SC105- Calculus with Complex Variables

Home Work on Differential Equations

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1. Show that the solution of the equation $\frac{d^2x}{dt^2} + k^2x = f(t)$ satisfying the initial conditions $x(0) - 0$ and $x'(0) = 0$ is of the form

$$x(t) = \frac{1}{k}\int_0^t f(u)\sin k(t-u)\,du$$

2. For what values of p and q all solutions of the equation $y'' + py' + qy = 0$ tend to zero as $x \to +\infty$?

3. For what values of p and q all solutions of the equation $y'' + py' + qy = 0$ (p, q are constant) periodic function of x ?

4. Are the following statements are true? If the statement is true then prove it otherwise provide a counter example showing it is false.

(a) “If ϕ_1 and ϕ_2 are linearly independent functions on an interval I, they are linearly independent on any interval J contained inside I.”

(b) “If ϕ_1 and ϕ_2 are linearly dependent functions on an interval I, they are linearly dependent on any interval J contained inside I.”

(c) “If ϕ_1 and ϕ_2 are linearly independent solutions of $L(y) = 0$ on an interval I, they are linearly independent on any interval J contained inside I.”

(d) “If ϕ_1 and ϕ_2 are linearly dependent solutions of $L(y) = 0$ on an interval I, they are linearly dependent on any interval J contained inside I.”

5 (a) Show that the functions ϕ_1 and ϕ_2 defined by $\phi_1(x) = x^2$, $\phi_2(x) = x|x|$, are linearly independent for $-\infty < x < \infty$.

(b) Compute the Wronskian of these functions.

(c) Do the results of part (a) and (b) contradict the following theorem. (Explain your answer.)

Two solutions ϕ_1 and ϕ_2 of $L(y) = 0$ are linearly independent on an interval I if, and only if, $W(\phi_1, \phi_2)(x) \neq 0$ for all $x \in I$.

6 Let ϕ_1 and ϕ_2 be two differentiable functions on an interval I, which are not necessarily solutions of an equation $L(y) = 0$. Prove the following:

(a) If ϕ_1 and ϕ_2 of $L(y) = 0$ are linearly dependent on an interval I, then $W(\phi_1, \phi_2)(x) = 0$ for all $x \in I$.

(b) If $W(\phi_1, \phi_2)(x) \neq 0$ for some $x_0 \in I$, then ϕ_1 and ϕ_2 are linearly dependent on an interval I.

(c) If $W(\phi_1, \phi_2)(x) = 0$ for all $x \in I$ does not imply that ϕ_1 and ϕ_2 are linearly dependent on I.

(d) If $W(\phi_1, \phi_2)(x) = 0$ for all $x \in I$ and $\phi_2(x) \neq 0$ on I, imply that ϕ_1 and ϕ_2 are linearly dependent on I.

7 Show that if $y_1(x) = \begin{cases} x^2 & \text{if } -1 \le x \le 0, \\ 0 & \text{if } 0 < x \le 1 \end{cases}$ and $y_2(x) = \begin{cases} 0 & \text{if } -1 \le x \le 0, \\ x^2 & \text{if } 0 < x \le 1, \end{cases}$ then $W(y_1, y_2)$ on $[-1, 1]$ but $Y = \{y_1, y_2\}$ is linearly independent on $[-1, 1]$.

8 Solve the following differential equation by the method of variation of parameters.

1. $y'' + y = \frac{1}{\sin x}$,
2. $y'' - 2y' + y = \frac{e^x}{x^2+1}$,
3. $y'' - y' = \frac{1}{e^x+1}$,
4. $y'' + 2y' + 2y = \frac{1}{e^x \sin x}$,
5. $y'' + y = \frac{1}{\cos^2 x}$,
6. $y'' - y' = e^{2x} \cos e^x$,
7. $y'' + y = \frac{1}{\sqrt{\sin^5 x \cos x}}$,
8. $y''' + y'' = \frac{x-1}{x^2}$,
9. $x \ln x \, y'' - y' = \ln^2 x$,
10. $y'' + y' \tan x = \cos x \cot x$

9 Prove that the functions $\sin x$, $\sin(x + \pi/8)$, $\sin(x - \pi/8)$ are linearly independent in the interval $(-\infty, \infty)$. Compute their wronskian also.
10 Examine whether the given the functions are linearly independent in their domain of definition.

$$(a)\ e^{-\frac{ax^2}{2}},\ e^{-\frac{ax^2}{2}} \int_0^x e^{-\frac{at^2}{2}} dt \qquad (b)\ x,\ a^{\log_a x}\ (x > 0) \qquad (c)\ 2\pi, \arctan \frac{x}{2\pi},\ \operatorname{arccot} \frac{x}{2\pi}$$

$$(d)\ x,\ x \int_{x_0}^1 \frac{e^t}{t^2} dt\ (x_0 > 0) \qquad (e)\ \log_a x^2,\ \log_a x\ (x > 0) \qquad (f)\ 5, \cos^2 x,\ \sin^2 x.$$

11 Consider the equation $y' = ky$ on $-\infty < x < \infty$, where k is some constant

- (a) Show that ϕ is only solution, and $\Psi(x) = \phi(x)e^{-kx}$, then $\Phi(x) = c$, where c is constant.
- (b) Prove that if Re $k < 0$, then every solution tends to zero as $x \to \infty$.
- (c) Prove that if Re $k > 0$, then the magnitude of every non trivial (not identically zero) solution tends to zero as $x \to \infty$.
- (d) What can you say about the magnitude of the solution if $k = 0$?

12 Solve the following systems.

- (a) $\frac{dx_1}{dt} = \frac{x_1 - x_2}{x_3 - t}$, $\frac{dx_2}{dt} = \frac{x_1 - x_2}{x_3 - t}$, $\frac{dx_1}{dt} = x_1 - x_2 + 1$.

- (b) $e^t \frac{dx}{dt} = \frac{1}{y}$, $e^t \frac{dy}{dt} = \frac{1}{x}$.
- (c) $\frac{dx}{dt} = \sin x \cos y$, $\frac{dy}{dt} = \sin y \cos x$.
- (d) $\frac{dx_1}{dt} = (x_1^2 + x_2^2)t$, $\frac{dx_2}{dt} = 4x_1x_2t$.
- (e) $\frac{dx}{dt} = \frac{y}{x-y}$, $\frac{dy}{dt} = \frac{x}{x-y}$.

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- (a) Give an example of an initial value problem which has no solution.
- (b) Give an example of a differential equation with no real solution.
- (c) Give an example of a system differential equation which has no solution.

14 Find the general solution of each of the following system:

$$(a)\ \begin{array}{l} \frac{dx}{dt} = -3x + 4y \\ \frac{dy}{dt} = -2x + 3y \end{array} \quad (b)\ \begin{array}{l} \frac{dx}{dt} = 4x - 3y \\ \frac{dx}{dt} = 8x - 6y \end{array} \quad (c)\ \begin{array}{l} \frac{dx}{dt} = x \\ \frac{dy}{dt} = y \end{array} \quad (d)\ \begin{array}{l} \frac{dx}{dt} = x + y \\ \frac{dx}{dt} = y \end{array}$$

15 Find the Wronskian of the following functions and cheek their linear dependence.

(a) $y_1(x) = \begin{cases} 0 & \text{if } 0 \le x \le \frac{1}{2}, \\ \left(x - \frac{1}{2}\right)^2 & \text{if } \frac{1}{2} < x \le 1 \end{cases}$ and $y_2(x) = \begin{cases} \left(x - \frac{1}{2}\right)^2 & \text{if } 0 \le x \le \frac{1}{2}, \\ 0 & \text{if } \frac{1}{2} < x \le 1, \end{cases}$

(b) $y_1(x) = \begin{cases} x^3 & \text{if } -2 \le x \le 0, \\ 0 & \text{if } 0 < x \le 1 \end{cases}$ and $y_2(x) = \begin{cases} 0 & \text{if } -2 \le x \le 0, \\ x^2 & \text{if } 0 < x \le 1, \end{cases}$

(c) π, $\arcsin x$ and $\arccos x$,

(d) $\frac{1}{x}$ and $e^{\frac{1}{x}}$,

(e) 4, $\sin^2 x$ and $\cos 2x$,

(f) $e^{\alpha^x} \cos \beta x$ and $e^{\alpha^x} \sin \beta x$,

(g) x and $\ln x$.

15 Suppose ϕ is the solution of $y^{(n)} + a_1 y^{(n-1)} + \ldots + a_n y = 0$ and $\Phi(x) = \phi(x) e^{\frac{a_1 x}{n}}$. Show that $\Phi(x)$ satisfying the linear homogeneous differential equation with constant coefficient $y^{(n)} + b_1 y^{(n-1)} + \ldots + b_n y = 0$ with $b_1 = 0$.