Calculus (SC216)
Tutorial 12 Solution
Second Order Linear Equations

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November 8, 2015

1 Topics covered:

- General solution of homogeneous and and non-homogeneous differential equation.
- Use of known solution to fins another.
- Homogeneous equation with constant coefficient.
- Method of undetermined coefficient.
- Method of variation of parameters.
- Higher order linear equation.

(1)(a) Verify that $y_1 = 1$ and $y_2 = x^2$ are solutions of the reduced equation $xy^{''} - y^{'} = 0$ and write down the general solution.
Solution:

$$y_2 = x^2 \Rightarrow y_2^{'} = 2x \; and \; y_2^{''} = 2.$$
$$\Rightarrow xy_2^{''} - y_2^{'} = 0 \Rightarrow y_2 \; is \; a \; solution.$$
$$G.S : C_1 + C_2x^2.$$

(b) Determine the value of a for which $y_p = ax^3$ is a particular solution of the complete equation $xy^{''} - y^{'} = 3x^2$. Use this solution and result of part (a) to write down the general solution of this equation.
Solution:

$$y_p = ax^3 \Rightarrow y_p^{'} = 3ax^2 \quad y_p^{''} = 6ax.$$
$$L.H.S \quad xy_p^{''} - y_p^{'} = 6ax^2 - 3ax^2 = 3ax^2 = 3x^2 \; (R.H.S).$$
$$\Rightarrow a = 1. \tag{1}$$

The equation is non-homogeneous.
The general solution of non-homogeneous equation = general solution of homogeneous equation + particular solution.
For general solution of homogeneous equation.

$$xy^{''} - y^{'} = 0.$$

As in (a) $y_1 = 1$ and $y_2 = x^2$ are the general solution.

$$\Rightarrow (G.S)_{nonhom} = C_1 + C_2x^2 + x^3.$$

(C) Can you discover y_1, y_2 and y_p by inspection?
Solution:

Yes.
(2) Very that $y_1 = 1$ and $y_2 = log(x)$ are the solutions of the equation $xy^{''} + y^{'} = 0$ and write down the general solution. Can you discover y_1 and y_2 by inspection?
Solution:

$$y_2 = log(x) \Rightarrow y_2^{''} = \frac{-1}{x^2}..$$

$$xy_2^{''} + y_2^{'} = \frac{-1}{x} + \frac{1}{x} = 0 \Rightarrow y_2 \; is \; the \; solution. \tag{2}$$

G.S: $C_1 + C_2 log(x)$.
(3). Use the inspection or experiment to find a particular solution for each of the following equation.
(a) $x^3y^{''} - x^2y^{'} + xy = 1$.
Solution:

$$y_p = \frac{1}{2x} \Rightarrow y_p^1 = \frac{-1}{2x^2} \Rightarrow y_p^{''} = \frac{1}{x^3}.$$

$$x^3y_p^{''} - x^2y_p^{'} + xy_p = 1 - \frac{1}{2} + \frac{1}{2} = 1.$$

$$\Rightarrow y_p = \frac{1}{2x}.$$

[we may start with $y_p = ax^n$]. (b) $y_{''} - 2y^{'} = 6$.
Solution:
$y_{''} - 2y^{'} = 6$ is not a function of $x \Rightarrow y^{''}$ & $y^{'}$ are not function of x.
Let

$$y = ax \Rightarrow y^{'} = a \Rightarrow y^{''} = 0.$$

$$y^{''} - 2y^{'} = -2a = 6 \Rightarrow a = -3.$$

(C) $y^{''} - 2y = sin(x)$.
Let $y = a \cdot sin(x) \Rightarrow y^{'} = a \cdot cos(x) \Rightarrow y^{''} = -a \cdot sin(x)$.
$y^{''} - 2y = -a \cdot sin(x) - 2a \cdot sin(x) = -3a \cdot sin(x) = sin(x)$.
$\Rightarrow a = \frac{-1}{3}$ & $y_p = \frac{-sin(x)}{3}$.

(4). If $Y_1(x)$ and $y_2(x)$ are two solution of equation (2) (homogeneous) on an interval [a,b], and have a common zero in this interval, show that one is a constant multiple of the other. [Recall that a print x_0 is said to be a zero of a function $f(x_0) = 0$].
Solution:
$y^{''} + P(x)y^{'} + Q(x)y = 0$ is the homogeneous equation.
$y = C_1y_1(x) + C_2y_2(x)$ is the general solution of it.
At some $x = x_0, x \in [a, b]$. $y_1(x_0) = y_2(x_0) = 0 = y(x_0)$.
Lemma 1 says, $w(y_1, y_2)$ is either identically zero or neither zero.
$\Rightarrow w(y_1, y_2)$ s identically zero here.
$y_1(x)$ and $y_2(x)$ are linearly dependent (lemma 2).

(5) By inspection or experiment, find the linearly independent solution of $x^2y^{''} - 2y = 0$ on the interval [1,2] and determine the particular solution satisfying the initial condition $y(1) = 1$, $y^{'}(1) = 8$.
Solution:
$x^2y^{''} - 2y = 0$.

$$y_1 = nx^2 \Rightarrow y_1^{'} = 2nx \Rightarrow y_1^{''} = 2n.$$

$$x^2y_1^{''} - 2y_1 = 2nx^2 - 2nx^2 = 0 \Rightarrow n = 1 \Rightarrow y_1 = x^2.$$

$y_2 = \frac{n}{x} \Rightarrow y_2^{'} = \frac{-n}{x^2} \Rightarrow y_2^{''} = \frac{2n}{x^3}$.

$$x^2y_2^{''} - 2y_2 = \frac{2n}{x^3}x^2 - \frac{2n}{x} = 0 \Rightarrow n = 1 \Rightarrow y_2 = \frac{1}{x}. \tag{3}$$

The G.S: $y = C_1x^2 + C_2\frac{1}{x}$.
$y(1) = 1$, $y^{'}(1) = 8$.
$y(1) = C_1 + C_2 = 1 \rightarrow (1)$
$y\prime(x) = 2C_1(x) - \frac{C_2}{x^2}$
$y^{'}(1) = 2C_1 - C_2 = 8 \rightarrow (2)$
from (1) and (2), $C_1 = 3$ and $C_2 = -2$.
The P.I : $y(x) = 3x^2 - \frac{2}{x}$.

(6) (a) $y'' + (y')^2 = 0$.
Method:1.

$$y' = p \Rightarrow y'' = p'.$$

$$p' + p^2 = 0$$

$$\frac{dp}{dx} = -p^2 \Rightarrow \frac{1}{p^2}dp = -dx$$

Take integration both side.

$$\frac{1}{p} = x + c$$

$$\frac{dx}{dy} = x + c$$

$$\frac{dx}{x + c} = dy$$

$$logx + c = y + c_2$$

$$y = logx + c + c_3$$

Method:2. independent variable x is missing:

$$y' = p \Rightarrow y'' = p'$$

$$\frac{dp}{dx} = \frac{dp}{dy}\frac{dy}{dx} = p\frac{dp}{dy}$$

The equation is:

$$y'' + (y')^2 = 0$$

$$\frac{dp}{dx} + p^2 = 0$$

$$p\frac{dp}{dy} + p^2 = 0$$

$$\frac{dp}{dy} + p = 0$$

This is the linear equation $\frac{dy}{dx} + P(x)y + Q(x) = 0$
Here $P(x) = 1, Q(x) = 0$
Solution:

$$e^y = c_1x + c_2$$

$$y = log(x + c_1) + c_3$$

(c) It is proved that $y_1 = 1$ and $y_2 = log(x)$ are linearly independent solutions. To verify whether, $y = c_1 + c_2log(x)$ is the solution

$$y' = \frac{c_2}{x}, y'' = \frac{-c_2}{x^2}$$

$$y'' + y'^2 = \frac{-c_2}{x^2} + \frac{c_2^2}{x^2} = \frac{c_2^2 - C_2}{x^2}$$

(7) Use the wronskian to prove that two solutions of the homogeneous equation on an interval [a,b] are linearly dependent if (a) They have a common zero x_0 in the interval.
(b) They have maxima or minima at the same point x_0 in the interval.

$$w(y_1, y_2) = y_1 y_2^{'} - y_2 y_1^{'}$$

Solution: (a) $y_1(x_0) = y_2(x_0) = 0 \Rightarrow w(y_1, y_2) = 0$.
From Lemma 1 $\rightarrow w(y_1, y_2)$ is identically zero.
From lemma 2 $\rightarrow y_1, y_2$ arelinearly dependent.

(b) $y_2^{'}(x_0) = 0 = y_1^{'}(x_0) \Rightarrow w(y_1, y_2) = 0$.
$\Rightarrow y_1 and y_2$ are linearly dependent.

(8)Consider the two functions $f(x) = x^3$ and $g(x) = x^2$ on the interval [-1,1].
(a) Show that their wronskian w(f,g) vanish identically.
(b) show that f and g are not linearly dependent.
(c) Do (a) and (b) contradict lemma 2? if not why not?
$\rightarrow w(f, g) = fg^{'} - gf^{'}$
Solution: (a) $f(x) = x^3$ and $g(x) = x^3 : x > 0$, and $g(x) = -x^3 : x < 0$.

$$w(f, g) = x^3(3x^2) - x^3(3x^2) = 0 \; if x > 0$$
$$w(f, g) = x^3(-3x^2) - x^3(-3x^2) = 0$$

$\Rightarrow w(f, g) = 0$
(b) $w(f, g) = fg^{'} - gf^{'} = 0$
$f(x) = x^3$ and $g(x) = x^2|x|$
let $f(x) = kf(y)$ then k=1 x¿0.
$f(x) = kf(y)$ then k=-1 x¡0.
(c) (a) and (b) are in support with lemma 2.
(9) The equation $xy'' + 3y' = 0$ has the one solution $y_1 = 1$. find y_2 and the general solution.

$$xy'' + 3y' = 0$$
$$y'' + \frac{3}{x}y' = 0$$

Here $p(x) = \frac{3}{x}$

$$V = \int \frac{1}{y_1^2} e^{-\int p dx} dx$$
$$V = \int e^{-\int \frac{3}{x} dx} dx$$
$$V = \int e^{-3log(x)} dx$$
$$V = \int \frac{1}{x^3} dx$$
$$V = \frac{-1}{2x^2}$$
$$y_2 = vy_1 \Rightarrow y_2 and \; y_1 are \; linearly \; independent$$

General solution $y = c_1 + c_2 \frac{1}{x^2}$
(10) Find the solution of $y'' - xf(x)y' + f(x)y = 0$
Solution:

$$y'' - xf(x)y' + f(x)y = 0$$

Where $P(x) = -xf(x) and Q(x) = f(x)$
As the linear equation the solution is By inspection method, We can say that $y_1 = x$ is one of the solution. let

other solution is $y_2 = vy_1$.

$$V = \int \frac{1}{y_1^2} e^{-\int p dx} dx$$

$$V = \int \frac{1}{x^2} e^{\int x f(x) dx} dx$$

General solution is $y = c_1 x + \int \frac{1}{x^2} e^{\int x f(x) dx} dx$

(11) Find the general solution of each of the following equation: (a) $y'' - 4y' + 4y = 0$
let $y = e^{mx}$ is the solution.

$$y' = me^{mx} \; and \; y'' = m^2 e^{mx}$$

$$(m^2 - 4m + 4)e^{mx} = 0$$

$$(m^2 - 4m + 4) = 0 \Rightarrow m = 2, 2$$

one of the solution is $y_1 = e^{2x}$. let other solution is $y_2 = vy_1$

$$V = \int \frac{1}{y_1^2} e^{-\int p dx} dx$$

$$V = \int \frac{1}{e^4 x} e^{-\int 4 dx} dx = x$$

Second solution is $y_2 = xe^{2x}$.
General solution: $y = (c_1 + c_2 x)e^{2x}$
(b) $4y'' - 12y' + 9y = 0$
let $y = e^{mx}$ is the solution.

$$y' = me^{mx} \; and \; y'' = m^2 e^{mx}$$

$$(m^2 - 3m + \frac{9}{4})e^{mx} = 0$$

$$(m^2 - 3m + \frac{9}{4}) = 0 \Rightarrow m = \frac{3}{2}$$

one of the solution is $y_1 = e^{\frac{3}{2}x}$. let other solution is $y_2 = vy_1$

$$V = \int \frac{1}{y_1^2} e^{-\int p dx} dx$$

$$V = \int e^{-\frac{3x}{2}} e^{-\int e^{3x/2} dx} dx \tag{4}$$

Second solution is $y_2 = xe^{\frac{3}{2}x}$.
General solution: $y = (c_1 + c_2 x)e^{\frac{3}{2}x}$
(c) $y'' - 9y' + 20y = 0$
let $y = e^{mx}$ is the solution.

$$y' = me^{mx} \; and \; y'' = m^2 e^{mx}$$

$$(m^2 - 9m + 20)e^{mx} = 0$$

$$(m^2 - 9m + 20) = 0 \Rightarrow m = 5, 4$$

General solution: $y = c_1 e^{4x} + c_2 e^{5x}$.

(12) In this problem we present another way of discovering the second linearly independent solution of $y'' + py' + qy = 0$ where p and q are constant. When the roots of the auxiliary equation are real and equal.
(a) If $m_1 = m_2$, verify that the differential equation $y'' - (m_1 + m_2)y' + m_1 m_2 y = 0$ has $y = \frac{e^{m_1 x} - e^{m_2 x}}{m_1 - m_2}$ as a solution.
(c) Verify that the limit in part (b) sastisfy the differential equation obtained from the equation in part(a) by replacing m_1 by m_2.

Solution: (a) $y'' + py' + qy = 0 \to (1)$.
$y'' - (m_1 + m_2)y' + m_1 m_2 y = 0 \to (2)$.
$y = e^{mx}$ be the solution of (1).
$(m^2 + pm + q)e^{mx} = 0 \Rightarrow (m^2 + pm + q) = 0$.
As, $p = -(m_1 + m_2)$ and $q = (-m_1)(-m_2)$.
m_1 and m_2 are solution of this differential equation.
General solution: $y = c_1 e^{m_1 x} + c_2 e^{m_2 x}$. Or $y = \frac{g}{m_1 m_2} e^{m_1 x} - e^{m_2 x}$ is also a solution.
Verify :

$$y = \frac{1}{m_1 - m_2} e^{m_1 x} - e^{m_2 x}$$

$$y' = \frac{1}{m_1 - m_2} m_1 e^{m_1 x} - m_2 e^{m_2 x}$$

$$y'' = \frac{1}{m_1 - m_2} {m_1}^2 e^{m_1 x} - {m_2}^2 e^{m_2 x}$$

$$\textit{Equation is } y'' - (m_1 + m_2)y' + m_1 m_2 y$$

$$y'' - (m_1 + m_2)y' + m_1 m_2 y = \frac{1}{m_1 - m_2} {m_1}^2 e^{m_1 x} - {m_2}^2 e^{m_2 x} - \frac{m_1 + m_2}{m_1 - m_2} m_1 e^{m_1 x} - m_2 e^{m_2 x} + \frac{m_1 m_2}{m_1 - m_2} e^{m_1 x} - e^{m_2 x}$$

$$= \frac{m_1^2 - (m_1 + m_2)m_1 + m_1 m_2}{m_1 - m_2} e^{m_1 x} - \frac{m_2^2 - (m_1 + m_2)m_2 + m_1 m_2}{m_1 - m_2} e^{m_1 x} = 0$$

y is the solution.
(c) $y'' - (m_1 + m_2)y' + m_1 m_2 y = 0$

$$y_2'' - (m_1 + m_2)y_2' + m_1 m_2 y_2 = (m_2 + m_2 + x m_2^2)e^{x m_2} - (m_1 + m_2)(1 + x m_2)e^{x m_2} + x m_1 m_2 e^{x m_2}$$

$$= (2m_2 + x m_2^2 - m_1 - m_2 - x m_1 m_2 - x m_2^2 + x m_1 m_2)$$

$$= (m_2 - m_1)e^{x m_2} = 0 (as m_1 \to m_2)$$

(13) If $y_1(x)$ and $y_2(x)$ are solution of $y'' + p(x)y' + Q(x)y = R_1(x)$ and $y'' + p(x)y' + Q(x)y = R_2(x)$. Show that $y(x) = y_1(x) + y_2(x)$ is a solution of $y'' + p(x)y' + Q(x)y = R_1(x) + R_2(x)$. This is called the principle of superposition use this principle to find the general solution of

(a) $y'' + 4y = 4\cos(2x) + 6\cos(x) + 8x^2 - 4x$. $y_1(x)$ and $y_2(x)$ are the solution of the given equations.
${y_1}'' + p(x){y_1}' + Q(x)y_1 = R_1(x)$.
${y_2}'' + p(x){y_2}' + Q(x)y_2 = R_2(x)$.
Let $y(x) = y_1(x) + y_2(x)$.

$$y'' + p(x)y' + Q(x)y = ({y_1}'' + {y_2}'') + p(x)({y_1}' + {y_2}') + Q(x)(y_1 + y_2)$$

$$(y_1('')) + p(x)y' + Q(x)y) + ({y_2}'' + p(x){y_2}' + Q(x)y) = R_1(x) + R_2(x)$$

$\Rightarrow y(x) = y_1(x) + y_2(x)$ is the solution.

(15). Find a particular solution of $y^{''} - 2y^{'} + y = 2x$ first by inspection and then by variation of parameters.
Solution:
$y^{''} - 2y^{'} + y = 2x$.
By inspection:

$$y_1 = ax + b \Rightarrow y_1^{'} = a \Rightarrow {y_1}'' = 0.$$

$$y^{''} - 2y^{'} + y = 2x = -2a + ax + b = 2x$$

$$\Rightarrow a = 2, b = 2a = 4..$$

$$so, y_p = 2x + 4..$$

By variation of parameter:
$y^{''} - 2y^{'} + y = 0 \rightarrow homo$ General solution: $y = e^{mx}$.

$$\begin{aligned}
&\Rightarrow (m^2 - 2m + 1)e^{mx} = 0 \\
&\Rightarrow (m-1) = 0 \Rightarrow m = 1. \\
&\Rightarrow y_1 = C_1 e^x + C_2 x e^x \\
&\Rightarrow y_1 = e^x \Rightarrow y_1^{'} = e^x \\
&\Rightarrow y_2 = xe^x \Rightarrow y_2^{'} = xe^x + e^x
\end{aligned}$$

$$w(y_1, y_2) = y_1 y_2^{'} - y_2 y_1^{'} = e^x[xe^x + e^x] - xe^x[e^x] = e^{2x}$$

Variation of parameter:

$$\begin{aligned}
v_1 &= \int \frac{-y_2 R(x)}{w(y_1, y_2)} dx = -\int \frac{xe^x 2x}{e^{2x}} dx \\
&= -\int 2x^2 e^{-x} dx \\
&= +2e^{-x}(x(x+2)+2). \\
v_1 y_1 &= 2(x^2 + 2x + 2).
\end{aligned} \tag{5}$$

$$\begin{aligned}
v_2 &= \int \frac{-y_1 R(x)}{w(y_1, y_2)} dx = -\int \frac{e^x 2x}{e^{2x}} dx \\
&= -\int 2xe^{-x} dx \\
&= -2e^{-x}(x+1). \\
v_2 y_2 &= -2x(x+1).
\end{aligned} \tag{6}$$

$$\begin{aligned}
y_p &= v_1 y_1 + v_2 y_2 \\
&= 2(x^2 + 2x + 2) - 2x(x+1) \\
y_p &= 2x + 4
\end{aligned} \tag{7}$$

(16)(a) Show that the method of variation of parameters applied to the equation $y^{''} + y = f(x)$ leads to the particular solution $y_p(x) = \int_0^x sin(x-t)dt$.
(b) Find the similar formula for a particular solution of equation $y^{''} + k^2 = f(k)$, where k is a positive constant.
Solution:
(a) The general solution to $y^{''} + y = 0$ is $y = C_1 sin(x) + C_2 cos(x)$.

$$y_1 = sin(x) \Rightarrow y_1^{'} = cos(x) \quad y_2 = cos(x) \Rightarrow y_2^{'} = -sin(x).$$

$$w(y_1, y_2) = y_1 y_2^{'} - y_2 y_1^{'} = -sin^2(x) - cos^2(x) = -1.$$

$$v_1 = \int \frac{-y_2 R(x)}{w(y_1, y_2)} dx = -\int \frac{cos(x) f(x)}{-1} dx = \int f(x) cos(x) dx.$$

$$v_2 = \int \frac{-y_1 R(x)}{w(y_1, y_2)} dx = \int f(x) sin(x) dx.$$

$$y_p = v_1 y_1 + v_2 y_2 = sin(x) \int f(x) cos(x) dx + cos(x) \int f(x) sin(x) dx.$$

(b). $y^{''} + k^2 y = f(x)$, k is a positive constant.
G.S to $y^{''} + k^2 y = 0$.

Let $y = e^{mx}$.
$y^{''} + k^2y = 0$.

$$\Rightarrow (m^2 + k^2)e^{mx} = 0.$$
$$\Rightarrow m^2 + k^2 = 0.$$
$$\Rightarrow m^2 = -k^2.$$
$$\Rightarrow m = \pm jk.$$

So $y_1 = e^{ikx}$ and $y_2 = e^{-jkx}$.

$$y_1 = (cos(kx) + isin(kx)) \ and \ y_2 = (cos(kx) - isin(kx)).$$

To get the real solution: $\frac{y_1+y_2}{2}$ and $\frac{y_1-y_2}{2}$.
G.S: $y = C_1cos(kx) + C_2sin(kx)$.

$y^{''} + k^2y = f(x)$.
Variation of parameter:

$$v_1 = \int \frac{y_2R(x)}{w(y_1, y_2}.$$

Here, $y_1 = cos(kx) \Rightarrow y_1^{'} = -ksin(kx)$.
$y_2 = sin(kx) \Rightarrow y_1^{'} = kcos(kx)$.
$w(y_1, y_2) = y_1y_2^{'} - y_2y_1^{'} = +k$.

$$v_1 = \int \frac{sin(kx)f(x)}{k}dx.$$
$$v_2 = \int \frac{1}{k}f(x)cos(kx)dx.$$

$$\begin{aligned} y_p &= v_1y_1 + v_2y_2. \\ &= \frac{1}{k}cos(kx) \int sin(kx)f(x)dx + \frac{1}{k}sin(kx) \int f(x)cos(kx)dx. \\ y_p &= \frac{1}{k} \int_0^k f(t)sin(tx)dt. \end{aligned}$$

(17). Find the general solution of the following.
$y^{'''} - y = 0$.
Solution:

$$y^{'''} - y = 0.$$

Let $y = e^{mx}$.

$$\Rightarrow (m^3 - 1)e^{mx} = 0.$$
$$\Rightarrow (m - 1)(m^2 + m + 1) = 0.$$
$$\Rightarrow (m - 1)(m + (\frac{1}{2} - j\frac{\sqrt{3}}{2}))(m - (\frac{-1}{2} - j\frac{\sqrt{3}}{2})) = 0.$$
$$m = 1, m = \frac{-1}{2} + j\frac{\sqrt{3}}{2} \ and \ m = \frac{-1}{2} - j\frac{\sqrt{3}}{2}.$$
$$y = C_1e^x + C_2e^{\frac{-x}{2}}\left[C_3cos(\frac{\sqrt{3}}{2}) + C_4sin(\frac{\sqrt{3}}{2})\right].$$

(18) Find the particular solution using successive integration method.
$y^{''} - y = e^{-x}$.
Solution:

$$(D^2 - 1)y = e^{-x}.$$

$$\Rightarrow y = \frac{1}{(D+1)(D-1)} e^{-x}.$$

$$\frac{1}{D-1} e^{-x} = e^{x} \int e^{-x} e^{-x} dx = e^{x} \int e^{-2x} dx = \frac{-1}{2} e^{-x}.$$

$$\frac{1}{D+1} \left(\frac{1}{D-1} e^{-x} \right) = e^{-x} \int e^{x} \left(\frac{-1}{2} e^{-x} \right) dx = e^{-x} \left(\frac{-1}{2} \right) x.$$

$$\Rightarrow y = \frac{-1}{2} (1+x) e^{-x}.$$

(19) Find a particular solution by using series expansions of operators method: $4y^{''} + y = x^4$.
Solution:

$$4y^{''} + y = x^4.$$
$$\Rightarrow (4D^2 + 1) y = x^4.$$

$$\begin{aligned} \Rightarrow y &= \frac{1}{1+4D^2} x^4. \\ &= (1 - 4D^2 + 16D^4 - 64D^6 + \cdots) x^4. \\ &= x^4 - 4 \cdot 4 \cdot 3 \cdot x^2 + 16 \cdot 4 \cdot 3 \cdot 2 \cdot 1. \end{aligned}$$

$$y = x^4 - 48x^2 + 384.$$