Second Order Linear Partial Differential Equations

Part I

Second linear partial differential equations; Separation of Variables; 2-point boundary value problems; Eigenvalues and Eigenfunctions

Introduction

We are about to study a simple type of partial differential equations (PDEs): the second order linear PDEs. Recall that a partial differential equation is any differential equation that contains two or more independent variables. Therefore the derivative(s) in the equation are partial derivatives. We will examine the simplest case of equations with 2 independent variables. A few examples of second order linear PDEs in 2 variables are:

$$\alpha^2 \, u_{xx} = u_t \qquad \text{(one-dimensional heat conduction equation)}$$

$$a^2 \, u_{xx} = u_{tt} \qquad \text{(one-dimensional wave equation)}$$

$$u_{xx} + u_{yy} = 0 \qquad \text{(two-dimensional Laplace/potential equation)}$$

In this class we will develop a method known as the method of Separation of Variables to solve the above types of equations.

(Optional topic) **Classification of Second Order Linear PDEs**

Consider the generic form of a second order linear partial differential equation in 2 variables with constant coefficients:

$$a\,u_{xx} + b\,u_{xy} + c\,u_{yy} + d\,u_x + e\,u_y + fu = g(x,y).$$

For the equation to be of second order, a, b, and c cannot all be zero. Define its *discriminant* to be $b^2 - 4ac$. The properties and behavior of its solution are largely dependent of its *type*, as classified below.

If $b^2 - 4ac > 0$, then the equation is called *hyperbolic*. The wave equation is one such example.

If $b^2 - 4ac = 0$, then the equation is called *parabolic*. The heat conduction equation is one such example.

If $b^2 - 4ac < 0$, then the equation is called *elliptic*. The Laplace equation is one such example.

In general, elliptic equations describe processes in equilibrium. While the hyperbolic and parabolic equations model processes which evolve over time.

Example: Consider the one-dimensional damped wave equation
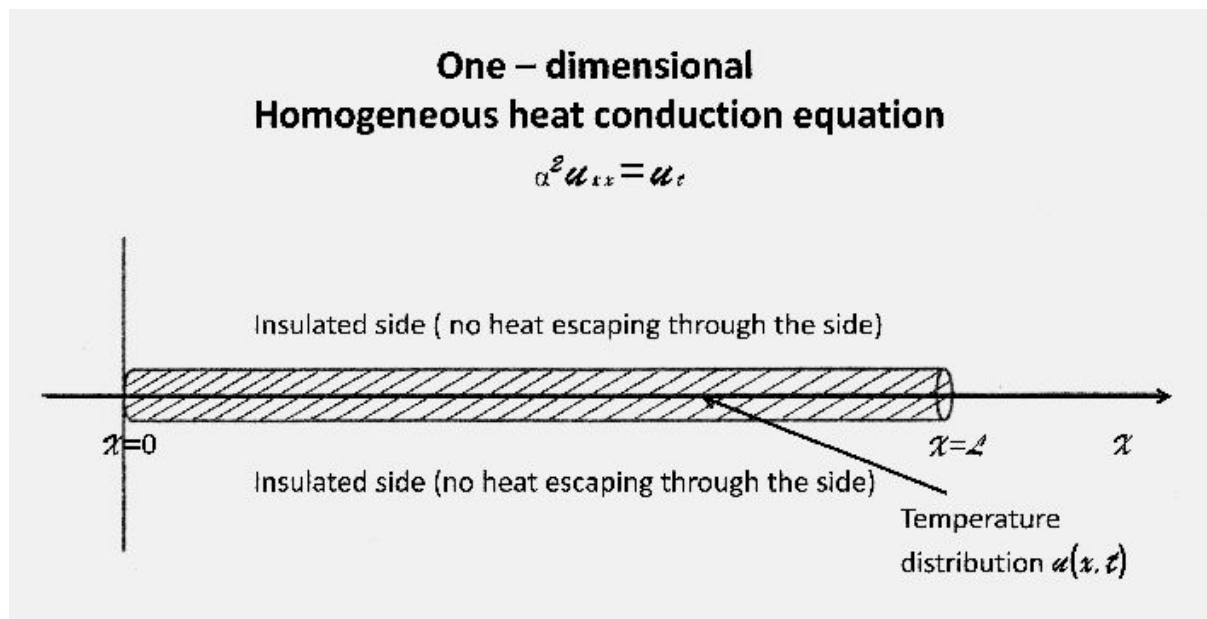$$9u_{xx} = u_{tt} + 6u_t.$$

It can be rewritten as: $9u_{xx} - u_{tt} - 6u_t = 0$. It has coefficients $a = 9$, $b = 0$, and $c = -1$. Its discriminant is $9 > 0$. Therefore, the equation is hyperbolic.

The One-Dimensional Heat Conduction Equation

Consider a thin bar of length L, of uniform cross-section and constructed of homogeneous material. Suppose that the side of the bar is perfectly insulated so no heat transfer could occur through it (heat could possibly still move into or out of the bar through the two ends of the bar). Thus, the movement of heat inside the bar could occur only in the x-direction. Then, the amount of heat content at any place inside the bar, $0 < x < L$, and at any time $t > 0$, is given by the temperature distribution function $u(x, t)$. It satisfies the *homogeneous one-dimensional heat conduction equation*:

$$\alpha^2 u_{xx} = u_t$$

Where the constant coefficient α^2 is the *thermo diffusivity* of the bar, given by $\alpha^2 = k / \rho s$. (k = thermal conductivity, ρ = density, s = specific heat, of the material of the bar.)

One – dimensional
Homogeneous heat conduction equation

$$\alpha^2 u_{xx} = u_t$$

Insulated side (no heat escaping through the side)

$x=0$ $x=L$ x

Insulated side (no heat escaping through the side)

Temperature distribution $u(x, t)$

Further, let us assume that both ends of the bar are kept constantly at 0 degree temperature (abstractly, by connecting them both to a heat reservoir

of the same temperature; more practically, say they are immersed in iced water). This assumption imposes explicit restriction on the bar's ends, in this case:

$$u(0, t) = 0, \text{ and } \quad u(L, t) = 0. \quad t > 0$$

Those two conditions are called the <mark>boundary conditions</mark> of this problem. They literally specify the conditions present at the boundaries between the bar and the outside. Think them as the "environmental factors" of the given problem.

In addition, there is an initial condition: the initial temperature distribution within the bar, $u(x, 0)$. It is a snapshot of the temperature everywhere inside the bar at $t = 0$. Therefore, it is an (arbitrary) function of the spatial variable x only. That is, the initial condition is $u(x, 0) = f(x)$.

Hence, what we have is a problem given by:

(Heat conduction eq.) $\alpha^2 u_{xx} = u_t$, $0 < x < L, \quad t > 0,$

(Boundary conditions) $u(0, t) = 0, \text{ and } \quad u(L, t) = 0,$

(Initial condition) $u(x, 0) = f(x).$

This is an example of what is known, formally, as an *initial-boundary value problem*. Although it is still true that we will find a general solution first, then apply the initial condition to find the particular solution. A major difference now is that <u>the general solution is dependent not only on the equation, but also on the boundary conditions</u>. In other words, the given partial differential equation will have different general solutions when paired with different sets of boundary conditions.

If the boundary conditions specify u, e.g. $u(0,t) = f(t)$ and $u(L,t) = g(t)$, then they are often called *Dirichlet conditions*. If they specify the (spatial) derivative, e.g. $u_x(0,t) = f(t)$ and $u_x(L,t) = g(t)$, then they are often called *Neumann conditions*. If the boundary conditions are linear combinations of u and its derivative, e.g. $\alpha u(0,t) + \beta u_x(0,t) = f(t)$, then they are called *Robin conditions*. Those are the 3 most common classes of boundary conditions. If the specified functions in a set of condition are all equal to zero, then they are *homogeneous*. Our current example, therefore, is a homogeneous Dirichlet type problem.

But before any of those boundary and initial conditions could be applied, we will first need to process the given partial differential equation. What can we do with it? There are other tools (by Laplace transforms, for example), but the most accessible method to us is called the method of *Separation of Variables*. The idea is to somehow de-couple the independent variables, therefore rewrite the single partial differential equation into 2 ordinary differential equations of one independent variable each (which we already know how to solve). We will solve the 2 equations individually, and then combine their results to find the general solution of the given partial differential equation. For a reason that should become clear very shortly, the method of Separation of Variables is sometimes called the method of *Eigenfunction Expansion*.

Separation of Variables

Start with the one-dimensional heat conduction equation $\alpha^2 u_{xx} = u_t$. Suppose that its solution $u(x, t)$ is such a function that it can be expressed as a product, $u(x, t) = X(x)T(t)$, where X is a function of x alone and T is a function of t alone. Then, its partial derivatives can also be expressed simply by:

$$u = X\,T \qquad u_{xx} = X''\,T$$

$$u_x = X'\,T \qquad u_{tt} = X\,T''$$

$$u_t = X\,T' \qquad u_{xt} = u_{tx} = X'\,T'$$

Hence, the heat conduction equation $\alpha^2 u_{xx} = u_t$ can be rewritten as

$$\alpha^2 X''\,T = X\,T'.$$

Dividing both sides by $\alpha^2 X\,T$:

$$\frac{X''}{X} = \frac{T'}{\alpha^2\,T}$$

(α^2 is a constant, so it could go to either side of the equation, but it is usually, and more conveniently, moved to the t side.) The equation is now "separated", as all the x-terms are on the left and t-terms are on the right.

Note: The above step would not have been possible if either $X = 0$ or $T = 0$. However, if either part is zero, then $u = XT = 0$, which will trivially satisfy the given equation $\alpha^2 u_{xx} = u_t$. This constant zero solution is called the *trivial solution* of the equation. We know this is going to be the case. Therefore, we will assume from this point onward that $X \neq 0$ and $T \neq 0$. We look for only the nonzero solutions.

But how do we completely pull it apart into 2 equations? The critical idea here is that, because the independent variables x and t can, and do, vary independently, in order for the above equation to hold for all values of x and t, the expressions on both sides of the equation must be equal to the same constant. Let us call the constant $-\lambda$. It is called the *constant of separation*. (The negative sign is optional, of course, since λ is an arbitrary number and it could be either positive or negative or even zero. But putting a negative sign here right now makes our later calculation a little easier.) Thus,

$$\frac{X''}{X} = \frac{T'}{\alpha^2 T} = -\lambda .$$

Why must the two sides be the same constant? Well, think what would happen if one of the sides isn't a constant. The equation would not be true for all x and t if that were the case, because then one side/variable could be held at a fixed value while the other side/variable changes.

Next, equate first the x-term and then the t-term with $-\lambda$. We have

$$\frac{X''}{X} = -\lambda \qquad \rightarrow \qquad X'' = -\lambda X \qquad \rightarrow \qquad X'' + \lambda X = 0,$$

and,

$$\frac{T'}{\alpha^2 T} = -\lambda \qquad \rightarrow \qquad T' = -\alpha^2 \lambda T \qquad \rightarrow \qquad T' + \alpha^2 \lambda T = 0.$$

Consequently, the single partial differential equation has now been separated into a simultaneous system of 2 ordinary differential equations. They are a second order homogeneous linear equation in terms of x, and a first order linear equation (it is also a separable equation) in terms of t. Both of them can be solved easily using what we have already learned in this class.

Lastly, now that the partial differential equation becomes two ordinary differential equations, we need to similarly rewrite the boundary conditions.

The boundary conditions can be rewritten as:

$u(0, t) = 0 \rightarrow X(0)T(t) = 0 \rightarrow X(0) = 0$ or $T(t) = 0$
$u(L, t) = 0 \rightarrow X(L)T(t) = 0 \rightarrow X(L) = 0$ or $T(t) = 0$

If we choose $T(t) = 0$, both conditions would be satisfied. However, it would mean that the temperature distribution function, $u(x, t) = X(x)T(t)$, would be the constant zero function (the trivial solution). That is a totally uninteresting solution that would not give us the general solution (it could not satisfy any initial condition, except when it is also constant zero). Hence, we have to let the new boundary conditions to be: $X(0) = 0$ and $X(L) = 0$.

Therefore, at the end of this process, we have two ordinary differential equations, together with a set of two boundary conditions that go with the equation of the spatial variable x:

$$X'' + \lambda X = 0, \quad X(0) = 0 \quad \text{and} \quad X(L) = 0,$$

$$T' + \alpha^2 \lambda T = 0.$$

The general solution (that satisfies the boundary conditions) shall be solved from this system of simultaneous differential equations. Then the initial condition $u(x, 0) = f(x)$ could be applied to find the particular solution.

Example: Separate $t^3 u_{xx} + x^3 u_{tt} = 0$ into an equation of x and an equation of t.

Let $u(x, t) = X(x)T(t)$ and rewrite the equation in terms of X and T:

$$t^3 X'' T + x^3 XT'' = 0,$$

$$t^3 X'' T = -x^3 XT''.$$

Divide both sides by $X'' T''$, we have separated the variables:

$$\frac{t^3 T}{T''} = \frac{-x^3 X}{X''}.$$

Now insert a constant of separation:

$$\frac{t^3 T}{T''} = \frac{-x^3 X}{X''} = -\lambda.$$

Finally, rewrite it into 2 equations:

$$t^3 T = -\lambda T'' \qquad \rightarrow \qquad \lambda T'' + t^3 T = 0,$$
$$-x^3 X = -\lambda X'' \qquad \rightarrow \qquad \lambda X'' - x^3 X = 0.$$

Example: Separate

$$u_x + 2\,u_{tx} - 10\,u_{tt} = 0, \qquad u(0, t) = 0, \ \ u_x(L, t) = 0.$$

Let $u(x, t) = X(x)T(t)$ and rewrite the equation in terms of X and T:

$$X'T + 2\,X'T' - 10\,XT'' = 0,$$

$$X'T + 2\,X'T' = 10\,XT''.$$

Divide both sides by $X'\,T''$, and insert a constant of separation:

$$\frac{T + 2T'}{T''} = \frac{10X}{X'} = -\lambda \ .$$

Rewrite it into 2 equations:

$$T + 2\,T' = -\lambda\,T'' \qquad \rightarrow \qquad \lambda\,T'' + 2\,T' + T = 0,$$
$$10\,X = -\lambda\,X' \qquad \rightarrow \qquad \lambda\,X' + 10\,X = 0.$$

The boundary conditions also must be separated:

$$u(0, t) = 0 \ \rightarrow \ X(0)T(t) = 0 \ \rightarrow \qquad X(0) = 0 \quad \text{or} \quad T(t) = 0$$
$$u_x(L, t) = 0 \ \rightarrow \ X'(L)T(t) = 0 \ \rightarrow \qquad X'(L) = 0 \quad \text{or} \quad T(t) = 0$$

As before, setting $T(t) = 0$ would result in the constant zero solution only. Therefore, we must choose the two (nontrivial) conditions in terms of x: $\qquad X(0) = 0, \quad$ and $\quad X'(L) = 0.$

The Two-Point Boundary Value Problems

What we have done thus far is to separate the heat conduction equation, with 2 independent variables, into 2 equations of one variable each. Meanwhile we have also rewritten the boundary conditions, so that they now associate with the spatial variable x only.

$$(1) \qquad X'' + \lambda X = 0, \qquad X(0) = 0 \quad \text{and} \quad X(L) = 0,$$
$$(2) \qquad T' + \alpha^2 \lambda T = 0.$$

The next task is to solve this system of two simultaneous ordinary differential equations, one of them with boundary conditions. We will look the 2 equations one at a time, and consolidate their solutions at the end. We will start off by solving the more interesting (and more complex) of the two, namely the second order linear equation where x is the independent variable. It is our first taste of a *boundary value problem* (BVP). (It is <u>not</u> an initial value problem; as mentioned earlier in this course, an initial value problem requires that both of its data points be taken at the same time/place, but here we have 2 data points taken at different instances of x: at $x = 0$ and $x = L$.)

A little background first: unlike an initial value problem of a second order linear equation, whose solution's existence and uniqueness (under certain well-understood conditions) are guaranteed, there is no such guarantee for a boundary value problem. Indeed, take an arbitrary pairing of a differential equation and a set of boundary conditions, the odds are good that there is not a solution satisfying them, or that there are multiple solutions satisfying them. Notice that the boundary value problem in (1) arises from a homogeneous linear differential equation, which always has at least one solution (the constant zero solution, or the *trivial solution*) which would also satisfy the homogeneous boundary conditions given. Thus, (in this case, at least) the <u>existence</u> of a solution is not the issue. The constant zero solution, $X(t) = 0$, however, is not usable for us. Because, if $X(t) = 0$, then $u(x, t) = X(x)T(t) = 0$, which is not the general solution. (Why not?) Hence, what we are looking to find presently is a second, nonzero, solution to the given boundary value problem. That is, we are looking for instance(s) where the <u>uniqueness of solution</u> fails to hold.

Somewhat fortunately for us, that in (1) above while the boundary conditions are fixed, the equation itself is not − note that the coefficient λ, which was just the arbitrary constant of separation, has not been determined. Thus, what we need to do here is somewhat a reverse of what you'd have expected to be doing. Namely, we will start with the fixed boundary conditions and try to find an equation (by finding an appropriate coefficient λ) that has a <u>nonzero solution</u> satisfying the given boundary conditions. (In other words, we are starting with the answer and then go looking for the correct question that would give that answer!) This kind of reversed boundary value problems is called an *Eigenvalue problem*. The specific value(s) of λ that would produce a nonzero solution of the boundary value problem is called an *eigenvalue* of the boundary value problem. The nonzero solution that arises from each eigenvalue is called a corresponding *eigenfunction* of the boundary value problem.

Note: Why are λ and $X(t)$ called, respectively, eigenvalue and eigenfunction? Let $\boldsymbol{D} = -\dfrac{d^2}{dx^2}$ denotes the second derivative differential operator. The boundary value problem (1) on the previous page can be rewritten into

$$\boldsymbol{D}X = \lambda X,$$

where X is restricted to functions that satisfy the boundary conditions $X(0) = 0$ and $X(L) = 0$. Notice the similarity with the defining relation, $\boldsymbol{A}\boldsymbol{x} = \lambda\boldsymbol{x}$, of eigenvalue and eigenvector in linear algebra, hence the similar naming scheme. Like eigenvectors, every constant multiple of an eigenfunction is another eigenfunction corresponding to the same eigenvalue.

Eigenvalues and Eigenfunctions of a Two-Point BVP

Hence, the next goal is to find the eigenvalues λ such that the boundary value problem (1)

$$X'' + \lambda X = 0, \qquad\qquad X(0) = 0 \quad \text{and} \quad X(L) = 0,$$

will have a *nonzero* solution satisfying both boundary conditions. Since the form of the general solution of the second order linear equation is dependent on the type of roots that its characteristic equation has. In this example, the characteristic equation is $r^2 + \lambda = 0$. The type of roots it has is dependent on its discriminant, which is simply -4λ. We will attempt to find λ by separately considering the 3 possible types of the solution arise from the different roots of the characteristic equation.

Case 1: If $\lambda < 0$ ($-4\lambda > 0$, distinct real roots of characteristic equation):

Let us denote $\lambda = -\sigma^2$, where $\sigma = \sqrt{-\lambda} > 0$. The characteristic equation becomes $r^2 + \lambda = r^2 - \sigma^2 = 0$, which has roots $r = \pm\,\sigma$.

The general solution is then $X(x) = C_1 e^{\sigma x} + C_2 e^{-\sigma x}$. Applying the first of the boundary conditions gives

$$X(0) = 0 = C_1 + C_2 \qquad \rightarrow \qquad C_2 = -\,C_1$$

The second boundary condition gives

$$X(L) = 0 = C_1 e^{\sigma L} + C_2\, e^{-\sigma L} = C_1(e^{\sigma L} - e^{-\sigma L})$$

Hence, either $C_1 = 0 = C_2$, (this would be the zero, or the trivial solution, which does not lead to the general solution later on); or $e^{\sigma L} - e^{-\sigma L} = 0$. But this second case does not have a solution (why?). Therefore, there is no negative value of λ that would give a nonzero solution. Thus, there is no negative eigenvalue for this problem.

Case 2: If $\lambda = 0$ $(-4\lambda = 0$, repeated real root of characteristic equation):

The equation becomes $X'' = 0$. It has the general solution (either by integrating both sides twice, or from the characteristic equation $r^2 = 0$)

$$X(x) = C_1 + C_2 x.$$

Applying boundary conditions to get

$$X(0) = 0 = C_1 \qquad\qquad \rightarrow \qquad C_1 = 0$$
$$X(L) = 0 = C_1 + C_2 L = C_2 L \qquad \rightarrow \qquad C_2 = 0 \ \ (\text{because } L > 0)$$

Hence, $X(x) = 0 + 0 x = 0$, the trivial solution is the only possibility. Therefore, zero is not an eigenvalue for this problem either.

Case 3: If $\lambda > 0$ $(-4\lambda < 0$, complex roots of characteristic equation):

Let us denote $\lambda = \sigma^2$, where $\sigma = \sqrt{\lambda} > 0$. The characteristic equation becomes $r^2 + \lambda = r^2 + \sigma^2 = 0$, which has roots $r = \pm\,\sigma i$.

The general solution is then $X(x) = C_1 \cos(\sigma x) + C_2 \sin(\sigma x)$. Applying boundary conditions to get

$$X(0) = 0 = C_1 \cos(0) + C_2 \sin(0) = C_1 \qquad\qquad \rightarrow \qquad C_1 = 0$$
$$X(L) = 0 = C_1 \cos(\sigma L) + C_2 \sin(\sigma L) = C_2 \sin(\sigma L)$$

The second equation has 2 possible solutions: $C_2 = 0$ $(= C_1)$, which results in the trivial solution again; or, more importantly, it could be that $\sin(\sigma L) = 0$, which means $\sigma L = \pi, 2\pi, 3\pi, \ldots, n\pi, \ldots$ That is, there are infinitely many values $\sigma = \pi/L, 2\pi/L, 3\pi/L, \ldots, n\pi/L, \ldots$ such that there exists a nonzero solution of this boundary value problem.

The (positive) values of λ for which the equation will have a solution satisfying the specified boundary conditions, i.e. the eigenvalues of this BVP, are

$$\lambda = \sigma^2 = \frac{n^2 \pi^2}{L^2}, \qquad\qquad n = 1, 2, 3, \ldots$$

All such values of λ are called the eigenvalues of the given boundary value problem. Recall that the actual solution of this problem corresponding to each eigenvalue λ is called an eigenfunction of the problem.

What are the eigenfunctions, then? Retracing our work above, we see that they occur only when λ are positive (therefore, the general solution in the form of $X(x) = C_1 \cos(\sigma x) + C_2 \sin(\sigma x)$), that C_1 must be zero, and C_2 could be any nonzero constant. Lastly, $\lambda = n^2\pi^2/L^2$, where $n = 1, 2, 3, \ldots$ are all the positive integers.

Therefore, the eigenfunctions corresponding to the eigenvalues found above − that is, they are the actual nonzero solutions that satisfy the given set of boundary conditions when the original differential equation has $\lambda = n^2\pi^2/L^2$ as its coefficient − are

$$X_n = \sin \frac{n \pi x}{L}, \qquad n = 1, 2, 3, \ldots$$

Example: Find the eigenvalues and eigenfunctions of the two-point boundary value problem

$$X'' + \lambda X = 0, \qquad X(0) = 0 \quad \text{and} \quad X'(L) = 0.$$

Case 1: If $\lambda < 0$:

Again denote $\lambda = -\sigma^2$, where $\sigma = \sqrt{-\lambda} > 0$. The characteristic equation becomes $r^2 + \lambda = r^2 - \sigma^2 = 0$, which has roots $r = \pm \sigma$.

The general solution is then $X(x) = C_1 e^{\sigma x} + C_2 e^{-\sigma x}$. Its derivative is $X'(x) = C_1 \sigma e^{\sigma x} - C_2 \sigma e^{-\sigma x}$. Apply the boundary conditions and we get

$$X(0) = 0 = C_1 + C_2 \qquad \rightarrow \qquad C_2 = -C_1$$

$$X'(L) = 0 = C_1 \sigma e^{\sigma L} - C_2 \sigma e^{-\sigma L} = C_1 \sigma (e^{\sigma L} + e^{-\sigma L})$$

Since $\sigma > 0$, once again we encounter the situation where either $C_1 = 0 = C_2$ (thus giving us the trivial solution), or $e^{\sigma L} + e^{-\sigma L} = 0$ (which is impossible, because each exponential term is always positive, and as a result their sum cannot be zero). As before, this second case does not have a solution, either. Therefore, there is no negative eigenvalue for this problem.

Case 2: If $\lambda = 0$:

Again, the equation becomes $X'' = 0$. The general solution is, therefore, $X(x) = C_1 + C_2 x$. Its derivative is $X'(x) = C_2$.

Applying boundary conditions to get

$$X(0) = 0 = C_1 \qquad \rightarrow \qquad C_1 = 0$$
$$X'(L) = 0 = C_2 \qquad \rightarrow \qquad C_2 = 0$$

Hence, $X(x) = 0 + 0x = 0$, the trivial solution once again. Therefore, zero is not an eigenvalue for this problem.

Case 3: If $\lambda > 0$:

As before, denote $\lambda = \sigma^2$, where $\sigma = \sqrt{\lambda} > 0$. The characteristic equation is then $r^2 + \lambda = r^2 + \sigma^2 = 0$, which has roots $r = \pm \sigma i$.

The general solution is $X(x) = C_1 \cos(\sigma x) + C_2 \sin(\sigma x)$. Its derivative is $X'(x) = -C_1 \sigma \sin(\sigma x) + C_2 \sigma \cos(\sigma x)$. Applying boundary conditions to get

$$X(0) = 0 = C_1 \cos(0) + C_2 \sin(0) = C_1 \qquad\qquad \rightarrow \qquad C_1 = 0$$
$$X'(L) = 0 = -C_1 \sigma \sin(\sigma L) + C_2 \sigma \cos(\sigma L) = C_2 \sigma \cos(\sigma L)$$

Since $\sigma > 0$ always, the second equation has 2 possible solutions: $C_2 = 0 \ (= C_1)$, which results in the trivial solution; or, it could be that $\cos(\sigma L) = 0$, which means $\sigma L = \pi/2, 3\pi/2, 5\pi/2, \ldots, (2n-1)\pi/2, \ldots$ That is, there are infinitely many values $\sigma = \pi/2L, 3\pi/2L, 5\pi/2L, \ldots, (2n-1)\pi/2L, \ldots$ such that there exists a nonzero solution of this boundary value problem.

Therefore, yes, there are positive eigenvalues in the form

$$\lambda = \sigma^2 = \frac{(2n-1)^2 \pi^2}{4L^2}, \qquad\qquad n = 1, 2, 3, \ldots$$

Retracing the steps in the above calculation, we see that the eigenvalues have corresponding eigenfunctions

$$X_n = \sin\frac{(2n-1)\pi x}{2L}, \qquad\qquad n = 1, 2, 3, \ldots$$

Let us return to solving the heat conduction problem. After finding the eigenvalues and corresponding eigenfunctions by solving the 2-point boundary value problem, the next step in the process is to solve the second equation (that of the time variable t):

$$T' + \alpha^2 \lambda T = 0. \qquad\qquad (2)$$

Since it is the second half of a system of simultaneous equations, the eigenvalues λ from the first equation have to be used here. Hence, the equation becomes:

$$T' + \alpha^2 \frac{n^2 \pi^2}{L^2} T = 0.$$

The equation is both a first order linear equation, as well as a separable equation. Solve it using either method (solving it as a separable equation is probably quicker) to get the general solution

$$T_n(t) = C_n\, e^{-\alpha^2 n^2 \pi^2 t / L^2}, \qquad\qquad n = 1, 2, 3, \ldots$$

We can now assemble the result from the two ordinary differential equations to find the solutions of the partial differential equation. Recall that the assumption in the separation of variables method is that the PDE has solutions in the form $u(x,t) = X(x)T(t)$. Therefore, we see that the solutions of the one-dimensional heat conduction equation, with the boundary conditions $u(0,t) = 0$ and $u(L,t) = 0$, are in the form

$$u_n(x,t) = X_n(x)\, T_n(t) = C_n\, e^{-\alpha^2 n^2 \pi^2 t / L^2} \sin\frac{n\pi x}{L},$$

$$n = 1, 2, 3, \ldots$$

The general solution, of the temperature distribution (recall that this general solution is only valid for the given set of boundary conditions) within a bar that has both ends kept at 0 degree, is just the linear combination of all the above (linearly independent) functions $u_n(x, t)$. That is,

$$u(x,t) = \sum_{n=1}^{\infty} C_n \, e^{-\alpha^2 n^2 \pi^2 t / L^2} \sin \frac{n \pi x}{L} .$$

Once the general solution has been found, we can now apply the initial condition in order to find the particular solution. Set $t = 0$ in the general solution above, and equate it with the initial condition $u(x,0) = f(x)$:

$$u(x,0) = \sum_{n=1}^{\infty} C_n \sin \frac{n \pi x}{L} = f(x) .$$

Take a few seconds to grasp what this equation says, and we quickly realize that we are not out of the woods yet. The equation above specifies that the (arbitrary) initial condition must be equal to an infinite series of sine terms, and there are infinitely many coefficients c_n that need to be solved. Can this equation even be solved, in general? The answer is a reassuring "yes". But to get there we still need to know several things. Namely, what kind of functions could be expressed as a series of sines (and, more generally, sines and/or cosines)? Even if such functions exist, how does the arbitrary initial condition $f(x)$ fit in? Lastly, how could we possibly solve for the infinitely many coefficients with just this one equation given?

Summary (thus far)

The Method of Separation of Variables:

1. Separate the PDE into ODEs of one independent variable each. Rewrite the boundary conditions so they associate with only one of the variables.

2. One of the ODEs is a part of a two-point boundary value problem. Solve this problem for its eigenvalues and eigenfunctions.

3. Solve the other ODE.

4. Multiply the results from steps (2) and (3), and sum up all the products to find the general solution.

$$u = X\,T \qquad\qquad u_{xx} = X''\,T$$

$$u_x = X'\,T \qquad\qquad u_{tt} = X\,T''$$

$$u_t = X\,T' \qquad\qquad u_{xt} = u_{tx} = X'\,T'$$
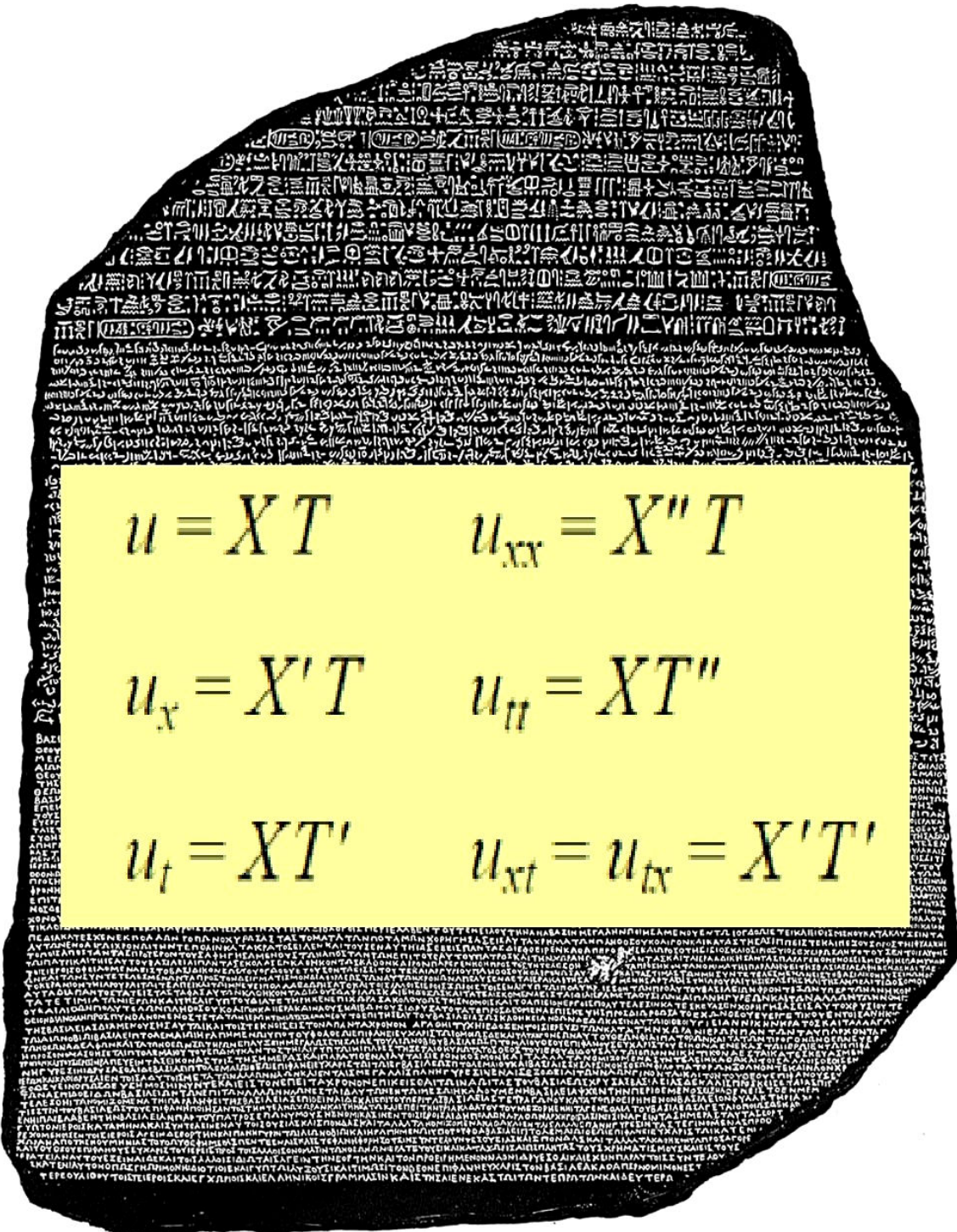
The "Rosetta Stone" of Separation of Variables.

Exercises E-1.1:

1 − 4 Determine whether each PDE can be separated. Then separate it into two ODEs if it is possible to separate.

1. $x^2 u_{xx} - t^2 u_{tt} = 0$

2. $x u_{xx} - \pi u_{tt} = 5 u_{xt}$

3. $u_{xx} - 3u = u_t$, $\qquad\qquad$ $u(0, t) = 0,$ $\quad$ $u(\pi, t) = 0.$

4. $u_{xx} + 2t u_{tx} = 4u,$ $\qquad\qquad$ $u_x(0, t) = 0,$ $\quad$ $u(9, t) = 0.$

5. Show that the following boundary value problem has no solutions:
$$y'' + 4y = 0, \qquad\qquad y(0) = 2, \quad y(\pi) = 3.$$

6 − 11 Find all eigenvalues and their corresponding eigenfunctions of each two-point boundary value problem.

6. $X'' + \lambda X = 0,$ $\qquad\qquad$ $X(0) = 0 ,$ $\quad$ $X(2\pi) = 0.$

7. $X'' + \lambda X = 0,$ $\qquad\qquad$ $X(0) = 0 ,$ $\quad$ $X'(2\pi) = 0.$

8. $X'' + \lambda X = 0,$ $\qquad\qquad$ $X'(0) = 0 ,$ $\quad$ $X(2\pi) = 0.$

9. $X'' + \lambda X = 0,$ $\qquad\qquad$ $X'(0) = 0 ,$ $\quad$ $X'(2\pi) = 0.$

10. $X'' - \lambda X = 0,$ $\qquad\qquad$ $X(0) = 0 ,$ $\quad$ $X(1) = 0.$

11. $X'' - \lambda X = 0,$ $\qquad\qquad$ $X'(0) = 0 ,$ $\quad$ $X(1) = 0.$

12. (a) Show that any positive eigenvalue of the boundary value problem
$$X'' + \lambda X = 0, \qquad\qquad X(0) + X'(0) = 0, \qquad X(L) = 0,$$
must be in the form $\lambda = \sigma^2$, where σ satisfies the equation $\sigma = \tan(\sigma L)$. (b) Is 0 an eigenvalue of this problem?

13. Show that 0 is not an eigenvalue, and that any positive eigenvalue of the boundary value problem
$$X'' + \lambda X = 0, \qquad\qquad X(0) - X'(0) = 0, \qquad X(L) + 2X'(L) = 0,$$
must be in the form $\lambda = \sigma^2$, where σ satisfies the equation $\cot(\sigma L) = \dfrac{2\sigma^2 - 1}{3\sigma}$.

Answers E-1.1:

1. Separable; there are many ways to separate, one possible result is
   $$x^2 X'' + \lambda X = 0, \text{ and } t^2 T'' + \lambda T = 0.$$
2. Not separable.
3. Separable; one possible result is
   $$X'' + (\lambda - 3)X = 0, \text{ and } T' + \lambda T = 0.$$
   The boundary conditions become $X(0) = 0$, $X(\pi) = 0$.
4. Separable; one possible result is
   $$X'' + \lambda X' - 4X = 0, \text{ and } 2t\,T' - \lambda T = 0.$$
   The boundary conditions become $X'(0) = 0$, $X(9) = 0$.

6. $\lambda = \dfrac{n^2}{4}$, $X_n = \sin\dfrac{n\,x}{2}$, $n = 1, 2, 3, \ldots$

7. $\lambda = \dfrac{(2n-1)^2}{16}$, $X_n = \sin\dfrac{(2n-1)\,x}{4}$, $n = 1, 2, 3, \ldots$

8. $\lambda = \dfrac{(2n-1)^2}{16}$, $X_n = \cos\dfrac{(2n-1)\,x}{4}$, $n = 1, 2, 3, \ldots$

9. $\lambda = 0$, $X_0 = 1$; and $\lambda = \dfrac{n^2}{4}$, $X_n = \cos\dfrac{n\,x}{2}$, $n = 1, 2, 3, \ldots$

10. $\lambda = -n^2\pi^2$, $X_n = \sin n\pi x$, $n = 1, 2, 3, \ldots$

11. $\lambda = \dfrac{-(2n-1)^2\pi^2}{4}$, $X_n = \cos\dfrac{(2n-1)\pi\,x}{2}$, $n = 1, 2, 3, \ldots$

12. (b) No, 0 is not an eigenvalue.