SC107- Calculus and Complex Variables

Home Work 11

(1) Verify that the following functions (explicit or implicit) are solution of the corresponding differential equations.

(a). $y^2 = e^{2x} + c \qquad yy^{'} = e^{2x}$

(b). $y = ce^{kx} \qquad y^{'} = ky$

(c). $y = c_1 \sin 2x + c_2 \cos 2x \qquad y^{''} = -4y$

(d). $y = c_1 e^{2x} + c_2 e^{-2x} \qquad y^{''} = 4y$

solution:

(a):

$$y^2 = e^{2x} + c \Rightarrow 2yy' = 2e^{2x} \Rightarrow yy' = e^{2x}$$

(b):

$$y = ce^{kx} \Rightarrow y' = cke^{kx} \Rightarrow y' = ky$$

(c):

$$\begin{array}{rcl} y & = & c_1 \sin 2x + c_2 \cos 2x \\ \Rightarrow \quad y' & = & 2c_1 \cos 2x - 2c_2 \sin 2x \\ \Rightarrow \quad y'' & = & -4c_1 \sin 2x - 4c_2 \cos 2x \\ \Rightarrow \quad y'' & = & -4y \\ \Rightarrow \quad y'' + 4y & = & 0 \end{array}$$

(d):

$$\begin{array}{rcl} y & = & c_1 e^{2x} + c_2 e^{-2x} \\ \Rightarrow \quad y' & = & 2c_1 e^{2x} - 2c_2 e^{-2x} \\ \Rightarrow \quad y'' & = & 4c_1 e^{2x} + 4c_2 e^{-2x} \\ \Rightarrow \quad y'' & = 4y & \\ \Rightarrow \quad y'' - 4y & = 0 & \end{array}$$

(2) Find the general solution.

(a). $xy^{'} = 1$

(b). $y^{'} = xe^{x^2}$

(c). $(1 + x^2)dy + (1 + y^2)dx = 0$

(d). $y\log y dx - xdy = 0$

solution:

(a):

$x\frac{dy}{dx} = 1 \Rightarrow dy = \frac{dx}{x}$

Integrating $y = \ln x + c$

(b):

$\frac{dy}{dx} = xe^{x^2} \Rightarrow \int dy = \int xe^{x^2}dx$

Let,$x^2 = t \Rightarrow 2xdx = dt$

$\Rightarrow y = \int e^t \frac{dt}{2} = \frac{1}{2}e^t + c$

$\Rightarrow y = \frac{1}{2}e^{x^2} + c$

(C):

$$\begin{array}{lccc} & (1+x^2)dy + (1+y^2)dx & = & 0 \\ \Rightarrow & \frac{dy}{1+y^2} & = & -\frac{dx}{1+x^2} \\ \Rightarrow & \tan^- 1y + \tan^- 1x & = & c_1 \\ \Rightarrow & \tan^- 1\frac{x+y}{1-xy} & = & c_1 \\ \Rightarrow & \frac{x+y}{1-xy} & = & \tan c_1 = c \\ \Rightarrow & x+y & = & c(1-xy) \\ \Rightarrow & y(1+cx) = c - x & & \\ \Rightarrow & y & = & \frac{c-x}{1+cx} \end{array}$$

(d):

$$\begin{array}{lccc} & y\log y dx - xdy & = & 0 \\ \Rightarrow & \frac{dy}{y\log y} & = & \frac{dx}{x} \\ \Rightarrow & \log\log y & = & \log x + c \\ \Rightarrow & \log y & = & e^{logc_1 x} = c_1 x \\ \Rightarrow & y & = & e^{cx} \end{array}$$

(3) Show that

$$y = e^{x^2}\int_0^x e^{-t^2}dt$$

is a solution of

$$y' = 2xy + 1$$

solution:

$$\begin{aligned} y &= e^{x^2}\int_0^x e^{-t^2}dt \\ \Rightarrow\ y' &= 2xe^{x^2}\int_0^x e^{-}t^2dt + e^{x^2}[e^{-x^2}] \\ \Rightarrow\ y' &= 2xy+1 \end{aligned}$$

(4)Verify that the following equations are homogeneous and solve them.

(a). $xy^{'} = y + 2xe^{-\frac{y}{x}}$

(b). $xy^{'} = \sqrt{x^2+y^2}$

solution:

(a):

It is homogeneous with degree 0

$y' = \frac{y}{x} + 2e^{-\frac{y}{x}}$

let $\frac{y}{x} = z$

$$\begin{aligned} \Rightarrow\ \frac{dy}{dx} &= z + x\frac{dz}{dx} \\ \Rightarrow\ z + x\frac{dz}{dx} &= z + 2e^{-z} \\ \Rightarrow\ \frac{dz}{2e^{-z}} &= -\frac{dx}{x} \\ \Rightarrow\ \frac{1}{2}e^z &= -\ln x + c_1 \\ \Rightarrow\ e^z &= -2\ln x + c_2 \\ \Rightarrow\ z &= \log(\log cx^2) \\ \Rightarrow\ y &= x\log(\log cx^2) \end{aligned}$$

(b):

It is homogeneous of degree 0

let $\frac{y}{x} = z$

$$\begin{aligned} x\frac{dy}{dx} &= \sqrt{x^2+y^2} \\ \Rightarrow\ \frac{dy}{dx} &= \sqrt{1+(\tfrac{y}{x})^2} \\ \Rightarrow\ z + x\frac{dz}{dx} &= \sqrt{1+z^2} \\ \Rightarrow\ \frac{dz}{\sqrt{1+z^2}-z} &= \frac{dx}{x} \\ \Rightarrow\ \frac{[\sqrt{1+z^2}+z]dz}{1+z^2-z^2} &= \frac{dx}{x} \\ \Rightarrow\ \frac{z}{2}\sqrt{1+z^2} + \frac{1}{2}\ln\left|z+\sqrt{1+z^2}\right| + \frac{z^2}{2} &= \ln x + c \\ \Rightarrow\ \frac{y}{2x}\sqrt{1+(\tfrac{y}{x})^2} + \frac{1}{2}\ln\left|\tfrac{y}{x}+\sqrt{1+(\tfrac{y}{x})^2}\right| + \frac{(\frac{y}{x})^2}{2} &= \ln x + c \\ \Rightarrow\ y\sqrt{x^2+y^2} + y^2 + x^2\ln\left|y+\sqrt{x^2+y^2}\right| - x\ln x &= 2x^2\ln x + cx^2 \\ \Rightarrow\ y\sqrt{x^2+y^2} + y^2 + x^2\ln\left|y+\sqrt{x^2+y^2}\right| - 3x^2\ln x &= cx^2 \end{aligned}$$

(5) Find the value of n for which each of the following equations is exact and solve the equation for that values of n.

(a). $(xy^2 + nx^2y)dx + (x^3 + x^2y)dy = 0$

(b). $(x + ye^{2xy})dx + (nxe^{2xy})dy = 0$

solution:

(a): $\frac{\partial M}{dy} = \frac{\partial N}{dx}$ **is the necessary condition for exactness**

$$\begin{aligned} M &= xy^2 + nx^2y \\ \frac{\partial M}{dy} &= 2xy + nx^2 \\ N &= x^3 + x^2y \\ \frac{\partial N}{dx} &= 3x^2 + 2xy \end{aligned}$$

By comparison $n = 3$ **As the equation follows the exactness**

$$\begin{array}{rcccl} & M & = & \frac{\partial f}{dx} & = xy^2 + 3xy^2 \\ & N & = & \frac{\partial f}{dy} & = x^3 + x^2y \\ \Rightarrow & f & = & \frac{x^2y^2}{2} + \frac{3x^3}{3}y + g(y) & \\ \Rightarrow & \frac{\partial f}{dy} & = & \frac{1}{2}x^2 2y + x^3 + g'(y) & \\ \Rightarrow & g'(y) & = & 0 & \\ \Rightarrow & g(y) & = & c_1 & \\ \Rightarrow & f(x,y) & = & \frac{1}{2}x^2y^2 + x^3y & = c_1 \\ \Rightarrow & x^2y^2 + 2x^3y & = & c & \end{array}$$

(b):

$$\begin{aligned} M &= x + ye^{2xy} \\ \frac{\partial M}{dy} &= e^{2xy} + 2xye^{2xy} \\ &= e^{2xy}(1 + 2xy) \\ N &= nxe^{2xy} \\ \frac{\partial N}{dx} &= ne^{2xy}[1 + 2xy] \end{aligned}$$

$n = 1$

As the equation follows exactness:

$$\begin{array}{rcccl} M & = & \frac{\partial f}{\partial x} & = & x + ye^{2xy} \\ N & = & \frac{\partial f}{\partial y} & = & xe^{2xy} \\ \Rightarrow & & f & = & \frac{x^2}{2} + \frac{ye^{2xy}}{2y} + g(y) \\ & & & = & \frac{x^2}{2} + \frac{e^{2xy}}{2} + g(y) \\ & & \frac{\partial f}{\partial y} & = & \frac{1}{2}2xe^{2xy} + g'(y) \\ & & & & \textbf{comparing } g'(y) = 0 \\ \Rightarrow & & f(x,y) & = & x^2 + e^{2xy} + c \end{array}$$

(6) Show that if $\frac{\left(\frac{\partial M}{\partial y} - \frac{\partial n}{\partial x}\right)}{N}$ **is a function of** $g(x)$ **then the integrating factor**

$$\mu = e^{\int g(x)dx}$$

solution:
Let $M(x,y)dx + N(x,y)dy = 0$ **is not exact and** $\mu M(x,y)dx + \mu N(x,y)dy = 0$ **is exact, where** μ **is an integrating factor**

$$\begin{aligned}
\Rightarrow \quad \frac{\partial(\mu M)}{dy} &= \frac{\partial(\mu N)}{dx} \\
\Rightarrow \quad \mu\frac{\partial M}{dy} + M\frac{\partial \mu}{dy} &= \mu\frac{\partial N}{dx} + N\frac{\partial u}{dx} \\
\Rightarrow \quad \frac{1}{\mu}\left(N\frac{\partial u}{dx} - M\frac{\partial \mu}{dy}\right) &= \frac{\partial M}{dy} - \frac{\partial N}{dx} \quad \to (i)
\end{aligned}$$

There are multiple solution for μ **are possilbe Let** μ **is a function of** x **only, then**

$$\frac{\partial u}{dx} = \frac{d\mu}{dx} \text{ and } \frac{\partial \mu}{dy} = 0$$
$$\Rightarrow \frac{1}{\mu}\frac{d\mu}{dx} = \frac{\left(\frac{\partial M}{dy} - \frac{\partial N}{dx}\right)}{N} = g(x)$$

$$\begin{aligned}
\textbf{integrating} \quad \log\mu &= \int g(x)dx \\
\Rightarrow \quad \mu &= e^{\int g(x)dx}
\end{aligned}$$

Same way if μ **is the function of** y **only then**

$$\frac{1}{\mu}\frac{d\mu}{dy} = \frac{\left(\frac{\partial M}{dy} - \frac{\partial N}{dx}\right)}{M} = g(y) \Rightarrow \mu = \mu = e^{\int g(y)dy}$$

(7) Solve each of the following equations by finding an integrating factor.

(a). $e^x dx + (e^x \cot y + 2y\csc y)dy = 0$

(b). $ydx + (x - 2x^2y^3)dy = 0$

(c). $(x + 3y^2)dx + 2xydy = 0$

solution:

(a):

$$\begin{aligned}
M &= e^x \\
\frac{\partial M}{dy} &= 0 \\
N &= e^x coty + 2y\csc y \\
\frac{\partial N}{dx} &= e^x \cot y \\
\frac{\frac{\partial M}{dy} - \frac{\partial N}{dx}}{-M} &= -\cot y &= h(y) \\
\mu &= e^{\int h(y)dy} \\
&= e^{-\int \cot y dy} \\
&= e^{\ln|\cos y|} &= \sin y
\end{aligned}$$

The function $\mu M dx + \mu N dy = 0$

$$\begin{aligned}
\frac{\partial f}{\partial y} &= e^x \cos y + 2y \\
\frac{\partial f}{\partial x} &= e^x \sin y \\
\Rightarrow \quad f &= e^x \sin y + g(y) \\
\Rightarrow \quad \frac{\partial f}{\partial y} &= e^x \cos y + g'(y) \\
\Rightarrow \quad g'(y) &= 2y \\
\Rightarrow \quad g(y) &= y^2 \\
\Rightarrow \quad f &= e^x \sin y + y^2 + c
\end{aligned}$$

(b):

$$\begin{aligned}
& ydx + (x - 2x^2y^3)dy &&= 0 \\
\textbf{we know} \quad & d(xy) &&= xdy + ydx \\
\Rightarrow \quad & d(xy) - (2x^2y^3)dy &&= 0 \\
\Rightarrow \quad & d(z) - 2yz^2dy &&= 0 \\
\Rightarrow \quad & \frac{dz}{z^2} &&= 2ydy \\
\Rightarrow \quad & -\frac{1}{3z^3} &&= \frac{y^2}{2} + c_1 \\
\Rightarrow \quad & \frac{y^2}{2} + \frac{1}{3x^3y^3} &&= c
\end{aligned}$$

(c):

$$\begin{aligned}
& (x + 3y^2)dx + 2xydy &&= 0 \\
& M &&= x + 3y^2 \\
\Rightarrow \quad & \frac{\partial M}{dy} &&= 6y \\
& N &&= 2xy \\
\Rightarrow \quad & \frac{\partial N}{dx} &&= 2y \\
\Rightarrow \quad & \frac{\partial M}{dy} - \frac{\partial N}{dx} &&= 4y \\
\Rightarrow \quad & \frac{\frac{\partial M}{dy} - \frac{\partial N}{dx}}{N} &&= \frac{4y}{2xy} = \frac{2}{x} = h(x) \\
\Rightarrow \quad & \mu &&= e^{\int h(x)dx} = e^{\int \frac{2}{x}dx} = e^{2\log x} \\
& &&= x^2 \\
& \mu M dx + \mu N dy &&= 0 \\
\Rightarrow \quad & (x^3 + 3x^2y^2)dx + 2x^3ydy &&= 0
\end{aligned}$$

This is exact

$$\begin{aligned}
\frac{\partial f}{\partial y} &= 2x^3y \\
\frac{\partial f}{\partial x} &= x^3 + 3x^2y \\
\Rightarrow \quad f &= \frac{x^4}{4} + 3y^2\frac{x^3}{3} + g(y) \\
\frac{\partial f}{\partial y} &= 2x^3y + g'(y) \\
\Rightarrow \quad g'(y) &= 0 \\
\Rightarrow \quad g(y) &= c \\
\Rightarrow \quad f(x,y) &= x^4 + 4y^2x^3 = c
\end{aligned}$$

(8) The equation $\frac{dy}{dx}+p(x)y = Q(x)y^n$ which is known as Bernoullis equation is linear when $n = 0$ *or* 1. Show that it can be reduced to a linear equation for any other value of n by the change of variable $z = y^{1-n}$ and apply this method to solve the following equation.

$$xy' + y = x^4y^3$$

solution:
$\frac{dy}{dx} + P(x)y = Q(x)y^n$
Let $z = y^{(1-n)} \Rightarrow z^{\frac{1}{1-n}} = y$ **or** $y^n = z^n1 - n$

$$\begin{aligned} \frac{1}{y^n}\frac{dy}{dx} + P(x)\frac{y}{y^n} &= Q(x) \\ \Rightarrow \quad \frac{dz}{dx} + P(x)z &= Q(x) \end{aligned}$$

which is a linear equation.

$$\begin{aligned} x\frac{dy}{dx} + y &= x^4y^3 \\ \frac{1}{y^3}\frac{dy}{dx} + \frac{1}{x}\frac{y}{y^3} &= x^3 \\ \frac{dz}{dx} + \frac{1}{x}z &= x^3 \quad \textbf{taking } z = y^{-2} \end{aligned}$$

$P = \frac{1}{x} \int Pdx = \log xe^{\int Pdx} = e^{\log x} = x$
$\frac{Q}{dx}(e^{\int pdx}z) = \frac{d}{dx}(xz) = Qe^{\int Pdx} = x^3x = x^4$

$$\begin{aligned} \Rightarrow \quad \frac{d}{dx}(xz) &= x^4 \\ \Rightarrow \quad xz &= \frac{1}{5}x^5 + c \\ \Rightarrow \quad \frac{x}{y^2} &= \frac{1}{5}x^5 + c \\ \Rightarrow \quad 1 &= \frac{1}{5}x^4y^2 + c\frac{y^2}{x} \\ \Rightarrow \quad x^5y^2 + 5cy^2 - 5x &= 0 \end{aligned}$$

which is solution

(9) Solve the following equations.

(a). $x^2y'' = 2xy' + (y')^2$

(b). $yy'' - (y')^2 = 0$

solution:
(a): As y **term is missing let** $p = \frac{dy}{dx}$

$$\begin{aligned} \Rightarrow \quad x^2\frac{dP}{dx} - 2xp - p^2 &= 0 \\ \Rightarrow \quad \frac{dp}{dx} - \frac{2p}{x} &= \frac{p}{x^2}^2 \end{aligned}$$

Let $z = p^{1-2} = \frac{1}{p} or \frac{p}{p^{-2}} = z$

$$\begin{array}{rcccl}
& \frac{1}{p^2}\frac{dp}{dx} - \frac{2}{x}\frac{p}{p^2} & = & \frac{1}{x^2} & \\
& \frac{dz}{dx} - \frac{2}{x}z & = & \frac{1}{x^2} & \\
& P(x) = -\frac{2}{x} & & Q(x) = \frac{1}{x^2} & \\
& e^{\int Pdx} & = & e^{-2\log x} & = \quad x^{-2} \\
& \frac{d}{dx}(e^{\int Pdx}z) & = & Qe^{\int pdx} & \\
\Rightarrow & \frac{d}{dx}(x^{-2}z) & = & \frac{1}{x^2}x^{-2} & = \quad 1
\end{array}$$

$\frac{z}{x^2} = c_1$ **or** $\frac{1}{px^2} = c_1$ **or** $px^2 = c_2$
As,$p = \frac{dy}{dx}$ $\frac{dy}{dx}x^2 = c_2$ **or** $dy = c_2\frac{dx}{x^2}$
$y = -\frac{c_2}{3x^3} + c_3$

(b):

$yy'' - (y')^2 = 0$
As there is no x **term: Let** $y' = p$ $y^2 = \frac{dp}{dx} = \frac{dp}{dy}\frac{dy}{dx} = p\frac{dp}{dy}$

$$\begin{array}{rcccl}
& yp\frac{dp}{dy} - p^2 & = & 0 & \\
\Rightarrow & ypdp & = & p^2dy & \\
\Rightarrow & \frac{pdp}{p^2} & = & \frac{dy}{y} & \\
\Rightarrow & \frac{dp}{p} & = & \frac{dy}{y} & \\
\Rightarrow & p & = & \ln y + c_1 & \\
\Rightarrow & p & = & c_2y & \\
\Rightarrow & \frac{dy}{dx} & = & c_2y & \\
\Rightarrow & \frac{dy}{y} & = & c_2dx & \\
\Rightarrow & \ln y & = & c_2x + c_3 & \\
\Rightarrow & y & = & e^{c_2x+c_3} & = \quad c_4e^{c_2x}
\end{array}$$