CCV-20

SC916 Calculus with Complex Variables

PARTIAL DIFFERENTIAL EQUATIONS (P.D.E.)

— $F(x, y, u, u_x, u_y, u_{xx}, u_{xy}, u_{yy}, \ldots) = 0$
general partial differential eqn
Soln. $\longrightarrow u = u(x,y)$

— order of the P.D.E. is the order of the highest derivative of u.

— Linear PDE: F is linear fn. of u & its derivatives.

Example 1 $\quad \dfrac{\partial u}{\partial x} = 0$, Solve for $u = u(x,y)$

> Note in the case of ODE $\dfrac{du}{dx} = 0$ Solve for $u = u(x)$
> $u = c$ (constt.)

Here we have soln. as
$u(x,y) = c(y) \leftarrow$ arbitrary fn. of y

$\Rightarrow$ General solns. of PDE's involve arbitrary fns.

— How to solve PDE? $\quad$ PDE $\rightarrow$ ODE?

Ex 2 $\quad$ PDEs

— $u_t - c\, u_x = 0$ (Kinematic eqn) $\quad c$ constt.
traffic flow, gas dynamics

— $u_{tt} - c^2 u_{xx} = 0$ (Wave eqn)

— $u_{xx} + u_{yy} = 0$ (Laplace eqn)
$\nabla^2 u = 0$

Application of PDE

Weather prediction, airplane design, shock waves, ---

Abel Prize : Peter D. Lax US \$1 million

$\boxed{\text{Ex 3}}$ $\begin{cases} u''+u=0 & u=u(x) \quad 0\le x\le 1 \\ u(0)=0 & \text{IVP for ODE} \\ u'(0)=1 \end{cases}$

$\boxed{\text{Ex-4}}$ $\begin{cases} u''+u=0 & u=u(x) \quad 0\le x\le 1 \\ u(0)=0 & \text{BVP for ODE} \\ u'(1)=1 \end{cases}$

What will happen to IVP/BVP for PDE?

$\boxed{\text{Ex-5}}$ Find P.D.E. that governs the family of surfaces $u(x,y)=(x-\alpha)^2+(y-\beta)^2$

$\square$ $\frac{\partial u}{\partial x}=2(x-\alpha)$ & $\frac{\partial u}{\partial y}=2(y-\beta)$

$\Rightarrow u=\frac{1}{4}\left(\frac{\partial u}{\partial x}\right)^2+\frac{1}{4}\left(\frac{\partial u}{\partial y}\right)^2$

$\Rightarrow$ P.D.E is $4u=\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2$ ▨

$\boxed{\text{Ex-6}}$ Find the P.D.E. from

$u(x,y)=ax+by+a^2b^2$

a & b are const.

Verify that $u=\frac{\partial u}{\partial x}x+\frac{\partial u}{\partial y}y+\left(\frac{\partial u}{\partial x}\right)^2\left(\frac{\partial u}{\partial y}\right)^2$

is the required P.D.E..

▨

$\boxed{\text{FIRST ORDER P.D.E.}}$

$$p = \frac{\partial z}{\partial x} \quad q = \frac{\partial z}{\partial y}$$

$$F(x, y, z, p, q) = 0$$

$\boxed{\text{SECOND ORDER P.D.E.}}$

Linear PDE of order 2 in 2 variables

$$a u_{xx} + 2b u_{xy} + c u_{yy} + d u_x + e u_y + f u = g \quad \text{---(1)}$$

$u \equiv u(x, y)$

& a, b, c, d, e, f are constt.

The characteristic polynomial is

$$P(\alpha, \beta) \equiv a\alpha^2 + 2b\alpha\beta + c\beta^2 + d\alpha + e\beta + f \quad \text{--(2)}$$

$\boxed{\text{Classification}}$:

P.D.E. (1) is said to be.

— hyperbolic if $b^2 - ac > 0$

— parabolic if $b^2 - ac = 0$

— elliptic if $b^2 - ac < 0$

Ex: P.D.E.

$$3u_{xx} + 2u_{xy} + 5u_{yy} + x u_y = 0$$

is elliptic $\because$ $b^2 - ac = 1^2 - 3.5 = -14 < 0$

Ex. The Tricomi eqn $u_{xx} + y u_{yy} = 0$

has $b^2 - 4ac = -y$

$\Rightarrow$ the eqn is elliptic for $y > 0$

parabolic for $y = 0$

& hyperbolic for $y < 0$

— The general linear PDE of order 2 in n variables has the form

$$\sum_{i,j=1}^{n} a_{ij} u_{x_i x_j} + \sum_{i=1}^{n} b_i u_{x_i} + cu = d$$

Special P.D.E.

① Parabolic eqn $\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}$ (One-dim, heat eqn)

$$\frac{\partial u}{\partial t} = c^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right)$$ (2-dim heat eqn)

② Hyperbolic eqn $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$ (1-dim wave eqn)

$$\frac{\partial^2 u}{\partial t^2} = c^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right)$$ (2-dim wave eqn)

③ elliptic eqn $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ (2-dim Laplace eqn)

Solving a P.D.E. using the method of separation of variables (Fourier Method)

$u(x,y) \rightarrow$ dep. variable

independ variables

We seek a soln. of the form $u(x,y) = X(x)Y(y)$

then we have $\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}(XY) = X'Y$

$$\frac{\partial u}{\partial y} = XY', \quad \frac{\partial^2 u}{\partial x^2} = X''Y$$

$$\frac{\partial^2 u}{\partial y^2} = XY'' \text{ \& so on}$$

$X' = \frac{dX}{dx}$

$Y' = \frac{dY}{dy}$ etc.

Example Solve $\frac{\partial^2 u}{\partial x^2} = 16 \frac{\partial u}{\partial y}$

□ Using $u(x,y) = X(x)\, Y(y)$ we obtain

$$X'' Y = 16 X Y'$$

$$\Rightarrow \frac{X''}{16X} = \frac{Y'}{Y}$$ → RHS is a fn. of y alone

L.H.S. is a fn. of x alone $\Rightarrow \frac{X''}{16X} = \frac{Y'}{Y} = \text{some constant} = \lambda$ (say)

Case ① $\lambda = 0$: $\frac{X''}{16X} = \frac{Y'}{Y} = 0 \Rightarrow X'' = 0$ & $Y' = 0$.

$$\Rightarrow X = (A_1 x + B_1) \ \& \ Y = C_1$$

$$\Rightarrow u(x,y) = (A_1 x + B_1)\, C_1 = Ax + B$$

Case ② when $\lambda = k^2$ (say) > 0

Then $X'' - 16k^2 X = 0$ & $Y' - k^2 Y = 0$

$$\Rightarrow X(x) = A_1 e^{4kx} + B_1 e^{-4kx}$$

$$\Rightarrow X(x) = A_2 \cosh(4kx) + B_2 \sinh(4kx)$$

& $Y(y) = c_1 e^{k^2 y}$

$$\Rightarrow u(x,y) = [A_2 \cosh(4kx) + B_2 \sinh(4kx)] \cdot c_1 e^{k^2 y}$$

$$= [A \cosh(4kx) + B \sinh(4kx)]\, e^{k^2 y}$$

Case ③ When $\lambda = -k^2 < 0$

$$X'' + 16k^2 X = 0 \ \& \ Y' + k^2 Y = 0$$

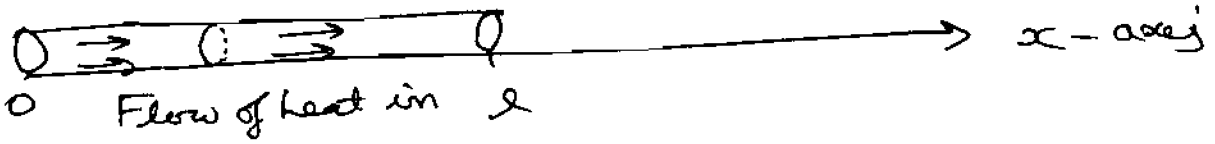
$\rightarrow X(x) = A_1 \cos(4kx) + B_1 \sin(4kx)$ & $Y(y) = c_1 e^{-k^2 y}$

$$\rightarrow u(x,y) = [A \cos(4kx) + B \sin(4kx)]\, e^{-k^2 y}$$

One diemensional heat eqn

Heat eqn arises in many context (such as financial math). Black-Scholes option pricing model D. E's can be transformed into heat eqn.

— Consider a thin homo. bar of length l.



Flow of heat in a bar

- $u(x,t)$ — temp. distribution or heat flow in the bar

— Assume that the initial temp. in the bar is $f(x)$ & the ends of the bar are at zero temp. all the time.

— The BVP modeling is given by

$$\frac{\partial u}{\partial t} = c^2 \frac{\partial^2 u}{\partial x^2}, \quad 0 < x < l, \; t > 0$$

with I.C. $\quad u(x,0) = f(x), \quad 0 < x < l$ &

B.C. $\quad u(0,t) = u(l,t) = 0, \quad t > 0$

$c^2 \to$ constt. (thermal diffusivity)

— Look at the Fourier series soln. of this at page 629 of the book using method of sep. of variables (Do it yourself!)

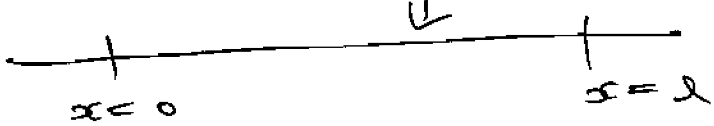
Hint: Let $u(x,t) = X(x)\,T(t)$

$$\Rightarrow \quad XT' = c^2 X'' T \quad \text{or} \quad \frac{X''}{X} = \frac{T'}{c^2 T} = \lambda \text{ (say)}$$

use 3 cases $\rightarrow$ ① $\lambda = 0$
② $\lambda = k^2$
③ $\lambda = -k^2$ etc. .....

One dimensional wave eqn

Elastic string of length l



fastened at its ends

— If the string is displaced & then released to vibrate in the x-t plane then let $u(x,t)$ denote the vertical displacement of the vibrating string.

— The modeling is given by

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \qquad 0 < x < l,\ t > 0$$

with B.C. $u(0,t) = 0$, $u(l,t) = 0$, $t > 0$

& I.C. $u(x,0) = f(x)$, $\frac{\partial u}{\partial t}(x,0) = g(x)$

Initial displacement at $t = 0$

initial velocity of the string.

— For solution see page 636 of the book

Hint: Let $u(x,t) = X(x)\,T(t)$

$$\Rightarrow \quad X T'' = c^2 X'' T \quad \text{or} \quad \frac{X''}{X} = \frac{T''}{c^2 T} = \lambda \text{ (say)}$$

Now one can show that (Verify!)

$\lambda \neq 0$ & $\lambda \neq k^2 \Rightarrow \lambda = -k^2$ only

& then solve.

Laplace eqⁿ

We want to study the steady-state temp. distribution in a thin, flat, rectangular plate

— Suppose boundaries of the plate be $x = 0$, $x = a$, $y = 0$, $y = b$

— The D.E. modeling is given by

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \quad 0 < x < a, \quad 0 < y < b$$

— Different B.C. will give different answer

— Let B.C. are

$$u(0, y) = 0, \quad u(a, y) = 0, \quad 0 < y < b$$

$$u(x, 0) = f(x), \quad u(x, b) = g(x), \quad 0 < x < a$$

— Look for soln. at page 646 of the books

Hint: Let $u(x, y) = X(x)Y(y)$

$$\Rightarrow \quad X''Y + XY'' = 0 \text{ or } \frac{X''}{X} = -\frac{Y''}{Y} = \lambda \text{ (say)}$$

Now

$- \lambda \neq 0$ & $\lambda \neq k^2 \Rightarrow \lambda = -k^2$ & solve