Tenth International Olympiad, 1968

1968/1.

Prove that there is one and only triangle whose side lengths are consecutive integers, and one of whose angles is twice as large as another.

1968/2.

Find all natural numbers x such that the product of their digits (in decimal notation) is equal to $x^2 - 10x - 22$.

1968/3.

Consider the system of equations:

$$ax_1^2 + bx_1 + c = x_2 \tag{1}$$

$$ax_2^2 + bx_2 + c = x_3 \tag{2}$$

$$\dots \tag{3}$$

$$ax_{n-1}^2 + bx_{n-1} + c = x_n \tag{4}$$

$$ax_n^2 + bx_n + c = x_1, \tag{5}$$

with unknowns $x_1, x_2, \dots, x_n$, where a, b, c are real and $a \neq 0$. Let $\triangle = (b-1)^2 - 4ac$. Prove that for this system

1. if $\triangle < 0$, there is no solution,
2. if $\triangle = 0$, there is exactly one solution,
3. if $\triangle > 0$, there is more than one solution.

1968/4.

Prove that in every tetrahedron there is a vertex such that the three edges meeting there have lengths which are the sides of a triangle.

1968/5.

Let f be a real-valued function defined for all real numbers x such that, for some positive constant a, the equation

$$f(x+a) = \frac{1}{2} + \sqrt{f(x) - [f(x)]^2} \tag{6}$$

holds for all x.

1. Prove that the function f is periodic (i.e., there exists a positive number b such that $f(x+b) = f(x)$ for all x).
2. For $a = 1$, give an example of a non-constant function with the required properties.

1968/6.

For every natural number n, evaluates the sum

$$\sum_{k=10}^{\infty} [\frac{n+2^k}{2^{k+1}}] = [\frac{n+1}{2}] + [\frac{n+2}{4}] + \ldots + [\frac{n+2^k}{2^{k+1}}] + \ldots \tag{7}$$

(The symbol $[x]$ denotes the greatest integer not exceeding x.)