

POWER SERIES SOLUTIONS AND SPECIAL FUNCTIONS

FUNCTIONS

ELEMENTARY FUNCTIONS

(Algebraic, trigonometric, inverse trig., exponential, logarithmic or a combination of these)

TRANSCENDENTAL FUNCTIONS

(They are solutions of D.E. other than elementary fn's)

Algebraic fn.

→ A fn. which satisfies an eqn of the

form

$$P_n(x)y^{(n)} + P_{n-1}y^{(n-1)} + \dots + P_1(x)y' + P_0(x)y = 0$$

where $P_i(x)$ is a polynomial

Example

$$y'' + y = 0$$

Solv: $y = \sin x$ & $y = \cos x$ (Elementary non algebraic fn's.)

Example

$$x y'' + y' + xy = 0$$

We can not solve this eqn in terms of elementary fn's.

We look for a soln. in terms of a power series

Recall An ∞ series $\sum_{n=0}^{\infty} a_n x^n = a_0 + a_1 x + a_2 x^2 + \dots$ --- (1)is called a power series in x .Series $\sum_{n=0}^{\infty} a_n (x-x_0)^n = a_0 + a_1(x-x_0) + a_2(x-x_0)^2 + \dots$ lower series in $(x-x_0)$ --- (2)

Series (1) is convergent if

 $\lim_{m \rightarrow \infty} \sum_{n=0}^m a_n x^n$ exist.

①

— (1) is always convergent for $x=0$.

$$\sum_{n=0}^{\infty} u_n = u_0 + u_1 + u_2 + \dots$$

If $\lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| = L$ (exist)

then from ratio test series is convergent

if $L < 1$, diverges if $L > 1$

— For power series $\sum_{n=0}^{\infty} a_n x^n$, if

each $a_n \neq 0$ & for a fixed pt. $x \neq 0$

$$\lim_{n \rightarrow \infty} \left| \frac{u_{n+1}}{u_n} \right| = \lim_{n \rightarrow \infty} \left| \frac{a_{n+1} x^{n+1}}{a_n x^n} \right|$$

$$= \lim_{n \rightarrow \infty} \left| \frac{a_{n+1}}{a_n} \right| |x| = L$$

then by ratio test series will be conv.

if $L < 1$ & div if $L > 1$

$\Rightarrow \sum_{n=0}^{\infty} a_n x^n$ is conv. if $|x| < R$
div. if $|x| > R$

where $R = \lim_{n \rightarrow \infty} \left| \frac{a_n}{a_{n+1}} \right|$

(radius of convergence.)

Example

$\sum_{n=0}^{\infty} n! x^n$ is divergent $\forall x \neq 0$ $R=0$

$\sum_{n=0}^{\infty} \frac{x^n}{n!}$ conv. $\forall x$ $R=\infty$

$\sum_{n=0}^{\infty} x^n$ is conv. for $|x| < 1 \Rightarrow R=1$

SERIES SOLUTIONS OF FIRST ORDER EQUATIONS

Consider $y' = y + f(x) \dots (1)$

Suppose $y = a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n \dots \text{initial condition } x < R, R > 0$ do we find

is a soln. of (1) (~~if~~ is analytic at $x=0$)
assume it

\Rightarrow we can differentiate term by term

$$\Rightarrow y' = a_1 + 2a_2 x + 3a_3 x^2 + \dots + (n+1)a_{n+1}x^n \dots (3)$$

$\therefore y' = y \Rightarrow$ comparing coefficients gives us

$$a_1 = a_0 \quad \text{and} \quad 3a_3 = a_2$$

$$2a_2 = a_1$$

$$(n+1)a_{n+1} = a_n \dots \text{so on}$$

\Rightarrow we can express a_n in terms of a_0

$$\Rightarrow a_1 = a_0, a_2 = \frac{a_1}{2} = \frac{a_0}{2}$$

$$a_3 = \frac{a_2}{3} = \frac{a_0}{2 \cdot 3}, \dots, a_n = \frac{a_0}{n!}$$

$$\Rightarrow y = a_0 \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots \right) \dots (4)$$

with no conditions on a_0

\therefore series in (4) conv. $\forall x$ and we know

$$y = a_0 e^x$$

\Rightarrow this is the required soln.

Example

$$(1+x)y' = by, y(0) = 1 \dots (5)$$

Show that $y = (1+x)^b$, b is arbitrary

is a P.S. of (5) using power series method.

(Verify yourself!)

SECOND ORDER LINEAR EQUATIONS

$$y'' + P(x)y' + Q(x)y = 0 \quad (1)$$

(Homo.)

— When P & Q are const. this eqn can be solved in terms of elementary fun. In most general case this can only be solved by means of power series.

— Behavior of the soln. of (1) near a pt. x_0 depends on the behavior of its coeff. fun. $P(x)$ & $Q(x)$ near this pt. x_0 .

— Suppose $P(x)$ & $Q(x)$ are "well behaved" in the sense of being analytic at x_0 . This means that each has a power series expansion valid in some neighborhood of the point. We call this x_0 as an ordinary pt. of $y''(1)$. \Rightarrow Every soln. of (1) is also analytic at this pt. A point which is not an ordinary pt. of (1) is called a singular point.

Example

$$y'' + y = 0 \quad (2)$$

Here $P(x) = 0$ & $Q(x) = 1$

\Rightarrow P & Q are analytic at all pts.

We seek a soln. of the form

$$y = a_0 + a_1 x + a_2 x^2 + \dots + a_n x^n + \dots$$

$$\Rightarrow y' = a_1 + 2a_2 x + 3a_3 x^2 + \dots + (n+1)a_{n+1}x^n + \dots$$

$$\text{and } y'' = 2a_2 + 3a_3 x + 3 \cdot 4 a_4 x^2 + \dots + (n+1)(n+2)a_{n+2}x^n + \dots$$

\Rightarrow (2) gives

$$(2a_2 + a_0) + (2 \cdot 3 a_3 + a_1)x + (3 \cdot 4 a_4 + a_2)x^2 + (4 \cdot 5 a_5 + a_3)x^3 + \dots + [(n+1)(n+2)a_{n+2} + a_{n+1}]x^{n+1} + \dots = 0$$

(4)

⇒ Equating the coefficients of successive powers of x gives

$$2a_2 + a_0 = 0 \quad 3 \cdot 4 \cdot a_4 + a_2 = 0$$

$$2 \cdot 3 a_3 + a_1 = 0 \quad 4 \cdot 5 a_5 + a_3 = 0$$

$$(n+1)(n+2) a_{n+2} + a_n = 0$$

- Express a_n in terms of a_0 or a_1 depending on n is even or odd we get

$$a_2 = -\frac{a_0}{2}, \quad a_3 = -\frac{a_1}{2 \cdot 3}, \quad a_4 = -\frac{a_2}{3 \cdot 4} = \frac{a_0}{2 \cdot 3 \cdot 4}$$

$$a_5 = -\frac{a_3}{4 \cdot 5} = \frac{a_1}{2 \cdot 3 \cdot 4 \cdot 5}, \dots$$

$$\begin{aligned} \Rightarrow y &= a_0 + a_1 x - \frac{a_0}{2} x^2 - \frac{a_1}{2 \cdot 3} x^3 + \frac{a_0}{2 \cdot 3 \cdot 4} x^4 \\ &\quad + \frac{a_1}{2 \cdot 3 \cdot 4 \cdot 5} x^5 \dots \\ &= a_0 \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \dots\right) + a_1 \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots\right) \\ &\quad y_1(x) \quad \quad \quad y_2(x) \end{aligned}$$

- Note by choosing $a_0 = 1$ & $a_1 = 0$ satisfying eqn (2)

& by choosing $a_0 = 0$ & $a_1 = 1$ we see y_2 also satisfying the eqn (2).

- Both the series for y_1 & y_2 conv. & ∞

In fact one can see that $y_1 \approx \cos x$

$$\Rightarrow y = a_0 \cos x + a_1 \sin x \quad \& \quad y_2 = \sin x$$

for arbitrary constt. a_0 & a_1 .

Example Legendre's Equation

$$(1-x^2)y'' - 2xy' + b(b+1)y = 0 \quad (1)$$

b is a constt.

□ Coeff coefficients frs. $P(x) = -\frac{2x}{1-x^2}$ & $Q(x) = \frac{b(b+1)}{1-x^2}$

are analytic at the origin. $\Rightarrow x=0$ is an ordinary pt. & we expect a solution of the form

$$y = \sum a_n x^n \Rightarrow y' = \sum (n+1) a_{n+1} x^n$$

$$\& y'' = \sum (n+1)(n+2) a_{n+2} x^n$$

$$\Rightarrow -x^2 y'' = \sum -(n-1)n a_n x^n$$

$$\Rightarrow -2xy' = \sum -2na_n x^n$$

$$\Rightarrow b(b+1)y = \sum b(b+1)a_n x^n$$

Putting all this in eqn (1) & so the coefficient of x^n must be zero & n :

$$(n+1)(n+2)a_{n+2} - (n-1)n a_n - 2na_n + b(b+1)a_n = 0$$

With some manipulation this becomes

$$a_{n+2} = -\frac{(b-n)(b+n+1)}{(n+1)(n+2)} a_n \quad \dots \quad (2)$$

\Rightarrow we can express a_n in terms of a_0 or a_1 according as n is even or odd.

$$a_2 = -\frac{b(b+1)}{1 \cdot 2} a_0$$

$$a_3 = -\frac{(b-1)(b+2)}{2 \cdot 3} a_1$$

$$a_4 = -\frac{(b-2)(b+3)}{3 \cdot 4} a_2 = \frac{b(b-1)(b+1)(b+3)}{4!} a_0$$

$$a_5 = \frac{(b-3)(b+4)}{4 \cdot 5} a_3 = \frac{(b-1)(b-3)(b+2)(b+4)}{5!} a_1$$

& so on.

$$\Rightarrow \text{Sett of two series no ed ex tel} \quad \boxed{\text{MEASUREMENT}}$$

$$y = a_0 \left[1 - \frac{b(b+1)}{2!} x^2 + \frac{b(b-2)(b+1)(b+3)}{4!} x^4 \right]$$

(or) \therefore L.H.S. $= 1 + E$ $\underbrace{= b(b-2)(b-4)(b+1)(b+3)(b+5)}_{\text{in } (2) \text{ go. mil. 2 to 4 to 5 to 6 to 7 to 8}} x^6 + \dots$

$$+ a_1 \left[x - \frac{(b-1)(b+2)}{3!} x^3 + \frac{(b-1)(b-3)(b+2)(b+4)}{5!} x^5 \right.$$

$- \frac{(b-1)(b-3)(b-5)(b+2)(b+4)(b+6)}{7!} x^7 + \dots$

as our formed solution. $\dashrightarrow \text{---} \quad (3)$

— When b is not an integer each series in brackets has ~~no~~ radius of convergence $R = 1$. We can prove this using recursion formula (2).

Replace n by $2n$ gives $\text{Recd } R = (\infty)^2$

$$\left| \frac{a_{2n+2} x^{2n+2}}{a_{2n} x^{2n}} \right| = \left| \frac{(b-2n)(b+2n+1)}{(2n+1)(2n+2)} \right| |x|^2$$

as $n \rightarrow \infty \rightarrow |x|^2$

Similarly for second series $\Rightarrow R = 1$

— One can also see that both the series are l.i.
Thus for arbitrary const. $a_0 + a_1$ (3) is a general soln. for $|x| < 1$.

THEOREM Let x_0 be an ordinary point of the D.E.
 $y'' + P(x)y' + Q(x)y = 0 \quad (4)$ and let $a_0 + a_1$
 be arbitrary constants. Then \exists a! fn. $y(x)$
 that is analytic at x_0 , is a soln. of (4) in
 a certain nbd. of this pt. and satisfying
 the initial conditions $y(x_0) = a_0$ & $y'(x_0) = a_1$.
 Further, if the power series expansion of
 $P(x)$ & $Q(x)$ are valid on an interval $|x - x_0| < R$
 $R > 0$, then the power series soln. y is
 also valid on the same interval of x .

□ Try yourself. \square

Try yourself \Rightarrow Biharmonic \Rightarrow Laplace's

Hints: $\Delta^2 u = 0$ $\Rightarrow \nabla^2 (\nabla^2 u) = 0$ $\Rightarrow \nabla^2 u = \text{const.}$

- ① Take $x_0 = 0$ (W.L.O.G.)
- ② $P(x) = \sum_{n=0}^{\infty} p_n x^n$ & $Q(x) = \sum_{n=0}^{\infty} q_n x^n$ and for $|x| < R$
- ③ Seek the solution of the form $y = \sum_{n=0}^{\infty} a_n x^n$ with R as radius of conv. (at least)

Substitutes in (u) & solve.

REGULAR SINGULAR POINT

Suppose $y'' + P(x)y' + Q(x)y = 0 \quad \dots \quad (1)$

- A pt. x_0 is a singular pt. of the D.E. if one or ~~the other~~ the other (or both) of the coefficient frs. $P(x)$ & $Q(x)$ fails to be analytic at x_0 .

Example

$$y'' + \frac{2}{x}y' - \frac{2}{x^2}y = 0$$

$x=0$ is a singular pt.

- A singular pt. x_0 of eqn (1) is said to be regular if the frs. $(x-x_0)^1 P(x)$ and $(x-x_0)^2 Q(x)$ are analytic and irregular otherwise.

Example

$$y'' - \frac{2x}{(1-x^2)}y' + \frac{b(b+1)}{(1-x^2)}y = 0$$

has $x=1$ & $x=-1$ as singular pts.

↓ regular ↓ regular. (Legendre's eqn)

Example

Bessel's eqn of order b ($b \geq 0$, const.)

$$x^2y'' + xy' + (x^2 - b^2)y = 0$$

$$\Rightarrow y'' + \frac{1}{x}y' + \frac{x^2 - b^2}{x^2}y = 0$$

$\Rightarrow x=0$ is a regular singular pts
 $\therefore xP(x) = 1$ & $x^2Q(x) = x^2 - b^2$.