SC 216 Tutorial 13

1. If p is an integer greater than or equal to zero. Show that the series

$$\sum_{n=1}^{\infty} \frac{p(p-1)(p-2)\ldots.(p-n+1)}{n!}.x^n$$

converges for $|x| < 1$ and diverges for $|x| > 1$.

2. Chebyshev's equation is

$$(1-x^2)y'' - xy' + p^2y = 0$$

Where p is a constant.

   a. Find the two linearly independent solution valid for $|x| < 1$.
   b. Show that if $p = n$ where $n \geq 0$, n is an integer, then there is a polynomial solution of degree n. When these are multiplied by suitable constants, they are called Chebyshev's polynomial.

3. When $p > 0$ Bessel's equation becomes $x^2y'' + xy' + x^2y = 0$. Show that its indicated equation has only one root, and use the method of this section to deduce that

$$y = \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{2n}(n!)^2} x^{2n}$$

4. Bessel equation of order $p = 1$ is $x^2y'' + xy' + (x^2 - 1)y = 0$ .Show that $m_1 - m_2 = 2$ and that the equation has only one Frobenius series solution.Then find it.

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SC216

Assignment - 13

— Power Series Solutions & Special Functions :—

Topics :-

1. Series solutions of first order equations.
2. Second order Linear eq.^{n}s, ordinary points.
3. Regular Singular Points. (method of Frobenius)

Q-1

If p is not-zero or a positive integer, show that the series $\sum_{n=1}^{\infty} \frac{p(p-1)(p-2)\cdots(p-n+1)}{n!} x^n$

converges for $|x|<1$ and diverges for $|x|>1$.

$\Rightarrow$ $a_n = \sum_{n=1}^{n} \frac{p(p-1)(p-2)\cdots(p-n+1)}{n!} \cdot x^n$

$a_{n+1} = \sum_{n=1}^{n+1} \frac{p(p-1)(p-2)\cdots(p-n)}{(n+1)!} \cdot x^{n+1}$

Ratio Test for convergence.

$$R = \lim_{n\to\infty} \left|\frac{a_n}{a_{n+1}}\right| = \left|\frac{(n+1)}{(p-n)}\right| = \left|\frac{1+1/n}{p/n - 1}\right| = |-1| = \frac{1}{|x|}$$

$\Rightarrow$ The sequence converges for $|x|<1$ and diverges for $|x|>1$.

Q-2. Chebychev's equation is $(1-x^2)y'' - xy' + p^2y = 0$ where p is a constant.

(a) Find two linearly independent series solutions valid for $|x|<1$

(b) Show that if $p=n$ where n is an integer ≥ 0, then there is a polynomial solution of degree n. When these are multiplied by suitable constants, they are called the Chebyshev polynomials. ~~the~~ ~~check~~

~~test~~

$\Rightarrow$ $(1-x^2)y'' - xy' + p^2 y = 0$

$P(x) = -\dfrac{x}{1-x^2}$ $\qquad Q(x) = \dfrac{p^2}{(1-x^2)}$

let $y = \sum a_n x^n$

$\Rightarrow y' = \sum (n+1) a_{n+1} x^n$

$y'' = \sum (n+2)(n+1) a_{n+2} x^n$

$(1-x^2) y'' = \sum (1-x^2)(n+2)(n+1) a_{n+2} x^n$

$x y' = \sum x(n+1) a_{n+1} x^n = \sum n a_n x^n$

$p^2 y = \sum p^2 a_n x^n$

$(1-x^2) y'' - xy' + p^2 y = 0$

$\Rightarrow \sum (1-x^2)(n+2)(n+1) a_{n+2} x^n - \sum \cancel{n(n+1)}\, n\, a_n x^n + \sum p^2 a_n x^n = 0$

coefficients of all x^n must be zero.

$(n+1)(n+2) a_{n+2} - (\cancel{(n+1) a_{n+1}})\, n a_n + p^2 a_n = 0$

$\Rightarrow a_{n+2} = \dfrac{n - p^2}{(n+1)(n+2)} \cdot a_n$

$\Rightarrow$ All a_n can be represented in terms of a_0 and a_1 according as n is even or odd.

$a_2 = -\dfrac{p^2}{2} a_0$; $\quad a_3 = \dfrac{1-p^2}{\cancel{(n+1)(n+2)}\ 2\cdot 3} \cdot a_1$

$a_4 = \dfrac{2-p^2}{3\cdot 4} a_2 = -\dfrac{p^2(2-p^2)}{2\cdot 3\cdot 4} a_0 = -\dfrac{p^2(2-p^2)}{4!} a_0$

$a_5 = \dfrac{3-p^2}{4\cdot 5} \cdot a_3 = \dfrac{(3-p^2)(1-p^2)}{5!} a_1$

$a_6 = \dfrac{4-p^2}{5\cdot 6} a_4 = -\dfrac{p^2(2-p^2)(4-p^2)}{6!} a_0$

$a_7 = \dfrac{5-p^2}{7\cdot 6} \cdot a_5 = \dfrac{(5-p^2)(3-p^2)(1-p^2)}{7!} a_1$

$$y = \Sigma a_n x^n$$

$$= a_0\left[1 - \frac{p^2}{2!}x^2 - \frac{p^2(2-p^2)}{4!}x^4 - \frac{p^2(2-p^2)(4-p^2)}{6!}x^6 - \dots\right]$$

$$+ a_1\left[x + \frac{(1-p^2)}{3!}x^3 + \frac{(3-p^2)(1-p^2)}{5!}x^5 + \frac{(5-p^2)(3-p^2)(1-p^2)}{7!}x^7 + \dots\right]$$

$$\left|\frac{a_{2n+2}\,x^{2n+2}}{a_{2n}\,x^{2n}}\right| = \left|\frac{2n-p^2}{(2n+1)(2n+2)}\right| |x|^2 < |x|^2 \Rightarrow \underline{\underline{|x|<1}}$$

$$\left|\frac{a_{(2n-1)+2}\,x^{(2n-1)}}{a_{2n-1}}\right| = \left|\frac{2n-1-p^2}{(2n-1+1)(2n-1+2)}\right| = \left|\frac{2n-(p^2)-1}{2n(2n-1)}\right| < 1$$

$\Rightarrow$ The series converges for $\underline{\underline{R=1}}$.

or $\underline{\underline{|x|<1}}$

Q.3 When $p=0$, Bessel's equation becomes

$x^2y'' + xy' + x^2y = 0$.

show that its indicial equation has only one root, and use the method of ~~of for~~ regular singular pts. to deduce that $y = \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{2n}(n!)^2} x^{2n}$

Method of Frobenius

$\Rightarrow$ $x^2y'' + xy' + x^2y = 0$

$\Rightarrow y'' + \frac{1}{x}y' + y = 0$

$P(x) = 1/x$ and $Q(x) = 1$ $\Rightarrow x=0$ is a regular singular pt

~~$y = x^m(a_0 + a_1x + a_2x^2 + \dots)$~~

~~$= a_0x^m + a_1x^{m+1} + a_2x^{m+2} + \dots$~~

~~$y' = ma_0x^{m-1} + (m+1)a_1x^m + (m+2)a_2x^{m+1} + \dots$~~

~~$y'' = m(m-1)a_0x^{m-2} + (m+1)m\,a_1x^{m-1} + (m+2)(m+1)a_2x^m + \dots$~~

$x P(x) = 1 = \sum_{n=0}^{\infty} p_n x^n \Rightarrow p_0 = 1$

$x^2 Q(x) = x^2 = \sum_{n=0}^{\infty} q_n x^n \Rightarrow q_0 = 0$

The indicial eq.^{n} is:

$m(m-1) + m p_0 + q_0 = 0$

$\Rightarrow m(m-1) + m = 0 \Rightarrow m^2 - m + m = 0$

$\Rightarrow m^2 = 0$ or $m = 0$.

$\Rightarrow$ The indicial eq.^{n} has only one root.

Sol.^{n}: $y = x^m (a_0 + a_1 x + a_2 x^2 + \ldots)$

$y = a_0 + a_1 x + a_2 x^2 + a_3 x^3$

$y' = a_1 + 2a_2 x + 3a_3 x^2 + 4a_4 x^3 + \ldots$

$y'' = 2a_2 + 6a_3 x + 4.3.a_4 x^2 + \ldots$

eq.^{n} is: $y'' + \frac{1}{x} y' + y = 0$

$\Rightarrow \cancel{2a_2}\ a_0 + a_1 x + a_2 x^2 + \ldots$

$+ \frac{a_1}{x} + 2a_2 + 3a_3 x + 4a_4 x^2 + \ldots$

$+ 2a_2 + 6a_3 x + 4.3.a_4 x^2 + \ldots = 0$

$\Rightarrow a_1 x^{-1} + (a_0 + 4a_2) + (a_1 + 3a_3 + 6a_3) x + \ldots = 0$

$+ (a_2 + 4a_4 + 4.3a_4) x^2$

$\Rightarrow a_0 + 4a_2 = 0 \Rightarrow a_2 = -a_0/4$

$a_1 = 0$

$a_1 + 9a_3 = 0 \Rightarrow a_3 = -a_1/9 = 0$

$a_2 + 16a_4 = 0 \Rightarrow a_4 = -\frac{1}{16} a_2 = \left(-\frac{1}{16}\right)\left(-\frac{1}{4}\right) a_0 = \frac{a_0}{64}$

$$\boxed{y = \sum_{n=0}^{\infty} \frac{(-1)^n}{2^{2n} (n!)^2} x^{2n}}$$

Q-4 Bessel's eq.^{n} of order $p = 1$ is

$x^2 y'' + xy' + (x^2-1)y = 0$.

show that $m_1 - m_2 = 2$ and that the eq.^{n} has only one Frobenius series sol.^{n}. Then find it.

$\Rightarrow$ $x^2 y'' + xy' + (x^2-1)y = 0$

$$\Rightarrow y'' + \frac{y'}{x} + \frac{(x^2-1)}{x^2} y = 0$$

$$P(x) = 1/x \qquad Q(x) = \frac{x^2-1}{x^2}$$

$$x P(x) = 1 = \sum_{n=0}^{\infty} p(n) x^n \Rightarrow p_0 = 1$$

$$x^2 Q(x) = -1 + x^2 = \sum_{n=0}^{\infty} q(n) x^n \Rightarrow q(0) = -1$$

The incidial eq.^{n} is:

$$m(m-1) + p_0 m + q_0 = 0$$

$$\Rightarrow m(m-1) + m - 1 = 0$$

$$\Rightarrow m^2 - 1 = 0$$

$$\Rightarrow m_1 = +1 \quad \text{and} \quad m_2 = -1$$

$$\Rightarrow m_1 - m_2 = 2.$$

$$\Rightarrow y_1 = x(a_0 + a_1 x + a_2 x^2 + \ldots)$$

$$y_2 = x^{-1}(a_0 + a_1 x + a_2 x^2 + \ldots)$$

$$y_1' = a_0 + 2a_1 x + 3a_2 x^2 + 4a_3 x^3 + \ldots$$

$$y_1'' = 2a_1 + 6a_2 x + 12 a_3 x^2 + \ldots$$

$$y_1'' + \frac{y_1'}{x} + \frac{(x^2-1)}{x^2} y = 0$$

$$\Rightarrow (2a_1 + 6a_2 x + 12 a_3 x^2 + \ldots)$$
$$+ (a_0 x^{-1} + 2a_1 + 3a_2 x + 4a_3 x^2 + \ldots)$$
$$+ \left(1 - \frac{1}{x^2}\right)(a_0 x + a_1 x^2 + a_2 x^3 + a_3 x^4 + \ldots) = 0$$

$\cancel{x\ f(x)} \; 1$

$$\left(a_0 - a_0\right)\frac{1}{x} + (2a_1 + 2a_1 - a_1) + \{(6a_2 + 3a_2 - a_2) + a_0\}x$$
$$+ \{(12+4-1)a_3 + a_1\}x^2 + \{(4\cdot 5\cdot 4 + 5 - 1)a_4 + a_2\}x^3 + \dots$$

$\Rightarrow \quad 3a_1 = 0 \quad \Rightarrow \quad a_1 = 0$

$8a_2 + a_0 = 0 \quad \Rightarrow \quad a_2 = -\frac{a_0}{8}$

$15a_3 + a_1 = 0 \quad \Rightarrow \quad a_3 = 0$

$24a_4 + a_2 = 0 \quad \Rightarrow \quad a_4 = -a_2/24 = \frac{1}{24}\cdot\frac{1}{8}a_0$

$$\Rightarrow y = x\left(1 - \frac{x^2}{2^2 2!} + \frac{x^4}{2^4 2!3!} - \dots\right)$$

$y_2 = a_0x^{-1} + a_1 + a_2x + a_3x^2 + \dots$

$y_2' = -a_0x^{-2} + a_2 + 2a_3x + 3a_4x^2 + 4a_5x^3 + \dots$

$y_2'' = 2a_0x^{-3} + 2a_3 + 6a_4x + 12a_5x^2 + \dots$

$$y_2'' + \frac{y_2'}{x} + \left(1 - \frac{1}{x^2}\right)y_2 = 0$$

$$\Rightarrow \left(2a_0x^{-3} + 2a_3 + 6a_4x + 12a_5x^2 + \dots\right)$$
$$+ \left(-a_0x^{-3} + a_2x^{-1} + 2a_3 + 3a_4x + 4a_5x^2 + \dots\right)$$
$$+ \left(a_0x^{-1} + a_1 + a_2x + a_3x^2 + \dots\right)$$
$$- \left(a_0x^{-3} + a_1x^{-2} + a_2x^{-1} + a_3 + a_4x + a_5x^2 + \dots\right) = 0$$

$$\Rightarrow (2a_0 - 2a_0)x^{-3} - a_1x^{-2} + (a_2 - a_2)x^{-1}$$
$$+ \{(2+2-1)a_3 + a_1\} + \{(6+3-1)a_4 + a_2\}x$$
$$+ \{(12+4-1)a_5 + a_3\}x^2 + \dots = 0$$

$\Rightarrow \quad a_1 = 0.$

$3a_3 + a_1 = 0 \quad \Rightarrow \quad a_3 = 0.$

$8a_4 + a_2 = 0 \Rightarrow a_4 = -a_2/8$

$24a_6 + a_4 = 0 \Rightarrow a_6 = -\frac{1}{24\cdot 8}\cdot a_2$

$$\Rightarrow y = x^{-1}\left[x^2 - \frac{1}{8}x^4 + \frac{x^6}{2^4\cdot 2!3!} - \dots\right] = x\left[1 - \frac{x^2}{2^2 2!} + \frac{x^4}{2^4 2!3!} - \dots\right]$$

$\Rightarrow$ Only one Frobenius Serie.