CCV-21

SC-216 Calculus with Complex Variables

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Laplace transform (L.T.)

$f(t)$ defined for $t \geq 0$

$$\mathcal{L}\{f(t)\} = F(s) = \int_0^\infty e^{-st} f(t)\,dt \quad \text{--- (1)}$$

Provided Integral exists as it is an improper integral

$$\mathcal{L}^{-1}\{F(s)\} = f(t) \quad \text{(Inverse L.T.) --- (2)}$$

Properties of L.T.

① Linearity $\mathcal{L}\{\alpha f(t) + \beta g(t)\} = \alpha \mathcal{L}\{f(t)\} + \beta \mathcal{L}\{g(t)\}$

(Verify All !)

② $\mathcal{L}\{e^{at}\} = \int_0^\infty e^{-st} e^{at}\,dt = \dfrac{1}{s-a} \quad (s>a)$

③ $\mathcal{L}\{1\} = \dfrac{1}{s} \quad (s>0) \Rightarrow \mathcal{L}^{-1}\left\{\dfrac{1}{s}\right\} = 1$

④ $\mathcal{L}\{t\} = \dfrac{1}{s^2} \quad (s>0) \Rightarrow \mathcal{L}^{-1}\left\{\dfrac{1}{s^2}\right\} = t$

& so on

⑤ $\mathcal{L}\{\sinh \omega t\} = \dfrac{\omega}{s^2 - \omega^2} \quad (s > \omega)$

⑥ $\mathcal{L}\{\cos at\} = \dfrac{s}{s^2 + a^2}$

— A fn. $f(t)$ is said to be of exponential order α if $\exists\, \alpha$ & $M > 0$ s.t. $|f(t)| \leq M e^{\alpha t}$, $t \geq 0$

— If $f(t)$ is a piecewise conts fn. on $[0, \infty)$ & is of exponential order α for $t \geq 0$ then $\mathcal{L}\{f(t)\}$ exists.

Type(2) — Give an example of a fn. $f(t)$ for which $\mathcal{L}\{f(t)\}$ does not exist

L.T. of Derivatives

① $\boxed{\mathcal{L}\{f'(t)\} = s\,\mathcal{L}\{f(t)\} - f(0)} = sF(s) - f(0)$

□ $\mathcal{L}\{f'(t)\} = \int_0^\infty e^{-st} f'(t)\,dt = \left[e^{-st} f(t)\right]_0^\infty$

$+ s\int_0^\infty e^{-st} f(t)\,dt$

$= -f(0) + s\,\mathcal{L}\{f(t)\}$, $s > \alpha$ ▨

② $$\boxed{\mathcal{L}\{f^{(n)}(t)\} = s^n \mathcal{L}\{f(t)\} - s^{n-1} f(0) - s^{n-2} f'(0) - \cdots - f^{(n-1)}(0)}$$

— Using L.T. of derivatives & inverse Laplace transform we can solve O.D.E.

Example $y'' + 4y = 0$, $y(0) = 1$, $y'(0) = 6$ (IVP)

□ Taking L.T. of the O.D.E.

$$\mathcal{L}\{y'' + 4y\} = \mathcal{L}\{0\} = 0$$

$$\Rightarrow \mathcal{L}\{y''\} + 4\,\mathcal{L}\{y\} = 0$$

$$\Rightarrow s^2 Y(s) - s\,y(0) - y'(0) + 4Y(s) = 0$$

using initial conditions we get

$$(s^2 + 4)\,Y(s) = s + 6$$

$$\Rightarrow Y(s) = \frac{s+6}{s^2+4} = \frac{s}{s^2+4} + \frac{6}{s^2+4}$$

$$\Rightarrow y(t) = \mathcal{L}^{-1}\{Y(s)\} = \mathcal{L}^{-1}\left\{\frac{s}{s^2+4}\right\} + \mathcal{L}^{-1}\left\{\frac{6}{s^2+4}\right\}$$

$$= \cos 2t + 3\sin 2t$$ ▨

→ Using Tables write ~~down~~ them directly.

Exercise: Solve the IVP using L.T.

$$y'' - 5y' + 4y = e^{2t}$$

Ans. $y = -\frac{1}{2}e^{2t} + \frac{14}{9}e^{t} + \frac{19}{36}e^{4t}$

$\boxed{\text{L.T. of Integral}}$

$$\boxed{\mathcal{L}\left\{\int_0^t f(\tau)\,d\tau\right\} = \frac{1}{s}\mathcal{L}\{f(t)\}} \quad s>0,\ s>\alpha$$

□ The result follows since $\frac{d}{dt}\left\{\underbrace{\int_0^t f(\tau)\,d\tau}_{=\ \phi(t)\ \text{(say)}}\right\} = f(t)$

$$\Rightarrow \mathcal{L}\{f(t)\} = \mathcal{L}\{\phi'(t)\}$$

$$= s\,\mathcal{L}\{\phi(t)\} - \underbrace{\phi(0)}_{=0} = s\,\mathcal{L}\{\phi(t)\}$$

$$\Rightarrow \mathcal{L}\left\{\int_0^t f(\tau)\,d\tau\right\} = \frac{1}{s}\mathcal{L}\{f(t)\}$$ ■

Example: ~~$\mathcal{L}^{-1}\left\{\frac{1}{s^2(s+4)}\right\}$~~ $\mathcal{L}^{-1}\left\{\frac{1}{s(s^2+9)}\right\}$

□ ~~$F(s) = \frac{1}{s^2(s+4)}$~~ $\mathcal{L}^{-1}\left\{\frac{1}{s}F(s)\right\} = \int_0^t \frac{1}{3}\sin 3\tau\,d\tau$

$$= \frac{1}{9}(1-\cos 3t)$$

since $\mathcal{L}\left\{\frac{\sin 3t}{3}\right\} = \frac{1}{s^2+9}$ ■

$\boxed{\text{Shifting formula}}$ If $\mathcal{L}\{f(t)\} = F(s)$ $s>\alpha\geq 0$, $\alpha\in\mathbb{R}$

$$\mathcal{L}\{e^{at}f(t)\} = F(s-a), \quad s>a+\alpha$$

$$\Rightarrow f(t) = e^{-at}\mathcal{L}^{-1}\{F(s-a)\} \quad \text{etc.}$$

$\boxed{\text{Heaviside fn.}}$

(Unit step fn.) $H(t) = \begin{cases} 0 & \text{if } t<0 \\ 1 & \text{if } t\geq 0 \end{cases}$

$$H(t-a) = u(t-a) := u_a(t) = \begin{cases} 0 & \text{if } t<a \\ 1 & \text{if } t\geq a \end{cases}$$

(7) $\mathcal{L}\{f(t-a)\,u_a(t)\} = e^{-as}F(s)$ $\quad s>\alpha,\ a\geq 0$

where $\mathcal{L}\{f(t)\} = F(s)$

(8) $\mathcal{L}\{u_a(t)\} = \dfrac{e^{-as}}{s}$ & $\mathcal{L}^{-1}\{e^{-as}F(s)\} = f(t-a)\,u_a(t)$

Dirac - Delta fn.

— δ- sequences

(1) $w_K(x) = \begin{cases} K/2, & |x| < \frac{1}{K} \\ 0, & |x| \geq 1/K \end{cases}$

then $\displaystyle\int_{-\infty}^{\infty} w_K(x)\,dx = \int_{-1/K}^{1/K} \frac{K}{2}\,dx = 1$

(2) $w_K(x) = \dfrac{K}{\pi(1+K^2x^2)}$, $K>0$

then also

$$\int_{-\infty}^{\infty} w_K(x) = \frac{K}{\pi}\int_{-\infty}^{+\infty} \frac{dx}{1+K^2x^2} = 1$$

Now we can define the Dirac Delta fn. using these δ-sequences as follows.

$$\boxed{\delta(x) = \lim_{K\to\infty} w_K(x)}$$

(3) Let $\delta_K(t) = \begin{cases} 0 & t<0 \\ 1/K & 0\leq t<K \\ 0 & t\geq K \end{cases}$

$$\delta_K(t) = \frac{1}{K}\left[u_0(t) - u_K(t)\right]$$
$$= \frac{1}{K}\left[H(t) - H(t-K)\right]$$

Let

$\delta(t) = \lim\limits_{K\to 0^+} \delta_K(t)$. $\Rightarrow$ $\delta(t) = 0$ for $t \neq 0$, at $t=0$ it is ∞

(4)

If we take $k \to 0$ in ~~the~~ $\delta_k(t)$

we arrive at $\boxed{H'(t) = \delta(t)}$

~~Filte~~ **Filtering property of δ-fn.**

— $\boxed{\int_0^{\infty} f(t)\, \delta(t-a)\, dt = f(a)}$

— $\mathcal{L}\{H'(t)\} = \mathcal{L}\{\delta(t)\} = 1$

— $\mathcal{L}\{t\, f(t)\} = -F'(s)$

— $\mathcal{L}\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n}\{F(s)\}$ $\quad s > \alpha$

— $\mathcal{L}\left\{\frac{f(t)}{t}\right\} = \int_s^{\infty} F(s^*)\, ds^*$, $s > \alpha$

$\mathcal{L}\{f(t)\} = F(s)$

Convolution Th.

If $f(t)$, $g(t)$ defined in $[0, \infty)$ then

$$(f * g)(t) = \int_0^t f(\tau)\, g(t-\tau)\, d\tau \quad (t \geq 0)$$

— $\mathcal{L}\{(f * g)(t)\} = \mathcal{L}\{f(t)\} \cdot \mathcal{L}\{g(t)\}$
$= F(s) \cdot G(s)$

— If $f(t)$ is piecewise conts on $[0, \infty)$, is of exp. order & periodic with period T then

$$\mathcal{L}\{f(t)\} = \frac{1}{1 - e^{-sT}} \int_0^T e^{-st} f(t)\, dt, \quad s > 0$$

— Please refer your book for a Table of L.T. of standard fns. !

Application of L.T. on P.D.E.

— $\mathcal{L}\left\{\frac{\partial u(x,t)}{\partial x}\right\} = \int_0^\infty \frac{\partial u}{\partial x} e^{-st}\, dt$

$$= \frac{\partial}{\partial x}\int_0^\infty e^{-st} u(x,t)\, dt = \frac{d}{dx}\{\mathcal{L}\{u(x,t)\}\}$$

— Also we write $U(x,s) = \mathcal{L}\{u(x,t)\}$

$\boxed{\text{Example}}$ Using L.T. find the soln. of the IVP.

$$x\frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} = xt, \quad u(x,0) = 0, \quad u(0,t) = t \qquad \text{— (1)}$$

□ Let $\mathcal{L}\{u(x,t)\} = U(x,s)$

Taking L.T. of (1) w.r.t. to t & the B.C. $u(0,t) = t$

we get

$$x\left[sU(x,s) - u(x,0)\right] + \frac{d}{dx}U(x,s) = \frac{x}{s^2}$$

$$U(0,s) = \frac{1}{s^2}$$

$$\Rightarrow \quad \frac{dU}{dx} + sxU = \frac{x}{s^2}$$

I.F. of the ODE is $e^{sx^2/2}$ ∴ soln

$$e^{sx^2/2}U = \int \frac{x}{s^2} e^{sx^2/2}\, dx + a(s) \quad \to \text{arbitrary fn. of } s$$

$$= \frac{1}{s^3} e^{sx^2/2} + a(s)$$

$$\Rightarrow \quad U(x,s) = \frac{1}{s^3} + a(s)\, e^{-sx^2/2}$$

using $U(0,s) = \frac{1}{s^2}$ we get $a(s) = \frac{1}{s^2} - \frac{1}{s^3}$

$$\Rightarrow \quad U(x,s) = \frac{1}{s^3} + \left(\frac{1}{s^2} - \frac{1}{s^3}\right) e^{-sx^2/2}$$

$$\Rightarrow u(x,t) = \frac{t^2}{2!} + \mathcal{L}^{-1}\left\{ \left(\frac{1}{s^2} - \frac{1}{s^3}\right) e^{-sx^2/2} \right\}$$

using shift ~~the~~ theorem

$$u(x,t) = \frac{t^2}{2!} + \left[\left(t - \frac{x^2}{2}\right) - \frac{1}{2}\left(t - \frac{x^2}{2}\right)^2 \right] u_{x^2/2}(t)$$

$$= \begin{cases} \dfrac{t^2}{2}, & t < \dfrac{x^2}{2} \\ \dfrac{t^2}{2} + \left(t - \dfrac{x^2}{2}\right) - \dfrac{1}{2}\left(t - \dfrac{x^2}{2}\right)^2, & t \geq \dfrac{x^2}{2} \end{cases}$$

Simly one can use F.T. to solve P.D.E $\blacksquare$
(See Example 9.36 page 664 of the text book)