SC 216 Calculus with Complex Variables

(O.D.E.)

$y = f(x)$

— The general O.D.E. of n^{th} order is

$$F\left(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}, \dots, \frac{d^ny}{dx^n}\right) = 0$$

or $\quad F(x, y, y', y'', \dots, y^{(n)}) = 0$

— Solutions could be of the form

$$y = f(x)$$

or

$$f(x, y) = 0 \quad \text{implicit}$$

Eg. $\quad y'' - 5y' + 6y = 0$

$$y = e^{2x}$$

$$\Rightarrow y'' = \frac{d}{dx}(2e^{2x}) = 4e^{2x} = 4y$$

$$\Rightarrow y'' - 5y' + 6y = 4y - 5 \times 2y + 6y = 0$$

$\therefore \; y = e^{2x}$ is a soln.

$y = e^{3x}$ is also a soln. (verify!)

More generally

$$y = c_1 e^{2x} + c_2 e^{3x}$$ a general soln. for some constt. c_1 & c_2

Example 2

$$\frac{dy}{dx} = \frac{y^2}{1 - xy}$$

has a soln. $\quad xy = \log y + c \quad$ (c – constt.)

(verify!)

— Differential Equations are very natural tool to model various ~~and~~ physical, chemical, biological, natural, engineering ~~the~~ processes.

$$
\begin{array}{c}
\text{D.E.} \\
\swarrow \qquad \searrow
\end{array}
$$

O.D.E. (Ordinary differential eq^n)

P.D.E. (Partial differential equations)

If $\omega = f(x, y, z, t)$

$$\frac{\partial^2 \omega}{\partial x^2} + \frac{\partial^2 \omega}{\partial y^2} + \frac{\partial^2 \omega}{\partial z^2} = 0$$

$$
\begin{array}{c}
\text{D.E.} \\
\swarrow \qquad \searrow
\end{array}
$$

Linear

Non linear (Most difficult to solve Often we don't know How to solve?)

~~O.D.E.~~

~~Linear~~

— Linear D.E. with constant cofficients

— " " " ~~coef~~ variable "

— Simplest D.E.

$$\frac{dy}{dx} = f(x)$$

$$\Rightarrow \text{Soln is } y = \int f(x)\,dx + c$$

— If $\frac{dy}{dx} = f(x)\, g(y)$

$$\Rightarrow \frac{dy}{g(y)} = f(x)\, dx$$

$$\Rightarrow \int \frac{dy}{g(y)} = \int f(x)\, dx + c$$

— The general 1st order D.E. is $(n=1)$

$$F\left(x, y, \frac{dy}{dx}\right) = 0$$

$\boxed{\text{Example}}$ $\left(\frac{dy}{dx}\right)^2 + 1 = 0$ has no real valued soln. (Why?)

$\boxed{\text{Example}}$ $\left(\frac{dy}{dx}\right)^2 + y^2 = 0$ has only single soln. $y = 0$

$\therefore$ Sometime soln. may not exist. How to find all that?

$\boxed{\text{Picard's Theorem}}$ If $f(x,y)$ and $\frac{\partial f}{\partial y}$ are cont.s fns. on a closed rectangle R then through each pt. (x_0, y_0) in the interior of R there passes a unique integral curve of the eqn $\frac{dy}{dx} = f(x,y)$

(What is the geometrical meaning?)

— The eqn of the family

$$y = y(x, c) \qquad c \to \text{const.}$$

$$\frac{dy}{dx} = f(x, y)$$

Solving it is very difficult.
No general formula exist. We give soln. for ~~sp~~ cases:

— If $\frac{dy}{dx} = g(x)\, h(y)$ (Variables are separable)

$$\Rightarrow \int \frac{dy}{h(y)} = \int g(x)\, dx + c \leftarrow \text{Soln.}$$

— We know that a fn. $f(x, y)$ is called homogeneous of degree n if $f(tx, ty) = t^n f(x,y)$.

— The D.E. $M(x, y)\, dx + N(x, y)\, dy = 0$ is said to be homo. if M & N are homo fns. of the same degree.

and in that case

$$\frac{dy}{dx} = -\frac{M(x, y)}{N(x, y)} \rightarrow \text{homo. of deg } 0$$

$$= f(x, y)$$

$\Rightarrow$ we can write f as a fn. of $z = \frac{y}{x}$ & then ~~use~~ ~~sep~~ the method of separation of variables

$$\Rightarrow \quad f(x, y) = f(1, y/x) = f(1, z)$$

& $\therefore y = zx$

$$\Rightarrow \frac{dy}{dx} = z + x\frac{dz}{dx}$$

& D.E. will be $z + x\frac{dz}{dx} = f(1, z)$

or $\frac{dz}{f(1,z) - z} = \frac{dx}{x}$

Example Solve $(x+y)\,dx - (x-y)\,dy = 0$

$\square$

$$\Rightarrow \frac{dy}{dx} = \frac{x+y}{x-y} \quad \text{(homo. of deg 0)}$$

$\Rightarrow$ Put $z = \frac{y}{x}$ to obtain

$$\frac{dy}{dx} = \frac{1+y/x}{1-y/x} = \frac{1+z}{1-z}$$

$$\Rightarrow \because \frac{dy}{dx} = z + x\frac{dz}{dx} \quad \left(\text{as } z = \frac{y}{x}\right)$$

we get
$$\frac{(1-z)\,dz}{1+z^2} = \frac{dx}{x}$$

$$\Rightarrow \tan^{-1} z - \frac{1}{2}\log(1+z^2) = \log x + c$$

$\because z = \frac{y}{x}$ we get

$$\tan^{-1}\left(\frac{y}{x}\right) = \log\sqrt{x^2+y^2} + c$$

Exact Equations

If $f(x,y) = c$

then $df = 0$

or $\frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = 0$

e.g. $x^2y^3 = c$ has $2xy^3\,dx + 3x^2y^2\,dy = 0$

Reverse the process suppose we have

$$M(x,y)\,dx + N(x,y)\,dy = 0 \quad \text{—}$$

if $\exists$ a fn. $f(x,y)$ s.t.

$\frac{\partial f}{\partial x} = M$ & $\frac{\partial f}{\partial y} = N$

then (1) can be written as

$$\frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = 0 \text{ or } df = 0$$

& so the general soln. will be

$$f(x,y) = c$$

(1) is called Exact D.E.

$\boxed{\text{Example}}$ $y\,dx + x\,dy = 0$

$$\Rightarrow d(xy = c) = 0$$

$$\Rightarrow xy = c \leftarrow \text{soln.}$$

$$\frac{1}{y}dx - \frac{x}{y^2}dy = 0$$

$$\Rightarrow d(x/y = c) = 0$$

$$\Rightarrow \frac{x}{y} = c \leftarrow \text{soln.}$$

— A eqn $M\,dx + N\,dy$ is exact iff

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$ & then the

general soln. is $f(x,y) = c$.

$\boxed{\text{Example}}$ Solve $e^y dx + (xe^y + 2y)\,dy = 0$

☐ $M = e^y$ & $N = xe^y + 2y$

$$\frac{\partial M}{\partial y} = e^y = \frac{\partial N}{\partial x}$$

$\Rightarrow$ Eqn is exact D.E.

$\Rightarrow \exists$ a fn. $f(x,y)$ s.t.

$$\frac{\partial f}{\partial x} = e^y \quad (2) \qquad \& \qquad \frac{\partial f}{\partial y} = xe^y + 2y \quad (3)$$

Integrating (2) w.r.t. to x gives

$$f = \int e^y \, dx + g(y) = xe^y + g(y)$$

$$\Rightarrow \frac{\partial f}{\partial y} = xe^y + g'(y) = \frac{\partial f}{\partial y}$$

$$= xe^y + 2y$$

$$\Rightarrow g'(y) = 2y$$

$$\Rightarrow g(y) = y^2$$

$$\Rightarrow f = xe^y + y^2$$

$\Rightarrow$ soln. is $\boxed{xe^y + y^2 = c}$

Integrating Factor

We can make a non-exact D.E. exact by multiplying it with a fn. of x & y called Integrating Factor (I.F.)

Example $\quad y\,dx + (x^2 y - x)\,dy = 0$

$$\frac{\partial M}{\partial y} = 1 \neq \frac{\partial N}{\partial x} = 2xy - 1$$

If we multiply the eqn by $\frac{1}{x^2}$ we get

$$\frac{y}{x^2}\,dx + \left(y - \frac{1}{x}\right) dy = 0$$

This is exact D.E. (Verify!)
Now we can solve it

— Suppose we want to develop a method for finding ~~a formula for~~ I.F. What we should do? (Exercise!)

$\boxed{\text{Linear Equations}}$

— General first order linear eq^n is

$$\frac{dy}{dx} = p(x)\, y + q(x) \qquad \text{——} (1)$$

— General second order linear eq^n is

$$\frac{d^2y}{dx^2} = p(x)\frac{dy}{dx} + q(x)\, y + r(x)$$

How to solve (1)?

$\boxed{\text{Note}}$ $$\frac{d}{dx}\left(e^{\int p\,dx}\, y\right) = e^{\int p\,dx}\frac{dy}{dx} + y\, p \cdot e^{\int p\,dx}$$

$$= e^{\int p\,dx}\left(\frac{dy}{dx} + py\right)$$

$\Rightarrow$
$$\frac{d}{dx}\left(e^{\int p\,dx}\, y\right) = q\, e^{\int p\,dx}$$

$$\Rightarrow \quad e^{\int p\,dx}\, y = \int q\, e^{\int p\,dx}\, dx + c$$

Example $\frac{dy}{dx} + \frac{1}{x} y = 3x$

$p = 1/x \quad \& \quad Q = 3x$

$\Rightarrow \int p\, dx = \ln x$

$\Rightarrow e^{\int p\, dx} = e^{\ln x} = x$

$\Rightarrow \frac{d}{dx}(xy) = 3x^2$

$\Rightarrow xy = x^3 + c$

Reduction of order

- If dependent variable y is missing

i.e. if eqn is $f(x, y', y'') = 0$ — (1)

then introduce $y' = p$ & $y'' = \frac{dp}{dx}$

(1) will give $f(x, p, \frac{dp}{dx}) = 0$

Example $x y'' - y' = 3x^3$

y is missing

$\Rightarrow$ put $y' = p$ to get

$\frac{dp}{dx} - \frac{1}{x} p = 3x$

which is linear and hence

soln. will

$p = \frac{dy}{dx} = 3x^2 + c_1 x$

$\Rightarrow y = x^3 + \frac{1}{2} c_1 x^2 + c_2$

– If independent variable x is missing
~~(i.e. x is miss~~

$$g(y, y', y'') = 0$$

Put $y' = p$

$$\& \Rightarrow y'' = \frac{dp}{dx} = \frac{dp}{dy}\frac{dy}{dx} = p\frac{dp}{dy}$$

& eq^n become

$$g\left(y, p, p\frac{dp}{dy}\right) = 0$$

$\boxed{\text{Example}}$ Solve $y'' + k^2 y = 0$

D x is missing

$\Rightarrow$ put $y' = p$ we get

$$p\frac{dp}{dx} + k^2 y = 0$$

or $$p\,dp + k^2 y\,dy = 0$$

Integrating yields

$$p^2 + k^2 y^2 = k^2 a^2$$

$$\Rightarrow p = \frac{dy}{dx} = \pm k\sqrt{a^2 - y^2}$$

$$\Rightarrow \frac{dy}{\sqrt{a^2 - y^2}} = \pm k\,dx$$

$$\Rightarrow \sin^{-1}\left(\frac{y}{a}\right) = \pm kx + b$$

$$\Rightarrow y = a\sin(\pm kx + b)$$

or $y = A\sin(kx + B)$

$\Rightarrow y = c_1 \sin kx + c_2 \cos kx$