Special functions

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SC-216 Calculus with complex variables

Euler's Beta fn.

$$B(x,y) = \int_0^1 t^{x-1}(1-t)^{y-1}\,dt \qquad \begin{array}{l} Re(x)>0 \\ Re(y)>0 \end{array}$$

① $B(x,y) = B(y,x)$ — (1)

Gamma fn.

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t}\,dt \qquad Re(z)>0$$

(Extension of factorial fn.) — (2)

② If $n>0$ ($\in \mathbb{Z}$) then $\Gamma(n) = (n-1)!$ — (3)

③ $\Gamma(z+1) = z\,\Gamma(z)$ (Verify!) — (4)

④ $\Gamma(1) = \int_0^\infty e^{-t}\,dt = 1$

Alternative def. of Gamma fn.

$$\Gamma(z) = \lim_{n\to\infty} \frac{n!\; n^z}{z(z+1)\cdots(z+n)} \quad \text{— (5)}$$

$$\& \quad \Gamma(z) = \frac{e^{-\gamma z}}{z} \cdot \prod_{n=1}^{\infty} \left(1+\frac{z}{n}\right)^{-1} e^{z/n} \quad \text{— (6)}$$

$\gamma \to$ Euler-Mascheroni constt.

Exercise Prove (4) using (5)!

⑤ $$\Gamma(1-z)\,\Gamma(z) = \frac{\pi}{\sin(\pi z)}$$

⑥ $$\Gamma(1/2) = \sqrt{\pi}$$

$\boxed{\text{Imp. Relation}}$

$$B(x,y) = \frac{\Gamma(x)\,\Gamma(y)}{\Gamma(x+y)}$$

$\boxed{\text{Properties}}$

① $B(m,n) = 2\int_0^{\pi/2} \sin^{2n-1}\theta \, \cos^{2m-1}(\theta)\, d\theta$

□ Put $t = \sin^2\theta$ in eqⁿ (1) & simplify.

② $B(m,n) = \int_0^{\infty} \frac{x^{m-1}}{(1+x)^{m+n}}\, dx$

□ Put $t = \frac{y}{1+y}$ in eqⁿ (1)

Prove that $B(x,y) = \frac{\Gamma(x)\,\Gamma(y)}{\Gamma(x+y)}$

□ $\Gamma(x) = \int_0^{\infty} t^{x-1} e^{-t}\, dt = 2\int_0^{\infty} u^{2x-1} e^{-u^2}\, du$

& $\Gamma(y) = \int_0^{\infty} t^{y-1} e^{-t}\, dt$ (set $t = u^2$) $= 2\int_0^{\infty} v^{2y-1} e^{-v^2}\, dv$

(set $t = v^2$)

$\Rightarrow \Gamma(x)\,\Gamma(y) = 4\int_0^{\infty}\int_0^{\infty} u^{2x-1} v^{2y-1} e^{-(u^2+v^2)}\, du\, dv$

Put $u = r\cos\theta$ & $v = r\sin\theta$

$\Gamma(x)\,\Gamma(y) = \int_{\theta=0}^{\pi/2}\int_{r=0}^{\infty} \cos^{2x-1}\theta \, \sin^{2y-1}\theta \, r^{2x+2y-1} e^{-r^2}\, dr\, d\theta$

$= 4\left[\int_0^{\infty} r^{2x+2y-1} e^{-r^2}\, dr\right]\left[\int_0^{\pi/2} \cos^{2x-1}\theta \, \sin^{2y-1}\theta \, d\theta\right]$

$= 2\, B(x,y) \int_0^{\infty} r^{2x+2y-1} e^{-r^2}\, dr$

$= \Gamma(x+y)$ (∵ set $t = r^2$ in ②)

$\boxed{\text{Legendre Polynomials}}$ (L.P.). Legendre D.E.

$$(1-x^2)y'' - 2xy' + n(n+1)y = 0$$

— One can solve using series soln.

~~Soln.~~

— Soln. is $P_n(x) = \frac{1}{2^n} \sum_{r=0}^{N} \frac{(-1)^r (2n-2r)!\, x^{n-2r}}{r!\,(n-r)!\,(n-2r)!}$

where $N = n/2$ or $\frac{(n-1)}{2}$ whichever is integer.

— $|P_n(x)| \leq 1$, $x \in [-1,1]$

— $\frac{1}{\sqrt{1-2xt+t^2}} = \sum_{n=0}^{\infty} t^n P_n(x)$, $t \neq 1$

Generating fn. of L.P.

— Recurrence relations of L.P.

① $(n+1)P_{n+1}(x) = (2n+1)xP_n(x) - nP_{n-1}(x)$

② $nP_n(x) = xP_n'(x) - P_{n-1}'(x)$ & so on

— These are orthogonal polynomials

$$\int_{-1}^{1} P_m(x)P_n(x)\,dx = \begin{cases} 0 & m \neq n \\ \frac{2}{2n+1}, & m = n \end{cases}$$

Chebyshev D.E. & Chebyshev Polynomials

$$(1-x^2)\,y'' - x\,y' + n^2 y = 0$$ n real. +ve integer

C.P. of first Kind

$$T_n(x) = \cos(n\theta) = \cos(n\cos^{-1}x)$$

$-1 \le x \le 1$

$|T_n(x)| \le 1$

① $T_{n+1}(x) - 2x\,T_n(x) + T_{n-1}(x) = 0$

② $$\frac{1-xt}{1-2xt+t^2} = \sum_{n=0}^{\infty} T_n(x)\,t^n$$

(G.F. of C.P. 1st)

③ Orthogonality $$\int_{-1}^{1} \frac{T_m(x)\,T_n(x)}{\sqrt{1-x^2}}\,dx = 0 \text{ if } m \ne n$$
$= \pi$ if $m = n = 0$
$= \pi/2$ $m = n \ne 0$

$\boxed{\text{Bessel's D.E. \& Bessel fns}}$

$$x^2 y'' + x y' + (x^2 - \nu^2) y = 0$$

(1) Soln. is Bessel fn. $J_n(x)$

(2) $$e^{\frac{1}{2} x (t - 1/t)} = \sum_{-\infty}^{\infty} J_n(x)\, t^n$$