REAL-LIFE APPLICATIONS OF ODES FOR UNDERGRADUATES

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Abstract. This study introduces real-life mathematical theories and models of international relationships suitable for undergraduate ordinary differential equations, by investigating conflicts between different nations or alliances. Based on the work of Richardson, systems of differential equations are constructed. The solutions and the stability of systems of ODEs are observed. One of the most interesting tasks is to analyse the coefficients in the constructed models.

1. INTRODUCTION

Ordinary differential equations (ODEs), especially systems of ODEs, have been applied in many fields such as physics, electronic engineering and population dynamics. This is a powerful tool for analysing the relationship between various dynamic quantities. As a real-life application in the teaching of ODE, the famous predator-prey model which describes the competition between several species of carnivore and vegetarian (see [4]) appears in most undergraduate textbooks. Another real-life application of ODE, which we suggest needs to be included in undergraduate textbooks, is the analysis of international relationships.

Artists often describe wars incisively and vividly in ways that impact on our senses. The Argentinian artist, Cándido López, painted "After the Battle of Curupaytí " with his left hand because he lost his right hand in the Battle of Curupaytí. Curupaytí was one of the battle fields of the Great War in La Plata (1865-1870), also known as the War of the Triple Alliance. It is ranked among the worst calamities in modern history fought between Paraguay, Uruguay, Argentina and Brazil. The painting is housed in the Museo Nacional de Bellas Artes in Buenos Aires. Figure 1 which was found on the Internet ([9]) depicts this great work of art.

The approach adopted in this paper is different from that of an artist in the sense that a scientific tool is used — mathematical modelling is applied in order to rationally analyse international relationships such as conflict or combat between different nations or alliances. It is a known fact that numerous factors affect the course or result of a war. Many nations endeavour to ease the strain of potential war or to fight more efficiently in order to lessen the suffering and possible economic loss caused by war. Many pioneers such as LF Richardson (see [12], [13], [14], [15]), FW Lanchester (see [8]), PM Morse (see [7]), GE Kimball (see [7]), provided mathematical models of conflicts or combats for military or commercial purposes.

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FIGURE 1. **"After the Battle of Curupaytí"**

FIGURE 2. Lewis Fry Richardson

In this paper we focus on the study of models describing international conflicts, which were originally derived by Lewis Fry Richardson, describing the relationship between two nations or two alliances that deem war to be imminent. For educational purposes, we provide mathematical history background. Lewis Fry Richardson (1881-1953, Figure 2, [15]) was one of the pioneers to use a mathematical model to analyse arms races and international relations. Richardson devoted half of his life to studying the mathematics of armed conflict in the hope that a mathematical understanding of the causes and dynamics of war would lead to an abatement of aggression.

He conducted an in-depth quantitative study of war and devoted half his life to the study of the mathematics of armed conflict. In his book, *Statistics of deadly quarrels* ([14]), he collected a vast amount of data about wars, ranging from World War I to the gang wars in Chicago. He published his achievements in war analysis, in the following works: *The mathematical psychology of war* in 1919, *Generalized foreign politics* ([12]), and *Arms and insecurity* ([13]).

In particular, he devised mathematical models of arms races using differential equations. One assumes that if one country increases its weapons, another country will do the same. Sequentially, the first country responds by storing more weapons. Richardson proposed that this kind of arms race can be represented by a pair of differential equations.

Richardson's model of international relations, which includes an arm race, used for discussing stability, is analogous to the differential equations in the predator-prey model. The most interesting work analyses or estimates the coefficients and stability of the system of differential equations. Some coefficients cannot be estimated, for example, measurement of the satisfaction or dissatisfaction of each nation.

In our opinion, the model first constructed by Richardson is an excellent application of ODEs (ordinary differential equations) and is usefull for practice for learning ODEs. One would expect this kind of model to be added to the material in textbooks as a typical example.

In the present paper, firstly, a simpler model of an arms race is depicted. More realistic models are constructed, and additional factors that influence the relationship between two nations or alliances, such as the cost of armaments, the grievances between nations and their ambitions, are considered. Examples and graphs analysis illustrating the solutions, trajectory and phase portrait of the system of ODEs are provided with the assistance of *Mathematica* and *Scientific Workplace*. We prefer to leave some basic work for students as exercises.

2. DIFFERENTIAL EQUATIONS FOR A SIMPLE ARMS RACE

It is a well-known fact that an increase in armaments is one of the primary reasons for war. Another reason is the unsolvable conflict of ambitions, such as occupying more territory or recovering tracts of land.

We assume that if one nation increases its armaments, then the opposing nation will do likewise because it assumes that the balance of power will be negatively affected. The result is fear. An arms race thus occurs.

Let $x(t)$ be the armaments of nation X, and $y(t)$ be the armaments of nation Y at time t. The rate of change of the armaments on one side depends on the number of armaments on the opposing side, because if one nation increases its armaments, the other will follow suit. That is, dx/dt (or dy/dt) is proportional to y (or x) . We assign constants of proportionally k and l to x and y, respectively, which represent the efficiency of increasing armaments.

Hence we can establish a system of differential equations in the following form:

$$\left\{\begin{array}{l}\dfrac{dx}{dt}=ky\\ \dfrac{dy}{dt}=lx\end{array}\right. . \tag{2.1}$$

This system can be used to describe the relationship between two nations or alliances, each of which decides to defend itself against possible attack by the other.

It is easy to obtain the solutions for the system (2.1) which we give as follows:

$$x(t)=\sqrt{\frac{k}{l}}\left(Ae^{t\sqrt{kl}}-Be^{-t\sqrt{kl}}\right),\ y(t)=Ae^{t\sqrt{kl}}+Be^{-t\sqrt{kl}} \tag{2.2}$$

Given the initial conditions,

$$x(0)=x_0,y(0)=y_0,$$

we can obtain

$$A=\frac{1}{2}\left(y_0+\sqrt{\frac{l}{k}}x_0\right),B=\frac{1}{2}\left(y_0-\sqrt{\frac{l}{k}}x_0\right) \tag{2.3}$$

It is possible to estimate the values of k and l. For example, when y remains a constant C, it follows from (2.1) that

$$\frac{1}{k}=\frac{C}{\frac{dx}{dt}}=C\frac{dt}{dx} \tag{2.4}$$

Solving (2.4), we obtain

$$\frac{1}{k}x=Ct+b \tag{2.5}$$

Assuming $x(0)=0$, it follows from (2.5) that $b=0$ and

$$\frac{1}{k}=\frac{C}{x}t \text{ for } x>0.$$

Hence when X has caught up to Y, which means $x=C$, we have $\frac{1}{k}=t$. Thus $1/k$ is the time required for nation X to catch up with the armaments of Y provided that y remains constant. Richardson also observed that k is proportional to the amount of industry in a country. In the more realistic model we shall provide a more detailed analysis with an example. Here we assume that the efficiency of increasing armaments for each nation is equal, and we take

$$k=l=0.9$$

as an example. It follows from (2.2) and (2.3), with the initial condition

$$x(0)=20,\quad y(0)=0$$

that

$$A=10,B=-10$$

and

$$x(t)=10e^{0.9t}+10e^{-0.9t},\ y(t)=10e^{0.9t}-10e^{-0.9t}$$

or

$$\begin{pmatrix}x\\y\end{pmatrix}=\begin{pmatrix}10\\10\end{pmatrix}e^{0.9t}+\begin{pmatrix}10\\-10\end{pmatrix}e^{-0.9t} \tag{2.6}$$

Figures 3 and 4 describe the relationship between the two nations with initial condition $x_0 = 20$, $y_0 = 0$. From Figures 3 and 4 we observe that, when A is

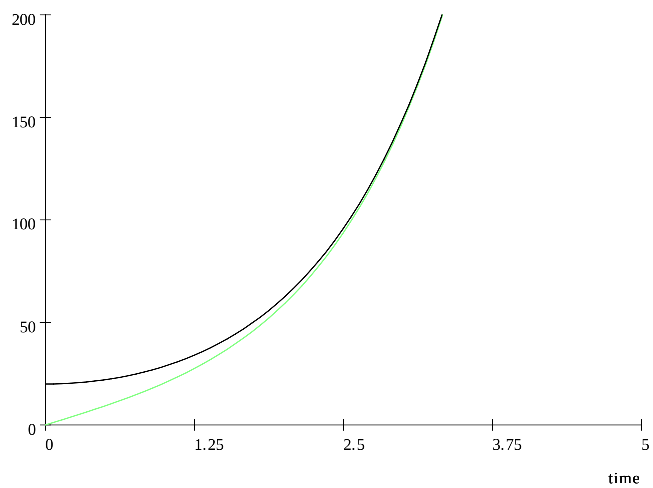

FIGURE 3. Solutions for the model of an arms race (2.1) ($x-$solid, $y-$grey).

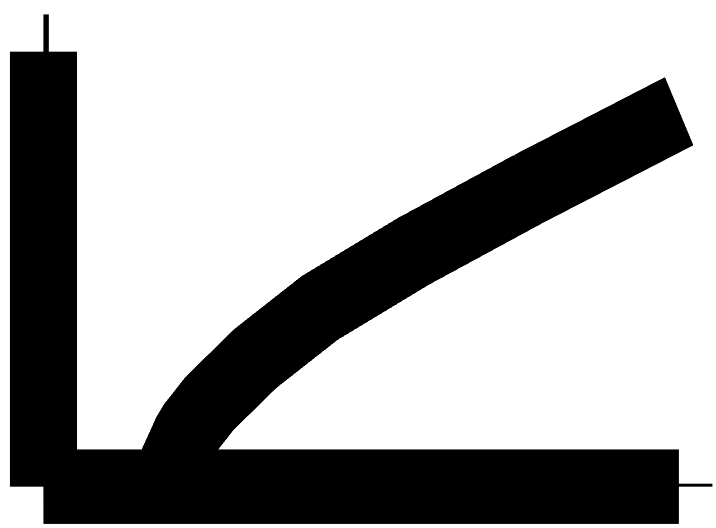

FIGURE 4. Trajectory for the model of an arms race (2.1).

positive, $x(t)$ and $y(t)$ tend to infinity which can be interpreted as an acceleration in the arms race, possibly leading to a war between the two nations, X and Y.

We note that since the system has two real eigenvalues $\lambda = \pm\sqrt{kl}$ with opposite signs, the critical point $(0, 0)$ is a saddle point which is always unstable. The phase portrait, which is constructed using many different initial conditions, is shown in Figure 5 below. The trajectory and phase portraits were plotted using the software

FIGURE 5. Phase portrait for the model of an arms race (2.1).

Mathematica® (see [10]), from which we can see that the armaments of each nation y and x increase simultaneously. The straight half line through the origin apparent in the phase portrait is the line of action of the "vector" $\begin{pmatrix} 10 \\ 10 \end{pmatrix}$ in the solution (2.6).

3. CONSTRUCTION OF A MORE REALISTIC MODEL

The relationship between nations or alliances in the real world is more complicated. We therefore need to modify the system (2.1) by considering more factors that affect the change rates dx/dt and dy/dt in an effort to adapt it to the real world. Richardson constructed differential equations of conflict, taking into account factors such as the cost of armaments, grievances or ambitions between nations, etc ([1], 1993; [11], 1957). The system constructed for describing the relationship between the nations or alliances, X and Y, is as follows:

$$\left\{ \begin{array}{l} \frac{dx}{dt} = ky - \alpha x + g \\ \frac{dy}{dt} = lx - \beta y + h \end{array} \right. , k, l, \alpha, \beta, g, h > 0 \tag{3.1}$$

where $x(t)$ (respectively $y(t)$) denotes the armaments of nation X (Y); k (l) is the efficiency of increasing the armaments of X (Y); g (h) is the ambitions of or grievances that X (Y) has towards Y (X), affecting dx/dt (dy/dt) positively; the influence of the cost of armaments is a restraining factor, represented by $-\alpha x$ $(-\beta y)$.

We encourage students to observe the stability of system (3.1) by finding and analysing the critical point. This is an excellent exercise for students to familiarise themselves with the classification of a critical point by stability and type, utilising the eigenvalues. The stability of the critical point of the system (3.1) depends on the logical relationship between kl and $\alpha\beta$.

It would entail a great deal of work to simplify the solutions to system (3.1). Given the initial conditions $x(0) = x_0, y(0) = y_0$, the unique solution to (3.1) can be written as follows:

$$x(t) = x_1 + \frac{kl}{2\mu(\mu+\omega)} A e^{\lambda_1 t} + \frac{1}{2}\frac{\mu+\omega}{\mu} B e^{\lambda_2 t} \tag{3.2}$$

$$y(t) = y_1 + \frac{l}{2\mu} A e^{\lambda_1 t} - \frac{l}{2\mu} B e^{-\lambda_2 t} \tag{3.3}$$

where

$$(x_1, y_1) = \left(\frac{\beta g + hk}{\alpha\beta - kl}, \frac{\alpha h + gl}{\alpha\beta - kl} \right)$$

is the critical point,

$$\lambda_1 = -\frac{1}{2}(\alpha+\beta) + \frac{1}{2}\sqrt{(\alpha-\beta)^2 + 4kl}$$

$$\lambda_2 = -\frac{1}{2}(\alpha+\beta) - \frac{1}{2}\sqrt{(\alpha-\beta)^2 + 4kl}$$

are eigenvalues, and

$$\omega = \frac{1}{2}(\alpha-\beta), \quad \mu = \sqrt{\omega^2 + kl} \tag{3.4}$$

$$A = \left(x_0 + \frac{g}{\lambda_1}\right) + \frac{\mu+\omega}{l}\left(y_0 + \frac{h}{\lambda_1}\right) \tag{3.5}$$

$$B = \left(x_0 + \frac{g}{\lambda_2}\right) - \frac{k}{\mu+\omega}\left(y_0 + \frac{h}{\lambda_2}\right). \tag{3.6}$$

Finally, we estimate the coefficients in our model. It is interesting to note that Richardson estimates α^{-1} $\left(\beta^{-1}\right)$ to be the lifetime of $X's$ $(Y's)$ parliament ([2]). For example, since the lifetime of Britain's parliament is five years, we obtain $\alpha = 0.2$ for that country.

To estimate k and l, for example, we consider $g = 0$ and $y = C$. Hence

$$\frac{dx}{dt} = kC - \alpha x.$$

Assuming $x(0) = 0$ and solving the above equation we obtain:

$$x = \frac{kC}{\alpha}\left(1 - e^{-\alpha t}\right). \tag{3.7}$$

Substituting the power series expansion

$$e^{-\alpha t} = \sum_{n=0}^{\infty} \frac{(-\alpha t)^n}{n!} = 1 - \alpha t + \cdots$$

in (3.7), since $\alpha t < 1$, we have

$$x \approx \frac{kC}{\alpha}\alpha t \text{ for } t > 0.$$

That is,

$$\frac{1}{k} = \frac{C}{x} t$$

Thus $1/k$ is the time required for X to catch up to Y, if $Y's$ armaments only remain a constant C. We recall that k represents the product efficiency of armaments of nation X. It is obvious that a useful exercise is to apply the knowledge to series

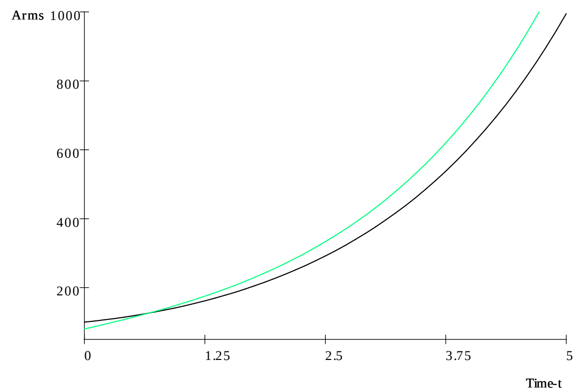

FIGURE 6. Solutions to the model (3.1) with $k = 0.6$, $l = 0.8$, $\alpha = \beta = 0.2, x_0 = 100, y_0 = 80$.

expansion of functions. We note that our method of estimating k is different from the method given in [2], in which the constant k is estimated in the following way, assuming $g = 0$ and $y = y_1$: when $x = 0$, $1/k = y_1/(dx/dt)$.

A future work could include the collection of data and information on historical wars to construct mathematical models. For example, during the Cold War, both the USA and the Soviet Union were involved in an arms race for conventional and nuclear weapons.

Here we only give some examples and graphs, choosing different values for the constants k, l, α, β. We assume $g = h = 0$, so $x_1 = y_1 = 0$. We use a solid line to represent the solution $x(t)$, and a grey line for the solution $y(t)$.

If $k = 0.6$, $l = 0.8$, $\alpha = \beta = 0.2, x_0 = 100, y_0 = 80$, the solutions and graphs are as follows (unstable case, $\alpha\beta < kl$, $\lambda_1 > 0 > \lambda_2$) (see Figure 6):

$$\begin{aligned} x(t) &= 84.641 \exp(0.49282t) + 15.359 \exp(-0.89282t) \\ y(t) &= 97.735 \exp(0.49282t) - 17.735 \exp(-0.89282t) \end{aligned}$$

If $k = 0.35, l = 0.4, \alpha = 0.3, \beta = 0.5$, $x_0 = 100$, $y_0 = 95$, the solutions and graphs are as follows (stable case, $\alpha\beta > kl$, $\lambda_2 < \lambda_1 < 0$) (see Figure 7):

$$\begin{aligned} x(t) &= 105.84 \exp\left(-1.270\,2 \times 10^{-2} t\right) - 5.8355 \exp(-0.7873t) \\ y(t) &= 8.1247 \exp(-0.7873t) + 86.875 \exp\left(-1.2702 \times 10^{-2} t\right) \end{aligned}$$

We believe that more research could be conducted into the mathematical modelling of international relationships at undergraduate level. Such modelling also has realistic applications in military, business and other fields. From an educational perspective, these mathematical models are also realistic applications of ordinary differential equations (ODEs) — hence the proposal that these models should be added to ODE textbooks as flexible and vivid examples to illustrate and study differential equations.

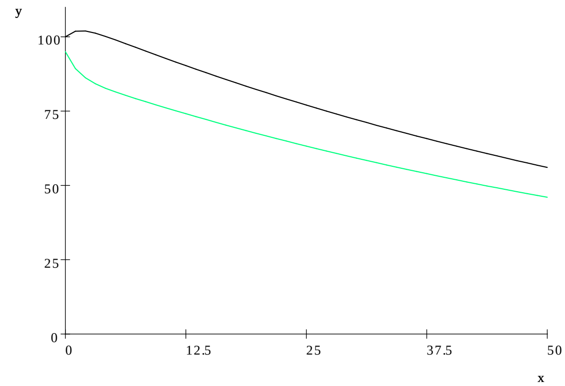

FIGURE 7. Solutions to the model (3.1) with $k = 0.6$, $l = 0.8$, $\alpha = \beta = 0.2, x_0 = 100, y_0 = 80$.

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