarrow \frac{1}{f(D)} x\, v(x) = \frac{x}{f(D)} v(x) + \left(\frac{\partial}{\partial x} \frac{1}{f(D)}\right)$

Que. $y'' + 4y' + 3y = x \sin 2x$

Soln. $\boxed{y_c = c_1 e^{-x} + c_2 e^{-3x}}$

$y_p = \frac{1}{f(D)} x \rightarrow \frac{1}{D^2+4D+3} x \sin 2x$

$\Rightarrow \frac{x}{D^2+4D+3} \cdot \sin 2x - \frac{\partial}{\partial x}\left[\frac{1}{D^2+4D+3}\right] \sin 2x$

$D^2 = -a^2 = -4$

$\Rightarrow \frac{x}{-4+4D+3} \sin 2x - \frac{1 \cdot (2D+4)}{(D^2+4D+3)^2} \sin 2x$

$\Rightarrow \frac{x}{4D-1} \sin 2x - \frac{2D+4}{(4D-1)^2} \sin 2x$

$\Rightarrow \frac{x}{4D-1} \times \frac{4D+1}{4D+1} \sin 2x - \frac{2D+4}{(4D-1)^2} \cdot \frac{(4D+1)^2}{(4D+1)^2} \sin 2x$

$\frac{x(4D+1)}{16D^2-1} \sin 2x - \frac{(2D+4)(4D+1)^2}{[(4D-1)(4D+1)]^2} \sin 2x$

$\Rightarrow \frac{-x}{65} (4D+1) \sin 2x - \frac{1}{65^2}[32D^3 + 2D + 4D^2 + 64D^2 + 4 + 32D] \sin 2x$

$\Rightarrow \frac{-x}{65}\left[\begin{matrix} 8\cos 2x \\ + \sin 2x \end{matrix}\right] - \frac{1}{4225}\left[\begin{matrix} -188 \cos 2x \\ - 316 \sin 2x \end{matrix}\right]$

$\therefore y = y_c + y_p$

$= c_1 e^{-x} + c_2 e^{-3x} - \frac{x}{65}\left[\begin{matrix} 8\cos 2x \\ + \sin 2x \end{matrix}\right]$

$-\frac{1}{4225}[-188 \cos 2x - 316 \sin 2x]$

Questions — Practise

Q. ① $y'' + 9y = x\sin x$

② $y'' + 2y' + y = x\sin x$

③ $y'' + y = x^2 \sin 2x$

④ $y^{IV} - y = x\sin x$

Sol^n.

①. $c_1 \cos 3x + c_2 \sin 3x + \frac{x}{8}\sin x - \frac{1}{32}\cos x$

②. $(c_1 + c_2 x)e^{-x} + \frac{1}{2}[(x-1)\sin x + \cos x]$

③. $c_1 \cos x + c_2 \sin x - \frac{1}{3}\left[\left(x^2 - \frac{26}{9}\right)\sin 2x + \frac{8}{9}x\cos 2x\right]$

④. $c_1 e^{x} + c_2 e^{-x} + c_3 \cos x + c_4 \sin x + \frac{1}{8}(x^2\cos x - 3x\sin x)$

Methods of variations of Parameters.

$$a_0(x)y'' + a_1(x)y' + a_2(x)y = r(x)$$

How to solve.

1. find y_c ?? $\Rightarrow$ $y_c = c_1 y_1(x) + c_2 y_2(x)$
2. Check solutions are Linearly independent or dependent by Wronskian $(W(x))$ $W(x) = ??$
3. Assume $\boxed{y_p = A(x)y_1(x) + B(x)y_2(x)}$; where $A(x)$ & $B(x)$ are fxn of x.

Now:

$$A(x) = -\int \frac{g(x)\, y_2(x)}{W(x)}\, dx \qquad \& \quad g(x) = \frac{r(x)}{a_0(x)}$$

$$B(x) = \int \frac{g(x)\, y_1(x)}{W(x)}\, dx$$

4. Now, write general sol^n as :-

$$\boxed{y = y_c + y_p}$$

Ques $y'' + 3y' + 2y = 2e^{x}$

Sol^n $D^2y + 3Dy + 2y = 2e^{x}$ (M.O V.P).

$\Rightarrow m^2 + 3m + 2 = 0$

$\Rightarrow \boxed{y_c = c_1 e^{-x} + c_2 e^{-2x}}$ — ①

So $y_1(x) = e^{-x}$ & $y_2(x) = e^{-2x}$

Now:

$$W(x) = \begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix} = \begin{vmatrix} e^{-x} & e^{-2x} \\ -e^{-x} & -2e^{-2x} \end{vmatrix}$$

$$= -2e^{-3x} + e^{-3x}$$

$$= -e^{-3x} \neq 0$$

$\therefore$ Solutions are Linearly ind.

Now: Let us assume $y_p = A(x) y_1(x) + B(x) y_2(x)$

$\Rightarrow \boxed{y_p = A(x) e^{-x} + B(x) e^{-2x}}$ — ②

Now: finding $A(x)$ & $B(x)$ as:-

- $g(x) = \dfrac{h(x)}{a_0(x)} = \dfrac{2e^{x}}{1} = 2e^{x}$
- $A(x) = -\int \dfrac{g(x)\, y_2(x)}{W(x)} = -\int \dfrac{2e^{x} \cdot e^{-2x}}{-e^{-3x}}\, dx$

$\Rightarrow 2\int e^{2x}\, dx = e^{2x}$.

- $B(x) = \int \dfrac{g(x) \cdot y_1(x)}{W(x)} = \int \dfrac{2e^{x} \cdot e^{-x}}{-e^{-3x}}\, dx$

$= -2\int e^{3x} = \dfrac{-2}{3} e^{3x}$

$\therefore y_p = e^{2x} \cdot e^{-x} + \left(\dfrac{-2}{3}\right) e^{3x} \cdot e^{-2x}$

$= e^{x} - \dfrac{2}{3} e^{x}$

$\Rightarrow \dfrac{e^{x}}{3}$

$\therefore y = y_c + y_p$

$\Rightarrow \boxed{y = c_1 e^{-x} + c_2 e^{-2x} + \dfrac{e^{x}}{3}}$

Ans.

③ $y'' + y = \operatorname{cosec} x$

Solⁿ

$m^2 + 1 = 0 \qquad m = \pm i$

$$\boxed{y_c = c_1 \cos x + c_2 \sin x}$$

$\therefore y_1(x) = \cos x \;\;\&\;\; y_2(x) = \sin x$

$$\therefore W(x) = \begin{vmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{vmatrix}$$

$\Rightarrow \cos^2 x + \sin^2 x \neq 0$

$\Rightarrow 1$

$\therefore$ Sol^n are Linearly ind.

Now.
Let us assume particular sol^n as

$y_p = A(x)\, y_1(x) + B(x)\, y_2(x)$ — ②

$= A(x) \cos x + B(x) \sin x$

$$g(x) = \frac{g(x)}{a_0(x)} = \frac{\operatorname{cosec} x}{1}$$

$= \operatorname{cosec} x$

$$A(x) = -\int \frac{g(x) \cdot y_2(x)}{W(x)} = -\int \frac{\operatorname{cosec} x \cdot \sin x}{1}\, dx \qquad \left| \frac{1}{\operatorname{cosec} x} = \sin x \right.$$

$= -\int 1\, dx = -x$

$$B(x) = \int \frac{g(x)\, y_1(x)}{W(x)} = \int \frac{\operatorname{cosec} x \cdot \cos x}{1}\, dx \qquad \left| \frac{\cos x}{\sin x} \right.$$

$$= \int \frac{\cos x}{\sin x}\, dx = \int \cot x\, dx$$

$= \log \sin x$

$\therefore$ GS $= y_c + y_p$

$= c_1 \cos x + c_2 \sin x + (-x) \cos x + \log(\sin x) \cdot \sin x$

Ans

Que. Given that $y_1(x) = x$, $y_2(x) = 1/x$ are two Linearly independent solutions of $x^2y'' + xy' - y = x$, $x \neq 0$ find general sol^n.

Sol^n. A/c $y_1(x) = x$ & $y_2(x) = 1/x$

$\Rightarrow \boxed{y_c = c_1 x + c_2 \cdot \frac{1}{x}}$

$$W(x) = \begin{vmatrix} x & 1/x \\ 1 & -1/x^2 \end{vmatrix} = \frac{-1}{x} - \frac{1}{x}$$

$$= \frac{-2}{x} \neq 0$$

$\therefore$ Sol^n are Linearly ind.

Now: Let us assume $y_p = A(x)y_1(x) + B(x)y_2(x)$

$\therefore g(x) = \frac{x}{x^2} = 1/x$

$$A(x) = +\int \frac{\frac{1}{x} \cdot \frac{1}{x}}{\frac{+2}{x}} = \frac{1}{2}\int \frac{1}{x}$$

$$= \frac{1}{2} \log x$$

$$B(x) = -\int \frac{\frac{1}{x} \cdot x}{\frac{-2}{x}} dx = -\frac{1}{2}\int x \, dx$$

$$= \frac{-1}{2} \frac{x^2}{2} = \frac{-x^2}{4}$$

$\therefore y_p = \frac{1}{2} \log x \cdot x + \left(\frac{-x^2}{4}\right) \cdot \frac{1}{x}$

$\therefore$ GS $= y_c + y_p$

$$= c_1 x + c_2 \cdot \frac{1}{x} + \frac{1}{2} x \log x - \frac{x}{4}$$

Ans.

Questions Practise.

Sol^n

① $x^2y'' + xy' - 4y = x^2\log(x)$; $y_1(x) = x^2$, $y_2(x) = 1/x^2$

① $c_1x^2 + \frac{c_2}{x^2} + \frac{x^2[\log(x)]^2}{8} - \frac{x^2\log(x)}{16} + \frac{x^2}{64}$

② $y'' + 4y' + 4y = e^{-2x}\sin x$

③ $(c_1 + c_2x)e^{-3x} + xe^{-3x}(\log(x) - 1)$

③ $y'' + 6y' + 9y = \frac{e^{-3x}}{x}$

② $(c_1 + c_2x)e^{-2x} - e^{-2x}\sin x$

④ $y'' + a^2y = \tan ax$

④ $c_1\cos ax + c_2\sin ax - \frac{1}{a^2}[\cos ax \cdot \log(\sec ax + \tan ax)]$

⑤ $y'' + 4y = \cot 2x$

⑤ $c_1\cos 2x + c_2\sin 2x + \frac{1}{4}\log\tan x$

⑥ $y'' - 4y' + 3y = \frac{e^x}{1+e^x}$

⑥ $c_1e^x + c_2e^{3x} + \frac{1}{2}(e^x - e^{3x})\log(1+e^{-x}) + \frac{1}{2}e^{2x}$

Que ① $y'' + a^2 y = \tan ax$ $\boxed{CA}$

* M.O variation of parameters

Sol^n $D^2 y + a^2 y = \tan ax$

$\Rightarrow m^2 + a^2 = 0$

$\Rightarrow \boxed{m = \pm a i}$

$\therefore \boxed{y_c = c_1 \cos ax + c_2 \sin ax}$

$\Rightarrow y_1(x) = \cos ax$ & $y_2(x) = \sin ax$

Now.
$$W(x) = \begin{vmatrix} \cos ax & \sin ax \\ -a\sin ax & a\cos ax \end{vmatrix}$$

$$= a\cos^2 ax + a\sin^2 ax = a \neq 0$$

$\therefore$ solutions are linearly independent

Now, let us assume $\boxed{y_p = A(x) y_1(x) + B(x) y_2(x)}$

$\Rightarrow y_p = A(x)\cos ax + B(x)\sin ax$

$\therefore$ finding $A(x)$ & $B(x)$ as :-

$$g(x) = \frac{R(x)}{a_1(x)} = \frac{\tan ax}{1} = \tan ax.$$

$$A(x) = -\int \frac{g(x)\, y_2(x)}{W(x)} = -\int \frac{\tan ax \cdot \sin ax}{a} = \frac{-1}{a}\int \frac{\sin ax \cdot \sin ax}{\cos ax}$$

$$= \frac{-1}{a}\int \left[\frac{1-\cos^2 ax}{\cos ax}\right] = \frac{-1}{a}\int \left[\sec x - \cos ax\right]$$

$$= \frac{-1}{a}\left[\frac{\log(\sec ax + \tan ax)}{a} - \frac{\sin ax}{a}\right]$$

$$B(x) = \int \frac{g(x)\, y_1(x)}{W(x)} = \int \frac{\tan ax \cdot \cos ax}{a} = \frac{1}{a}\int [\sin ax]$$

$$= \frac{-1}{a^2}\cos ax$$

$\therefore$
$$y_p = \frac{-\cos ax}{a^2}\left[\log(\sec ax + \tan ax) - \sin ax\right] - \frac{\sin ax}{a^2}[\cos ax]$$

$\therefore$
$$\boxed{y = y_c + y_p = c_1 \cos ax + c_2 \sin ax - \frac{\cos ax}{a^2}\left[\log(\sec ax + \tan ax) - \sin ax\right] - \frac{\sin ax}{a^2}[\cos ax]}$$

Ans.

Que ② $y'' + 4y' + 4y = e^{-2x}\sin x.$

Soln. $D^2y + 4Dy + 4y = e^{-2x}\sin x$

$\Rightarrow m^2 + 4m + 4 = 0$

$\Rightarrow (m+2)^2 = 0$ $\boxed{m = -2, -2}$

$\therefore y_c = (c_1 + c_2 x)e^{-2x}$

$= c_1 e^{-2x} + c_2 x e^{-2x}.$

$\therefore y_1 = e^{-2x}$ & $y_2 = x e^{-2x}$

$\therefore W(x) = \begin{vmatrix} e^{-2x} & x e^{-2x} \\ -2e^{-2x} & -2x e^{-2x} + e^{-2x} \end{vmatrix}$

$= -2x e^{-4x} + e^{-4x} + 2x e^{-4x}$

$= e^{-4x} \neq 0$

$\therefore$ soln. are Linearly independent.

Now. Let us assume $y_p = A(x) y_1(x) + B(x) y_2(x)$

$\therefore \boxed{y_p = A(x) e^{-2x} + B(x) . x . e^{-2x}}$

$g(x) = \frac{h(x)}{a_0(x)} = e^{-2x}\sin x$

$A(x) = -\int \frac{e^{-2x}\sin x . x . e^{-2x}}{e^{-4x}} = -\int x . \sin x$

$= +x\cos x - \sin x.$

$B(x) = \int \frac{e^{-2x}\sin x . e^{-2x}}{e^{-4x}} = \int \sin x = -\cos x.$

$\therefore y_p = e^{-2x}[-\sin x + x\cos x] + x . e^{-2x}[-\cos x]$

$= -e^{-2x}[\sin x]$

$\therefore \boxed{y = y_c + y_p}$

$= (c_1 + c_2 x)e^{-2x} - e^{-2x}[\sin x]$

Ans.

Que. ③ $y'' + 6y' + 9y = \frac{e^{-3x}}{x}$

Soln. $D^2y + 6Dy + 9y = \frac{e^{-3x}}{x}$

$\Rightarrow m^2 + 6m + 9 = 0$

$\Rightarrow (m+3)^2 = 0 \Rightarrow \boxed{m_1 = m_2 = -3}$

$\therefore y_c = c_1 e^{-3x} + c_2 x e^{-3x}$

Let $y_1(x) = e^{-3x}$ & $y_2(x) = x e^{-3x}$

$\therefore W(x) = \begin{vmatrix} e^{-3x} & x e^{-3x} \\ -3e^{-3x} & -3xe^{-3x} + e^{-3x} \end{vmatrix}$

$= -3e^{-6x} + e^{-6x} + 3xe^{-6x}$

$= e^{-6x} \neq 0$

$\therefore$ soln. are Linearly ind.

Now, Let us assume

$\boxed{y_p = A(x)y_1(x) + B(x)y_2(x)}$

Now. $g(x) = \frac{a(x)}{a_0(x)} = \frac{e^{-3x}}{x}$

$A(x) = -\int \frac{g(x) \cdot y_2(x)}{W(x)} = -\int \frac{e^{-3x}}{x} \cdot \frac{x \cdot e^{-3x}}{e^{-6x}} = -\int 1$

$= -x$

$B(x) = \int \frac{g(x) y_1(x)}{W(x)} = \int \frac{e^{-3x}}{x} \cdot \frac{e^{-3x}}{e^{-6x}} = \int \frac{1}{x}$

$= \log x$

$\therefore y_p = -x e^{-3x} + \log x \cdot x e^{-3x}$

$\therefore y = y_c + y_p$

$= (c_1 + c_2 x) e^{-3x} + x e^{-3x} [-1 + \log x]$

Ans.

Method of undetermined coefficients. / (trial sol)

It is an another method to find out the particular integral of $f(D)y = X$

We can apply this method if X contains the terms as

① $\boxed{x^n \;/\; a^n x^n \;/\; a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n} \rightarrow A_0 + A_1 x + \cdots A_n x^n$

eg. ① $x^2 \Rightarrow A_0 + A_1 x + A_2 x^2$

② $x^3 \Rightarrow A_0 + A_1 x + A_2 x^2 + A_3 x^3$

② e^{ax} or $pe^{ax} \rightarrow Ae^{ax}$

③ $a_n x^n e^{ax}$ or $e^{ax}(a_0 + a_1 x + \cdots + a_n x^n) \rightarrow e^{ax}(A_0 + A_1 x + \cdots A_n x^n)$

④ $p\sin ax$ or $q\cos ax$ or $p\sin ax + q\cos ax \rightarrow A\sin ax + B\cos ax$

⑤ $pe^{bx}\sin ax$ or $pe^{bx}\cos ax$ or $e^{bx}(p\sin ax + q\cos ax) \rightarrow e^{bx}(A\sin ax + B\cos ax)$

⑥ $x^n \sin ax \;/\; a_n x^n \sin ax \;/\; (a_0 + \cdots a_n x^n)\sin ax \rightarrow (A_0 + A_1 x + \cdots A_n x^n)\sin ax$

or

$x^n \cos ax \;/\; a_n x^n \cos ax$ or $(a_0 + a_1 x + \cdots a_n x^n)\cos ax.$ $+ (A_0' + A_1' x + \cdots A_n' x^n)\cos ax$

Eg. $y'' + 4y = x^2$

Soln. $D^2y + 4 = x^2$ — ①

$\Rightarrow m^2 + 4 = 0 \Rightarrow m = \pm 2i$

$\boxed{y_c = c_1 \cos 2x + c_2 \sin 2x}$

Let the trial solution $(y^*) = A_0 + A_1 x + A_2 x^2$

Now, y^* must satisfy eqn ①

$\Rightarrow (D^2 + 4) y^* = x^2$

$\Rightarrow D^2 y^* + 4y^* = x^2$

$\Rightarrow D^2(A_0 + A_1 x + A_2 x^2) + 4(A_0 + A_1 x + A_2 x^2) = x^2$

$\Rightarrow 2A_2 + 4A_0 + 4A_1 x + 4A_2 x^2 = x^2$

On comparing like terms, we get

$4A_2 = 1 \Rightarrow \boxed{A_2 = 1/4}$

$2A_2 + 4A_0 = 0$, $\boxed{4A_1 = 0}$

$\Rightarrow \boxed{A_0 = -1/8}$

$\therefore y^* = \frac{-1}{8} + 0 + \frac{1}{4}x^2$

$= \frac{1}{8}(2x^2 - 1)$

$\therefore$ G.S = C.F + P.I $(y_c + y_p)$

$= c_1 \cos 2x + c_2 \sin 2x + \frac{1}{8}(2x^2 - 1)$

Ans.

Que. $y'' + 2y' + y = x - e^x$ — ①

Soln. $D^2y + 2Dy + y = x - e^x$

$\Rightarrow m^2 + 2m + 1 = 0$

$\Rightarrow (m+1)^2 = 0$

$\therefore \boxed{y_c = (c_1 + c_2 x) e^{-x}}$

Let trial soln $y^* = A_0 + A_1 x + A_2 e^x$

Now y^* must satisfy eqn ①

$\Rightarrow f(D) y^* = x$

$\therefore (D^2 + 2D + 1) y^* = x - e^x$

$\Rightarrow D^2[A_0 + A_1 x + A_2 e^x]$
$+ 2D[A_0 + A_1 x + A_2 e^x] = x - e^x$
$+ A_0 + A_1 x + A_2 e^x$

$\Rightarrow A_2 e^x + 2A_1 + 2A_2 e^x + A_0 + A_1 x + A_2 e^x$
$= x - e^x$

$e^x[A_2 + 2A_2 + A_2]$
$+ x[A_1] + A_0 + 2A_1 = x - e^x$

on comparing like terms we get

$4A_2 = -1 \Rightarrow \boxed{A_2 = -1/4}$ $\boxed{A_1 = 1}$

$A_0 + 2A_1 = 0$

$\Rightarrow A_0 + 2 = 0 \Rightarrow \boxed{A_0 = -2}$

$\therefore y^* = -2 + x - \frac{1}{4} e^x$

$\therefore$ GS $= y_c + y_p$

$= (c_1 + c_2 x) e^{-x}$
$- 2 + x - \frac{1}{4} e^x$

Ans.

Ans: $(D^2-2D+3)y = x^3 + \sin x$ — ①

Sol^n. $D^2y - 2Dy + 3 = 0$

$\Rightarrow m^2 - 2m + 3 = 0$

$\Rightarrow \boxed{m = 1 \pm i\sqrt{3}}$

$\therefore y_c = e^x[c_1\cos\sqrt{3}x + c_2\sin\sqrt{3}x]$

Let us assume trial sol^n

$y^* = A_0 + A_1x + A_2x^2 + A_3x^3 + A_4\cos x + A_5\sin x$

Now y^* must satisfy eq^n ①.

$\therefore f(D)y^* = X = x^3 + \sin x$

$\Rightarrow (D^2-2D+3)y^* = x^3 + \sin x$

$\Rightarrow (D^2-2D+3)[A_0 + A_1x + A_2x^2 + A_3x^3 + A_4\cos x + A_5\sin x] = x^3 + \sin x$

$\Rightarrow 2A_2 + 6A_3x - A_4\cos x - A_5\sin x$
$- 2A_1 - 4A_2x - 6A_3x^2 + 2A_4\sin x - 2A_5\cos x$
$+ 3A_0 + 3A_1x + 3A_2x^2 + 3A_3x^3 + 3A_4\cos x + 3A_5\sin x = x^3 + \sin x$

On comparing like terms we get.

$3A_3 = 1$; $\boxed{6A_3 + 3A_1 - 4A_2 = 0}$

$\Rightarrow \boxed{A_3 = \frac{1}{3}}$ $\boxed{3A_4 - 2A_5 - A_4 = 0}$

$\boxed{3A_2 - 6A_3 = 0}$ $\boxed{2A_2 - 2A_1 + 3A_0 = 0}$

$\boxed{3A_5 - A_5 + 2A_4 = 1}$

$\Rightarrow \boxed{A_0 = \frac{-8}{27}}$, $\boxed{A_1 = \frac{2}{9}}$, $\boxed{A_2 = 2/3}$

$\boxed{A_3 = 1/3}$ $\boxed{A_4 = A_5 = 1/4}$

$\therefore y^* = P.I \Rightarrow \frac{-8}{27} + \frac{2}{9}x + \frac{2}{3}x^2 + \frac{1}{3}x^3 + \frac{1}{4}(\sin x + \cos x)$

$\therefore G.S = y_c + y_p$

$= e^x[c_1\cos\sqrt{3}x + c_2\sin\sqrt{3}x]$
$\frac{-8}{27} + \frac{2}{9}x + \frac{2}{3}x^2 + \frac{1}{3}x^3$
$+ \frac{1}{4}(\sin x + \cos x)$

Questions — Practise

Q ① $(D^2-4D+4)y = x^3e^{2x} + xe^{2x} \rightarrow (c_1+c_2x)e^{2x} + \frac{1}{20}x^5e^{2x} + \frac{1}{6}x^3e^{2x}$

② $(D^2+4)y = x^2\sin 2x \rightarrow c_1\cos 2x + c_2\sin 2x - \frac{1}{12}x^3\cos 2x + \frac{1}{16}x^2\sin 2x + \frac{1}{32}x\cos 2x$

③ $y'' - 6y' + 9y = x^2e^{3x} \rightarrow (c_1+c_2x)e^{3x} + \frac{x^4}{12}e^{3x}$

④ $y'' - 2y' + y = x^2e^x \rightarrow (c_1+c_2x)e^x + \frac{x^4}{12}e^x$

⑤ $y''' - 3y'' + 2y' = x^2e^{3x} \rightarrow c_1 + c_2e^x + c_3e^{2x} - 2xe^x - \frac{x^3}{3}e^x$

⑥ $y''' - 3y'' + 2y' = 3xe^{2x} + 5x^2 \rightarrow c_1 + c_2e^x + c_3e^{2x} + \frac{5}{6}x^3 + \frac{15}{4}x^2 + \frac{35}{4}x + \frac{3}{4}e^{2x}(x^2-3x)$

⑦ $y'' + y = \sin x$

$c_1\cos x + c_2\sin x - \frac{x}{2}\cos x$

⑧ $y'' - y = e^x\sin 2x$

$c_1e^x + c_2e^{-x} - e^x(\sin 2x + \cos 2x)/8$

Q21. $y'' + 4y = 4e^{2x}$ a.) Ae^{x} b.) Ae^{2x} c.) e^{2x} d.) $4e^{2x}$

Q22. $y'' + 10y' + 25y = e^{-5x}$ a.) Be^{-5x} b.) Ae^{5x} c.) xe^{-5x} d.) NOT

Q23. $y'' - 2y' + y = e^{x} + x$. a.) $Ae^{x} + Bx + C$ b.) $Axe^{x} + Bx + C$

c.) $x(Ae^{x} + Bx + C)$ d.) NOT.

Q24. $y'' + a^2y = \tan(ax)$?: a.) M. of und. coeff. b.) Operator method

c.) V. of parameters d.) ~~NOT~~

d.) All of the above

Cauchy - Euler - Equations.

Working rule

Step ① put $x = e^z$ or $z = \log x$; $x > 0$

Step ② Assume that $D_1 = \frac{\partial}{\partial z}$ & $D = \frac{\partial}{\partial x}$ then.

$xD = D_1$, $x^2D^2 = D_1(D_1 - 1)$; $x^3D^3 = D_1(D_1-1)(D_1-2)$ & so on.

reduced to $f(D_1)y = z$

Step ③ Now, solve by operator method according to z-fxn or method of undeterminant coefficients and method of variation of parameters.

Eg ① Solve $x^2y'' + xy' - 4y = 0$

Solⁿ. $(x^2D^2 + xD - 4)y = 0$ — ①

Let $z = \log x$ & $xD = D_1$; $x^2D^2 = D_1(D_1 - 1)$

So, eqⁿ ① becomes.

$(D_1(D_1 - 1) + D_1 - 4)y = 0$

$\Rightarrow (D_1^2 - D_1 + D_1 - 4)y = 0$

$\Rightarrow (D_1^2 - 4)y = 0$

$\boxed{D_1 = \pm 2}$

$\therefore y = c_1 e^{2z} + c_2 e^{-2z}$

$\Rightarrow c_1 e^{2\log x} + c_2 e^{-2\log x}$

$\Rightarrow c_1 e^{\log x^2} + c_2 e^{\log x^{-2}}$

$\boxed{y \Rightarrow c_1 x^2 + c_2 x^{-2}}$

Ans.

Q2. $x^2y'' - 3xy' + 4y = 0$

$[D_1(D_1 - 1) - 3D_1 + 4]y = 0$

$\Rightarrow (D_1 - 2)^2 = 0$

$\boxed{D_1 = 2, 2}$

$y_c = (c_1 + c_2 z) e^{2z}$

$= (c_1 + c_2 \log x) e^{2\log x}$

$= (c_1 + c_2 \log x) x^2$

Ans.

Q3. $x^3 y''' + 2x^2 y'' + 3x y' - 3y = 0$

Solⁿ $z = \log x$

$D_1(D_1-1)(D_1-2) + 2D_1(D_1-1) + 3D_1 - 3 = 0$

$\Rightarrow (D_1 - 1)(D_1^2 + 3) = 0$

$y = c_1 e^z + c_2 \sin(\sqrt{3} z) + c_3 \cos(\sqrt{3} z)$

$\Rightarrow y = c_1 e^{\log x} + c_2 \sin(\sqrt{3} \log x) + c_3 \cos(\sqrt{3} \log x)$

$\Rightarrow y = c_1 x + c_2 \sin(\sqrt{3} \log x) + c_3 \cos(\sqrt{3} \log x)$

Ans.

Q4. $(x^3 D^3 + 3x^2 D^2 - 2x D + 2) y = 0$

Let $z = \log x$

$D_1(D_1-1)(D_1-2) + 3D_1(D_1-1) - 2D_1 + 2 = 0$

$\Rightarrow (D_1 - 1)(D_1 - 1)(D_1 + 2) = 0$

$\Rightarrow y = (c_1 + c_2 z) e^z + c_3 (e^z)^{-2}$

$= (c_1 + c_2 \log x) e^{\log x} + c_3 (e^{\log x})^{-2}$

$$\Rightarrow \boxed{y = (c_1 + c_2 \log x)\, x + c_3 x^{-2}}$$

Ans.

Ques. $x^2y'' + y = 3x^2$

$D_1(D_1-1) + 1 = 3x^2$

$\Rightarrow D_1^2 - D_1 + 1 = 0$

$\Rightarrow D_1 = \dfrac{1 \pm \sqrt{3}i}{2}$

$CF = e^{z/2}\left[c_1 \cos \dfrac{z\sqrt{3}}{2} + c_2 \sin \dfrac{z\sqrt{3}}{2}\right]$

$= e^{\log x/2}\left[c_1 \cos \dfrac{\sqrt{3}}{2}\log x + c_2 \sin \dfrac{\sqrt{3}}{2}\log x\right]$

$P.I = \dfrac{1}{D_1^2 - D_1 + 1} 3e^{2z} = \dfrac{3}{4-2+1} e^{2z} = e^{2z}$

$(D_1 \to 2)$

$= e^{2\log x}$

$= x^2$

$x = e^z$

$x^2 = (e^z)^2$

$= e^{2z}$

$\therefore y = e^{\log x/2}\left[c_1 \cos \dfrac{\sqrt{3}}{2}\log x + c_2 \sin \dfrac{\sqrt{3}}{2}\log x\right]$

$+ x^2$

Ans.

Practise Questions.

① $x^2y'' + 2xy' = \log x \rightarrow c_1 + \frac{c_2}{x} + \frac{1}{2}[\log(x)]^2 - \log x$

② $(x^2D^2 + 7xD + 13)y = \log x \rightarrow x^{-3}[c_1\cos(2\log x) + c_2\sin(2\log x)] + \frac{1}{169}(13\log x - 6)$

③ $x^3y''' + 3x^2y'' + xy' + y = \log x + x \rightarrow \frac{c_1}{x} + x^{1/2}[c_2\cos(\frac{\sqrt{3}}{2}\log x) + c_3\sin(\frac{\sqrt{3}}{2}\log x)$

④ $(x^2D^2 - 3xD + 5)y = \sin(\log x)$ $x^2[c_1\cos(\log x) + c_2\sin(\log x)] + \frac{x}{2} + \log x + \frac{1}{8}[\sin(\log x) + \cos(\log x)]$

⑤ $[x^2D^2 - (2m-1)xD + (m^2+n^2)]y = n^2x^m\log x \rightarrow x^m[c_1\cos(n\log x) + c_2\sin(n\log x)] + x^m\log x$

⑥ $x^2y'' - 2xy' + 2y = x + x^2\log x + x^3 \rightarrow c_1x + c_2x^2 - x\log x + \frac{x^3}{2} + (x^2/2)[(\log x)^2 - 2\log x]$

⑦ $x^2D^2y - 3xDy + 5y = x^2\sin\log x$ $x^2[c_1\cos\log x + c_2\sin\log x) - \frac{x^2}{2}\log x\cos\log x.$

Ques. ① $x^2y'' - 3xy' + 5y = x^2 \sin(\log x) + x$ —①

Sol^n Let $x = e^z$ or $z = \log x$

& $xD = D_1$; $x^2D^2 = D_1(D_1 - 1)$

$\therefore$ Eq^n ① Becomes.

$D_1(D_1-1) - 3D_1 + 5 = 0$

$\Rightarrow D_1^2 - D_1 - 3D_1 + 5 = 0$

$\Rightarrow \boxed{D_1^2 - 4D_1 + 5 = 0}$

$\therefore \boxed{D_1 = 2 \pm i}$

$\therefore y_c = e^{2z}[c_1 \cos z + c_2 \sin z]$

$= x^2 [c_1 \cos(\log x) + c_2 \sin \log(x)]$

$\therefore y_p = \dfrac{1}{D_1^2 - 4D_1 + 5}[e^{2z}\sin(z) + e^z]$

$\Rightarrow \dfrac{e^{2z}\sin(z)}{D_1^2 - 4D_1 + 5} + \dfrac{1}{D_1^2 - 4D_1 + 5}e^z$

$D \to D + a \to D + 2$

$D \to a \to 1$

$\Rightarrow \dfrac{1}{(D_1+2)^2 - 4(D_1+2) + 5} e^{2z}\sin(z) + \dfrac{1}{D_1^2 - 4D_1 + 5}e^z$

$\Rightarrow \dfrac{e^{2z}}{D_1^2 + 4 + 4D_1 - 4D_1 - 8 + 5}\sin(z) + \dfrac{e^z}{1 - 4 + 5}$

$\Rightarrow \dfrac{e^{2z}}{D_1^2 + 1}\sin(z) + \dfrac{e^z}{2}$

$\Rightarrow \boxed{D_1^2 = -a^2 = -1}$

$\dfrac{e^{2z}}{-1+1}\sin z + \dfrac{e^z}{2}$

$\Rightarrow \dfrac{z \cdot e^{2z}\sin z}{2D_1} + \dfrac{e^z}{2}$

$\Rightarrow -\dfrac{z \cdot e^{2z}}{2}\cos z + \dfrac{e^z}{2} \Rightarrow \dfrac{-\log x \cdot x^2 \cos(\log x)}{2} + x/2$

$\therefore \boxed{y = y_c + y_p}$

$= x^2[c_1 \cos(\log x) + c_2 \sin(\log x)] - \dfrac{x^2}{2}\log x \cos(\log x) + x/2$

Que. ① $x^3y''' + 3x^2y'' + xy' + y = \log x + x^2 + \sin(\log x)$

Sol^n. $x^3D^3y + 3x^2D^2y + xDy + y = \log x + x^2 + \sin(\log x)$ — ①

Let assume that $z = \log x$ | $x = e^z$

∴ By cauchy euler's theorem, eq^n ① becomes

$D_1(D_1-1)(D_1-2) + 3D_1(D_1-1) + D_1 + 1 = 0$

$D_1 = \frac{-b \pm \sqrt{4ac}}{2a}$

$\Rightarrow (D_1-1)[D_1^2 - 2D_1 + 3D_1] + D_1 + 1 = 0$

$\Rightarrow (D_1-1)[D_1^2 + D_1] + D_1 + 1 = 0$

$\Rightarrow D_1(D_1-1)(D_1+1) + D_1 + 1 = 0$

$D_1 = \frac{1 \pm \sqrt{1-4}}{2}$

$\frac{1 \pm \sqrt{3}i}{2}$

$\Rightarrow D_1^3 - D_1 + D_1 + 1 = 0$

$\Rightarrow D_1^3 + 1 = 0$

$\Rightarrow (D_1+1)(D_1^2 - D_1 + 1) = 0$

$\Rightarrow$

$\therefore y_c = c_1e^{-z} + e^{z/2}[c_2\cos\frac{\sqrt{3}}{2}(z) + c_3\sin\frac{\sqrt{3}}{2}(z)]$

$= \frac{c_1}{x} + x^{1/2}[c_2\cos\frac{\sqrt{3}}{2}\log x + c_3\sin\frac{\sqrt{3}}{2}\log x]$

Now.

$$y_p \Rightarrow \frac{z + e^{2z} + \sin z}{D_1^3 + 1}$$

$$\Rightarrow \frac{z}{D_1^3+1} + \frac{1}{D_1^3+1} e^{2z} + \frac{1}{D_1^3+1} \sin z$$

$(D \to a \to 2)$ $\quad D_1^2 \to -a^2 = -1$

$$\Rightarrow [1 + (D_1^3)]^{-1} (z) + \frac{1}{8+1} e^{2z} + \frac{1}{-D_1+1} \sin z$$

$$\Rightarrow [1 - (D_1^3) + (D_1^3)^2 \ldots] z + \frac{e^{2z}}{9} + \frac{1}{(1-D_1)} \times \frac{(1+D_1)}{(1+D_1)} \sin z$$

$$\frac{1+D_1}{1-D_1^2} \sin z \quad (D_1^2 = -1)$$

$$\frac{1+D_1}{2} \sin z$$

$$\Rightarrow \frac{\sin z}{2} + \frac{\cos z}{2}$$

$$\Rightarrow z + \frac{e^{2z}}{9} + \frac{\sin z}{2} + \frac{\cos z}{2}$$

$$y_p \Rightarrow \log x + \frac{x^2}{9} + \frac{\sin \log(x)}{2} + \frac{\cos \log(x)}{2}$$

$$\therefore \quad y = y_c + y_p$$

$$\Rightarrow \boxed{y = \frac{c_1}{x} + x^{1/2}\left[c_2 \cos \frac{\sqrt{3}}{2} \log x + c_3 \sin \frac{\sqrt{3}}{2} \log x\right] + \log x + \frac{x^2}{9} + \frac{\sin \log(x)}{2} + \frac{\cos \log(x)}{2}}$$

Ans

Que $x^2y'' + xy' - 4y = x^2\log(x)$ CA PYQ

Sol^n. $z = \log x \quad ; x = e^z$

$D_1(D_1-1) + D_1 - 4 = 0$

$\Rightarrow D_1^2 - D_1 + D_1 - 4 = 0$

$\Rightarrow D_1^2 - 4 = 0$

$\Rightarrow \boxed{D_1 = \pm 2}$

$\therefore y_c = c_1 e^{2z} + c_2 e^{-2z}$

$\boxed{y_c = c_1 x^2 + c_2 x^{-2}}$

$y_p = \frac{1}{D_1^2 - 4} \cdot e^{2z} \cdot z$

$\Rightarrow D_1 \Rightarrow D_1 + a \Rightarrow D_1 + 2$

$\frac{1}{D_1^2 + 4 + 4D_1 - 4} e^{2z} \cdot z$

$\Rightarrow \frac{e^{2z}}{D_1^2 + 4D_1} \cdot z$

$\frac{e^{2z}}{D_1^2 + 4D_1} \cdot z \Rightarrow \frac{e^{2z}}{D_1^2 + 4D_1 + 1 - 1} \cdot z$

$\Rightarrow e^{2z}\left[1 + (D_1^2 + 4D_1 - 1)\right]^{-1} z$

$\Rightarrow e^{2z}\left[1 - (D_1^2 + 4D_1 - 1) + (D_1^2 + 4D_1 - 1)^2 \cdots\right] z$

$\Rightarrow e^{2z}\left[z - (D_1^2 + 4D_1 - 1)z + (D_1^2 + 4D_1 - 1)^2 z\right]$

$\Rightarrow e^{2z}\left[z - (0 + 4 - z) + [0 + 0 + 1 - 8D_1 + 0] z\right]$

$\Rightarrow e^{2z}\left[z - 4 + z + z - 8\right]$

$\Rightarrow e^{2z}\left[3z - 12\right]$

$3e^{2z}[z - 4] = y_p \Rightarrow 3x^2[\log x - 4]$

$\therefore y = y_c + y_p$

$\Rightarrow y = c_1 x^2 + c_2 x^{-2} + 3x^2[\log x - 4]$

Ans.

Ques ③ $x^3y''' + 3x^2y'' + xy' + y = \log x + x^2$

Ques ④ $x^2y'' - 2xy' + 2y = x + x^2\log x + x^3$

Soln. $x^2D^2y - 2xDy + 2y = x + x^2\log x + x^3$ —①

Let assume that $z = \log x$ / $x = e^z$

By Cauchy Euler's theorem, eqn ① becomes

$D_1(D_1 - 1) - 2D_1 + 2 = 0$

$\Rightarrow D_1^2 - D_1 - 2D_1 + 2 = 0$

$\Rightarrow D_1^2 - 3D_1 + 2 = 0$

$\Rightarrow D_1^2 - 2D_1 - D_1 + 2 = 0$

$\Rightarrow D_1(D_1 - 2) - 1(D_1 - 2) = 0$

$\Rightarrow \boxed{D_1 = 2 \ ; \ D_1 = 1}$

$$\boxed{y_c = c_1e^{2z} + c_2e^z}$$

$$y_p \Rightarrow \frac{e^z + e^{2z}\cdot z + e^{3z}}{D_1^2 - 3D_1 + 2}$$

Rough.

$$\frac{e^{2z}}{D_1^2 + D_1 + 1 - 1}\cdot z$$

$= e^{2z}[1 + (D_1^2 + D_1 - 1)]^{-1} z$

$\Rightarrow e^{2z}[z + (-D_1^2 - D_1 + 1)z + (0 + 0 + 1 + 0 - 2D_1)z.$

$\Rightarrow e^{2z}[z + z - 1 - 2 + z]$

$e^{2z}[3z - 3]$

$\boxed{3e^{2z}(z - 1)}$

$$\frac{e^z}{D_1^2 - 3D_1 + 2} + \frac{e^{2z}\cdot z}{D_1^2 - 3D_1 + 2} + \frac{e^{3z}}{D_1^2 - 3D_1 + 2}$$

$(D \to 1)$ $(D \to D+2)$ $(D \to 3)$

$$\frac{e^z}{1 - 3 + 2} + \frac{e^{2z}\cdot z}{(D_1+2)^2 - 3(D_1+2) + 2} + \frac{e^{3z}}{9 - 9 + 2}$$

$$\Rightarrow \frac{e^z}{0} + \frac{e^{2z}}{D_1^2 + 4 + 4D_1 - 3D_1 - 6 + 2} + \frac{e^{3z}}{2}$$

$$\Rightarrow \frac{z\cdot e^z}{2D_1 - 3} + \frac{e^{2z}}{D_1^2 + D_1}\cdot z + \frac{e^{3z}}{2}$$

$$\Rightarrow \frac{z\cdot e^z}{-1} + 3e^{2z}(z - 1) + \frac{e^{3z}}{2}$$

$\Rightarrow -z\cdot e^z + 3e^{2z}(z - 1) + \frac{e^{3z}}{2}$

$\Rightarrow -\log x\cdot x + 3x^2(\log x - 1) + x^3/2$

$\therefore y = y_c + y_p$

$\Rightarrow y = c_1x^2 + c_2x + \frac{x^3}{2} + 3x^2(\log x - 1) - x\log x$

Ans