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UNIT - 01

Introduction To Image Processing System

* **Introduction:**

- Image: An image is a two dimensional function that represents a measure of some characteristic such as brightness or colour of a viewed scene. It is a projection of a 3D scene into a 2D projection plane. It can be defined as a two variable function $f(x,y)$ where each position (x,y) in the projection plane has $F(x,y)$ which defines the light intensity at that point.
- Analog Image: An analog image can be mathematically represented as a continuous range of values representing position and intensity. An analog image is characterised by a physical magnitude varying continuously in space.
- Digital Image: A digital image is composed of picture elements called pixels. Pixels are the smallest sample of an image which represents the brightness at one point. A digital image is a numerical representation of an object / scene.
- Digital Image Processing:

  The processing of an image by means of a processor is generally termed as digital image processing.

  advantages:
  - Flexibility and adaptability.
  - Data storage and transmission

  Different image processing techniques include:

- image acquisition	- image denoising
- image enhancement	- image fusion
- image segmentation	- image watermarking
- image compression	- colour image processing
- image restoration	and so on....

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– Digital Image from Analog Image:

Analog Image → Sampling → Quantization → Digital Image

Advantages:
- Faster and cost effective
- Digital images can be effectively stored and transmitted.
- copying a digital image is easy.
- Reproduction of the image is faster and cheaper

Disadvantages:
- A digital file cannot be enlarged beyond a certain size without compromising on quality.
- misuse of image.
- Memory required to store and process good quality images is very high.
- A good processor is required for manipulation

A digital image is represented as an array of pixel.

– Neighbours of pixels

- 4 neighbours:

NORTH

		x		
WEST	x	P	x	EAST
		x		

SOUTH

- 8 neighbours

NORTH

NW	x	x	x	NE
WEST	x	P	x	EAST
SW	x	x	x	SE

SOUTH

– Digital Image Representation

A digital image is a two dimensional discrete signal which is also an N x N array of elements. Each element in the array is a number which represents the sampled intensity.

pixel

$$\begin{bmatrix} 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \end{bmatrix}_{4\times4}$$

★ Image Sampling:

Sampling is the process of measuring the brightness information only at a discrete spatial location. A continuous image function $f(x,y)$ can be sampled using a discrete grid of sampling points in the plane.

– 2D Sampling

Method I: Let $f(x,y)$ represent an analog image, that is $f(x,y)$ is defined at each and every value of x and y. The discrete version of $f(x,y)$ is obtained by defining $f(x,y)$ at specific instants.

Mathematically,

discrete $f(x,y) \longrightarrow f(m,n)$

$$f(m,n) = f(m\Delta x, n\Delta y) \quad \text{——— (1)}$$

where Δx and Δy are positive real constants which are also known as sampling intervals.

To obtain $f(m,n)$, $f(x,y)$ needs to be 2D sampled

i.e., $f(x,y) \xrightarrow{\text{2D sampling}} f(m,n)$

On taking Fourier Transform of $f(x,y)$ we get the spectrum of $f(x,y)$ which is denoted by $F(\Omega_1, \Omega_2)$, where

$$F(\Omega_1, \Omega_2) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\, e^{-j\Omega_1 x}\, e^{-j\Omega_2 y}\, dx\, dy \quad \text{——— (2)}$$

Here $e^{-j\Omega_1 x}$ and $e^{-j\Omega_2 x}$ represent Fourier basis function

On taking inverse Fourier transform of eq (2), we get

$$f(x,y) = \frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(\Omega_1, \Omega_2)\, e^{j\Omega_1 x}\, e^{j\Omega_2 y}\, d\Omega_1\, d\Omega_2 \quad \text{——— (3)}$$

Performing sampling of the analog signal $f(x,y)$

$$f(m,n) = f(m\Delta x, n\Delta y) \quad \text{——— (4)}$$

On taking the inverse Fourier transform of this sampled signal

$$f(m,n) = \frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(\Omega_1, \Omega_2)\, e^{j\Omega_1 m\Delta x}\, e^{j\Omega_2 n\Delta y}\, d\Omega_1\, d\Omega_2 \quad \text{——— (5)}$$

For a discrete signal

$$\omega_1 = \Omega_1 \Delta x \quad \text{——— (6)} \qquad \text{and} \qquad \omega_2 = \Omega_2 \Delta y \quad \text{——— (7)}$$

where ω_1 and ω_2 are in radians.

From eq ⑥ and eq ⑦

$$\Omega_1 = \frac{\omega_1}{\Delta x} \quad \text{and} \quad \Omega_2 = \frac{\omega_2}{\Delta y}$$

Differentiating Ω_1, we get

$$\frac{d\omega_1}{\Delta x} = d\Omega_1 \quad \text{——} \quad (8)$$

Similarly: $$\frac{d\omega_2}{\Delta y} = d\Omega_2 \quad \text{——} \quad (9)$$

Substituting eq ⑧ and eq ⑨ in eq ⑤

$$f(m,n) = \frac{1}{4\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} F\left(\frac{\omega_1}{\Delta x}, \frac{\omega_2}{\Delta y}\right) e^{j\omega_1 n} e^{j\omega_2 n} \frac{d\omega_1}{\Delta x} \frac{d\omega_2}{\Delta y} \quad \text{——} \quad (10)$$

$$f(m,n) = \frac{1}{4\pi^2} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{1}{\Delta x \Delta y} F\left(\frac{\omega_1}{\Delta x}, \frac{\omega_2}{\Delta y}\right) e^{j\omega_1 n} e^{j\omega_2 n} d\omega_1 d\omega_2 \quad \text{——} \quad (11)$$

The signal $f(m,n)$ is discretized, hence the integral can be replaced by summation. The double integral over the entire (ω_1, ω_2) plane can be broken into an infinite series of integrals, each of which is over a square of area $4\pi^2$.

The range of ω_1 and ω_2 is given by

$$-\pi + 2\pi k_1 \leq \omega_1 < \pi + 2\pi k_1$$

$$-\pi + 2\pi k_2 \leq \omega_2 < \pi + 2\pi k_2$$

Therefore

$$f(m,n) = \frac{1}{4\pi^2} \iint_{sq(k_1,k_2)} \frac{1}{\Delta x \Delta y} \sum_{k_1} \sum_{k_2} F\left(\frac{\omega_1}{\Delta x}, \frac{\omega_2}{\Delta y}\right) e^{j\omega_1 n} e^{j\omega_2 n} d\omega_1 d\omega_2 \quad \text{——} \quad (12)$$

In order to change the limits of the integral so as to remove the dependence of the limits of integration on k_1 and k_2, hence replace

ω_1 by $\omega_1 - 2\pi k_1$

ω_2 by $\omega_2 - 2\pi k_2$

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On including this modification in eq ⑫, we get

$$f(m,n) = \frac{1}{4\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\frac{1}{\Delta x\Delta y}\sum_{k_1}\sum_{k_2}F\left(\frac{\omega_1-2\pi k_1}{\Delta x},\frac{\omega_2-2\pi k_2}{\Delta y}\right) e^{j(\omega_1-2\pi k_1)m}\, e^{j(\omega_2-2\pi k_2)n}\, d\omega_1\, d\omega_2 \quad \text{— (13)}$$

$$f(m,n) = \frac{1}{4\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\frac{1}{\Delta x\Delta y}\sum_{k_1}\sum_{k_2}F\left(\frac{\omega_1-2\pi k_1}{\Delta x},\frac{\omega_2-2\pi k_2}{\Delta y}\right) e^{j\omega_1 m}\, e^{-j2\pi k_1 m}\, e^{j\omega_2 n}\, e^{-j2\pi k_2 n}\, d\omega_1\, d\omega_2 \quad \text{— (14)}$$

The exponential term in the above equation is equal to one for all values of the integer variables m, k_1, n and k_2. Then from eq ⑭

$$f(m,n) = \frac{1}{4\pi^2}\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\frac{1}{\Delta x\Delta y}\sum_{k_1}\sum_{k_2}F\left(\frac{\omega_1-2\pi k_1}{\Delta x},\frac{\omega_2-2\pi k_2}{\Delta y}\right) e^{j\omega_1 m}\, e^{j\omega_2 n}\, d\omega_1\, d\omega_2 \quad \text{— (15)}$$

The above equation represents inverse fourier transform of $F(\omega_1, \omega_2)$, where

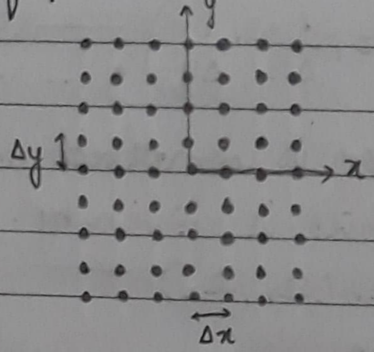
$$F(\omega_1,\omega_2) = \frac{1}{\Delta x\Delta y}\sum_{k_1}\sum_{k_2}F\left(\frac{\omega_1-2\pi k_1}{\Delta x},\frac{\omega_2-2\pi k_2}{\Delta y}\right) \quad \text{— (16)}$$

The above equation can be written as,

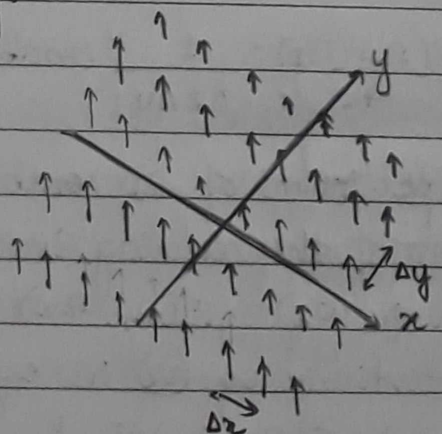
$$F(\omega_1,\omega_2) = \frac{1}{\Delta x\Delta y}\sum_{k_1}\sum_{k_2}F\left(\Omega_1-\frac{2\pi k_1}{\Delta x},\ \Omega_2-\frac{2\pi k_2}{\Delta y}\right) \quad \text{— (17)}$$

Method 2: Alternate Method

In this method analog image is multiplied by a 2-D comb function. The 2D comb function is a rectangular grid of points as shown in fig.



2D view of a comb function



3D view of a comb function

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The spaces between successive grid points in the x and y direction are Δx and Δy respectively. The 2D comb function is otherwise known as bed of nail function.

2D comb function is defined as:

$$comb(x, y, \Delta x, \Delta y) = \sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty} \delta(x - k_1 \Delta x, y - k_2 \Delta y) \quad \text{— (18)}$$

After multiplying the analog image $f(x, y)$ with the 2D comb function we get the discrete version of the analog image which is given by:

$$f(m, n) = f(x, y)\, comb(x, y, \Delta x, \Delta y) \quad \text{— (19)}$$

$$f(m, n) = \sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty} f(k_1 \Delta x, k_2 \Delta y)\, \delta(x - k_1 \Delta x, y - k_2 \Delta y) \quad \text{— (20)}$$

We know that convolution in time domain is equal to multiplication in the frequency domain and vice versa.

For the purpose of analysis in frequency domain, let us take the Fourier transform of the input analog image $f(x, y)$ and the 2D comb function $comb(x, y, \Delta x, \Delta y)$.

The Fourier Transform of the signal $f(x, y)$ is $F(\Omega_1, \Omega_2)$. Similarly the Fourier transform of comb function is another comb function which is represented by

$$comb(\Omega_1, \Omega_2) = FT\{comb(x, y, \Delta x, \Delta y)\} \quad \text{— (21)}$$

$$comb(\Omega_1, \Omega_2) = \frac{1}{\Delta x \Delta y} \sum_{p=-\infty}^{\infty} \sum_{q=-\infty}^{\infty} \delta\left(\Omega_1 - \frac{p}{\Delta x}, \Omega_2 - \frac{q}{\Delta y}\right) \quad \text{— (22)}$$

$$comb(\Omega_1, \Omega_2) = \frac{1}{\Delta x \Delta y}\, comb\left(\Omega_1, \Omega_2, \frac{1}{\Delta x}, \frac{1}{\Delta y}\right) \quad \text{— (23)}$$

The spectrum of 2D comb function is convolved with the spectrum of the analog image which is given by.

$$F(\omega_1, \omega_2) = F(\Omega_1, \Omega_2) \circledast comb(\Omega_1, \Omega_2) \quad \text{— (24)}$$

Substituting eq (22) in eq (24), we get

$$F(\omega_1, \omega_2) = F(\Omega_1, \Omega_2) \circledast \frac{1}{\Delta x \Delta y} \sum_{p=-\infty}^{\infty} \sum_{q=-\infty}^{\infty} \delta\left(\Omega_1 - \frac{p}{\Delta x}, \Omega_2 - \frac{q}{\Delta y}\right) \quad \text{— (25)}$$

Upon convolving the two functions, we get

$$F(\omega_1,\omega_2) = \frac{1}{\Delta x \Delta y} \sum_{k=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} F(\Omega_1-k, \Omega_2-l) \sum_{p=-\infty}^{\infty} \sum_{q=-\infty}^{\infty} \delta\left(k-\frac{p}{\Delta x}, l-\frac{q}{\Delta y}\right) \quad (26)$$

As summation is a linear operator, interchanging the order of summation, we get

$$F(\omega_1,\omega_2) = \frac{1}{\Delta x \Delta y} \sum_{p=-\infty}^{\infty} \sum_{q=-\infty}^{\infty} \sum_{k=-\infty}^{\infty} \sum_{l=-\infty}^{\infty} F(\Omega_1-k, \Omega_2-l)\, \delta\left(k-\frac{p}{\Delta x}, l-\frac{q}{\Delta y}\right) \quad (27)$$

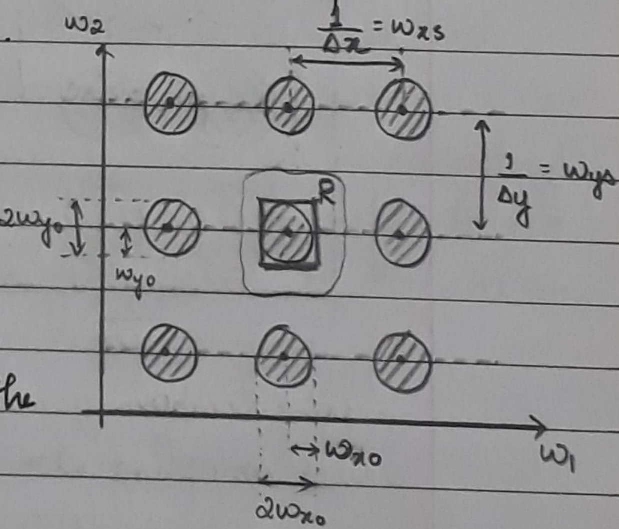
$$F(\omega_1,\omega_2) = \frac{1}{\Delta x \Delta y} \sum_{p=-\infty}^{\infty} \sum_{q=-\infty}^{\infty} F\left(\Omega_1-\frac{p}{\Delta x}, \Omega_2-\frac{q}{\Delta y}\right) \quad (28)$$

Equation (28) resembles equation (17)

- Retrieving the image from its samples:

Discreteness in one domain leads to periodicity in another domain. Hence sampling in time domain leads to periodic spectrum in the frequency domain.

In order to retrieve the original signal from the sampled spectrum the following conditions have to be satisfied.



- $\omega_{xs} > 2\omega_{x0}$ — (29) where $\omega_{xs} = 1/\Delta x$ and $2\omega_{x0}$ is bandwidth of the spectrum in the ω_1 direction.
- $\omega_{ys} > 2\omega_{y0}$ — (30) where $\omega_{ys} = 1/\Delta y$ and $2\omega_{y0}$ is bandwidth of the spectrum in ω_2 direction.

Eq (29) and eq (30) implies that the sampling frequency should be greater than twice the maximum signal frequency, which is generally termed as the sampling theorem.

Here $2\omega_{x0}$ and $2\omega_{y0}$ are called Nyquist rates.

A low pass filter is normally employed in order to extract the desired spectrum. The transfer function of the low pass filter is given as follows:

$$H(\omega_1,\omega_2) = \begin{cases} 1/\omega_{1s}\omega_{2s} & ; (\omega_1,\omega_2) \in \text{region of support } R \\ 0 & ; \text{otherwise.} \end{cases} \quad (31)$$

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The region of support is indicated as R. The continuous image can be obtained from the sampled spectrum by multiplying the sampled spectrum with the low pass filter which is given as

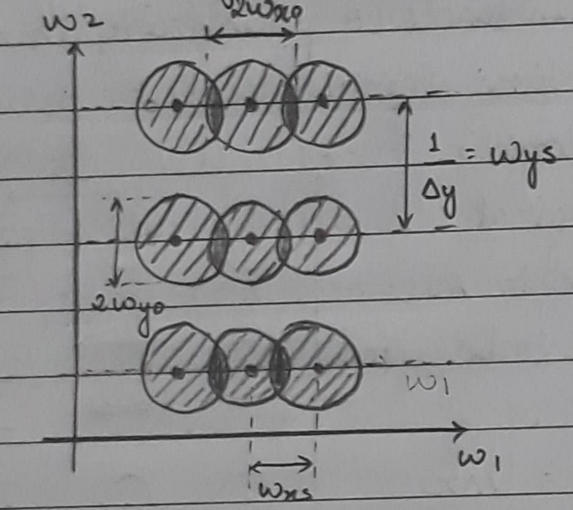
$$\hat{F}(\omega_1, \omega_2) = H(\omega_1, \omega_2)\, F(\omega_1, \omega_2) \quad \text{--- (32)}$$

By taking inverse Fourier transform, we get the continuous image

$$\hat{f}(x, y) = F^{-1}\{\hat{F}(\omega_1, \omega_2)\} \quad \text{--- (33)}$$

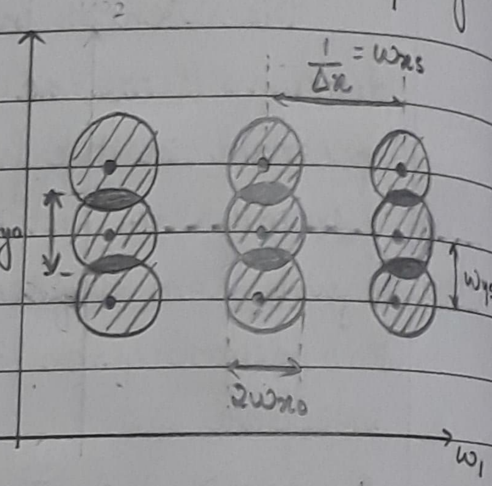
- Violation of Sampling Criterion

Violation of sampling criterion in eq (29) and eq (30) leads to aliasing which basically occurs due to under sampling.



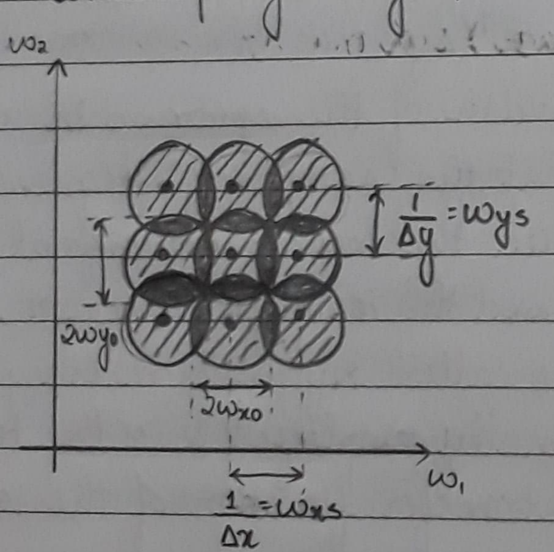
$\omega_{xs} < 2\omega_{x0}$; $\omega_{ys} > 2\omega_{y0}$

under sampling along ω_1 direction



$\omega_{xs} > 2\omega_{x0}$; $\omega_{ys} < 2\omega_{y0}$

under sampling along ω_2 direction



$\omega_{xs} < 2\omega_{x0}$; $\omega_{ys} < 2\omega_{y0}$

Under sampling along both ω_1 and ω_2 direction.

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* **Quantization:**

Quantization involves representing the sampled data by a finite number of levels based on some criteria such as minimisation of quantizer distortion.

Quantizer classification

- Scalar Quantizer
  - → Uniform Quantizer : Ex: midtread and midrise
  - → Non Uniform Quantizer: Ex: Lloyd-Max Quantizer
- Vector Quantizer

* **Resolution:**

Resolution gives the degree of distinguishable details. Resolution can be broadly classified into

- Spatial Resolution: Spatial resolution is the smallest discernible detail in an image. It depends on the number of pixels. The principle factor determining spatial resolution is sampling.
- Gray Level Resolution: Gray level resolution refers to the smallest discernible change in the gray level. It depends on the number of gray levels.

* **Classification of Digital Images:**

- **Raster Images:** (Bitmap Image) (dot matrix data structure)
  - It is generally defined as a rectangular array of regularly sampled values known as pixels.
  - It is resolution dependent because it contains a fixed number of pixels that are used to create the image, hence it will lose its quality if it is enlarged beyond the number of pixels.
  - Pixels are represented by bits.
  - It can be viewed on paper, monitor or any other display units.
  - Spatial resolution depends on acquisition device such as CCDs, optical scanners etc.

Ex: jpeg, gif, png, bmp, tiff

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- Vector Image:
  - It is defined by objects which are made of lines and curves that are mathematically defined by expression.
  - A vector can have various attributes such as line thickness, length and colour.
  - Scaling can be done without losing quality. Hence it can be printed at any size, on any output device, at any resolution without losing the detail.
  - They are thus suitable for typography, line art and illustrations.

★ Image Types:

- Black and White or binary images:
  - They take only two values, i.e., either '0' or '1'.
  - Brightness graduation cannot be differentiated.
- Gray Scale Images:
  - They contain only brightness information.
  - Brightness graduation can be differentiated.
  - Each pixel value represents the light intensity at that point in the image.

  Ex: An 8 bit image will have a brightness variation from 0 to 255 (2^8). 0 - black and 255 - white
- Colour Image:
  - It has three values per pixel and they measure the intensity and chrominance of light.
  - It can be modeled as 3 band monochrome image data where each band of data corresponds to a different colour.
  - Common colour spaces are:

    RGB: Red, Green and Blue

    HSV: Hue, Saturation, Value

    CMYK: Cyan, Magenta, Yellow, Black.

- Volume Image
  - It is a three dimensional image.
  - It can be obtained from some medical imaging equipment in which the individual data points are called voxels: volume pixels. Ex: CAT scan.
- Range Image
  - Each pixel expresses the distance between a known reference frame and a visible point in the screen.
  - It is also referred to as depth images as it reproduces the 3D structure of a scene.
- Multispectral Image
  - Images of the same object is taken in different bands of visible or infrared regions of the electromagnetic spectrum. Most of the remote sensing images are multispectral.
  - They contain information outside the normal human perceptual range (It is mapped to RGB components from the different spectral bands to represent in visual form).

★ Elements of Image Processing System:

The different elements in an image processing system are
- Image acquisition element (image sensors: CCD and CMOS sensors)
- Image storage devices
- Image processing elements
- Image display devices

• Image Sensor and Acquisition

The term image acquisition refers to the process of capturing real world images and storing them into a computer. We use a charge-coupled device or CMOS device as the image sensor that converts light into electrical charges. An image sensor is a 2D array of light sensitive elements that convert photons to electrons.

= CCD sensor:

CCD (charge-coupled device) is basically a series of closely spaced MOS capacitor. CCD imaging is performed in three steps:

Step 1: Exposure: The sensor is exposed to the incident light. Upon exposure, light is converted into an electric charge at discrete sites called pixels.

Step 2: Charge Transfer: Charge transfer moves the packets of charge within the silicon substrate.

Step 3: Charge-to-voltage conversion and output amplification

- CCD formats:

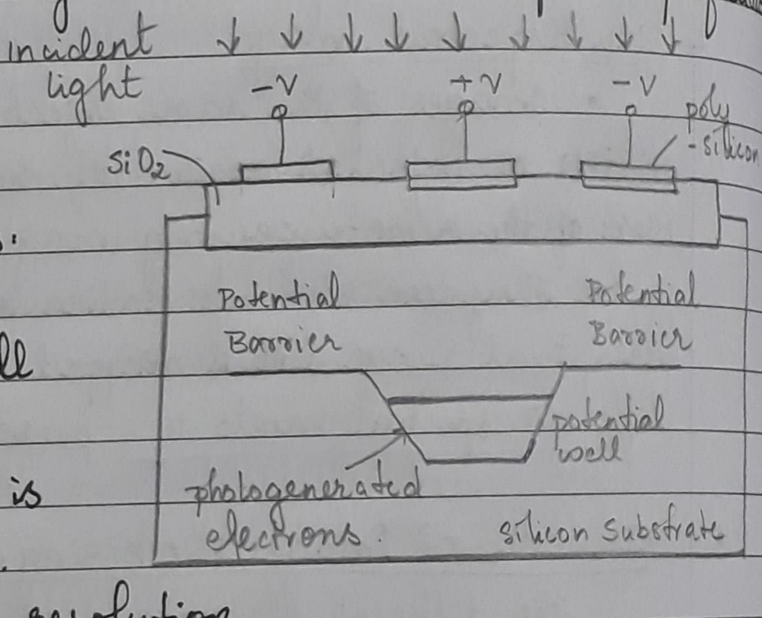
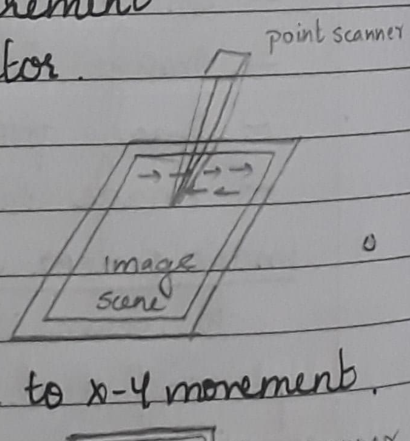


Image sensing can be performed using three basic techniques.

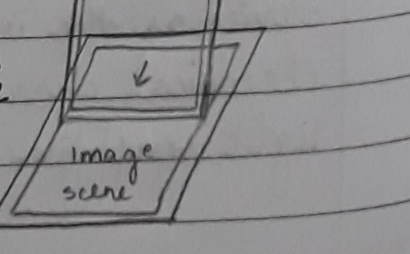
- Point scanning
  - It uses a single-cell detection or pixel.
  - Scene information is from xy coordinates.

Advantages: - high resolution
- uniformity of measurement
- simplicity of the detector.



Disadvantages: - registration errors due to the x-y movement of the scene or detector.
- repeated exposure
- system complexity due to x-y movement.

- Line scanning



- An array of single-cell detectors can be placed along a single axis such that scanning takes place in only one direction.

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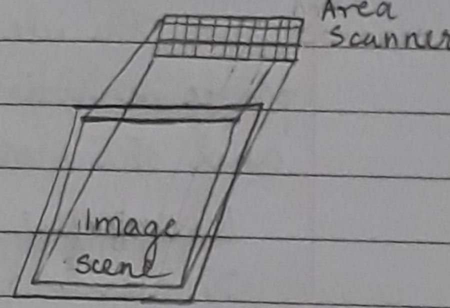
- CCD line sensor length is limited, hence we use multiple CCD along the line.

Advantages: - time of scanning is less when compared to point scanning.
- High resolution
- Less complicated scanning mechanism

Disadvantages: - using multiple CCDs increases the cost
- Though there is saving in time, it is undesirable for many applications.
- Non-uniformity results in less measurement accuracy.

• <u>Area Scanning</u>

- A two dimensional array of detectors can be used to capture the entire image in a single exposure.
- no need of movement in the detector or the scene.



Advantages: - Highest frame rates
- Highest registration accuracy between pixels.
- minimum system complexities.

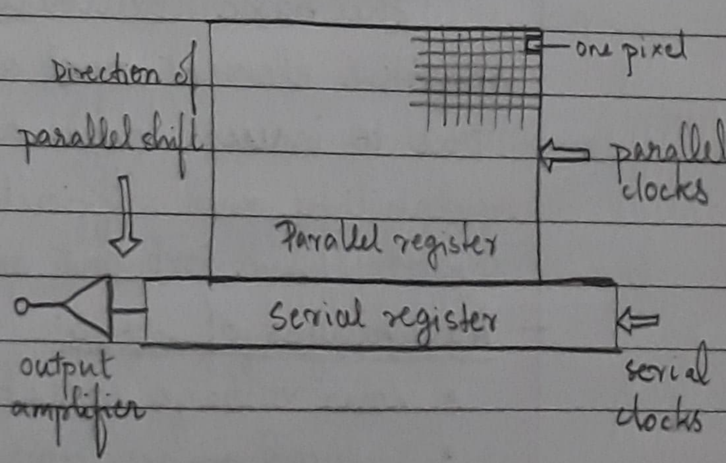
Disadvantages: - Lower SNR
- higher cost

- <u>Architectures of CCD</u>

There are two different types of CCD architectures.

• <u>Full Frame CCD:</u>

- Incoming photons fall the parallel array that acts as the imge plane. Then the device partitions the image into discrete elements that are defined by the number of pixels.



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The resulting rows of scene information are then shifted in a parallel fashion to the serial register which subsequently shifts the row of information to the output as a serial stream of data. Then process repeats until all rows are transferred off the chip.

- The parallel register is used for both scene detection and read out.

- **Frame-Transfer CCD:**
  - They are similar to full frame CCDs except that they have additional identical parallel register called a "storage array" which is not light sensitive.
  - The idea is to shift a captured scene from the photo sensitive image array quickly to the storage register and the storage array is integrated with the next frame.

  Advantage: - Faster

  Disadvantage: - smearing of images.

- **Advantages of CCD:**
  - High resolution
  - do not introduce noise or non uniformity.
- **Disadvantages of CCD:**
  - High power requirement (due to high speed shifting clocks)
  - Frame rat is limited

= **CMOS Image sensor:**

The basic structure of a pixel consists of a light sensitive element and a MOS transistor acting as a switch. Due to issues such as Fixed Pattern Noise (FPN), scalability and difficult to obtain fast read out, MOS sensors have lost out to CCD sensor as choice.

- **Advantages of CMOS:**
  - Low voltage operation (5V)
  - Reduced power consumption.

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- System integration on a single chip is very good.
- Random access of image data that allows electronic windowing and pan and zoom.
- Low cost fabrication of silicon.

- Disadvantages:
  - Each pixel in a CMOS sensor has its own amplifier, hence gain changes in amplifier results in Fixed Pattern Noise (FPN).

Applications of CCD and CMOS sensors:

CCD sensors:

- consumer: CCD camcorder, low end digital cameras.
- medical: endoscopy, Xrays.
- scientific fields: Astronomical, spectroscopical sensing applications.
- military: enemy detection etc.
- commercial: catalogue preparations, advertizing, photojournalism and studio photography.

CMOS sensors:

- because of advantages in low cost, high level of integration, low power it is widely used in multimedia applications.
- low cost cameras
- interactive gaming
- video conferencing
- It is basically used in applications where high resolution is not the major requirement.

NOTE:

significant improvement in scaling and compact integration of imaging system has contributed to renewed interest in CMOS sensors.

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= Comparison of CCD and CMOS Image sensors

CCD sensors	CMOS sensors
1. Power consumption	
The supply is 5-15V. Hence consumption is high	The supply required is 5V Hence consumption is less
2. System Integration	
It is not possible.	Signal processing can be directly integrated on the chip
3. Image Quality	
It creates high quality low noise images.	They are more susceptable to noise.
4. Read-out mode	
Read out is performed by transferring the charge from pixel to pixel that requires the entire array of pixels.	Various modes of readout - windowed readout - scanning readout - accelerated read out
5. Fixed Pattern Noise	
Lesser Fixed Pattern Noise.	Higher Fixed Pattern Noise.
6. Dynamic Range	
Larger Dynamic Range	Lesser Dynamic Range
7. Fill Factor	
20-30% fill factor	80% fill factor
8. Cost	
High cost	Low cost

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- **Scanning Mechanism:**
  - scanner is used to convert light into 0's and 1's ie, conversion of analog images into digital image files.
  - scanner allows one to capture docments that are printed on paper and turn them into digital or online format to be viewed on a computer.
  - speed and colourimetric accuracy are important criteria
  - CCD with RGB colour filters are used for the colourimetric accuracy.
- **components of a scanner:**
  - an optical system
  - light sensor
  - an interface
  - driver software
- **Drum scanner:**
  - mainly used in advertising agencies and printing
  - used to obtain high quality scans.

It consists of a cylindrical drum scanner made of translucent plastic like material. The image to be scanned, usually a film is wet-mounted by means of a fluid (oil based or alchol-based) on the drum. The sensing element is a photo multiplier tube is placed at the center of the drum. In order to obtain colour images the multiplier splits beam of light into three beams.

- **Flying spot scanner:**
  - mainly used for scanning still images or pictures from the cinema film.
  - simple method
  - high quality images
  - complex implementation

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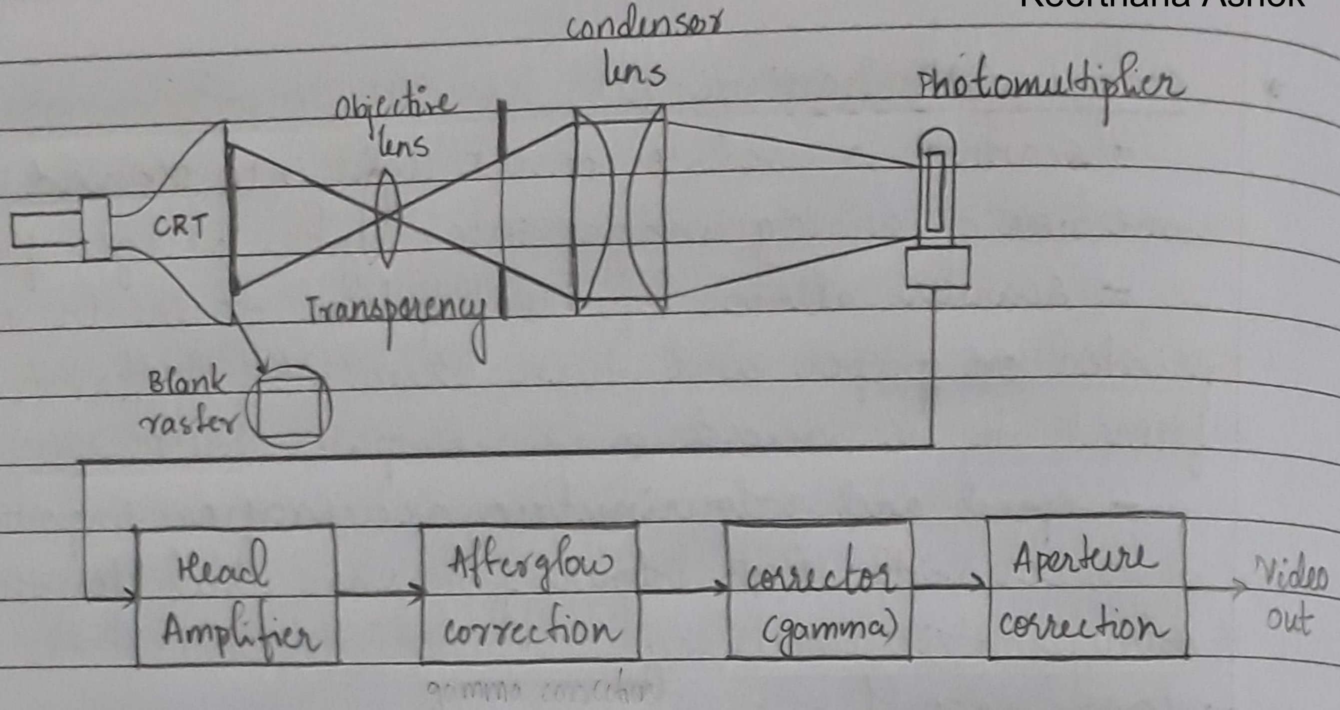

The principle is modulation of a scanning light beam by the density of the photographic film.

The blank raster from CRT is projected onto photographic film. Modulated signal from the film is collected by condenser lens and then focused on to photomultiplier. Photons are converted to electrons and then to accelerated electrons. The output of photo multiplier is then converted to video output using head amplifier, gamma corrector, afterglow corrector and aperture corrector.

Head Amplifier

The photomultiplier output current is converted into the standard video voltage level by the head amplifier.

Afterglow Correction

The afterglow corrector processes the signal to negate the effect of non-zero after-glow from the CRT phosphor.

Gamma Correction

It is to compensate the CRT display response as usual, since the photomultiplier is a linear device. It is for correction of non linear luminescence.

Aperture correction:
there is a high frequency boost inserted because of the finite size of the scanning spot.

- Vidicon

the vidicon is a storage-type camera tube in which a charge density pattern is formed by the image scene radiation on a photoconductive surface which is then scanned by a beam of low-velocity electrons. The fluctuating voltage coupled out to a video amplifier can be used to reproduce the scene being imaged.

- Single Lens Reflex (SLR) cameras and Point-and-shoot cameras:

The difference between the two depends on how the photographer sees the scene via the view-finder.
In a point and shoot camera, the viewfinder is a simple window (hole) through the body of the camera.
In a SNR camera, the actual image to be filmed can be seen.

- Digital camera

The key difference between digital and film based cameras is that digital cameras use no film instead they use a sensor to convert light into electrical charges. Like a conventional camera, the digital camera also has lens through which light from objects enter the camera but instead of falling on a film, the light falls on the sensor. The image sensor employed by most digital cameras is either CCD or CMOS.

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* Applications of Digital Image Processing

1. **Medicine**

Techniques like image segmentation and pattern recognition used in digital mammography to identify tumors.

2. **Forensics:**

Biometrics used in personal identification are face, fingerprint, iris etc. commonly used preprocessing techniques are edge enhancement, denoising, skeletonization

3. **Remote sensing**

It is used for remote observation to make useful inferences about a target. Techniques used are image enhancement, image merging and image classification.

4. **communications**

Multimedia technology – video conferencing.

5. **Automotives**

Night vision system helps to identify obstacles during night to avoid accidents. Techniques commonly used are image enhancement, boundary detection and object recognition.

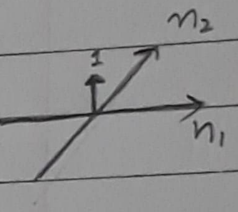
UNIT - 02

2D Signals and Systems

2D discrete signals are represented as $x(n_1, n_2)$, where
n_1, n_2 – pair of integers and
x – real or complex value
In the case of an image $x(n_1, n_2)$ represents the value of pixel intensity at the location (n_1, n_2)

* 2D Signals:

1. **2D Unit Impulse Sequence:**



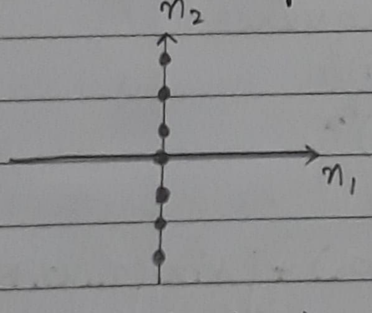
The 2D impulse sequence is given by:

$$x(n_1, n_2) = \delta(n_1, n_2) = \begin{cases} 1 & n_1 = n_2 = 0 \\ 0 & \text{else} \end{cases}$$

2. **Line Impulse:**

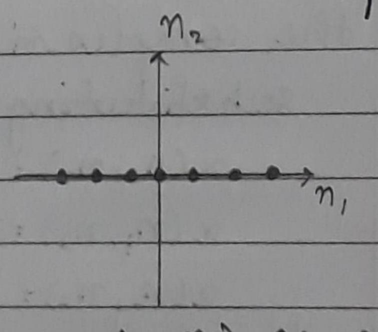
There are three types of line impulses:

i. Vertical line impulse



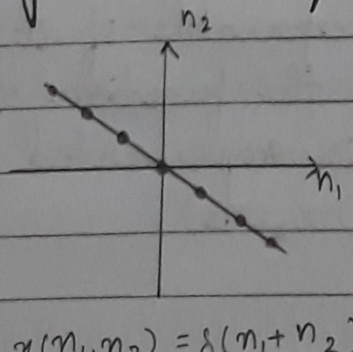
$x(n_1, n_2) = \delta(n_1)$

ii. Horizontal line impulse

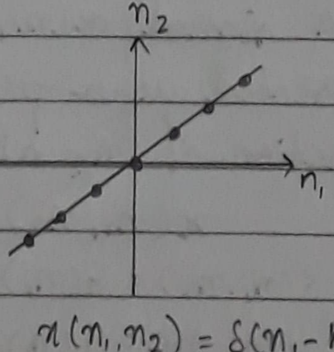


$x(n_1, n_2) = \delta(n_2)$

iii. Diagonal line impulse



$x(n_1, n_2) = \delta(n_1 + n_2)$



$x(n_1, n_2) = \delta(n_1 - n_2)$

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Q. sketch the sequence

(a) $x(n_1, n_2) = \delta(2n_1 - n_2)$

$2n_1 - n_2 = 0$

$\Rightarrow 2n_1 = n_2$.

(b) $x(n_1, n_2) = \delta(n_1 + n_2 - 1)$

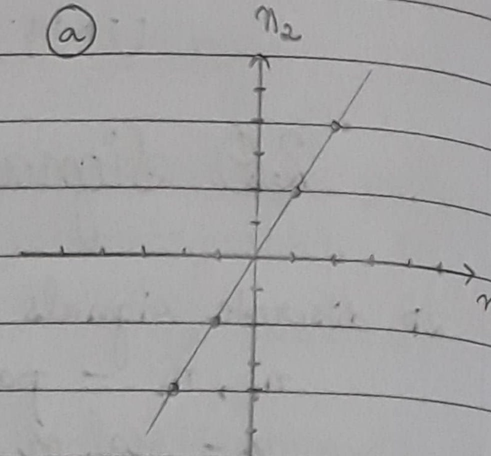
$n_1 + n_2 - 1 = 0$

$n_2 = 1 - n_1$

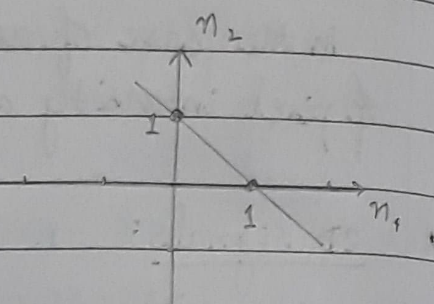
when $n_1 = 0$: $n_2 = 1$

when $n_2 = 0$: $n_1 = 1$

(a)



(b)



3. <u>Exponential Sequence</u>

The exponential sequences are defined by

$$x(n_1, n_2) = a^{n_1} b^{n_2}, \quad -\infty < n_1, n_2 < \infty \quad \text{--- (1)}$$

where a and b are complex numbers

When a and b have unit magnitude they can be written as:

$$a = e^{j\omega_1} \quad \text{and} \quad b = e^{j\omega_2} \quad \text{--- (2)}$$

in which case the exponential sequence becomes the complex sinusoidal sequence

substituting eq (2) in eq (1)

$$x(n_1, n_2) = (e^{j\omega_1})^{n_1} (e^{j\omega_2})^{n_2}$$

$$x(n_1, n_2) = e^{j(n_1\omega_1 + n_2\omega_2)}$$

$$x(n_1, n_2) = \cos(n_1\omega_1 + n_2\omega_2) + j\sin(n_1\omega_1 + n_2\omega_2)$$

* <u>separable Sequence</u>:

A signal $x(n_1, n_2)$ is separable if it can be represented as a product of the function of n_1 alone and a function of n_2 alone. It is represented as.

$$x(n_1, n_2) = x_1(n_1)\, x_2(n_2)$$

<u>EX</u>: $\delta(n_1, n_2) = \delta(n_1)\delta(n_2)$

$u(n_1, n_2) = u(n_1)u(n_2$

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* Periodic sequence:

A 2D sequence is periodic if it repeats itself in a regularly spaced interval. A 2D sequence $x(n_1, n_2)$ is periodic with a period $N_1 \times N_2$ if the following equalities hold for all integers n_1 and n_2:

$$x(n_1, n_2) = x(n_1 + N_1, n_2)$$
$$= x(n_1, n_2 + N_2)$$

where N_1 and N_2 are positive integers

Q: Determine whether the 2D signal $x(n_1, n_2) = \cos\left[\frac{\pi}{4} n_1 + \frac{\pi}{2} n_2\right]$ is periodic or not. If periodic, determine the fundamental period.

given: $x(n_1, n_2) = \cos\left(\frac{\pi}{4} n_1 + \frac{\pi}{2} n_2\right)$

which is of the form

$$x(n_1, n_2) = \cos(\omega_1 n_1 + \omega_2 n_2)$$

For the signal to be periodic, the following condition has to be satisfied.

$$\frac{\omega_1}{2\pi} = \frac{\omega_2}{2\pi} = \text{ratio of two integers.}$$

Here $\omega_1 = \frac{\pi}{4}$ and $\omega_2 = \frac{\pi}{2}$

$$\frac{\omega_1}{2\pi} = \frac{\pi}{4} \times \frac{1}{2\pi} = \frac{1}{8} \Rightarrow N_1 = 8$$

$$\frac{\omega_2}{2\pi} = \frac{\pi}{2} \times \frac{1}{2\pi} = \frac{1}{4} \Rightarrow N_2 = 4$$

Therefore $x(n_1, n_2)$ is a periodic signal.
The fundamental period is given by

$$N = N_1 \times N_2 = 8 \times 4 = \underline{\underline{32}}$$

Q: Determine whether the 2D signal $x(n_1, n_2) = \cos(n_1 + n_2)$ is periodic or not. If periodic find the fundamental period.

— given: $x(n_1, n_2) = \cos(n_1 + n_2)$

It is of the form

$$x(n_1, n_2) = \cos(\omega_1 n_1 + \omega_2 n_2)$$

$\therefore \omega_1 = 1$ and $\omega_2 = 1$

$\frac{\omega_1}{2\pi} = \frac{1}{2\pi}$ not integer

$\frac{\omega_2}{2\pi} = \frac{1}{2\pi}$ not integer.

also the given sequence cannot be expressed as.

$$\cos(n_1 + N_1, n_2) = \cos(n_1, n_2 + N_2)$$

Hence the given sequence is not periodic.

★ **2D systems:**

If $x(n_1, n_2)$ is the input signal to a system and $y(n_1, n_2)$ is the output of the system then the relationship between the input and output of the system is given by:

$$y(n_1, n_2) = T[x(n_1, n_2)]$$

where T denotes the transformation or processing performed by the system on the input $x(n_1, n_2)$ to produce the output $y(n_1, n_2)$.

= **Classification of 2D systems:**

1. **Linear versus Non-linear systems:**

A linear system is the one that satisfies the superposition principle. The linearity of the system T is defined as:

$$T[a x_1(n_1, n_2) + b x_2(n_1, n_2)]$$
$$= T[a x_1(n_1, n_2)] + T[b x_2(n_1, n_2)]$$
$$= a T[x_1(n_1, n_2)] + b T[x_2(n_1, n_2)]$$
$$= a y_1(n_1, n_2) + b y_2(n_1, n_2)$$

$$x_1(n_1,n_2) \longrightarrow \boxed{h(n_1,n_2)} \longrightarrow y_1(n_1,n_2) \qquad x_2(n_1,n_2) \longrightarrow \boxed{h(n_1,n_2)} \longrightarrow y_2(n_1,n_2)$$

$$a x_1(n_1,n_2) + b x_2(n_1,n_2) \longrightarrow \boxed{h(n_1,n_2)} \longrightarrow a y_1(n_1,n_2) + b y_2(n_1,n_2)$$

Q: Determine whether the system described by the following input-output relations is linear or not.

1. $y(n_1,n_2) = n\,x(n_1,n_2)$

For input $x_1(n_1,n_2)$, the output is given by:

$$y_1(n_1,n_2) = n\,x_1(n_1,n_2)$$

For input $x_2(n_1,n_2)$, the output is given by:

$$y_2(n_1,n_2) = n\,x_2(n_1,n_2)$$

Therefore for the input as the linear combination of the sequences $x_1(n_1,n_2)$ and $x_2(n_1,n_2)$, the output is:

$$\begin{aligned} y(n_1,n_2) &= T[a\,x_1(n_1,n_2) + b\,x_2(n_1,n_2)] \\ &= n[a\,x_1(n_1,n_2) + b\,x_2(n_1,n_2)] \\ &= a n\,x_1(n_1,n_2) + b n\,x_2(n_1,n_2) \\ &= a\,y_1(n_1,n_2) + b\,y_2(n_1,n_2) \end{aligned}$$

Hence the system is a linear system.

2. $y(n_1,n_2) = A x(n_1,n_2) + B$

For a linear system, when the input to the system is zero the output of the system is zero.

Here for $x(n_1,n_2) = 0$

$$y(n_1,n_2) = A(0) + B = B \neq 0$$

Hence the system is not a linear system.

3. $y(n_1,n_2) = e^{x(n_1,n_2)}$

For a linear system, when the input to the system is zero the output of the system is zero.

$\therefore$ Here for $x(n_1,n_2) = 0$

$$y(n_1,n_2) = e^0 = 1 \neq 0$$

Hence the system is not a linear system.

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2. Shift-variant versus Shift-Invariant systems:

A system is a shift invariant system if its input-output characteristics do not change with time. Shift invariance is given by:

$$T[x(n_1-m_1, n_2-m_2)] = y(n_1-m_1, n_2-m_2)$$

Q: Determine whethere the system described by the following input-output relation is shift invariant or not.

1. $y(n_1, n_2) = 5x(n_1, n_2)$

– The time shift in the input and output produces

$y_1(n_1, n_2) = 5x(n_1-k, n_2-k)$ (replacing n by n-k)

$y_2(n_1, n_2) = 5x(n_1-k, n_2-k)$ respectively.

This shows that time shift in the input is equal to the time shift in output. Hence, the system is a shift-invariant system.

2. $y(n_1, n_2) = n\,x(n_1, n_2)$

– The time shift in the input is given by

$y_1(n_1, n_2) = n\,x(n_1-k, n_2-k)$

The time shift in the output is given by

$y_2(n_1, n_2) = (n-k)\,x(n_1-k, n_2-k)$

Therefore $y_1(n_1, n_2) \neq y_2(n_1, n_2)$

Hence, the system is a shift-variant system.

3. Static versus Dynamic Systems:

A system is static or memoryless if its output at any instant depends at most on the input sample but not on the past or future samples of the input.

The system is dynamic or has memory if its output depends not only on the present input sample but also past or future samples of the input.

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Q: Determine whether the system given by the input-output relation $y(n_1, n_2) = n\, x(n_1, n_2)$ is static or not.

— The output of the system depends only on the present value of the input and not on the past or future value of the input. Hence the system is a static or a memoryless system.

4. Stable versus unstable system:

A 2D LSI system is Bounded Input Bounded Output (BIBO) stable if and only if its impulse response is absolutely summable which is given by:

$$\sum_{n_1=-\infty}^{\infty} \sum_{n_2=-\infty}^{\infty} |h(n_1, n_2)| < \infty.$$

★ 2D Convolution:

— Properties:

1. Commutative property.

$$x(n_1, n_2) * y(n_1, n_2) = y(n_1, n_2) * x(n_1, n_2)$$

2. Associative property

$$(x(n_1, n_2) * y(n_1, n_2)) * z(n_1, n_2) = x(n_1, n_2) * (y(n_1, n_2) * z(n_1, n_2))$$

3. Distributive property

$$x(n_1, n_2) * (y(n_1, n_2) + z(n_1, n_2))$$
$$= x(n_1, n_2) * y(n_1, n_2) + x(n_1, n_2) * z(n_1, n_2)$$

4. Convolution with shifted impulses.

$$x(n_1, n_2) * \delta(n_1 - m_1, n_2 - m_2) = x(n_1 - m_1, n_2 - m_2)$$

★ 2D Z Transform:

The z-Transform of a 2D sequence $x(n_1, n_2)$ is given by:

$$X(z_1, z_2) = \sum_{n_1=-\infty}^{\infty} \sum_{n_2=-\infty}^{\infty} x(n_1, n_2)\, z_1^{-n_1} z_2^{-n_2} \quad \text{—— (1)}$$

where z_1 and z_2 are complex variables.

This equation can be reduced to the Fourier Transform when:

$$z_1 = e^{j\omega_1} \quad \text{and} \quad z_2 = e^{j\omega_2}$$

$$\therefore X(\omega_1, \omega_2) = \sum_{n_1=-\infty}^{\infty} \sum_{n_2=-\infty}^{\infty} x(n_1, n_2) e^{-j\omega_1 n_1} e^{-j\omega_2 n_2} \quad \text{--- (2)}$$

Region of convergence (ROC)

The sum given in eq (1) may or may not converge for all values of z_1 and z_2. The values for which the sum converges is known as the region of convergence (ROC) of the transform.

Q: compute the 2D z-transform of the 2D impulse sequence:

1. $\delta(n_1, n_2)$

- The 2D z-transform of impulse sequence is given by:

$$X(z_1, z_2) = \sum_{n_1=-\infty}^{\infty} \sum_{n_2=-\infty}^{\infty} \delta(n_1, n_2) z_1^{-n_1} z_2^{-n_2}$$

wkt $\delta(n_1, n_2) = 1$ for $n_1 = 0, n_2 = 0$
$\qquad\qquad\quad 0$ else

Therefore

$$X(z_1, z_2) = 1\, z_1^0\, z_2^0 = 1$$

2. $\delta(n_1 - 1, n_2)$

- The 2D z-transform of impulse sequence is given by:

$$X(z_1, z_2) = \sum_{n_1=-\infty}^{\infty} \sum_{n_2=-\infty}^{\infty} \delta(n_1 - 1, n_2) z_1^{-n_1} z_2^{-n_2}$$

here $\delta(n_1 - 1, n_2) = 1$ for $n_1 = 1$ and $n_2 = 0$
$\qquad\qquad\qquad\quad 0$ else

Therefore

$$X(z_1, z_2) = 1\, z_1^{-1}\, z_2^0 = z_1^{-1}$$

3. $\delta(n_1-1, n_2-1)$

The 2D Z-transform of the impulse sequence is given by:

$$X(z_1,z_2) = \sum_{n_1=-\infty}^{\infty} \sum_{n_2=-\infty}^{\infty} \delta(n_1-1, n_2-1)\, z_1^{-n_1} z_2^{-n_2}$$

here $\delta(n_1-1, n_2-1) = 1$ for $n_1 = 1$ and $n_2 = 1$
0 otherwise

Therefore

$$X(z_1,z_2) = 1\, z_1^{-1} z_2^{-1} = (z_1 z_2)^{-1}$$

Q: Compute the 2D Z transform of:

1. $x(n_1, n_2) = u(n_1, n_2)$

The 2D Z-transform is given by

$$X(z_1,z_2) = \sum_{n_1=-\infty}^{\infty} \sum_{n_2=-\infty}^{\infty} u(n_1, n_2)\, z_1^{-n_1} z_2^{-n_2}$$

here $u(n_1, n_2) = 1$ for $n_1 \geq 0$ and $n_2 \geq 0$
0 otherwise

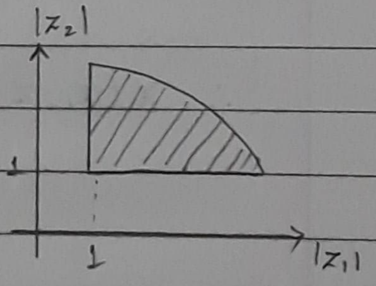
Therefore

$$X(z_1,z_2) = \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} 1\, z_1^{-n_1} z_2^{-n_2}$$

$$= \sum_{n_1=0}^{\infty} z_1^{-n_1} \sum_{n_2=0}^{\infty} z_2^{-n_2}$$

$$= \left[\frac{1}{1-z_1^{-1}}\right]\left[\frac{1}{1-z_2^{-1}}\right]$$

$$= \frac{z_1 z_2}{(z_1-1)(z_2-1)}$$



ROC is $|z_1| > 1$ and $|z_2| > 1$.

2. $x(n_1, n_2) = a^{n_1} b^{n_2} u(n_1, n_2)$

The 2D Z-transform is given by

$$X(z_1,z_2) = \sum_{n_1=-\infty}^{\infty} \sum_{n_2=-\infty}^{\infty} a^{n_1} b^{n_2} u(n_1, n_2)\, z_1^{-n_1} z_2^{-n_2}$$

here $u(n_1, n_2) = 1$ for $n_1 \geq 0$ and $n_2 \geq 0$
0 otherwise

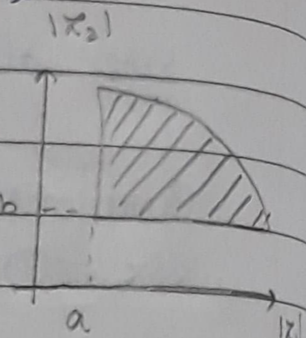
Therefore

$$X(z_1,z_2) = \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty} a^{n_1} b^{n_2} z_1^{-n_1} z_2^{-n_2}$$

$$= \sum_{n_1=0}^{\infty}(az_1^{-1})^{n_1} \sum_{n_2=0}^{\infty}(bz_2^{-1})^{n_2}$$

$$= \left[\frac{1}{1-az_1^{-1}}\right]\left[\frac{1}{1-bz_2^{-1}}\right]$$

$$= \frac{z_1 z_2}{(z_1-a)(z_2-b)}$$



ROC is $|z_1| > a$ and $|z_2| > b$

3. $x(n_1,n_2) = a^{n_1}\delta(n_1-n_2)u(n_1,n_2)$

The 2D Z transform is given by:

$$X(z_1,z_2) = \sum_{n_1=-\infty}^{\infty}\sum_{n_2=-\infty}^{\infty} a^{n_1}\delta(n_1-n_2)u(n_1,n_2)z_1^{-n_1}z_2^{-n_2}$$

here $u(n_1,n_2) = 1$ for $n_1 \geq 0$ and $n_2 \geq 0$
0 else

$$\therefore X(z_1,z_2) = \sum_{n_1=0}^{\infty}\sum_{n_2=0}^{\infty} a^{n_1}\delta(n_1-n_2) z_1^{-n_1} z_2^{-n_2}$$

here $\delta(n_1-n_2) = 1$ for $n_1 = n_2$
0 otherwise

$$\therefore X(z_1,z_2) = \sum_{n_1=0}^{\infty}\sum_{n_1=0}^{\infty} a^{n_1}\, 1\, z_1^{-n_1} z_2^{-n_1}$$

$$= \sum_{n_1=0}^{\infty}(az_1^{-1}z_2^{-1})^{n_1}$$

$$= \frac{1}{1-az_1^{-1}z_2^{-1}}$$

$$= \frac{z_1 z_2}{z_1 z_2 - a}$$

ROC is $|z_1 z_2| > a$

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4. $x(n_1, n_2) = (-a^{n_1})(-b^{n_2})\,u(-n_1-1, -n_2-1)$

The 2D Z transform is given by:

$$X(z_1, z_2) = \sum_{n_1=-\infty}^{\infty} \sum_{n_2=-\infty}^{\infty} (-a^{n_1})(-b^{n_2})\,u(-n_1-1, -n_2-1)\, z_1^{-n_1} z_2^{-n_2}$$

here $u(-n_1-1, -n_2-1) = 1$ for $n_1 \le -1$ and $n_2 \le -1$
0 otherwise

$$\therefore X(z_1, z_2) = \sum_{n_1=-\infty}^{-1} \sum_{n_2=-\infty}^{-1} a^{n_1} b^{n_2}\, 1\, z_1^{-n_1} z_2^{-n_2}$$

$$= \sum_{n_1=-\infty}^{-1} (a z_1^{-1})^{n_1} \sum_{n_2=-\infty}^{-1} (b z_2^{-1})^{n_2}$$

Substitute $n_1 = -m_1$ and $n_2 = -m_2$

$$\therefore X(z_1, z_2) = \sum_{m_1=1}^{\infty} (a z_1^{-1})^{-m_1} \sum_{m_2=1}^{\infty} (b z_2^{-1})^{-m_2}$$

$$= \sum_{m_1=1}^{\infty} (a^{-1} z_1)^{m_1} \sum_{m_2=1}^{\infty} (b^{-1} z_2)^{m_2}$$

$$= \left[a^{-1} z_1 + (a^{-1} z_1)^2 + (a^{-1} z_2)^3 + \cdots\right] \left[b^{-1} z_2 + (b^{-1} z_2)^2 + (b^{-1} z_2)^3 + \cdots\right]$$

$$= a^{-1} z_1 \left[1 + a^{-1} z_1 + (a^{-1} z_2)^2 + \cdots\right] b^{-1} z_2 \left[1 + b^{-1} z_2 + (b^{-1} z_2)^2 + \cdots\right]$$

$$= \left[a^{-1} z_1 \sum_{m_1=0}^{\infty} (a^{-1} z_1)^{m_1}\right] \left[b^{-1} z_2 \sum_{m_2=0}^{\infty} (b^{-1} z_2)^{m_2}\right]$$

$$= \left[\frac{a^{-1} z_1}{1 - a^{-1} z_1}\right] \left[\frac{b^{-1} z_2}{1 - b^{-1} z_2}\right]$$

$$= \frac{z_1}{a - z_1} \frac{z_2}{b - z_2}$$

$$= \frac{z_1 z_2}{(a - z_1)(b - z_2)}$$

ROC is $|z_1| < a$ and $|z_2| < b$

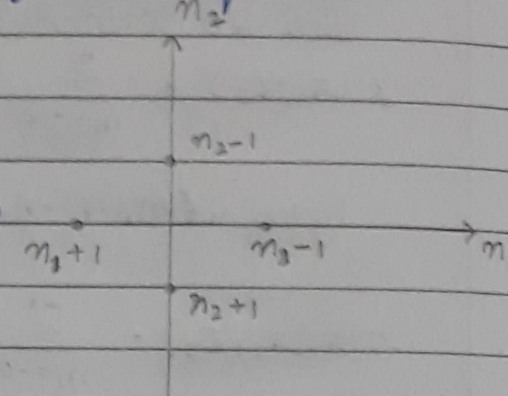
Date:

Keerthana Ashok

Q: Determine the 2D z-transform of the sequence and the frequency response:

1. The 2D sequence is given by

$x(n_1, n_2) = \delta(n_1 - 1) + \delta(n_1 + 1) + \delta(n_2 - 1) + \delta(n_2 + 1)$



The 2D z transform is

$$X(z_1, z_2) = \sum_{n_1=-\infty}^{\infty} \sum_{n_2=-\infty}^{\infty} [\delta(n_1 - 1) + \delta(n_1 + 1) + \delta(n_2 - 1) + \delta(n_2 + 1)] z_1^{-n_1} z_2^{-n_2}$$

$$X(z_1, z_2) = \sum_{n_1=-\infty}^{\infty} [\delta(n_1 - 1) + \delta(n_1 + 1)] z_1^{-n_1} z_2^{-n_2} + \sum_{n_2=-\infty}^{\infty} [\delta(n_2 - 1) + \delta(n_2 + 1)] z_1^{-n_1} z_2^{-n_2}$$

$$= z_1^{-1} + z_1^{1} + z_2^{-1} + z_2^{1} \quad //$$

Frequency Response

substituting $z_1 = e^{j\omega_1}$ and $z_2 = e^{j\omega_2}$

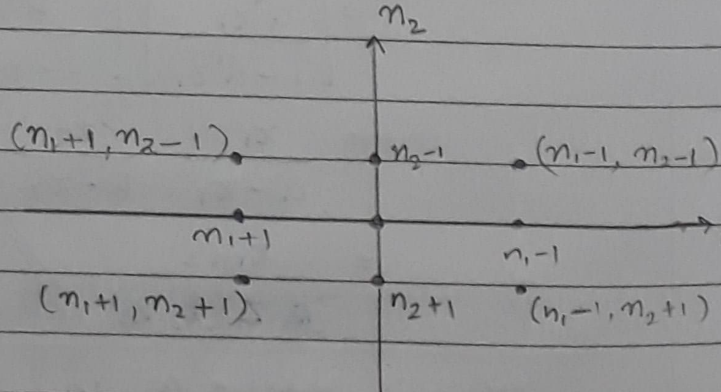
$$X(\omega_1, \omega_2) = e^{-j\omega_1} + e^{j\omega_1} + e^{-j\omega_2} + e^{j\omega_2}$$

$$= 2\left[\frac{e^{j\omega_1} + e^{-j\omega_1}}{2}\right] + 2\left[\frac{e^{j\omega_2} + e^{-j\omega_2}}{2}\right]$$

$$= 2\cos\omega_1 + 2\cos\omega_2 \quad //$$

2. The 2D sequence is given by:

$x(n_1, n_2) = \delta(n_1, n_2) + \delta(n_2 - 1) + \delta(n_2 + 1) + \delta(n_1 - 1) + \delta(n_1 + 1) + \delta(n_1 + 1, n_2 - 1) + \delta(n_1 - 1, n_2 - 1) + \delta(n_1 + 1, n_2 + 1) + \delta(n_1 - 1, n_2 + 1)$



The z transform is given by:

$$X(z_1, z_2) = 1 + z_2^{-1} + z_2^{1} + z_1^{-1} + z_1^{1} + z_1^{1} z_2^{-1} + z_1^{-1} z_2^{-1} + z_1^{1} z_2^{1} + z_1^{-1} z_2^{1}$$

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Frequency response

substituting $z_1 = e^{j\omega_1}$ and $z_2 = e^{j\omega_2}$

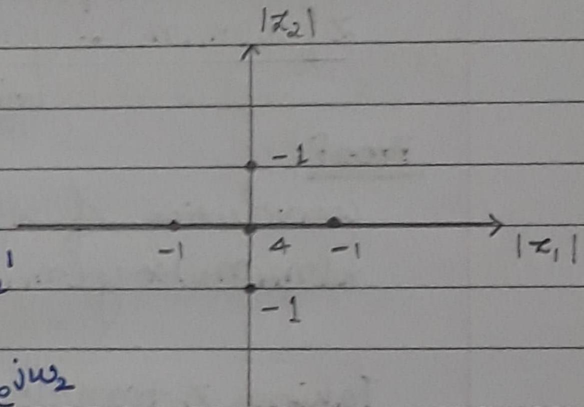
$$X(\omega_1, \omega_2) = 1 + e^{-j\omega_1} + e^{j\omega_1} + e^{-j\omega_2} + e^{j\omega_2} + e^{-j\omega_1} e^{-j\omega_2} + e^{-j\omega_1} e^{j\omega_2} + e^{j\omega_1} e^{-j\omega_2} + e^{j\omega_1} e^{j\omega_2}$$

$$= 1 + 2\left[\frac{e^{j\omega_1} + e^{-j\omega_1}}{2}\right] + 2\left[\frac{e^{j\omega_2} + e^{-j\omega_2}}{2}\right] + e^{-j\omega_1} 2\left[\frac{e^{j\omega_2} + e^{-j\omega_2}}{2}\right] + 2e^{j\omega_1}\left[\frac{e^{j\omega_2} + e^{-j\omega_2}}{2}\right]$$

$$= 1 + 2\cos\omega_1 + 2\cos\omega_2 + 2e^{-j\omega_1}\cos\omega_2 + 2e^{j\omega_1}\cos\omega_2$$

3. The 2D sequence is given by:

$$x(n_1, n_2) = 4\delta(n_1, n_2) - \delta(n_1 - 1) - \delta(n_1 + 1) - \delta(n_2 - 1) - \delta(n_2 + 1)$$



Taking z-transform

$$X(z_1, z_2) = 4 - z_1^{-1} - z_1^{1} - z_2^{-1} - z_2^{1}$$

Frequency response

substituting $z_1 = e^{j\omega_1}$ and $z_2 = e^{j\omega_2}$

$$X(\omega_1, \omega_2) = 4 - e^{-j\omega_1} - e^{j\omega_1} - e^{-j\omega_2} - e^{j\omega_2}$$

$$= 4 - 2\left[\frac{e^{j\omega_1} + e^{-j\omega_1}}{2}\right] - 2\left[\frac{e^{j\omega_2} + e^{-j\omega_2}}{2}\right]$$

$$= 4 - 2\cos\omega_1 - 2\cos\omega_2 \quad //$$

* Properties of z-transform:

1. Linearity Property:

If $x(n_1, n_2) \xrightarrow{ZT} X(z_1, z_2)$ ROC : R_x

and $y(n_1, n_2) \xrightarrow{ZT} Y(z_1, z_2)$ ROC : R_y

then

$$Z[a x(n_1, n_2) + b y(n_1, n_2)] = a X(z_1, z_2) + b Y(z_1, z_2)$$

ROC = $R_x \cap R_y$

Proof:

considering LHS

$$Z[a x(n_1, n_2) + b y(n_1, n_2)]$$

$$= \sum_{n_1=-\infty}^{\infty} \sum_{n_2=-\infty}^{\infty} [a x(n_1, n_2) + b y(n_1, n_2)] z_1^{-n_1} z_2^{-n_2}$$

For a linear system

$$= \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} [a x(n_1, n_2) + b y(n_1, n_2)] z_1^{-n_1} z_2^{-n_2}$$

$$= \sum_{n_1=0}^{\infty} a x(n_1, n_2) z_1^{-n_1} + \sum_{n_2=0}^{\infty} b y(n_1, n_2) z_2^{-n_2}$$

$$= a \sum_{n_1=0}^{\infty} x(n_1, n_2) z_1^{-n_1} + b \sum_{n_2=0}^{\infty} y(n_1, n_2) z_2^{-n_2}$$

$$= a X(z_1, z_2) + b Y(z_1, z_2)$$

with ROC : $R_x \cap R_y$

2. Convolution Property:

$$Z[x(n_1, n_2) * y(n_1, n_2)] = X(z_1, z_2) \cdot Y(z_1, z_2)$$

with ROC : $R_x \cap R_y$

Proof:

Convolution of $x(n_1, n_2)$ and $y(n_1, n_2)$

$$x(n_1, n_2) * y(n_1, n_2) = \sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty} [x(n_1, n_2) * y(n_1, n_2)] z_1^{-n_1} z_2^{-n_2}$$

Taking z transform on both sides

$$Z[x(n_1, n_2) * y(n_1, n_2)] = Z\left[\sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty} (x(k_1, k_2)\, y(n_1-k_1, n_2-k_2))\right]$$

$$= \sum_{n_1=-\infty}^{\infty} \sum_{n_2=-\infty}^{\infty} \sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty} [x(k_1, k_2) y(n_1-k_1, n_2-k_2)] z_1^{-n_1} z_2^{-n_2}$$

Let $n_1 - k_1 = l_1$ and $n_2 - k_2 = l_2$

$$= \sum_{l_1=-\infty}^{\infty} \sum_{l_2=-\infty}^{\infty} \sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty} [x(k_1, k_2) y(l_1, l_2)] z_1^{-(l_1+k_1)} z_2^{-(l_2+k_2)}$$

$$= \sum_{l_1=-\infty}^{\infty} \sum_{l_2=-\infty}^{\infty} \sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty} [x(k_1, k_2) y(l_1, l_2)] z_1^{-l_1} z_1^{-k_1} z_2^{-l_2} z_2^{-k_2}$$

$$= \sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty} x(k_1, k_2) z_1^{-k_1} z_2^{-k_2} \sum_{l_1=-\infty}^{\infty} \sum_{l_2=-\infty}^{\infty} y(l_1, l_2) z_1^{-l_1} z_2^{-l_2}$$

$$= X(z_1, z_2)\, Y(z_1, z_2)$$

$\therefore$ Convolution in spatial domain is equal to multiplication in the frequency domain.

3. **Delay Property or Shift Property:**

$$Z[x(n_1-m_1, n_2-m_2)] = z_1^{-m_1} z_2^{-m_2} X(z_1, z_2) \qquad \text{ROC}: R_x$$

Proof:

considering LHS

$$Z[x(n_1-m_1, n_2-m_2)] = \sum_{n_1=-\infty}^{\infty} \sum_{n_2=-\infty}^{\infty} x(n_1-m_1, n_2-m_2)\, z_1^{-n_1} z_2^{-n_2}$$

substituting $n_1 - m_1 = k_1$ and $n_2 - m_2 = k_2$

$$= \sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty} x(k_1, k_2)\, z_1^{-(k_1+m_1)} z_2^{-(k_2+m_2)}$$

$$= \sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty} x(k_1, k_2)\, z_1^{-k_1} z_1^{-m_1} z_2^{-k_2} z_2^{-m_2}$$

$$= z_1^{-m_1} z_2^{-m_2} \sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty} x(k_1, k_2)\, z_1^{-k_1} z_2^{-k_2}$$

$$= z_1^{-m_1} z_2^{-m_2} X(z_1, z_2)$$

4. **Conjugation Property:**

$$Z[x^*(n_1, n_2)] = X^*(z_1^*, z_2^*)$$

Proof:

The z-transform of sequence $x(n_1, n_2)$ is given by

$$X(z_1, z_2) = \sum_{n_1=-\infty}^{\infty} \sum_{n_2=-\infty}^{\infty} x(n_1, n_2)\, z_1^{-n_1} z_2^{-n_2}$$

Taking conjugate on both sides.

$$X^*(z_1, z_2) = \sum_{n_1=-\infty}^{\infty} \sum_{n_2=-\infty}^{\infty} x^*(n_1, n_2)\, (z_1^*)^{-n_1} (z_2^*)^{-n_2}$$

substituting $z_1 = z_1^*$ and $z_2 = z_2^*$

$$X^*(z_1, z_2) = \sum_{n_1=-\infty}^{\infty} \sum_{n_2=-\infty}^{\infty} x^*(n_1, n_2)\, z_1^{-n_1} z_2^{-n_2}$$

$$= Z[x^*(n_1, n_2)]$$

5. **Reflection Property:**

$$Z[x(-n_1, -n_2)] = X(z_1^{-1}, z_2^{-1})$$

Proof:

The z-transform of sequence $x(-n_1, -n_2)$ is given by:

$$\sum_{n_1=-\infty}^{\infty} \sum_{n_2=-\infty}^{\infty} x(-n_1, -n_2)\, z_1^{-n_1} z_2^{-n_2}$$

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Replacing $-n_1$ by n_1 and $-n_2$ by n_2

$$= \sum_{n_1=-\infty}^{\infty} \sum_{n_2=-\infty}^{\infty} x(n_1,n_2) z_1^{n_1} z_2^{n_2}$$

$$= \sum_{n_1=-\infty}^{\infty} \sum_{n_2=-\infty}^{\infty} x(n_1,n_2) (z_1^{-1})^{-n_1} (z_2^{-1})^{-n_2}$$

$$= X(z_1^{-1}, z_2^{-1})$$

6. Differential Property:

$$-n_1 x(n_1,n_2) \longleftrightarrow z_1 \frac{\partial X(z_1,z_2)}{\partial z_1} \qquad \text{ROC: } R_x$$

$$-n_2 x(n_1,n_2) \longleftrightarrow z_2 \frac{\partial X(z_1,z_2)}{\partial z_2} \qquad \text{ROC: } R_y$$

$$-n_1 n_2 x(n_1,n_2) \longleftrightarrow z_1 z_2 \frac{\partial^2 X(z_1,z_2)}{\partial z_1 z_2}$$

Proof:

The z transform of $x(n_1,n_2)$ is given by:

$$X(z_1,z_2) = \sum_{n_1=-\infty}^{\infty} \sum_{n_2=-\infty}^{\infty} x(n_1,n_2) z_1^{-n_1} z_2^{-n_2}$$

Partially differentiating the above equation wrt z_1

$$\frac{\partial X(z_1,z_2)}{\partial z_1} = \sum_{n_1=-\infty}^{\infty} \sum_{n_2=-\infty}^{\infty} x(n_1,n_2)(-n_1) z_1^{-n_1} z_1^{-1} z_2^{-n_2}$$

$$\frac{\partial X(z_1,z_2)}{\partial z_1} = \frac{-n_1}{z_1} \sum_{n_1=-\infty}^{\infty} \sum_{n_2=-\infty}^{\infty} x(n_1,n_2) z_1^{-n_1} z_2^{-n_2}$$

$$z_1 \frac{\partial X(z_1,z_2)}{\partial z_1} = -n_1 \sum_{n_1=-\infty}^{\infty} \sum_{n_2=-\infty}^{\infty} x(n_1,n_2) z_1^{-n_1} z_2^{-n_2}$$

This implies that z-transform of $n_1 x(n_1,n_2)$ is $-z_1 \frac{\partial X(z_1,z_2)}{\partial z_1}$

similarly.

for $n_2 x(n_1,n_2)$ is $-z_2 \frac{\partial X(z_1,z_2)}{\partial z_2}$

and $n_1 n_2 x(n_1,n_2)$ is $-z_1 z_2 \frac{\partial^2 X(z_1,z_2)}{\partial z_1 z_2}$

7. Modulation Property:

$$Z[\alpha^{-n_1}\beta^{-n_2}x(n_1,n_2)] = X(\alpha z_1, \beta z_2)$$

Proof:

considering LHS

$$Z[\alpha^{-n_1}\beta^{-n_2}x(n_1,n_2)] = \sum_{n_1=-\infty}^{\infty}\sum_{n_2=-\infty}^{\infty} \alpha^{-n_1}\beta^{-n_2}x(n_1,n_2)z_1^{-n_1}z_2^{-n_2}$$

$$= \sum_{n_1=-\infty}^{\infty}\sum_{n_2=-\infty}^{\infty} x(n_1,n_2)(\alpha z_1)^{-n_1}(\beta z_2)^{-n_2}$$

$$= X(\alpha z_1, \beta z_2)$$

★ 2D Inverse Z-transform:

The Z transform of $x(k_1,k_2)$ is

$$X(z_1,z_2) = \sum_{k_1=-\infty}^{\infty}\sum_{k_2=-\infty}^{\infty} x(k_1,k_2)z_1^{-k_1}z_2^{-k_2} \quad \text{—} (1)$$

Multiplying $z_1^{n_1-1}z_2^{n_2-1}$ on both sides and integrate over a closed contour with in the ROC of $X(z_1,z_2)$.

$$\oint_{C_1}\oint_{C_2} X(z_1,z_2)z_1^{n_1-1}z_2^{n_2-1}dz_1dz_2$$

$$= \oint_{C_1}\oint_{C_2}\sum_{k_1=-\infty}^{\infty}\sum_{k_2=-\infty}^{\infty} x(k_1,k_2)z_1^{-k_1}z_2^{-k_2}z_1^{n_1-1}z_2^{n_2-1}dz_1dz_2 \quad \text{—} (2)$$

$$\oint_{C_1}\oint_{C_2} X(z_1,z_2)z_1^{n_1-1}z_2^{n_2-1}dz_1dz_2$$

$$= \sum_{k_1=-\infty}^{\infty}\sum_{k_2=-\infty}^{\infty} x(k_1,k_2)\oint_{C_1}\oint_{C_2} z_1^{n_1-1-k_1}z_2^{n_2-1-k_2}dz_1dz_2 \quad \text{—} (3)$$

Cauchy integral theorem, states that

$$\left(\frac{1}{2\pi j}\right)^2\oint_{C_1}\oint_{C_2} z_1^{n_1-1-k_1}z_2^{n_2-1-k_2}dz_1dz_2$$

$$= \begin{cases} 1 & k_1 = n_1 \text{ and } k_2 = n_2 \\ 0 & k_1 \neq n_1 \text{ and } k_2 \neq n_2 \end{cases} \quad \text{—} (4)$$

Multiplying the term $\left(\frac{1}{2\pi j}\right)^2$ on both sides in eq (3)

$$\left(\frac{1}{2\pi j}\right)^2\oint_{C_1}\oint_{C_2} X(z_1,z_2)z_1^{n_1-1}z_2^{n_2-1}dz_1dz_2$$

$$= \left(\frac{1}{2\pi j}\right)^2\sum_{k_1=-\infty}^{\infty}\sum_{k_2=-\infty}^{\infty} x(k_1,k_2)\oint_{C_1}\oint_{C_2} z_1^{n_1-1-k_1}z_2^{n_2-1-k_2}dz_1dz_2 \quad \text{—} (5)$$

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Using eq ④.

$$\left(\frac{1}{2\pi j}\right)^2 \oint_{c_1}\oint_{c_2} X(z_1, z_2)\, z_1^{n_1-1} z_2^{n_2-1}\, dz_1\, dz_2 = x(n_1, n_2) \quad — ⑥$$

1. Product of two sequences

If $x_1(n_1, n_2) \xleftrightarrow{z} X_1(z_1, z_2)$

$x_2(n_1, n_2) \xleftrightarrow{z} X_2(z_1, z_2)$

then

$$x(n_1, n_2) = x_1(n_1, n_2)\, x_2(n_1, n_2) \xleftrightarrow{z} X(z_1, z_2)$$

$$= \left(\frac{1}{2\pi j}\right)^2 \oint_{c_1}\oint_{c_2} X_1(u, v)\, X_2\left(\frac{z_1}{u}, \frac{z_2}{v}\right) u^{-1} v^{-1}\, du\, dv$$

Proof:

The z transform of $x(n_1, n_2)$ is

$$X(z_1, z_2) = \sum_{n_1=-\infty}^{\infty} \sum_{n_2=-\infty}^{\infty} x(n_1, n_2)\, z_1^{-n_1} z_2^{-n_2}$$

$$= \sum_{n_1=-\infty}^{\infty} \sum_{n_2=-\infty}^{\infty} x_1(n_1, n_2)\, x_2(n_1, n_2)\, z_1^{-n_1} z_2^{-n_2} \quad — ①$$

The inverse z transform of $x_1(n_1, n_2)$ is

$$x_1(n_1, n_2) = \left(\frac{1}{2\pi j}\right)^2 \oint_{c_1}\oint_{c_2} X_1(u, v)\, u^{n_1-1} v^{n_2-1}\, du\, dv \quad — ②$$

substituting eq ② in eq ①

$$X(z_1, z_2) = \sum_{n_1=-\infty}^{\infty} \sum_{n_2=-\infty}^{\infty} \left(\frac{1}{2\pi j}\right)^2 \oint_{c_1}\oint_{c_2} X_1(u, v)\, u^{n_1-1} v^{n_2-1}\, du\, dv\; x_2(n_1, n_2)\, z_1^{-n_1} z_2^{-n_2}$$

$$= \left(\frac{1}{2\pi j}\right)^2 \oint_{c_1}\oint_{c_2} X_1(u, v) \sum_{n_1=-\infty}^{\infty} \sum_{n_2=-\infty}^{\infty} x_2(n_1, n_2)\, u^{n_1-1} v^{n_2-1} z_1^{-n_1} z_2^{-n_2}\, du\, dv$$

$$= \left(\frac{1}{2\pi j}\right)^2 \oint_{c_1}\oint_{c_2} X_1(u, v) \sum_{n_1=-\infty}^{\infty} \sum_{n_2=-\infty}^{\infty} x_2(n_1, n_2) \left(\frac{z_1}{u}\right)^{-n_1} u^{-1} \left(\frac{z_2}{v}\right)^{-n_2} v^{-1}\, du\, dv$$

$$\therefore X(z_1, z_2) = \left(\frac{1}{2\pi j}\right)^2 \oint_{c_1}\oint_{c_2} X_1(u, v) \sum_{n_1=-\infty}^{\infty} \sum_{n_2=-\infty}^{\infty} x_2(n_1, n_2) \left(\frac{z_1}{u}\right)^{-n_1} \left(\frac{z_2}{v}\right)^{-n_2} u^{-1} v^{-1}\, du\, dv$$

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2. <u>Parseval's Theorem</u>

If $x_1(n_1,n_2)$ and $x_2(n_1,n_2)$ are complex-valued sequence then the Parseval's relation is given by:

$$\sum_{n_1=-\infty}^{\infty}\sum_{n_2=-\infty}^{\infty} x_1(n_1,n_2)\, x_2^*(n_1,n_2) = \left(\frac{1}{2\pi j}\right)^2 \oint_{C_1}\oint_{C_2} X_1(u,v)\, X_2^*\left(\frac{1}{u^*},\frac{1}{v^*}\right) u^{-1}v^{-1}\,du\,dv \quad \text{---} (1)$$

<u>Proof:</u>

The inverse ~~z~~ transform

$$x_1(n_1,n_2) = \left(\frac{1}{2\pi j}\right)^2 \oint_{C_1}\oint_{C_2} X_1(u,v)\, u^{n_1-1} v^{n_2-1}\,du\,dv \quad \text{---} (2)$$

Substituting eq (2) in eq (1) by considering LHS

$$\sum_{n_1=-\infty}^{\infty}\sum_{n_2=-\infty}^{\infty} x_1(n_1,n_2)\, x_2^*(n_1,n_2)$$

$$= \sum_{n_1=\infty}^{\infty}\sum_{n_2=-\infty}^{\infty} \left(\frac{1}{2\pi j}\right)^2 \oint_{C_1}\oint_{C_2} X_1(u,v)\, u^{n_1-1} v^{n_2-1}\,du\,dv\; x_2^*(n_1,n_2)$$

$$= \left(\frac{1}{2\pi j}\right)^2 \oint_{C_1}\oint_{C_2} X_1(u,v) \sum_{n_1=-\infty}^{\infty}\sum_{n_2=-\infty}^{\infty} x_2^*(n_1,n_2)\, u^{n_1-1} v^{n_2-1}\,du\,dv$$

$$= \left(\frac{1}{2\pi j}\right)^2 \oint_{C_1}\oint_{C_2} X_1(u,v) \sum_{n_1=-\infty}^{\infty}\sum_{n_2=-\infty}^{\infty} x_2^*(n_1,n_2) \left(\frac{1}{u}\right)^{-n_1}\left(\frac{1}{v}\right)^{-n_2} u^{-1}v^{-1}\,du\,dv$$

Applying conjugation property.

$$= \left(\frac{1}{2\pi j}\right)^2 \oint_{C_1}\oint_{C_2} X_1(u,v)\, X_2^*\left(\frac{1}{u^*},\frac{1}{v^*}\right) u^{-1}v^{-1}\,du\,dv$$

This shows that energy in spatial domain = energy in the frequency domain.

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UNIT - 03

Image Transforms

Image transforms are extensively used in image processing and image analysis. Transform is basically a mathematical tool which allows us to move from one domain to another domain (time domain to frequency domain).

Image transforms are useful for fast computation of convolution and correlation. The transform changes the representation of a signal by projecting it onto a set of basic functions but the information content present in the signal doesnot change.

- Need for transform:
  - Mathematical convenience

    The complex convolution operation in time domain is equal to simple multiplication operation in the frequency domain.
  - To extract more information

    Transforms allow us to extract more relevant information. It doesnot change the information content present in a signal.

★ Image Transforms:

The two reasons for transforming an image from one representation to another are:

- It isolates critical components of the image pattern so that they are directly accessible for analysis.
- The transformation may place the image data in a more compact form so that they can be stored and transmitted efficiently.

- Classification of Image Transforms:

Image transforms can be classified based on the nature of the basis functions.

IMAGE TRANSFORMS

- Orthogonal sinusoidal basis function
  - Fourier Transform
  - Discrete Cosine Transform
  - Discrete sine Transform
- Non-sinusoidal orthogonal Basis Function
  - Haar Transform
  - Walsh Transform
  - Hadamard Transform
  - Slant Transform
- Basis function depending on statistics of input signal
  - KL Transform
  - Singular value decomposition
- Directional Transformation
  - Hough Transform
  - Radon transform
  - Ridgelet Transform
  - Contourlet Transform

★ FOURIER TRANSFORM:

A Fourier Transform is used to transform an intensity image into the domain of spatial frequency.

For a continuous time signal $x(t)$, the Fourier transform is defined as $X(\Omega)$

$$x(t) \xrightarrow{CTFT} X(\Omega)$$

Continuous Time Fourier Transform $X(\Omega)$ is defined as:

$$X(\Omega) = \int_{-\infty}^{\infty} x(t)\, e^{-j\Omega t}\, dt$$

A continuous time signal $x(t)$ is converted into discrete time signal $x(nT)$ by sampling process, where T is the sampling interval.

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$$x(t) \xrightarrow{\text{sampling}} x(nT)$$

The Fourier transform of a finite energy discrete time signal $x(nT)$ is given by:

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x(nT)\, e^{-j\Omega nt}$$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j(\Omega t)n}$$

where $X(e^{j\omega})$ is known as Discrete Time Fourier Transform (DTFT) and is a continuous function of ω.

The relation between ω and Ω is given by:

$$\omega = \Omega T$$

Replacing Ω by $2\pi f$

$$\omega = 2\pi f T$$

where T is sampling interval and $T = 1/f_s$

$$\therefore \omega = \frac{2\pi f}{f_s}$$

Here $f/f_s = k$

$$\therefore \omega = 2\pi k$$

To limit the infinite number of values to a finite number

$$\frac{\omega}{2\pi} = \frac{k}{N}$$

Therefore the Discrete Fourier Transform (DFT) of a finite duration sequence $x(n)$ is defined as:

$$X(k) = \sum_{n=0}^{N-1} x(n)\, e^{-j\frac{2\pi}{N}kn} \quad \text{where } k = 0, 1, 2 \ldots N-1$$

The DFT is a discrete frequency representation that projects a discrete signal onto a basis of complex sinusoids.

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- Unitary Transform

A discrete linear transform is unitary if its transform matrix conforms to the unitary condition:

$$A \times A^H = I$$

where A = transformation matrix

A^H represents Hermitian matrix

$$A^H = A^{*T}$$

I = Identity matrix.

When the transform matrix A is unitary, the defined transform is called unitary transform.

Q: Check whether the DFT matrix is unitary or not:

- Determination of the matrix A

For 4 point DFT ($N=4$)

$$X(k) = \sum_{n=0}^{3} x(n) e^{-j\frac{2\pi}{4}kn} \quad \text{where } k=0,1,2,3$$

$$X(0) = \sum_{n=0}^{3} x(n) = x(0)+x(1)+x(2)+x(3)$$

$$X(1) = \sum_{n=0}^{3} x(n) e^{-j\frac{\pi}{2}n} = x(0) + x(1) e^{-j\frac{\pi}{2}} + x(2)e^{-j\pi} + x(3)e^{-j\frac{3\pi}{2}}$$
$$= x(0) - jx(1) - x(2) + jx(3)$$

$$X(2) = \sum_{n=0}^{3} x(n) e^{-j\pi n} = x(0)+x(1) e^{-j\pi} + x(2) e^{-j2\pi} + x(3)e^{-j3\pi}$$
$$= x(0) - x(1) + x(2) - x(3)$$

$$X(3) = \sum_{n=0}^{3} x(n) e^{-j\frac{3\pi}{2}n} = x(0)+x(1)e^{-j\frac{3\pi}{2}} + x(2)e^{-j3\pi} + x(3) e^{-j\frac{9\pi}{2}}$$
$$= x(0)+jx(1) - x(2) - jx(3)$$

Collecting the coefficients of $X(0), X(1), X(2)$ and $X(3)$.

$$X[K] = A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix}$$

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Computation of A^H

$$A \xrightarrow{\text{conjugate}} A^* \xrightarrow{\text{Transpose}} A^H$$

conjugate of $A = A^*$

$$A^* = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & j & -1 & -j \\ 1 & -1 & 1 & -1 \\ 1 & -j & -1 & j \end{bmatrix}$$

Transpose of $A^* = A^{*T}$

$$A^{*T} = A^H = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & j & -1 & -j \\ 1 & -1 & 1 & -1 \\ 1 & -j & -1 & j \end{bmatrix}$$

Determination of $A \times A^H$

$$A \times A^H = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & j & -1 & -j \\ 1 & -1 & 1 & -1 \\ 1 & -j & -1 & j \end{bmatrix}$$

$$= \begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 4 \end{bmatrix} = 4 \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

The result is an identity matrix, which shows that Fourier Transform satisfies unitary condition.

NOTE:

Sequency is the number of sign changes.

For a DFT matrix of order 4.

$$x[k] = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{array}{l} \text{Sequency} \\ \rightarrow \text{Zero sign change} \\ \rightarrow \text{Two sign change} \\ \rightarrow \text{Three sign change} \\ \rightarrow \text{One sign change} \end{array}$$

- 2D Discrete Fourier Transform

- The 2D DFT of a rectangular image $f(m,n)$ of size $M \times N$ is given by $F(k,l)$.

$$f(m,n) \xrightarrow{\text{2D-DFT}} F(k,l)$$

where $F(k,l)$ is defined as

$$F(k,l) = \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} f(m,n)\, e^{-j\frac{2\pi}{M}mk}\, e^{-j\frac{2\pi}{N}nl}$$

$$F(k,l) = \sum_{m=0}^{M-1} \sum_{n=0}^{N-1} f(m,n)\, e^{-j\frac{2\pi}{M}nl}\, e^{-j\frac{2\pi}{N}mk}$$

- For a square image $f(m,n)$ of size $N \times N$, the 2D DFT is defined as:

$$F(k,l) = \sum_{m=0}^{N-1} \sum_{n=0}^{N-1} f(m,n)\, e^{-j\frac{2\pi}{N}km}\, e^{-j\frac{2\pi}{N}nl}$$

Inverse 2D DFT is given by

$$f(m,n) = \frac{1}{N^2} \sum_{k=0}^{N-1} \sum_{l=0}^{N-1} F(k,l)\, e^{j\frac{2\pi}{N}mk}\, e^{j\frac{2\pi}{N}nl}$$

- The Fourier transform $F(k,l)$ is given by:

$$F(k,l) = R(k,l) + jI(k,l)$$

where $R(k,l)$ represents the real part and $I(k,l)$ represents the imaginary part of the spectrum.

★ Properties of 2D-DFT

1. Separable Property:

The separable property allows a 2D transform to be computed in two steps by successive 1D operations on rows and columns of an image.

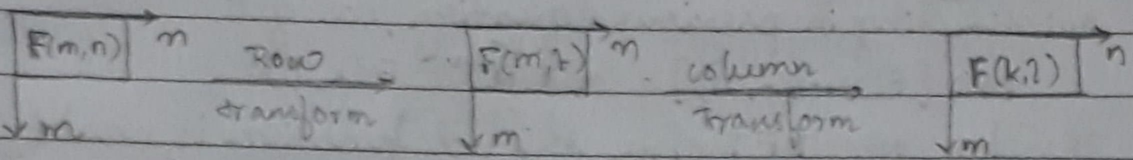
Proof:

Mathematically

$$F(k,l) = \sum_{m=0}^{N-1} \sum_{n=0}^{N-1} f(m,n)\, e^{-j\frac{2\pi}{M}mk}\, e^{-j\frac{2\pi}{N}nl}$$

$$F(k,l) = \sum_{m=0}^{M-1}\left[\sum_{n=0}^{N-1} f(m,n)\, e^{-j\frac{2\pi}{N}nl}\right] e^{-j\frac{2\pi}{M}mk}$$

$$F(k,l) = \sum_{m=0}^{M-1} f(m,l)\, e^{-j\frac{2\pi}{M}mk} = F(k,l)$$



Thus performing a 2D Fourier Transform is equivalent to performing two 1D transforms as:

→ performing a 1D transform on each row of image $f(m,n)$ to get $F(m,l)$

→ performing a 1D transform on each column of $F(m,l)$ to get $F(k,l)$.

The main advantage of separable property is that a Fourier Transform of any dimension can be performed by applying a 1D transform on each dimension.

2. <u>Spatial shift Property</u>:

The 2D DFT of a shifted version of image $f(m,n)$ is $f(m-m_0,n)$ which is given by:

$$f(m-m_0,n) \xrightarrow{DFT} \sum_{m=0}^{N-1}\sum_{n=0}^{N-1} f(m-m_0,n)\, e^{-j\frac{2\pi}{N}mk}\, e^{-j\frac{2\pi}{N}nl}$$

where m_0 represents the number of times that the function $f(m,n)$ is shifted.

<u>PROOF</u>:

$$DFT[f(m-m_0,n)] = \sum_{m=0}^{N-1}\sum_{n=0}^{N-1} f(m-m_0,n)\, e^{-j\frac{2\pi}{N}mk}\, e^{-j\frac{2\pi}{N}nl}$$

adding and subtracting m_0 to $e^{-j\frac{2\pi}{N}mk}$, we get

$$= \sum_{m=0}^{N-1}\sum_{n=0}^{N-1} f(m-m_0,n)\, e^{-j\frac{2\pi}{N}(m-m_0+m_0)k}\, e^{-j\frac{2\pi}{N}nl}$$

$$= \sum_{m=0}^{N-1}\sum_{n=0}^{N-1} f(m-m_0,n)\, e^{-j\frac{2\pi}{N}(m-m_0)k}\, e^{-j\frac{2\pi}{N}m_0k}\, e^{-j\frac{2\pi}{N}nl}$$

$$= e^{-j\frac{2\pi}{N}m_0k} \sum_{m=0}^{N-1}\sum_{n=0}^{N-1} f(m-m_0,n)\, e^{-j\frac{2\pi}{N}(m-m_0)k}\, e^{-j\frac{2\pi}{N}nl}$$

From the defination of forward two-dimensional discrete fourier transform, we can write.

$$\sum_{m=0}^{N-1}\sum_{n=0}^{N-1} f(m-m_0,n)\, e^{-j\frac{2\pi}{N}(m-m_0)k}\, e^{-j\frac{2\pi}{N}nl} = F(k,l)$$

substituting in previous equation, we get

$$DFT[f(m-m_0,n)] = e^{-j\frac{2\pi}{N}m_0k}\, F(k,l)$$

Therefore the DFT of a shifted function is unaltered except for a linearly varying phase factor.

3. Periodicity Property

The 2D DFT of a function $f(m,n)$ is said to be periodic with a period N if:

$$F(k,l) \longrightarrow F(k+pN,\, l+qN)$$

Proof:

$$F(k+pN, l+qN) = \sum_{m=0}^{N-1}\sum_{n=0}^{N-1} f(m,n)\, e^{-j\frac{2\pi}{N}(k+pN)m}\, e^{-j\frac{2\pi}{N}n(l+qN)}$$

$$F(k+pN, l+qN) = \sum_{m=0}^{N-1}\sum_{n=0}^{N-1} f(m,n)\, e^{-j\frac{2\pi}{N}mk}\, e^{-j2\pi mp}\, e^{-j\frac{2\pi}{N}nl}\, e^{-j2\pi nq}$$

$$F(k+pN, l+qN) = F(k,l)$$

4. convolution Property:

convolution in spatial domain is equal to multiplication in the frequency domain. convolution of two sequences $x(n)$ and $h(n)$ is defined as

$$x(n) * h(n) = \sum_{k=-\infty}^{\infty} x(k)\, h(n-k)$$

For two dimensional convolution of two arrays or matrices $f(m,n)$ and $g(m,n)$ is given by:

$$f(m,n) * g(m,n) = \sum_{a=0}^{N-1}\sum_{b=0}^{N-1} f(a,b)\, g(m-a, n-b)$$

Proof:

DFT of the convolution of two sequences $f(m,n)$ and $g(m,n)$ is

$$DFT\{f(m,n)*g(m,n)\} = \sum_{m=0}^{N-1}\sum_{n=0}^{N-1}\left[\sum_{a=0}^{N-1}\sum_{b=0}^{N-1} f(a,b)\, g(m-a,n-b)\right] e^{-j\frac{2\pi}{N}mk}\, e^{-j\frac{2\pi}{N}nl}$$

$$DFT\{f(m,n) * g(m,n)\} = \sum_{m=0}^{N-1}\sum_{n=0}^{N-1}\sum_{a=0}^{N-1}\sum_{b=0}^{N-1} f(a,b)\, g(m-a, n-b)\, e^{-j\frac{2\pi}{N}(m-a+a)k}\, e^{-j\frac{2\pi}{N}(n-b+b)l}$$

$$= \sum_{m=0}^{N-1}\sum_{n=0}^{N-1}\sum_{a=0}^{N-1}\sum_{b=0}^{N-1} f(a,b) g(m-a,n-b) e^{-j\frac{2\pi}{N}(m-a)k} e^{-j\frac{2\pi}{N}ak} e^{-j\frac{2\pi}{N}(n-b)l} e^{-j\frac{2\pi}{N}bl}$$

$$= \sum_{a=0}^{N-1}\sum_{b=0}^{N-1} f(a,b) e^{-j\frac{2\pi}{N}ak} e^{-j\frac{2\pi}{N}bl} \sum_{m=0}^{N-1}\sum_{n=0}^{N-1} g(m-a,n-b) e^{-j\frac{2\pi}{N}(m-a)k} e^{-j\frac{2\pi}{N}(n-b)l}$$

Therefore

$$DFT\{f(m,n) * g(m,n)\} = F(k,l) \times G(k,l)$$

5. **Correlation Property:**

Correlation is basically used to find the relative similarity between two signals. The process of finding similarity of a signal to itself is autocorrelation whereas the process of finding the similarity between two different signals is cross correlation.

The cross correlation of two sequences $x(n)$ and $h(n)$ is equivalent to performing the convolution of one sequence with the folded version of the other sequence.

Proof:

The DFT of correlation of two sequences $x(n)$ and $h(n)$ is defined as:

$$DFT\{R_{x,h}\} = \sum_{m=0}^{N-1}\left\{\sum_{n=0}^{N-1} x(n) h(n+m)\right\} e^{-j\frac{2\pi}{N}mk}$$

where,

$R_{x,h}$ denotes the correlation between the signals $x(n)$ and $h(n)$

By adding and subtracting n to the power of the exponential term

$$DFT\{R_{x,h}\} = \sum_{m=0}^{N-1}\left\{\sum_{n=0}^{N-1} x(n) h(n+m)\right\} e^{-j\frac{2\pi}{N}(m+n-n)k}$$

$$= \sum_{m=0}^{N-1}\left\{\sum_{n=0}^{N-1} x(n) h(n+m)\right\} e^{-j\frac{2\pi}{N}(m+n)k} e^{j\frac{2\pi}{N}nk}$$

$$= \sum_{m=0}^{N-1} h(n+m) e^{-j\frac{2\pi}{N}(m+n)k} \sum_{n=0}^{N-1} x(n) e^{-j\frac{2\pi}{N}n(-k)}$$

Therefore,

$$DFT\{R_{x,h}\} = H(k) X(-k)$$

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Hence, the correlation of two sequences in time domain is equal to the multiplication of DFT of one sequence and time reversal of the DFT of another sequence in the frequency domain.

6. Scaling Property:

Scaling is basically used to increase or decrease the size of an image. According to this property, the expansion of a signal in one domain is equal to compression of the signal in another domain.

If $f(m,n) \xrightarrow{DFT} F(k,l)$

then $f(am,bn) \xrightarrow{DFT} \frac{1}{|ab|} F\left(\frac{k}{a}, \frac{l}{b}\right)$

Proof:

The DFT of a function $f(am,bn)$ is given by:

$$DFT\{f(am,bn)\} = \sum_{m=0}^{N-1}\sum_{n=0}^{N-1} f(am,bn)\, e^{-j\frac{2\pi}{N}mk}\, e^{-j\frac{2\pi}{N}nl}$$

By multiplying and dividing a and b to the respective powers of the exponential term, we get

$$DFT\{f(am,bn)\} = \sum_{m=0}^{N-1}\sum_{n=0}^{N-1} f(am,bn)\, e^{-j\frac{2\pi}{N}mk\frac{a}{a}}\, e^{-j\frac{2\pi}{N}nl\frac{b}{b}}$$

$$= \sum_{m=0}^{N-1}\sum_{n=0}^{N-1} f(am,bn)\, e^{-j\frac{2\pi}{N}ma\frac{k}{a}}\, e^{-j\frac{2\pi}{N}nb\frac{l}{b}}$$

$$= \frac{1}{|ab|} F\left(\frac{k}{a}, \frac{l}{b}\right)$$

7. Conjugate Symmetry:

If the $f(m,n) \xrightarrow{DFT} F(k,l)$

then $f^*(m,n) \xrightarrow{DFT} F^*(-k,-l) = F(k,l)$

Proof:

The DFT of the function $f(m,n)$ is defined as:

$$F(k,l) = \sum_{m=0}^{N-1}\sum_{n=0}^{N-1} f(m,n)\, e^{-j\frac{2\pi}{N}mk}\, e^{-j\frac{2\pi}{N}nl}$$

By applying complex conjugate to $F(k,l)$, we get

$$F^*(k,l) = \sum_{m=0}^{N-1}\sum_{n=0}^{N-1} f(m,n)\, e^{j\frac{2\pi}{N}mk}\, e^{j\frac{2\pi}{N}nl}$$

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By applying reversal to $F^*(k,l)$

$$F^*(-k,-l) = \sum_{m=0}^{N-1}\sum_{n=0}^{N-1} f(m,n)\, e^{j\frac{2\pi}{N}m(-k)}\, e^{j\frac{2\pi}{N}n(-l)} = F(k,l)$$

8. <u>Orthogonality Property:</u>

The orthogonality property of 2D DFT is given by:

$$\frac{1}{N^2}\sum_{m=0}^{N-1}\sum_{n=0}^{N-1} a_{k,l}(m,n)\, a^*_{k',l'}(m,n) = \delta(k-k', l-l')$$

where $\delta(k-k', l-l')$ is the kronecker delta.

This orthogonality condition can be used to derive the formula for the IDFT from the definition of the DFT.

9. <u>Multiplication by Exponential:</u>

If $f(m,n) \xrightarrow{DFT} F(k,l)$

then, $DFT\left[e^{j\frac{2\pi}{N}mk_0}\, e^{j\frac{2\pi}{N}nl_0} f(m,n)\right]$

$$= \sum_{m=0}^{N-1}\sum_{n=0}^{N-1} e^{j\frac{2\pi}{N}mk_0}\, e^{j\frac{2\pi}{N}nl_0} f(m,n)\, e^{-j\frac{2\pi}{N}mk}\, e^{-j\frac{2\pi}{N}nl}$$

<u>Proof:</u>

The definition of a 2D DFT, we can write

$$DFT\left[e^{j\frac{2\pi}{N}mk_0}\, e^{j\frac{2\pi}{N}nl_0} f(m,n)\right] = \sum_{m=0}^{N-1}\sum_{n=0}^{N-1} e^{j\frac{2\pi}{N}mk_0}\, e^{j\frac{2\pi}{N}nl_0} f(m,n)\, e^{-j\frac{2\pi}{N}mk}\, e^{-j\frac{2\pi}{N}nl}$$

$$= \sum_{m=0}^{N-1}\sum_{n=0}^{N-1} f(m,n)\, e^{-j\frac{2\pi}{N}m(k-k_0)}\, e^{-j\frac{2\pi}{N}n(l-l_0)}$$

$$DFT\left[e^{j\frac{2\pi}{N}mk_0}\, e^{j\frac{2\pi}{N}nl_0} f(m,n)\right] = F(k-k_0, l-l_0)$$

Thus multiplication of a function $f(m,n)$ with an exponential in spatial domain leads to a frequency shift.

10. <u>Rotation Property:</u>

The rotation property states that if a function is rotated by the angle, its Fourier transform also rotates by an equal amount.

$$f(m,n) \rightarrow f(r\cos\theta, r\sin\theta)$$

$$DFT[f(r\cos\theta, r\sin\theta)] \rightarrow F[R\cos\phi, R\sin\phi]$$

$$DFT[f(r\cos(\theta+\theta_0), r\sin(\theta+\theta_0))] \rightarrow F[R\cos(\phi+\phi_0), R\sin(\phi+\phi_0)]$$

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Q. Compute the 2D DFT of the 4x4 grayscale image given.

$$f(m,n) = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

- The 2D DFT of the image $f(m,n)$ is represented as $F(k,l)$

$$F(k,l) = \text{basis} \times f(m,n) \times \text{basis}^T$$

The basis of the Fourier transform for $N=4$ is given by

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix}$$

Therefore

$$F(k,l) = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix}$$

$$F(k,l) = \begin{bmatrix} 4 & 4 & 4 & 4 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix}$$

$$F(k,l) = \begin{bmatrix} 16 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} //$$

To find the inverse 2D DFT

$$f(m,n) = \frac{1}{N^2} \times \text{basis} \times F(k,l) \times \text{basis}^T$$

$$f(m,n) = \frac{1}{4^2} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} 16 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix}$$

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$$f(m,n) = \frac{1}{16}\begin{bmatrix}16 & 0 & 0 & 0\\ 16 & 0 & 0 & 0\\ 16 & 0 & 0 & 0\\ 16 & 0 & 0 & 0\end{bmatrix}\begin{bmatrix}1 & 1 & 1 & 1\\ 1 & -j & -1 & j\\ 1 & -1 & 1 & -1\\ 1 & j & -1 & -j\end{bmatrix}$$

$$f(m,n) = \frac{1}{16}\begin{bmatrix}16 & 16 & 16 & 16\\ 16 & 16 & 16 & 16\\ 16 & 16 & 16 & 16\\ 16 & 16 & 16 & 16\end{bmatrix} = \begin{bmatrix}1 & 1 & 1 & 1\\ 1 & 1 & 1 & 1\\ 1 & 1 & 1 & 1\\ 1 & 1 & 1 & 1\end{bmatrix}$$

★ **WALSH TRANSFORM:**

Fourier analysis is basically the representation of a signal by a set of orthogonal sinusoidal waveforms. The coefficients of this representation are called frequency components and the waveforms are orded by frequency.

Walsh introduced a complete set of orthogonal square-wave functions to represent these functions. The computational simplicity of the Walsh function is due to the fact that Walsh functions are real and they take only two values which are either +1 or -1.

The one dimensional Walsh transform basis can be given by:

$$g(n,k) = \frac{1}{N}\sum_{i=0}^{m-1}(-1)^{b_i(n)\, b_{m-1-i}(k)}$$

where n represents time index

k represents frequency index $\quad m = \log_2 N$

N represents the order

m represents the number bits to represent a number

$b_i(n)$ represents the i^{th} (from LSB) bit of the binary value of n decimal number represented in binary.

The two-dimensional Walsh transform of a function $f(m,n)$ is given by:

$$F(k,l) = \frac{1}{N}\sum_{m}\sum_{n} f(m,n)\prod_{i=0}^{p-1}(-1)^{b_i(m)b_{p-1-i}(k) + b_i(n)b_{p-1-i}(k)}$$

Q: Find the 1D Walsh basis for the fourth-order system ($N=4$)

→ Given $N=4$

wkt $m = \log_2 N = \log_2 4 = 2$

also n and k vary from 0 to $N-1$ i.e., 0, 1, 2, 3

i varies from 0 to $m-1$ i.e., 0, 1

When n or k is zero, the basis value will be $1/N$

i.e, $g(n,k) = \frac{1}{N}$ for $n=0$ or $k=0$

construction of Walsh basis for $N=4$

Decimal value	Binary values	
n	$b_1(n)$	$b_0(n)$
0	$b_1(0)=0$	$b_0(0)=0$
1	$b_1(1)=0$	$b_0(1)=1$
2	$b_1(2)=1$	$b_0(2)=0$
3	$b_1(3)=1$	$b_0(3)=1$

Walsh transform basis for $N=4$

k \ n	0	1	2	3	sequency
0	1/4	1/4	1/4	1/4	zero sign change
1	1/4	1/4	-1/4	-1/4	one sign change
2	1/4	-1/4	1/4	-1/4	three sign change
3	1/4	-1/4	-1/4	1/4	two sign change

Ex: $g(1,2)$

$$g(1,2) = \frac{1}{4} \prod_{i=0} (-1)^{b_i(2)\, b_{m-1-i}(1)}$$

$$= \frac{1}{4}\left[(-1)^{b_0(2) b_1(1)} \times (-1)^{b_1(2) b_0(1)}\right]$$

$$= \frac{1}{4}\left[(-1)^{0\times 0} \times (-1)^{1\times 1}\right] = -1/4$$

Therefore, the basis for $N=4$ is

$$g(n,k) = \begin{bmatrix} 1/4 & 1/4 & 1/4 & 1/4 \\ 1/4 & 1/4 & -1/4 & 1/4 \\ 1/4 & -1/4 & 1/4 & -1/4 \\ 1/4 & -1/4 & -1/4 & 1/4 \end{bmatrix}$$

★ **HADAMARD TRANSFORM:**

The Hadamard transform is basically the same as the walsh transform except the rows of the transform matrix are reordered. The elements of the mutually orthogonal basis vectors of a Hadamard transform are either +1 or -1, which results in very low computational complexity in the calculation of the transform coefficients. Hadamard matrices are easily constructed for $N = 2^n$ by the following procedure

The order N=2 Hadamard matrix is given as:

$$H_2 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

The Hadamard matrix of order 2N can be generated by Kronecker product operation:

$$H_{2N} = \begin{bmatrix} H_N & H_N \\ H_N & -H_N \end{bmatrix}$$

For N=2

$$H_4 = \begin{bmatrix} H_2 & H_2 \\ H_2 & -H_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}$$

★ **HAAR TRANSFORM:**

The Haar Transform is based on a class of orthogonal matrices whose elements are either 1, -1 or 0 multiplied by powers of $\sqrt{2}$. The Haar transform is a computationally efficient transform as the transform of an N-point vector requires only 2(N-1) additions and N multiplications.

- Algorithm to generate Haar Basis:

Step 1: Determine the order of N of the Haar basis

Step 2: Determine n where $n = \log_2 N$

Step 3: Determine p and q

i. $0 \le p < n-1$

ii. If $p=0$ then $q=0$ or $q=1$

iii. If $p\neq 0$ then $1 \leq q \leq 2^p$

step 4: Determine k

$$k = 2^p + q - 1$$

step 5: Determine z

$$z \to [0,1) \Rightarrow \left\{ \frac{0}{N}, \frac{1}{N}, \dots \frac{N-1}{N} \right\}$$

step 6:

If $k=0$ then $H(z) = 1/\sqrt{N}$

otherwise

$$H_k(z) = H_{pq}(z) = \frac{1}{\sqrt{N}} \begin{cases} +2^{p/2}; & \frac{q-1}{2^p} \leq z < \frac{q-1/2}{2^p} \\ -2^{p/2}; & \frac{q-1/2}{2^p} \leq z < \frac{q}{2^p} \\ 0 & ; \text{otherwise} \end{cases}$$

Q: generate one Haar Basis for N=2.

– step 1: $N=2$

step 2: $n = \log_2 N = \log_2 2 = 1$

step 3:

i. since $n=1$, $0 \leq p < n-1 \Rightarrow p=0$

ii. when $p=0$, $q=0$ or $q=1$

step 4: $k = 2^p + q - 1$

p	q	k
0	0	0
0	1	1

step 5: $z \to [0,1) \Rightarrow \left\{ \frac{0}{2}, \frac{1}{2} \right\} = \left\{ 0, \frac{1}{2} \right\}$

step 6: $p=0 \to q=0$, $q=1$

case 1: If $k=0$ then

$$H(z) = \frac{1}{\sqrt{N}} = \frac{1}{\sqrt{2}} \in \forall z$$

k \ n	0	1
0	$1/\sqrt{2}$	$1/\sqrt{2}$
1	–	–
z	0	1

case 2: $k=1$; $p=0$; $q=1$

i. $\frac{q-1}{2^p} \le z < \frac{q-1/2}{2^p} \Rightarrow 0 \le z < 1/2$

ii. $\frac{q-1/2}{2^p} \le z < \frac{q}{2^p} \Rightarrow 1/2 \le z < 1$

Therefore

$$H_k(z) = H_{pq}(z) = \frac{1}{\sqrt{2}} \begin{cases} +2^{p/2} & 0 \le z < 1/2 \\ -2^{p/2} & 1/2 \le z < 1 \\ 0 & \text{otherwise} \end{cases}$$

For $z=0$; $H(z) = \frac{1}{\sqrt{2}}(2^{p/2}) = \frac{1}{\sqrt{2}}$ //

For $z = 1/2$; $H(z) = \frac{1}{\sqrt{2}}(-2^{p/2}) = \frac{-1}{\sqrt{2}}$ //

Therefore the Haar basis for $N=2$ is

k \ n	0	1
0	$1/\sqrt{2}$	$1/\sqrt{2}$
1	$1/\sqrt{2}$	$-1/\sqrt{2}$

Q: compute the Haar basis for $N=6$.

– Step 1: $N=4$

Step 2: $n = \log_2 N = \log_2 4 = 2$

Step 3: i. $0 \le p < n-1 \Rightarrow 0 \le p < 1$

ii. If $p=0$ then $q=0$ or $q=1$

iii. If $p \ne 0$ then $1 \le q \le 2^p$

therefore

$p=0 \rightarrow q=0$, $q=1$ $\qquad$ $p=1 \rightarrow q=1$, $q=2$

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step 4: $k = 2^p + q - 1$

p	q	k
0	0	0
0	1	1
1	1	2
1	2	3

Step 5: $z \rightarrow [0,1) \Rightarrow \left\{\frac{0}{4}, \frac{1}{4}, \frac{2}{4}, \frac{3}{4}\right\} = \left\{0, \frac{1}{4}, \frac{1}{2}, \frac{3}{4}\right\}$

step 6: If $k=0$ then $H_0(z) = \frac{1}{\sqrt{N}} = \frac{1}{\sqrt{4}} = \frac{1}{2}$

otherwise,

$$H_k(z) = H_{pq}(z) = \frac{1}{\sqrt{4}} \begin{cases} +2^{p/2} & \frac{q-1}{2^p} \leq z < \frac{q-1/2}{2^p} \\ -2^{p/2} & \frac{q-1/2}{2^p} \leq z < \frac{q}{2^p} \\ 0 & \text{otherwise} \end{cases}$$

case 1: When $k=1$, $p=0$ and $q=1$

conditions:

i. $\frac{q-1}{2^p} \leq z < \frac{q-1/2}{2^p} \Rightarrow 0 \leq z \leq 1/2$

ii. $\frac{q-1/2}{2^p} \leq z < \frac{q}{2^p} \Rightarrow 1/2 \leq z < 1$

For $z=0$: $H_1(z) = (1/2)\, 2^{p/2} = 1/2 \quad (\because 0 \leq z < 1/2)$

For $z = 1/4$: $H_1(z) = (1/2)\, 2^{p/2} = 1/2$

For $z = 1/2$: $H_1(z) = (1/2)(-2^{p/2}) = -1/2 \quad (\because 1/2 \leq z < 1)$

For $z = 3/4$: $H_1(z) = (1/2)(-2^{p/2}) = -1/2$

case 2: When $k=2$, $p=1$, $q=1$

conditions:

i. $\frac{q-1}{2^p} \leq z < \frac{q-1/2}{2^p} \Rightarrow 0 \leq z < 1/4$

ii. $\frac{q-1/2}{2^p} \leq z < \frac{q}{2^p} \Rightarrow 1/4 \leq z < 1/2$

For $z=0$: $H_2(z) = (1/2)(2^{p/2}) = 1/\sqrt{2}$

For $z = 1/4$: $H_2(z) = (1/2)(-2^{p/2}) = -1/\sqrt{2}$

For $z = 1/2$ and $z = 3/4$: $H_2(z) = 0$

case 3: When $k=3$; $p=1$ and $q=2$

conditions :

i. $\frac{q-1}{2^p} \leq z < \frac{q-1/2}{2^p} \Rightarrow 1/2 \leq z < 3/4$

ii. $\frac{q-1/2}{2^p} \leq z < \frac{q}{2^p} \rightarrow 3/4 \leq z < 1$

For $z=0$: $H_3(z) = 0$

For $z = 1/4$: $H_3(z) = 0$

For $z = 1/2$: $H_3(z) = (1/2)(2^{p/2}) = 1/\sqrt{2}$

For $z = 3/4$: $H_3(z) = (1/2)(-2^{p/2}) = -1/\sqrt{2}$

Therefore, the Haar transform basis for $N=4$ is.

k \ n	0	1	2	3
0	$1/2$	$1/2$	$1/2$	$1/2$
1	$1/2$	$1/2$	$-1/2$	$-1/2$
2	$1/\sqrt{2}$	$-1/\sqrt{2}$	0	0
3	0	0	$1/\sqrt{2}$	$-1/\sqrt{2}$

★ SLANT TRANSFORM:

The slant transform is an orthogonal transform containing sawtooth waveforms or 'slant' basis vectors. A slant basis vector that is monotonically decreasing in constant steps from maximum to minimum has the sequency property and has a fast computational algorithm. Let S_N denote as $N \times N$ slant matrix with $N = 2^n$. Then

$$S_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

The S_4 matrix is obtained by the following operation:

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$$S_4 = \begin{bmatrix} 1 & 0 & 1 & 0 \\ a & b & -a & b \\ 0 & 1 & 0 & -1 \\ -b & a & b & a \end{bmatrix} \begin{bmatrix} S_2 & 0 \\ 0 & S_2 \end{bmatrix}$$

If $a = 2b$ and $b = \frac{1}{\sqrt{5}}$

then

$$S_4 = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 2/\sqrt{5} & 1/\sqrt{5} & -2/\sqrt{5} & 1/\sqrt{5} \\ 0 & 1 & 0 & -1 \\ -1/\sqrt{5} & 2/\sqrt{5} & 1/\sqrt{5} & 2/\sqrt{5} \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

$$S_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 3/\sqrt{5} & 1/\sqrt{5} & -1/\sqrt{5} & -3/\sqrt{5} \\ 1 & -1 & -1 & 1 \\ 1/\sqrt{5} & -3/\sqrt{5} & 3/\sqrt{5} & -1/\sqrt{5} \end{bmatrix} \begin{matrix} \text{zero sign change} \\ \text{one sign change} \\ \text{two sign change} \\ \text{three sign change} \end{matrix}$$

From the sequency property, it is clear that the rows are ordered by the number of sign changes.

The slant transform produces linear variations of brightness very well. But its performance at edges is not optimal because of the slant nature of lower order coefficients its effect is to smear the edges.

★ **DISCRETE COSINE TRANSFORM:**

A discrete cosine transform consists of a set of basis vectors that are sampled cosine functions.

If $x(n)$ is the signal of length N, the Fourier transform of the signal $x(n)$ is given by $X[k]$ where

$$X[k] = \sum_{n=0}^{N-1} x(n) e^{-j\frac{2\pi}{N}kn} \quad \text{where } k \to 0 \dots N-1$$

Now consider the extention of the signal $x(n)$ is denoted by $x_e(n)$ so that the length of the extended sequence is 2N. The sequence can be extended in two ways:

- by simply copying the original sequence again
- by folding the sequence (preferable)

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The Discrete Fourier Transform (DFT) of the extended sequence is given by $X_e(k)$ where

$$X_e(k) = \sum_{n=0}^{2N-1} x_e(n)\, e^{-j\frac{2\pi}{2N}kn}$$

by splitting the summation:

$$X_e(k) = \sum_{n=0}^{N-1} x(n)\, e^{-j\frac{2\pi}{2N}kn} + \sum_{n=N}^{2N-1} x[2N-1-n]\, e^{-j\frac{2\pi}{2N}kn}$$

let $m = 2N-1-n$

$$X_e(k) = \sum_{n=0}^{N-1} x(n)\, e^{-j\frac{2\pi}{2N}kn} + \sum_{m=0}^{N-1} x(m)\, e^{-j\frac{2\pi}{2N}k(2N-1-m)}$$

$$X_e(k) = \sum_{n=0}^{N-1} x(n)\, e^{-j\frac{2\pi}{2N}kn} + \sum_{m=0}^{N-1} x(m)\, \cancelto{1}{e^{-j\frac{2\pi}{2N}k2N}}\; e^{j\frac{2\pi}{2N}k(m+1)}$$

$$X_e(k) = \sum_{n=0}^{N-1} x(n)\, e^{-j\frac{2\pi}{2N}kn} + \sum_{m=0}^{N-1} x(m)\, e^{j\frac{2\pi}{2N}(m+1)k}$$

replacing m by n

$$X_e(k) = \sum_{n=0}^{N-1} x(n)\, e^{-j\frac{2\pi}{2N}kn} + \sum_{n=0}^{N-1} x(n)\, e^{j\frac{2\pi}{2N}k(n+1)}$$

multiplying $e^{-j\frac{\pi k}{2N}}$ on both sides

$$X_e(k)\, e^{-j\frac{\pi k}{2N}} = \sum_{n=0}^{N-1} x(n) \left[e^{-j\frac{\pi k}{2N}(2n+1)} + e^{j\frac{\pi k}{2N}(2n+1)} \right]$$

$$X_e(k)\, e^{-j\frac{\pi k}{2N}} = \sum_{n=0}^{N-1} 2x(n) \cos\left(\frac{\pi k}{2N}(2n+1)\right)$$

$$\frac{X_e(k)\, e^{-j\frac{\pi k}{2N}}}{2} = \sum_{n=0}^{N-1} x(n) \cos\left(\frac{\pi k}{2N}(2n+1)\right)$$

Thus the basis of a one dimensional discrete cosine transform is given by:

$$X(k) = \alpha(k) \sum_{n=0}^{N-1} x(n) \cos\left(\frac{\pi k}{2N}(2n+1)\right) \quad \text{where } 0 \le k \le N-1$$

$$\alpha(k) = \sqrt{\frac{1}{N}} \text{ if } k = 0$$

$$\alpha(k) = \sqrt{\frac{2}{N}} \text{ if } k \neq 0$$

The process of reconstructing a set of spacial domain samples from the DCT coefficients is called the inverse discrete cosine transform (IDCT)

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Date:

The Inverse Discrete Cosine Transform is given by:

$$x(n) = \alpha(k) \sum_{k=0}^{N-1} X(k) \cos\left[\frac{(2n+1)\pi k}{2N}\right] \qquad 0 \le n \le N-1$$

The forward 2D discrete cosine transform of a signal $f(m,n)$ is given by:

$$F[k,l] = \alpha(k)\alpha(l) \sum_{m=0}^{N-1} \sum_{n=0}^{N-1} f(m,n) \cos\left(\frac{(2m+1)\pi k}{2N}\right) \cos\left[\frac{(2n+1)\pi l}{2N}\right]$$

where $\alpha(k) = \begin{cases} \sqrt{\frac{1}{N}} & \text{if } k=0 \\ \sqrt{\frac{2}{N}} & \text{if } k \neq 0 \end{cases}$

$$\alpha(l) = \begin{cases} \sqrt{\frac{1}{N}} & \text{if } l=0 \\ \sqrt{\frac{2}{N}} & \text{if } l \neq 0 \end{cases}$$

The 2D inverse discrete cosine transform is given by:

$$f(m,n) = \sum_{k=0}^{N-1} \sum_{l=0}^{N-1} \alpha(k)\alpha(l) F(k,l) \cos\left[\frac{(2m+1)\pi k}{2N}\right] \cos\left[\frac{(2n+1)\pi l}{2N}\right]$$

Q: Compute the discrete cosine transform (DCT) matrix for $N=4$.

– The formula to compute the DCT matrix is given by

$$X(k) = \alpha(k) \sum_{n=0}^{N-1} x(n) \cos\left(\frac{(2n+1)\pi k}{2N}\right) \quad \text{where } 0 \le k \le N-1$$

$$\alpha(k) = \begin{cases} \sqrt{\frac{1}{N}} & \text{if } k=0 \\ \sqrt{\frac{2}{N}} & \text{if } k \neq 0 \end{cases}$$

Here $N = 4$.

$$\therefore X(k) = \alpha(k) \sum_{n=0}^{3} x(n) \cos\left(\frac{(2n+1)\pi k}{8}\right)$$

For $k=0$

$$X(0) = \sqrt{\frac{1}{4}} \sum_{n=0}^{3} x(n) \cos\left(\cancelto{1}{\frac{(2n+1)(0)}{8}}\right)$$

$$X(0) = \frac{1}{2}\sum_{n=0}^{3} x(n) = \frac{x(0)}{2} + \frac{x(1)}{2} + \frac{x(2)}{2} + \frac{x(3)}{2}$$

For $k=1$

$$X(1) = \sqrt{\frac{2}{4}}\sum_{n=0}^{3} x(n)\cos\left(\frac{(2n+1)(\pi)}{8}\right)$$

$$= \frac{1}{\sqrt{2}}\left[x(0)\cos\frac{\pi}{8} + x(1)\cos\frac{3\pi}{8} + x(2)\cos\frac{5\pi}{8} + x(3)\cos\frac{7\pi}{8}\right]$$

$$= 0.6532x(0) + 0.2706x(1) - 0.2706x(2) - 0.6532x(3)$$

For $k=2$

$$X(2) = \sqrt{\frac{2}{4}}\sum_{n=0}^{3} x(n)\cos\left(\frac{(2n+1)(2\pi)}{8}\right)$$

$$= \frac{1}{\sqrt{2}}\left[x(0)\cos\frac{\pi}{4} + x(1)\cos\frac{3\pi}{4} + x(2)\cos\frac{5\pi}{4} + x(3)\cos\frac{7\pi}{4}\right]$$

$$= 0.5x(0) - 0.5x(1) - 0.5x(2) + 0.5x(3)$$

For $k=3$

$$X(3) = \sqrt{\frac{2}{4}}\sum_{n=0}^{3} x(n)\cos\left(\frac{(2n+1)3\pi}{8}\right)$$

$$= \frac{1}{\sqrt{2}}\left[x(0)\cos\frac{3\pi}{8} + x(1)\cos\frac{9\pi}{8} + x(2)\cos\frac{15\pi}{8} + x(3)\cos\frac{21\pi}{8}\right]$$

$$= 0.2706x(0) - 0.6533x(1) + 0.6533x(2) - 0.2706x(3)$$

Therefore

$$\begin{bmatrix} X(0) \\ X(1) \\ X(2) \\ X(3) \end{bmatrix} = \begin{bmatrix} 0.5 & 0.5 & 0.5 & 0.5 \\ 0.6532 & 0.2706 & -0.2706 & -0.6532 \\ 0.5 & -0.5 & -0.5 & 0.5 \\ 0.2706 & -0.6533 & 0.6533 & -0.2706 \end{bmatrix} \begin{bmatrix} x(0) \\ x(1) \\ x(2) \\ x(3) \end{bmatrix}$$

Properties of DCT:

- The discrete cosine transform is real and orthogonal.

  If A is a DCT matrix of order N, and if the matrix A is orthogonal then

  $$A \times A^T = 1$$

Proof: Let A be a DCT matrix of order four.

ie $N=4$.

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Then Matrix A is given by:

$$A = \begin{bmatrix} 0.5 & 0.5 & 0.5 & 0.5 \\ 0.6532 & 0.2706 & -0.2706 & -0.6532 \\ 0.5 & -0.5 & -0.5 & 0.5 \\ 0.2706 & -0.6533 & 0.6533 & -0.2706 \end{bmatrix}$$

$$A^T = \begin{bmatrix} 0.5 & 0.6532 & 0.5 & 0.2706 \\ 0.5 & 0.2706 & -0.5 & -0.6533 \\ 0.5 & -0.2706 & -0.5 & 0.6533 \\ 0.5 & -0.6532 & 0.5 & -0.2706 \end{bmatrix}$$

$$A \times A^T = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = I$$

$\Rightarrow \underline{A^{-1} = A^T}$

- Separable property

A 2D DCT can be computed as two separate 1D transforms because of separable property. The 2D DCT of an image can be computed in two steps by successive 1D operations on rows and columns of an image.

★ <u>KL TRANSFORN</u>: (Karhunen - Loeve Transform)

The KL transform is a reversible linear transform that exploits the statistical properties of a vector representation. The basic functions of the KL transform are orthogonal eigen vectors of the covariance matrix of a data set. A KL transform optimally decorrelates the input data. After a KL transform, most of the 'energy' of the transform coefficients is concentrated within the first two components. This is the energy compaction property of a KL transform.

- <u>Drawbacks</u>

- A KL transform is input dependent and the basis function has to be calculated for each signal model on which it operates. The KL bases have no specific mathematical structure that leads to fast implementations.

- The KL transform requires $O(m^2)$ multiply / add operations. The DFT and DCT require $O(\log_2 m)$ multiplications.

- Applications
  - Cluster Analysis:
  - Image compression

Q: Perform KL transform for the following matrix:

$$X = \begin{bmatrix} 4 & -2 \\ -1 & 3 \end{bmatrix}$$

- Step 1: Formation of vectors from the given matrix

The given matrix is a 2x2 matrix hence two vectors can be extracted from the given matrix.

Let it be x_0 and x_1

$$x_0 = \begin{bmatrix} 4 \\ -1 \end{bmatrix} \quad x_1 = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$$

Step 2: Determination of covariance matrix:

The formula to compute covariance of the matrix is

$$\text{cov}(x) = E[xx^T] - \bar{x}\bar{x}^T$$

where $\bar{x}$: mean of input matrix

$$\bar{x} = \frac{1}{M} \sum_{k=0}^{M-1} x_k$$

where M: number of vectors in x

$$\therefore \bar{x} = \frac{1}{2} \sum_{k=0}^{1} x_k = \frac{1}{2}\{x_0 + x_1\}$$

$$\bar{x} = \frac{1}{2}\left\{\begin{bmatrix} 4 \\ -1 \end{bmatrix} + \begin{bmatrix} -2 \\ 3 \end{bmatrix}\right\} = \frac{1}{2}\left\{\begin{bmatrix} 2 \\ 2 \end{bmatrix}\right\} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\bar{x}\bar{x}^T = \begin{bmatrix} 1 \\ 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

$$E[xx^T] = \frac{1}{M} \sum_{k=0}^{M-1} x_k x_k^T$$

$$E[xx^T] = \frac{1}{2}\sum_{k=0}^{1} x_k x_k^T = \frac{1}{2}\left\{\begin{bmatrix}4\\-1\end{bmatrix}[4\ \ -1] + \begin{bmatrix}-2\\3\end{bmatrix}[-2\ \ 3]\right\}$$

$$= \frac{1}{2}\left\{\begin{bmatrix}16 & -4\\-4 & 1\end{bmatrix} + \begin{bmatrix}4 & -6\\-6 & 9\end{bmatrix}\right\}$$

$$E[xx^T] = \frac{1}{2}\begin{bmatrix}20 & -10\\-10 & 10\end{bmatrix} = \begin{bmatrix}10 & -5\\-5 & 5\end{bmatrix} = \begin{bmatrix}2 & -1\\-1 & 1\end{bmatrix}$$

Therefore

$$\mathrm{cov}(x) = E[xx^T] - \bar{x}\bar{x}^T$$

$$= \begin{bmatrix}2 & -1\\-1 & 1\end{bmatrix} - \begin{bmatrix}1 & 1\\1 & 1\end{bmatrix} = \begin{bmatrix}1 & -2\\-2 & 0\end{bmatrix}$$

<u>Step 3</u>: <u>Determination of eigen values of the covariance matrix</u>

The solution to the characteristic equation gives the eigen values λ.

$$|\mathrm{cov}(x) - \lambda I| = 0$$

$$\det\left(\begin{bmatrix}1 & -2\\-2 & 0\end{bmatrix} - \lambda\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}\right) = 0$$

$$\det\left(\begin{bmatrix}1-\lambda & -2\\-2 & -\lambda\end{bmatrix}\right) = 0$$

$$(1-\lambda)(-\lambda) - (-2)(-2) = 0$$

$$-\lambda + \lambda^2 - 4 = 0$$

$$\therefore \lambda = \frac{1 \pm \sqrt{1+16}}{2} = \frac{1 \pm \sqrt{17}}{2} = \frac{1 \pm 4.1231}{2}$$

<u>Eigen values</u>

$$\Rightarrow \lambda_0 = \frac{1+4.1231}{2} = 2.5615 \ ; \ \lambda_1 = \frac{1-4.1231}{2} = -1.5615$$

<u>Step 4</u>: <u>Determination of eigen vectors of the covariance matrix</u>:

First eigen vector ϕ_0

$$(\mathrm{cov}(x) - \lambda_0 I)\phi_0 = 0$$

$$(\mathrm{cov}(x) - \lambda_0 I)\phi_0 = \left[\begin{bmatrix}1 & -2\\-2 & 0\end{bmatrix} - \begin{bmatrix}2.5615 & 0\\0 & 2.5615\end{bmatrix}\right]\begin{bmatrix}\phi_{00}\\\phi_{01}\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$$

$$= \begin{bmatrix}-1.5615 & -2\\-2 & -2.5615\end{bmatrix}\begin{bmatrix}\phi_{00}\\\phi_{01}\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix}$$

ϕ_{01} is a free variable $\Rightarrow \phi_{01} = 1$

$-1.5615\,\phi_{00} - 2\phi_{01} = 0$

Let $\phi_{01} = 1$

$\phi_{00} = \dfrac{-2}{1.5615} = -1.2808$

Therefore

$$\phi_0 = \begin{bmatrix} \phi_{00} \\ \phi_{01} \end{bmatrix} = \begin{bmatrix} -1.2808 \\ 1 \end{bmatrix}$$

Second eigen vector ϕ_1

$$(cov(x) - \lambda_1 I)\,\phi_1 = 0$$

$$(cov(x) - \lambda_1 I)\phi_1 = \left[\begin{bmatrix} 1 & -2 \\ -2 & 0 \end{bmatrix} - \begin{bmatrix} -1.5615 & 0 \\ 0 & -1.5615 \end{bmatrix}\right]\begin{bmatrix} \phi_{10} \\ \phi_{11} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 2.5615 & -2 \\ -2 & -1.5615 \end{bmatrix}\begin{bmatrix} \phi_{10} \\ \phi_{11} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

ϕ_{11} is a free variable $\Rightarrow \phi_{11} = 1$

$2.5615\,\phi_{10} - 2\phi_{11} = 0$

$\phi_{10} = \dfrac{2}{2.5615} = 0.7808$

Therefore

$$\phi_1 = \begin{bmatrix} \phi_{10} \\ \phi_{11} \end{bmatrix} = \begin{bmatrix} 0.7808 \\ 1 \end{bmatrix}$$

Step 5: Normalisation of the eigen vectors:

The normalization of the eigen vector ϕ_0 is

$$\frac{\phi_0}{\|\phi_0\|} = \frac{1}{\sqrt{\phi_{00}^2 + \phi_{01}^2}}\begin{bmatrix} \phi_{00} \\ \phi_{01} \end{bmatrix}$$

$$= \frac{1}{\sqrt{(-1.2808)^2 + 1^2}}\begin{bmatrix} -1.2808 \\ 1 \end{bmatrix} = \begin{bmatrix} -0.7882 \\ 0.6154 \end{bmatrix}$$

The normalization of the eigen vector ϕ_1 is

$$\frac{\phi_1}{\|\phi_1\|} = \frac{1}{\sqrt{\phi_{10}^2 + \phi_{11}^2}}\begin{bmatrix} \phi_{10} \\ \phi_{11} \end{bmatrix}$$

$$= \frac{1}{\sqrt{(0.7808)^2 + 1^2}}\begin{bmatrix} 0.7808 \\ 1 \end{bmatrix} = \begin{bmatrix} 0.6154 \\ 0.7882 \end{bmatrix}$$

Step 6: KL transformation matrix from the eigen vector of the covariance matrix:

The transformation matrix is

$$T = \begin{bmatrix} -0.7882 & 0.6154 \\ 0.6154 & 0.7882 \end{bmatrix}$$

To check if orthogonal

i.e., $TT^T = TT^{-1} = I$

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$$TT' = \begin{bmatrix} -0.7882 & 0.6154 \\ 0.6154 & 0.7882 \end{bmatrix} \begin{bmatrix} -0.7882 & 0.6154 \\ 0.6154 & 0.7882 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 1$$

step 7: KL transformation of the input matrix:

$$y = T[x]$$

$$y_0 = T[x_0] = \begin{bmatrix} -0.7882 & 0.6154 \\ 0.6154 & 0.7882 \end{bmatrix} \begin{bmatrix} 4 \\ -1 \end{bmatrix} = \begin{bmatrix} -3.7682 \\ 1.6734 \end{bmatrix}$$

$$y_1 = T[x_1] = \begin{bmatrix} -0.7882 & 0.6154 \\ 0.6154 & 0.7882 \end{bmatrix} \begin{bmatrix} -2 \\ 3 \end{bmatrix} = \begin{bmatrix} 3.4226 \\ 1.1338 \end{bmatrix}$$

The final transform matrix is

$$y = \begin{bmatrix} -3.7682 & 3.4226 \\ 1.6734 & 1.1338 \end{bmatrix}$$

step 8: Reconstruction of input values from the transformed coefficients:

$$X = T^T y$$

$$X_0 = T^T y_0 = \begin{bmatrix} -0.7882 & 0.6154 \\ 0.6154 & 0.7882 \end{bmatrix} \begin{bmatrix} -3.7682 \\ 1.6734 \end{bmatrix} = \begin{bmatrix} 4 \\ -1 \end{bmatrix}$$

$$X_1 = T^T y_1 = \begin{bmatrix} -0.7882 & 0.6154 \\ 0.6154 & 0.7882 \end{bmatrix} \begin{bmatrix} 3.4226 \\ 1.1338 \end{bmatrix} = \begin{bmatrix} -2 \\ 3 \end{bmatrix}$$

$$\therefore X = \begin{bmatrix} 4 & -2 \\ -1 & 3 \end{bmatrix} //$$

Q: compute the basis of the KL transform for the input data

$$x_1 = \begin{bmatrix} 4 \\ 4 \\ 5 \end{bmatrix};\ x_2 = \begin{bmatrix} 3 \\ 2 \\ 5 \end{bmatrix};\ x_3 = \begin{bmatrix} 5 \\ 7 \\ 6 \end{bmatrix};\ x_4 = \begin{bmatrix} 6 \\ 7 \\ 7 \end{bmatrix}$$

– **step 1: calculation of mean of the input vectors**

$$\bar{x} = \frac{1}{M} \sum_{k=0}^{M-1} x_k = \frac{1}{4} \sum_{k=0}^{3} x_k$$

$$\bar{x} = \frac{1}{4} \left\{ \begin{bmatrix} 4 \\ 4 \\ 5 \end{bmatrix} + \begin{bmatrix} 3 \\ 2 \\ 5 \end{bmatrix} + \begin{bmatrix} 5 \\ 7 \\ 6 \end{bmatrix} + \begin{bmatrix} 6 \\ 7 \\ 7 \end{bmatrix} \right\} = \frac{1}{4} \begin{bmatrix} 18 \\ 20 \\ 23 \end{bmatrix} = \begin{bmatrix} 4.5 \\ 5 \\ 5.75 \end{bmatrix}$$

Step 2: computation of covariance matrix

$$\mathrm{cov}(x) = E[x x^T] - \bar{x}\bar{x}^T$$

$$= \frac{1}{M}\sum_{k=0}^{M-1} x_k x_k^T - \bar{x}\bar{x}^T$$

$$= \frac{1}{4}\sum_{k=0}^{3} x_k x_k^T - \bar{x}\bar{x}^T$$

$$\frac{1}{4}\sum_{k=0}^{3} x_k x_k^T = \frac{1}{4}\left\{\begin{bmatrix}4\\4\\5\end{bmatrix}\begin{bmatrix}4 & 4 & 5\end{bmatrix} + \begin{bmatrix}3\\2\\5\end{bmatrix}\begin{bmatrix}3 & 2 & 5\end{bmatrix} + \begin{bmatrix}5\\7\\6\end{bmatrix}\begin{bmatrix}5 & 7 & 6\end{bmatrix} + \begin{bmatrix}6\\7\\7\end{bmatrix}\begin{bmatrix}6 & 7 & 7\end{bmatrix}\right\}$$

$$= \frac{1}{4}\left\{\begin{bmatrix}16 & 16 & 20\\16 & 16 & 20\\20 & 20 & 25\end{bmatrix} + \begin{bmatrix}9 & 6 & 15\\6 & 4 & 10\\15 & 10 & 25\end{bmatrix} + \begin{bmatrix}25 & 35 & 30\\35 & 49 & 42\\30 & 42 & 36\end{bmatrix} + \begin{bmatrix}36 & 42 & 42\\42 & 49 & 49\\42 & 49 & 49\end{bmatrix}\right\}$$

$$= \frac{1}{4}\begin{bmatrix}86 & 99 & 107\\99 & 118 & 121\\107 & 121 & 135\end{bmatrix} = \begin{bmatrix}21.5 & 24.75 & 26.75\\24.75 & 29.5 & 30.25\\26.75 & 30.25 & 33.75\end{bmatrix}$$

$$\bar{x}\bar{x}^T = \begin{bmatrix}4.5\\5\\5.75\end{bmatrix}\begin{bmatrix}4.5 & 5 & 5.75\end{bmatrix} = \begin{bmatrix}20.25 & 22.5 & 25.875\\22.5 & 25 & 28.75\\25.875 & 28.75 & 33.0625\end{bmatrix}$$

$$\therefore \mathrm{cov}(x) = \begin{bmatrix}21.5 & 24.75 & 26.75\\24.75 & 29.5 & 30.25\\26.75 & 30.25 & 33.75\end{bmatrix} - \begin{bmatrix}20.25 & 22.5 & 25.875\\22.5 & 25 & 28.75\\25.875 & 28.75 & 33.0625\end{bmatrix}$$

$$= \begin{bmatrix}1.25 & 2.25 & 0.875\\2.25 & 4.5 & 1.5\\0.875 & 1.5 & 0.6875\end{bmatrix}$$

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Step 3: Determination of eigen values

$$|cov(x) - \lambda I| = 0$$

$$\Rightarrow \left| \begin{bmatrix} 1.25 & 2.25 & 0.875 \\ 2.25 & 4.5 & 1.5 \\ 0.875 & 1.5 & 0.6875 \end{bmatrix} - \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} \right| = 0$$

$$\Rightarrow \begin{vmatrix} 1.25-\lambda & 2.25 & 0.875 \\ 2.25 & 4.5-\lambda & 1.5 \\ 0.875 & 1.5 & 0.6875-\lambda \end{vmatrix} = 0$$

$$\Rightarrow (1.25-\lambda)[(4.5-\lambda)(0.6875-\lambda) - (1.5)(1.5)]$$
$$- 2.25[2.25(0.6875-\lambda) - 1.5(0.875)]$$
$$+ 0.875[2.25(1.5) - 0.875(4.5-\lambda)] = 0$$

$$\Rightarrow (1.25-\lambda)[3.094 - 4.5\lambda - 0.6875\lambda + \lambda^2 - 2.25]$$
$$-2.25[1.547 - 2.25\lambda - 1.3125]$$
$$+0.875[3.375 - 3.9375 + 0.875\lambda] = 0$$

$$\Rightarrow (1.25-\lambda)[\lambda^2 - 5.1875\lambda + 0.844] - 2.25[-2.25\lambda + 0.2345]$$
$$+ 0.875[-0.5625 + 0.875\lambda] = 0$$

$$\Rightarrow 1.25\lambda^2 - 6.484\lambda + 1.055 - \lambda^3 + 5.1875\lambda^2 - 0.844\lambda$$
$$+ 5.0625\lambda - 0.5276 - 0.4922 + 0.766\lambda = 0$$

$$\Rightarrow \lambda^3 - 6.4375\lambda^2 + 1.5\lambda - 0.0352 = 0$$

Therefore

$$\lambda_0 = 6.1963 \; ; \; \lambda_1 = 0.0264 \; ; \; \lambda_2 = 0.2147$$

Step 4: Determination of eigen vectors

$$\phi_0 = \begin{bmatrix} 0.4385 \\ 0.8471 \\ 0.3003 \end{bmatrix} \quad \phi_1 = \begin{bmatrix} 0.4460 \\ -0.4952 \\ 0.7456 \end{bmatrix} \quad \phi_2 = \begin{bmatrix} -0.7803 \\ 0.1929 \\ 0.5949 \end{bmatrix}$$

Step 5: KL transform basis

$$T = \begin{bmatrix} 0.4385 & 0.8471 & 0.3003 \\ 0.4460 & -0.4952 & 0.7456 \\ -0.7803 & 0.1929 & 0.5949 \end{bmatrix}$$