Twelfth International Mathematical Olympiad, 1970

1970/1.

Let M be a point on the side AB of $\triangle ABC$. Let r_1, r_2, and r be the radii of the inscribed circles of triangles AMC, BMC, and ABC, respectively. Let q_1, q_2, and q be the radii of the escribed circles of the same triangles that lie in the angle ACB. Prove that

$$\frac{r_1}{q_1} \cdot \frac{r_2}{q_2} = \frac{r}{q}. \tag{1}$$

1970/2.

Let a, b, and n be integers greater than 1, and let a and b be the bases of two number systems. $A_n - 1$ and A_n are numbers in the system with base a, and $B_n - 1$ and B_n are numbers in the system with base b; these are related as follows:

$$A_n = x_n x_{n-1} \cdots x_0, A_{n-1} = x_{n-1} x_{n-2} \cdots x_0, \tag{2}$$

$$B_n = x_n x_{n-1} \cdots x_0, B_{n-1} = x_{n-1} x_{n-2} \cdots x_0, \tag{3}$$

$$x_n \neq 0, x_{n-1} \neq 0. \tag{4}$$

Prove that:

$$\frac{A_{n-1}}{A_n} < \frac{B_{n-1}}{B_n} \quad \text{if and only if} \quad a > b. \tag{5}$$

1970/3.

The real numbers $a_0, a_1, \ldots, a_n, \ldots$ satisfy the condition:

$$1 = a_0 \le a_1 \le a_2 \le \cdots \le a_n \le \cdots . \tag{6}$$

The numbers $b_1, b_2, \ldots, b_n, \ldots$ are defined by:

$$b_n = \sum_{k=1}^{n} \left(1 - \frac{a_{k-1}}{a_k}\right) \frac{1}{\sqrt{a_k}}. \tag{7}$$

1. Prove that $0 \le b_n < 2$ for all n.

2. Given c with $0 \le c < 2$, prove that there exist numbers $a_0, a_1, \ldots$ with the above properties such that $b_n > c$ for large enough n.

1970/4.

Find the set of all positive integers n such that the set $n, n+1, n+2, n+3, n+4, n+5$ can be partitioned into two sets such that the product of the numbers in one set equals the product of the numbers in the other set.

1970/5.

In the tetrahedron $ABCD$, angle BDC is a right angle. Suppose that the foot H of the perpendicular from D to the plane ABC is the intersection of the altitudes of ΔABC. Prove that:

$$(AB + BC + CA)^2 \le 6(AD^2 + BD^2 + CD^2). \tag{8}$$

For what tetrahedra does equality hold?

1970/6.

In a plane, there are 100 points, no three of which are collinear. Consider all possible triangles having these points as vertices. Prove that no more than 70% of these triangles are acute-angled.