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UNIT - 1

Fourier Series

★ **Periodic Function:**

Any function that repeats for certain time interval is called periodic function.

Ex: sine and cosine are periodic functions of period 2π.

$f(x+nT) = f(x)$ with period T.

NOTE:

1. $\int e^{ax}\cos bx\, dx = \dfrac{e^{ax}}{a^2+b^2}\left[a\cos bx + b\sin bx\right]$

2. $\int e^{ax}\sin bx\, dx = \dfrac{e^{ax}}{a^2+b^2}\left[a\sin bx - b\cos bx\right]$

3. $\cos n\pi = \cos(-n\pi) = (-1)^n$

4. $\sin n\pi = \sin(-n\pi) = 0$

★ **Fourier Series:**

Fourier series Expansion for the period l in the interval $[-l, l]$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right)$$

where $a_0 = \dfrac{1}{l}\displaystyle\int_{-l}^{l} f(x)\,dx$

$$a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\left(\frac{n\pi x}{l}\right)dx$$

$$b_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin\left(\frac{n\pi x}{l}\right)dx$$

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★ **Bernoulli's Rule:**

$$\int uv\,dx = u\int v\,dx - u'\iint v\,dx + u''\iiint v\,dx - \dots$$

Q1: $\int x\cos nx\,dx$

sol: $= x\int \cos nx\,dx - \iint \sin nx\,dx$

$= x\left(\frac{\sin nx}{n}\right) - \left(-\frac{\cos nx}{n^2}\right)$

$= \frac{x\sin nx}{n} + \frac{\cos nx}{n^2}$ //

Q2: $\int x^2 \sin nx\,dx$

sol: $= x^2\int \sin nx\,dx - 2x\iint \sin nx\,dx + 2\iiint \sin nx\,dx$

$= x^2\left(-\frac{\cos nx}{n}\right) - 2x\left(-\frac{\sin nx}{n^2}\right) + 2\left(\frac{\cos nx}{n^3}\right)$

$= -\frac{x^2\cos nx}{n} + \frac{2x\sin nx}{n^2} + \frac{2\cos nx}{n^3}$ //

Q3: $\int e^{2x}\cos nx\,dx$

sol: $= \frac{e^{2x}}{4+n^2}\left[2\cos nx + n\sin nx\right]$ //

Q4: $\int (x-x^2)\sin nx\,dx$

sol: $= (x-x^2)\int \sin nx\,dx - (1-2x)\iint \sin nx\,dx + (-2)\iiint \sin$

$= (x-x^2)\left(-\frac{\cos nx}{n}\right) - (1-2x)\left(-\frac{\sin nx}{n^2}\right) - 2\left(\frac{\cos nx}{n^3}\right)$

$= -\frac{(x-x^2)\cos nx}{n} + \frac{(1-2x)\sin nx}{n^2} - \frac{2\cos nx}{n^3}$ //

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Q1: Obtain the Fourier series for the following functions:
$f(x) = x - x^2$ in $(-1, 1)$

Sol: The Fourier series is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right) \quad (1)$$

where $a_0 = \frac{1}{l}\int_{-l}^{l} f(x)\,dx$

$$a_0 = \frac{1}{1}\int_{-1}^{1} (x - x^2)\,dx$$

$$a_0 = \left[\frac{x^2}{2} - \frac{x^3}{3}\right]_{-1}^{1}$$

$$a_0 = \left[\cancel{\frac{1}{2}} - \frac{1}{3} - \cancel{\frac{1}{2}} - \frac{1}{3}\right]$$

$$a_0 = \frac{-2}{3} \Rightarrow \frac{a_0}{2} = \frac{-1}{3}$$

$$a_n = \frac{1}{l}\int_{-1}^{1} f(x)\cos\left(\frac{n\pi x}{l}\right)dx$$

$$= \frac{1}{1}\int_{-1}^{1} (x - x^2)\cos n\pi x\,dx$$

$$= \left[(x - x^2)\left(\frac{\sin n\pi x}{n\pi}\right) - (1 - 2x)\left(-\frac{\cos n\pi x}{n^2\pi^2}\right) + (-2)\left(\frac{-\sin n\pi x}{n^3\pi^3}\right)\right]_{-1}^{1}$$

$$= \frac{1}{n\pi}\left[\cancel{(x - x^2)\sin n\pi x} + \frac{(1 - 2x)\cos n\pi x}{n\pi} + \cancel{\frac{2\sin n\pi x}{n^2\pi^2}}\right]_{-1}^{1}$$

$$= \frac{1}{n^2\pi^2}\left[(1 - 2x)\cos n\pi x\right]_{-1}^{1}$$

$$= \frac{1}{n^2\pi^2}\left[(-1)(-1)^n - (3)(-1)^n\right]$$

$$a_n = \frac{-4(-1)^n}{n^2\pi^2} = \frac{4(-1)^{n+1}}{n^2\pi^2}$$

$$b_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin\left(\frac{n\pi x}{l}\right)dx$$

$$= \frac{1}{1}\int_{-1}^{1}(x-x^2)\sin n\pi x \, dx$$

$$= \left[(x-x^2)\left(\frac{-\cos n\pi x}{n\pi}\right) - (1-2x)\left(\frac{-\sin n\pi x}{n^2\pi^2}\right) + (-2)\left(\frac{\cos n\pi x}{n^3\pi^3}\right)\right]_{-1}^{1}$$

$$= \frac{1}{n\pi}\left[-(x-x^2)\cos n\pi x + \frac{(1-2x)\sin n\pi x}{n\pi} - \frac{2\cos n\pi x}{n^2\pi^2}\right]_{-1}^{1}$$

$$= \frac{1}{n\pi}\left[0 - \frac{2(-1)^n}{n^2\pi^2} - 2(-1)^n + \frac{2(-1)^n}{n^2\pi^2}\right]$$

$$b_n = \frac{2(-1)^{n+1}}{n\pi} \quad //$$

Substituting a_0, a_n and b_n in eq ①

$$f(x) = \frac{-1}{3} + \sum_{n=1}^{\infty}\frac{4(-1)^{n+1}}{n^2\pi^2}\cos n\pi x + \sum_{n=1}^{\infty}\frac{2(-1)^{n+1}}{n\pi}\sin n\pi x \quad //$$

Q2: $f(x) = e^{-x}$ in $-l \le x \le l$

sol: The Fourier series is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right) \quad \text{——} ①$$

$$a_0 = \frac{1}{l}\int_{-l}^{l} f(x)\,dx$$

$$= \frac{1}{l}\int_{-l}^{l} e^{-x}\,dx$$

$$= \frac{1}{l}\left[-e^{-x}\right]_{-l}^{l}$$

$$= \frac{1}{l}\left[-e^{-l} + e^{l}\right]$$

$$\frac{a_0}{2} = \frac{\sinh l}{l} \quad //$$

$$a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\left(\frac{n\pi x}{l}\right)dx$$

$$= \frac{1}{l}\int_{-l}^{l} e^{-x}\cos\left(\frac{n\pi x}{l}\right)dx$$

$$= \frac{1}{l}\left[\frac{e^{-x}}{1+(n\pi/l)^2}\left(-\cos\left(\frac{n\pi x}{l}\right) + \cancel{\frac{n\pi}{l}\sin\left(\frac{n\pi x}{l}\right)}^{\,0}\right)\right]_{-l}^{l}$$

$$= \frac{1}{l}\left[\frac{l^2}{l^2+n^2\pi^2}\left(-e^{-x}\cos\left(\frac{n\pi x}{l}\right)\right|_{-l}^{l}\right]$$

$$= \frac{l}{l^2+n^2\pi^2}\left[-e^{-l}(-1)^n + e^{l}(-1)^n\right]$$

$$= \frac{2l(-1)^n}{l^2+n^2\pi^2}\left[\frac{e^{l}-e^{-l}}{2}\right]$$

$$a_n = 2\sinh l \cdot \frac{l(-1)^n}{l^2+n^2\pi^2}$$

$$b_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin\left(\frac{n\pi x}{l}\right)dx$$

$$= \frac{1}{l}\int_{-l}^{l} e^{-x}\sin\left(\frac{n\pi x}{l}\right)dx$$

$$= \frac{1}{l}\left[\frac{e^{-x}}{1+(n\pi/l)^2}\left(\cancel{-\sin\left(\frac{n\pi x}{l}\right)}^{\,0} - \frac{n\pi}{l}\cos\left(\frac{n\pi x}{l}\right)\right)\right]_{-l}^{l}$$

$$= \frac{n\pi}{l^2}\left[\frac{l^2}{l^2+n^2\pi^2}\left(-e^{-x}\cos\left(\frac{n\pi x}{l}\right)\right)_{-l}^{l}\right]$$

$$= \frac{n\pi}{l^2+n^2\pi^2}\left[-e^{l}(-1)^n + e^{-l}(-1)^n\right]$$

$$= \frac{-2n\pi(-1)^n}{l^2+n^2\pi^2}\left[\frac{e^{l}-e^{-l}}{2}\right]$$

$$b_n = \frac{-2n\pi(-1)^n\sinh l}{l^2+n^2\pi^2}$$

Substituting a_0, a_n and b_n in eq ①

$$f(x) = \frac{\sinh l}{l} + \sum_{n=1}^{\infty}\frac{2l(-1)^n}{l^2+n^2\pi^2}\sinh l\cos\frac{n\pi x}{l} - \sum_{n=1}^{\infty}\frac{2n\pi(-1)^n}{l^2+n^2\pi^2}\sinh l\sin\frac{n\pi x}{l}$$

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Q3: $f(x) = x^2 - 2$ in $(-2 \le x \le 2)$

Sol: The Fourier series is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right)$$

$$a_0 = \frac{1}{l}\int_{-l}^{l} f(x)\,dx$$

$$= \frac{1}{2}\int_{-2}^{2} (x^2-2)\,dx$$

$$= \frac{1}{2}\left[\frac{x^3}{3} - 2x\right]_{-2}^{2}$$

$$= \frac{1}{2}\left[\frac{8}{3} - 4 + \frac{8}{3} - 4\right]$$

$$= \frac{1}{2}\left[\frac{-8}{3}\right]$$

$$\frac{a_0}{2} = \frac{-2}{3}$$

$$a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\left(\frac{n\pi x}{l}\right)dx$$

$$= \frac{1}{2}\int_{-2}^{2} (x^2-2)\cos\left(\frac{n\pi x}{2}\right)dx$$

$$= \frac{1}{2}\left[(x^2-2)\left(\sin\left(\frac{n\pi x}{2}\right)\right)\frac{2}{n\pi} - 2x\left(-\cos\left(\frac{n\pi x}{2}\right)\right)\frac{4}{n^2\pi^2} + 2\left(-\sin\left(\frac{n\pi x}{2}\right)\right.\right.$$

$$= \frac{1}{n\pi}\left[\cancel{(x^2-2)\sin\left(\frac{n\pi x}{2}\right)} + \frac{4x}{n\pi}\cos\left(\frac{n\pi x}{2}\right) - \cancel{\frac{8}{n^2\pi^2}\sin\left(\frac{n\pi x}{2}\right)}\right]_{-2}^{2}$$

$$= \frac{4}{n^2\pi^2}\left[x\cos\left(\frac{n\pi x}{2}\right)\right]_{-2}^{2}$$

$$= \frac{4}{n^2\pi^2}\left[2(-1)^n + 2(-1)^n\right]$$

$$a_n = \frac{16(-1)^n}{n^2\pi^2}$$

$$b_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin\left(\frac{n\pi x}{l}\right)dx$$

$$= \frac{1}{2}\int_{-2}^{2} (x^2-2)\sin\left(\frac{n\pi x}{2}\right)dx$$

$$= \frac{1}{2}\left[(x^2-2)\left(-\cos\left(\frac{n\pi x}{2}\right)\right)\frac{2}{n\pi} - 2x\left(-\sin\left(\frac{n\pi x}{2}\right)\right)\frac{4}{n^2\pi^2} + 2\left(\cos\left(\frac{n\pi x}{2}\right)\right)\frac{8}{n^3\pi^3}\right]_{-2}^{2}$$

$$= \frac{1}{n\pi}\left[-(x^2-2)\cos\left(\frac{n\pi x}{2}\right) + \frac{4x}{n^2\pi^2}\cancel{\sin\left(\frac{n\pi x}{2}\right)} + \frac{8}{n^2\pi^2}\cos\left(\frac{n\pi x}{2}\right)\right]_{-2}^{2}$$

$$= \frac{1}{n\pi}\left[-(2)(-1)^n + \frac{8}{n^2\pi^2}(-1)^n + (2)(-1)^n - \frac{8}{n^2\pi^2}(-1)^n\right]$$

$b_n = 0$

Substituting a_0, a_n and b_n

$$f(x) = \frac{-2}{3} + \sum_{n=1}^{\infty}\frac{16(-1)^n}{n^2\pi^2}\cos\left(\frac{n\pi x}{2}\right) \;//$$

NOTE:

- It is also possible to deduce a particular series from the Fourier series of the given function $f(x)$ in a given interval. We have to substitute a suitable value for x in a given interval. Normally it takes the value to be either of the end points or the middle point. The resulting series will be equal to the value discussed in convergence.
- If $f(x)$ is discontinous at x, then the series converges to $\frac{1}{2}\left[f(x^+) + f(x^-)\right]$, where $f(x^+)$ and $f(x^-)$ are respectively right hand limit and left hand limit of $f(x)$.

Q1: Obtain the Fourier series for the function

$$f(x) = \begin{cases} -l & \text{in } -l<x<0 \\ x & \text{in } 0<x<l \end{cases}$$

Hence deduce the sum of the reciprocal squares of the odd integers is equal to $\frac{\pi^2}{8}$.

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sol: The Fourier series is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right)$$

$$a_0 = \frac{1}{l}\int_{-l}^{l} f(x)\,dx$$

$$= \frac{1}{l}\int_{-l}^{0} f(x)\,dx + \frac{1}{l}\int_{0}^{l} f(x)\,dx$$

$$= \frac{1}{l}\int_{-l}^{0} -l\,dx + \frac{1}{l}\int_{0}^{l} x\,dx$$

$$= -1\left[x\right|_{-l}^{0} + \frac{1}{l}\left[\frac{x^2}{2}\right|_{0}^{l}$$

$$= -1(0+l) + \frac{1}{l}\left(\frac{l^2}{2}\right)$$

$$= -l + \frac{l}{2}$$

$$\frac{a_0}{2} = \frac{-l}{4}$$

$$a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\left(\frac{n\pi x}{l}\right)dx$$

$$= \frac{1}{l}\left[\int_{-l}^{0} f(x)\cos\left(\frac{n\pi x}{l}\right)dx + \int_{0}^{l} f(x)\cos\left(\frac{n\pi x}{l}\right)dx\right]$$

$$= \frac{1}{l}\left[\int_{-l}^{0} -l\cos\left(\frac{n\pi x}{l}\right)dx + \int_{0}^{l} x\cos\left(\frac{n\pi x}{l}\right)dx\right]$$

$$= \frac{1}{l}\left[-l\left(\cancel{\sin\left(\frac{n\pi x}{l}\right)}\right)\frac{l}{n\pi}\Big|_{-l}^{0} + x\left(\cancel{\sin\left(\frac{n\pi x}{l}\right)}\right)\frac{l}{n\pi} + \left(\cos\left(\frac{n\pi x}{l}\right)\right)\frac{l^2}{n^2\pi^2}\Big|_{0}^{l}\right]$$

$$= \frac{l}{n^2\pi^2}\left[\cos\left(\frac{n\pi x}{l}\right)\right]_{0}^{l}$$

$$= \frac{l}{n^2\pi^2}\left[(-1)^n - 1\right]$$

$$a_n = \frac{[(-1)^n - 1]\,l}{n^2\pi^2}$$

$$b_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin\left(\frac{n\pi x}{l}\right)dx$$

$$= \frac{1}{l}\left[\int_{-l}^{0} f(x)\sin\left(\frac{n\pi x}{l}\right)dx + \int_{0}^{l} f(x)\sin\left(\frac{n\pi x}{l}\right)dx\right]$$

$$= \frac{1}{l}\left[\int_{-l}^{0} -l\sin\left(\frac{n\pi x}{l}\right)dx + \int_{0}^{l} x\sin\left(\frac{n\pi x}{l}\right)dx\right]$$

$$= \frac{1}{l}\left[-l\left(-\cos\left(\frac{n\pi x}{l}\right)\right)\frac{l}{n\pi}\Big|_{-l}^{0} + \left[x\left(-\cos\left(\frac{n\pi x}{l}\right)\right)\frac{l}{n\pi} - \cancel{\left(-\sin\left(\frac{n\pi x}{l}\right)\right)}^{0}\frac{l^2}{n^2\pi^2}\Big|_{0}^{l}\right]\right]$$

$$= \frac{1}{n\pi}\left[l\cos\left(\frac{n\pi x}{l}\right)\Big|_{-l}^{0} - x\cos\left(\frac{n\pi x}{l}\right)\Big|_{0}^{l}\right]$$

$$= \frac{1}{n\pi}\left[l(1) - l(-1)^n - l(-1)^n + 0\right]$$

$$b_n = \frac{l}{n\pi}\left[1-2(-1)^n\right]$$

Substituting a_0, a_n and b_n

$$f(x) = \frac{-l}{4} + \sum_{n=1}^{\infty}\frac{[(-1)^n-1]l}{n^2\pi^2}\cos\left(\frac{n\pi x}{l}\right) + \sum_{n=1}^{\infty}\frac{[1-2(-1)^n]l}{n\pi}\sin\left(\frac{n\pi x}{l}\right)$$

To deduce the given series, in the Fourier series choose x value equal to 0, since the given function is discontinous at $x=0$ the series converges at $x=0$

$$f(0) = \frac{f(x^+)+f(x^-)}{2} = \frac{x-l}{2} = \frac{0-l}{2} = \frac{-l}{2}$$

$$f(0) = \frac{-l}{4} + \frac{l}{\pi^2}\sum_{n=1}^{\infty}\frac{[(-1)^n-1]}{n^2} + \frac{l}{\pi}\sum_{n=1}^{\infty}\frac{[1-2(-1)^n]}{n}(0)$$

$$\frac{-l}{2} = \frac{-l}{4} + \frac{l}{\pi^2}\sum_{n=1}^{\infty}\frac{[(-1)^n-1]}{n^2}$$

But $1-(-1)^n = \begin{cases} 0 & \text{for n is even} \\ 2 & \text{for n is odd} \end{cases}$

$$\frac{-l}{2} = \frac{-l}{4} - \frac{l}{\pi^2}\sum_{n=1,3,4}\frac{2}{n^2}$$

$$\frac{l}{4} = \frac{2l}{\pi^2}\left[\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\right]$$

$$\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots \quad //$$ Hence Proved

* Even and Odd Function : (in interval $[-l, l]$)

If $f(-x) = f(x)$ is an even function such as x^2, cos
and if $f(-x) = -f(x)$ it is an odd function such as x^3

Ex : $f(x) = x^2$ in $-1 < x < 1$
replace x by $(-x)$

$$f(-x) = (-x)^2$$
$$f(-x) = x^2$$
$$f(-x) = f(x)$$

Hence x^2 is an even function

NOTE:

- Even function x Even function, Odd function x Odd fuction } Even function
- Even function x Odd function, Odd function x Even function } Odd function
- $$\int_{-a}^{a} f(x)\,dx = \begin{cases} 2\int_0^a f(x)\,dx & \text{when } f(x) \text{ is even function} \\ 0 & \text{when } f(x) \text{ is odd function} \end{cases}$$

We know that

$$a_0 = \frac{1}{l}\int_{-l}^{l} f(x)\,dx$$

$$a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\left(\frac{n\pi x}{l}\right)dx$$

$$b_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin\left(\frac{n\pi x}{l}\right)dx$$

CASE 1: When $f(x)$ is an even function

$$a_0 = \frac{2}{l}\int_0^l f(x)\,dx$$

$$a_n = \frac{2}{l}\int_0^l f(x)\cos\left(\frac{n\pi x}{l}\right)dx$$

$$b_n = 0$$

CASE 2: When $f(x)$ is an odd function

$$a_0 = 0$$

$$a_n = 0$$

$$b_n = \frac{2}{l}\int_0^l f(x)\sin\left(\frac{n\pi x}{l}\right)dx$$

NOTE:

- If $f(x)$ is a discontinous function such that

$$f(x) = \begin{cases} \phi(x) & \text{in } -l<x<0 \\ \psi(x) & \text{in } 0<x<l \end{cases}$$

then if $\phi(-x) = \psi(x)$ then it is even function and if $\phi(-x) = -\psi(x)$ then it is odd function.

Q1: Obtain the Fourier series for the function $f(x) = x^2$ in $-l<x<l$. Hence deduce $\frac{\pi^2}{12} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \dots$

Sol: $f(x) = x^2$

Replacing x by $-x$

$f(-x) = (-x)^2$

$f(-x) = x^2$

$f(-x) = f(x)$ $\therefore$ The given function is even function

Hence:

$$a_0 = \frac{2}{l}\int_0^l f(x)\,dx$$

$$= \frac{2}{l}\int_0^l x^2\,dx$$

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$$a_0 = \frac{2}{l}\left[\frac{x^3}{3}\right]_0^l$$

$$= \frac{2}{l}\left[\frac{l^3}{3}\right]$$

$$a_0 = \frac{2l^2}{3} \Rightarrow \frac{a_0}{2} = \frac{l^2}{3}$$

$$a_n = \frac{2}{l}\int_0^l f(x)\cos\left(\frac{n\pi x}{l}\right)dx$$

$$= \frac{2}{l}\int_0^l x^2\cos\left(\frac{n\pi x}{l}\right)dx$$

$$= \frac{2}{l}\left[x^2\left(\sin\frac{n\pi x}{l}\right)\frac{l}{n\pi} - 2x\left(-\cos\frac{n\pi x}{l}\right)\frac{l^2}{n^2\pi^2} + 2\left(-\sin\right.\right.$$

$$= \frac{2}{l}\left[\frac{2xl^2}{n^2\pi^2}\cos\frac{n\pi x}{l}\right]_0^l$$

$$= \frac{4l}{n^2\pi^2}\left[l\cos n\pi - 0\right]$$

$$a_n = \frac{4l^2(-1)^n}{n^2\pi^2}$$

$$b_n = 0$$

The Fourier series is given by

$$f(x) = \frac{l^2}{3} + \sum_{n=1}^{\infty}\frac{4l^2(-1)^n}{n^2\pi^2}\cos\left(\frac{n\pi x}{l}\right)$$

Let $x = 0$

$$f(0) = 0$$

$$f(0) = \frac{l^3}{3} + \frac{4l^2}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\cos 0$$

But $(-1)^n = \begin{cases} 1 & \text{for } n \text{ is even} \\ -1 & \text{for } n \text{ is odd}\end{cases}$

$$\therefore 0 = \frac{l^3}{3} + \frac{4l^2}{\pi^2}\left[\sum_{n=1,3,5}^{\infty}\frac{-1}{n^2} + \sum_{n=2,4,6}^{\infty}\frac{1}{n^2}\right]$$

$$-\frac{\cancel{l^2}}{3} = \frac{4\cancel{l^2}}{\pi}\left[-\frac{1}{1^2} - \frac{1}{3^2} + \frac{1}{2^2} + \frac{1}{4^2} \cdots\right]$$

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$$\frac{-\pi}{12} = \frac{-1}{1^2} + \frac{1}{2^2} - \frac{1}{3^2} + \frac{1}{4^2} - \dots$$

$$\therefore \frac{\pi}{12} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \dots$$

Q2: $f(x) = |x|$ in $-l < x < l$, deduce $\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots$

sol: $f(x) = |x|$

replacing x by $-x$

$f(-x) = |-x|$

$f(-x) = |x|$

$f(-x) = f(x)$ $\therefore$ The given function is even function.

Hence

$$a_0 = \frac{2}{l}\int_0^l f(x)\,dx$$

considering $f(x) = \begin{cases} -x & \text{in } -l<x<0 \\ x & \text{in } 0<x<l \end{cases}$

$$= \frac{2}{l}\int_0^l x\,dx$$

$$= \frac{\cancel{2}}{l}\left[\frac{x^2}{\cancel{2}}\right]_0^l$$

$$= \frac{l^2}{l}$$

$$a_0 = l \Rightarrow \frac{a_0}{2} = \frac{l}{2}$$

$$a_n = \frac{2}{l}\int_0^l f(x)\cos\left(\frac{n\pi x}{l}\right)dx$$

$$= \frac{2}{l}\int_0^l x\cos\left(\frac{n\pi x}{l}\right)dx$$

$$= \frac{2}{l}\left[x\left(\cancel{\sin\frac{n\pi x}{l}}\right)^0\frac{l}{n\pi} - \left(-\cos\frac{n\pi x}{l}\right)\frac{l^2}{n^2\pi^2}\right]_0^l$$

$$= \frac{2l}{n^2\pi^2}\left[\cos\frac{n\pi x}{l}\right]_0^l$$

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$$= \frac{2l}{n^2\pi^2}\left[\cos n\pi - \cos 0\right]$$

$$a_n = \frac{2l\left[(-1)^n - 1\right]}{n^2\pi^2}$$ //

$b_n = 0$ //

Therefore the Fourier series is given by

$$f(x) = \frac{l}{2} + \sum_{n=1}^{\infty} \frac{2l\left[(-1)^n-1\right]}{n^2\pi^2}\cos\left(\frac{n\pi x}{l}\right)$$

Let $x = 0$

$f(0) = 0$

$$f(0) = \frac{l}{2} + \frac{2l}{\pi^2}\sum_{n=1}^{\infty}\frac{\left[(-1)^n-1\right]}{n^2}$$

But $$\left[(-1)^n - 1\right] = \begin{cases} -2 & \text{for } n \text{ is odd} \\ 0 & \text{for } n \text{ is even} \end{cases}$$

$$\therefore 0 = \frac{l}{2} + \frac{2l}{\pi^2}\left[\sum_{n=1,3,5\ldots}^{\infty}\frac{-2}{n^2}\right]$$

$$\frac{l}{2} = \frac{4l}{\pi^2}\left[\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\right]$$

$$\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots$$ //

Q3: $$f(x) = \begin{cases} -k & \text{in } -2<x<0 \\ k & \text{in } 0<x<2 \end{cases}$$ Deduce: $\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} + \cdots$

Sol: consider

$\phi(x) = -k$

replacing x by $-x$

$\phi(-x) = -k$

$\phi(-x) = -\psi(x)$ $\therefore$ The given function is odd function

Hence

$a_0 = 0$ //

$a_n = 0$ //

$$b_n = \frac{2}{l}\int_0^l f(x)\sin\left(\frac{n\pi x}{l}\right)dx$$

$$= \frac{2}{2}\int_0^2 k\sin\left(\frac{n\pi x}{2}\right)dx$$

$$= k\left[\left(-\cos\frac{n\pi x}{2}\right)\cdot\frac{2}{n\pi}\right]_0^2$$

$$= \frac{-2k}{n\pi}\left[\cos n\pi - \cos 0\right]$$

$$b_n = \frac{-2k}{n\pi}\left[(-1)^n - 1\right]$$

Therefore the Fourier series is

$$f(x) = \sum_{n=1}^{\infty} \frac{-2k}{n\pi}\left[(-1)^n - 1\right]\sin\left(\frac{n\pi x}{2}\right)$$

Let $x = 1$

But $[(-1)^n - 1] = \begin{cases} -2 & \text{for } n \text{ is odd} \\ 0 & \text{for } n \text{ is even} \end{cases}$

$f(1) = k$

$$f(1) = \frac{-2k}{\pi}\sum_{n=1,3,5,\ldots}^{\infty} \frac{-2}{n}\sin\frac{n\pi}{2}$$

$$\therefore k = \frac{+4k}{\pi}\left[\frac{1}{1}\sin\frac{\pi}{2} + \frac{1}{3}\sin\frac{3\pi}{2} + \frac{1}{5}\sin\frac{5\pi}{2} + \cdots\right]$$

$$\frac{\pi}{4} = 1 + \frac{1}{3}(-1) + \frac{1}{5}(1) + \cdots$$

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} \cdots\cdots$$

Q4: $f(x) = \begin{cases} 1 + 4x/3 & \text{in } -3/2 < x < 0 \\ 1 - 4x/3 & \text{in } 0 < x < -3/2 \end{cases}$ Deduce: $\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots$

Sol: consider

$\phi(x) = 1 + 4x/3$

replacing x by $-x$

$\phi(-x) = 1 - 4x/3$

$\phi(-x) = \psi(x)$ $\therefore$ The given function is even func

Hence

$$a_0 = \frac{2}{l}\int_0^l f(x)\,dx$$

$$= \frac{2}{3/2}\int_0^{3/2}\left(1 - \frac{4x}{3}\right)dx$$

$$= \frac{4}{3}\left[x - \frac{2x^2}{3}\right|_0^{3/2}$$

$$= \frac{4}{3}\left[\frac{3}{2} - \frac{3}{2}\right]$$

$$a_0 = 0 //$$

$$a_n = \frac{2}{l}\int_0^l f(x)\cos\left(\frac{n\pi x}{l}\right)dx$$

$$= \frac{2}{3/2}\int_0^{3/2}\left(1 - \frac{4x}{3}\right)\cos\left(\frac{3n\pi x}{2}\right)\,dx$$

$$= \frac{4}{3}\left[\left(1 - \frac{4x}{3}\right)\left(\sin\frac{2n\pi x}{3}\right)\frac{3}{2n\pi} + \frac{4}{3}\left(-\cos\frac{2n\pi x}{3}\right)\frac{9}{4n^2\pi^2}\right.$$

$$= \frac{-4}{n^2\pi^2}\left[\cos\frac{2n\pi x}{3}\right]_0^{3/2}$$

$$= \frac{-4}{n^2\pi^2}\left[\cos n\pi - \cos 0\right]$$

$$a_n = \frac{4}{n^2\pi^2}\left[1 - (-1)^n\right] //$$

$$b_n = 0 //$$

Therefore the Fourier series is

$$f(x) = \sum_{n=1}^{\infty} \frac{4}{n^2\pi^2}\left[1-(-1)^n\right]\cos\frac{2n\pi x}{3}$$

But $[1-(-1)^n] = \begin{cases} 0 & \text{for } n \text{ is even} \\ 2 & \text{for } n \text{ is odd} \end{cases}$

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$$f(x) = \frac{4}{\pi^2} \sum_{n=1,3,5}^{\infty} \frac{2}{n^2} \cos \frac{2n\pi x}{2}$$

$$f(0) = 1$$

$$f(0) = \frac{4}{\pi^2} \sum_{n=1,3,5}^{\infty} \frac{2}{n^2}$$

$$1 = \frac{8}{\pi^2} \left[\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots \right]$$

$$\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots$$

★ <u>Fourier series in $[0, 2l]$</u>

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right)$$

where $a_0 = \frac{1}{l} \int_0^{2l} f(x)\,dx$

$$a_n = \frac{1}{l} \int_0^{2l} f(x) \cos\left(\frac{n\pi x}{l}\right) dx$$

$$b_n = \frac{1}{l} \int_0^{2l} f(x) \sin\left(\frac{n\pi x}{l}\right) dx$$

Q1: $f(x) = 2x - x^2$ in $0 < x < 3$

<u>Sol:</u> The Fourier series is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right) \quad \text{—— (1)}$$

where $a_0 = \frac{1}{l} \int_0^{2l} f(x)\,dx$

$$a_0 = \frac{1}{3/2} \int_0^3 (2x - x^2)\,dx$$

$$= \frac{2}{3} \left[x^2 - \frac{x^3}{3} \right]_0^3$$

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$$a_0 = 0$$

$$a_n = \frac{1}{l}\int_0^{2l} f(x)\cos\left(\frac{n\pi x}{l}\right)dx$$

$$= \frac{1}{3/2}\int_0^3 (2x-x^2)\cos\left(\frac{2n\pi x}{3}\right)dx$$

$$= \frac{2}{3}\left[(2x-x^2)\left(\sin\frac{2n\pi x}{3}\right)\frac{3}{2n\pi} - (2-2x)\left(-\cos\frac{2n\pi x}{3}\right)\frac{9}{4n^2\pi^2}\right.$$

$$\left. + (-2)\left(-\sin\frac{2n\pi x}{3}\right)\frac{27}{8n^2\pi^2}\right]_0^3$$

$$= \frac{3}{2n^2\pi^2}\left[(1-x)\cos\frac{2n\pi x}{3}\right]_0^3$$

$$= \frac{3}{n^2\pi^2}\left[(1-3)\cos 2n\pi - (1)\cos 0\right]$$

$$= \frac{3}{n^2\pi^2}\left[-2(1)-1\right]$$

$$a_n = \frac{-9}{n^2\pi^2}$$

$$b_n = \frac{1}{l}\int_0^{2l} f(x)\sin\left(\frac{n\pi x}{l}\right)dx$$

$$= \frac{1}{3/2}\int_0^3 (2x-x^2)\sin\left(\frac{2n\pi x}{3}\right)dx$$

$$= \frac{2}{3}\left[(2x-x^2)\left(-\cos\frac{2n\pi x}{3}\right)\frac{3}{2n\pi} - (2-2x)\left(-\sin\frac{2n\pi x}{3}\right)\frac{9}{4n^2\pi^2}\right.$$

$$\left. + (-2)\left(\cos\frac{2n\pi x}{3}\right)\frac{27}{8n^3\pi^3}\right]_0^3$$

$$= \frac{\cancel{2}}{\cancel{3}}\left[\frac{\cancel{3}}{2n\pi}\left[(x^2-2x)\cos\frac{2n\pi x}{3} - \frac{9}{4n^2\pi^2}\cos\frac{2n\pi x}{3}\right|_0^3\right.$$

$$= \frac{1}{n\pi}\left[(9-6)\cos 2n\pi - \frac{9}{4n^2\pi^2}\cos 2n\pi - 0 + \frac{9}{4n^2\pi^2}\cos 0\right]$$

$$= \frac{1}{n\pi}\left[3(1)\right]$$

$$b_n = \frac{3}{n\pi}$$

Substituting a_0, a_n and b_n in eq ①

$$f(x) = \sum_{n=1}^{\infty} \frac{-9}{n^2\pi^2}\cos\left(\frac{2n\pi x}{3}\right) + \sum_{n=1}^{\infty} \frac{3}{n\pi}\sin\left(\frac{2n\pi x}{3}\right)$$

★ **Even and Odd Functions in $(0, 2l)$:**

A function is said to be an even function in $(0, 2l)$ if $f(2l-x) = f(x)$

A function is said to be an odd function in $(0, 2l)$ if $f(2l-x) = f(x)$

Q1: $f(x) = 2x - x^2$ in $0 \le x \le 2$

Sol: Replacing x by $2l - x$ where $l = 1$.

$f(2-x) = 2(2-x) - (2-x)^2$

$f(2-x) = 4 - 2x - x^2 - 4 + 4x$

$f(2-x) = -x^2 + 2x$

$f(2-x) = f(x)$ ∴ the given function is an even function

Hence

$$a_0 = \frac{2}{l}\int_0^l f(x)\,dx$$

$$= \frac{2}{1}\int_0^1 (2x - x^2)\,dx$$

$$= 2\left[x^2 - \frac{x^3}{3}\right]_0^1$$

$$= 2\left[1 - \frac{1}{3}\right]$$

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$\frac{a_0}{2} = \frac{2}{3}$

$a_n = \frac{2}{l} \int_0^l f(x) \cos\left(\frac{n\pi x}{l}\right) dx$

$= \frac{2}{1} \int_0^1 (2x - x^2) \cos n\pi x \, dx$

$= 2\left[(2x-x^2)\left(\frac{\sin n\pi x}{n\pi}\right) - (2-2x)\left(\frac{-\cos n\pi x}{n^2\pi^2}\right) + (-2)\left(\cancel{\frac{-\sin n\pi x}{n^3\pi^3}}\right)\right.$

$= \frac{4}{n^2\pi^2}\left[(1-x)\cos n\pi x\right|_0^1$

$= \frac{4}{n^2\pi^2}\left[0 - (1)\cos 0\right]$

$a_n = \frac{-4}{n^2\pi^2}$

$b_n = 0$

Hence the fourier series is

$f(x) = \frac{2}{3} - \sum_{n=1}^{\infty} \frac{4}{n^2\pi^2} \cos n\pi x$

Q2: $f(x) = \frac{l-x}{2}$ in $0 < x < 2l$

Hence deduce $\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \cdots$

Sol: The fourier series is given by

$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right)$ —(1)

Replace x by $2l - x$

$f(2l-x) = \frac{l-(2l-x)}{2}$

$f(2l-x) = \frac{l-2l+x}{2}$

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$f(2l-x) = \frac{-l+x}{2}$

$f(2l-x) = -f(x)$ ∴ the given function is odd function

Hence

$a_0 = 0$ //

$a_n = 0$ //

$b_n = \frac{2}{l}\int_0^l f(x)\sin\left(\frac{n\pi x}{l}\right)dx$

$= \frac{2}{l}\int_0^l \frac{l-x}{2}\sin\left(\frac{n\pi x}{l}\right)dx$

$= \frac{2}{l}\left[\frac{l-x}{2}\left(-\cos\frac{n\pi x}{l}\right)\frac{l}{n\pi} + \frac{1}{2}\left(-\cancel{\sin\frac{n\pi x}{l}}\right)\frac{l^2}{n^2\pi^2}\right|_0^l$

$= \frac{1}{l}\left[\frac{-l}{n\pi}(l-x)\cos\frac{n\pi x}{l}\Big|_0^l\right]$

$= \frac{-1}{n\pi}[(0)\cos n\pi - l\cos 0]$

$b_n = \frac{-1}{n\pi}[0 - l]$ ∴ $b_n = \frac{l}{n\pi}$ //

Substituting a_0, a_n and b_n in eq ①

$f(x) = \sum_{n=1}^{\infty}\frac{l}{n\pi}\sin\frac{n\pi x}{l}$

$f(x) = \frac{l}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin\frac{n\pi x}{l}$

Let $x = l/2$

$f(l/2) = \frac{l - l/2}{2} = \frac{l}{4}$

$f(l/2) = \frac{l}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin\frac{n\pi}{2}$

$\frac{\cancel{l}}{4} = \frac{\cancel{l}}{\pi}\sum_{n=1}^{\infty}\frac{1}{n}\sin\frac{n\pi}{2}$

$\frac{\pi}{4} = \left[\frac{1}{1}\sin\frac{\pi}{2} + \cancel{\frac{1}{2}\sin\pi} + \frac{1}{3}\sin\frac{3\pi}{2} + \cancel{\frac{1}{4}\sin 2\pi} + \cdots\right]$

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \cdots$$

Q3: $f(x) = \left(\frac{1-x}{2}\right)^2$ in $[0, 2]$

Deduce : $\frac{\pi^2}{6} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots$

Sol: The Fourier series is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right) \quad \text{—①}$$

Given $f(x) = \left(\frac{1-x}{2}\right)^2$

Replacing x by $(2l - x)$ where $l = 1$

$$f(2-x) = \left(\frac{1-(2-x)}{2}\right)^2$$

$$f(2-x) = \left(\frac{1-2+x}{2}\right)^2$$

$$f(2-x) = \left(\frac{-1+x}{2}\right)^2$$

$f(2-x) = f(x)$ $\therefore$ The given function is even function

Hence

$$a_0 = \frac{2}{l}\int_0^l f(x)\,dx$$

$$= 2\int_0^1 \left(\frac{1-x}{2}\right)^2 dx$$

$$= 2\int_0^1 \left(\frac{1}{4} + \frac{x^2}{4} - \frac{x}{2}\right) dx$$

$$= 2\left[\frac{x}{4} + \frac{x^3}{12} - \frac{x^2}{4}\right]_0^1$$

$$= \frac{2}{4}\left[1 + \frac{1}{3} - 1\right]$$

$$\frac{a_0}{2} = \frac{1}{12}$$

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$$a_n = \frac{2}{l}\int_0^l f(x)\cos\left(\frac{n\pi x}{l}\right)dx$$

$$= 2\int_0^1 \left(\frac{1-x}{2}\right)^2 \cos n\pi x \, dx$$

$$= \frac{1}{2}\left[(1-x)^2\left(\frac{\sin n\pi x}{n\pi}\right) - 2(1-x)(-1)\left(-\frac{\cos n\pi x}{n^2\pi^2}\right) + 2\left(-\frac{\sin n\pi x}{n^3\pi^3}\right)\right]_0^1$$

$$= \frac{1}{2}\left[\frac{-2}{n^2\pi^2}\left[(1-x)\cos n\pi x\Big|_0^1\right]\right]$$

$$= \frac{-1}{n^2\pi^2}\left[0-(1)\cos 0\right]$$

$$a_n = \frac{1}{n^2\pi^2}$$

$$b_n = 0$$

substituting a_0, a_n and b_n in eq (1)

$$f(x) = \frac{1}{12} + \sum_{n=1}^{\infty}\frac{1}{n^2\pi^2}\cos n\pi x$$

$$f(x) = \frac{1}{12} + \frac{1}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\cos n\pi x$$

Let $x = 0$

$$f(0) = \frac{1}{4}$$

$$f(0) = \frac{1}{12} + \frac{1}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}(1)$$

$$\therefore \frac{1}{4} = \frac{1}{12} + \frac{1}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}$$

$$\frac{1}{4} - \frac{1}{12} = \frac{1}{\pi^2}\left[\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots\right]$$

$$\frac{\pi^2}{6} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots$$

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Q4: $f(x) = \begin{cases} \pi x & \text{in } 0 \le x \le 1 \\ \pi(2-x) & \text{in } 1 \le x \le 2 \end{cases}$

Hence deduce $\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots$

or $\frac{\pi^2}{8} = \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2}$

Sol: The Fourier series is given by

$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right)$ —

Let $\phi(x) = \pi x$ and $\psi(x) = \pi(2-x)$

Replacing x by $2l - x$ in $\phi(x)$ where $l = 1$

$\phi(2-x) = \pi(2-x)$

$\phi(2-x) = \psi(x)$ $\therefore$ The given function is even func

Hence

$a_0 = \frac{2}{l}\int_0^l f(x)\,dx$

$= \frac{2}{1}\int_0^1 \pi x\,dx$

$= 2\pi\left[\frac{x^2}{\cancel{2}}\right]_0^1$

$a_0 = \pi \Rightarrow \frac{a_0}{2} = \frac{\pi}{2}$ //

$a_n = \frac{2}{l}\int_0^l f(x)\cos\left(\frac{n\pi x}{l}\right)dx$

$= \frac{2}{1}\int_0^1 \pi x \cos n\pi x\,dx$

$= 2\pi\left[x\left(\cancel{\frac{\sin n\pi x}{n\pi}}\right) - \left(\frac{-\cos n\pi x}{n^2\pi^2}\right)\right]_0^1$

$= \frac{2\pi}{n^2\pi^2}\left[\cos n\pi x\right]_0^1$

$= \frac{2}{n^2\pi}\left[\cos n\pi - \cos 0\right]$

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$$a_n = \frac{2[(-1)^n - 1]}{n^2\pi}$$

$b_n = 0$

Substituting a_0, a_n and b_n in eq ①

$$f(x) = \frac{\pi}{2} + \sum_{n=1}^{\infty} 2\frac{[(-1)^n - 1]}{n^2\pi} \cos n\pi x$$

but $[(-1)^n - 1] = \begin{cases} -2 & \text{for } n \text{ is odd} \\ 0 & \text{for } n \text{ is even} \end{cases}$

$$f(x) = \frac{\pi}{2} + \frac{2}{\pi} \sum_{n=1,3}^{\infty} \frac{-2}{n^2} \cos n\pi x$$

Let $x = 0$

$f(0) = 0$

$$f(0) = \frac{\pi}{2} - \frac{4}{\pi} \sum_{n=1,3}^{\infty} \frac{1}{n^2}$$

$$\therefore 0 = \frac{\pi}{2} - \frac{4}{\pi} \sum_{n=1,3}^{\infty} \frac{1}{n^2}$$

$$\frac{\pi}{2} = \frac{4}{\pi}\left[\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\right]$$

$$\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots$$

or $\frac{\pi^2}{8} = \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2}$

Q5: $f(x) = \begin{cases} 2-x & \text{in } 0 \le x \le 4 \\ 6-x & \text{in } 4 \le x \le 8 \end{cases}$

Hence deduce $\frac{\pi^2}{8} = \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2}$

sol: The Fourier series is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right) \quad \text{—— ①}$$

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Replacing x by $2l-x$ in $\phi(x)$ where [illegible]

$\phi(8-x) = 2-(8-x)$

$\phi(8-x) = 2-8+x$

$\phi(8-x) = -6+x$

$\phi(8-x) = -\psi(x)$ ∴ It is an odd function.

Hence

$a_0 = 0$ //

$a_n = 0$ //

$$b_n = \frac{2}{l}\int_0^l f(x)\sin\left(\frac{n\pi x}{l}\right)dx$$

$$= \frac{2}{4}\int_0^4 (2-x)\sin\left(\frac{n\pi x}{4}\right)dx$$

$$= \frac{1}{2}\left[(2-x)\left(-\cos\frac{n\pi x}{4}\right)\frac{4}{n\pi} - (-1)\left(-\sin\frac{n\pi x}{4}\right)\frac{16}{n^2\pi^2}\right]$$

$$= \frac{1}{2}\left[\frac{-4}{n\pi}\left[(2-x)\cos\frac{n\pi x}{4}\right]_0^4\right]$$

$$= \frac{-2}{n\pi}\left[(-2)\cos n\pi - 2\cos 0\right]$$

$$= \frac{-2}{n\pi}\left[-2(-1)^n - 2(1)\right]$$

$$b_n = \frac{4}{n\pi}\left[(-1)^n+1\right]$$ //

Substituting a_0, a_n and b_n in eq ①

$$f(x) = \sum_{n=1}^{\infty}\frac{4}{n\pi}\left[(-1)^n+1\right]\sin\left(\frac{n\pi x}{4}\right)$$

but $[(-1)^n+1] = \begin{cases} 0 & \text{for } n \text{ is odd} \\ 2 & \text{for } n \text{ is even}\end{cases}$

$$f(x) = \sum_{n=2,4}^{\infty}\frac{4}{n\pi}(2)\sin\left(\frac{n\pi x}{4}\right)$$

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Q6: $f(x) = \begin{cases} 2-x & \text{in } 0 \le x \le 4 \\ x-6 & \text{in } 4 \le x \le 8 \end{cases}$

Deduce: $\frac{\pi^2}{8} = \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2}$

sol: The Fourier series is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right) \quad \text{---} (1)$$

Here $l = 4$

Let $\phi(x) = 2-x$ and $\psi(x) = x-6$

Replacing x by $2l-x$ in $\phi(x)$

$\phi(8-x) = 2-(8-x)$

$\phi(8-x) = 2-8+x$

$\phi(8-x) = x-6$

$\phi(8-x) = \psi(x)$ $\therefore$ The given function is an even function

Hence

$$a_0 = \frac{2}{l}\int_0^l f(x)\,dx$$

$$a_0 = \frac{2}{4}\int_0^4 (2-x)\,dx$$

$$= \frac{1}{2}\left[2x - \frac{x^2}{2}\right|_0^4$$

$$= \frac{1}{2}\left[8-8\right]$$

$$a_0 = 0$$

$$a_n = \frac{2}{l}\int_0^l f(x)\cos\left(\frac{n\pi x}{l}\right)dx$$

$$= \frac{2}{4}\int_0^4 (2-x)\cos\frac{n\pi x}{4}\,dx$$

$$= \frac{1}{2}\left[(2-x)\left(\cancel{\sin\frac{n\pi x}{4}}\right)\frac{4}{n\pi} + \left(-\cos\frac{n\pi x}{4}\right)\frac{16}{n^2\pi^2}\right|_0^4$$

$$= \frac{1}{2}\left[\frac{-16}{n^2\pi^2}\left(\cos\frac{n\pi x}{4}\right|_0^4\right]$$

$$= \frac{-8}{n^2\pi^2}\left[\cos n\pi - \cos 0\right]$$

$$= \frac{-8}{n^2\pi^2}\left[(-1)^n - 1\right]$$

$$a_n = \frac{8[1-(-1)^n]}{n^2\pi^2}$$

$$b_n = 0$$

Substituting a_0, a_n and b_n in eq ①

$$f(x) = \sum_{n=1}^{\infty} \frac{8}{n^2\pi^2}[1-(-1)^n]\cos\frac{n\pi x}{4}$$

but $[1-(-1)^n] = \begin{cases} 2 & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even} \end{cases}$

$$f(x) = \frac{8}{\pi^2}\sum_{n=1,3}^{\infty}\frac{2}{n^2}\cos\frac{n\pi x}{4}$$

Let $x=0$

$f(0) = 2$

$$f(0) = \frac{16}{\pi^2}\sum_{n=1,3}^{\infty}\frac{1}{n^2}$$

$$\therefore 2 = \frac{16}{\pi^2}\sum_{n=1,3}^{\infty}\frac{1}{n^2}$$

$$\frac{\pi^2}{8} = \frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\dots$$

Since 1, 3, 5 .. can be denoted as $(2n-1)$

$$\frac{\pi^2}{8} = \sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}$$

Q7: $f(x) = \begin{cases} 1-x & \text{in } 0<x<1 \\ 0 & \text{in } 1<x<2 \end{cases}$

Hence deduce i) $\frac{\pi^2}{8} = \sum_{n=1}^{\infty}\frac{1}{(2n-1)^2}$; ii) $\frac{\pi}{4} = 1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\dots$

Sol: The Fourier series is given by

$$f(x) = \frac{a_0}{2}+\sum_{n=1}^{\infty}a_n\cos\left(\frac{n\pi x}{l}\right)+\sum_{n=1}^{\infty}b_n\sin\left(\frac{n\pi x}{l}\right) \quad \text{---} (1)$$

Here $f(x)$ is neither an even function nor an odd function.

$$a_0 = \frac{1}{l}\int_0^{2l}f(x)\,dx$$

$$= \frac{1}{1}\int_0^{2}f(x)\,dx$$

$$= \int_0^1 f(x)\,dx + \int_1^2 f(x)\,dx$$

$$= \int_0^1 (1-x)\,dx + 0$$

$$= x-\frac{x^2}{2}\Big|_0^1$$

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$$\frac{a_0}{2} = \frac{1}{4}$$

$$a_n = \frac{1}{l}\int_0^{2l} f(x)\cos\left(\frac{n\pi x}{l}\right)dx$$

$$= \frac{1}{1}\int_0^2 f(x)\cos n\pi x\, dx$$

$$= \int_0^1 f(x)\cos n\pi x\, dx + \int_1^2 f(x)\cos n\pi x\, dx$$

$$= \int_0^1 (1-x)\cos n\pi x\, dx + 0$$

$$= \left[(1-x)\left(\frac{\sin n\pi x}{n\pi}\right) - (-1)\left(\frac{-\cos n\pi x}{n^2\pi^2}\right)\right]_0^1$$

$$= \frac{-1}{n^2\pi^2}\left[\cos n\pi x\right]_0^1$$

$$= \frac{-1}{n^2\pi^2}\left[\cos n\pi - \cos 0\right]$$

$$= \frac{-1}{n^2\pi^2}\left[(-1)^n - 1\right]$$

$$a_n = \frac{[1-(-1)^n]}{n^2\pi^2}$$

$$b_n = \frac{1}{l}\int_0^{2l} f(x)\sin\left(\frac{n\pi x}{l}\right)dx$$

$$= \frac{1}{1}\int_0^2 f(x)\sin n\pi x\, dx$$

$$= \int_0^1 f(x)\sin n\pi x\, dx + \int_1^2 f(x)\sin n\pi x\, dx$$

$$= \int_0^1 (1-x)\sin n\pi x\, dx + 0$$

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$$= \left[(1-x)\left(-\frac{\cos n\pi x}{n\pi}\right) - (-1)\left(-\frac{\sin n\pi x}{n^2\pi^2}\right)\right]_0^1$$

$$= \left[\frac{x-1}{n\pi} \cos n\pi x \right]_0^1$$

$$= \left[0 + \frac{\cos 0}{n\pi} \right]$$

$$b_n = \frac{1}{n\pi}$$

Substituting a_0, a_n and b_n in eq ①

$$f(x) = \frac{1}{4} + \sum_{n=1}^{\infty} \frac{[1-(-1)^n]}{n^2\pi^2} \cos n\pi x + \sum_{n=1}^{\infty} \frac{1}{n\pi} \sin n\pi x$$

but $[1-(-1)^n] = \begin{cases} 2 & \text{for } n \text{ is odd} \\ 0 & \text{for } n \text{ is even} \end{cases}$

$$f(x) = \frac{1}{4} + \sum_{n=1,3}^{\infty} \frac{2}{n^2\pi^2} \cos n\pi x + \sum_{n=1}^{\infty} \frac{1}{n\pi} \sin n\pi x \quad \text{— ②}$$

i] In eq ②, let $x = 1$

$f(1) = 0$

$$f(1) = \frac{1}{4} + \frac{2}{\pi^2} \sum_{n=1,3}^{\infty} \frac{\cos n\pi}{n^2}$$

$$0 = \frac{1}{4} + \frac{2}{\pi^2} \sum_{n=1,3}^{\infty} \frac{1}{n^2} \cos n\pi$$

$$-\frac{\pi^2}{8} = \sum_{n=1,3}^{\infty} \frac{(-1)^n}{n^2}$$

$$-\frac{\pi^2}{8} = \frac{-1}{1^2} - \frac{1}{3^2} - \frac{1}{5^2} - \cdots$$

$$\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots$$

$$\therefore \frac{\pi^2}{8} = \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2}$$

ii] Let $x = 1/2$

$$f(1/2) = 1 - 1/2 = 1/2$$

from eq ② $f(1/2) = \frac{1}{4} + \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \sin \frac{n\pi}{2}$

$$\therefore \frac{1}{2} = \frac{1}{4} + \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \sin \frac{n\pi}{2}$$

$$\frac{1}{4} = \frac{1}{\pi}\left[\frac{1}{1}\sin\frac{\pi}{2} + \frac{1}{2}\sin\pi + \frac{1}{3}\sin\frac{3\pi}{2} + \frac{1}{4}\sin 2\pi + \dots\right]$$

$$\frac{\pi}{4} = 1 + \frac{1}{3}(-1) + \frac{1}{5}(1) + \frac{1}{7}(-1) + \dots$$

$$\therefore \frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots$$

* **Half Range Fourier Series in (0, 1):**

In an interval of length $2l$ in general a periodic function of x will have a Fourier expansion containing cosine and sine terms. Many times it becomes necessary to have an expression containing only cosine or sine terms.

In the half range Fourier series, the function must be defined in the interval of the form $(0, l)$ which is to be regarded as half the interval.

Half range cosine series:

It is defined as $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right)$

where $a_0 = \frac{2}{l}\int_0^l f(x)\,dx$

$$a_n = \frac{2}{l}\int_0^l f(x)\cos\left(\frac{n\pi x}{l}\right)dx$$

Half range sine series:

It is defined as $f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right)$

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where $b_n = \frac{2}{l}\int_0^l f(x)\sin\left(\frac{n\pi x}{l}\right) dx$.

Q1: Obtain the sine half range Fourier series of $f(x) = x^2$ for $0 < x < \pi$.

Sol: Half range sine series is given by
$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right)$

where $b_n = \frac{2}{l}\int_0^l f(x)\sin\left(\frac{n\pi x}{l}\right) dx$

Here $l = \pi$

$$b_n = \frac{2}{\pi}\int_0^{\pi} x^2 \sin nx \, dx$$

$$= \frac{2}{\pi}\left[x^2\left(\frac{-\cos nx}{n}\right) - 2x\left(\frac{-\sin nx}{n^2}\right) + 2\left(\frac{\cos nx}{n^3}\right)\right]_0^{\pi}$$

$$= \frac{2}{n\pi}\left[-x^2\cos nx + \frac{2\cos nx}{n^2}\right]_0^{\pi}$$

$$= \frac{2}{n\pi}\left[-\pi^2\cos n\pi + \frac{2}{n^2}\cos n\pi + 0 - \frac{2}{n^2}\cos 0\right]$$

$$= \frac{2}{n\pi}\left[-\pi^2(-1)^n + \frac{2}{n^2}(-1)^n - \frac{2}{n^2}\right]$$

$$b_n = \frac{2}{\pi}\left[\frac{-\pi^2(-1)^n}{n} + \frac{2}{n^3}\left[(-1)^n - 1\right]\right]$$

$$\therefore f(x) = \sum_{n=1}^{\infty} \frac{2}{\pi}\left[\frac{-\pi^2(-1)^n}{n} + \frac{2}{n^3}\left[(-1)^n - 1\right]\right] \sin nx$$

Q2: Expand $f(x) = 2x - 1$ as a cosine half range Fourier series in $0 < x < 1$.

Sol: Half range cosine series is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right)$$ Here $l = 1$

where $a_0 = \frac{2}{l}\int_0^l f(x)\,dx$

$$a_0 = \frac{2}{1}\int_0^1 (2x-1)\,dx$$

$$= 2\left[x^2 - x\right]_0^1$$

$$= 2\left[1-1\right]$$

$$\underline{a_0 = 0}$$

$$a_n = \frac{2}{l}\int_0^l f(x)\cos\left(\frac{n\pi x}{l}\right)dx$$

$$= \frac{2}{1}\int_0^1 (2x-1)\cos n\pi x\,dx$$

$$= 2\left[(2x-1)\left(\cancel{\frac{\sin n\pi x}{n\pi}}\right) - 2\left(-\frac{\cos n\pi x}{n^2\pi^2}\right)\right]_0^1$$

$$= \frac{4}{n^2\pi^2}\left[\cos n\pi x\right]_0^1$$

$$= \frac{4}{n^2\pi^2}\left[\cos n\pi - \cos 0\right]$$

$$a_n = \frac{4}{n^2\pi^2}\left[(-1)^n - 1\right] \quad //$$

$$\therefore f(x) = \sum_{n=1}^{\infty} \frac{4}{n^2\pi^2}\left[(-1)^n - 1\right]\cos n\pi x \quad //$$

Q3: Find the cosine half range Fourier series of $f(x) = (x-1)^2$ in $0 \le x \le 1$.

Hence deduce: $\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots$

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Sol: Half range cosine series is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1} a_n \cos\left(\frac{n\pi x}{l}\right)$$

where $a_0 = \frac{2}{l}\int_0^l f(x)\,dx$ Here $l = 1$

$$a_0 = \frac{2}{1}\int_0^1 (x-1)^2\,dx$$

$$= 2\int_0^1 (x^2-2x+1)\,dx$$

$$= 2\left[\frac{x^3}{3} - x^2 + x\right]_0^1$$

$$= 2\left[\frac{1}{3} \cancel{-1} \cancel{+1}\right]$$

$$\frac{a_0}{2} = \frac{1}{3}$$

$$a_n = \frac{2}{l}\int_0^l f(x)\cos\left(\frac{n\pi x}{l}\right)dx$$

$$= \frac{2}{1}\int_0^1 (x-1)^2 \cos n\pi x\,dx$$

$$= 2\left[(x-1)^2\left(\frac{\sin n\pi x}{n\pi}\right) - 2(x-1)\left(\frac{-\cos n\pi x}{n^2\pi^2}\right) + 2\left(\frac{-\sin n\pi x}{n^3\pi^3}\right)\right]_0^1$$

$$= \frac{4}{n^2\pi^2}\left[(x-1)\cos n\pi x\right]_0^1$$

$$= \frac{4}{n^2\pi^2}\left[0 + \cos 0\right]$$

$$a_n = \frac{4}{n^2\pi^2}$$

$$\therefore f(x) = \frac{1}{3} + \sum_{n=1}^{\infty} \frac{4}{n^2\pi^2}\cos n\pi x$$

Let $x = 0$

$f(0) = 1$

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$$f(0) = \frac{1}{3} + \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}$$

$$\therefore 1 = \frac{1}{3} + \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}$$

$$\frac{2}{3} = \frac{4}{\pi^2}\left[\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots\right] \quad \text{——— (1)}$$

$$\frac{\pi^2}{6} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \cdots$$

Let $x = 1$

$f(1) = 0$

$$f(1) = \frac{1}{3} + \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{1}{n^2}\cos n\pi$$

$$\therefore 0 = \frac{1}{3} + \frac{4}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}$$

$$\frac{-1}{3} = \frac{4}{\pi^2}\left[\frac{-1}{1^2} + \frac{1}{2^2} - \frac{1}{3^2} + \frac{1}{4^2} - \cdots\right] \quad \text{——— (2)}$$

$$\frac{\pi^2}{12} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} \cdots$$

Adding eq (1) and eq (2)

$$\frac{\pi^2}{6} + \frac{\pi^2}{12} = 2\left[\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots\right]$$

$$\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \cdots$$

Q4: Find half range cosine series for the function

$$f(x) = \begin{cases} kx & \text{in } 0 \le x \le \pi/2 \\ k(\pi - x) & \text{in } \pi/2 \le x \le \pi \end{cases}$$

Deduce: $f(x) = \frac{\pi k}{4} - \frac{2k}{\pi}\left(\frac{\cos 2x}{1^2} + \frac{\cos 6x}{3^2} + \frac{\cos 10x}{5^2}\right)$

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sol: Half range cosine series is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right) \qquad \text{here } l = \pi$$

where $a_0 = \frac{2}{l}\int_0^l f(x)\,dx$

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx$$

$$= \frac{2}{\pi}\left[\int_0^{\pi/2} f(x)\,dx + \int_{\pi/2}^{\pi} f(x)\,dx\right]$$

$$= \frac{2}{\pi}\left[\int_0^{\pi/2} kx\,dx + \int_{\pi/2}^{\pi} k(\pi - x)\,dx\right]$$

$$= \frac{2}{\pi}\left[\frac{kx^2}{2}\Big|_0^{\pi/2} + \left(k\pi x - \frac{kx^2}{2}\right)\Big|_{\pi/2}^{\pi}\right]$$

$$= \frac{2k}{\pi}\left[\frac{\pi^2}{8} + \pi^2 - \frac{\pi^2}{2} - \frac{\pi^2}{2} + \frac{\pi^2}{8}\right]$$

$$= \frac{2k}{\pi}\left[\frac{\pi^2}{4}\right]$$

$$\frac{a_0}{2} = \frac{k\pi}{4}$$

$$a_n = \frac{2}{l}\int_0^l f(x)\cos\left(\frac{n\pi x}{l}\right)dx$$

$$= \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx$$

$$= \frac{2}{\pi}\left[\int_0^{\pi/2} f(x)\cos nx + \int_{\pi/2}^{\pi} f(x)\cos nx\right]$$

$$= \frac{2}{\pi}\left[\int_0^{\pi/2} kx\cos nx\,dx + \int_{\pi/2}^{\pi} k(\pi - x)\cos nx\,dx\right]$$

$$= \frac{2k}{\pi}\left[\left[x\left(\frac{\sin nx}{n}\right) - \left(\frac{-\cos nx}{n^2}\right)\right]_0^{\pi/2} + \left[(\pi - x)\left(\frac{\sin nx}{n}\right) + \left(\frac{-\cos nx}{n^2}\right)\right]_{\pi/2}^{\pi}\right]$$

$$= \frac{2k}{\pi}\left[\frac{\pi}{2n}\sin\frac{n\pi}{2} + \frac{1}{n^2}\cos\frac{n\pi}{2} - 0 - \frac{\cos 0}{n^2} + 0 - \frac{\cos n\pi}{n^2} - \frac{\pi}{2n}\sin\frac{n\pi}{2} + \frac{1}{n^2}\cos\frac{n\pi}{2}\right]$$

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$$= \frac{2k}{\pi}\left[\frac{2}{n^2}\cos\frac{n\pi}{2} - \frac{1}{n^2} - \frac{(-1)^n}{n^2}\right]$$

$$a_n = \frac{-2k}{\pi}\left[\frac{1}{n^2} + \frac{(-1)^n}{n^2} - \frac{2}{n^2}\cos\frac{n\pi}{2}\right]$$

$$\therefore f(x) = \frac{k\pi}{4} - \frac{2k}{\pi}\sum_{n=1}^{\infty}\left[\frac{1+(-1)^n - 2\cos n\pi/2}{n^2}\right]\cos nx$$

but $1+(-1)^n = \begin{cases} 0 & \text{for } n \text{ is odd} \\ 2 & \text{for } n \text{ is even} \end{cases}$

$$f(x) = \frac{k\pi}{4} - \frac{2k}{\pi}\sum_{n=2,4..}^{\infty}\left[\frac{2-2\cos n\pi/2}{n^2}\right]\cos nx$$

$$f(x) = \frac{k\pi}{4} - \frac{4k}{\pi}\sum_{n=2,4..}^{\infty}\left[\frac{1-\cos n\pi/2}{n^2}\right]\cos nx$$

but $1-\cos\frac{n\pi}{2} = \begin{cases} 2 & \text{for } n = 2,6,10\cdots \\ 0 & \text{for } n = 4,8,12\cdots \end{cases}$

$$\therefore f(x) = \frac{k\pi}{4} - \frac{4k}{\pi}\sum_{n=2,6..}^{\infty}\frac{2}{n^2}\cos nx$$

$$f(x) = \frac{k\pi}{4} - \frac{8k}{\pi}\sum_{n=2,6..}^{\infty}\frac{1}{n^2}\cos nx$$

$$f(x) = \frac{k\pi}{4} - \frac{8k}{\pi}\left[\frac{1}{2^2}\cos 2x + \frac{1}{6^2}\cos 6x + \frac{1}{10^2}\cos 10x + \cdots\right]$$

$$f(x) = \frac{k\pi}{4} - \frac{2k}{\pi}\left[\frac{\cos 2x}{1^2} + \frac{\cos 6x}{3^2} + \frac{\cos 10x}{5^2} + \cdots\right]$$

Q5: Find the Half range cosine series for

$$f(x) = \begin{cases} 1 & \text{in } 0<x<1 \\ x & \text{in } 1<x<2 \end{cases}$$

Sol: Half range cosine series is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right)$$ Here $l = 2$

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$$a_0 = \frac{2}{l}\int_0^l f(x)\,dx$$

$$= \frac{2}{2}\int_0^2 f(x)\,dx$$

$$= \int_0^1 f(x)\,dx + \int_1^2 f(x)\,dx$$

$$= \int_0^1 1\,dx + \int_1^2 x\,dx$$

$$= x\Big|_0^1 + \frac{x^2}{2}\Big|_1^2$$

$$= 1 + 2 - \frac{1}{2}$$

$$a_0 = \frac{5}{2} \Rightarrow \frac{a_0}{2} = \frac{5}{4}$$

$$a_n = \frac{2}{l}\int_0^l f(x)\cos\left(\frac{n\pi x}{l}\right)dx$$

$$= \frac{2}{2}\int_0^2 f(x)\cos\left(\frac{n\pi x}{2}\right)dx$$

$$= \int_0^1 f(x)\cos\left(\frac{n\pi x}{2}\right)dx + \int_1^2 f(x)\cos\left(\frac{n\pi x}{2}\right)dx$$

$$= \int_0^1 \cos\left(\frac{n\pi x}{2}\right)dx + \int_1^2 x\cos\left(\frac{n\pi x}{2}\right)dx$$

$$= \left(\sin\frac{n\pi x}{2}\right)\frac{2}{n\pi}\Big|_0^1 + \left[x\left(\sin\frac{n\pi x}{2}\right)\frac{2}{n\pi} + \left(\cos\frac{n\pi x}{2}\right)\frac{4}{n^2\pi^2}\right]\Big|_1^2$$

$$= \frac{2}{n\pi}\left[\sin\frac{n\pi x}{2}\Big|_0^1 + \left(x\sin\frac{n\pi x}{2} + \frac{2}{n\pi}\cos\frac{n\pi x}{2}\Big|_1^2\right)\right]$$

$$= \frac{2}{n\pi}\left[\cancel{\sin\frac{n\pi}{2}} - 0 + 2\overset{0}{\sin n\pi} + \frac{2}{n\pi}\cos n\pi - \cancel{\sin\frac{n\pi}{2}} - \frac{2}{n\pi}\cos\frac{n\pi}{2}\right]$$

$$= \frac{2}{n\pi}\left[\frac{2}{n\pi}\left(\cos n\pi - \cos\frac{n\pi}{2}\right)\right]$$

$$a_n = \frac{4}{n^2\pi^2}\left[(-1)^n - \cos\frac{n\pi}{2}\right]$$

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$$\therefore f(x) = \frac{5}{4} + \frac{4}{\pi^2} \sum_{n=1}^{\infty} \frac{[(-1)^n - \cos n\pi/2]}{n^2} \cos \frac{n\pi x}{2}$$

Q6: Find the half range sine series for the function

$$f(x) = \begin{cases} 1 & \text{in } 0 < x < \pi/2 \\ -1 & \text{in } \pi/2 < x < \pi \end{cases}$$

Sol: The half range sine series is given by

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right) \quad \text{here } l = \pi$$

where $b_n = \frac{2}{l} \int_0^l f(x) \sin\left(\frac{n\pi x}{l}\right) dx$

$$b_n = \frac{2}{\pi} \int_0^{\pi} f(x) \sin nx \, dx$$

$$= \frac{2}{\pi} \left[\int_0^{\pi/2} f(x) \sin nx \, dx + \int_{\pi/2}^{\pi} f(x) \sin nx \, dx \right]$$

$$= \frac{2}{\pi} \left[\int_0^{\pi/2} \sin nx \, dx - \int_{\pi/2}^{\pi} \sin nx \, dx \right]$$

$$= \frac{2}{\pi} \left[\frac{-\cos nx}{n} \Big|_0^{\pi/2} + \frac{\cos nx}{n} \Big|_{\pi/2}^{\pi} \right]$$

$$= \frac{2}{n\pi} \left[-\cos \frac{n\pi}{2} + \cos 0 + \cos n\pi - \cos \frac{n\pi}{2} \right]$$

$$b_n = \frac{2}{n\pi} \left[1 + (-1)^n - 2\cos \frac{n\pi}{2} \right]$$

$$\therefore f(x) = \sum_{n=1}^{\infty} \frac{2}{n\pi} \left[1 + (-1)^n - 2\cos \frac{n\pi}{2} \right] \sin nx$$

Q7: Find the half range sine series

$$f(x) = \begin{cases} \frac{1}{4} - x & \text{in } 0 < x < \frac{1}{2} \\ x - \frac{3}{4} & \text{in } \frac{1}{2} < x < 1 \end{cases}$$

Sol: The half range sine series is given by

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right) \quad \text{here } l = 1$$

where $b_n = \frac{2}{l}\int_0^l f(x)\sin\left(\frac{n\pi x}{l}\right)dx$

$$b_n = \frac{2}{1}\left[\int_0^1 f(x)\sin n\pi x\,dx\right]$$

$$= 2\left[\int_0^{1/2} f(x)\sin n\pi x\,dx + \int_{1/2}^1 f(x)\sin n\pi x\,dx\right]$$

$$= 2\left[\int_0^{1/2}\left(\frac{1}{4}-x\right)\sin n\pi x\,dx + \int_{1/2}^1\left(x-\frac{3}{4}\right)\sin n\pi x\,dx\right]$$

$$= 2\left[\left(\frac{1}{4}-x\right)\left(\frac{-\cos n\pi x}{n\pi}\right) + \left(\frac{-\sin n\pi x}{n^2\pi^2}\right)\Big|_0^{1/2} + \left(x-\frac{3}{4}\right)\left(\frac{-\cos n\pi x}{n\pi}\right) - \left(\frac{-\sin n\pi x}{n^2\pi^2}\right)\Big|_{1/2}^1\right]$$

$$= 2\left[\cancel{\frac{1}{4n\pi}\cos\frac{n\pi}{2}} - \frac{1}{n^2\pi^2}\sin\frac{n\pi}{2} + \frac{1}{4n\pi}\cos 0 + 0 - \frac{1}{4n\pi}\cos n\pi + \cancel{\frac{1}{n^2\pi^2}\sin n\pi}^{\,0} - \cancel{\frac{1}{4n\pi}\cos\frac{n\pi}{2}} - \frac{1}{n^2\pi^2}\sin\frac{n\pi}{2}\right]$$

$$= \frac{2}{n\pi}\left[\frac{1}{4} - \frac{(-1)^n}{4} - \frac{2\sin n\pi/2}{n\pi}\right]$$

$$b_n = \frac{1}{2n\pi}\left[1-(-1)^n - \frac{8}{n\pi}\sin\frac{n\pi}{2}\right]$$ //

$$\therefore f(x) = \sum_{n=1}^{\infty}\left[\frac{1}{2\pi n}\left(1-(-1)^n\right) - \frac{4}{n^2\pi^2}\sin\frac{n\pi}{2}\right]\sin n\pi x$$ //

Q8: Find the half range sine series

$$f(x) = \begin{cases} kx & \text{in } 0 \le x \le l/2 \\ k(l-x) & \text{in } l/2 \le x \le l \end{cases}$$

Sol: Half range sine series is given by

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Sol: $f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right)$

where $b_n = \frac{2}{l}\int_0^l f(x)\sin\left(\frac{n\pi x}{l}\right)dx$

$b_n = \frac{2}{l}\left[\int_0^{l/2} f(x)\sin\left(\frac{n\pi x}{l}\right)dx + \int_{l/2}^{l} f(x)\sin\left(\frac{n\pi x}{l}\right)dx\right]$

$= \frac{2}{l}\left[\int_0^{l/2} kx\sin\left(\frac{n\pi x}{l}\right)dx + \int_{l/2}^{l} k(l-x)\sin\left(\frac{n\pi x}{l}\right)dx\right]$

$= \frac{2k}{l}\left[\left[x\left(-\cos\frac{n\pi x}{l}\right)\frac{l}{n\pi} + \left(\sin\frac{n\pi x}{l}\right)\frac{l^2}{n^2\pi^2}\right|_0^{l/2}\right.$

$\left. + \left[(l-x)\left(-\cos\frac{n\pi x}{l}\right)\frac{l}{n\pi} + \left(-\sin\frac{n\pi x}{l}\right)\frac{l^2}{n^2\pi^2}\right|_{l/2}^{l}\right]$

$= \frac{2k}{l}\left[\frac{l}{n\pi}\left[\frac{-l}{2}\cos\frac{n\pi}{2} + \frac{l}{n\pi}\sin\frac{n\pi}{2} + 0 - 0 - 0 - 0 + \frac{l}{2}\cos\frac{n\pi}{2} + \dots\right.\right.$

$= \frac{2k}{n\pi}\left[\frac{2l}{n\pi}\sin\frac{n\pi}{2}\right]$

$b_n = \frac{4kl}{n^2\pi^2}\sin\frac{n\pi}{2}$ //

$\therefore f(x) = \sum_{n=1}^{\infty}\frac{4kl}{n^2\pi^2}\sin\frac{n\pi}{2}\sin\frac{n\pi x}{l}$ //

Q7: Find the half range cosine and sine series for the following functions:

i. $f(x) = \begin{cases} x & \text{in } 0<x<\pi/2 \\ \pi - x & \text{in } \pi/2<x<\pi \end{cases}$

Sol: a. Half range cosine series is given by

$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n\cos\frac{n\pi x}{l}$ Here $l = \pi$

where $a_0 = \frac{2}{l}\int_0^l f(x)dx$

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$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx$$

$$= \frac{2}{\pi}\left[\int_0^{\pi/2} f(x)\,dx + \int_{\pi/2}^{\pi} f(x)\,dx\right]$$

$$= \frac{2}{\pi}\left[\int_0^{\pi/2} x\,dx + \int_{\pi/2}^{\pi} (\pi - x)\,dx\right]$$

$$= \frac{2}{\pi}\left[\frac{x^2}{2}\Big|_0^{\pi/2} + \left[\pi x - \frac{x^2}{2}\right]_{\pi/2}^{\pi}\right]$$

$$= \frac{2}{\pi}\left[\frac{\pi^2}{8} + \cancel{\pi^2} - \frac{\cancel{\pi^2}}{\cancel{2}} - \frac{\cancel{\pi^2}}{\cancel{2}} + \frac{\pi^2}{8}\right]$$

$$= \frac{2}{\pi}\left[\frac{\pi^2}{4}\right]$$

$$\frac{a_0}{2} = \frac{\pi}{4}$$

$$a_n = \frac{2}{l}\int_0^{l} f(x)\cos\frac{n\pi x}{l}\,dx$$

$$= \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx$$

$$= \frac{2}{\pi}\left[\int_0^{\pi/2} f(x)\cos nx\,dx + \int_{\pi/2}^{\pi} f(x)\cos nx\,dx\right]$$

$$= \frac{2}{\pi}\left[\int_0^{\pi/2} x\cos nx\,dx + \int_{\pi/2}^{\pi} (\pi - x)\cos nx\,dx\right]$$

$$= \frac{2}{\pi}\left[\left[x\left(\frac{\sin nx}{n}\right) - \left(-\frac{\cos nx}{n^2}\right)\right]_0^{\pi/2} + \left[(\pi - x)\left(\frac{\sin nx}{n}\right) + \left(\frac{-\cos nx}{n^2}\right)\right]_{\pi/2}^{\pi}\right]$$

$$= \frac{2}{\pi}\left[\frac{\pi}{2n}\sin\frac{n\pi}{2} + \frac{1}{n^2}\cos\frac{n\pi}{2} - 0 - \frac{1}{n^2}\cos 0 + 0 - \frac{1}{n^2}\cos n\pi - \frac{\pi}{2n}\sin\frac{n\pi}{2} + \frac{1}{n^2}\cos\frac{n\pi}{2}\right]$$

$$a_n = \frac{2}{n^2\pi}\left[2\cos\frac{n\pi}{2} - 1 - (-1)^n\right]$$

$$\therefore f(x) = \frac{\pi}{4} + \sum_{n=1}^{\infty} \frac{2}{n^2\pi}\left[2\cos\frac{n\pi}{2} - 1 - (-1)^n\right]$$

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b. Half range sine series is given by

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right) \quad \text{here } l = \pi$$

where $b_n = \frac{2}{l}\int_0^l f(x) \sin\left(\frac{n\pi x}{l}\right) dx$

$$b_n = \frac{2}{\pi}\int_0^{\pi} f(x) \sin nx \, dx$$

$$= \frac{2}{\pi}\left[\int_0^{\pi/2} f(x) \sin nx \, dx + \int_{\pi/2}^{\pi} f(x) \sin nx \, dx\right]$$

$$= \frac{2}{\pi}\left[\int_0^{\pi/2} x \sin nx \, dx + \int_{\pi/2}^{\pi} (\pi - x) \sin nx \, dx\right]$$

$$= \frac{2}{\pi}\left[\left[x\left(\frac{-\cos nx}{n}\right) - \left(\frac{-\sin nx}{n^2}\right)\right|_0^{\pi/2} + \left[(\pi - x)\left(\frac{-\cos nx}{n}\right) + \left(\frac{-\sin nx}{n^2}\right)\right|_{\pi/2}^{\pi}\right]$$

$$= \frac{2}{\pi}\left[\frac{-\pi}{2n}\cos\frac{n\pi}{2} + \frac{1}{n^2}\sin\frac{n\pi}{2} + 0 - 0 + 0 - 0 + \frac{\pi}{2n}\cos\frac{n\pi}{2} + \frac{1}{n^2}\sin\frac{n\pi}{2}\right]$$

$$= \frac{2}{\pi}\left[\frac{2}{n^2}\sin\frac{n\pi}{2}\right]$$

$$b_n = \frac{4}{n^2\pi}\sin\frac{n\pi}{2}$$

$$\therefore f(x) = \sum_{n=1}^{\infty} \frac{4}{n^2\pi}\sin\frac{n\pi}{2} \sin nx$$

ii. $f(x) = lx - x^2$ in $0 \le x \le 1$

Sol: a. Half range cosine series is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right) \quad \text{Here } l = 1$$

$$a_0 = \frac{2}{l}\int_0^l f(x)\,dx$$

$$= \frac{2}{1}\int_0^1 (lx - x^2)\,dx$$

$$= 2\left[\frac{lx^2}{2} - \frac{x^3}{3}\right]_0^1$$

$$= 2\left[\frac{l}{2} - \frac{1}{3}\right]$$

$$\frac{a_0}{2} = \frac{l}{2} - \frac{1}{3}$$

$$a_n = \frac{2}{l}\int_0^l f(x)\cos\left(\frac{n\pi x}{l}\right)dx$$

$$= \frac{2}{1}\int_0^1 (lx - x^2)\cos n\pi x\,dx$$

$$= 2\left[(lx-x^2)\frac{\sin n\pi x}{n\pi} - (l-2x)\left(\frac{-\cos n\pi x}{n^2\pi^2}\right) + (-2)\left(\frac{-\sin n\pi x}{n^3\pi^3}\right)\right]_0^1$$

$$= 2\left[0 + \frac{(l-2)}{n^2\pi^2}\cos n\pi + 0 - 0 - \frac{l}{n^2\pi^2}\cos 0 - 0\right]$$

$$a_n = \frac{2}{n^2\pi^2}\left[(l-2)(-1)^n - l\right]$$

$$\therefore f(x) = \frac{l}{2} - \frac{1}{3} + \sum_{n=1}^{\infty}\frac{2}{n^2\pi^2}\left[(l-2)(-1)^n - l\right]$$

b. Half range sine series is given by

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right) \quad \text{here } l = 1$$

where $b_n = \frac{2}{l}\int_0^l f(x)\sin\left(\frac{n\pi x}{l}\right)dx$

$$b_n = \frac{2}{1}\int_0^1 (lx - x^2)\sin n\pi x\,dx$$

$$= 2\left[(lx-x^2)\left(-\frac{\cos n\pi x}{n\pi}\right) - (l-2x)\left(-\frac{\sin n\pi x}{n^2\pi^2}\right) + (-2)\left(\frac{\cos n\pi x}{n^3\pi^3}\right)\right]_0^1$$

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$$= 2\left[-\frac{(l-1)}{n\pi}\cos n\pi + 0 \, \frac{-2}{n^2\pi^2}\cos n\pi + 0 - 0 + \frac{2}{n^2\pi^2}\cos 0\right]$$

$$= \frac{2}{n\pi}\left[-(l-1)(-1)^n + \frac{2(-1)^n}{n^2\pi^2} + \frac{2}{n^2\pi^2}\right]$$

$$b_n = \frac{2}{n\pi}\left[\frac{2}{n^2\pi^2}\left[(-1)^n+1\right] - (l-1)\right]$$

$$\therefore f(x) = \sum_{n=1}^{\infty} \frac{2}{n\pi}\left[\frac{2}{n^2\pi^2}\left[(-1)^n+1\right] - (l-1)\right] \sin n\pi x$$

★ Complex form of Fourier Series : $(c, c+2l)$

The complex form of Fourier series is defined as

$$f(x) = \sum_{n=-\infty}^{\infty} C_n e^{in\pi x/l}$$

where $C_n = \frac{1}{2l}\int_c^{c+2l} f(x) e^{-in\pi x/l}\, dx$

Q1: Find the complex form of Fourier series for the function $f(x) = e^x$ in $-\pi < x < \pi$

Sol: $C_n = \frac{1}{2l}\int_c^{c+2l} f(x) e^{-in\pi x/l}\, dx$ here $l = \pi$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^x e^{-in\pi x/\pi}\, dx$$

$$= \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{x(1-in)}\, dx$$

$$= \frac{1}{2\pi}\left[\frac{e^{x(1-in)}}{1-in}\right]_{-\pi}^{\pi}$$

$$= \frac{1}{2\pi}\left[\frac{1}{1-in} \times \frac{1+in}{1+in}\right]\left(e^{\pi(1-in)} - e^{-\pi(1-in)}\right)$$

$= \frac{1}{2\pi}\left[\frac{1+in}{1+n^2}\right]\left(e^{\pi}\cdot e^{-in\pi} - e^{-\pi}e^{in\pi}\right)$

wkt $e^{i\theta} = \cos\theta + i\sin\theta$

$e^{-i\theta} = \cos\theta - i\sin\theta$

$= \frac{1+in}{2\pi(1+n^2)}\left[e^{\pi}(\cos n\pi - i\sin n\pi) - e^{-\pi}(\cos n\pi + i\sin n\pi)\right]$

$= \frac{1+in}{2\pi(1+n^2)}\left[e^{\pi}\cos n\pi - e^{-\pi}\cos n\pi\right]$

$= \frac{1+in}{\pi(1+n^2)}\cos n\pi\left[\frac{e^{\pi}-e^{-\pi}}{2}\right]$

$C_n = \frac{1+in}{\pi(1+n^2)}(-1)^n \sinh n\pi$ //

$\therefore f(x) = \sum_{n=-\infty}^{\infty} \frac{1+in}{\pi(1+n^2)}(-1)^n \sinh n\pi \; e^{inx}$ //

Q2: Find the complex form of Fourier series for the function $f(x) = e^{-x}$ in $(-1, 1)$.

Sol: $f(x) = \sum_{n=-\infty}^{\infty} C_n e^{in\pi x/l}$

where $C_n = \frac{1}{2l}\int_c^{c+2l} f(x)\, e^{-in\pi x/l}\, dx$ Here $l = 1$

$C_n = \frac{1}{2}\int_{-1}^{1} e^{-x} e^{-in\pi x}\, dx$

$= \frac{1}{2}\int_{-1}^{1} e^{-x(1+in\pi)}\, dx$

$= \frac{1}{2}\left[\frac{e^{-x(1+in\pi)}}{-(1+in\pi)}\right]_{-1}^{1}$

$= \frac{-1}{2(1+in\pi)}\left[e^{-(1+in\pi)} - e^{(1+in\pi)}\right]$

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$$= \frac{-1}{2(1+in\pi)}\left[e^{-1}e^{in\pi} - e^{1}e^{in\pi}\right]$$

$$= \frac{-1}{2(1+in\pi)} e^{in\pi}\left[e^{-1} - e^{1}\right]$$

$$= \frac{1}{(1+in\pi)}(\cos n\pi + i\,\cancel{\sin n\pi})\left[\frac{e^{1}-e^{-1}}{2}\right]$$

$$= \left[\frac{1}{1+in\pi} \times \frac{1-in\pi}{1-in\pi}\right]\cos n\pi \sinh 1.$$

$$c_n = \left[\frac{1-in\pi}{1+n^2\pi^2}\right](-1)^n \sinh 1$$

$$\therefore f(x) = \sum_{n=-\infty}^{\infty} \frac{\sinh 1 (-1)^n (1-in\pi)}{1+n^2\pi^2} e^{in\pi x}$$

Q1:

sol:

★ **Practical Harmonic Analysis:**

TYPE 1: x is in degree

Period 2π

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

First Harmonic

$$f(x) = \frac{a_0}{2} + a_1\cos x + b_1 \sin x$$

Second Harmonic

$$f(x) = \frac{a_0}{2} + a_1\cos x + a_2\cos 2x + b_1 \sin x + b_2 \cos 2x$$

Third Harmonic

$$f(x) = \frac{a_0}{2} + a_1\cos x + a_2\cos 2x + a_3\cos 3x + b_1\sin x + b_2 \sin 2x + b_3 \sin 3x$$

where $a_0 = \frac{2}{N}\Sigma y$

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$$a_n = \frac{2}{N} \sum y \cos nx$$

$$b_n = \frac{2}{N} \sum y \sin nx$$

Q1: Determine the constant term and first cosine and sine terms of the fourier series.

x°	0	45	90	135	180	225	270	315
y	2	3/2	1	1/2	0	1/2	1	3/2

Sol:

$y\cos x$	2	1.0606	0	-0.3535	0	-0.3535	0	1.0606
$y\sin x$	0	1.0606	1	0.3535	0	-0.3535	-1	-1.0606

First harmonic of Fourier series is given by

$$f(x) = \frac{a_0}{2} + a_1 \cos x + b_1 \sin x$$

where $a_0 = \frac{2}{N} \sum y$

$$a_0 = \frac{2}{8}(8) \Rightarrow a_0 = 2 \Rightarrow \frac{a_0}{2} = 1$$

$$a_1 = \frac{2}{N} \sum y \cos x$$

$$a_1 = \frac{2}{8}(3.4142) \Rightarrow a_1 = 0.8535$$

$$b_1 = \frac{2}{N} \sum y \sin x$$

$$b_1 = \frac{2}{8}(0) \Rightarrow b_1 = 0$$

$$\therefore f(x) = 1 + 0.8535 \cos x$$

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Q2: Express y as Fourier series upto the third harmonic

x	0	$\pi/3$	$2\pi/3$	π	$4\pi/3$	$5\pi/3$
y	1.98	1.30	1.05	1.30	-0.88	-0.25

Sol: Here the interval of x is $0 \le x \le 2\pi$ and the value at $x=0$ and $x=2\pi$ must be same by the periodic properties i.e., $f(x+2\pi) = f(x)$.

In the given problem value of y at $x=0$ and $x=2\pi$ both are given thus we must take any one of the

x	0	$\pi/3$	$2\pi/3$	π	$4\pi/3$	$5\pi/3$
x°	0	60	120	180	240	300
y	1.98	1.30	1.05	1.30	-0.88	-0.25
$y\cos x$	1.98	0.65	-0.525	-1.30	0.44	-0.125
$y\cos 2x$	1.98	-0.65	-0.525	-1.30	0.44	0.125
$y\cos 3x$	1.98	-1.3	1.05	-1.30	-0.88	0.25
$y\sin x$	0	1.1258	0.9093	0	0.7621	0.2165
$y\sin 2x$	0	1.1258	-0.9093	0	-0.7621	0.2165
$y\sin 3x$	0	0	0	0	0	0

Third harmonic of fourier series is given by

$$f(x) = \frac{a_0}{2} + a_1\cos x + a_2\cos 2x + a_3\cos 3x + b_1\sin x + b_2\sin 2x + \ldots$$

$$a_0 = \frac{2}{N}\Sigma y = \frac{2}{6}(4.5) = 1.5$$

$$a_1 = \frac{2}{N}\Sigma y\cos x = \frac{2}{6}(1.12) = 0.3733$$

$$a_2 = \frac{2}{N}\Sigma y\cos 2x = \frac{2}{6}(2.67) = 0.89$$

$$a_3 = \frac{2}{N}\Sigma y\cos 3x = \frac{2}{6}(-0.2) = -0.0667$$

$$b_1 = \frac{2}{N}\Sigma y\sin x = \frac{2}{6}(3.0137) = 1.0045$$

$b_2 = \frac{2}{N} \Sigma y \sin 2x = \frac{2}{6}(-0.3291) = -0.1097$

$b_3 = \frac{2}{N} \Sigma y \sin 3x = \frac{2}{6}(0) = 0$

$\therefore f(x) = 0.75 + 0.3733 \cos x + 0.89 \cos 2x - 0.6617 \cos 3x$
$+ 1.0045 \sin x - 0.1097 \sin 2x$

TYPE 2: Period of $2l$.

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right)$$

put $\theta = \frac{\pi x}{l}$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\theta + \sum_{n=1}^{\infty} b_n \sin n\theta$$

First Harmonic

$$f(x) = \frac{a_0}{2} + a_1 \cos\theta + b_1 \sin\theta$$

Second Harmonic

$$f(x) = \frac{a_0}{2} + a_1 \cos\theta + a_2 \cos 2\theta + a_3 \cos 3\theta + b_1 \sin\theta + b_2 \sin 2\theta + b_3 \sin 3\theta$$

where $a_0 = \frac{2}{N} \Sigma y$

$a_n = \frac{2}{N} \Sigma y \cos n\theta$

$b_n = \frac{2}{N} \Sigma y \sin n\theta$

Q1: Obtain the constant term and the coefficients of the first cosine and sine terms in the Fourier expansion of y from the following data:

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x	0	1	2	3	4	5
y	9	18	24	28	26	20

Sol: By the data x value lies in the interval $0 \le x < 6$ i.e., Therefore length of the interval is 6 i.e, $N=6$, comparing the period $2l$

we have $2l = 6 \Rightarrow l = 3$

The Fourier series of period $2l$ is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right)$$

put $\theta = \frac{\pi x}{l}$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\theta + \sum_{n=1}^{\infty} b_n \sin n\theta$$

First harmonic of Fourier series is

$$f(x) = \frac{a_0}{2} + a_1 \cos\theta + b_1 \sin\theta \qquad \text{where } \theta = \frac{\pi x}{3}$$

x	0	1	2	3	4	5	
$\theta = \pi x/3$	0°	60°	120°	180°	240°	300°	
y	9	18	24	28	26	20	125
$y\cos\theta$	9	9	-12	-28	-13	10	-25
$y\sin\theta$	0	15.5884	20.7846	0	-22.5166	-17.3205	-13.464

$$a_0 = \frac{2}{N}\Sigma y = \frac{2}{6}(125) = \underline{\underline{41.6667}}$$

$$a_1 = \frac{2}{N}\Sigma y\cos\theta = \frac{2}{6}(-25) = \underline{\underline{-8.3333}}$$

$$b_1 = \frac{2}{N}\Sigma y\sin\theta = \frac{2}{6}(-13.4641) = \underline{\underline{-4.488}}$$

$$\therefore f(x) = 20.8333 - 8.3333\cos\theta - 4.488\sin\theta$$

Find the Fourier series of y upto second harmonic from the following data.

x	0	2	4	6	8	10	12
y	9.0	18.2	24.4	27.8	27.5	22.0	9.0

The values of y at $x=0$ and $x=12$ are same hence the interval of x is $(0,12)$ i.e, $0 \le x \le 12$ and we shall neglect the value of y for $x=12$ in the process of calculation, comparing this with the period $2l$. the length of the interval is 12. comparing with the period of $2l$

we have $2l = 12 \Rightarrow l = 6$ //

The Fourier series of period $2l$ is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right)$$

put $\theta = \frac{\pi x}{l}$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\theta + \sum_{n=1}^{\infty} b_n \sin n\theta$$

Second harmonic of Fourier series is

$$f(x) = \frac{a_0}{2} + a_1 \cos\theta + a_2 \cos\theta + b_1 \sin\theta + b_2 \sin\theta$$

x	0	2	4	6	8	10	
$\theta = \pi x/6$	0°	60°	120°	180°	240°	300°	
y	9.0	18.2	24.4	27.8	27.5	22.0	128.9
$y\cos\theta$	9.0	9.1	−12.2	−27.8	−13.75	11.0	−24.65
$y\cos 2\theta$	9.0	−9.1	−12.2	27.8	−13.75	−11.0	−9.25
$y\sin\theta$	0	15.7616	21.1310	0	−23.8156	−19.0525	−5.9755
$y\sin 2\theta$	0	15.7616	−21.1310	0	23.8156	−19.0525	−0.6063

$a_1 = \frac{2}{N} \Sigma y \cos\theta = \frac{2}{6}(-24.65) = \underline{\underline{-8.2164}}$

$a_2 = \frac{2}{N} \Sigma y \cos 2\theta = \frac{2}{6}(-9.25) = \underline{\underline{-3.0833}}$

$b_1 = \frac{2}{N} \Sigma y \sin\theta = \frac{2}{6}(-5.9755) = \underline{\underline{-1.9918}}$

$b_2 = \frac{2}{N} \Sigma y \sin 2\theta = \frac{2}{6}(-0.6063) = \underline{\underline{-0.2021}}$

$\therefore f(x) = 21.4833 - 8.2164 \cos\theta - 3.0833 \cos 2\theta - 1.9918 \sin\theta - 0.2021 \sin 2\theta$ //

Q3: Express y as a fourier series upto third harmonic from the given data.

x	0	1	2	3	4	5
y	4	8	15	7	6	2

Sol: By the data x value lies on the interval $0 \le x < 6$. $[0,6)$

Therefore by comparing with period $2l$

$2l = 6 \Rightarrow \underline{l = 3}$

Fourier series of period $2l$ is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right)$$

put $\theta = \frac{\pi x}{l} = \frac{\pi x}{3}$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\theta + \sum_{n=1}^{\infty} b_n \sin n\theta$$

Third harmonic of Fourier series is given by

$$f(x) = \frac{a_0}{2} + a_1 \cos\theta + a_2 \cos 2\theta + a_3 \cos 3\theta + b_1 \sin\theta$$

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x	0	1	2	3	4	5	
$\theta = \pi x/3$	$0°$	$60°$	$120°$	$180°$	$240°$	$300°$	
y	4	8	15	7	6	2	42
$y\cos\theta$	4	4	-7.5	-7	-3	1	-8.5
$y\cos 2\theta$	4	-4	-7.5	+7	-3	-1	-4.5
$y\cos 3\theta$	4	-8	15	-7	6	-2	8
$y\sin\theta$	0	6.9282	12.99	0	-5.1961	-1.732	12.9901
$y\sin 2\theta$	0	6.9282	-12.99	0	5.1961	-1.732	-2.5941
$y\sin 3\theta$	0	0	0	0	0	0	0

$$a_0 = \frac{2}{N}\Sigma y = 14$$

$$a_1 = \frac{2}{N}\Sigma y\cos\theta = \frac{2}{6}(-8.5) = -2.833$$

$$a_2 = \frac{2}{N}\Sigma y\cos 2\theta = \frac{2}{6}(-4.5) = -1.5$$

$$a_3 = \frac{2}{N}\Sigma y\cos 3\theta = \frac{2}{6}(8) = 2.667$$

$$b_1 = \frac{2}{N}\Sigma y\sin\theta = \frac{2}{6}(12.9901) = 4.33$$

$$b_2 = \frac{2}{N}\Sigma y\sin 2\theta = \frac{2}{6}(-2.5941) = -0.8659$$

$$b_3 = \frac{2}{N}\Sigma y\sin 3\theta = \frac{2}{6}(0) = 0$$

$$\therefore f(x) = 7 - 2.833\cos\theta - 1.5\cos 2\theta + 2.667\cos 3\theta + 4.33\sin\theta - 0.8659\sin 2\theta$$

Q4: Obtain the constant term and coefficient of $\sin\theta$ and $\sin 2\theta$ in the fourier expansion of y from the given data

$\theta°$	0	60	120	180	240	300	360
y	0	9.2	14.4	17.8	17.3	11.7	0

x°	0	60	120	180	240	300	
y	0	9.2	14.4	17.8	17.3	11.4	70.4
$y\sin\theta$	0	-7.9674	12.4707	0	-14.9822	-10.1324	-4.6765
$y\sin 2\theta$	0	7.9674	-12.4709	0	14.9822	-10.1324	0.3465

$$a_0 = \frac{2}{N}\Sigma y = \frac{2}{6}(70.4) = 23.467$$

$$a_1 = \frac{2}{N}\Sigma y\sin\theta = \frac{2}{6}(-4.6765) = -1.558$$

$$a_2 = \frac{2}{N}\Sigma y\sin 2\theta = \frac{2}{6}(0.3465) = 0.1151$$

Q5: Given a following table

θ°	0	60	120	180	240	300
y	7.9	7.2	3.6	0.5	0.9	6.8

Obtain the Fourier series neglecting the terms higher than first harmonic

Sol:

θ°	0	60	120	180	240	300	
y	7.9	7.2	3.6	0.5	0.9	6.8	26.9
$y\cos\theta$	7.9	3.6	-1.8	-0.5	-0.45	3.4	12.25
$y\sin\theta$	0	6.2354	3.1177	0	-0.7794	-5.8889	2.6848

$$a_0 = \frac{2}{N}\Sigma y = \frac{2}{6}(26.9) = 8.967$$

$$a_1 = \frac{2}{N}\Sigma y\cos\theta = \frac{2}{6}(12.25) = 4.083$$

$$a_2 = \frac{2}{N}\Sigma y\sin\theta = \frac{2}{6}(2.6848) = 0.895$$

$$\therefore f(x) = 4.4835 + 4.083\cos\theta + 0.895\sin\theta$$

Q6:

Sol:

Q6: Obtain the constant term and first three coefficients in the Fourier cosine series for y using the following data:

x	0	1	2	3	4	5
y	4	8	15	7	6	2

Sol: Fourier cosine series: By the data x values lies in the interval $0 \le x \le 6$ i.e., $(0, 6)$ Half range series

comparing $(0, l) \Rightarrow l = 6$

Fourier series of period l is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right)$$

$$\text{put } \theta = \frac{\pi x}{l} = \frac{\pi x}{6}$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\theta + \sum_{n=1}^{\infty} b_n \sin n\theta$$

The third harmonic is given by

$$f(x) = \frac{a_0}{2} + a_1 \cos\theta + a_2 \cos 2\theta + a_3 \cos 3\theta \cancel{+ b_1 \sin\theta} \cancel{+ b_2 \sin 2\theta + b_3 \sin 3\theta}$$

x	0	1	2	3	4	5	
$\theta = \pi x/6$	0	30	60	90	120	150	
y	4	8	15	7	6	2	42
$y\cos\theta$	4	6.928	7.5	0	−3	−1.732	13.696
$y\cos 2\theta$	4	4	−7.5	−7	−3	1	−8.5
$y\cos 3\theta$	4	0	−15	0	6	0	−5
~~$y\sin\theta$~~							
~~$y\sin 2\theta$~~							
~~$y\sin 3\theta$~~							

$$a_0 = \frac{2}{N}\sum y = \frac{2}{6}(42) = 14$$

$$a_1 = \frac{2}{N}\sum y\cos\theta = \frac{2}{6}(13.696) = 4.565$$

$$a_2 = \frac{2}{N}\sum y\cos 2\theta = \frac{2}{6}(-8.5) = -2.833$$

$$a_3 = \frac{2}{N} \sum y \cos 3\theta = \frac{2}{6}(-5) = -1.667$$

UNIT - 2

Partial Differential Equations

* **Order and Degree:**

i. $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0$ order = 2 degree = 1

ii. $\frac{\partial^2 z}{\partial x \partial y} = xy$ order = 2 degree = 1

iii. $\frac{\partial^2 z}{\partial x^2} + \frac{\partial z}{\partial y} = xy$ order = 2 degree = 1

* **Formation of a PDE by eliminating arbitrary constant and arbitrary functions:**

Many problems in vibration of strings, heat conductions, electro statics involve two or more variables, analysis of these problems leads to partial derivatives and equations involving them.

An equation involving one or more partial derivatives for a function of two or more variables is called a partial differential equations.

The order of the PDE is the order of the highest derivative and the degree of the PDE is the degree of the highest order derivative after clearing the equation of fractional powers.

Given a relation of the form $f(x, y, z, a, b) = 0$, where z is a function of (x, y) and a, b are arbitrary constants, we differentiate the given relation with respect to x and y partially and eliminate the arbitrary constants a, b to form the PDE. In case the number of arbitrary constants are more then the number of independent variables we need appropriate number of partial derivatives of second order or higher also.

Suppose z is a function of two arbitrary functions we have to find partial derivatives upto the second order and use the necessary partial derivatives of the second order to form the PDE by eliminating the arbitrary functions.

The following are standard notations when z is a function of two independent variables

$$p = \frac{\partial z}{\partial x}, \quad q = \frac{\partial z}{\partial y}, \quad r = \frac{\partial^2 z}{\partial x^2}$$

$$s = \frac{\partial^2 z}{\partial x \partial y}, \quad t = \frac{\partial^2 z}{\partial y^2}$$

Q1: Find the PDE by eliminating arbitrary constants

$$z = (x+a)(y+b) \quad \text{—— (1)}$$

Sol: diff partially wrt x, y

$$p = \frac{\partial z}{\partial x} = (y+b) \quad \text{—— (2)}$$

$$q = \frac{\partial z}{\partial y} = (x+a) \quad \text{—— (3)}$$

Substitute eq (2) and (3) in (1)

$z = pq$ is the required PDE.

Q2: $$2z = \frac{x^2}{a^2} + \frac{y^2}{b^2} \quad \text{—— (1)}$$

Sol: diff eq (1) partially wrt x and y

$$2\frac{\partial z}{\partial x} = \frac{2x}{a^2} \Rightarrow p = \frac{x}{a^2} \Rightarrow a^2 = \frac{x}{p}$$

$$2\frac{\partial z}{\partial y} = \frac{2y}{b^2} \Rightarrow q = \frac{y}{b^2} \Rightarrow b^2 = \frac{y}{q}$$

substituting a^2 and b^2 in eq ①

$$2z = \frac{x^2}{x/p} + \frac{y^2}{y/q}$$

$2z = xp + yq$ // is the required PDE

Q5. $ax^2 + by^2 + z^2 = 1$

Sol: $z^2 = 1 - ax^2 - by^2$ —— ①

diff eq ① w.r.t x and y

$$2z\frac{\partial z}{\partial x} = -2ax$$

$$zp = -ax$$

$$a = \frac{-zp}{x}$$

$$2z\frac{\partial z}{\partial y} = -2by$$

$$zq = -by$$

$$b = -\frac{zq}{y}$$

substituting a and b in eq ①

$$z^2 = 1 - \left(\frac{-zp}{x}\right)x^2 - \left(\frac{-zq}{y}\right)y^2$$

$z^2 = 1 + zpx + zqy$ // is the required PDE.

Q4: $z = a\log(x^2+y^2) + b^2$ —— ①

Sol: diff eq ① w.r.t x

$$\frac{\partial z}{\partial x} = \frac{a}{x^2+y^2}(2x)$$

$$p = \frac{2ax}{x^2+y^2} \text{ —— ②} \quad \cancel{\Rightarrow a = \frac{p(x^2+y^2)}{2x}}$$

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diff eq ① wrt y

$$\frac{\partial z}{\partial y} = \frac{a}{(x^2+y^2)}(2y)$$

$$q = \frac{2ay}{(x^2+y^2)} \quad \text{—— ③}$$

Dividing eq ② and eq ③

$$\frac{p}{q} = \frac{\cancel{2a}x}{\cancel{(x^2+y^2)}} \cdot \frac{\cancel{(x^2+y^2)}}{\cancel{2a}y}$$

$$\frac{p}{q} = \frac{x}{y} \Rightarrow \underline{py = xq}$$ is the required PDE

Q5: $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$

Sol: $\frac{z^2}{c^2} = 1 - \frac{x^2}{a^2} - \frac{y^2}{b^2}$ —— ①

diff eq ① wrt x

$$\frac{2z}{c^2}\left(\frac{\partial z}{\partial x}\right) = \frac{-2x}{a^2}$$

$$\frac{zp}{c^2} = \frac{-x}{a^2} \Rightarrow \frac{1}{a^2} = \frac{-zp}{xc^2} \quad \text{—— ②}$$

diff eq ① wrt y

$$\frac{2z}{c^2}\left(\frac{\partial z}{\partial y}\right) = \frac{-2y}{b^2}$$

$$\frac{zq}{c^2} = \frac{-y}{b^2} \Rightarrow \frac{1}{b^2} = \frac{-zq}{yc^2} \quad \text{—— ③}$$

~~Substituting eq ② and ③ in eq ①~~

$$\cancel{\frac{z^2}{c^2} = 1 - x^2\left(\frac{-zp}{xc^2}\right) - y^2\left(\frac{-zq}{yc^2}\right)}$$

$$\cancel{z^2 = c^2 + zpx + zqy}$$

Q6:

Sol:

Q7:

Sol:

diff eq ② wrt x

$$\frac{-1}{a^2} = \frac{1}{c^2}\left[zr + pp\right]$$

$$\frac{-1}{a^2} = \frac{1}{c^2}\left[zr + p^2\right]$$

$$\frac{zp}{xc^2} = \frac{1}{c^2}\left[zr + p^2\right] \quad \text{from eq ④}$$

$zp = xzr + xp^2$ is the required PDE

Q: $z = xy + y\sqrt{x^2-a^2} + b$ —— ①

Sol: diff eq ① wrt x

$$p = \frac{\partial z}{\partial x} = y + y\frac{1}{\cancel{2}\sqrt{x^2-a^2}}(\cancel{2}x)$$

$$p = y + \frac{yx}{\sqrt{x^2-a^2}} \quad \text{—— ②}$$

diff eq ① wrt y

$$q = \frac{\partial z}{\partial y} = x + \sqrt{x^2-a^2}$$

$$q - x = \sqrt{x^2-a^2} \quad \text{—— ③}$$

Substituting eq ③ in eq ②

$$p = y + \frac{xy}{q-x}$$

Q7 Find the PDE by eliminating arbitrary function

$$z = f(x^2+y^2) \quad \text{—— ①}$$

Sol: diff eq ① wrt x

$$p = \frac{\partial z}{\partial x} = f'(x^2+y^2)\,2x \quad \text{—— ②}$$

diff eq ① wrt y

$q = \frac{\partial z}{\partial y} = f'(x^2+y^2).2y$ —③

Dividing eq ② by eq ③

$\frac{p}{q} = \frac{f'(x^2+y^2)2x}{f'(x^2+y^2)2y}$

$\underline{py = qx}$

Q8: $z = y^2 + 2f\left(\frac{1}{x} + \log y\right)$ —①

sol: diff wrt x

$p = \frac{\partial z}{\partial x} = 2f'\left(\frac{1}{x} + \log y\right) \cdot \frac{-1}{x^2}$

$px^2 = -2f'\left(\frac{1}{x} + \log y\right)$ —②

diff wrt y

$q = \frac{\partial z}{\partial y} = 2y + 2f'\left(\frac{1}{x} + \log y\right)\frac{1}{y}$

$(q-2y)y = 2f'\left(\frac{1}{x} + \log y\right)$ —③

Substituting eq ③ in eq ②

$\underline{px^2 = -(q-2y)y}$

Q9: $z = e^{ax+by} f(ax-by)$ —①

sol: diff eq ① wrt x

$\frac{\partial z}{\partial x} = e^{ax+by} f'(ax-by).(+a) + f(ax-by)e^{ax+by}(a)$

$p = ae^{ax+by}\left[f(ax-by) + f'(ax-by)\right]$ —②

diff eq ① wrt y

$\frac{\partial z}{\partial y} = e^{ax+by} f'(ax-by)(-b) + f(ax-by)e^{ax+by}(b)$

$q = be^{ax+by}\left[f(ax-by) - f'(ax-by)\right]$ —— ③

Multiplying a to eq ③ and b to eq ②

eq ②: $bp = abe^{ax+by}\left[f(ax-by) + f'(ax-by)\right]$ —— ④

eq ③: $aq = abe^{ax+by}\left[f(ax-by) - f'(ax-by)\right]$ —— ⑤

Adding eq ④ and eq ⑤

~~$pb = abe^{ax+by}$~~

$pb + aq = 2abe^{ax+by} f(ax-by)$

$pb + aq = 2ab\, e^{ax+by} \frac{z}{e^{ax+by}}$ from eq ①

$\underline{pb + aq = 2abz}$

Q10: $lx + my + nz = \phi(x^2+y^2+z^2)$ —— ①

sol: diff eq ① wrt x

$l + n\frac{\partial z}{\partial x} = \phi'(x^2+y^2+z^2)\left(2x + 2z\frac{\partial z}{\partial x}\right)$

$l + np = 2\phi'(x^2+y^2+z^2)(x+zp)$ —— ②

diff eq ② wrt y

$m + n\frac{\partial z}{\partial y} = \phi'(x^2+y^2+z^2)\left(2y + 2z\frac{\partial z}{\partial y}\right)$

$m + nq = 2\phi'(x^2+y^2+z^2)(y+zq)$ —— ③

Dividing eq ② by eq ③

$\frac{l+np}{m+nq} = \frac{x+zp}{y+zq}$

$$(l+np)(y+zq) = (x+zp)(m+nq)$$

$$ly + lzq + npy + \cancel{npzq} = xm + xnq + mzp + \cancel{nzpq}$$

$$p(ny-mz) + q(lz-nx) = mx - ly$$

- NOTE:

To form the PDE from $\phi(u,v)=0$ where u and v are functions of (x,y) are as follows.

$$\frac{\partial\phi}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial\phi}{\partial v}\frac{\partial v}{\partial x} = 0$$

$$\frac{\partial\phi}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial\phi}{\partial v}\frac{\partial v}{\partial y} = 0$$

Q1: $\phi(x+y+z,\ x^2+y^2-z^2) = 0$

Sol: It is of the form $\phi(u,v)=0$

Here $u = x+y+z$

$v = x^2+y^2-z^2$

$$\frac{\partial\phi}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial\phi}{\partial v}\frac{\partial v}{\partial x} = 0 \quad \text{--- (1)}$$

$$\frac{\partial\phi}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial\phi}{\partial v}\frac{\partial v}{\partial y} = 0 \quad \text{--- (2)}$$

From eq (1) $\frac{\partial\phi}{\partial u}\frac{\partial u}{\partial x} = -\frac{\partial\phi}{\partial v}\frac{\partial v}{\partial x}$ --- (3)

From eq (2) $\frac{\partial\phi}{\partial u}\frac{\partial u}{\partial y} = -\frac{\partial\phi}{\partial v}\frac{\partial v}{\partial y}$ --- (4)

Dividing eq (3) by eq (4)

$$\frac{\partial u/\partial x}{\partial u/\partial y} = \frac{\partial v/\partial x}{\partial v/\partial y} \quad \text{--- (5)}$$

diff u w.r.t x

$$\frac{\partial u}{\partial x} = 1 + \frac{\partial z}{\partial x} = 1 + p$$

diff u w.r.t y

$$\frac{\partial u}{\partial y} = 1 + \frac{\partial z}{\partial y} = 1 + q$$

diff v w.r.t x

$$\frac{\partial v}{\partial x} = 2x - 2z\frac{\partial z}{\partial x} = 2x - 2pz$$

diff v w.r.t y

$$\frac{\partial v}{\partial y} = 2y - 2z\frac{\partial z}{\partial y} = 2y - 2qz$$

substituting in eq 5

$$\frac{1+p}{1+q} = \frac{2x-2pz}{2y-2qz}$$

$$(1+p)(y-qz) = (1+q)(x-pz)$$

$$y + py - qz - \cancel{pqz} = x - pz - \cancel{pqz} + qx$$

$$\underline{p(y+z) - q(x+z) = x - y}$$

Q2: $\phi(xy + x^2, x+y+z) = 0$

sol: It is of the form $\phi(u,v) = 0$

Here $u = xy + x^2$

$v = x + y + z$

$$\frac{\partial \phi}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial \phi}{\partial v}\frac{\partial v}{\partial x} = 0 \quad \text{—— (1)}$$

$$\frac{\partial \phi}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial \phi}{\partial v}\frac{\partial v}{\partial y} = 0 \quad \text{—— (2)}$$

From eq (1) $\frac{\partial \phi}{\partial u}\frac{\partial u}{\partial x} = -\frac{\partial \phi}{\partial v}\frac{\partial v}{\partial x}$ —— (3)

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From eq ② $\frac{\partial\phi}{\partial u}\frac{\partial u}{\partial y} = -\frac{\partial\phi}{\partial v}\frac{\partial v}{\partial y}$ —— ④

Dividing eq ③ by eq ④

$$\frac{\partial u/\partial x}{\partial u/\partial y} = \frac{\partial v/\partial x}{\partial v/\partial y} \quad —— ⑤$$

diff u wrt x

$$\frac{\partial u}{\partial x} = y + 2z\frac{\partial z}{\partial x} = y + 2zp$$

diff u wrt y

$$\frac{\partial u}{\partial y} = x + 2z\frac{\partial z}{\partial y} = x + 2zq$$

diff v wrt x

$$\frac{\partial v}{\partial x} = 1 + \frac{\partial z}{\partial x} = 1 + p$$

diff v wrt y

$$\frac{\partial v}{\partial y} = 1 + \frac{\partial z}{\partial y} = 1 + q$$

Substituting in eq ⑤

$$\frac{y+2zp}{x+2zq} = \frac{1+p}{1+q}$$

$$(y+2zp)(1+q) = (x+2zq)(1+p)$$

$$y + qy + \cancel{2pqz} + 2pz = x + xp + 2qz + \cancel{2pqz}$$

$$\cancel{z(2p-2q)}\ p(x-2z) - q(y-2z) = y-x$$

Q3: $f(x^2+2yz,\ y^2+2zx) = 0$

sol: It is of the form $\phi(u,v) = 0$

where $u = x^2+2yz$

$v = y^2+2zx$

$$\frac{\partial f}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial x} = 0 \Rightarrow \frac{\partial f}{\partial u}\frac{\partial u}{\partial x} = -\frac{\partial f}{\partial v}\frac{\partial v}{\partial x} \quad \text{--- (1)}$$

$$\frac{\partial f}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial f}{\partial v}\frac{\partial v}{\partial y} = 0 \Rightarrow \frac{\partial f}{\partial u}\frac{\partial u}{\partial y} = -\frac{\partial f}{\partial v}\frac{\partial v}{\partial y} \quad \text{--- (2)}$$

Dividing eq (1) by eq (2)

$$\frac{\partial u/\partial x}{\partial u/\partial y} = \frac{\partial v/\partial x}{\partial v/\partial y} \quad \text{--- (3)}$$

diff u wrt x

$$\frac{\partial u}{\partial x} = 2x + 2y\frac{\partial z}{\partial x} = 2(x+yp)$$

diff u wrt y

$$\frac{\partial u}{\partial y} = 2z + 2y\frac{\partial z}{\partial y} = 2(z+yq)$$

diff v wrt x

$$\frac{\partial v}{\partial x} = 2z + 2x\frac{\partial z}{\partial x} = 2(z+xp)$$

diff v wrt y

$$\frac{\partial v}{\partial y} = 2y + 2x\frac{\partial z}{\partial y} = 2(y+xq)$$

substituting in eq (3)

$$\frac{2(x+yp)}{2(z+yq)} = \frac{2(z+xp)}{2(y+xq)}$$

$$(x+yp)(y+xq) = (z+xp)(z+yq)$$

$$xy + x^2q + y^2p + xyqp = z^2 + zyq + zxp + xypq$$

$$p(y^2 - zx) + q(x^2 - zy) = z^2 - xy$$

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Q4: $z = yf(x) + x\phi(y)$ — ①

Sol: diff z wrt x

$p = \frac{\partial z}{\partial x} = yf'(x) + \phi(y)$ — ②

diff z wrt y

$q = \frac{\partial z}{\partial y} = f(x) + x\phi'(y)$ — ③

diff z wrt x^2

$\frac{\partial p}{\partial x} = r = yf''(x)$ — ④

diff z wrt y^2

$\frac{\partial q}{\partial y} = t = x\phi''(y)$ — ⑤

diff p wrt y

$s = f'(x) + \phi'(y)$ — ⑥

From eq ②

$f'(x) = \frac{p - \phi(y)}{y}$

From eq ③

$\phi'(x) = \frac{q - f(x)}{x}$

Substituting in eq ⑥

$s = \frac{p - \phi(y)}{y} + \frac{q - f(x)}{x}$

$xys = xp - x\phi(y) + yq - yf(x)$

$xys = xp + yq - z$. (From eq ①)

$z = xp + yq - xys$

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$z = x f_1(x+y) + f_2(x+y)$ —— ①

diff z wrt x

$p = \frac{\partial z}{\partial x} = f_1(x+y) + x f_1'(x+y) + f_2'(x+y)$ —— ②

diff z wrt y

$q = \frac{\partial z}{\partial y} = x f_1'(x+y) + f_2'(x+y)$ —— ③

diff z wrt x^2

$r = \frac{\partial p}{\partial x} = f_1'(x+y) + x f_1''(x+y) + f_1'(x+y) + f_2''(x+y)$

$r = 2 f_1'(x+y) + x f_1''(x+y) + f_2''(x+y)$ —— ④

diff z wrt y^2

$t = \frac{\partial q}{\partial y} = x f_1''(x+y) + f_2''(x+y)$ —— ⑤

diff z wrt x and y

$s = \frac{\partial p}{\partial y} = f_1'(x+y) + x f_1''(x+y) + f_2''(x+y)$ —— ⑥

Substituting eq ⑤ in eq ④

$r = 2 f_1'(x+y) + t$

Substituting eq ⑤ in eq ⑥

$s = f_1'(x+y) + t$

$r = 2 f_1'(x+y) + t$

$2s = 2 f_1'(x+y) + 2t$

$r - 2s = -t$

$$\frac{\partial^2 z}{\partial x^2} - 2\frac{\partial^2 z}{\partial x \partial y} + \frac{\partial^2 z}{\partial y^2} = 0$$

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Q6: $z = f(y+x) + g(y+2x)$ — ①

sol: diff z wrt x

$p = \frac{\partial z}{\partial x} = f'(y+x) + 2g'(y+2x)$ — ②

diff z wrt y

$q = \frac{\partial z}{\partial y} = f'(y+x) + g'(y+2x)$ — ③

diff z wrt x^2

$r = \frac{\partial p}{\partial x} = f''(y+x) + 4g''(y+2x)$ — ④

diff z wrt y^2

$t = \frac{\partial q}{\partial y} = f''(y+x) + g''(y+2x)$ — ⑤

diff z wrt xy

$s = \frac{\partial p}{\partial y} = f''(y+x) + 2g''(y+2x)$ — ⑥

Subtracting eq ④ and eq ⑥

$r - t = 2g''(y+2x)$

Subtracting eq ⑥ and eq ⑤

$s - t = g''(y+2x)$

$\Rightarrow r - t = 2(s-t)$

$r - t = 2s - 2t$

$r = 2s - t$

$r - 2s + t = 0$ $\qquad \frac{\partial^2 z}{\partial x^2} - 2\frac{\partial^2 z}{\partial x \partial y} + \frac{\partial^2 z}{\partial y^2} = 0$

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$z = f(y-2x) + g(2y-x)$

diff z wrt x

$p = \frac{\partial z}{\partial x} = -2f'(y-2x) - g'(2y-x)$ — (1)

diff z wrt y

$q = \frac{\partial z}{\partial y} = f'(y-2x) + 2g'(2y-x)$ — (2)

diff z wrt x^2

$r = \frac{\partial p}{\partial x} = 4f''(y-2x) + g''(2y-x)$ — (3)

diff z wrt y^2

$t = \frac{\partial q}{\partial y} = f''(y-2x) + 4g''(2y-x)$ — (4)

diff z wrt x and y

$s = \frac{\partial p}{\partial y} = -2f''(y-2x) - 2g''(2y-x)$ — (5)

$r + 2s = -3g''(2y-x)$

similarly $2t + s = 6g''(2y-x)$

$\therefore \frac{r+2s}{2t+s} = \frac{-1}{2}$

$2r + 4s = -2t - s$

$2r + 5s + 2t = 0$

$2\frac{\partial^2 z}{\partial x^2} + 5\frac{\partial^2 z}{\partial x \partial y} + 2\frac{\partial^2 z}{\partial y^2} = 0$

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★ **Solution of Non-Homogeneous PDE by direct Integration Method**

In this method we find the dependent variable which be the solution, by removing the differential operator through the process of integration

Ex: If u is function of x, y, z, then the solution $u(x, y, z)$

$$\frac{\partial u}{\partial x} = x^2 \Rightarrow u = \frac{x^3}{3} + f(y, z)$$

Q1: Solve

$$\frac{\partial^2 u}{\partial x^2} = x + y$$

Sol: Given $\frac{\partial^2 u}{\partial x^2} + x + y$

$$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) = x + y$$

Integrating wrt x by keeping y as constant.

$$\frac{\partial u}{\partial x} = \frac{x^2}{2} + xy + f(y)$$

Integrating wrt x by keeping y as constant

$$u = \frac{x^3}{6} + \frac{x^2 y}{2} + x f(y) + g(y)$$

Q2 Solve: $\frac{\partial^2 z}{\partial x \partial y} = \frac{x}{y} + a$

Sol: Given $\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{x}{y} + a$

Integrating wrt x by keeping y as constant

$$\int \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) dx = \int \frac{x}{y}\, dx + \int a\, dx$$

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$$\frac{\partial z}{\partial y} = \frac{x^2}{2y} + ax + f(y)$$

Integrating wrt y by keeping x as constant.

$$\int \frac{\partial z}{\partial y} dy = \int \frac{x^2}{2y} dy + \int ax\, dy + \int f(y)\, dy$$

$$z = \frac{x^2}{2} \log y + axy + y F(y) + g(x) \qquad \text{where } F(y) = \int f(y) dy$$

Q: Solve: $\frac{\partial^2 u}{\partial x \partial t} = e^{-t} \cos x$ given that $u = 0$ when $t = 0$

and $\frac{\partial u}{\partial t} = 0$ at $x = 0$. Also show that $u \to \sin x$ as $t \to \infty$.

Sol: Given: $\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial t}\right) = e^{-t} \cos x$

Integrating wrt x by keeping t as constant

$$\int \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial t}\right) dx = \int e^{-t} \cos x\, dx$$

$$\frac{\partial u}{\partial t} = e^{-t} \sin x + f(t) \quad \text{--- ①}$$

Integrating wrt t keeping x as constant

$$\int \frac{\partial u}{\partial t} dt = \int e^{-t} \sin x\, dt + \int f(t)\, dt$$

where $\int f(t) dt = F(t)$

$$u = -e^{-t} \sin x + tF(t) + g(x) \quad \text{--- ②}$$

given that $\frac{\partial u}{\partial t} = 0$ at $x = 0$

From eq ① $\quad 0 = e^{-t} \sin 0 + f(t)$

$\Rightarrow f(t) = 0$

given that $u = 0$ at $t = 0$

From eq ② $\quad 0 = -e^{0} \sin x + 0(F(t)) + g(x)$

$g(x) = \sin x$

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Substituting $f(t)$ in eq 1

$$\frac{\partial u}{\partial t} = e^{-t}\sin x$$

Substituting $g(x)$ in eq 2

$$u = -e^{-t}\sin x + \sin x$$

$$u = \sin x(1 - e^{-t})$$

when

$t \to \infty$ it implies $e^{-t} = 0$

$\therefore u = \sin x$ Hence $u \to \sin x$

Q4: Solve: $\frac{\partial^2 z}{\partial x \partial y} = \frac{x}{y}$ subjected to the conditions

i. $\frac{\partial z}{\partial x} = \log x$ when $y = 1$ ~~and $z=0$~~

ii. $z = 0$ ~~$\frac{\partial z}{\partial x} = \log x$~~ when $x = 1$.

Sol: Given: $\frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = \frac{x}{y}$

integrating wrt y and keeping x as constant

$$\int \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) dy = \int \frac{x}{y}\, dy$$

$$\frac{\partial z}{\partial x} = x \log y + f(x) \quad \text{—— } ①$$

Integrating wrt x and keeping y as constant

$$\int \frac{\partial z}{\partial x} dx = \int x \log y \, dx + \int f(x)\, dx \qquad \text{where } \int f(x)\,dx = F(x)$$

$$z = \frac{x^2}{2}\log y + F(x) + g(y) \quad \text{—— } ②$$

i. Given that $\frac{\partial z}{\partial x} = \log x$ when $y = 1$

From eq 1 $\log x = x \log 1 + f(x)$

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$f(x) = \log x$

$F(x) = \int f(x)\,dx \quad \int \log x\,dx = x\log x - x$

$\therefore z = 0$ when $x = 1$

From eq ② $\quad 0 = \frac{1}{2}\log y + 1\log 1 - 1 + g(y)$

$$g(y) = 1 - \frac{1}{2}\log y$$

Substituting in eq ②

$$z = \frac{x^2}{2}\log y + x\log x - x + 1 - \frac{1}{2}\log y$$

Solve: $\frac{\partial^2 z}{\partial x\,\partial y} = \sin x \sin y$ for which $\frac{\partial z}{\partial y} = -2\sin y$ when $x = 0$ and $z = 0$ if $y = (2n+1)\frac{\pi}{2}$.

<u>Given</u>: $\frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \sin x \sin y$

Integrating wrt x by keeping y as constant

$$\int \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) dx = \int \sin x \sin y\,dx$$

$$\frac{\partial z}{\partial y} = -\sin y\cos x + f(y) \quad \text{—— ①}$$

Integrating wrt y by keeping x as constant

$$\int \frac{\partial z}{\partial y}\,dy = \int -\sin y\cos x\,dy + \int f(y)\,dy \quad \text{where } \int f(y)\,dy = F(y)$$

$$z = \cos x\cos y + F(y) + g(x) \quad \text{—— ②}$$

~~For~~ Given that

$\frac{\partial z}{\partial y} = -2\sin y$ when $x = 0$

From eq ① $\quad -2\sin y = -\sin y\cos 0 + f(y)$

$$f(y) = -\sin y \implies F(y) = \cos y$$

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Given that $z=0$ if $y=(2n+1)\frac{\pi}{2}$

From eq ②

$$0 = \cos x \cos(2n+1)\frac{\pi}{2} + \cos(2n+1)\frac{\pi}{2} + g(x)$$

$$g(x) = 0$$

Substituting in eq ②

$$z = \cos x \cos y + \cos y$$

Q6: $\frac{\partial^2 z}{\partial x^2} = xy$

i. $\frac{\partial z}{\partial x} = \log(1+y)$ when $x=1$

ii. $z=0$ when $x=0$

Sol: Given: $\frac{\partial^2 z}{\partial x^2} = xy \Rightarrow \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = xy$

Integrating wrt x keeping y as constant

$$\int \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) dx = \int xy \, dx$$

$$\frac{\partial z}{\partial x} = \frac{x^2 y}{2} + f(y) \quad \text{--- ①}$$

Integrating wrt x keeping y as constant.

$$\int \frac{\partial z}{\partial x} dx = \int \left(\frac{x^2 y}{2} + f(y)\right) dx$$

$$z = \frac{x^3 y}{6} + x f(y) \quad \text{--- ②}$$

Given that $\frac{\partial z}{\partial x} = \log(1+y)$ when $x=1$

From eq ① $\log(1+y) = \frac{y}{2} + f(y) \Rightarrow f(y) = \log(1+y) - \frac{y}{2}$

Given that $z = 0$ when $x = 0$

From eq ② $\quad 0 = 0$

Substituting in eq ⑥

$$z = \frac{x^3 y}{6} + x\left(\log(1+y) - \frac{y}{2}\right)$$

$$z = \frac{x^3 y}{6} + x\log(1+y) - \frac{xy}{2}$$

Solution of Homogeneous PDE involving derivatives with respect to one independent variable:

Suppose that the dependent variable has been differentiated partially with respect to the independent variable, then the PDE can be treated as ordinary differential equations. The arbitrary constant in the solution are replaced by arbitrary function and of the other variable. This gives a solution for the PDE.

$$\frac{\partial^2 z}{\partial x^2} + z = 0, \text{ Given that } x = 0, z = e^y \text{ and } \frac{\partial z}{\partial x} = 1.$$

Given $\frac{\partial^2 z}{\partial x^2} + z = 0$

$D^2 z + z = 0$

$(D^2 + 1)z = 0$

The auxillary equation is given by

$m^2 + 1 = 0$

$m^2 = -1$

$m = \pm i$ are the roots.

The solution is given by

$z = c_1 \cos x + c_2 \sin x$

$z = f(y)\cos x + g(y)\sin x \quad$ ①

diff eq ① wrt x

$$\frac{\partial z}{\partial x} = -f(y)\sin x + g(y)\cos x \quad \text{—— ②}$$

Given conditions. $z = e^y$ at $x=0$ in eq ①

$$e^y = f(y)\cos 0 + g(y)\sin 0$$
$$e^y = f(y)$$

Substituting $f(y)$ in eq ②

$$\frac{\partial z}{\partial x} = -e^y \sin x + g(y)\cos x$$

Given $\frac{\partial z}{\partial x} = 1$ at $x=0$ in eq ②

$$\Rightarrow 1 = +e^y \sin 0 + g(y)\cos 0$$
$$\therefore \; g(y) = 1$$

Substituting $f(y)$ and $g(y)$ in eq ①

$$\underline{z = e^y \cos x + \sin x}$$

Q2: $\frac{\partial^2 z}{\partial y^2} = z$ given that $y=0$, $z = e^x$ and $\frac{\partial z}{\partial y} = e^{-x}$

<u>Sol</u>: Given $\frac{\partial^2 z}{\partial y^2} - z = 0$

$$D^2 z - z = 0$$
$$(D^2 - 1)z = 0$$

The auxillary equation is

$$m^2 - 1 = 0$$
$$m^2 = 1$$

$\underline{m = \pm 1}$ are the roots.

The solution is given by,

$$z = c_1 e^y + c_2 e^{-y}$$
$$z = f(x)e^y + g(x)e^{-y} \quad \text{—— ①}$$

Q3:

diff eq ① wrt y

$$\frac{\partial z}{\partial y} = f(x)e^{y} - g(x)e^{-y} \quad \text{——— ②}$$

Given condition $y=0$ and $z=e^{x}$

From eq ① $e^{x} = f(x)e^{0} + g(x)e^{-0}$

$$e^{x} = f(x) + g(x) \quad \text{——— ③}$$

Given condition $\frac{\partial z}{\partial y} = e^{-x}$ at $y=0$

From eq ②

$$e^{-x} = f(x)e^{0} - g(x)e^{0}$$

$$e^{-x} = f(x) - g(x) \quad \text{——— ④}$$

Adding eq ③ and eq ④

$$2f(x) = e^{x} + e^{-x}$$

$$f(x) = \frac{e^{x}+e^{-x}}{2} = \cosh x$$

Substituting in eq ④

$$g(x) = \frac{e^{x}-e^{-x}}{2} = \sinh x$$

Substituting $f(x)$ and $g(x)$ in eq ①

$$\cancel{z = \frac{e^{x}e^{y}}{2} + \frac{e^{x}e^{-y}}{2}}$$

$$\cancel{z = \frac{e^{x+y}+e^{x-y}}{2}}$$

$$\underline{z = \cosh x\, e^{y} + \sinh e^{-y}}$$

Q3: $\frac{\partial^2 u}{\partial x^2} + u = 0$ given conditions

i. $u(0,y) = \sqrt{e}$

ii. $\frac{\partial u}{\partial x}(0,y) = 1$

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Sol: Given: $\frac{\partial^2 u}{\partial x^2} + u = 0$

$D^2 u + u = 0$

$(D^2+1)u = 0$

The auxillary equation is

$m^2 + 1 = 0$

$m^2 = -1$

$m = \pm i$ are the roots

$\therefore$ The solution is given by

$u = c_1 \cos x + c_2 \sin x$

$u = f(y)\cos x + g(y)\sin x$ —— (1)

diff eq (1) wrt x

$\frac{\partial u}{\partial x} = -f(y)\sin x + g(y)\cos x$ —— (2)

Given condition $u = \sqrt{e}$ at $x = 0$

From eq (1)

$\sqrt{e} = f(y)\cos 0 + g(y)\sin 0$

$f(y) = \sqrt{e}$

Given condition $\frac{\partial u}{\partial x} = 1$ at $x = 0$

From eq (2)

$1 = -f(y)\sin 0 + g(y)\cos 0$

$g(y) = 1$

Substituting in eq (1)

$u = \sqrt{e}\cos x + \sin x$

Q4: Solve: $\frac{\partial^3 z}{\partial x^3} + 4\frac{\partial z}{\partial x} = 0$, given that $z = 0$, $\frac{\partial z}{\partial x} = 0$, $\frac{\partial^2 z}{\partial x^2} = 4$

when $x = 0$.

Sol: Given $\frac{\partial^3 z}{\partial x^3} + 4\frac{\partial z}{\partial x} = 0$

$D^3z + 4Dz = 0$

$D(D^2+4)z = 0$

The auxillary equation is

$m(m^2+4) = 0$

$\underline{m=0}$ $\quad m^2+4=0$ $\qquad \therefore 0, \pm 2i$ are the roots.

$m^2 = -4$

$\underline{m = \pm 2i}$

$\therefore$ the solution is

$z = c_1e^0 + c_2\cos 2x + c_3\sin 2x$

$z = f(y) + g(y)\cos 2x + h(y)\sin 2x \quad \text{——} ①$

diff eq ① wrt x

$\frac{\partial z}{\partial x} = -2g(y)\sin 2x + 2h(y)\cos 2x \quad \text{——} ②$

~~[illegible]~~

diff wrt x (eq ②)

$\frac{\partial^2 z}{\partial x^2} = -4g(y)\cos 2x - 4h(y)\sin 2x \quad \text{——} ③$

~~diff eq ③ wrt x~~

$\cancel{\frac{\partial^3 z}{\partial x^3} = 8g(y)\sin 2x - 8h(y)\cos 2x} \quad \cancel{④}$

Given condition $z=0$ at $x=0$

From eq ①

$0 = f(y) + g(y)\cos 0 + h(y)\sin 0$

$0 = f(y) + g(y) \quad \text{——} ④$

Given condition $\frac{\partial z}{\partial x} = 0$ at $x = 0$

From eq ②

$0 = -2g(y)\sin 0 + 2h(y)\cos 0$

$\Rightarrow h(y) = 0$ — (5)

Given condition $\frac{\partial^2 x}{\partial x^2} = 4$ at $x = 0$

From eq (3)

$4 = -4g(y)\cos 0 - 4h(y)\sin 0$

$\Rightarrow g(y) = -1$ — (6)

Substituting eq (6) in eq (4)

$f(y) = 1$

Substituting $f(y)$, $g(y)$ and $h(y)$ in eq (1)

$x = 1 - \cos 2x$

Q5: solve: $\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial u}{\partial x}$ using the substitution $\frac{\partial u}{\partial x} = v$

sol: Given: $\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial u}{\partial x}$

$\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right) = \frac{\partial u}{\partial x}$

Given $\frac{\partial u}{\partial x} = v$

$\therefore \frac{\partial u}{\partial x} = v$

$\Rightarrow \frac{\partial v}{\partial y} - v = 0$

$Dv - v = 0$

$(D-1)v = 0$

The auxillary equation is

$m - 1 = 0$

$m = 1$ is the root

$\therefore$ The solution is given by

$v = ce^y$

$v = f(x)e^y$

but $v = \frac{\partial u}{\partial x}$

$\therefore \frac{\partial u}{\partial x} = f(x)e^{y}$

Integrating wrt x

$\int \frac{\partial u}{\partial x} dx = \int f(x) e^{y} dx$

$u = e^{y}F(x) + g(y)$ where $\int f(x)dx = F(x)$

Q6: Solve: $\frac{\partial^3 u}{\partial x^2 \partial y} - \frac{\partial u}{\partial y} = 0$ using the substitution $\frac{\partial u}{\partial y} = v$

Sol: Given: $\frac{\partial^2}{\partial x^2}\left(\frac{\partial u}{\partial y}\right) - \frac{\partial u}{\partial y} = 0$

$\Rightarrow \frac{\partial^2 v}{\partial x^2} - v = 0$

$D^2 v - v = 0$

$(D^2 - 1)v = 0$

The auxillary equation is

$m^2 - 1 = 0$

$m^2 = 1$

$m = \pm 1$ are the roots

$\therefore$ The solution is given by

$v = c_1 e^{x} + c_2 e^{-x}$

$v = f(y)e^{x} + g(y)e^{-x}$

but $v = \frac{\partial u}{\partial y}$

$\therefore \frac{\partial u}{\partial y} = f(y)e^{x} + g(y)e^{-x}$

Integrating wrt y

$\int \frac{\partial u}{\partial y} dy = \int f(y) e^{x} dy + \int g(y) e^{-x} dy$

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$u = F(y)e^{x} + G(y)e^{-x} + h(y)$ where $F(y) = \int f(y)\,dy$

$G(y) = \int g(y)\,dy$

Q: Solve: $\frac{\partial^2 z}{\partial x^2} = a^2 z$; given that when $x=0$, $\frac{\partial z}{\partial x} = a\sin y$

and $\frac{\partial z}{\partial y} = 0$, ~~$\frac{\partial^2 z}{\partial x^2} = a^2 z$~~

Sol: Given: $\frac{\partial^2 z}{\partial x^2} - a^2 z = 0$

$D^2 z - a^2 z = 0$

$(D^2 - a^2) z = 0$

The auxillary equation is

$m^2 - a^2 = 0$

$m^2 = a^2$

$m = \pm a$ are the roots

$\therefore$ The solution is given by

$z = C_1 e^{ax} + C_2 e^{-ax}$

$z = f(y)e^{ax} + g(y)e^{-ax}$ —— (1)

diff eq (1) wrt x

$\frac{\partial z}{\partial x} = af(y)e^{ax} - ag(y)e^{-ax}$ —— (2)

diff eq (1) wrt y

$\frac{\partial z}{\partial y} = e^{ax} f'(y) + e^{-ax} g'(y)$ —— (3)

at $x=0$ $\frac{\partial z}{\partial x} = a\sin y$ (given)

From eq (2)

$a\sin y = af(y)e^{0} - ag(y)e^{-0}$

$f(y) - g(y) = \sin y$ —— (4)

at $x=0$ $\frac{\partial z}{\partial y} = 0$ (given

From eq (3)

$0 = e^{0} f'(y) + e^{0} g'(y)$

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$f'(y) + g'(y) = 0$

Integrating w.r.t. y

$f(y) + g(y) = k$ —— ⑤

Adding eq ④ and eq ⑤

$2f(y) = k + \sin y$

$f(y) = \frac{k + \sin y}{2}$

Substituting in eq ⑤

$g(y) = k - \frac{k + \sin y}{2} = \frac{k - \sin y}{2}$

Substituting f(y) and g(y) in eq ①

$z = \frac{k + \sin y}{2} e^{ax} + \frac{k - \sin y}{2} e^{-ax}$

* <u>Method of Seperation of Variables:</u>

This method is applicable for solving a linear homogenous PDE involving derivatives with respect to independent variables. Solution of the PDE is determined by ordinary differential equation.

Q: Solve: $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 2(x+y)u$ by the method of separation of variables

Sol: Given $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 2(x+y)u$

Let us assume $u = XY$ —— ①

$\frac{\partial (XY)}{\partial x} + \frac{\partial (XY)}{\partial y} = 2(x+y)XY$

$Y\frac{\partial X}{\partial x} + X\frac{\partial Y}{\partial y} = 2(x+y)XY$

Dividing both sides by XY

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$$\frac{1}{X}\frac{dX}{dx} + \frac{1}{Y}\frac{dY}{\partial y} = 2(x+y)$$

$$\frac{1}{X}\frac{dX}{dx} - 2x = 2y - \frac{1}{Y}\frac{dY}{dy}$$

Equating both sides to constant k

$$\frac{1}{X}\frac{dX}{dx} - 2x = k \qquad 2y - \frac{1}{Y}\frac{dY}{dy} = k$$

$$\frac{1}{X}\frac{dX}{dx} = k+2x \qquad \frac{1}{Y}\frac{dY}{dy} = 2y-k$$

$$\frac{1}{X}dX = (k+2x)dx \qquad \frac{1}{Y}dY = (2y-k)dy$$

Integrating $\qquad$ Integrating

$$\int\frac{1}{X}dX = \int(k+2x)dx \qquad \int\frac{1}{Y}dY = \int(2y-k)dy$$

$$\log X = x^2+kx+c_1 \qquad \log Y = y^2-ky+c_2$$

$$\Rightarrow X = e^{(x^2+kx+c_1)} \qquad \Rightarrow Y = e^{(y^2-ky+c_2)}$$

Substituting X and Y in eq (1)

$$u = e^{(x^2+kx+c_1)} e^{(y^2-ky+c_2)}$$

$$u = e^{(c_1+c_2)} e^{(x^2+y^2-kx-ky)}$$

$$\therefore u = c\, e^{(x^2+y^2-kx-ky)} \quad \text{where } e^{(c_1+c_2)} = c$$

Q2: Solve by the method of separation of variables $\frac{\partial u}{\partial x} = 2\frac{\partial u}{\partial t} + u$ where $u(x,0) = 6e^{-3x}$.

Sol: Given: $\frac{\partial u}{\partial x} = 2\frac{\partial u}{\partial t} + u$

Let us assume $u = XT$ —— (1)

$$\therefore \frac{\partial(XT)}{\partial x} = 2\frac{\partial(XT)}{\partial t} + XT$$

$$T\frac{dX}{dx} = 2X\frac{dT}{dt} + XT$$

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Dividing both sides by XT

$$\frac{1}{X}\frac{dX}{dx} = \frac{2}{T}\frac{dT}{dt} + 1$$

Equating both sides to constant k

$$\frac{1}{X}\frac{dX}{dx} = k$$

$$\frac{1}{X}\,dX = k\,dx$$

Integrating

$$\int \frac{1}{X}\,dX = \int k\,dx$$

$$\log X = kx + c_1$$

$$\Rightarrow X = e^{kx+c_1}$$

$$\frac{2}{T}\frac{dT}{dt} + 1 = k$$

$$\frac{1}{T}\frac{dT}{dt} = \frac{k-1}{2}$$

$$\frac{1}{T}\,dT = \frac{k-1}{2}\,dt$$

Integrating

$$\int \frac{1}{T}\,dT = \int \left(\frac{k-1}{2}\right) dt$$

$$\log T = \left(\frac{k-1}{2}\right)t + c_2$$

$$\Rightarrow T = e^{\left(\frac{k-1}{2}\right)t + c_2}$$

Substituting X and T in eq ①

$$u = e^{kx+c_1} \cdot e^{\left(\frac{k-1}{2}\right)t + c_2}$$

$$= e^{kx + \left(\frac{k-1}{2}\right)t} \cdot e^{c_1 + c_2}$$

$$u = c\,e^{kx + \left(\frac{k-1}{2}\right)t} \quad \text{where } c = e^{c_1 + c_2}$$

given $u(x, 0) = 6e^{-3x}$

i.e., $u = 6e^{-3x}$ at $t = 0$

$\therefore 6e^{-3x} = c\,e^{kx}$

$\Rightarrow c = 6$ and $k = -3$

$$\therefore u = 6\,e^{-3x-2t}$$

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Q3: solve $py^3 + qx^2 = 0$ by method of separation of variables.

sol: given: $y^3 \frac{\partial z}{\partial x} + x^2 \frac{\partial z}{\partial y} = 0$

Assume $z = XY$ —— ①

$$y^3 \frac{\partial (XY)}{\partial x} + x^2 \frac{\partial (XY)}{\partial y} = 0$$

$$y^3 Y \frac{dX}{dx} + x^2 X \frac{dY}{dy} = 0$$

dividing both sides by $x^2 y^3 XY$

$$\frac{1}{x^2 X} \frac{dX}{dx} = -\frac{1}{y^3 Y} \frac{dY}{dy}$$

Equating both sides to constant k.

$$\frac{1}{x^2 X} \frac{dX}{dx} = k \qquad \frac{-1}{y^3 Y} \frac{dY}{dy} = k$$

$$\frac{1}{X} \frac{dX}{dx} = x^2 k \qquad \frac{1}{Y} dY = -y^3 k\, dy$$

Integrating

$$\int \frac{1}{X} dX = \int x^2 k\, dx$$

Integrating

$$\int \frac{1}{Y} dY = -\int y^3 k\, dy$$

$$\log X = \frac{kx^3}{3} + c_1 \qquad \log Y = -\frac{ky^4}{4} + c_2$$

$$X = e^{\frac{kx^3}{3} + c_1} \qquad Y = e^{\frac{-ky^4}{4} + c_2}$$

Substituting X and Y in eq ①

$$z = e^{kx^3/3 + c_1} \times e^{-kx^4/4 + c_2}$$

$$= e^{k(x^3/3 - x^4/4)} \cdot e^{c_1 + c_2}$$

$$z = c\, e^{k(x^3/3 - x^4/4)}$$

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$\frac{\partial^2 z}{\partial x^2} - 2\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = 0$ by methods of variable separation.

given: $\frac{\partial^2 z}{\partial x^2} - 2\frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = 0$

Assume $z = XY$ —— ①

$$\frac{\partial^2 (XY)}{\partial x^2} - 2\frac{\partial (XY)}{\partial x} + \frac{\partial (XY)}{\partial y} = 0$$

$$Y\frac{d^2X}{dx^2} - 2Y\frac{dX}{dx} + X\frac{dY}{dy} = 0$$

dividing both sides by XY

$$\frac{1}{X}\frac{d^2X}{dx^2} - \frac{2}{X}\frac{dX}{dx} + \frac{1}{Y}\frac{dY}{dy} = 0$$

$$\frac{1}{X}\frac{d^2X}{dx^2} - \frac{2}{X}\frac{dX}{dx} = \frac{-1}{Y}\frac{dY}{dy}$$

Equating both sides to constant k

$$\frac{1}{X}\frac{d^2X}{dx^2} - \frac{2}{X}\frac{dX}{dx} = k \qquad \frac{-1}{Y}\frac{dY}{dy} = k$$

$$\frac{d^2X}{dx^2} - 2\frac{dX}{dx} - kX = 0 \qquad \frac{1}{Y}dY = -k\,dy$$

$D^2X - 2DX - kX = 0$

Integrating

$X(D^2 - 2D - k) = 0 \quad \because \frac{d^2}{dx^2} = D^2$

$\int \frac{1}{Y}dY = -\int k\,dy$

Auxillary equation

$m^2 - 2m - k = 0$

$\log Y = -ky + c_1$

$Y = e^{-ky + c_1}$

$$m = \frac{2 \pm \sqrt{4+4k}}{2}$$

$$m = 1 \pm \sqrt{1+k}$$

$\therefore$ the solution is given by

$$X = c_2 e^{(1+\sqrt{1+k})x} + c_3 e^{(1-\sqrt{1+k})x}$$

Substituting X and Y in eq ①

$$z = \left(c_2 e^{(1+\sqrt{1+k})x} + c_3 e^{(1-\sqrt{1+k})x}\right)\left(e^{-ky+c_1}\right)$$

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$$z = \left[A e^{(1+\sqrt{1+k})x} + B e^{(1-\sqrt{1+k})x}\right] e^{-ky}$$ where $A = c_2 e^{c_4}$, $B = c_3 e^{c_4}$

Q5: solve by the method of separation of variables $\frac{\partial^2 z}{\partial x^2} = \frac{\partial z}{\partial y} + 2z$ subjected to the conditions

given: $z(0,y)=0$, $z_x(0,y) = e^{2y}$

Sol: given $\frac{\partial^2 z}{\partial x^2} = \frac{\partial z}{\partial y} + 2z$

Assume $z = XY$ — ①

$$\frac{\partial^2 (XY)}{\partial x^2} = \frac{\partial (XY)}{\partial y} + 2XY$$

$$Y \frac{d^2 X}{dx^2} = X \frac{dY}{dy} + 2XY$$

dividing both sides by XY

$$\frac{1}{X}\frac{d^2 X}{dx^2} = \frac{1}{Y}\frac{dY}{dy} + 2$$

Equating both sides to constant k

$$\frac{1}{X}\frac{d^2 X}{dx^2} = k$$

$$D^2 X - kX = 0$$

$$(D^2 - k)X = 0$$

Auxiliary equation

$m^2 - k = 0$

$m^2 = k \Rightarrow m = \pm\sqrt{k}$

∴ the solution is

$$X = c_1 e^{-\sqrt{k}x} + c_2 e^{\sqrt{k}x}$$

$$\frac{1}{Y}\frac{dY}{dy} + 2 = k$$

$$\frac{1}{Y}\frac{dY}{dy} = k-2$$

$$\frac{1}{Y} dY = (k-2)dy$$

Integrating

$$\int \frac{1}{Y} dY = \int (k-2) dy$$

$$\log Y = (k-2)y + c_3$$

$$Y = e^{(k-2)y + c_3}$$

Substituting x and y in eq ①

$$z = (c_1 e^{-\sqrt{k}x} + c_2 e^{\sqrt{k}x})(e^{(k-2)y + c_3})$$

$$z = [A e^{-\sqrt{k}x} + B e^{\sqrt{k}x}] e^{(k-2)y} \quad \text{— ②} \quad \text{where } A = c_1 e^{c_3}, \; B = c_2 e^{c_3}$$

diff wrt x

$$z_x = [-A\sqrt{k} e^{-\sqrt{k}x} + \sqrt{k} B e^{\sqrt{k}x}] e^{(k-2)y} \quad \text{— ③}$$

given that $z = 0$ when $x = 0$

From eq ②

$$0 = (A e^0 + B e^0) e^{(k-2)y}$$

$$0 = e^{(k-2)y}(A+B)$$

Since $e^{(k-2)y} \neq 0$

$$\therefore A + B = 0 \Rightarrow B = -A$$

Substituting in eq ②

$$z = [A e^{-\sqrt{k}x} - A e^{\sqrt{k}x}] e^{(k-2)y}$$

$$z = A[e^{-\sqrt{k}x} - e^{\sqrt{k}x}] e^{(k-2)y} \quad \text{— ④}$$

given that $z_x = e^{2y}$ at $x = 0$

From eq ③

$$e^{2y} = [-A\sqrt{k} e^0 - A\sqrt{k} e^0] e^{(k-2)y}$$

$$e^{2y} = -2A\sqrt{k} \, e^{(k-2)y}$$

$$\Rightarrow 2 = k - 2 \qquad \Rightarrow -2A\sqrt{k} = 1$$

$$\therefore k = 4 \qquad A = \frac{-1}{2\sqrt{k}} = \frac{-1}{2\sqrt{4}} = \frac{1}{4}$$

Substituting k and A in eq ④

$$z = \frac{1}{4}[e^{-2x} - e^{2x}] e^{+2y}$$

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Q6: solve by method of separation of variables

$$4\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 3u \quad \text{given that } u(0,y) = 2e^{5y}$$

Sol: given $4\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 3u$

Assume $u = XY$ —— (1)

$$\therefore 4\frac{\partial (XY)}{\partial x} + \frac{\partial (XY)}{\partial y} = 3XY$$

$$4Y\frac{dX}{dx} + X\frac{dY}{dy} = 3XY$$

dividing both sides by XY.

$$\frac{4}{X}\frac{dX}{dx} + \frac{1}{Y}\frac{dY}{dy} = 3$$

$$\frac{4}{X}\frac{dX}{dx} = 3 - \frac{1}{Y}\frac{dY}{dy}$$

Equating both sides to constant k

$$\frac{4}{X}\frac{dX}{dx} = k \qquad 3 - \frac{1}{Y}\frac{dY}{dy} = k$$

$$\frac{4}{X}dX = k\,dx \qquad \frac{1}{Y}\frac{dY}{dy} = 3 - k$$

Integrating

$$4\log X = kx + c_1$$

$$\log X = \frac{kx + c_1}{4}$$

$$X = e^{\frac{kx}{4} + c_2}$$

Integrating

$$\int \frac{1}{Y}dY = \int (3-k)\,dy$$

$$\log Y = (3-k)y + c_3$$

$$Y = e^{(3-k)y + c_3}$$

Substituting X and Y in eq (1)

$$u = \left(e^{\frac{kx}{4} + c_2}\right)\left(e^{(3-k)y + c_3}\right)$$

$$u = e^{c_2 + c_3}\, e^{\left(\frac{kx}{4} + (3-k)y\right)}$$

$$u = c\, e^{\left(\frac{kx}{4} + (3-k)y\right)} \qquad \text{where } c = e^{c_2 + c_3}$$

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given $u = 2e^{5y}$ at $x = 0$

$2e^{5y} = ce^{(0+(3-k)y)}$

$\Rightarrow c = 2 \quad \Rightarrow (3-k)y = 5y$

$3-k = 5$

$k = -2$

$\therefore \underline{u = 2e^{-x/2+5y}}$

- <u>Various possible solution of one dimensional wave equation:</u>

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial u}{\partial x^2}$$

Assume $u = XT$

$$\frac{\partial^2(XT)}{\partial t^2} = c^2\frac{\partial^2(XT)}{\partial x^2}$$

$$X\frac{d^2T}{dt^2} = c^2 T\frac{d^2X}{dx^2}$$

Dividing both sides by Xc^2T

$$\frac{1}{c^2T}\frac{d^2T}{dt^2} = \frac{1}{X}\frac{d^2X}{dx^2}$$

Equating both sides to constant k

$$\frac{1}{c^2T}\frac{d^2T}{dt^2} = k \qquad \frac{1}{X}\frac{d^2X}{dx^2} = k$$

~~Integrating~~

$$\frac{d^2T}{dt^2} = kc^2T \qquad \frac{d^2X}{dx^2} = kX$$

$$D^2X - kX = 0$$

$$D^2T - kc^2T = 0 \qquad (D^2-k)X = 0 \quad \text{——— (2)}$$

$$(D^2-kc^2)T = 0 \quad \text{——— (1)}$$

<u>CASE 1</u>: Suppose $k = 0$ in eq (1) and eq (2)

$$D^2T = 0 \qquad D^2X = 0$$

Auxillary equation $\qquad$ Auxillary equation

$$m^2 = 0 \qquad m^2 = 0$$

$$\underline{m = 0, 0} \qquad \underline{m = 0, 0}$$

$\therefore T = (c_1 + c_2 t)e^{0t}$ $\quad X = (c_3 + c_4 x)e^{0x}$

wkt $u = XT$

$\therefore u = (c_3 + c_4 x)(c_1 + c_2 t)$ is the solution.

- CASE 2: suppose k is positive, that is $k = p^2$ in eq ① and eq ②

$(D^2 - p^2c^2)T = 0$ $\quad (D^2 - p^2)T = 0$

Auxillary equation $\quad$ Auxillary equation

$m^2 - p^2c^2 = 0$ $\quad m^2 - p^2 = 0$

$m^2 = p^2c^2$ $\quad m^2 = p^2$

$m = \pm pc$ $\quad m = \pm p$

$\therefore T = c_1 e^{pct} + c_2 e^{-pct}$ $\quad \therefore X = c_3 e^{px} + c_4 e^{-px}$

wkt $u = XT$

$\therefore u = (c_3 e^{px} + c_4 e^{-px})(c_1 e^{pct} + c_2 e^{-pct})$ is the solution.

- CASE 3: suppose k is negative, that is $k = -p^2$ in eq ① and eq ②

$(D^2 + p^2c^2)T = 0$ $\quad (D^2 + p^2)T = 0$

Auxillary equation $\quad$ Auxillary equation

$m^2 + p^2c^2 = 0$ $\quad m^2 + p^2 = 0$

$m^2 = -p^2c^2$ $\quad m^2 = -p^2$

$m = \pm pci$ $\quad m = \pm pi$

$\therefore T = c_1 \cos pct + c_2 \sin pct$ $\quad \therefore X = c_3 \cos px + c_4 \sin px$

wkt $u = XT$

$\therefore u = (c_1 \cos pct + c_2 \sin pct)(c_3 \cos px + c_4 \sin px)$ is the solution

LAGRANGE'S LINEAR PDE:

The equation of the form $Pp + Qq = R$ is called Lagrange's linear equation which has a solution of the form $\phi(u, v) = 0$

Procedure:

- Given the PDE of the form $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$ known as the auxillary equation. This system of equation can be solved as follows.
- We can consider suitable pairs which can be put in the forms like
$$f(x)dx = g(y)dy$$
$$g(y)dy = h(z)dz$$
$$h(z)dz = f(x)dx$$
so that by integration we can get the relations in the form (x, y) (y, z) (z, x).
- We have a property in the ratio and proportion that a ratio
$$\frac{a_1}{b_1} = \frac{a_2}{b_2} = \frac{a_3}{b_3} = \frac{k_1a_1 + k_2a_2 + k_3a_3}{k_1b_1 + k_2b_2 + k_3b_3}.$$

With reference to the auxillary equation we try to find multipliers k_1, k_2, k_3 such that $\frac{k_1dx + k_2dy + k_3dz}{k_1P + k_2Q + k_3R}$

- Integrating the 2 new expressions we obtain two relations connecting x, y, z.

Suppose $u(x, y, z) = c_1$ and $v(x, y, z) = c_2$ are the two relations obtained then $\phi(u, v) = 0$ is the general solution of the PDE.

Q1. solve $xp + yq = z$.

Sol. The given equation is of the form $Pp + Qq = R$.
Auxillary equation is given by

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

$$\therefore \frac{dx}{x} = \frac{dy}{y} = \frac{dz}{z}$$

Considering

$$\frac{dx}{x} = \frac{dy}{y}$$

Integrating

$$\int \frac{dx}{x} = \int \frac{dy}{y}$$

$$\log x = \log y + \log c_1$$

$$\log x - \log y = \log c_1$$

$$\log \frac{x}{y} = \log c_1$$

$$\therefore u = \frac{x}{y} = c_1$$

considering

$$\frac{dy}{y} = \frac{dz}{z}$$

Integrating

$$\int \frac{dy}{y} = \int \frac{dz}{z}$$

$$\log y = \log z + \log c_2$$

$$\log y - \log z = \log c_2$$

$$\log \frac{y}{z} = \log c_2$$

$$\therefore v = \frac{y}{z} = c_2$$

$$\phi(u, v) = 0$$

$\therefore \phi\left(\frac{x}{y}, \frac{y}{z}\right) = 0$ is the solution for the given PDE.

Q2. $p\cot x + q\cot y = \cot x$

Sol: The given equation is of the form $Pp + Qq = R$
Auxillary equation is given by

$$\frac{dx}{x} = \frac{dy}{y} = \frac{dz}{R}$$

$$\frac{dx}{\cot x} = \frac{dy}{\cot y} = \frac{dz}{\cot z}$$

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considering

$$\frac{dx}{\cot x} = \frac{dy}{\cot y}$$

$\tan x \, dx = \tan y \, dy$

integrating

$\int \tan x \, dx = \int \tan y \, dy$

$-\log|\cos x| = -\log|\cos y| + \log c_1$

$\log|\cos y| - \log|\cos x| = \log c_1$

$$\log \frac{\cos y}{\cos x} = \log c_1$$

$$\therefore u = \frac{\cos y}{\cos x} = c_1$$

considering

$$\frac{dy}{\cot y} = \frac{dz}{\cot z}$$

$\tan y \, dy = \tan z \, dz$

Integrating

$\int \tan y \, dy = \int \tan z \, dz$

$-\log|\cos y| = -\log|\cos z| + \log c_2$

$\log|\cos z| - \log|\cos y| = \log c_2$

$$\log \frac{\cos z}{\cos y} = \log c_2$$

$$\therefore v = \frac{\cos z}{\cos y} = c_2$$

$\phi(u, v) = 0$

$\phi\left(\frac{\cos y}{\cos x}, \frac{\cos x}{\cos y}\right) = 0$ is the solution for the given PDE.

Q3: solve:

$$(y-z)p + (z-x)q = (x-y)$$

Sol: The given equation is of the form $Pp + Qq = R$

Auxiliary equation is given by

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

$$\frac{dx}{y-z} = \frac{dy}{z-x} = \frac{dz}{x-y}$$

considering multiplier $k_1 = k_2 = k_3 = 1$ (To get denominator $= 0$)

$$\therefore \frac{dx}{y-z} = \frac{dy}{z-x} = \frac{dz}{x-y} = \frac{dx+dy+dz}{(y-z)+(z-x)+(x-y)}$$

$$= \frac{dx+dy+dz}{0}$$

$$\int dx + \int dy + \int dz = 0$$

$$u = x+y+z = c_1$$

considering multipliers $k_1 = x$ and $k_2 = y$ and $k_3 = z$

$$\frac{dx}{y-z} = \frac{dy}{z-x} = \frac{dz}{x-y} = \frac{x\,dx + y\,dy + z\,dz}{xy - xz + yz - xy + xz - yz}$$

$$= \frac{x\,dx + y\,dy + z\,dz}{0}$$

$$\int x\,dx + \int y\,dy + \int z\,dz = 0$$

$$v = \frac{x^2}{2} + \frac{y^2}{2} + \frac{z^2}{2} = c_2$$

Thus $\phi(u,v) = 0$

$$\therefore \phi\left(x+y+z,\ \frac{x^2+y^2+z^2}{2}\right) = 0$$

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Q4: $x(y^2-z^2)p + y(z^2-x^2)q = z(x^2-y^2)$

Sol: The given equation is of the form $Pp+Qq = R$
The auxillary equation is given by

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

$$\frac{dx}{x(y^2-z^2)} = \frac{dy}{y(z^2-x^2)} = \frac{dz}{z(x^2-y^2)}$$

considering multipliers $k_1 = x$, $k_2 = y$ and $k_3 = z$

$$\frac{dx}{x(y^2-z^2)} = \frac{dy}{y(z^2-x^2)} = \frac{dz}{z(x^2-y^2)} = \frac{xdx+ydy+zdz}{x^2y^2-x^2z^2+y^2z^2-x^2y^2+x^2z^2-y^2z^2}$$

$$= \frac{xdx+ydy+zdz}{0}$$

$$\therefore \int x\,dx + \int y\,dy + \int z\,dz = 0$$

$$u = \frac{x^2}{2} + \frac{y^2}{2} + \frac{z^2}{2} = C_1$$

considering multipliers $k_1 = 1/x$, $k_2 = 1/y$, $k_3 = 1/z$

$$\frac{dx}{x(y^2-z^2)} = \frac{dy}{y(z^2-x^2)} = \frac{dz}{z(x^2-y^2)} = \frac{dx/x+dy/y+dz/z}{y^2-z^2+z^2-x^2+x^2-y^2}$$

$$= \frac{dx/x+dy/y+dz/z}{0}$$

$$\therefore \int \frac{dx}{x} + \int \frac{dy}{y} + \int \frac{dz}{z} = 0$$

$$v = \log x + \log y + \log z = C_2$$

$$v = \log xyz = C_2$$

wkt $\phi(u,v) = 0$

$$\therefore \phi\left(\frac{x^2+y^2+z^2}{2}, \log xyz\right) = 0$$

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$$(mz-ny)\frac{\partial z}{\partial x} + (nx-lz)\frac{\partial z}{\partial y} = (ly-mx)$$

The given equation is of the form $Pp+Qq=R$

The auxillary equation is given by

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

$$\frac{dx}{mz-ny} = \frac{dy}{nx-lz} = \frac{dz}{ly-mx} = 0$$

considering multipliers $k_1 = x$, $k_2 = y$, $k_3 = z$

$$\frac{dx}{mz-ny} = \frac{dy}{nx-lz} = \frac{dz}{ly-mx} = \frac{xdx+ydy+zdz}{mxz-nxy+nxy-lyz+lyz-mxz}$$

$$= \frac{xdx+ydy+zdz}{0}$$

$$\therefore \int xdx + \int ydy + \int zdz = 0$$

$$u = \frac{x^2}{2} + \frac{y^2}{2} + \frac{z^2}{2} = c_1$$

considering multipliers $k_1 = l$, $k_2 = m$, $k_3 = n$

$$\frac{dx}{mz-ny} = \frac{dy}{nx-lz} = \frac{dz}{ly-mx} = \frac{ldx+mdy+ndz}{mlz-nly+mnx-mlz+lny-mnx}$$

$$= \frac{ldx+mdy+ndz}{0}$$

$$\therefore \int ldx + \int mdy + \int ndz = 0$$

$$v = lx + my + nz = c_2$$

Solution is given by $\phi(u,v) = 0$

$$\therefore \phi\left(\frac{x^2+y^2+z^2}{2}, lx+my+nz\right) = 0$$

* Applications of PDE: (continued)

Befitting solution of one dimensional wave equation

$u(x,t) = (A\cos px + B\sin px)(C\cos pct + D\sin pct)$

and

Two dimensional Laplace equation

$u(x,t) = (A\cos px + B\sin px)(Ce^{py} + De^{-py})$

Q1: Solve the wave equation $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$ given that $u(0,t) = 0$, $u(l,t) = 0$,

$\frac{\partial u}{\partial t} = 0$ when $t = 0$ and $u(x,0) = u_0 \sin \frac{\pi x}{l}$

Sol: The solution of wave equation is given by

$$u(x,t) = (A\cos px + B\sin px)(C\cos pct + D\sin pct) \quad (1)$$

Applying the given condition $u(0,t) = 0$ in eq (1)

$u(0,t) = (A\cos 0 + B\sin 0)(C\cos pct + D\sin pct) = 0$

$\Rightarrow A(C\cos pct + D\sin pct) = 0$

$\therefore A = 0$

Applying the given condition $u(l,t) = 0$ in eq (1)

$u(l,t) = (A\cos lp + B\sin lp)(C\cos pct + D\sin pct) = 0$

$\Rightarrow B\sin lp(C\cos pct + D\sin pct) = 0$

$\therefore B\sin lp = 0$

Since $A = 0 \Rightarrow B \neq 0$

$\therefore \sin lp = 0$

$\sin lp = \sin n\pi$

$lp = n\pi$

$p = \frac{n\pi}{l}$

Substituting p and A in eq (1)

$$u(x,t) = \left(B\sin \frac{n\pi x}{l}\right)\left(C\cos \frac{n\pi ct}{l} + D\sin \frac{n\pi ct}{l}\right) \quad (2)$$

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partially diff eq ② wrt t

$$\frac{\partial u}{\partial t} = B\sin\frac{n\pi x}{l}\left(-\frac{Cn\pi c}{l}\sin\frac{n\pi ct}{l} + \frac{Dn\pi c}{l}\cos\frac{n\pi ct}{l}\right)$$

$$\frac{\partial u}{\partial t} = \frac{Bn\pi c}{l}\sin\frac{n\pi x}{l}\left(D\cos\frac{n\pi ct}{l} - C\sin\frac{n\pi ct}{l}\right)$$

Applying the condition $\frac{\partial u}{\partial t} = 0$ when $t=0$

$$0 = \frac{Bn\pi c}{l}\sin\frac{n\pi x}{l}\left(D\cos 0 - C\sin 0\right)$$

$$\frac{DBn\pi c}{l}\sin\frac{n\pi x}{l} = 0$$

Since $B \neq 0 \Rightarrow \underline{D=0}$

Substituting D in eq ②

$$u(x,t) = B\sin\frac{n\pi x}{l} \cdot C\cos\frac{n\pi ct}{l}$$

Let $BC = b_n$

$$u(x,t) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}\cos\frac{n\pi ct}{l} \quad \text{——} ③$$

Applying the given condition $u(x,0) = u_0 \sin\frac{\pi x}{l}$ to eq. ③

$$u(x,0) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}\cos 0$$

$$u_0\sin\frac{\pi x}{l} = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}$$

$$u_0\sin\frac{\pi x}{l} = b_1\sin\frac{\pi x}{l} + b_2\sin\frac{2\pi x}{l} + b_3\sin\frac{3\pi x}{l} + \cdots$$

$$\therefore u_0\sin\frac{\pi x}{l} = b_1\sin\frac{\pi x}{l}$$

$$\therefore \underline{b_1 = u_0}$$

Substituting b_1 in eq ③

$$u(x,t) = u_0\sin\frac{\pi x}{l}\cos\frac{\pi ct}{l}$$

Q2: Solve the wave equation $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$

given that

$u(0,t)=0$, $u(l,t)=0$, $\frac{\partial u}{\partial t}=0$ when $t=0$ and $u(x,0) = u_0 \sin^3 \frac{\pi x}{l}$

Sol: * Applying the given condition $u(x,0) = u_0 \sin^3 \frac{\pi x}{l}$ in eq ③

$$u_0 \sin^3 \frac{\pi x}{l} = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l}$$

$$u_0\left(\frac{3}{4}\sin\frac{\pi x}{l} - \frac{1}{4}\sin\frac{3\pi x}{l}\right) = b_1 \sin\frac{\pi x}{l} + b_2 \sin\frac{2\pi x}{l} + b_3 \sin\frac{3\pi x}{l} + \ldots$$

Equating the coefficients

$$\frac{3}{4} u_0 \sin\frac{\pi x}{l} = b_1 \sin\frac{\pi x}{l}$$

$$\Rightarrow b_1 = \frac{3}{4} u_0$$

$$-\frac{1}{4} u_0 \sin\frac{3\pi x}{l} = b_3 \sin\frac{3\pi x}{l}$$

$$\Rightarrow b_3 = \frac{-u_0}{4}$$

Substituting in eq ③

$$u(x,t) = \frac{3u_0}{4} \sin\frac{\pi x}{l} \cos\frac{\pi c t}{l} - \frac{u_0}{4} \sin\frac{3\pi x}{l} \cos\frac{3\pi c t}{l}$$

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UNIT 3

Fourier Transforms

The infinite Fourier transform or complex fourier transform of a real value function $f(x)$ is defined as

$$F[f(x)] = \int_{-\infty}^{\infty} f(x)\, e^{iux}\, dx = F(u)$$

On integration we obtain a function of 'u' which is denoted by $F(u)$. The inverse fourier tranrform of $F(u)$ is defined by the integral,

$$F^{-1}[F(u)] = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(u)\, e^{-iux}\, du = f(x)$$

On integration we obtain a function of 'x'.

Q: Find the complex fourier transform of the function

$$f(x) = \begin{cases} 1 & \text{for } |x| \le a \\ 0 & \text{for } |x| > a \end{cases}$$

Deduce $\int_0^{\infty} \frac{\sin x}{x}\, dx$.

Sol: By defination

$$F(u) = \int_{-\infty}^{\infty} f(x)\, e^{iux}\, dx$$

$$= \int_{-a}^{a} 1 \cdot e^{iux}\, dx$$

$$= \left[\frac{e^{iux}}{iu}\right]_{-a}^{a}$$

$$= \frac{1}{iu}\left[e^{iux}\right]_{-a}^{a}$$

$$= \frac{-i}{u}\left[e^{iau} - e^{-iau}\right]$$

NOTE:

1) $e^{iu} + e^{-iu} = (\cos u + i\sin u) + (\cos u - i\sin u)$

$e^{iu} + e^{-iu} = 2\cos u$

2) $e^{iu} - e^{-iu} = (\cos u + i\sin u) - (\cos u - i\sin u)$

$e^{iu} - e^{-iu} = 2i\sin u$

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$$F(u) = \frac{-i}{u}(2i \sin au)$$

$$F(u) = \frac{2\sin au}{u}$$

By inverse Fourier transform.

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(u)\, e^{-iux}\, du$$

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{2\sin au}{u}\, e^{-iux}\, du$$

Let $x = 0$

$$f(0) = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\sin au}{u}\, du$$

Since $x = 0$ is a point of continuity of $f(x)$, the value of $f(x)$ at $x=0$ implies $f(0) = 1$ because $f(x)$ value is 1 for $|x| \leq a$

$$1 = \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\sin au}{u}\, du$$

$$\int_{-\infty}^{\infty} \frac{\sin au}{u}\, du = \pi$$

$$2\int_{0}^{\infty} \frac{\sin au}{u}\, du = \pi \qquad (\because f(x) \text{ is even})$$

put $a = 1$, change u to x

$$\int_{0}^{\infty} \frac{\sin x}{x}\, dx = \frac{\pi}{2}$$

Q2: $f(x) = \begin{cases} x & \text{for } |x| \leq \alpha \\ 0 & \text{for } |x| > \alpha \end{cases}$

Find the complex Fourier transform for the function given where α is a positive constant.

Sol: By defination

$$F(u) = \int_{-\infty}^{\infty} f(x)\, e^{iux}\, dx$$

$$F(u) = \int_{-\alpha}^{\alpha} x \cdot e^{iux}\, dx$$

$$F(u) = \left[x \frac{e^{iux}}{iu} - \frac{e^{iux}}{i^2u^2} \right|_{-\alpha}^{\alpha}$$

$$F(u) = \frac{1}{iu} \left[x e^{iux} \right|_{-\alpha}^{\alpha} - \frac{1}{i^2u^2} \left[e^{iux} \right|_{-\alpha}^{\alpha}$$

$$F(u) = \frac{-i}{u} \left[\alpha e^{iu\alpha} + \alpha e^{-iu\alpha} \right] + \frac{1}{u^2} \left[e^{iu\alpha} - e^{-iu\alpha} \right]$$

$$F(u) = \frac{-i}{-u} \left[\alpha (e^{iu\alpha} + e^{-iu\alpha}) \right] + \frac{1}{u^2} (2i \sin u\alpha)$$

$$F(u) = \frac{-i\alpha}{u} (2\cos u\alpha) + \frac{1}{u^2} (2i \sin u\alpha)$$

Q3: $f(x) = \begin{cases} 1-x^2, & \text{for } |x| \le 1 \\ 0, & \text{for } |x| > 1 \end{cases}$

Find the Fourier tranform of $f(x)$ and hence find the value of

i. $\displaystyle\int_0^{\infty} \frac{x\cos x - \sin x}{x^3}\, dx$

ii. $\displaystyle\int_0^{\infty} \frac{x\cos x - \sin x}{x^3} \cos\left(\frac{x}{2}\right) dx$

Sol: By defination

$$F(u) = \int_{-\infty}^{\infty} f(x)\, e^{iux}\, dx$$

$$F(u) = \int_{-1}^{1} (1-x^2)\, e^{iux}\, dx$$

$$= \left[\frac{e^{iux}}{iu} - \left(x^2 \frac{e^{iux}}{iu} - \int 2x \frac{e^{iux}}{iu} dx + 2\int \frac{e^{iux}}{i^2u^2} dx \right) \right]_{-1}^{1}$$

$$= \left[\frac{e^{iux}}{iu} - \frac{x^2 e^{iux}}{iu} + 2x\frac{e^{iux}}{i^2u^2} - \frac{2e^{iux}}{i^3u^3} \cancel{-\frac{2e^{iux}}{i^3u^3}} \right]_{-1}^{1}$$

$$= \left[\frac{e^{iux}}{iu}(1-x^2) - \frac{2x\, e^{iux}}{u^2} - \frac{2}{i^3u^3} e^{iux} \right]_{-1}^{1}$$

$$= \left[\frac{e^{iu}}{iu}(0) - \frac{2e^{iu}}{u^2} - \frac{2e^{iu}}{i^3u^3} - \frac{e^{-iu}}{iu}(0) - \frac{2e^{-iu}}{u^2} + \frac{2e^{-iu}}{i^3u^3} \right]$$

$$= \frac{-2}{u^2}(e^{iu}+e^{-iu}) - \frac{2}{i^3u^3}(e^{iu}-e^{-iu})$$

$$= \frac{-2}{u^2}(2\cos u) - \frac{2}{i^3u^3}(2i\sin u)$$

$$= \frac{-4\cos u}{u^2} + \frac{4\sin u}{u^3}$$

$$F(u) = \frac{-4u\cos u + 4\sin u}{u^3} = \frac{4(\sin u - u\cos u)}{u^3}$$

i] By inverse Fourier transform

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(u)\, e^{-iux}\, du$$

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} 4\,\frac{(-u\cos u + \sin u)}{u^3}\, e^{-iux}\, du \quad \text{——} \ (1)$$

let $x = 0$

$$f(0) = \frac{2}{\pi}\int_{-\infty}^{\infty} \frac{-u\cos u + \sin u}{u^3}\, du$$

Since $x=0$ is a point of continuity of $f(x)$, the value of $f(x)$ at $x=0$ implies $f(0)=1$ because $f(x)$ value is $1-x^2$ for $|x| \le 1$

$$\therefore 1 = \frac{2}{\pi}\int_{-\infty}^{\infty} \frac{-u\cos u + \sin u}{u^3}\, du$$

$$1 = \frac{4}{\pi}\int_{0}^{\infty} \frac{-u\cos u + \sin u}{u^3}\, du \qquad (\because f(x) \text{ is an even function})$$

change u to x

$$\int_0^\infty \frac{-x\cos x+\sin x}{x^3}\,dx = \frac{\pi}{4}$$

$$\therefore \int_0^\infty \frac{x\cos x-\sin x}{x^3}\,dx = \frac{-\pi}{4}$$

ii] let us substitute $x = 1/2$ in eq ①

$$f\left(\frac{1}{2}\right) = \frac{2}{\pi}\int_{-\infty}^{\infty} \frac{-u\cos u+\sin u}{u^3}\, e^{-iu/2}\,du$$

$$1-\frac{1}{4} = \frac{2}{\pi}\int_{-\infty}^{\infty} \frac{-u\cos u+\sin u}{u^3}\left[\cos\frac{u}{2} - i\sin\frac{u}{2}\right]du$$

$$\frac{3\pi}{8} = \int_{-\infty}^{\infty} \frac{-u\cos u+\sin u}{u^3}\cos\frac{u}{2}\,du$$ (considering only real part)

$$\frac{3\pi}{8} = 2\int_0^\infty \frac{-u\cos u+\sin u}{u^3}\cos\frac{u}{2}\,du$$ ($\because f(x)$ is an even function)

$$\frac{3\pi}{16} = \int_0^\infty \frac{-u\cos u+\sin u}{u^3}\cos\frac{u}{2}\,du$$

changing u by x

$$\int_0^\infty \frac{x\cos x-\sin x}{x^3}\cos\frac{x}{2}\,dx = \frac{-3\pi}{16}$$

Q4: Find Fourier transform of

$$f(x) = \begin{cases} 1-|x| & \text{for } |x|<1 \\ 0 & \text{for } |x|>1 \end{cases}$$

Hence deduce $\displaystyle\int_0^\infty \frac{\sin^2 t}{t^2}\,dt = \frac{\pi}{2}$

Sol: By defination

$$F(\alpha) = \int_{-\infty}^{\infty} f(x)\, e^{i\alpha x}\,dx$$

$$= \int_{-1}^{1} 1-|x|\, e^{i\alpha x}\,dx$$

$$= \int_{-1}^{0}(1+x)e^{i\alpha x}\,dx + \int_0^1 (1-x)\,e^{i\alpha x}\,dx$$

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$$= \left[\frac{e^{iux}}{iu} + x\frac{e^{iux}}{iu} - \frac{e^{iux}}{i^2u^2}\right]_{-1}^{0} + \left[\frac{e^{iux}}{iu} - x\frac{e^{iux}}{iu} + \frac{e^{iux}}{i^2u^2}\right]_{0}^{1}$$

$$= \left[\cancel{\frac{1}{iu}} + 0 + \frac{1}{u^2} - \cancel{\frac{e^{-iu}}{iu}} + \cancel{\frac{e^{-iu}}{iu}} - \frac{e^{-iu}}{u^2}\right]$$

$$+ \left[\cancel{\frac{e^{iu}}{iu}} - \cancel{\frac{e^{iu}}{iu}} - \frac{e^{iu}}{u^2} - \cancel{\frac{1}{iu}} + 0 + \frac{1}{u^2}\right]$$

$$= \frac{2}{u^2} - \frac{1}{u^2}\left(e^{iu} + e^{-iu}\right)$$

$$= \frac{2 - 2\cos u}{u^2}$$

$$F(u) = \frac{2(1-\cos u)}{u^2} \quad //$$

$$F(u) = \frac{2\left(2\sin^2 u/2\right)}{u^2} = \frac{4}{u^2}\sin^2\frac{u}{2} \quad //$$

By inverse fourier transform

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(u)\, e^{-iux}\, dx$$

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{4}{u^2}\sin^2\frac{u}{2}\, e^{iux}\, dx$$

$$f(x) = \frac{1}{2\pi}\, 2\int_{0}^{\infty} \frac{4}{u^2}\sin^2\frac{u}{2}\, e^{iux}\, dx \qquad (\because F(u) \text{ is an even function})$$

$$f(x) = \frac{4}{\pi}\int_{0}^{\infty} \frac{1}{u^2}\sin^2\frac{u}{2}\, e^{iux}\, dx$$

Let $x = 0$

$$f(0) = \frac{4}{\pi}\int_{0}^{\infty} \frac{1}{u^2}\sin^2\frac{u}{2}\, dx$$

$$1 = \frac{4}{\pi}\int_{0}^{\infty} \frac{1}{u^2}\sin^2\frac{u}{2}\, dx$$

~~replace u by x~~

$$\cancel{\frac{\pi}{4} = \int_{0}^{\infty} \frac{1}{x^2}\sin^2\frac{x}{2}\, dx}$$

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Replacing x by $2t$

$dx = 2dt$

as $x \to \infty$ $t \to \infty$

$x \to 0$ $t \to 0$

$$\frac{\pi}{4} = 2\int_0^\infty \frac{4}{t^2} \sin^2 t \, dt$$

$$\frac{\pi}{2} = \int_0^\infty \frac{\sin^2 t}{t^2} dt$$

* NOTE:

1. $F_s(u) = \int_0^\infty f(x) \sin ux \, dx$
2. $f(x) = \frac{2}{\pi}\int_0^\infty F_s(u) \sin ux \, du$
3. $F_c(u) = \int_0^\infty f(x) \cos ux \, dx$
4. $f(x) = \frac{2}{\pi}\int_0^\infty F_c(u) \cos ux \, du$

If $f(x)$ is defined for all positive values of x then

1. Fourier sine transform
2. Inverse Fourier sine transform
3. Fourier cosine transform
4. Inverse Fourier cosine transform

is given as.

Q: Find the fourier sine and cosine transform of

$$f(x) = \begin{cases} x & 0 < x < 2 \\ 0 & \text{elsewhere} \end{cases}$$

A: Fourier sine transform

$$F_s(u) = \int_0^\infty f(x) \sin ux \, dx$$

$$F_s(u) = \int_0^2 x \sin ux \, dx$$

$$= \left[x\left(\frac{-\cos ux}{u}\right) - \left(-\frac{\sin ux}{u^2}\right)\right]_0^2$$

$$F_s(u) = -\frac{2\cos 2u}{u} + \frac{\sin 2u}{u^2} = \frac{\sin 2u - 2u\cos 2u}{u^2}$$

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Fourier cosine transform

$$F_c(u) = \int_0^\infty f(x)\cos ux\, dx$$

$$F_c(u) = \int_0^2 x\cos ux\, dx$$

$$= \left[x\left(\frac{\sin ux}{u}\right) - \left(\frac{-\cos ux}{u^2}\right)\right]_0^2$$

$$= \frac{2\sin 2u}{u} + \frac{\cos 2u}{u^2} - \frac{1}{u^2}$$

$$F_c(u) = \frac{\cos 2u + 2u\sin 2u - 1}{u^2}$$

Q2: Find the Fourier sine and cosine transform of $e^{-\alpha x}$, $\alpha > 0$.

Sol: Fourier sine transform

$$F_s(u) = \int_0^\infty f(x)\sin ux\, dx$$

$$F_s(u) = \int_0^\alpha e^{-\alpha x}\sin ux\, dx$$

$$= \frac{e^{-\alpha x}}{\alpha^2+u^2}\left[\cancel{\alpha}\cos ux - u\sin ux\right|_0^\alpha$$

$$= \frac{e^{-\alpha x}}{\alpha^2+u^2}\left[-\alpha\cos\alpha u - u\sin\alpha u + \alpha\right]$$

Fourier cosine transform

$$F_c(u) = \int_0^\infty f(x)\cos ux\, dx$$

$$F_c(u) = \int_0^\alpha e^{-\alpha x}\cos ux\, dx$$

$$= \frac{e^{-\alpha x}}{\alpha^2+u^2}\left[-\alpha\cos \alpha u + u\sin ux\right|_0^\alpha$$

$$= \frac{e^{-\alpha x}}{\alpha^2+u^2}\left[-\alpha\cos\alpha u + u\sin\alpha u - u\right]$$

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Find the equivalent Fourier transform of e^{-x^2}

Fourier cosine transform

$F_c(u) = \int_0^\infty f(x) \cos ux \, dx$

$F_c(u) = \int_0^\infty e^{-x^2} \cos ux \, dx$

The integral is evaluated using Lebnix rule (diff wrt u)

$F_c'(u) = \int_0^\infty e^{-x^2} (-x \sin ux) \, dx$

$F_c'(u) = \int_0^\infty -x e^{-x^2} \sin ux \, dx$

$= \frac{1}{2} \int_0^\infty -2x e^{-x^2} \sin ux \, dx$

$= \frac{1}{2} \left[\left(\sin ux \left(e^{-x^2} \right) \right)_0^\infty - \int_0^\infty e^{-x^2} (u \cos ux) \, dx \right]$

$F_c'(u) = \frac{1}{2} \left[-u F_c(u) \right]$

$F_c'(u) = \frac{-u F_c(u)}{2}$

$\frac{F_c'(u)}{F_c(u)} = \frac{-u}{2}$

Integrating wrt u

$\int \frac{F_c'(u)}{F_c(u)} du = \int \frac{-u}{2} du$

$\log F_c(u) = \frac{-u^2}{4} + c$

If $u = 0$

$F_c(u) = e^{-u^2/4} e^{c}$

$F_c(u) = k e^{-u^2/4}$

$F_c(0) = k e^0$

$\therefore F_c(0) = \int_0^\infty e^{-x^2} \cos(0) dx = k$

$\frac{\sqrt{\pi}}{2} = k$

$\therefore F_c(u) = \frac{\sqrt{\pi}}{2} e^{-u^2/4}$

Q4: Find the fourier cosine transform of $\frac{1}{1+x^2}$

sol: Fourier cosine transform

$$F_c(u) = \int_0^\infty f(x) \cos ux \, dx$$

$$F_c(u) = \int_0^\infty \frac{1}{1+x^2} \cos ux \, dx \quad \text{--- (1)}$$

diff eq (1) wrt u (lebnix rule)

$$F_c'(u) = \int_0^\infty \frac{1}{1+x^2} (-x \sin ux) \, dx$$

$$F_c'(u) = -\int_0^\infty \frac{x}{1+x^2} \sin ux \, dx$$

$$F_c'(u) = -\int_0^\infty \frac{1+x^2-1}{x(1+x^2)} \sin ux \, dx$$

$$F_c'(u) = -\int_0^\infty \frac{\cancel{1+x^2}}{x(\cancel{1+x^2})} \sin ux \, dx + \int_0^\infty \frac{\sin ux}{x(1+x^2)} dx$$

$$F_c'(u) = -\int_0^\infty \frac{\sin ux}{x} dx + \int_0^\infty \frac{\sin ux}{x(1+x^2)} dx$$

$$F_c'(u) = \frac{-\pi}{2} + \int_0^\infty \frac{\sin ux}{x(1+x^2)} dx \quad \text{--- (2)}$$

diff eq (2) wrt u

$$F_c''(u) = \int_0^\infty \frac{\cancel{x} \cos ux}{\cancel{x}(1+x^2)} dx$$

From eq (1)

$$F_c''(u) = F_c(u)$$

$$F_c''(u) - F_c(u) = 0$$

$$(D^2 - 1) F_c(u) = 0$$

AE: $m^2 - 1 = 0$

$m^2 = 1$

$m = \pm 1$

$F_c(u) = c_1 e^{u} + c_2 e^{-u}$ —— ③

diff eq ③ w.r.t u

$F_c'(u) = c_1 e^{u} - c_2 e^{-u}$ —— ④

put $u=0$ in eq ③ and eq ④

$F_c(0) = c_1 + c_2$

$$\int_0^\infty \frac{\cos ux}{1+x^2}\,dx = \int_0^\infty \frac{\cos 0}{1+x^2}\,dx = c_1 + c_2$$

$$\int_0^\infty \frac{1}{1+x^2}\,dx = \left[\tan^{-1}x\right]_0^\infty = c_1 + c_2$$

$$\tan^{-1}(\infty) - \tan^{-1}0 = c_1 + c_2$$

$$\frac{\pi}{2} - 0 = c_1 + c_2$$

$\therefore c_1 + c_2 = \dfrac{\pi}{2}$ —— ⑤

From eq ② and eq ④

$F_c'(0) = c_1 - c_2$

$$-\frac{\pi}{2} + \int_0^\infty \frac{\sin 0}{x(1+x^2)}\,dx = c_1 - c_2$$

$c_1 - c_2 = -\dfrac{\pi}{2}$ —— ⑥

Adding eq ⑤ and eq ⑥

$2c_1 = 0$

$c_1 = 0$

From eq ⑤ $c_2 = \dfrac{\pi}{2}$

Substituting c_1 and c_2 in eq ③

$$F_c(u) = \frac{\pi}{2} e^{-u}$$

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Q5: $\frac{1}{x(1+x^2)}$. Find the fourier sine transform.

sol: Fourier sine transform

$$F_s(u) = \int_0^\infty f(x) \sin ux \, dx$$

$$F_s(u) = \int_0^\infty \frac{1}{x(1+x^2)} \sin ux \, dx \quad \text{—} ①$$

diff eq ① wrt u

$$F_s'(u) = \int_0^\infty \frac{1}{\cancel{x}(1+x^2)} \cancel{x} \cos ux \, dx$$

$$F_s'(u) = \int_0^\infty \frac{\cos ux}{1+x^2} \, dx \quad \text{—} ②$$

diff eq ② wrt u

$$F_s''(u) = \int_0^\infty \frac{-x \sin ux}{1+x^2} \, dx$$

$$F_s''(u) = -\int_0^\infty \frac{1+x^2-1}{x(1+x^2)} \sin ux \, dx$$

$$F_s''(u) = -\int_0^\infty \frac{\cancel{1+x^2}}{x\cancel{(1+x^2)}} \sin ux \, dx + \int_0^\infty \frac{\sin ux}{x(1+x^2)} \, dx$$

$$F_s''(u) = -\int_0^\infty \frac{\sin ux}{x} \, dx + \int_0^\infty \frac{\sin ux}{x(1+x^2)} \, dx$$

$$F_s''(u) = -\frac{\pi}{2} + \int_0^\infty \frac{\sin ux}{x(1+x^2)} \, dx \quad \text{—} ③$$

Substituting eq ① in eq ③

$$F_s''(u) = -\frac{\pi}{2} + F_s(u)$$

$$F_s''(u) - F_s(u) = -\frac{\pi}{2}$$

$$(D^2-1) F_s(u) = -\pi/2$$

AE: $m^2 - 1 = 0$

$m^2 = 1 \Rightarrow m = \pm 1$

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$\therefore CF = C_1 e^{u} + C_2 e^{-u}$

$PI = \frac{-\pi/2}{D^2 - 1}$ Replacing D by 0

$PI = \frac{-\pi/2}{-1} = \frac{\pi}{2}$

$F_s(u) = CF + PI$

$F_s(u) = C_1 e^{u} + C_2 e^{-u} + \frac{\pi}{2}$ —— ③

diff eq ③ wrt u

$F'_s(u) = C_1 e^{u} - C_2 e^{-u}$ —— ④

Put $u = 0$

From eq ③ $F_s(0) = C_1 + C_2 + \frac{\pi}{2}$ —— ⑤ $\Rightarrow C_1 + C_2 = -\pi/2$

From eq ④ $F'_s(0) = C_1 - C_2$ —— ⑥ $\Rightarrow C_1 - C_2 = \pi/2$

Adding eq ⑤ and eq ⑥

$2C_1 = 0$ $\quad$ $C_2 = -\pi/2$

$C_1 = 0$

Substituting ~~eq~~ C_1 and C_2 in eq ③

$F_s(u) = -\frac{\pi}{2} e^{-u} + \frac{\pi}{2}$

$F_s(u) = \frac{\pi}{2}(1 - e^{-u})$ //

26: Find the fourier sine transform of $f(x) = e^{-|x|}$ and also evaluate $\int_0^\infty \frac{x \sin mx}{1+x^2}\,dx$, $m > 0$.

A: Fourier sine transform

$F_s(u) = \int_0^\infty f(x) \sin ux \, dx$

$F_s(u) = \int_0^\infty e^{-x} \sin ux \, dx$

$$F_s(u) = \left[\frac{e^{-x}}{1+u^2}(-\sin ux - u\cos ux)\right]_0^\infty$$

$$F_s(u) = \frac{1}{1+u^2}\left[0 - e^0(-u)\right]$$

$$F_s(u) = \frac{u}{1+u^2}$$

By inverse fourier sine transform

$$f(x) = \frac{2}{\pi}\int_0^\infty F_s(u)\sin ux\,du$$

$$f(x) = \frac{2}{\pi}\int_0^\infty \frac{u\sin ux}{1+u^2}\,du$$

$$e^{-x} = \frac{2}{\pi}\int_0^\infty \frac{u\sin ux}{1+u^2}\,du$$

$$\int_0^\infty \frac{u\sin ux}{1+u^2}\,du = \frac{e^{-x}\pi}{2}$$

Replacing u by x and x by m

$$\int_0^\infty \frac{x\sin mx}{1+x^2}\,dx = \frac{e^{-m}\pi}{2}$$

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UNIT - 4

Numerical Methods 1

* **Gauss - Seidel Method**

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1$$
$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2$$
$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3$$

system of linear Equations

$$\Rightarrow x_1 = \frac{1}{a_{11}}\left[b_1 - a_{12}x_2 - a_{13}x_3\right]$$

$$x_2 = \frac{1}{a_{22}}\left[b_2 - a_{21}x_1 - a_{23}x_3\right]$$

$$x_3 = \frac{1}{a_{33}}\left[b_3 - a_{31}x_1 - a_{32}x_2\right]$$

system of equations must be diagonally dominant.

$$|a_{11}| > |a_{12}| + |a_{13}|$$
$$|a_{22}| > |a_{21}| + |a_{23}|$$
$$|a_{33}| > |a_{31}| + |a_{32}|$$

Initial Approximation

$x_1^{(0)} = 0 \quad x_2^{(0)} = 0 \quad x_3^{(0)} = 0$

First Iteration

$$x_1^{(1)} = \frac{1}{a_{11}}\left[b_1 - a_{12}x_2^{(0)} - a_{13}x_3^{(0)}\right]$$

$$x_2^{(1)} = \frac{1}{a_{22}}\left[b_2 - a_{21}x_1^{(1)} - a_{23}x_3^{(0)}\right]$$

$$x_3^{(1)} = \frac{1}{a_{33}}\left[b_3 - a_{31}x_1^{(1)} - a_{32}x_2^{(1)}\right]$$

Q1: Solve the system of equations using Gauss-Seidel Method

$x + y + 54z = 110$

$27x + 6y - z = 85$

$6x + 15y + 2z = 72$

Sol: Rearranging

$27x + 6y - z = 85$

$6x + 15y + 2z = 72$

$x + y + 54z = 110$

Let $x^{(0)} = 0$; $y^{(0)} = 0$; $z^{(0)} = 0$

$$x = \frac{1}{27}[85 - 6y + z]$$

$$y = \frac{1}{15}[72 - 6x - 2z]$$

$$z = \frac{1}{54}[110 - x - y]$$

First Iteration

$$x^{(1)} = \frac{1}{27}[85 - 6y^{(0)} + z^{(0)}] = \frac{1}{27}[85 - 0 + 0] = \underline{3.1481}$$

$$y^{(1)} = \frac{1}{15}[72 - 6x^{(1)} - 2z^{(0)}] = \frac{1}{15}[72 - 6(3.1481) - 0] = \underline{3.5407}$$

$$z^{(1)} = \frac{1}{54}[110 - x^{(1)} - y^{(1)}] = \frac{1}{54}[110 - 3.1481 - 3.5407] = \underline{1.9131}$$

Second Iteration

$$x^{(2)} = \frac{1}{27}[85 - 6y^{(1)} + z^{(1)}] = \underline{2.4321}$$

$$y^{(2)} = \frac{1}{15}[72 - 6x^{(2)} - 2z^{(1)}] = \underline{3.5720}$$

$$z^{(2)} = \frac{1}{54}[110 - x^{(2)} - y^{(2)}] = \underline{1.9258}$$

Third Iteration

$$x^{(3)} = \frac{1}{27}\left[85 - 6y^{(2)} + z^{(2)}\right] = 2.4256$$

$$y^{(3)} = \frac{1}{15}\left[72 - 6x^{(3)} - 2z^{(2)}\right] = 3.5729$$

$$z^{(3)} = \frac{1}{54}\left[110 - x^{(3)} - y^{(3)}\right] = 1.9259$$

Fourth Iteration

$$x^{(4)} = \frac{1}{27}\left[85 - 6y^{(3)} + z^{(3)}\right] = 2.4255$$

$$y^{(4)} = \frac{1}{15}\left[72 - 6x^{(4)} - 2z^{(3)}\right] = 3.5730$$

$$z^{(4)} = \frac{1}{54}\left[110 - x^{(4)} - y^{(4)}\right] = 1.9259$$

Hence $x = 2.425$ $y = 3.573$ $z = 1.925$

Q2: $5x + 2y + z = 12$ consider the initial approximation as $(1, 0, 3)$.

$x + 4y + 2z = 15$

$x + 2y + 5z = 20$

Sol: ~~Rearran~~

$$x = \frac{1}{5}\left[12 - 2y - z\right]$$

$$y = \frac{1}{4}\left[15 - x - 2z\right]$$

$$z = \frac{1}{5}\left[20 - x - 2y\right]$$

given: $x^{(0)} = 1$ $y^{(0)} = 0$ $z^{(0)} = 3$

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First Iteration

$$x^{(1)} = \frac{1}{5}\left[12-2y^{(0)}-z^{(0)}\right] = \frac{1}{5}\left[12-2(0)-3\right] = \underline{1.8000}$$

$$y^{(1)} = \frac{1}{4}\left[15-x^{(1)}-2z^{(0)}\right] = \frac{1}{4}\left[15-1.8-2(3)\right] = \underline{1.8000}$$

$$z^{(1)} = \frac{1}{5}\left[20-x^{(1)}-2y^{(1)}\right] = \frac{1}{5}\left[20-1.8-2(1.8)\right] = \underline{2.9200}$$

Second Iteration

$$x^{(2)} = \frac{1}{5}\left[12-2y^{(1)}-z^{(1)}\right] = \underline{1.0960}$$

$$y^{(2)} = \frac{1}{4}\left[15-x^{(2)}-2z^{(1)}\right] = \underline{2.0160}$$

$$z^{(2)} = \frac{1}{5}\left[20-x^{(2)}-2y^{(2)}\right] = \underline{2.9744}$$

Third Iteration

$$x^{(3)} = \frac{1}{5}\left[12-2y^{(2)}-z^{(2)}\right] = \underline{0.9987}$$

$$y^{(3)} = \frac{1}{4}\left[15-x^{(3)}-2z^{(2)}\right] = \underline{2.0131}$$

$$z^{(3)} = \frac{1}{5}\left[20-x^{(3)}-2y^{(3)}\right] = \underline{2.9950}$$

Fourth Iteration

$$x^{(4)} = \frac{1}{5}\left[12-2y^{(3)}-z^{(3)}\right] = \underline{0.9957}$$

$$y^{(4)} = \frac{1}{4}\left[15-x^{(4)}-2z^{(3)}\right] = \underline{2.0035}$$

$$z^{(4)} = \frac{1}{5}\left[20-x^{(4)}-2y^{(4)}\right] = \underline{2.9994}$$

Hence $\underline{x = 0.995}$ $\underline{y = 2.003}$ $\underline{z = 2.999}$

★ **Relaxation Method**:

Consider a system of equations

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1$$
$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2$$
$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3$$

This system of linear equations must be diagonally dominant.

Define residual as

$$R_1 = a_{11}x_1 + a_{12}x_2 + a_{13}x_3 - b_1$$
$$R_2 = a_{21}x_1 + a_{22}x_2 + a_{23}x_3 - b_2$$
$$R_3 = a_{31}x_1 + a_{32}x_2 + a_{33}x_3 - b_3$$

Relaxation method aims in reducing the values of the residuals as close to zero as possible by modifying the value of the variables at every stage.

1: Solve the system of equation by relaxation method.

$$12x_1 + x_2 + x_3 = 31$$
$$2x_1 + 8x_2 - x_3 = 24$$
$$3x_1 + 4x_2 + 10x_3 = 58$$

: Considering residuals

$$R_1 = 12x_1 + x_2 + x_3 - 31$$
$$R_2 = 2x_1 + 8x_2 - x_3 - 24$$
$$R_3 = 3x_1 + 4x_2 + 10x_3 - 58$$

Δx_1	Δx_2	Δx_3	R_1	R_2	R_3
0	0	0	-31	-24	$\boxed{-58}$
0	0	6	-25	$\boxed{-30}$	2
0	4	0	$\boxed{-21}$	2	18
2	0	0	3	6	$\boxed{24}$
0	0	-2	1	$\boxed{8}$	4
0	-1	0	0	0	0

$x_1 = \Delta x_1 = 2$ //

$x_2 = \Delta x_2 = 3$ //

$x_3 = \Delta x_3 = 4$ //

Q2: $10x - 2y - 3z = 205$
$-2x + 10y - 2z = 154$
$-2x - y + 10z = 120$

sol: Residuals are
$R_1 = 10x - 2y - 3z - 205$
$R_2 = -2x + 10y - 2z - 154$
$R_3 = -2x - y + 10z - 120$

$x = \Sigma \Delta x = 32$
$y = \Sigma \Delta y = 26$
$z = \Sigma \Delta z = 21$

Δx_0	Δy_0	Δz_0	R_1	R_2	R_3
0	0	0	$\boxed{-205}$	-154	-120
20	0	0	-5	$\boxed{-194}$	-160
0	20	0	-45	6	$\boxed{-180}$
0	0	18	$\boxed{-99}$	-30	0
10	0	0	1	$\boxed{-50}$	-20
0	5	0	-9	0	$\boxed{-25}$
0	0	2	$\boxed{-15}$	-4	-5
1	0	0	-5	-6	$\boxed{-7}$
0	0	1	$\boxed{-8}$	-8	3
1	0	0	2	$\boxed{-10}$	1
0	1	0	0	0	0

Q3: $9x - 2y + z = 50$
$x + 5y - 3z = 18$
$-2x + 2y + 7z = 19$

sol: Residuals are
$R_1 = 9x - 2y + z - 50$
$R_2 = x + 5y - 3z - 18$
$R_3 = -2x + 2y + 7z - 19$

Δx	Δy	Δz	R_1	R_2	R_3
0	0	0	$\boxed{-50}$	-18	-19
6	0	0	4	-12	$\boxed{-31}$
0	0	4	8	$\boxed{-24}$	-3
0	5	0	-2	1	$\boxed{7}$
0	0	-1	-3	$\boxed{4}$	0
0	-1	0	-1	-1	$\boxed{-2}$
0	0				
0	0				
0	0				

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★ **Rayleigh's Power Method**

It is used to find the largest Eigen value and its corresponding Eigen vector.

Initial Eigen vector can be considered as

$$\begin{bmatrix}1\\0\\0\end{bmatrix} \text{ or } \begin{bmatrix}0\\1\\0\end{bmatrix} \text{ or } \begin{bmatrix}0\\0\\1\end{bmatrix} \text{ or } \begin{bmatrix}1\\1\\1\end{bmatrix}$$

Q1: Find the largest Eigen value and corresponding Eigen vector for the following matrices by Rayleigh's Power method.

$$A = \begin{bmatrix}2 & 0 & 1\\0 & 2 & 0\\1 & 0 & 2\end{bmatrix}$$

λ - Eigen Value

X - Eigen Vector

Sol: Let $X^{(0)} = [1\ 0\ 0]'$

$$AX^{(0)} = \begin{bmatrix}2 & 0 & 1\\0 & 2 & 0\\1 & 0 & 2\end{bmatrix}\begin{bmatrix}1\\0\\0\end{bmatrix} = \begin{bmatrix}2\\0\\1\end{bmatrix} = 2\begin{bmatrix}1\\0\\0.5\end{bmatrix} = \lambda^{(1)}X^{(1)}$$

$$AX^{(1)} = \begin{bmatrix}2 & 0 & 1\\0 & 2 & 0\\1 & 0 & 2\end{bmatrix}\begin{bmatrix}1\\0\\0.5\end{bmatrix} = \begin{bmatrix}2.5\\0\\2\end{bmatrix} = 2.5\begin{bmatrix}1\\0\\0.8\end{bmatrix} = \lambda^{(2)}X^{(2)}$$

$$AX^{(2)} = \begin{bmatrix}2 & 0 & 1\\0 & 2 & 0\\1 & 0 & 2\end{bmatrix}\begin{bmatrix}1\\0\\0.8\end{bmatrix} = \begin{bmatrix}2.8\\0\\2.6\end{bmatrix} = 2.8\begin{bmatrix}1\\0\\0.928\end{bmatrix} = \lambda^{(3)}X^{(3)}$$

$$AX^{(3)} = \begin{bmatrix}2 & 0 & 1\\0 & 2 & 0\\1 & 0 & 2\end{bmatrix}\begin{bmatrix}1\\0\\0.928\end{bmatrix} = \begin{bmatrix}2.928\\0\\2.856\end{bmatrix} = 2.928\begin{bmatrix}1\\0\\0.975\end{bmatrix} = \lambda^{(4)}X^{(4)}$$

$$AX^{(4)} = \begin{bmatrix}2 & 0 & 1\\0 & 2 & 0\\1 & 0 & 2\end{bmatrix}\begin{bmatrix}1\\0\\0.975\end{bmatrix} = \begin{bmatrix}2.975\\0\\2.95\end{bmatrix} = 2.975\begin{bmatrix}1\\0\\0.991\end{bmatrix} = \lambda^{(5)}X^{(5)}$$

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$$Ax^{(5)} = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0.991 \end{bmatrix} = \begin{bmatrix} 2.991 \\ 0 \\ 2.98 \end{bmatrix} = 2.991 \begin{bmatrix} 1 \\ 0 \\ 0.996 \end{bmatrix} = \lambda^{(6)} x^{(6)}$$

$$Ax^{(6)} = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0.996 \end{bmatrix} = \begin{bmatrix} 2.996 \\ 0 \\ 2.992 \end{bmatrix} = 2.996 \begin{bmatrix} 1 \\ 0 \\ 0.998 \end{bmatrix} = \lambda^{(7)} x^{(7)}$$

$\therefore$ The largest eigen value $= 2.996$
and eigen vector $= [1 \ 0 \ 0.998]'$

Q2: $A = \begin{bmatrix} 25 & 1 & 2 \\ 1 & 3 & 0 \\ 2 & 0 & -4 \end{bmatrix}$

Sol: Let $x^{(0)} = [1 \ 0 \ 0]'$

$$Ax^{(0)} = \begin{bmatrix} 25 & 1 & 2 \\ 1 & 3 & 0 \\ 2 & 0 & -4 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 25 \\ 1 \\ 2 \end{bmatrix} = 25 \begin{bmatrix} 1 \\ 0.04 \\ 0.08 \end{bmatrix} = \lambda^{(1)} x^{(1)}$$

$$Ax^{(1)} = \begin{bmatrix} 25 & 1 & 2 \\ 1 & 3 & 0 \\ 2 & 0 & -4 \end{bmatrix} \begin{bmatrix} 1 \\ 0.04 \\ 0.08 \end{bmatrix} = \begin{bmatrix} 25.2 \\ 1.12 \\ 1.68 \end{bmatrix} = 25.2 \begin{bmatrix} 1 \\ 0.044 \\ 0.067 \end{bmatrix} = \lambda^{(2)} x^{(2)}$$

$$Ax^{(2)} = \begin{bmatrix} 25 & 1 & 2 \\ 1 & 3 & 0 \\ 2 & 0 & -4 \end{bmatrix} \begin{bmatrix} 1 \\ 0.044 \\ 0.067 \end{bmatrix} = \begin{bmatrix} 25.176 \\ 1.132 \\ 1.732 \end{bmatrix} = 25.176 \begin{bmatrix} 1 \\ 0.044 \\ 0.068 \end{bmatrix} = \lambda^{(3)} x^{(3)}$$

$$Ax^{(3)} = \begin{bmatrix} 25 & 1 & 2 \\ 1 & 3 & 0 \\ 2 & 0 & -4 \end{bmatrix} \begin{bmatrix} 1 \\ 0.044 \\ 0.068 \end{bmatrix} = \begin{bmatrix} 25.18 \\ 1.132 \\ 1.728 \end{bmatrix} = 25.18 \begin{bmatrix} 1 \\ 0.044 \\ 0.068 \end{bmatrix} = \lambda^{(4)} x^{(4)}$$

$$Ax^{(4)} = \begin{bmatrix} 25 & 1 & 2 \\ 1 & 3 & 0 \\ 2 & 0 & -4 \end{bmatrix} \begin{bmatrix} 1 \\ 0.044 \\ 0.068 \end{bmatrix} = \begin{bmatrix} 25.18 \\ 1.132 \\ 1.728 \end{bmatrix} = 25.18 \begin{bmatrix} 1 \\ 0.044 \\ 0.068 \end{bmatrix} = \lambda^{(5)} x^{(5)}$$

$\therefore$ The largest Eigen value $= 25.18$
and eigen vector $= [1 \ 0.044 \ 0.068]'$

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Q3: $A = \begin{bmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

Sol: Let $x^{(0)} = [1 \ 0 \ 0]'$

$$Ax^{(0)} = \begin{bmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = 1\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \lambda^{(1)}x^{(1)}$$

$$Ax^{(1)} = \begin{bmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 7 \\ 3 \\ 0 \end{bmatrix} = 7\begin{bmatrix} 1 \\ 0.428 \\ 0 \end{bmatrix} = \lambda^{(2)}x^{(2)}$$

$$Ax^{(2)} = \begin{bmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 0.428 \\ 0 \end{bmatrix} = \begin{bmatrix} 3.568 \\ 1.856 \\ 0 \end{bmatrix} = 3.568\begin{bmatrix} 1 \\ 0.52 \\ 0 \end{bmatrix} = \lambda^{(3)}x^{(3)}$$

$$Ax^{(3)} = \begin{bmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 0.52 \\ 0 \end{bmatrix} = \begin{bmatrix} 4.12 \\ 2.04 \\ 0 \end{bmatrix} = 4.12\begin{bmatrix} 1 \\ 0.495 \\ 0 \end{bmatrix} = \lambda^{(4)}x^{(4)}$$

$$Ax^{(4)} = \begin{bmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 0.495 \\ 0 \end{bmatrix} = \begin{bmatrix} 3.97 \\ 1.99 \\ 0 \end{bmatrix} = 3.97\begin{bmatrix} 1 \\ 0.501 \\ 0 \end{bmatrix} = \lambda^{(5)}x^{(5)}$$

$$Ax^{(5)} = \begin{bmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 0.501 \\ 0 \end{bmatrix} = \begin{bmatrix} 4.006 \\ 2.002 \\ 0 \end{bmatrix} = 4.006\begin{bmatrix} 1 \\ 0.499 \\ 0 \end{bmatrix} = \lambda^{(6)}x^{(6)}$$

$$Ax^{(6)} = \begin{bmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 0.499 \\ 0 \end{bmatrix} = \begin{bmatrix} 3.994 \\ 1.998 \\ 0 \end{bmatrix} = 3.994\begin{bmatrix} 1 \\ 0.500 \\ 0 \end{bmatrix} = \lambda^{(7)}x^{(7)}$$

$$Ax^{(7)} = \begin{bmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 0.5 \\ 0 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \\ 0 \end{bmatrix} = 4\begin{bmatrix} 1 \\ 0.5 \\ 0 \end{bmatrix} = \lambda^{(8)}x^{(8)}$$

$$Ax^{(8)} = \begin{bmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 0.5 \\ 0 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \\ 0 \end{bmatrix} = 4\begin{bmatrix} 1 \\ 0.5 \\ 0 \end{bmatrix} = \lambda^{(9)}x^{(9)}$$

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$\therefore$ The largest Eigen value $= 4$
and eigen vector $= [1 \; 0.5 \; 0]'$

Q4: $A = \begin{bmatrix} 5 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & 5 \end{bmatrix}$ Take $[1 \; 0 \; 0]'$

Sol: Here $X^{(0)} = [0 \; 1 \; 0]'$

$$AX^{(0)} = \begin{bmatrix} 5 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & 5 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 5 \\ 0 \\ 1 \end{bmatrix} = 5\begin{bmatrix} 1 \\ 0 \\ 0.2 \end{bmatrix} = \lambda^{(1)} X^{(1)}$$

$$AX^{(1)} = \begin{bmatrix} 5 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & 5 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0.2 \end{bmatrix} = \begin{bmatrix} 5.2 \\ 0 \\ 2 \end{bmatrix} = 5.2\begin{bmatrix} 1 \\ 0 \\ 0.384 \end{bmatrix} = \lambda^{(2)} X^{(2)}$$

$$AX^{(2)} = \begin{bmatrix} 5 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & 5 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0.384 \end{bmatrix} = \begin{bmatrix} 5.384 \\ 0 \\ 2.92 \end{bmatrix} = 5.384\begin{bmatrix} 1 \\ 0 \\ 0.542 \end{bmatrix} = \lambda^{(3)} X^{(3)}$$

$$AX^{(3)} = \begin{bmatrix} 5 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & 5 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0.542 \end{bmatrix} = \begin{bmatrix} 5.542 \\ 0 \\ 3.71 \end{bmatrix} = 5.542\begin{bmatrix} 1 \\ 0 \\ 0.669 \end{bmatrix} = \lambda^{(4)} X^{(4)}$$

$$AX^{(4)} = \begin{bmatrix} 5 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & 5 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0.669 \end{bmatrix} = \begin{bmatrix} 5.669 \\ 0 \\ 4.345 \end{bmatrix} = 5.669\begin{bmatrix} 1 \\ 0 \\ 0.766 \end{bmatrix} = \lambda^{(5)} X^{(5)}$$

$$AX^{(5)} = \begin{bmatrix} 5 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & 5 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0.766 \end{bmatrix} = \begin{bmatrix} 5.766 \\ 0 \\ 4.83 \end{bmatrix} = 5.766\begin{bmatrix} 1 \\ 0 \\ 0.837 \end{bmatrix} = \lambda^{(6)} X^{(6)}$$

$$AX^{(6)} = \begin{bmatrix} 5 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & 5 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \\ 0.837 \end{bmatrix} = \begin{bmatrix} 5.837 \\ 0 \\ 5.185 \end{bmatrix} = 5.837\begin{bmatrix} 1 \\ 0 \\ 0.999 \end{bmatrix} = \lambda^{(7)} X^{(7)}$$

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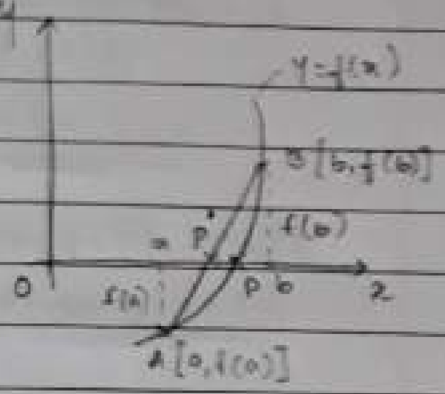
★ **Regula-Falsi Method / Method of False Position:**

Let $f(x)=0$ posses a real root in (a,b).

Let $f(a)<0$ and $f(b)>0$.

Consider two points $A[a,f(a)]$ and $B[b,f(b)]$ on the curve.



Line joining two points is given by: $\dfrac{y-y_1}{x-x_1} = \dfrac{y_2-y_1}{x_2-x_1}$

$$\Rightarrow \frac{y-f(a)}{x-a} = \frac{f(b)-f(a)}{b-a}$$

Since P' crosses x axis, $y=0$

$$\therefore \frac{-f(a)}{x-a} = \frac{f(b)-f(a)}{b-a}$$

$$(x-a)[f(b)-f(a)] = -f(a)[b-a]$$

$$x[f(b)-f(a)] - af(b) + af(a) = -bf(a) + af(a)$$

$$\therefore \boxed{x = \frac{af(b)-bf(a)}{f(b)-f(a)}}$$

Q1: Find the approximate root of $x^3-3x+4=0$ using the method of False Position.

Sl: $f(x) = x^3-3x+4=0$

$f(0)=4$ $f(-1)=6$ Roots lies in between $(-2,3)$

$f(1)=2$ $f(-2)=2>0$ $f(-2.1) = 1.039$

$f(2)=6$ $f(-3)=-14<0$ $f(-2.2) = -0.048$

Roots lies in between $(-2.1, -2.2)$

$\therefore a = -2.1$; $f(a) = 1.039$

$b = -2.2$; $f(b) = -0.048$

$\cancel{x_1 = \dfrac{af(b)-bf(a)}{f(b)-f(a)}}$

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1^{st} ~~applicaren~~ Approximation

$$x_1 = \frac{af(b) - bf(a)}{f(b) - f(a)} = \frac{-2.1(-0.048) - (-2.2)(1.039)}{-0.048 - 1.039}$$

$$x_1 = -2.1955$$

$f(x_1) = f(-2.1955)$

$f(x_1) = 0.0037$

$f(-2.2) = -0.048 < 0$

$\therefore$ The root lies in $(-2.1955, -2.2)$

2^{nd} ~~applicaion~~ approximation

$$x_2 = \frac{af(b) - bf(a)}{f(b) - f(a)} = \frac{-2.1955(-0.048) - (-2.2)(0.0037)}{-0.048 - 0.0037}$$

$$x_2 = -2.1958$$

Q2: $\tanh x + \tan x = 0$ in $(2,3)$. Consider 3 iterations.

Sol: $f(x) = \tan x + \tanh x$

$a = 2 \quad f(a) = -1.221$

$b = 3 \quad f(b) = 0.852$

1^{st} Approximation

$$x_1 = \frac{af(b) - bf(a)}{f(b) - f(a)} = \frac{2(0.852) - 3(-1.221)}{0.852 + 1.221}$$

$$x_1 = 2.5890$$

$f(x_1) = f(2.5890) = 0.3727$

$f(2) = -1.221 < 0$

$\therefore$ The root lies between $[\overset{a}{2.589}, \overset{b}{2}]$

$a = 2.589 \quad f(a) = 0.3727$

$b = 2 \quad f(b) = -1.221$

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2nd Approximation

$$x_2 = \frac{af(b)-bf(a)}{f(b)-f(a)} = \frac{2.589(-1.221)-2(0.3428)}{-1.221-0.3428}$$

$x_2 = 2.4514$

$f(x_2) = f(2.4514) = 0.1595 > 0$

$f(2) = -1.221 < 0$

$\therefore$ The root lies between $(2.4514, 2)$

$a = 2.4514 \quad f(a) = 0.1595$

$b = 2 \quad f(b) = -1.221$

3rd Approximation

$$x_3 = \frac{af(b)-bf(a)}{f(b)-f(a)} = \frac{2.4514(-1.221)-2(0.1595)}{-1.221-0.1595}$$

$x_3 = 2.3992$

Q3: $x\log_{10}x - 1.2 = 0$

Sol: $f(x) = x\log_{10}x - 1.2$

$f(1) = -1.2$

$f(2) = -0.597 < 0$

$f(3) = 0.281 > 0$

Root lies between $(2,3)$

$f(2.7) = -0.0353 \qquad a = 2.7$

$f(2.8) = 0.0520 \qquad b = 2.8$

The root lies in between $(2.7, 2.8)$

1st Approximation

$$x_1 = \frac{af(b)-bf(a)}{f(b)-f(a)} = \frac{2.7(0.0520)-2.8(-0.0353)}{0.052+0.0353}$$

$x_1 = 2.7404$

$f(x_1) = f(2.7404) = -0.0003 < 0$

$\therefore$ The root lies between $(2.7404, 2.8)$

2nd Approximation

$$x_2 = \frac{af(b) - bf(a)}{f(b) - f(a)} = \frac{2.7404(+0.0520) - 2.8(-0.0003)}{0.0520 + 0.0003}$$

$$x_2 = 2.7405$$

$f(x_2) = f(2.7405) = -0.00012 < 0$

$\therefore$ The root lies between $(2.7405, 2.8)$

3rd Approximation

$$x_3 = \frac{af(b) - bf(a)}{f(b) - f(a)} = \frac{2.7405(+0.052) - 2.8(-0.0001)}{0.052 + 0.0001}$$

$$x_3 = 2.7406$$

Q4: $x^4 + 2x^2 - 16x + 5 = 0$ in $(0, 1)$.

Sol: $f(x) = x^4 + 2x^2 - 16x + 5$

$a = 0 \quad f(0) = 5$

$b = 1 \quad f(1) = -8$

$f(0.3) = 0.3881 > 0$

$f(0.4) = -1.0544 < 0$

$\therefore$ The root lies between $(0.3, 0.4)$

1st Approximation

$$x_1 = \frac{af(b) - bf(a)}{f(b) - f(a)} = \frac{0.3(-1.0544) - 0.4(0.3881)}{-1.0544 - 0.3881}$$

$$x_1 = 0.3269$$

$f(x_1) = f(0.3269) = -0.0052 < 0$

$\therefore$ the root lies in between $(0.3, 0.3269)$

2nd Approximation

$$x_2 = \frac{af(b) - bf(a)}{f(b) - f(a)} = \frac{0.3(-0.0052) - 0.3269(0.3881)}{-0.0052 - 0.3881}$$

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$x_2 = 0.3265$

$f(x_2) = f(0.3265) = 0.0005 > 0$

The roots lies between $(0.3265, 0.3269)$

3rd Approximation

$$x_3 = \frac{af(b) - bf(a)}{f(b) - f(a)} = \frac{0.3265(-0.0052) - 0.3269(0.0005)}{-0.0052 - 0.0052}$$

$x_3 = 0.3265$

★ Newton - Raphson Method:

Let $f(x) = 0$

$x_0 \rightarrow$ approximate value

$h \rightarrow$ step length

$x_1 = x_0 + h \longrightarrow$ exact value

$f(x_0 + h) = 0$

$$f(x_0) + hf'(x_0) + \frac{h^2}{2!} f''(x_0) + \cdots = 0 \quad \text{(Taylor's series)}$$

Since h is very small, $h^2, h^3 \ldots$ can be neglected

$f(x_0) + hf'(x_0) = 0$

$$h = -\frac{f(x_0)}{f'(x_0)}$$

$x_1 = x_0 + h$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$\vdots$

$$\boxed{x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}}$$

Q1: Use Newton - Raphson method to find a real root.

$x^4 + x^3 - 7x^2 - x + 5 = 0$

$f'(x) = 4x^3 + 3x^2 - 14x - 1$

sol: $f(1) = -1$

$f(2) = -1 < 0$

$f(3) = 47 > 0$

The root lies in between $(2,3)$

Let $x_0 = 2$

$f(x_0) = f(2) = -1$

$f'(x_0) = f'(2) = 15$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$x_1 = 2 - \frac{-1}{15} = \frac{31}{15} = 2.0666$$

$f(x_1) = f(2.0666) = 0.1036$

$f'(x_1) = f'(2.0666) = 18.1845$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$x_2 = 2.0666 - \frac{0.1036}{18.1845} = 2.0609$$

$f(x_2) = f(2.0609) = 0.0008$

$f'(x_2) = f'(2.0609) = 17.9024$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$$

$$x_3 = 2.0609 - \frac{0.0008}{17.9024} = 2.0608$$

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Q2: Derive an iterative formula for finding k^{th} root of a positive number and hence find 4^{th} root of 22.

$x = \sqrt[k]{N}$

$\Rightarrow x = N^{1/k}$

$x^k = N \quad \Rightarrow \quad x^k - N = 0$

$\therefore f(x) = x^k - N \quad ; \quad f'(x) = k x^{k-1}$

Newton - Raphson Formula

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$= x_n - \frac{x_n^k - N}{k x_n^{k-1}}$$

$$= \frac{k x_n^{k-1} x_n - x_n^k + N}{k x_n^{k-1}}$$

$$= \frac{k x_n^k - x_n^k + N}{k x_n^{k-1}}$$

$$\therefore \boxed{x_{n+1} = \frac{(k-1) x_n^k + N}{k x_n^{k-1}}} \quad \text{Iterative formula}$$

4^{th} root of 22: $\sqrt[4]{22}$

$\sqrt[4]{16} < \sqrt[4]{22} < \sqrt[4]{81}$

$\sqrt[4]{22}$ is nearer to $\sqrt[4]{16} = 2$

$\sqrt[4]{22}$: $x_0 = 2$; $N = 22$; $k = 4$

$$x_{n+1} = \frac{(4-1) x_n^4 + 22}{4 x_n^{4-1}} \qquad \because x_{n+1} = \frac{(k-1) x_n^k + N}{k x_n^{4-1}}$$

$$x_{n+1} = \frac{3 x_n^4 + 22}{4 x_n^3}$$

$$x_{n+1} = \frac{1}{4}\left[3 x_n + \frac{22}{x_n^3}\right]$$

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1st iteration

$$x_1 = \frac{1}{4}\left[3x_0 + \frac{22}{x_0^3}\right]$$

$$x_1 = \frac{1}{4}\left[3(2) + \frac{22}{8}\right]$$

$$\underline{x_1 = 2.1875}$$

2nd iteration

$$x_2 = \frac{1}{4}\left[3x_1 + \frac{22}{x_1^3}\right]$$

$$x_2 = \frac{1}{4}\left[3(2.1875) + \frac{22}{(2.1875)^3}\right]$$

$$\underline{x_2 = 2.1660}$$

3rd iteration

$$x_3 = \frac{1}{4}\left[3x_2 + \frac{22}{x_2^3}\right]$$

$$x_3 = \frac{1}{4}\left[3(2.166) + \frac{22}{(2.166)^3}\right]$$

$$\underline{x_3 = 2.1657}$$

4th iteration

$$x_4 = \frac{1}{4}\left[3x_3 + \frac{22}{x_3^3}\right]$$

$$x_4 = \frac{1}{4}\left[3(2.1657) + \frac{22}{(2.1657)^3}\right]$$

$$\underline{x_4 = 2.1657}$$

$$\therefore \sqrt[4]{22} = 2.1657$$

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Q3: Derive an iterative formula to find reciprocal of square root of a positive number (22).

$$x = \frac{1}{\sqrt{N}}$$

$$x = \frac{1}{N^{1/2}}$$

$$\Rightarrow x^2 = \frac{1}{N} \quad \Rightarrow \quad x^2 - \frac{1}{N} = 0$$

$$\therefore f(x) = x^2 - \frac{1}{N} \; ; \; f'(x) = 2x$$

Newton Raphson Formula

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$x_{n+1} = x_n - \frac{x_n^2 - 1/N}{2x_n}$$

$$x_{n+1} = \frac{2x_n^2 - x_n^2 + 1/N}{2x_n}$$

$$x_{n+1} = \frac{x_n^2 + 1/N}{2x_n}$$

$$\boxed{x_{n+1} = \frac{1}{2}\left[x_n + \frac{1}{Nx_n}\right]}$$ Iterative formula

$$\frac{1}{\sqrt{16}} < \frac{1}{\sqrt{22}} < \frac{1}{\sqrt{25}}$$

$\frac{1}{\sqrt{22}}$ is nearer to $\frac{1}{\sqrt{25}} = \frac{1}{5} = 0.2$

$\therefore x_0 = 0.2, \; N = 22$

1st iteration

$$x_1 = \frac{1}{2}\left[x_0 + \frac{1}{Nx_0}\right]$$

$$x_1 = \frac{1}{2}\left[0.2 + \frac{1}{22(0.2)}\right]$$

$$\underline{\underline{x_1 = 0.2136}}$$

2nd iteration

$$x_2 = \frac{1}{2}\left[x_1 + \frac{1}{Nx_1}\right]$$

$$x_2 = \frac{1}{2}\left[0.2136 + \frac{1}{22(0.2136)}\right]$$

$$\underline{\underline{x_2 = 0.2132}}$$

3rd iteration

$$x_3 = \frac{1}{2}\left[x_2 + \frac{1}{Nx_2}\right]$$

$$x_3 = \frac{1}{2}\left[0.2132 + \frac{1}{22(0.2132)}\right]$$

$$\underline{\underline{x_3 = 0.2132}}$$

$$\therefore \frac{1}{\sqrt{22}} = 0.2132$$

Q4: Find an iterative formula for cube root of a positive number and hence find cube root of 12.

$$x = \sqrt[3]{N}$$

$$x = N^{1/3}$$

$$\Rightarrow x^3 = N \quad \Rightarrow x^3 - N = 0$$

$$\therefore f(x) = x^3 - N \quad \Rightarrow \quad f'(x) = 3x^2$$

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Newton Raphson Formula

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$x_{n+1} = x_n - \frac{x_n^3 - N}{3x_n^2}$$

$$x_{n+1} = \frac{3x_n^3 - x_n^3 + N}{3x_n^2}$$

$$x_{n+1} = \frac{2x_n^3 + N}{3x_n^2}$$

$$\boxed{x_{n+1} = \frac{1}{3}\left[2x_n + \frac{N}{x_n^2}\right]}$$ Iterative formula

$\sqrt[3]{8} < \sqrt[3]{12} < \sqrt[3]{27}$

$\sqrt[3]{12}$ is nearer to $\sqrt[3]{8} = 2$

$\therefore x_0 = 2$; $N = 12$

1st iteration

$$x_1 = \frac{1}{3}\left[2x_0 + \frac{N}{x_0^2}\right]$$

$$x_1 = \frac{1}{3}\left[2(2) + \frac{12}{4}\right]$$

$$x_1 = 2.3333$$

2nd Iteration

$$x_2 = \frac{1}{3}\left[2x_1 + \frac{N}{x_1^2}\right]$$

$$x_2 = \frac{1}{3}\left[2(2.3333) + \frac{12}{(2.3333)^2}\right]$$

$$x_2 = 2.2902$$

3rd iteration.

$$x_3 = \frac{1}{3}\left[2x_2 + \frac{N}{x_2^2}\right]$$

$$x_3 = \frac{1}{3}\left[2(2.2902) + \frac{12}{(2.2902)^2}\right]$$

$$\underline{\underline{x_3 = 2.2894}}$$

4th iteration

$$x_4 = \frac{1}{3}\left[2x_3 + \frac{N}{x_3^2}\right]$$

$$x_4 = \frac{1}{3}\left[2(2.2894) + \frac{12}{(2.2894)^2}\right]$$

$$\underline{\underline{x_4 = 2.2894}}$$

Hence $\underline{\underline{\sqrt[3]{12} = 2.2894}}$

UNIT - 5

Numerical Methods - 2

★ **FORWARD AND BACKWORD DIFFERENCE:**

consider $y = f(x)$, for different values of x i.e., $x_1, x_2, x_3 \cdots x_n$ corresponding y values will be $y_1, y_2, y_3 \cdots y_n$.

Interpolation is a process where we find a particular value in the set of values given. We consider two differences to find the particular value.

- i. **Forward Difference**: denoted as Δ

Forward difference table

x	y	Δy	$\Delta^2 y$	$\cdots\cdots \Delta^n y$
x_0	y_0			
		$y_1 - y_0 = \Delta y_0$		
x_1	y_1		$\Delta^2 y_0$	
		$y_2 - y_1 = \Delta y_1$		
x_2	y_2		$\Delta^2 y_1$	
		$y_3 - y_2 = \Delta y_2$		
x_3	y_3			$\Delta^n y_0$
$\vdots$	$\vdots$			
x_{n-1}	y_{n-1}		$\Delta^2 y_{n-2}$	
		$y_n - y_{n-1} = \Delta y_{n-1}$		
x_n	y_n			

Newton's Forward Interpolation Formula

$$y_x = y(x_0 + rh) = y_0 + r\Delta y_0 + \frac{r(r-1)}{2!}\Delta^2 y_0 + \frac{r(r-1)(r-2)}{3!}\Delta^3 y_0 + \cdots$$

where $r = \dfrac{x - x_0}{h}$

$x \rightarrow$ value to be found

$x_0 \rightarrow$ initial value

$h \rightarrow$ step length

ii. Backward Difference : Denoted as ∇

Backward Difference Table

x	y	∇y	$\nabla^2 y$	..... $\nabla^n y$
x_0	y_0	$y_1 - y_0 = \nabla y_1$	$\nabla^2 y_2$	
x_1	y_1	$y_2 - y_1 = \nabla y_2$	$\nabla^2 y_3$	
x_2	y_2	$y_3 - y_2 = \nabla y_3$		$\nabla^n y_n$
x_3	y_3	$\vdots$		
$\vdots$	$\vdots$	$\vdots$		
			$\nabla^2 y_n$	
x_{n-1}	y_{n-1}	$y_n - y_{n-1} = \nabla y_n$		
x_n	y_n			

Newton's Backward Interpolation Formula

$$y_r = y(x_n + rh) = y_n + r\nabla y_n + \frac{r(r+1)}{2!}\nabla^2 y_n + \frac{r(r+1)(r+2)}{3!}\nabla^3 y_n + \cdots$$

where $r = \dfrac{x - x_n}{h}$

$x \rightarrow$ value to be found

$x_n \rightarrow$ final value

$h \rightarrow$ step length

Newton's Forward Interpolation Formula

$$y_r = y(x_0 + rh) = y_0 + r\Delta y_0 + \frac{r(r-1)}{2!}\Delta^2 y_0 + \frac{r(r-1)(r-2)}{3!}\Delta^3 y_0 + \cdots$$

diff wrt r

$$y_r' = y'(x_0 + rh)h = 0 + \Delta y_0 + \frac{2r-1}{2!}\Delta^2 y_0 + \frac{3r^2-6r+2}{3!}\Delta^3 y_0 + \cdots$$

$$y'(x_0 + rh) = \frac{1}{h}\left[\Delta y_0 + \frac{2r-1}{2}\Delta^2 y_0 + \frac{3r^2-6r+2}{6}\Delta^3 y_0 + \cdots\right]$$

diff wrt r

$$y_r'' = y''(x_0 + rh).h = \frac{1}{h}\left[0 + \frac{2}{2}\Delta^2 y_0 + \frac{6r-6}{6}\Delta^3 y_0 + \cdots\right]$$

$$y''(x_0 + rh) = \frac{1}{h^2}\left[\Delta^2 y_0 + (r-1)\Delta^3 y_0 + \cdots\right]$$

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Q1: Use appropriate interpolation formula to compute as indicated for the following data

x	0.1	0.2	0.3	0.4	0.5	0.6
y	2.68	3.04	3.38	3.68	3.96	4.21

compute y(0.25),
(value lies in the first half hence forward polation is considered)

Sol:

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
0.1	2.68					
		0.36				
0.2	3.04		-0.02			
		0.34		-0.02		
0.3	3.38		-0.04		0.04	
		0.30		0.02		-0.07
0.4	3.68		-0.02		-0.03	
		0.28		-0.01		
0.5	3.96		-0.03			
		0.25				
0.6	4.21					

Newton's Forward Interpolation Formula

$$y_r = y(x_0+rh) = y_0 + r\Delta y_0 + \frac{r(r-1)}{2!}\Delta^2 y_0 + \frac{r(r-1)(r-2)}{3!}\Delta^3 y_0$$

$$+ \frac{r(r-1)(r-2)(r-3)}{4!}\Delta^4 y_0 + \frac{r(r-1)(r-2)(r-3)(r-4)}{5!}\Delta^5 y_0$$

$$r = \frac{x-x_0}{h} = \frac{0.25-0.1}{0.1} = 1.5$$

$$y_x = 2.68 + (1.5)(0.36) + \frac{(1.5)(1.5-1)}{2}(-0.02) + \frac{(1.5)(1.5-1)(1.5-2)}{6}(-0.02)$$

$$+ \frac{(1.5)(1.5-1)(1.5-2)(1.5-3)}{24}(0.04) + \frac{(1.5)(1.5-1)(1.5-2)(1.5-3)(1.5-4)}{120}(-0.07)$$

$$= 2.68 + 0.54 - 0.0075 + 0.0012 + 0.0009 + 0.0008$$

$$y_0 = 3.215$$

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Q2: Compute $f(1.1)$ and $f'(1.1)$

x	1	1.2	1.4	1.6	1.8	2.0
y	0.0	0.128	0.544	1.296	2.432	4.00

Sol:

x	y	Δy	$\Delta^2 y$	$\Delta^3 y$	$\Delta^4 y$	$\Delta^5 y$
1	0.0					
		0.128				
1.2	0.128		0.288			
		0.416		0.048		
1.4	0.544		0.336		0	
		0.752		0.048		0
1.6	1.296		0.384		0	
		1.136		0.048		
1.8	2.432		0.432			
		1.568				
2.0	4.00					

Newton's Forward Interpolation formula

$$r = \frac{x - x_0}{h} = \frac{1.1 - 1.0}{0.2} = 0.5$$

$$y_r = y(x_0 + rh) = y_0 + r\Delta y_0 + \frac{r(r-1)}{2!}\Delta^2 y_0 + \frac{r(r-1)(r-2)}{3!}\Delta^3 y_0$$

$$y = 0 + 0.5(0.128) + \frac{0.5(0.5-1)}{2}(0.288) + \frac{0.5(0.5-1)(0.5-2)}{6}(0.048)$$

$$y = 0.064 - 0.0285 + 0.003$$

$$f(1.1) = y = 0.0325$$

$$y' = f'(y) = \frac{1}{h}\left[\Delta y_0 + \frac{2r-1}{2}\Delta^2 y_0 + \frac{3r^2 - 6r + 2}{3}\Delta^3 y_0\right]$$

$$y' = f'(1.1) = \frac{1}{0.2}\left[0.128 + \frac{2(0.5)-1}{2}(0.288) + \frac{3(0.5)^2 - 6(0.5) + 2}{3}(0.048)\right]$$

$$f'(1.1) = \frac{1}{0.2}\left[0.128 + 0 - 0.004\right]$$

$$f'(1.1) = 0.6200$$

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Q3. $f(0)=0$, $f(2)=4$, $f(4)=56$, $f(6)=204$, $f(8)=496$, $f(10)=980$
Find the interpolating polynomial and Hence evaluate $f(3)$ and $f(7)$.

Sol:

x	y	I diff	II diff	III diff	IV diff	V diff
0	0					
		4				
2	4		48			
		52		48		
4	56		96		0	
		148		48		0
6	204		144		0	
		292		48		
8	496		192			
		484				
10	980					

Newton's Forward Interpolation formula

$$y_r = y(x_0 + rh) = y_0 + r\Delta y_0 + \frac{r(r-1)}{2!}\Delta^2 y_0 + \frac{r(r-1)(r-2)}{3!}\Delta^3 y_0$$

$$r = \frac{x - x_0}{h} = \frac{x-0}{2} = \frac{x}{2}$$

$$y = 0 + \frac{x}{\cancel{2}}(\cancel{4}^{2}) + \frac{x}{2}\left(\frac{x}{2}-1\right)\frac{1}{2!}(\cancel{48}^{6}) + \frac{x}{2}\left(\frac{x}{2}-1\right)\left(\frac{x}{2}-2\right)\frac{1}{6}(48)$$

$$y = 2x + 6x(x-2) + x(x-2)(x-4)$$

$$y = 2x + 6x^2 - 12x + (x^2-2x)(x-4)$$

$$y = 6x^2 + x^3 - 4x^2 - 2x^2 + 8x - 10x$$

$$\cancel{y = x^3 - 5x^2 + 8x}$$

$$y = x^3 - 2x$$

$$f(x) = y = x^3 - 2x$$

$$f(3) = 3^3 - 2(3) = 21$$

$$\cancel{f(7) = 7^3 - 2(7) = 329}$$

For $f(7)$ consider backward difference

Keerthana Ashok

Q1: Find the number of students securing
a. less than 45 marks
b. between 40 and 45 marks

Marks	30-40	40-50	50-60	60-70	70-80
No of students	31	42	51	35	31

Sol: Number of students securing less than 40 is 31 students

$\therefore$ $<40 = 31$

similarly $<50 = 31+42 = 73$

$<60 = 73+51 = 124$

$<70 = 124+35 = 159$

$<80 = 159+31 = 190$

Newton's forward difference table

x	y	Δy_0	$\Delta^2 y_0$	$\Delta^3 y_0$	$\Delta^4 y_0$
40	31				
		42			
50	73		9		
		51		-25	
60	124		-16		37
		35		12	
70	159		-4		
		31			
80	190				

Newton's forward interpolation formula.

a. $y_r = y(x_0+rh) = y_0 + r\Delta y_0 + \frac{r(r-1)}{2!}\Delta^2 y_0 + \frac{r(r-1)(r-2)}{3!}\Delta^3 y_0$

$+ \frac{r(r-1)(r-2)(r-3)}{4!}\Delta^4 y_0$

$r = \frac{x-x_0}{h} = \frac{45-40}{10} = 0.5$

$y_r = 31 + 0.5(42) + \frac{0.5(0.5-1)}{2}(9) + \frac{0.5(0.5-1)(0.5-2)}{6}(-25)$

$+ \frac{0.5(0.5-1)(0.5-2)(0.5-3)}{24}(37)$

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$y_7 = 31 + 21 - 1.125 - 1.5625 - 1.4453$

$y_7 = 47.8672$

$y_7 \approx 48$

b. $f(45) - f(40) = 48 - 31 = 17$

17 students have secured between 40 and 45.

Q5: A rod is rotating in a plane. The following table is the angle θ radians through which the rod has turned for various values of the time 't' sec.

t	0	0.2	0.4	0.6	0.8	1.0	1.2
θ	0	0.12	0.49	1.12	2.02	3.20	4.67

Calculate the angular velocity and angular acceleration of the rod when t = 0.6 sec

Sol:

t	θ	$\Delta\theta_0$	$\Delta^2\theta_0$	$\Delta^3\theta_0$	$\Delta^4\theta_0$	$\Delta^5\theta_0$	$\Delta^6\theta_0$
0	0						
		0.12					
0.2	0.12		0.25				
		0.37		0.01			
0.4	0.49		0.26		0		
		0.63		0.01		0	
0.6	1.12		0.27		0		0
		0.90		0.01		0	
0.8	2.02		0.28		0		
		1.18		0.01			
1.0	3.20		0.29				
		1.47					
1.2	4.67						

$r = \dfrac{x - x_0}{h} = \dfrac{0.6 - 0}{0.2} = 3$

$v = \dfrac{d\theta}{dt} = y_0' = \dfrac{1}{h}\left[\Delta\theta_0 + \dfrac{(2r-1)}{2!}\Delta^2\theta_0 + \dfrac{(3r^2-6r+2)}{3!}\Delta^3\theta_0\right]$

$v = \dfrac{1}{0.2}\left[0.12 + \dfrac{(2\times3-1)}{2}(0.25) + \dfrac{(3(3)^2-6(3)+2)}{6}(0.01)\right]$

$v = \dfrac{1}{0.2}\left[0.12 + 0.6250 + 0.0183\right]$

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$v = 3.816$ rad/sec

$$a = \frac{d^2\theta}{dt^2} = y_r'' = \frac{1}{h^2}\left[\Delta^2\theta_0 + \frac{(6r-6)}{3!}\Delta^3\theta_0\right]$$

$$a = \frac{1}{0.2^2}\left[0.25 + \frac{(6(3)-6)}{6}(0.01)\right]$$

$$a = \frac{1}{0.2^2}\left[0.25 + 0.02\right]$$

$a = 6.75$ rad/sec^2

★ Lagrange's Interpolation Formula:

$$y = f(x) = \frac{(x-x_1)(x-x_2)\cdots(x-x_n)}{(x_0-x_1)(x_0-x_2)\cdots(x_0-x_n)}y_0 + \frac{(x-x_0)(x-x_2)(x-x_3)\cdots(x-x_n)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)\cdots(x_1-x_n)}y_1 + \cdots + \frac{(x-x_0)(x-x_1)\cdots(x-x_{n-1})}{(x_n-x_0)(x_n-x_1)\cdots(x_n-x_{n-1})}y_n$$

Q1: Use Lagrange's interpolation formula to find an interpolating polynomial. Hence find $f(5)$

x	$1\ x_0$	$3\ x_1$	$4\ x_2$	$6\ x_3$
$f(x)$	$3\ y_0$	$9\ y_1$	$30\ y_2$	$132\ y_3$

Sol:

$$y = f(x) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)}y_0 + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)}y_1 + \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)}y_2 + \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)}y_3$$

$$f(x) = \frac{(x-3)(x-4)(x-6)}{(1-3)(1-4)(1-6)}(3) + \frac{(x-1)(x-4)(x-6)}{(3-1)(3-4)(3-6)}(9) + \frac{(x-1)(x-3)(x-6)}{(4-1)(4-3)(4-6)}(30) + \frac{(x-1)(x-3)(x-4)}{(6-1)(6-3)(6-4)}(132)$$

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$$f(x) = -0.1(x-2)(x-4)(x-6) + 1.5(x-1)(x-4)(x-6) - 5(x-1)(x-2)(x-6) + 4.4(x-1)(x-2)(x-4)$$

wkt

$$(x-a)(x-b)(x-c) = x^3 - (a+b+c)x^2 + (ab+bc+ca)x - abc$$

$$\therefore f(x) = -0.1[x^3 - 13x^2 + 54x - 72]$$
$$+1.5[x^3 - 11x^2 + 34x - 24]$$
$$-5[x^3 - 10x^2 + 27x - 18]$$
$$+4.4[x^3 - 8x^2 + 19x - 12]$$

$$f(x) = -0.1x^3 + 1.3x^2 - 5.4x + 7.2$$
$$+1.5x^3 - 16.5x^2 + 51x - 36$$
$$-5x^3 + 50x^2 - 135x + 90$$
$$+4.4x^3 - 35.2x^2 + 83.6x - 52.8$$

$$\underline{f(x) = 0.8x^3 - 0.4x^2 - 5.8x + 8.4}$$

$$f(5) = (0.8)5^3 - 0.4(5)^2 - 5.8(5) + 8.4 = 69.4$$

Q2:

x	$0\ x_0$	$1\ x_1$	$2\ x_2$	$5\ x_3$
$f(x)$	$2\ y_0$	$3\ y_1$	$12\ y_2$	$147\ y_3$

Find $f(3)$

Sol:

$$y = f(x) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)}y_0 + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)}y_1$$
$$+ \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)}y_2 + \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)}y_3$$

$$f(x) = \frac{(x-1)(x-2)(x-5)}{(0-1)(0-2)(0-5)}(2) + \frac{(x-0)(x-2)(x-5)}{(1-0)(1-2)(1-5)}(3)$$
$$+ \frac{(x-0)(x-1)(x-5)}{(2-0)(2-1)(2-5)}(12) + \frac{(x-0)(x-1)(x-2)}{(5-0)(5-1)(5-2)}(147)$$

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$$f(x) = -0.2(x-1)(x-2)(x-5) + 0.75(x-0)(x-2)(x-5) - 2(x-0)(x-1)(x-5) + 2.45(x-0)(x-1)(x-2)$$

since

$$(x-a)(x-b)(x-c) = x^3 - (a+b+c)x^2 + (ab+bc+ca)x - abc$$

$$\begin{aligned} f(x) = & -0.2\,[x^3 - 8x^2 + 17x - 10] \\ & +0.75\,[x^3 - 7x^2 + 10x - 0] \\ & -2\,[x^3 - 6x^2 + 5x - 0] \\ & +2.45\,[x^3 - 3x^2 + 2x - 0] \end{aligned}$$

$$\begin{aligned} f(x) = & -0.2x^3 + 1.6x^2 - 3.4x + 2 \\ & +0.75x^3 - 5.25x^3 + 7.5x + 0 \\ & -2x^3 + 12x^2 - 10x + 0 \\ & +2.45x^3 - 7.35x^2 + 4.9x - 0 \end{aligned}$$

$$f(x) = x^3 + x^2 - x + 2$$

$$f(3) = 3^3 + 3^2 - 3 + 2 = 35$$

Q3:

	x_0	x_1	x_2	x_3
x	654	658	659	661
$f(x)$	2.8156	2.8182	2.8189	2.8202
	y_0	y_1	y_2	y_3

compute $y(656)$

Sol:

$$y = f(x) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} y_0 + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} y_1 + \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} y_2 + \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} y_3$$

$$\begin{aligned} f(656) = & \frac{(656-658)(656-659)(656-661)}{(654-658)(654-659)(654-661)}(2.8156) \\ & + \frac{(656-654)(656-659)(656-661)}{(658-654)(658-659)(658-661)}(2.8182) \\ & + \frac{(656-654)(656-658)(656-661)}{(659-654)(659-658)(659-661)}(2.8189) \end{aligned}$$

$$+ \frac{(656-654)(656-658)(656-659)}{(661-654)(661-658)(661-659)}(2.8202)$$

$$f(656) = \frac{(-2)(-3)(-5)(2.8156)}{(-4)(-5)(-7)} + \frac{(2)(-3)(-5)(2.8182)}{(4)(-1)(-3)}$$

$$+ \frac{(2)(-2)(-5)}{(5)(1)(-2)}(2.8189) + \frac{(2)(-2)(-3)}{(7)(3)(2)}(2.8202)$$

$$f(656) = 0.6033 + 7.0455 - 5.6378 + 0.8057$$

$$\underline{f(656) = 2.8167}$$

* **Newton's Divided difference Formula:**

Divided difference table.

x	$y = f(x)$	I DD	II DD	
x_0	$y_0 = f(x_0)$			
		$f(x_0,x_1) = \frac{f(x_1)-f(x_0)}{x_1-x_0}$		
x_1	$y_1 = f(x_1)$		$f(x_0,x_1,x_2) = \frac{f(x_1,x_2)-f(x_0,x_1)}{x_2-x_0}$	
		$f(x_1,x_2) = \frac{f(x_2)-f(x_1)}{x_2-x_1}$		
x_2	$y_2 = f(x_2)$		$f(x_1,x_2,x_3) = \frac{f(x_2,x_3)-f(x_1,x_2)}{x_3-x_1}$	
		$f(x_2,x_3) = \frac{f(x_3)-f(x_2)}{x_3-x_2}$		$f(x_0,x_1,\ldots,x_{n-1},x_n)$
x_3	$y_3 = f(x_3)$	$\vdots$	$\vdots$	
$\vdots$	$\vdots$	$\vdots$	$\vdots$	
x_{n-1}	$y_{n-1} = f(x_{n-1})$		$f(x_{n-2},x_{n-1},x_n) = \frac{f(x_{n-1},x_n)-f(x_{n-2},x_{n-1})}{x_n-x_{n-2}}$	
		$f(x_{n-1},x_n) = \frac{f(x_n)-f(x_{n-1})}{x_n-x_{n-1}}$		
x_n	$y_n = f(x_n)$			

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Newton's Divided Difference Formula

$$y = f(x_0) + (x - x_0) f(x_0, x_1) + (x - x_0)(x - x_1) f(x_0, x_1, x_2) + (x - x_0)(x - x_1)(x - x_2) f(x_0, x_1, x_2, x_3) + \dots$$

Q1: Use Newton's general interpolation formula to compute: $f(310)$

x	300	304	305	307
$f(x)$	2.4771	2.4829	2.4843	2.4871

Sol:

x	$f(x)$	I DD	II DD	III DD
300	2.4771			
		0.00145		
304	2.4829		-0.00001	
		0.0014		0.000001
305	2.4843		0	
		0.0014		
307	2.4871			

$$y = f(x_0) + (x - x_0) f(x_0, x_1) + (x - x_0)(x - x_1) f(x_0, x_1, x_2) + (x - x_0)(x - x_1)(x - x_2) f(x_0, x_1, x_2, x_3) + \dots$$

$$y = 2.4771 + (310 - 300)[0.00145] + (310 - 300)(310 - 304)(-0.00001) + (310 - 300)(310 - 304)(310 - 305)(0.000001)$$

$$y = 2.4771 + 0.0145 - 0.0006 + 0.0003$$

$$y = 2.4913$$

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Q.2:

x	4	5	7	10	11	13
$f(x)$	48	100	294	900	1210	2028

compute $f(8)$ and $f'(8)$ and $f''(8)$.

Sol:

x	$f(x)$	I DD	II DD	III DD	IV DD	V DD
4	48					
		52				
5	100		15			
		97		1		
7	294		21		0	0
		202		1		
10	900		27		0	
		310		1		
11	1210		33			
		409				
13	2028					

$$y = f(x_0) + (x-x_0)f(x_0,x_1) + (x-x_0)(x-x_1)f(x_0,x_1,x_2) + (x-x_0)(x-x_1)(x-x_2)f(x_0,x_1,x_2,x_3)$$

$$y = 48 + (8-4)52 + (8-4)(8-5)15 + (8-4)(8-5)(8-7)(1)$$

$$y = 48 + 208 + 180 + 12$$

$$\underline{y = 448}$$

$$f(x) = f(x_0) + (x-x_0)f(x_0,x_1) + (x-x_0)(x-x_1)f(x_0,x_1,x_2) + (x-x_0)(x-x_1)(x-x_2)f(x_0,x_1,x_2,x_3).$$

diff w.r.t x

$$f'(x) = 0 + f(x_0,x_1) + [(x-x_0)+(x-x_1)]f(x_0,x_1,x_2) + [(x-x_0)(x-x_1) + (x-x_2)(x-x_0) + (x-x_1)(x-x_2)]f(x_0,x_1,x_2,x_3).$$

$$f'(8) = 52 + [(8-4)+(8-5)]15 + [(8-4)(8-5) + (8-7)(8-4) + (8-5)(8-7$$

$$f'(8) = 52 + 105 + 19$$

$$\underline{f'(8) = 176}$$

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diff wrt x

$$f''(x) = 2f(x_0, x_1, x_2) + [(x-x_1)+(x-x_0)+(x-x_1)+(x-x_2) + (x-x_2)+(x-x_0)]\, f(x_0, x_1, x_2, x_3)$$

$$f''(18) = 2(15) + [(18-5)+(18-4)+(18-5)+(18-7)+(18-7)+(18-4)]$$

$$f''(18) = 30 + 76$$

$$f''(18) = 106$$

Q3:

x	-4	-1	0	2	5
y	1245	33	5	9	1335

Determine $f(x)$ as a polynomial.

Sol:

x	y	I DD	II DD	III DD	IV DD
-4	1245				
		-404			
-1	33		94		
		-28		-14	3
0	5		10		
		2		13	
2	9		88		
		442			
5	1335				

$$y = f(x) = f(x_0) + (x-x_0) f(x_0, x_1) + (x-x_0)(x-x_1) f(x_0, x_1, x_2) + (x-x_0)(x-x_1)(x-x_2) f(x_0, x_1, x_2, x_3) + (x-x_0)(x-x_1)(x-x_2)(x-x_3) f(x_0, x_1, x_2, x_3, x_4)$$

$$f(x) = 1245 + (x+4)(-404) + (x+4)(x+1)(94) + (x+4)(x+1)x(-14) + (x+4)(x+1)x(x-2)(3)$$

$$f(x) = 1245 - 404x - 1616 + 94x^2 + 470x + 376 - 14x^3 - 70x^2 - 56x + 3x^4 + 15x^3 + 12x^2 - 6x^3 - 30x^2 - 24$$

$$f(x) = 3x^4 - 5x^3 + 6x^2 - 14x + 5$$

* Numerical Integration:

1 - **Simpson's 1/3rd Rule:** $I = \int_a^b f(x)\,dx$

$$I = \frac{h}{3}\left[(y_0 + y_n) + 4(y_1 + y_3 + y_5 + \cdots) + 2(y_2 + y_4 + y_6 + \cdots)\right]$$

where $h = \dfrac{b-a}{n}$

Q1: Use Simpson's 1/3 rule to evaluate

$$\int_0^5 \frac{dx}{4x+5}$$ taking 10 equal parts.

Hence find $\log_e 5$.

Sol: $h = \dfrac{b-a}{n}$ $\quad n = 10$ $\quad y = \dfrac{1}{4x+5}$

$h = \dfrac{5-0}{10} = 0.5$

x	0	0.5	1	1.5	2	2.5	3	3.5	4	4.5	5
$y = \frac{1}{4x+5}$	0.2	0.1428	0.1111	0.0909	0.0769	0.0666	0.0588	0.0526	0.0476	0.0434	0.04
	y_0	y_1	y_2	y_3	y_4	y_5	y_6	y_7	y_8	y_9	y_{10}

$$I = \cancel{\frac{0.5}{3}}\ \frac{h}{3}\left[(y_0 + y_{10}) + 4(y_1 + y_3 + y_5 + y_7 + y_9) + 2(y_2 + y_4 + y_6 + y_8)\right]$$

$$I = \frac{0.5}{3}\left[(0.2 + 0.04) + 4(0.1428 + 0.0909 + 0.0666 + 0.0526 + 0.0434) + 2(0.1111 + 0.0769 + 0.0588 + 0.0476)\right]$$

$$I = 0.1666\left[0.24 + 1.5852 + 0.5888\right]$$

$$\underline{I = 0.4021}$$

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$$\int_0^5 \frac{dx}{4x+5} = \frac{1}{4}\int_0^5 \frac{1}{(x+5/4)}\,dx = \frac{1}{4}\left[\log\left(x+\frac{5}{4}\right)\right]_0^5$$

$$= \frac{1}{4}\left[\log\frac{25}{4} - \log\frac{5}{4}\right]$$

$$= \frac{1}{4}\left[\frac{\log(25/4)}{\log(5/4)}\right]$$

$$= \frac{1}{4}\log 5$$

comparing with numerical solution

$$\frac{1}{4}\log_e 5 = 0.4023$$

$$\log_e 5 = 1.6092$$

Q2: Evalute $\int_0^{0.6} e^{-x^2}\,dx$ taking $n=6$

Sol: $h = \frac{b-a}{n}$ here $n=6$

$$h = \frac{0.6-0}{6} = 0.1$$

x	0	0.1	0.2	0.3	0.4	0.5	0.6
$y=e^{-x^2}$	1	0.9900	0.9607	0.9139	0.8521	0.7788	0.6976
	y_0	y_1	y_2	y_3	y_4	y_5	y_6

$$I = \frac{h}{3}\left[(y_0+y_6) + 4(y_1+y_3+y_5) + 2(y_2+y_4)\right]$$

$$= \frac{0.1}{3}\left[(1+0.6976) + 4(0.99+0.9139+0.7788) + 2(0.9607+0.85\ldots\right.$$

$$= \frac{0.1}{3}\left[1.6976 + 10.7308 + 3.6256\right]$$

$$I = 0.5351$$

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Q3: $I = \int_0^{\pi/2} \sqrt{\cos\theta}\, d\theta \quad n = 6$

Sol: $h = \frac{b-a}{n}$ here $n = 6$

$h = \frac{\pi/2 - 0}{6} = \frac{\pi}{12} = 15^\circ$

x	0°	15°	30°	45°	60°	75°	90°
$y = \sqrt{\cos\theta}$	1	0.9828	0.9306	0.8408	0.7071	0.5087	0
	y_0	y_1	y_2	y_3	y_4	y_5	y_6

$I = \frac{h}{3}\left[(y_0 + y_6) + 4(y_1 + y_3 + y_5) + 2(y_2 + y_4)\right]$

$= \frac{3.14}{12 \times 3}\left[(1+0) + 4(0.9828 + 0.8408 + 0.5087) + 2(0.9306 + 0.7071)\right]$

$= \frac{3.14}{36}\left[1 + 9.3292 + 3.2754\right]$

$I = 1.1866$

2 - Simpson's 3/8 th Rule:

$I = \frac{3h}{8}\left[(y_0 + y_n) + 3(y_1 + y_2 + y_4 + y_5 + y_7 + \cdots) + 2(y_3 + y_6 \cdots)\right]$

Q1: Use simpson's 3/8 th rule to evaluate $\int_0^{0.3} \sqrt{1 - 8x^3}\, dx$ by taking 7 ordinates : 6 equal parts.

Sol: $h = \frac{b-a}{n} = \frac{0.3-0}{6} = 0.05$

x	0	0.05	0.1	0.15	0.2	0.25	0.3
$y = \sqrt{1-8x^3}$	1	0.9994	0.9959	0.9864	0.9674	0.9354	0.8854
	y_0	y_1	y_2	y_3	y_4	y_5	y_6

$I = \frac{3h}{8}\left[(y_0 + y_6) + 3(y_1 + y_2 + y_4 + y_5) + 2(y_3)\right]$

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$$I = \frac{3(0.06)}{8}\left[(1+0.8854)+3(0.9994+0.9959+0.9674+0.9356) + 2(0.9864)\right]$$

$$I = \frac{0.15}{8}\left[1.8854 + 11.6943 + 1.9728\right]$$

$$\underline{I = 0.2916}$$

Q2: $\int_2^8 \frac{dx}{\log_{10} x}$ taking $h=1$

Sol: $h = \frac{b-a}{n} \Rightarrow n = \frac{b-a}{h} = \frac{8-2}{1} = \underline{6}$

x	~~8~~	~~8~~	2	3	4	5	6	7	8
$h = \frac{1}{\log_{10} x}$	~~8~~		3.3219	2.0959	1.6609	1.4306	1.2850	1.1832	1.107
			y_0	y_1	y_2	y_3	y_4	y_5	y_6

$$I = \frac{3h}{8}\left[(y_0+y_6)+3(y_1+y_2+y_4+y_5)+2(y_3)\right]$$

$$= \frac{3(1)}{8}\left[(3.3219+1.1073)+3(2.0959+1.6609+1.2850+1.1832)+2(1.4306)\right]$$

$$= \frac{3}{8}\left[4.4292+18.675+2.8612\right]$$

$$\underline{I = 9.7370}$$

Q3: $\int_0^3 \frac{dx}{(1+x)^2}$ taking $n=3$

Sol: $h = \frac{b-a}{n} = \frac{3-0}{3} = 1$

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x	0	1	2	3
$y = \frac{1}{(1+x^2)}$	1 y_0	0.25 y_1	0.1111 y_2	0.0625 y_3

$$I = \frac{3h}{8}\left[(y_0 + y_3) + 3(y_1 + y_2)\right]$$

$$I = \frac{3(1)}{8}\left[(1+0.0625) + 3(0.25+0.1111)\right]$$

$$I = 0.8046$$

* **Weddle's Rule**

$$I = \frac{3h}{10}\left[y_0 + 5y_1 + y_2 + 6y_3 + y_4 + 5y_5 + y_6\right]$$

or

$$I = \frac{3h}{10}\sum k y_i \quad \text{where } k = 1,5,1,6,1,5,2,5,1,6,1,5.$$

Q: Use weddle's rule to evalute the following $\int_4^{5.2} \log_e x \, dx$ taking 6 equal strips.

Sol: $h = \frac{b-a}{n} = \frac{5.2-4}{6} = 0.2$

x	4	4.2	4.4	4.6	4.8	5.0	5.2
$y = \log_e x$	1.3862 y_0	1.4350 y_1	1.4816 y_2	1.5260 y_3	1.5686 y_4	1.6094 y_5	1.6486 y_6

$$I = \frac{3h}{10}\left[y_0 + 5y_1 + y_2 + 6y_3 + y_4 + 5y_5 + y_6\right]$$

$$= \frac{3(0.2)}{10}\left[1.3862 + 5(1.4350) + 1.4816 + 6(1.5260) + 1.5686 + 5(1.6094) + 1.6486\right]$$

$$I = 1.8277$$

Q2: $\int_0^1 \frac{x\,dx}{1+x^2}$ taking 7 ordinates and hence find $\log_e 2$

Sol: $h = \frac{b-a}{n} = \frac{1-0}{6} = 0.1666 = \frac{1}{6}$

x	0	1/6	2/6	3/6	4/6	5/6	1
$y = \frac{x}{1+x^2}$	0	0.1621	0.3000	0.4000	0.4615	0.4918	0.5000
	y_0	y_1	y_2	y_3	y_4	y_5	y_6

$$I = \frac{3h}{10}\left[y_0 + 5y_1 + y_2 + 6y_3 + y_4 + 5y_5 + y_6\right]$$

$$I = \frac{3}{6\times 10}\left[0 + 5(0.1621) + 0.3 + 6(0.4) + 0.4615 + 5(0.4918) + 0.5\right]$$

$$\underline{I = 0.3465}$$

$$\int_0^1 \frac{x\,dx}{1+x^2} = \frac{1}{2}\int_0^1 \frac{2x}{1+x^2}\,dx$$

$$0.3465 = \frac{1}{2} \log |1+x^2| \Big|_0^1$$

$$0.6930 = \left[\log_e 2 - \log 1\right]$$

$$\Rightarrow \underline{\log_e 2 = 0.6930}$$

Q3: $\int_0^{\pi/2} \frac{\sqrt{\sin x}}{\sqrt{\sin x} + \sqrt{\cos x}}\,dx$ taking $n = 6$

Sol: $h = \frac{b-a}{n} = \frac{\pi/2 - 0}{6} = \frac{\pi}{12}$

x	0	$\pi/12$	$\pi/6$	$\pi/4$	$\pi/3$	$5\pi/12$	$\pi/2$
$y = \frac{\sqrt{\sin x}}{\sqrt{\sin x} + \sqrt{\cos x}}$	0	0.3410	0.4317	0.5	0.5682	0.6589	1
	y_0	y_1	y_2	y_3	y_4	y_5	y_6

$$I = \frac{3h}{10}\left[y_0 + 5y_1 + y_2 + 6y_3 + y_4 + 5y_5 + y_6\right]$$

$$= \frac{3\pi}{12\times 10}\left[0 + 5(0.3410) + 0.4317 + 6(0.5) + 0.5682 + 5(0.6589) + 1\right]$$

$$\underline{I = 0.7853}$$

Q4: A curve is drawn to pass through the points given by the following table.

x	1	1.5	2	2.5	3	3.5	4
y	2	2.4	2.7	2.8	3	2.6	2.1

Estimate the area bounded by the curve between the x axis and the lines $x=1$, $x=4$ by applying Weddle's rule

Sol:

$$I = \frac{3h}{10}\left[y_0 + 5y_1 + y_2 + 6y_3 + y_4 + 5y_3 + y_6\right]$$

$$I = \frac{3(0.5)}{10}\left[2 + 5(2.4) + 2.7 + 6(2.8) + 3 + 5(2.6) + 2.1\right]$$

$$\underline{I = 7.74} \text{ sq units}$$

5: A solid of revolution is formed by rotating about the x axis, The area between the x axis, the lines $x=0$ and x and a curve through the points with the coordinates

x	0.00	0.25	0.50	0.75	1.0
y	1	0.9896	0.9589	0.9089	0.8415

Estimate the volume of the solid formed ~~using~~

sol: $h = 0.25$

Simpson's $1/3^{rd}$ rule

$$I = \frac{h}{3}\left[(y_0 + y_4) + 4(y_1 + y_3) + 2(y_2)\right]$$

$$I = \frac{0.25}{3}\left[(1 + 0.8415) + 4(0.9896 + 0.9089) + 2(0.9589)\right]$$

$\underline{I = 0.9461}$ (is the solution) here $n = 4$ multiple of 2.

Simpson's $3/8^{th}$ rule

$$I = \frac{3h}{8}\left[(y_0 + y_4) + 3(y_1 + y_2) + 2(y_3)\right]$$

$$I = \frac{3(0.25)}{8}\left[(1 + 0.8415) + 3(0.9896 + 0.9589) + 2(0.9089)\right]$$

$\underline{I = 0.8910}$ X

For $1/3^{rd}$: n is multiples of 2

For $3/8^{th}$: n is multiples of 3

For Weddle's : n is multiples of 5.

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UNIT - 6

Z Transforms

If $u_n = f(n)$ defined for all $n = 0, 1, 2 \ldots$ and $u_n = 0$ if $n < 0$, then z transforms of u_n denoted by $Z_T[u_n]$ and is defined as

$$Z_T[u_n] = \sum_{n=0}^{\infty} u_n z^{-n} = \bar{u}(z)$$

$$Z_T[u_n] = \bar{u}(z)$$

$$\Rightarrow u_n = Z_T^{-1}[\bar{u}(z)]$$ Inverse z-transforms.

★ **z-transforms of some standard functions:**

1 $$Z_T[k^n] = \sum_{n=0}^{\infty} k^n z^{-n}$$

$$= \sum_{n=0}^{\infty} \left(\frac{k}{z}\right)^n$$

$$= 1 + \frac{k}{z} + \left(\frac{k}{z}\right)^2 + \left(\frac{k}{z}\right)^3 + \cdots$$

wkt $1 + x + x^2 + \cdots = 1/1-x$ (GP)

$$Z_T[k^n] = \frac{1}{1 - k/z}$$

$$\boxed{Z_T[k^n] = \frac{z}{z-k}}$$

2 $$Z_T[1] = \sum_{n=0}^{\infty} 1 \cdot z^{-n}$$

$$= \sum_{n=0}^{\infty} \frac{1}{z^n}$$

$$= 1 + \frac{1}{z} + \frac{1}{z^2} + \cdots$$

$$\therefore Z_T[1] = \frac{1}{1 - 1/z}$$

$$\boxed{Z_T[1] = \frac{z}{z-1}}$$

3. Show that $\boxed{Z_T[n^k] = -z \frac{d}{dz} Z_T(n^{k-1})}$

considering RHS

$$\text{RHS}: -z \frac{d}{dz} Z_T(n^{k-1}) = -z \frac{d}{dz} \sum_{n=0}^{\infty} n^{k-1} z^{-n}$$

$$= -z \sum_{n=0}^{\infty} n^{k-1} \frac{d}{dz} z^{-n}$$

$$= -z \sum_{n=0}^{\infty} n^{k-1} (-n) z^{-n-1}$$

$$= z \sum_{n=0}^{\infty} n^k z^{-n-1}$$

$$= \cancel{z} \sum_{n=0}^{\infty} \frac{n^k z^{-n}}{\cancel{z}}$$

$$= \sum_{n=0}^{\infty} n^k z^{-n}$$

$$= \underline{Z_T[n^k]} = \text{LHS}$$ Hence proved

4. $$Z_T[n] = \sum_{n=0}^{\infty} n z^{-n}$$

but $$\sum_{n=0}^{\infty} n^k z^{-n} = -z \frac{d}{dz} Z_T(n^{k-1})$$

Taking $k = 1$

$$Z_T[n] = -z \frac{d}{dz} Z_T[1]$$

$$Z_T[n] = -z\frac{d}{dz}\left[\frac{z}{z-1}\right]$$

$$= -z\left[\frac{(\cancel{z}-1)-\cancel{z}}{(z-1)^2}\right]$$

$$\boxed{Z_T[n] = \frac{z}{(z-1)^2}}$$

5. $Z_T[n^2]$

wkt $Z_T[n^k] = -z\frac{d}{dz} Z_T[n^{k-1}]$

Here $n = 2$

$$Z_T[n^2] = -z\frac{d}{dz} Z_T[n]$$

wkt $Z_T[n] = \frac{z}{(z-1)^2}$

$$\therefore Z_T[n^2] = -z\frac{d}{dz}\left[\frac{z}{(z-1)^2}\right]$$

$$= -z\left[\frac{(z-1)^2 - 2(z-1)z}{(z-1)^4}\right]$$

$$= -z\left[\frac{z-1-2z}{(z-1)^3}\right]$$

$$\boxed{Z_T[n^2] = \frac{z^2+z}{(z-1)^3}}$$

6. $Z_T[n^3]$

wkt $Z_T[n^k] = -z\frac{d}{dz} Z_T[n^{k-1}]$

Here $n = 3$

$$Z_T[n^3] = -z\frac{d}{dz} Z_T[n^2]$$

wkt $Z_T[n^2] = \frac{z^2+z}{(z-1)^3}$

$$\therefore Z_T[n^3] = -z \frac{d}{dz}\left[\frac{z^2+z}{(z-1)^3}\right]$$

$$= -z\left[\frac{(2z+1)(z-1)^3 - 3(z-1)^2(z^2+z)}{(z-1)^6}\right]$$

$$= -z\left[\frac{2z^2-z-1-3z^2-3z}{(z-1)^4}\right]$$

$$\boxed{Z_T[n^3] = \frac{z^3+4z^2+z}{(z-1)^4}}$$

★ **Linearity Property:**

If u_n and v_n be any two discreate valued functions

then $\boxed{Z_T[c_1 u_n + c_2 v_n] = c_1 Z_T[u_n] + c_2 Z_T[v_n]}$

considering

$$LHS = Z_T[c_1 u_n + c_2 v_n]$$

$$= \sum_{n=0}^{\infty} [c_1 u_n + c_2 v_n] z^{-n}$$

$$= \sum_{n=0}^{\infty} c_1 u_n z^{-n} + \sum_{n=0}^{\infty} c_2 v_n z^{-n}$$

$$= c_1 \sum_{n=0}^{\infty} u_n z^{-n} + c_2 \sum_{n=0}^{\infty} v_n z^{-n}$$

$$= c_1 Z_T[u_n] + c_2 Z_T[v_n] = RHS$$

★ **Damping Rule:**

If $Z_T[u_n] = \bar{u}(z)$, then:

1. $Z_T[k^n u_n] = \bar{u}\left[\frac{z}{k}\right]$

2. $Z_T[k^{-n} u_n] = \bar{u}[kz]$

Proof 1 :

$$Z_T[k^n U_n] = \sum_{n=0}^{\infty} [k^n U_n] z^{-n}$$

$$= \sum_{n=0}^{\infty} u_n \left[\frac{z}{k}\right]^{-n}$$

$$\boxed{Z_T[k^n U_n] = \bar{u}\left[\frac{z}{k}\right]}$$

Proof 2 :

$$Z_T[k^{-n} U_n] = \sum_{n=0}^{\infty} [k^{-n} U_n] z^{-n}$$

$$= \sum_{n=0}^{\infty} u_n [zk]^{-n}$$

$$\boxed{Z_T[k^{-n} U_n] = \bar{u}[kz]}$$

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★ Shifting Property:

Right shifting property:

If $Z_T[u_n] = \bar{u}(z)$, then $Z_T[u_{n-k}] = z^{-k}\bar{u}(z)$

considering

$$LHS = Z_T[u_{n-k}] = \sum_{n=k}^{\infty} u_{n-k} z^{-n}$$ for all $n<k$ $u_{n-k} = 0$

$$= u_0 z^{-k} + u_1 z^{-k-1} + u_2 z^{-k-2} + \cdots$$

$$= z^{-k}[u_0 + u_1 z^{-1} + u_2 z^{-2} + \cdots]$$

$$= z^{-k}\left[\sum_{n=0}^{\infty} u_n z^{-n}\right]$$

$$\boxed{Z_T[u_{n-k}] = z^{-k}\bar{u}(z)} = RHS.$$

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Left Shifting Property:

If $Z_T[u_n] = \bar{u}(z)$ then

then $Z_T[u_{n+k}] = z^k[\bar{U}(z) - u_0 - u_1 z^{-1} - u_2 z^{-2} - \cdots - u_{k-1} z^{-(k-1)}]$

considering

LHS : $Z_T[U_{n+k}] = \sum_{n=0}^{\infty} u_{n+k} z^{-n}$

$= z^k \sum_{n=0}^{\infty} u_{n+k} z^{-(n+k)}$

$= z^k [u_k z^{-k} + u_{k+1} z^{-(k+1)} + u_{k+2} z^{-(k+2)} + \cdots\cdots]$

Adding and subtracting $u_0 + u_1 z^{-1} + u_2 z^{-2} + \cdots$

$= z^k [u_0 + u_1 z^{-1} + u_2 z^{-2} + \cdots + u_k z^{-k} + u_{k+1} z^{-(k+1)} + \cdots$
$- u_0 - u_1 z^{-1} - u_2 z^{-2} \cdots - u_{k-1} z^{-(k-1)}]$

$= z^k [\sum u_n z^{-n} - (u_0 + u_1 z^{-1} + u_2 z^{-2} + \cdots)]$

$$\boxed{Z_T[u_{n+k}] = z^k[\bar{u}(z) - u_0 - u_1 z^{-1} - u_2 z^{-2} - \cdots - u_{k-1} z^{-(k-1)}]}$$

CASE 1: $Z_T[u_{n+1}] = z[\bar{U}(z) - u_0]$

CASE 2: $Z_T[U_{n+2}] = z^2[\bar{u}(z) - u_0 - u_1 z^{-1}]$

$\vdots \quad \vdots \quad \vdots$

★ $Z_T[(n+1)^2]$

wkt : $Z_T[n^2] = \dfrac{z^2+z}{(z-1)^2} = \bar{u}(z)$

$u_n = n^2 \Rightarrow u_{n+1} = (n+1)^2$

$Z_T[U_n] = \bar{u}(z)$

$Z_T[u_{n+1}] = z[\bar{U}(z) - u_0]$

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$$Z_T[(n+1)^2] = z\left[\frac{z^2+z}{(z-1)^3} - 0\right]$$

$$Z_T[(n+1)^2] = \frac{z^3+z^2}{(z-1)^3}$$

* show that $Z_T\left[\frac{1}{n!}\right] = e^{1/z}$

LHS

$$Z_T\left[\frac{1}{n!}\right] = \sum_{n=0}^{\infty} \frac{1}{n!} z^{-n}$$

$$= 1 + \frac{z^{-1}}{1!} + \frac{z^{-2}}{2!} + \frac{z^{-3}}{3!} + \cdots$$

$$= 1 + \frac{1}{z} + \frac{1}{2z^2} + \frac{1}{6z^3} + \cdots$$

~~RHS~~ $e^t = 1 + \frac{t}{1!} + \frac{t^2}{2!} + \frac{t^3}{3!}$

$$\therefore Z_T\left[\frac{1}{n!}\right] = e^{1/z}$$

* Obtain Z_T of $\cos n\theta$ and $\sin n\theta$. Hence deduce Z_T of $a^n\cos n\theta$, $a^n \sin n\theta$.

wkt $e^{in\theta} = \cos n\theta + i\sin n\theta$

~~$Z_T[\cos n\theta]$~~

$e^{in\theta} = [e^{i\theta}]^n = a^n$ where $a = e^{i\theta}$

$$Z_T[a^n] = \frac{z}{z-a}$$

$$Z_T[(e^{i\theta})^n] = \frac{z}{z-e^{i\theta}} \times \frac{z-e^{-i\theta}}{z-e^{-i\theta}}$$

$$= \frac{z^2 - ze^{-i\theta}}{z^2 - ze^{-i\theta} - ze^{i\theta} + 1}$$

$$= \frac{z[z-(\cos\theta - i\sin\theta)]}{z^2 - z(e^{-i\theta}+e^{i\theta})+1}$$

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$$= \frac{z[(z-\cos\theta)+i\sin\theta]}{z^2-2z\cos\theta+1}$$

$$Z_T[\cos n\theta + i\sin n\theta] = \frac{z[(z-\cos\theta)]}{z^2-2z\cos\theta+1} + i\frac{z\sin\theta}{z^2-2z\cos\theta+1}$$

$$Z_T[\cos n\theta] + i\,Z_T[\sin n\theta] = \frac{z[(z-\cos\theta)]}{z^2-2z\cos\theta+1} + i\,\frac{z\sin\theta}{z^2-2z\cos\theta+1}$$

$$\therefore Z_T[\cos n\theta] = \frac{z[z-\cos\theta]}{z^2-2z\cos\theta+1} \quad //$$

$$Z_T[\sin n\theta] = \frac{z\sin\theta}{z^2-2z\cos\theta+1} \quad //$$

Let $Z_T[\cos n\theta] = \bar{u}(z)$

$Z_T[\sin n\theta] = \bar{v}(z)$

$$Z_T[a^n\cos n\theta] = \bar{u}\left[\frac{z}{a}\right]$$ By damping rule

$$Z_T[a^n\sin n\theta] = \bar{v}\left[\frac{z}{a}\right]$$

$$\therefore Z_T[a^n\cos n\theta] = \frac{z/a\,[z/a-\cos\theta]}{(z/a)^2-2z\cos\theta/a+1}$$

$$= \frac{az[z-a\cos\theta]}{z^2-2az\cos\theta+a^2} \quad //$$

$$\therefore Z_T[a^n\sin n\theta] = \frac{z\sin\theta/a}{(z/a)^2-2z\cos\theta/a+1}$$

$$= \frac{az\sin\theta}{z^2-2az\cos\theta+a^2} \quad //$$

* Obtain Z_T of $\cosh n\theta$ and $\sinh n\theta$

$$\cosh n\theta = \frac{e^{n\theta}+e^{-n\theta}}{2} = \frac{1}{2}\left[(e^{\theta})^n + (e^{-\theta})^n\right]$$

$$\frac{1}{2} Z_T\left[(e^{\theta})^n+(e^{-\theta})^n\right] = \frac{1}{2}\left[Z_T[(e^{\theta})^n] + Z_T[(e^{-\theta})^n]\right]$$

$$= \frac{1}{2}\left[\frac{z}{z-e^{\theta}} + \frac{z}{z-e^{-\theta}}\right]$$

$$= \frac{z}{2}\left[\frac{z-e^{-\theta}+z-e^{\theta}}{z^2-ze^{-\theta}-ze^{\theta}+1}\right]$$

$$= \frac{z}{2}\left[\frac{2z-(e^{\theta}+e^{-\theta})}{z^2-z(e^{\theta}+e^{-\theta})+1}\right]$$

$$= \frac{z}{2}\left[\frac{2z-2\cosh\theta}{z^2-2z\cosh\theta+1}\right] = z\left[\frac{z-\cosh\theta}{z^2-2z\cosh\theta+1}\right] //$$

similarly

$$\sinh n\theta = \frac{e^{n\theta}-e^{-n\theta}}{2} = \frac{1}{2}\left[(e^{\theta})^n - (e^{-\theta})^n\right]$$

$$\frac{1}{2} Z_T\left[(e^{\theta})^n-(e^{-\theta})^n\right] = \frac{1}{2}\left[Z_T[(e^{\theta})^n - (e^{-\theta})^n]\right]$$

$$= \frac{1}{2}\left[\frac{z}{z-e^{\theta}} - \frac{z}{z-e^{-\theta}}\right]$$

$$= \frac{z}{2}\left[\frac{z-e^{-\theta}-z+e^{\theta}}{z^2-ze^{-\theta}-ze^{\theta}+1}\right]$$

$$= \frac{z}{2}\left[\frac{e^{\theta}-e^{-\theta}}{z^2-z(e^{\theta}+e^{-\theta})+1}\right]$$

$$= \frac{z}{2}\left[\frac{2\sinh\theta}{z^2-2z\cosh\theta+1}\right] = z\left[\frac{\sinh\theta}{z^2-2z\cosh\theta+1}\right] //$$

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* Find Z_T of $\cos\left(\frac{n\pi}{2}+\frac{\pi}{4}\right)$

$$\cos\left(\frac{n\pi}{2}+\frac{\pi}{4}\right) = \cos\frac{n\pi}{2}\cos\frac{\pi}{4} - \sin\frac{n\pi}{2}\sin\frac{\pi}{4}$$

$$\cos\left(\frac{n\pi}{2}+\frac{\pi}{4}\right) = \cos\frac{n\pi}{2}\cdot\frac{1}{\sqrt{2}} - \sin\frac{n\pi}{2}\frac{1}{\sqrt{2}}$$

$$= \frac{1}{\sqrt{2}}\left(\cos\frac{n\pi}{2} - \sin\frac{n\pi}{2}\right)$$

$$Z_T\left[\cos\left(\frac{n\pi}{2}+\frac{\pi}{4}\right)\right] = \frac{1}{\sqrt{2}}\left[Z_T\left[\cos\frac{n\pi}{2}\right] - Z_T\left[\sin\frac{n\pi}{2}\right]\right]$$

$$\cancel{= \frac{1}{\sqrt{2}}\left[Z_T\left[e^{in\pi/2}\right]\right]}$$

$$e^{in\pi/2} = \cos\frac{n\pi}{2} + i\sin\frac{n\pi}{2}$$

$$Z_T\left[(e^{i\pi/2})^n\right] = \frac{z}{z-e^{i\pi/2}} \times \frac{z-e^{-i\pi/2}}{z-e^{-i\pi/2}}$$

$$= \frac{z\left[z-e^{-i\pi/2}\right]}{z^2-ze^{-i\pi/2}-ze^{i\pi/2}+1}$$

$$= \frac{z\left[z-e^{-i\pi/2}\right]}{z^2-z(e^{i\pi/2}+e^{-i\pi/2})+1} = \frac{z\left[z-e^{i\pi/2}\right]}{z^2-2z\cos\pi/2+1}$$

$$= \frac{z\left[z-e^{i\pi/2}\right)}{z^2+1}$$

$$= \frac{z^2}{z^2+1} - \frac{ze^{i\pi/2}}{z^2+1}$$

$$= \frac{z^2}{z^2+1} - \frac{z(\cos\pi/2 - i\sin\pi/2)}{z^2+1}$$

$$= \frac{z^2}{z^2+1} + i\frac{z}{z^2+1}$$

$$\therefore Z_T\left[\cos\frac{n\pi}{2}\right] = \frac{z^2}{z^2+1} \qquad \therefore Z_T\left[\sin\frac{n\pi}{2}\right] = \frac{z}{z^2+1}$$

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$$\therefore Z_T\left[\cos\left(\frac{n\pi}{2}+\frac{\pi}{4}\right)\right] = \frac{1}{\sqrt{2}}\left[\frac{z^2}{z^2+1} - \frac{z}{z^2+1}\right]$$

$$= \frac{1}{\sqrt{2}}\left[\frac{z^2-z}{z^2+1}\right]$$

* Find Z_T of $\sin(3n+5)$

$Z_T[\sin(3n+5)]$

$\sin(3n+5) = \sin 3n \cos 5 + \cos 3n \sin 5$

$Z_T[\sin(3n+5)] = Z_T[\sin 3n \cos 5] + Z_T[\cos 3n \sin 5]$

$= \cos 5\, Z_T[\sin 3n] + \sin 5\, Z_T[\cos 3n]$ —— (1)

$e^{i3n} = \cos 3n + i \sin 3n$

$$Z_T[(e^{3i})^n] = \frac{z}{z-e^{3i}} \times \frac{z-e^{-3i}}{z-e^{-3i}}$$

$$= \frac{z[z-e^{-3i}]}{z^2-ze^{-3i}-ze^{3i}+1}$$

$$= \frac{z[z-e^{-3i}]}{z^2-z(e^{3i}+e^{-3i})+1}$$

$$= \frac{z[z-e^{-3i}]}{z^2-2z\cos 3+1}$$

$$= \frac{z[z-(\cos 3 - i\sin 3)]}{z^2-2z\cos 3+1}$$

$$= \frac{z^2-z\cos 3}{z^2-2z\cos 3+1} + i\,\frac{z\sin 3}{z^2-2z\cos 3+1}$$

$$\therefore Z_T[\cos 3n] = \frac{z(z-\cos 3)}{z^2-2z\cos 3+1}$$

$$Z_T[\sin 3n] = \frac{z\sin 3}{z^2-2z\cos 3+1}$$

Substituting in eq (1)

$$\therefore Z_T[\sin(3n+5)] = \cos 5 \frac{z\sin 3}{z^2-2z\cos 3+1} + \sin 5 \frac{(z(z-\cos 3))}{z^2-2z\cos 3+1}$$

$$= \frac{z\sin 3\cos 5 + z^2\sin 5 - z\cos 3\sin 5}{z^2-2z\cos 3+1}$$

$$= \frac{z(\sin 3\cos 5-\cos 3\sin 5)+z^2\sin 5}{z^2-2z\cos 3+1}$$

$$= \frac{z\sin(3-5)+z^2\sin 5}{z^2-2z\cos 3+1}$$

$$= \frac{z(z\sin 5-\sin 2)}{z^2-2z\cos 3+1}$$

* **Initial Value Theorem:**

State and Prove initial value theorem also find expressions for u_1, u_2, u_3.

Statement:

If $Z_T[u_n] = \bar{u}(z)$ then $\lim_{z\to\infty} \bar{u}(z) = u_0$

Proof:

wkt $Z_T[u_n] = \sum_{n=0}^{\infty} u_n z^{-n}$

$$Z_T[u_n] = \bar{u}(z) = u_0 + u_1 z^{-1} + u_2 z^{-2} + u_3 z^{-3} + \cdots$$

$$\bar{u}(z) = u_0 + \frac{u_1}{z} + \frac{u_2}{z^2} + \frac{u_3}{z^3} + \cdots \quad \text{—(1)}$$

Applying limit

$$\lim_{z\to\infty} \bar{u}(z) = \lim_{z\to\infty}\left[u_0 + \frac{u_1}{z} + \frac{u_2}{z^2} + \frac{u_3}{z^3} + \cdots\right]$$

$$\therefore \boxed{\lim_{z\to\infty} \bar{u}(z) = u_0}$$

considering eq ①

$$\bar{u}(z) - u_0 = \frac{u_1}{z} + \frac{u_2}{z^2} + \frac{u_3}{z^3} + \cdots\cdots$$

$$z\left[\bar{u}(z) - u_0\right] = u_1 + \frac{u_2}{z} + \frac{u_3}{z^2} + \cdots\cdots$$

Applying limit

$$\lim_{z\to\infty} z\left[\bar{u}(z) - u_0\right] = \lim_{z\to\infty}\left[u_1 + \frac{u_2}{z} + \frac{u_3}{z^2} + \cdots\right]$$

$$\boxed{\lim_{z\to\infty} z\left[\bar{u}(z) - u_0\right] = u_1}$$

From eq ①

$$\bar{u}(z) - u_0 - \frac{u_1}{z} = \frac{u_2}{z^2} + \frac{u_3}{z^3} + \frac{u_4}{z^4} + \cdots\cdots$$

$$z^2\left[\bar{u}(z) - u_0 - \frac{u_1}{z}\right] = u_2 + \frac{u_3}{z} + \frac{u_4}{z^2} + \cdots\cdots$$

Applying limit

$$\lim_{z\to\infty} z^2\left[\bar{u}(z) - u_0 - \frac{u_1}{z}\right] = \lim_{z\to\infty}\left[u_2 + \frac{u_3}{z} + \frac{u_4}{z^2} + \cdots\right]$$

$$\boxed{\lim_{z\to\infty} z^2\left[\bar{u}(z) - u_0 - \frac{u_1}{z}\right] = u_2}$$

From eq ①

$$\bar{u}(z) - u_0 - \frac{u_1}{z} - \frac{u_2}{z^2} = \frac{u_3}{z^3} + \frac{u_4}{z^4} + \cdots\cdots$$

$$z^3\left[\bar{u}(z) - u_0 - \frac{u_1}{z} - \frac{u_2}{z^2}\right] = u_3 + \frac{u_4}{z} + \cdots\cdots$$

Applying limit

$$\lim_{z\to\infty} z^3\left[\bar{u}(z) - u_0 - \frac{u_1}{z} - \frac{u_2}{z^2}\right] = \lim_{z\to\infty}\left[u_3 + \frac{u_4}{z} + \cdots\right]$$

$$\boxed{\lim_{z\to\infty} z^3\left[\bar{u}(z) - u_0 - \frac{u_1}{z} - \frac{u_2}{z^2}\right] = u_3}$$

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* Final Value Theorem:

State and prove final value theorem

Statement:

If $Z_T[U_n] = \bar{u}(z)$ then $\lim_{z\to 1} [(z-1)\bar{u}(z)] = \lim_{n\to\infty} u_n$

Proof:

wkt $Z_T[U_{n+1}] = z[\bar{u}(z) - u_0]$ —— ①

$Z_T[U_n] = \bar{u}(z)$ —— ②

Subtracting eq ② from eq ①

$Z_T[U_{n+1}] - Z_T[U_n] = z[\bar{U}(z) - u_0] - \bar{u}(z)$

$Z_T[U_{n+1} - U_n] = \bar{u}(z)(z-1) - zU_0$

$\sum_{n=0}^{\infty} [U_{n+1} - U_n] z^{-n} = (z-1)\bar{U}(z) - z u_0$

Applying limit $z \to 1$ on both sides

$\lim_{z\to 1} \sum_{n=0}^{\infty} [U_{n+1} - U_n] z^{-n} = \lim_{z\to 1} (z-1)\bar{u}(z) - z u_0$

$\sum_{n=0}^{\infty} [U_{n+1} - U_n] = \lim_{z\to 1} [(z-1)\bar{u}(z)] - u_0$

$\lim_{z\to 1} [(z-1)\bar{u}(z)] = \sum_{n=0}^{\infty} [U_{n+1} - U_n] + u_0$

$= u_0 + \lim_{m\to\infty} \sum_{n=0}^{m} [U_{n+1} - u_n]$

$= U_0 + \lim_{m\to\infty} [u_1 + U_2 + \cdots + U_{m+1} - U_0 - U_1 - U_2 \cdots - U_m]$

$= u_0 + \lim_{m\to\infty} [-u_0 + U_{m+1}]$

$\lim_{z\to 1} [(z-1)\bar{U}(z)] = \lim_{m\to\infty} U_{m+1}$

as $m \to \infty$

$m+1 \to \infty$

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$\therefore$ ~~[illegible]~~

$\therefore \lim_{m \to \infty} U_{m+1} \approx \lim_{m+1 \to \infty} U_{m+1}$

$$\therefore \boxed{\lim_{z \to 1} \left[(z-1)\bar{u}(z) \right] = \lim_{n \to \infty} u_n}$$

Q: If $\bar{u}(z) = \dfrac{2z^2 + 3z + 4}{(z-3)^3}$, find u_1, u_2, u_3

Sol: Initial value theorem

$$u_0 = \lim_{z \to \infty} \bar{u}(z) = \lim_{z \to \infty} \left[\frac{2z^2 + 3z + 4}{(z-3)^3} \right]$$

$$u_0 = \lim_{z \to \infty} \frac{z^2 \left[2 + 3/z + 4/z^2 \right]}{z^3 (1 - 3/z)^3}$$

$$u_0 = \lim_{z \to \infty} \frac{2 + 3/z + 4/z^2}{z(1 - 3/z)^3} = 0$$

$$u_1 = \lim_{z \to \infty} z \left[\bar{U}(z) - u_0 \right]$$

$$u_1 = \lim_{z \to \infty} z \left[\frac{2z^2 + 3z + 4}{(z-3)^3} - 0 \right]$$

$$u_1 = \lim_{z \to \infty} \frac{z \left[z^2 (2 + 3/z + 4/z^2) \right]}{z^3 (1 - 3/z)^3} = 2$$

$$u_2 = \lim_{z \to \infty} z^2 \left[\bar{u}(z) - u_0 - \frac{u_1}{z} \right]$$

$$u_2 = \lim_{z \to \infty} z^2 \left[\frac{2z^2 + 3z + 4}{(z-3)^3} - 0 - \frac{2}{z} \right]$$

$$\cancel{u_2 = \lim_{z \to \infty} \frac{z^4}{z^3} \left[\frac{2 + 3/z + 4/z^2}{z(1-3/z)^3} - \frac{2}{z} \right]}$$

$$u_2 = \lim_{z \to \infty} \frac{2z^4 + 3z^3 + 4z^2 - 2z(z-3)^3}{z(z-3)^3}$$

$$u_2 = \lim_{z \to \infty} \frac{2z^4 + 3z^3 + 4z^2 - 2z(z^3 - 9z^2 + 27z - 27)}{(z-3)^2}$$

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$$u_2 = \underset{z\to\infty}{Lt} \frac{\cancel{2z^4} + 3z^3 + 4z^2 - \cancel{2z^4} + 18z^3 - 54z^2 + 54z}{(z-3)^3}$$

$$u_2 = \underset{z\to\infty}{Lt} \frac{21z^3 - 50z^2 + 54z}{(z-3)^3}$$

$$u_2 = \underset{z\to\infty}{Lt} \frac{z^3(21 - 50/z + 54/z^2)}{z^3(1-3/z)^3} = 21$$

$$u_3 = \underset{z\to\infty}{Lt} z^3\left[\bar{u}(z) - u_0 - \frac{u_1}{z} - \frac{u_2}{z^2}\right]$$

$$u_3 = \underset{z\to\infty}{Lt} z^3\left[\frac{2z^2+3z+4}{(z-3)^3} - 0 - \frac{2}{z} - \frac{21}{z^2}\right]$$

$$u_3 = \underset{z\to\infty}{Lt} \left[\frac{2z^5+3z^4+4z^3 - 2z^2(z-3)^3 - 21z(z-3)^3}{(z-3)^3}\right]$$

$$u_3 = \underset{z\to\infty}{Lt} \left[\frac{2z^5+3z^4+4z^3-(2z^2+21z)(z^3-9z^2+21z-27)}{(z-3)^3}\right]$$

$$u_3 = \underset{z\to\infty}{Lt} \left[\frac{\cancel{2z^5} + 3z^4 + 4z^3 - \cancel{2z^5} + 18z^4 - 42z^3 + 54z^2 - 21z^4 + 189z^3 - 231z^2 - 567z}{(z-3)^3}\right]$$

$$u_3 = \underset{z\to\infty}{Lt} \left[\frac{139z^3 - 177z^2 - 567z}{(z-3)^3}\right]$$

$$u_3 = \underset{z\to\infty}{Lt} z^3\left[\frac{139 - 177/z - 567/z^2}{z^3(1-3/z)^3}\right] = 139$$

Q: If $\bar{u}(z) = \dfrac{2z^2+5z+14}{(z-1)^4}$, show that $u_0 + 6u_2 + 1 = u_3$

Sol: $u_0 = \underset{z\to\infty}{Lt} \bar{u}(z)$

$$u_0 = \underset{z\to\infty}{Lt} \left[\frac{2z^2+5z+14}{(z-1)^4}\right]$$

$$u_0 = \underset{z\to\infty}{Lt} \cancel{z^2}\left[\frac{2+5/z+14/z^2}{\cancel{z^4}(1-1/z)^4}\right] = 0$$

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$$u_1 = \lim_{z\to\infty} z\left[\bar{u}(z) - u_0\right]$$

$$u_1 = \lim_{z\to\infty} z\left[\frac{2z^2+5z+14}{(z-1)^4} - 0\right]$$

$$u_1 = \lim_{z\to\infty} \cancel{z}\left[\frac{z^{\cancel{2}}(2+5/z+14/z^2)}{z^4(\cancel{1}-1/z)^4}\right] = 0$$

$$u_2 = \lim_{z\to\infty} z^2\left[\bar{U}(z) - u_0 - \frac{u_1}{z}\right]$$

$$u_2 = \lim_{z\to\infty} z^2\left[\frac{2z^2+5z+14}{(z-1)^4} - 0 - 0\right]$$

$$u_2 = \lim_{z\to\infty} \cancel{z^4}\left[\frac{2+5/z+14/z^2}{\cancel{z^4}(1-1/z)^4}\right] = 2$$

$$u_3 = \lim_{z\to\infty} z^3\left[\bar{U}(z) - u_0 - \frac{u_1}{z} - \frac{u_2}{z^2}\right]$$

$$u_3 = \lim_{z\to\infty} z^3\left[\frac{2z^2+5z+14}{(z-1)^4} - 0 - 0 - \frac{2}{z^2}\right]$$

$$u_3 = \lim_{z\to\infty}\left[\frac{2z^5+5z^4+14z^3-2z(z-1)^4}{(z-1)^4}\right]$$

$$u_3 = \lim_{z\to\infty}\left[\frac{2z^5+5z^4+14z^3-2z(z^4-4z^3+6z^2-4z+1)}{(z-1)^4}\right]$$

$$u_3 = \lim_{z\to\infty}\left[\frac{\cancel{2z^5}+5z^4+14z^3-\cancel{2z^5}+8z^4-12z^3+8z^2-2z}{(z-1)^4}\right]$$

$$u_3 = \lim_{z\to\infty}\left[\frac{13z^4+2z^3+8z^2-2z}{(z-1)^4}\right]$$

$$u_3 = \lim_{z\to\infty} \cancel{z^4}\left[\frac{13+2/z+8/z^2-2/z^3}{\cancel{z^4}(1-1/z)^4}\right] = 13$$

$$u_0 + 6u_2 + 1 = u_3$$

Substituting

$$0 + 6(2) + 1 = 13 = u_3$$ Hence proved.

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Prove that $Z_T[nu_n] = -z\frac{d}{dz}[Z_T(u_n)]$

Hence show that

$$Z_T[n\cos n\theta] = \frac{z\cos\theta(z^2+1)-2z^2}{(z^2-2z\cos\theta+1)^2}$$

Considering RHS

$$-z\frac{d}{dz}[Z_T(u_n)] = -z\frac{d}{dz}\left[\sum_{n=0}^{\infty} u_n z^{-n}\right]$$

$$= -z\frac{d}{dz}\left[u_0 + \frac{u_1}{z} + \frac{u_2}{z^2} + \cdots\cdots\right]$$

$$= -z\left[-\frac{u_1}{z^2} - \frac{2u_2}{z^3} - \frac{3u_3}{z^4} + \cdots\cdots\right]$$

$$= \frac{u_1}{z} + \frac{2u_2}{z^2} + \frac{3u_3}{z^3} + \cdots\cdots$$

$$= \sum_{n=0}^{\infty} n u_n z^{-n} = Z_T[nu_n] = \text{LHS}$$

$$\therefore Z_T[n\cos n\theta] = -z\frac{d}{dz}[Z_T(\cos n\theta)]$$

wkt $Z_T[\cos n\theta] = \dfrac{z[z-\cos\theta]}{z^2-2z\cos\theta+1}$

$$= -z\frac{d}{dz}\left[\frac{z^2-z\cos\theta}{z^2-2z\cos\theta+1}\right]$$

$$= -z\left[\frac{(z^2-2z\cos\theta+1)(2z-\cos\theta)-(z^2-z\cos\theta)(2z-2\cos\theta)}{(z^2-2z\cos\theta+1)^2}\right]$$

$$= -z\left[\frac{2z^3 - z^2\cos\theta - \cancel{4z^2\cos\theta} + \cancel{2z\cos^2\theta} + 2z - \cos\theta - 2z^3 + \cancel{2z^2\cos\theta} + 2z^2\cos\theta - \cancel{2z\cos^2\theta}}{(z^2-2z\cos\theta+1)^2}\right]$$

$$= \frac{z^3\cos\theta - 2z^2 + z\cos\theta}{(z^2-2z\cos\theta+1)^2}$$

$$= \frac{z\cos\theta(z^2+1)-2z^2}{(z^2-2z\cos\theta+1)^2}$$

Hence proved

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* Inverse Z-Transforms

Z-Transform is given by

$Z_T[u_n] = \bar{u}(z)$

Taking inverse

$u_n = Z_T^{-1}[\bar{u}(z)]$

Formulas

1. $Z_T^{-1}\left[\frac{z}{z-1}\right] = 1$

2. $Z_T^{-1}\left[\frac{z}{z-k}\right] = k^n$

3. $Z_T^{-1}\left[\frac{kz}{(z-k)^2}\right] = nk^n$

4. $Z_T^{-1}\left[\frac{kz^2+zk^2}{(z-k)^3}\right] = n^2k^n$

Q. Find inverse z transform of $\frac{8z-z^3}{(4-z)^3}$

Sol: $Z_T^{-1}\left[\frac{8z-z^3}{(4-z)^3}\right]$

$\Rightarrow \frac{8z-z^3}{(4-z)^3} = \frac{z^3-8z}{(z-4)^3}$

$Z_T^{-1}\left[\frac{z}{z-4}\right] = 4^n$

$Z_T^{-1}\left[\frac{4z}{(z-4)^2}\right] = n4^n$

$Z_T^{-1}\left[\frac{4z^2+16z}{(z-4)^3}\right] = n^2 4^n$

$\frac{z^3-8z}{(z-4)^3} = A\frac{z}{z-4} + B\frac{4z}{(z-4)^2} + C\frac{(4z^2+16z)}{(z-4)^3}$ —— (1)

$\frac{z^3-8z}{(z-4)^3} = \frac{Az(z-4)^2 + B4z(z-4) + C(4z^2+16z)}{(z-4)^3}$

$z^3 - 8z = Az(z-4)^2 + 4Bz(z-4) + C(4z^2+16z)$

$z^2 - 8 = A(z-4)^2 + 4B(z-4) + C(4z+16)$

comparing the coefficients

$A = 1$

~~$16A - 16B - 16C = 8$~~

~~$16(B+C) = 8$~~

~~$B + C = \frac{1}{2}$~~

put $z = 4$

$32C = 8$

$C = \frac{1}{4}$

$0 = -8A + 4B + 4C$

$B = \frac{7}{4}$

Substituting in eq (1)

$$\frac{z^3-8z}{(z-4)^3} = \frac{z}{z-4} + \frac{7}{4}\,\frac{4z}{(z-4)^2} + \frac{1}{4}\,\frac{4z+16z}{(z-4)^3}$$

$$Z_T^{-1}\left[\frac{z^3-8z}{(z-4)^3}\right] = Z_T^{-1}\left[\frac{z}{z-4}\right] + \frac{7}{4} Z_T^{-1}\left[\frac{4z}{(z-4)^2}\right] + \frac{1}{4} Z_T^{-1}\left[\frac{4z+16z}{(z-4)^3}\right]$$

$$= 4^n + \frac{7}{4}4^n n + \frac{1}{4}4^n n^2$$

$$= \frac{4^n}{4}\left[4 + 7n + n^2\right]$$

★ **Difference equation:**

An equation of the form

$a_r y_{n+r} + a_{r-1} y_{n+r-1} + a_{r-2} y_{n+r-2} + \dots + a_2 y_{n+2} + a_1 y_{n+1} + a_0 y_n$

where $n = 0, 1, 2, 3 \dots$ $a_0, a_1, a_2 \dots a_r$ are constants,

It represents linear difference equation.

Q: $u_{n+2} - 5u_{n+1} + 6u_n = 2^n$

Sol: Applying z-transform

$Z_T[u_{n+2}] - 5Z_T[u_{n+1}] + 6Z_T[u_n] = Z_T[2^n]$

$z^2[\bar{u}(z) - u_0 - u_1 z^{-1}] - 5z[\bar{u}(z) - u_0] + 6\bar{u}(z) = \frac{z}{z-2}$

$$[-z^2+5z]u_0 + [6+z^2-5z]\bar{u}(z) - u_1 z = \frac{z}{z-2}$$

$$\bar{u}(z) = \frac{\frac{z}{z-2} + u_1 z + (z^2-5z)u_0}{z^2-5z+6}$$

$$\cancel{= \frac{z + u_1 z(z-2) + (z^2-5z)(z-2)u_0}{z-3}}$$

$$= \frac{z}{(z-2)(z^2-5z+6)} + u_1 \frac{z}{z^2-5z+6} + u_0 \frac{z(z-5)}{z^2-5z+6} \quad \text{—(1)}$$

$$\bar{u}(z) = p(z) + u_1 q_1(z) + u_0 r(z)$$

$$p(z) = \frac{z}{(z-2)(z-2)(z-3)} = \frac{z}{(z-2)^2(z-3)}$$

$$\frac{P(z)}{z} = \frac{1}{(z-2)^2(z-3)} = \frac{A}{z-3} + \frac{B}{(z-2)} + \frac{C}{(z-2)^2}$$

$$1 = A(z-2)^2 + B(z-3)(z-2) + C(z-3)$$

Put $z=2$

$1 = -C \Rightarrow C = -1$

Put $z = 3$

$1 = A \Rightarrow A = 1$

Equating coefficients of z^2

$A + B = 0$

$B = -A = -1$

Substituting

$$\frac{1}{(z-2)^2(z-3)} = \frac{1}{z-3} - \frac{1}{z-2} - \frac{1}{(z-2)^2}$$

$$p(z) = \frac{z}{(z-2)^2(z-3)} = \frac{z}{z-3} - \frac{z}{z-2} - \frac{z}{(z-2)^2}$$

$$Z_T^{-1}\left[\frac{z}{(z-2)^2(z-3)}\right] = Z_T^{-1}\left[\frac{z}{z-3}\right] - Z_T^{-1}\left[\frac{z}{z-2}\right] - Z_T^{-1}\left[\frac{z}{(z-2)^2}\right]$$

$$= 3^n - 2^n - \frac{2^n n}{2}$$

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$$q(z) = \frac{z}{z^2 - 5z + 6}$$

$$\frac{q(z)}{z} = \frac{1}{(z-2)(z-3)} = \frac{A}{(z-2)} + \frac{B}{(z-3)}$$

$$1 = A(z-3) + B(z-2)$$

Put $z = 3$ Put $z = 2$

$\underline{\underline{B = 1}}$ $\underline{\underline{A = -1}}$

Substituting

$$\frac{z}{(z-2)(z-3)} = \frac{-z}{z-2} + \frac{z}{z-3}$$

$$Z_T^{-1}\left[\frac{z}{(z-2)(z-3)}\right] = -Z_T^{-1}\left[\frac{z}{z-2}\right] + Z_T^{-1}\left[\frac{z}{z-3}\right]$$

$$= -2^n + 3^n$$

$$r(z) = \frac{z(z-2)}{z^2-5z+6} = \frac{z(z-2)}{(z-2)(z-3)}$$

$$r(z) = \frac{z}{z-3}$$

$$Z_T\left[\frac{z}{z-3}\right] = 3^n$$

Substituting in eq 1

$$u_n = \left(3^n - 2^n - \frac{2^n n}{2}\right) + \left(-2^n + 3^n\right)u_1 + 3^n u_0$$