Unit - 1

NUMERIAL METHODS

* **Numerical solution of First Order First Degree Differential Equations:**

We consider a differential equation of the form $\frac{dy}{dx} = f(x, y)$ with the initial condition $y = y_0$ at $x = x_0$.

The process of finding the value of y at any required value of x is called the numerical solution of differential equations.

Several methods are discussed in order to solve such initial value problems.

1. Taylor Series Method

Consider a differential equation of the form $\frac{dy}{dx} = f(x, y)$ with the initial, $y(x_0) = y_0$, in order to find $y(x_0 + h)$ we use the Taylor expansion as follows

$$y(x_0+h) = y(x_0) + hy'(x_0) + \frac{h^2}{2!} y''(x_0) + \frac{h^3}{3!} y'''(x_0) + \cdots\cdots$$

To obtain the Taylor's Power Series, we use the formula

$$y(x) = y(x_0) + (x - x_0) y'(x_0) + \frac{(x-x_0)^2}{2!} y''(x_0) + \frac{(x-x_0)^3}{3!} y'''(x_0) + \cdots\cdots$$

where $y(x)$ is expansion of the function about the point $x = x_0$.

Q1: Use the taylor series method to find $y(0.1)$ ~~to~~ at $\frac{dy}{dx} = x + y$ and $y(0) = 1$.

Sol: Taylor series

$$y(x_0+h) = y(x_0) + hy'(x_0) + \frac{h^2}{2!} y''(x_0) + \frac{h^3}{3!} y'''(x_0) + \cdots\cdots \quad \text{--- (1)}$$

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$x_0 = 0$ so we take $h = 0.1$

Given : $y_0 = 1$ at $x_0 = 0$

~~Substitution~~

we have $\frac{dy}{dx} = x + y$

$y' = x + y$

$y'(x_0=0) = x_0 + y_0 = 0 + 1 = 1$

$\boxed{y'(x_0=0) = 1}$

$y'' = 1 + y'$

$y''(x_0=0) = 1 + y'(x_0) = 1 + 1 = 2$

$\boxed{y''(x_0=0) = 2}$

$y''' = 0 + y''$

$y'''(x_0=0) = y''(x_0=0) = 2$

$\boxed{y'''(x_0=0) = 2}$

$y'''' = y'''$

$y''''(x_0=0) = y'''(x_0=0) = 2$

$\boxed{y''''(x_0=0) = 2}$

Substituting in eq (1), we get

$$y(0+0.1) = 1 + 0.1(1) + \frac{(0.1)^2}{2!}(2) + \frac{(0.1)^3}{3!}(2) + \frac{(0.1)^4}{4!}(2)$$

$y(0.1) = 1 + 0.1 + 0.01 + 0.0003 + 0.000008$

$y(0.1) = 1.1103$

Q2: $\frac{dy}{dx} = x^2 y - 1$; $y = 1$ at $x = 0$. Find y when $x = 0.03$

Sol: Taylor series

$$y(x_0+h) = y(x_0) + h y'(x_0) + \frac{h^2}{2!} y''(x_0) + \frac{h^3}{3!} y'''(x_0) + \cdots$$

we have $y_0 = 1$ at $x_0 = 0$

so we take $h = 0.03$

given $\frac{dy}{dx} = x^2y - 1$

$y' = x^2y - 1$

$y'(x_0=0) = x_0^2y_0 - 1 = 0 - 1 = -1$

$\boxed{y'(x_0=0) = -1}$

$y'' = 2xy + x^2y'$

$y''(x_0=0) = 2x_0y_0 + x_0^2y'(x_0) = 0 + 0 = 0$

$\boxed{y''(x_0=0) = 0}$

$y''' = 2y + 2xy' + 2xy' + x^2y''$

$y'''(x_0=0) = 2y_0 + 4x_0y'(x_0) + x_0^2y''(x_0) = 2 + 0 + 0 = 2$

$\boxed{y'''(x_0=0) = 2}$

$y'''' = 2y' + 2y' + 2xy'' + 2y' + 2xy'' + 2xy'' + x^2y'''$

$y''''(x_0=0) = 6y'(x_0) + 4x_0y''(x_0) + x_0^2y'''(x_0)$

$= 6(-1) + 0 + 0 = -6$

$\boxed{y''''(x_0=0) = -6}$

Substituting in eq ① we get

$$y(0+0.03) = y(0.03) = 1 + 0.03(-1) + \frac{(0.03)^2}{2!}(0) + \frac{(0.03)^3}{3!}(2) + \frac{(0.03)^4}{4!}(-6)$$

$y(0.03) = 1 - 0.03 + 0 + 0.000009 - 0.0000004$

$y(0.03) = 0.9700$

Q3: $\frac{dy}{dx} = x^2 + y$, $y(0) = 10$ find y at $x = 0.2$ by taking $h = 0.1$

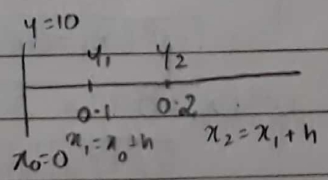
sol: Taylor series

$$y(x_0+h) = y(x_0) + hy'(x_0) + \frac{h^2}{2!}y''(x_0) + \frac{h^3}{3!}y'''(x_0) + \dots \quad \text{——— ①}$$

Stage 1

we have $y_0 = 10$ at $x_0 = 0$

here $h = 0.1$ (given)



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given $y' = x^2+y$

$y'(x_0=0) = x_0^2+y_0 = 0+10 = 10$

$\boxed{y'(x_0=0)=10}$

$y'' = 2x+y'$

$y''(x_0=0) = 2x_0+y'(x_0) = 0+10 = 10$

$\boxed{y''(x_0=0)=10}$

$y''' = 2+y''$

$y'''(x_0=0) = 2+y''(x_0) = 2+10 = 12$

$\boxed{y'''(x_0=0)=12}$

$y'''' = y'''$

$y''''(x_0=0) = y'''(x_0) = 12$

$\boxed{y''''(x_0=0)=12}$

Substituting in eq ①, we get

$$y(0+0.1) = 10 + 0.1(10) + \frac{(0.1)^2}{2!}(10) + \frac{(0.1)^3}{3!}(12) + \frac{(0.1)^4}{4!}(12)$$

$y(0.1) = 10 + 1 + 0.05 + 0.002 + 0.00005$

$\underline{y(0.1) = 11.0520} = y(x_1=0.1) = y_1$

Stage 2:

Now the initial condition is

$y_1 = 11.052$ at $x_1 = 0.1$ we take $h=0.1$

We find y_2 at $x_2 = x_1+h = 0.1+0.1 = 0.2$

~~given $y' = x^2+y$~~

$$y(x_1+h) = y(x_1) + hy'(x_1) + \frac{h^2}{2!}y''(x_1) + \frac{h^3}{3!}y'''(x_1) + \cdots \quad ②$$

$y'(x_1=0.1) = x_1^2+y_1 = (0.1)^2 + 11.052 = 11.062$

$\boxed{y'(x_1=0.1) = 11.062}$

$y''(x_1=0.1) = 2x_1+y'(x_1) = 2(0.1)+11.062 = 11.262$

$\boxed{y''(x_1=0.1) = 11.262}$

$y'''(x_1=0.1) = 2+y''(x_1) = 2+11.262 = 13.262$

$\boxed{y'''(x_1=0.1) = 13.262}$

$$\boxed{y''''(x_1=0.1) = y'''(x_1) = 13.262}$$

Substituting in eq ②, we get

$$y(0.1+0.1) = 11.052 + 0.1(11.062) + \frac{(0.1)^2}{2!}(11.262) + \frac{(0.1)^3}{3!}(13.262) + \frac{(0.1)^4}{4!}(13.262)$$

$$y(0.2) = 11.052 + 1.1062 + 0.05631 + 0.00221 + 0.000055$$

$$y(0.2) = 12.216$$

Q4: $y' = x - y^2$, $y(0) = 1$ find $y(0.1)$

sol: Taylor series

$$y(x_0+h) = y(x_0) + hy'(x_0) + \frac{h^2}{2!}y''(x_0) + \frac{h^3}{3!}y'''(x_0) + \dots \quad \text{①}$$

we have $y_0 = 1$ at $x_0 = 0$, we take $h = 0.1$

given $y' = x - y^2$

$$y'(x_0=0) = x_0 - y_0^2 = 0 - 1^2 = -1$$

$$\boxed{y'(x_0=0) = -1}$$

$$y'' = 1 - 2yy'$$

$$y''(x_0=0) = 1 - 2y_0 y'(x_0) = 1 - 2(1)(-1) = 3$$

$$\boxed{y''(x_0=0) = 3}$$

$$y''' = -2yy'' - 2y'^2$$

$$y'''(x_0=0) = -2y_0 y''(x_0) - 2y'(x_0)^2 = -2(1)(3) - 2(-1)^2 = -8$$

$$\boxed{y'''(x_0=0) = -8}$$

$$y'''' = -2yy''' - 2y'y'' - 4y'y'' = -2yy''' - 6y'y''$$

$$y''''(x_0=0) = -2y_0 y'''(x_0) - 6y'(x_0)y''(x_0) = -2(1)(-8) - 6(-1)(3) = 34$$

$$\boxed{y''''(x_0=0) = 34}$$

Substituting in eq ①, we get

$$y(0+0.1) = 1 + 0.1(-1) + \frac{(0.1)^2}{2!}(3) + \frac{(0.1)^3}{3!}(-8) + \frac{(0.1)^4}{4!}(34)$$

$$y(0.1) = 1 - 0.1 + 0.015 - 0.0013 + 0.00014$$

$$y(0.1) = 0.9138$$

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Q5: Use taylor series method to obtain a power series in $(x-1)$ upto 4 degree term for $\frac{dy}{dx} = x^3 + y$, $y(1) = 1$. Also find $y(1.2)$.

sol: By Taylor series method we have

$$y(x_0+h) = y(x_0) + hy'(x_0) + \frac{h^2}{2!}y''(x_0) + \frac{h^3}{3!}y'''(x_0) + \dots$$

put $x_0 + h = x \Rightarrow h = x - x_0$

Substituting in eq ①

$$y(x) = y(x_0) + (x-x_0)y'(x_0) + \frac{(x-x_0)^2}{2!}y''(x_0) + \frac{(x-x_0)^3}{3!}y'''(x_0)$$

Here $x_0 = 1$

we have $y' = x^3 + y$ $y(1) = 1 \Rightarrow y_0 = 1$ at $x_0 = 1$

at $x = x_0 = 1$

$y'(x_0=1) = x_0^3 + y_0 = 1^3 + 1 = 2$

$\boxed{y'(x_0=1) = 2}$

$y'' = 3x^2 + y'$

$y''(x_0=1) = 3x_0^2 + y'(x_0) = 3(1) + 2 = 5$

$\boxed{y''(x_0=1) = 5}$

$y''' = 6x + y''$

$y'''(x_0=1) = 6x_0 + y''(x_0) = 6(1) + 5 = 11$

$\boxed{y'''(x_0=1) = 11}$

$y'''' = 6 + y'''$

$y''''(x_0=1) = 6 + y'''(x_0) = 6 + 11 = 17$

$\boxed{y''''(x_0=1) = 17}$

Substituting in eq ②, we get

$$y(x) = 1 + (x-1)2 + \frac{(x-1)^2}{2!}5 + \frac{(x-1)^3}{3!}11 + \frac{(x-1)^4}{4!}17$$

$$y(1.2) = 1 + (1.2-1)2 + \frac{(1.2-1)^2 5}{2} + \frac{(1.2-1)^3 11}{6} + \frac{(1.2-1)^4 17}{24}$$

$$y(1.2) = 1 + 0.4 + 0.1 + 0.01467 + 0.0011 = \underline{1.5158}$$

Q 6: $x = 4.1, 4.2, 4.3$

$$5x\frac{dy}{dx} + y^2 - 2 = 0 \quad y = 1 \text{ at } x = 4$$

sol: By taylor series we have

$$y(x_0+h) = y(x_0) + h\,y'(x_0) + \frac{h^2}{2!}y''(x_0) + \frac{h^3}{3!}y'''(x_0) + \cdots \quad \text{— (1)}$$

we have we take $h = 0.1$

$$5xy' + y^2 - 2 = 0$$

$$5xy' = 2 - y^2$$

$$y' = \frac{2-y^2}{5x}, \quad y_0 = 1 \text{ at } x_0 = 4.$$

$$y'(x_0=4) = \frac{2-y_0^2}{5x_0} = \frac{2-1}{5(4)} = \frac{1}{20} = 0.05$$

$$\boxed{y'(x_0=4) = 0.05}$$

$$y'' = \frac{5x(-2yy') - (2-y^2)(5)}{(5x)^2}$$

$$y''(x_0=4) = \frac{5(4)(-2)(1)(0.05) - (2-1)(5)}{25(4)^2}$$

$$\boxed{y''(x_0=4) = -0.0175}$$

Substituting in eq (1)

$$y(4+0.1) = 1 + (0.1)(0.05) + \frac{(0.1)^2}{2}(-0.0175)$$

$$\underline{y(4.1) = 1.0049}$$

taking $h = 0.2$ in eq (1)

$$y(4+0.2) = 1 + (0.2)(0.05) + \frac{(0.2)^2}{2}(-0.0175)$$

$$\underline{y(4.2) = 1.0096}$$

taking $h=0.3$ in eq ①

$$y(4+0.3) = 1 + (0.3)(0.05) + \frac{(0.3)^2}{2!}(-0.0175)$$

$$\underline{y(4.3) = 1.0142}$$

* Simultaneous differential equations (Taylor series)

Q1: $\frac{dy}{dx} = z$; $\frac{dz}{dx} = -xz - y$; $y(0)=1$, $z(0)=0$

compute $y(0.1)$ and $z(0.1)$

by taylor series we have

sol: $$y(x_0+h) = y(x_0) + hy'(x_0) + \frac{h^2}{2!}y''(x_0) + \frac{h^3}{3!}y'''(x_0) + \ldots \quad ①$$

$$z(x_0+h) = z(x_0) + hz'(x_0) + \frac{h^2}{2!}z''(x_0) + \frac{h^3}{3!}z'''(x_0) + \ldots \quad ②$$

we have $y_0 = 1$ at $x_0 = 0$ we take $h = 0.1$.
and $z_0 = 0$ at $x_0 = 0$.

we have

$y' = z$

$y'(x_0=0) = z_0$

$\boxed{y'(x_0=0) = 0}$

$y'' = z'$

$\boxed{y''(x_0=0) = z'(x_0) = -1}$

$y''' = z''$

$\boxed{y'''(x_0=0) = z''(x_0) = 0}$

$y'''' = z'''$

$\boxed{y''''(x_0=0) = z'''(x_0) = 3}$

$z' = -xz - y$

$z'(x_0=0) = -x_0z_0 - y_0 = -1$

$\boxed{z'(x_0=0) = -1}$

$z'' = -z - xz' - y'$

$z''(x_0=0) = -z_0 - x_0z'(x_0) - y'(x_0)$

$\boxed{z''(x_0=0) = 0}$

$z''' = -z' - xz'' - z' - y''$

$z'''(x_0=0) = -z'(x_0) - x_0z''(x_0) - z'(x_0) - \ldots$

$z'''(x_0=0)\ 1 - 0 + 1 + 1$

$\boxed{z'''(x_0=0) = 3}$

$z'''' = -2z'' - xz''' - z'' - y'''$

$z''''(x_0=0) = -2(0) - 0 - 0 - 0$

$\boxed{z''''(x_0=0) = 0}$

substituting in eq ① and eq ②

$y(0+0.1) = 1 + (0.1)(0) + \frac{(0.1)^2}{2!}(-1) + \frac{(0.1)^3}{3!}(0) + \frac{(0.1)^4}{4!}(3)$

$y(0.1) = 1 - 0.005 + 0.00001$

$\underline{y(0.1) = 0.9950}$

$z(0+0.1) = 0 + (0.1)(-1) + \frac{(0.1)^2}{2!}(0) + \frac{(0.1)^3}{3!}(3) + \frac{(0.1)^4}{4!}(0)$

$z(0.1) = -0.1 + 0.0005$

$\underline{z(0.1) = -0.0995}$

Q2: Given $\frac{dx}{dt} = ty+1$, $\frac{dy}{dt} = -tx$, $x=0, y=1$ at $t=0$.

compute x and y at $t=0.2$.

By taylor series we have

sol: $x(t_0+h) = x(t_0) + hx'(t_0) + \frac{h^2}{2!}x''(t_0) + \frac{h^3}{3!}x'''(t_0) + \cdots$ ——①

$y(t_0+h) = y(t_0) + hy'(t_0) + \frac{h^2}{2!}y''(t_0) + \frac{h^3}{3!}y'''(t_0) + \cdots$ ——②

we have $x_0 = 0$ at $t_0 = 0$ we take $h = 0.2$

$y_0 = 1$ at $t = 0$

we have

$x' = ty+1$ $y' = -tx$

$x'(t_0=0) = t_0y_0+1 = 0+1$ $y'(t_0=0) = -t_0x_0 = 0$

$\boxed{x'(t_0=0) = 1}$ $\boxed{y'(t_0=0) = 0}$

$x'' = ty'+y$ $y'' = -tx'-x$

$x''(t_0=0) = t_0y'(t_0)+y_0$ $y''(t_0=0) = -t_0x'(t_0) - x_0$

$x''(t_0=0) = 0+1$ $y''(t_0=0) = 0-0$

$\boxed{x''(t_0=0) = 1}$ $\boxed{y''(t_0=0) = 0}$

$x''' = ty'' + y' + y'$
$x'''(t_0=0) = t_0 y''(t_0) + 2y'(t_0)$
$x'''(t_0=0) = 0 + 2(0)$
$\boxed{x'''(t_0=0) = 0}$
$x'''' = ty''' + y'' + y'' + y''$
$x''''(t_0=0) = t_0 y'''(t_0) + 3y''(t_0)$
$\boxed{x''''(t_0=0) = 0}$

$y''' = -tx'' - x' - x'$
$y'''(t_0=0) = -t_0 x''(t_0) - 2x'$
$y'''(t_0=0) = 0 - 2(1)$
$\boxed{y'''(t_0=0) = -2}$
$y'''' = -tx''' - x'' - x'' - x''$
$y''''(t_0=0) = -t_0 x'''(t_0) - 3x''$
$y''''(t_0=0) = 0 - 3(1)$
$\boxed{y''''(t_0=0) = -3}$

Substituting in eq ① and ② we get
$$x(0+0.2) = 0 + (0.2)1 + \frac{(0.2)^2}{2!}(1) + \frac{(0.2)^3}{3!}(0) + \frac{(0.2)^4}{4!}(0)$$
$$x(0.2) = 0.2 + 0.02$$
$$x(0.2) = 0.22$$

$$y(0+0.2) = 1 + (0.2)(0) + \frac{(0.2)^2}{2!}(0) + \frac{(0.2)^3}{3!}(-2) + \frac{(0.2)^4}{4!}(-3)$$
$$y(0.2) = 1 - 0.00267 - 0.0002$$
$$y(0.2) = 0.997$$

Taylor Series

Q7: $y' = 1 - 2xy$, $y(0.2) = 0.1948$ find $y(0.4)$

Sol: Taylor series is given by
$$y(x_0+h) = y(x_0) + hy'(x_0) + \frac{h^2}{2!}y''(x_0) + \frac{h^3}{3!}y'''(x_0) + \dots \quad \text{—①}$$

we have $y_0 = 0.1948$ at $x_0 = 0.2$
taking $h = 0.2$.
we have
$$y' = 1 - 2xy$$
$$y'(x_0=0.2) = 1 - 2x_0y_0 = 1 - 2(0.2)(0.1948)$$
$$\boxed{y'(x_0=0.2) = 0.922}$$

$y'' = -2xy' - 2y$

$y''(x_0=0.2) = -2x_0 y'(x_0) - 2y_0 = -2(0.2)(0.922) - 2(0.1948)$

$\boxed{y''(x_0=0.2) = -0.7584}$

$y''' = -2x_0 y'' - 2y' - 2y'$

$y''' = -2xy'' - 4y'$

$y'''(x_0=0.2) = -2x_0 y''(x_0) - 4y'(x_0) = -2(0.2)(-0.7584) - 4(0.922)$

$\boxed{y'''(x_0=0.2) = -3.3846}$

$y'''' = -2xy''' - 2y'' - 4y''$

$y'''' = -2xy''' - 6y''$

$y''''(x_0=0.2) = -2x_0 y'''(x_0) - 6y''(x_0) = -2(0.2)(-3.3846) - 6(-0.7584)$

$\boxed{y''''(x_0=0.2) = 5.9042}$

Substituting in eq ① we get

$$y(0.2+0.2) = 0.1948 + (0.2)(0.922) + \frac{(0.2)^2}{2!}(-0.7584) + \frac{(0.2)^3}{3!}(-3.3846) + \frac{(0.2)^4}{4!}(5.9042)$$

$y(0.4) = 0.1948 + 0.1844 - 0.0152 - 0.0045 + 0.00039$

$\underline{y(0.4) = 0.3599}$

★ **Modified Euler's Method:**

Consider a differential equation of the form $\frac{dy}{dx} = f(x, y)$ with the initial condition $y(x_0) = y_0$.

We first calculate $y_1 = y(x_0 + h)$ by using the Euler's formula as follows:

$$\boxed{y_1^{(0)} = y_0 + hf(x_0, y_0)}$$

The value thus obtained is improved by using the modified Euler's formula as follows.

$$y_1^{(1)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(0)})\right]$$

$$y_1^{(2)} + y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(1)})\right]$$

This process is continued till the consistent values of y is obtained.

Q1: Given $\frac{dy}{dx} = 3x + \frac{y}{2}$ with $y(0) = 1$, taking $h = 0.1$ perform three iterations at each step to find $y(0.2)$ by modified Euler's method.

Sol: We have $f(x,y) = 3x + \frac{y}{2}$; $y_0 = 1$ at $x_0 = 0$; $h = 0.1$

Stage 1:

We find y_1 at $x_1 = x_0 + h = 0.1$

by Euler's formula

$$y_1^{(0)} = y_0 + h f(x_0, y_0)$$

$$y_1^{(0)} = 1 + (0.1)\left[3(0) + \frac{1}{2}\right]$$

$$\boxed{y_1^{(0)} = 1.05}$$

We improve this value by using the Modified Euler's formula

$$y_1^{(1)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(0)})\right]$$

$$y_1^{(1)} = 1 + \frac{0.1}{2}\left[f(0, 1) + f(0.1, 1.05)\right]$$

$$y_1^{(1)} = 1 + 0.05\left[3(0) + \frac{1}{2} + 3(0.1) + \frac{1.05}{2}\right]$$

$$\boxed{y_1^{(1)} = 1.06625}$$

$$y_1^{(2)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(1)})\right]$$

$$y_1^{(2)} = 1 + \frac{0.1}{2}\left[f(0, 1) + f(0.1, 1.06625)\right]$$

$$y_1^{(2)} = 1 + 0.05\left[3(0) + \frac{1}{2} + 3(0.1) + \frac{1.06625}{2}\right]$$

$$\boxed{y_1^{(2)} = 1.06665}$$

$$y_1^{(3)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1 + y_1^{(2)})\right]$$

$$y_1^{(3)} = 1 + \frac{0.1}{2}\left[f(0,1) + f(0.1 + 1.06665)\right]$$

$$y_1^{(3)} = 1 + 0.05\left[3(0) + \frac{1}{2} + 3(0.1) + \frac{1.06665}{2}\right]$$

$$\boxed{y_1^{(3)} = 1.06666}$$

$$\therefore y_1 = \underline{1.0666 \text{ at } x_1 = 0.1}$$

Stage 2:

we find y_2 at $x_2 = x_1 + h = 0.1 + 0.1 = 0.2$

by Euler's formula

$$y_2^{(0)} = y_1 + h f(x_1, y_1)$$

$$y_2^{(0)} = 1.0666 + 0.1\left[3(0.2) + \frac{1.0666}{2}\right]$$

$$\boxed{y_2^{(0)} = 1.15}$$

we improve this value by using the Modified Euler's formula

$$y_2^{(1)} = y_1 + \frac{h}{2}\left[f(x_1, y_1) + f(x_2, y_2^{(0)})\right]$$

$$y_2^{(1)} = 1.0666 + \frac{0.1}{2}\left[f(0.1, 1.0666) + f(0.2, 1.15)\right]$$

$$y_2^{(1)} = 1.0666 + 0.05\left[3(0.1) + \frac{1.0666}{2} + 3(0.2) + \frac{1.15}{2}\right]$$

$$\boxed{y_2^{(1)} = 1.1670}$$

$$y_2^{(2)} = y_1 + \frac{h}{2}\left[f(x_1, y_1) + f(x_2, y_2^{(1)})\right]$$

$$y_2^{(2)} = 1.0666 + \frac{0.1}{2}\left[f(0.1, 1.0666) + f(0.2, 1.167)\right]$$

$$y_2^{(2)} = 1.0666 + 0.05\left[3(0.1) + \frac{1.0666}{2} + 3(0.2) + \frac{1.167}{2}\right]$$

$$\boxed{y_2^{(2)} = 1.1674}$$

$$y_2^{(3)} = y_1 + \frac{h}{2}\left[f(x_1, y_1) + f(x_2, y_2^{(2)})\right]$$

$$y_2^{(3)} = 1.0666 + \frac{0.1}{2}\left[f(0.1, 1.0666) + f(0.2, 1.1674)\right]$$

$$y_2^{(3)} = 1.0666 + 0.05\left[3(0.1) + \frac{1.0666}{2} + 3(0.2) + \frac{1.1674}{2}\right]$$

$$\boxed{y_2^{(3)} = 1.16745} \qquad \therefore y = 1.6745 \text{ at } x = 0.2$$

Q2: $\frac{dy}{dx} = \log_{10}\frac{x}{y}$ in the range $20 \le x \le 20.4$; $y = 5$ at $x = 2$ take $h = 0.2$.

sol: we have $f(x,y) = \log_{10}\frac{x}{y}$; $y_0 = 5$ at $x_0 = 20$; $h = 0.2$

stage 1:

we find y_1 at $x_1 = x_0 + h = 20 + 0.2 = 20.2$

by Euler's formula

$$y_1^{(0)} = y_0 + hf(x_0, y_0)$$

$$y_1^{(0)} = 5 + 0.2\left[\log_{10}\left(\frac{20}{5}\right)\right]$$

$$\boxed{y_1^{(0)} = 5.1204}$$

We improve this value by using the Modified Euler's for

$$y_1^{(1)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(0)})\right]$$

$$y_1^{(1)} = 5 + \frac{0.2}{2}\left[f(20, 5) + f(20.2, 5.1204)\right]$$

$$y_1^{(1)} = 5 + 0.1\left[\log_{10}\frac{20}{5} + \log_{10}\frac{20.2}{5.1204}\right]$$

$$\boxed{y_1^{(1)} = 5.1198}$$

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$$y_1^{(2)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(1)})\right]$$

$$y_1^{(2)} = 5 + \frac{0.2}{2}\left[f(20, 5) + f(20.2, 5.1198)\right]$$

$$y_1^{(2)} = 5 + 0.1\left[\log_{10}\frac{20}{5} + \log_{10}\frac{20.2}{5.1198}\right]$$

$$\boxed{y_1^{(2)} = 5.1198} \qquad \therefore y_1 = 5.1198 \text{ at } x_1 = 20.2$$

Stage 2:

We find y_2 at $x_2 = x_1 + h = 20.2 + 0.2 = 20.4$ by Euler's formula

$$y_2^{(0)} = y_1 + hf(x_1, y_1)$$

$$y_2^{(0)} = 5.1198 + 0.2\left[\log_{10}\frac{20.2}{5.1198}\right]$$

$$\boxed{y_2^{(0)} = 5.239}$$

We improve this value by using the Modified Euler's formula.

$$y_2^{(1)} = y_1 + \frac{h}{2}\left[f(x_1, y_1) + f(x_2, y_2^{(0)})\right]$$

$$y_2^{(1)} = 5.1198 + \frac{0.2}{2}\left[f(20.2, 5.1198) + f(20.4, 5.239)\right]$$

$$y_2^{(1)} = 5.1198 + 0.1\left[\log_{10}\frac{20.2}{5.1198} + \log_{10}\frac{20.4}{5.239}\right]$$

$$\boxed{y_2^{(1)} = 5.2384}$$

$$y_2^{(2)} = y_1 + \frac{h}{2}\left[f(x_1, y_1) + f(x_2 + y_2^{(1)})\right]$$

$$y_2^{(2)} = 5.1198 + \frac{0.2}{2}\left[f(20.2, 5.1198) + f(20.4, 5.2384)\right]$$

$$y_2^{(2)} = 5.1198 + 0.1\left[\log_{10}\frac{20.2}{5.1198} + \log_{10}\frac{20.4}{5.2384}\right]$$

$$\boxed{y_2^{(2)} = 5.2384} \qquad \therefore y_2 = 5.2384 \text{ at } x_2 = 20.4$$

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Q3: Given $\frac{dy}{dx} = 2 + \sqrt{xy}$; $y(1) = 1$. Find $y(1.2)$ in steps correct to 3 decimal places

Sol: We have $f(x,y) = 2 + \sqrt{xy}$; $y_0 = 1$ at $x_0 = 1$; $h = 0.1$

Stage 1:

We find y_1 at $x_1 = x_0 + h = 1 + 0.1 = 1.1$

By Euler's formula

$y_1^{(0)} = y_0 + hf(x_0, y_0)$

$y_1^{(0)} = 1 + 0.1(2 + \sqrt{1 \times 1})$

$\boxed{y_1^{(0)} = 1.3}$

We improve this value by Modified Euler's formula.

$y_1^{(1)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(0)})\right]$

$y_1^{(1)} = 1 + \frac{0.1}{2}\left[f(1,1) + f(1.1, 1.3)\right]$

$y_1^{(1)} = 1 + 0.05\left[2 + 1 + 2 + 1.1958\right]$

$\boxed{y_1^{(1)} = 1.3097}$

$y_1^{(2)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(1)})\right]$

$y_1^{(2)} = 1 + \frac{0.1}{2}\left[f(1,1) + f(1.1, 1.3097)\right]$

$y_1^{(2)} = 1 + 0.05\left[2 + 1 + 2 + 1.0489\right]$

$\boxed{y_1^{(2)} = 1.3013}$

$y_1^{(3)} = y_0 + \frac{h}{2}\left[f(x_0, y_0) + f(x_1, y_1^{(1)})\right]$

$y_1^{(3)} = 1 + \frac{0.1}{2}\left[f(1,1) + f(1.1, 1.3006)\right]$

$y_1^{(3)} = 1 + 0.05 [2 + 1 + 2 + 1.2004]$

$\boxed{y_1^{(3)} = 1.3098}$ $\therefore y_1 = 1.3100$ at $x_1 = 1.1$

Stage 2 :

we find y_2 at $x_2 = x_1 + h = 1.1 + 0.1 = 1.2$

By Euler's formula

$y_2^{(0)} = y_1 + h f(x_1, y_1)$

$y_2^{(0)} = 1.31 + 0.1 [2 + \sqrt{1.1 \times 1.31}]$

$\boxed{y_2^{(0)} = 1.6300}$

we improve this value by Modified Euler's formula

$y_2^{(1)} = y_1 + \frac{h}{2} [f(x_1, y_1) + f(x_2, y_2^{(0)})]$

$y_2^{(1)} = 1.31 + \frac{0.1}{2} [f(1.1, 1.31) + f(1.2, 1.63)]$

$y_2^{(1)} = 1.31 + 0.05 [2 + \sqrt{1.1 \times 1.31} + 2 + \sqrt{1.2 \times 1.63}]$

$\boxed{y_2^{(1)} = 1.6399}$

$y_2^{(2)} = y_1 + \frac{h}{2} [f(x_1, y_1) + f(x_2, y_2^{(1)})]$

$y_2^{(2)} = 1.31 + \frac{0.1}{2} [f(1.1, 1.31) + f(1.2, 1.6399)]$

$y_2^{(2)} = 1.31 + 0.05 [2 + 1.2004 + 2 + 1.4028]$

$\boxed{y_2^{(2)} = 1.64016}$

$y_2^{(3)} = y_1 + \frac{h}{2} [f(x_1, y_1) + f(x_2, y_2^{(2)})]$

$y_2^{(3)} = 1.31 + \frac{0.1}{2} [f(1.1, 1.31) + f(1.2, 1.64016)]$

$y_2^{(3)} = 1.31 + 0.05[2 + 1.2004 + 2 + 1.4029]$

$\boxed{y_2^{(3)} = 1.64016}$ $\therefore y_2 = 1.64016$ at $x_2 = 1.2$

* **Runge kutta Method of IV order:**

Consider a differential equation of the type $\frac{dy}{dx} = f(x,y)$ with the initial condition $y = y_0$ at $x = x_0$. To compute $y(x_0+h)$ we first calculate the following:

$k_1 = hf(x_0, y_0)$

$k_2 = hf(x_0 + h/2, y_0 + k_1/2)$

$k_3 = hf(x_0 + h/2, y_0 + k_2/2)$

$k_4 = hf(x_0 + h, y_0 + k_3)$

then $y(x_0+h) = y(x_0) + \frac{1}{6}[k_1 + 2k_2 + 2k_3 + k_4]$

Q1: Apply RK method of IV order to find the value of y when $x = 0.1$. Given that $\frac{dy}{dx} = 3x + \frac{y}{2}$; $y(0) = 1$.

sol: We have $f(x_0, y_0) = 3x + \frac{y}{2}$; $y_0 = 1$ at $x_0 = 0$; we take $h = 0.1$.

$k_1 = hf(x_0, y_0)$

$k_1 = 0.1 f(0, 1)$

$k_1 = 0.1\left[3(0) + \frac{1}{2}\right]$ $\boxed{k_1 = 0.05}$

$k_2 = hf(x_0 + h/2, y_0 + k_1/2)$

$k_2 = 0.1 f(0 + 0.05, 1 + 0.025)$

$k_2 = 0.1\left[3(0.05) + \frac{1.025}{2}\right]$ $\therefore$ $\boxed{k_2 = 0.06625}$

$k_3 = hf(x_0 + h/2, y_0 + k_2/2)$

$k_3 = 0.1 f(0 + 0.05, 1 + 0.03312)$

$k_3 = 0.1\left[3(0.05) + \frac{1.03312}{2}\right]$ $\boxed{k_3 = 0.06665}$

$k_4 = hf(x_0+h, y_0+k_3)$

$k_4 = 0.1f(0+0.1, 1+0.06665)$

$k_4 = 0.1\left[3(0.1) + \frac{1.06665}{2}\right]$ $\boxed{k_4 = 0.0833}$

$y(x_0+h) = y(x_0) + \frac{1}{6}\left[k_1 + 2k_2 + 2k_3 + k_4\right]$

$y(0+0.1) = 1 + \frac{1}{6}\left[0.05 + 2(0.06625) + 2(0.06665) + 0.08333\right]$

$y(0.1) = 1.0665$

Q2: Given that $\frac{dy}{dx} = \frac{y-x}{y+x}$; $y(0)=1$, find y at $x=0.5$

sol: We have $f(x,y) = \frac{y-x}{y+x}$; $y_0 = 1$ at $x_0 = 0$

we take $h = 0.5$.

$k_1 = hf(x_0, y_0)$

$k_1 = 0.5f(0,1)$

$k_1 = 0.5\left[\frac{1-0}{1+0}\right]$ $\boxed{k_1 = 0.5}$

$k_2 = hf(x_0+h/2, y_0+k_1/2)$

$k_2 = 0.5f(0+0.25, 1+0.25)$

$k_2 = 0.5\left[\frac{1.25-0.25}{1.25+0.25}\right]$

$k_2 = 0.5\left[\frac{1}{1.5}\right] = 0.3333$ $\boxed{k_2 = 0.3333}$

$k_3 = hf(x_0+h/2, y_0+k_2/2)$

$k_3 = 0.5f(0+0.25, 1+0.1667)$

$k_3 = 0.5\left[\frac{1.1667-0.25}{1.1667+0.25}\right] = 0.5\left[\frac{0.9167}{1.4167}\right]$

$\boxed{k_3 = 0.3235}$

$k_4 = hf(x_0+h, y_0+k_3)$

$k_4 = 0.5 f(0+0.5, 1+0.3235)$

$k_4 = 0.5\left[\frac{1.3235-0.5}{1.3235+0.5}\right] = 0.5\left[\frac{0.8235}{1.8235}\right]$

$\boxed{k_4 = 0.2258}$

$y(x_0+h) = y(x_0) + \frac{1}{6}\left[k_1 + 2k_2 + 2k_3 + k_4\right]$

$y(0+0.5) = 1 + \frac{1}{6}\left[0.5 + 2(0.3333) + 2(0.3235) + 0.2258\right]$

$y(0.5) = 1 + \frac{1}{6}\left[2.0394\right]$

$\underline{\underline{y(0.5) = 1.3399}}$

Q3: Given: $\frac{dy}{dx} = x+y$, $y(0)=1$. find $y(0.2)$ taking $h=0.1$

sol: we have $f(x,y) = x+y$; $y_0 = 1$ at $x_0 = 0$; $h = 0.1$

Stage 1:

we find y at $x = x_0 + h = 0 + 0.1 = 0.1$

$\Rightarrow$ y_1 at $x_1 = 0.1$

$y_1 = y(x_0+h) = y(x_0) + \frac{1}{6}\left[k_1 + 2(k_2+k_3) + k_4\right]$ —— (1)

where $k_1 = hf(x_0, y_0)$

$k_1 = 0.1 f(0, 1)$

$k_1 = 0.1(0+1)$

$\boxed{k_1 = 0.1}$

$k_2 = hf(x_0 + h/2, y_0 + k_1/2)$

$k_2 = 0.1 f(0+0.05, 1+0.05)$

$k_2 = 0.1(0.05 + 1.05)$

$\boxed{k_2 = 0.11}$

$k_3 = hf(x_0 + h/2,\ y_0 + k_2/2)$

$k_3 = 0.1 f(0 + 0.05,\ 1 + 0.055)$

$k_3 = 0.1(0.05 + 1.055)$

$\boxed{k_3 = 0.1105}$

$k_4 = hf(x_0 + h,\ y_0 + k_3)$

$k_4 = 0.1 f(0 + 0.1,\ 1 + 0.1105)$

$k_4 = 0.1(0.1 + 1.1105)$

$\boxed{k_4 = 0.12105}$

Substituting in eq ①, we get

$y_1 = y(0 + 0.1) = 1 + \frac{1}{6}\left[0.1 + 2(0.11 + 0.1105) + 0.12105\right]$

$\underline{y(0.1) = 1.1103}$ $\quad \therefore y_1 = 1.1103$ at $x_1 = 0.1$

<u>Stage 2:</u>

we find y at $x = x_1 + h = 0.1 + 0.1 = 0.2$

$\Rightarrow y_2$ at $x_2 = 0.2$

$y_2 = y(x_1 + h) = y(x_1) + \frac{1}{6}\left[k_1 + 2(k_2 + k_3) + k_4\right]$ —— ②

where $k_1 = hf(x_1, y_1)$

$k_1 = 0.1 f(0.1,\ 1.1103)$

$k_1 = 0.1(0.1 + 1.1103)$

$\boxed{k_1 = 0.12103}$

$k_2 = hf(x_1 + h/2,\ y_1 + k_1/2)$

$k_2 = 0.1 f(0.1 + 0.05,\ 1.1103 + 0.0605)$

$k_2 = 0.1(0.15 + 1.1708)$

$\boxed{k_2 = 0.13208}$

$k_3 = hf(x_1 + h/2,\ y_1 + k_2/2)$

$k_3 = 0.1 f(0.1 + 0.05,\ 1.1103 + 0.0660)$

$k_3 = 0.1(0.15 + 1.1790)$

$\boxed{k_3 = 0.1329}$

$k_4 = hf(x_1+h, y_1+k_3)$

$k_4 = 0.1f(0.1+0.1, 1.1103+0.1329)$

$k_4 = 0.1(0.2+1.2432)$

$\boxed{k_4 = 0.14432}$

substituting in eq ②, we get

$y_2 = y(0.1+0.1) = 1.1103 + \frac{1}{6}[0.121 + 2(0.132 + 0.1329) + 0.144$

$y(0.2) = 1.2427$ $\qquad \therefore y_2 = 1.2427$ at $x_2 = 0.2$

★ **Simultaneous differential equations** (Runge - Kutta Method)

Q1: $\frac{dy}{dx} = xz + 1$; $\frac{dz}{dx} = -xy$; $y(0) = 0$, $z(0) = 1$.

Compute y and z at $x = 0.2$.

Sol: We have $\frac{dy}{dx} = xz + 1 = f(x, y, z)$ independent variable

$\frac{dz}{dx} = -xy = g(x, y, z)$

~~$\frac{dy}{dx}$~~ we have $y_0 = 0$, $z_0 = 1$ at $x_0 = 0$.

taking $h = 0.2$.

$\frac{dy}{dx} = xz + 1 = f(x, y, z)$ $\qquad$ $\frac{dz}{dx} = -xy = g(x, y, z)$

By RK method we have

$y(x_0+h) = y(0+0.2) = y(0.2)$

$= y(x_0) + \frac{1}{6}(k_1 + 2(k_2+k_3) + k_4)$ —— ①

By RK method we have

$z(x_0+h) = z(0+0.2) = z(0.2)$

$= z(x_0) + \frac{1}{6}(k_1' + 2(k_2' + k_3') + k_4')$ —— ②

$k_1 = hf(x_0, y_0, z_0)$

$k_1 = 0.2 f(0, 0, 1)$

$k_1 = 0.2[0(1)+1]$

$\boxed{k_1 = 0.2}$

$k_1' = hg(x_0, y_0, z_0)$

$k_1' = 0.2 g(0, 0, 1)$

$k_1' = 0.2[-0(0)]$

$\boxed{k_1' = 0}$

$k_2 = hf(x_0 + h/2, y_0 + k_1/2, z_0 + k_1'/2)$

$k_2 = 0.2 f(0.1, 0.1, 1)$

$k_2 = 0.2[0.1(1)+1]$

$\boxed{k_2 = 0.22}$

$k_2' = hg(x_0 + h/2, y_0 + k_1/2 + z_0 + k_1'/2)$

$k_2' = 0.2g(0.1, 0.1, 1)$

$k_2' = 0.2[-0.1(0.1)]$

$\boxed{k_2' = -0.002}$

$k_3 = hf(x_0 + h/2, y_0 + k_2/2, z_0 + k_2'/2)$

$k_3 = 0.2 f(0.1, 0.11, 1 - 0.001)$

$k_3 = 0.2[0.1(0.999)+1]$

$\boxed{k_3 = 0.21998}$

$k_3' = hg(x_0 + h/2, y_0 + k_2/2, z_0 + k_2'/2)$

$k_3' = 0.2g(0.1, 0.11, 1 - 0.001)$

$k_3' = 0.2[-0.1(0.11)]$

$\boxed{k_3' = -0.0022}$

$k_4 = hf(x_0 + h, y_0 + k_3, z_0 + k_3')$

$k_4 = 0.2 f(0.2, 0.21998, 1 - 0.0022)$

$k_4 = 0.2[0.2(0.9978)+1]$

$\boxed{k_4 = 0.2399}$

$k_4' = hg(x_0 + h, y_0 + k_3, z_0 + k_3')$

$k_4' = 0.2g(0.2, 0.21998, 1 - 0.0022)$

$k_4' = 0.2[-0.2(0.21998)]$

$\boxed{k_4' = -0.00879}$

Substituting in eq ①, we get

$$y(0.2) = 0 + \frac{1}{6}[0.2 + 2(0.22 + 0.21998) + 0.2399]$$

$$\underline{y(0.2) = 0.21997}$$

Substituting in eq ②, we get

$$z(0.2) = 1 + \frac{1}{6}[0 + 2(-0.002 - 0.0022) - 0.00879]$$

$$\underline{z(0.2) = 0.99713}$$

Q2: $\frac{dx}{dt} = y - t$, $\frac{dy}{dt} = x + t$; $x(0) = 0$, $y(0) = 1$.

Find $x(0.1)$, $y(0.1)$

Sol: We have

$\frac{dx}{dt} = y - t = f(t, x, y)$ $\qquad$ $\frac{dy}{dt} = x + t = g(t, x, y)$

We have $x_0 = 0$ and $y_0 = 1$ at $t = 0$. take $h = 0.1$.
By RK method, we have

$x(t_0 + h) = x(0 + 0.1) = x(0.1)$
$= x(t_0) + \frac{1}{6}[k_1 + 2(k_2 + k_3) + k_4]$ —— (1)

$y(t_0 + h) = y(0 + 0.1) = y(0.1)$
$= y(t_0) + \frac{1}{6}[k_1' + 2(k_2' + k_3') + k_4']$ —— (2)

$k_1 = hf(t_0, x_0, y_0)$
$k_1 = 0.1 f(0, 0, 1)$
$k_1 = 0.1[1 - 0]$
$\boxed{k_1 = 0.1}$

$k_1' = hg(t_0, x_0, y_0)$
$k_1' = 0.1 g(0, 0, 1)$
$k_1' = 0.1[0 + 0]$
$\boxed{k_1' = 0}$

$k_2 = hf(t_0 + h/2, x_0 + k_1/2, y_0 + k_1'/2)$
$k_2 = 0.1 f(0.05, 0.05, 1)$
$k_2 = 0.1[1 - 0.05]$
$\boxed{k_2 = 0.095}$

$k_2' = hg(t_0 + h/2, x_0 + k_1/2, y_0 + k_1'/2)$
$k_2' = 0.1 g(0.05, 0.05, 1)$
$k_2' = 0.1[0.05 + 0.05]$
$\boxed{k_2' = 0.01}$

$k_3 = hf(t_0 + h/2, x_0 + k_2/2, y_0 + k_2'/2)$
$k_3 = 0.1 f(0.05, 0.0475, 1 + 0.005)$
$k_3 = 0.1[1.005 - 0.05]$
$\boxed{k_3 = 0.0955}$

$k_3' = hg(t_0 + h/2, x_0 + k_2/2, y_0 + k_2'/2)$
$k_3' = 0.1 g(0.05, 0.0475, 1 + 0.005)$
$k_3 = 0.1[0.0475 + 0.05]$
$\boxed{k_3' = 0.00975}$

$k_4 = hf(t_0 + h, x_0 + k_3, y_0 + k_3')$
$k_4 = 0.1 f(0.1, 0.0955, 1 + 0.00975)$
$k_4 = 0.1[1.00975 - 0.1]$
$\boxed{k_4 = 0.0909}$

$k_4' = hg(t_0 + h, x_0 + k_3, y_0 + k_3')$
$k_4' = 0.1 g(0.1, 0.0955, 1 + 0.00975)$
$k_4' = 0.2[0.0955 + 0.1]$
$\boxed{k_4' = 0.01955}$

Substituting in eq ①, we get

$$z(0.1) = 0 + \frac{1}{6}\left[0.1 + 2(0.095 + 0.0955) + 0.0909\right]$$

$$z(0.1) = 0.0953$$

Substituting in eq ②, we get

$$y(0.1) = 1 + \frac{1}{6}\left[0 + 2(0.01 + 0.00975) + 0.01955\right]$$

$$y(0.1) = 1.0098$$

Q3: $\frac{dy}{dx} = z$, $\frac{dz}{dx} = -xz - y$, $y(0) = 1$, $z(0) = 0$

compute $y(0.1)$ and $z(0.1)$

sol: We have

$$\frac{dy}{dx} = z = f(x, y, z) \qquad \frac{dz}{dx} = -xz - y = g(x, y, z)$$

We have $y_0 = 1$ and $z_0 = 0$ at $x_0 = 0$. take $h = 0.1$

By RK method we have

$$y(x_0+h) = y(0+0.1) = y(0.1) = y(x_0) + \frac{1}{6}\left[k_1 + 2(k_2+k_3) + k_4\right] \quad \text{—— ①}$$

$$z(x_0+h) = z(0+0.1) = z(0.1) = z(x_0) + \frac{1}{6}\left[k_1' + 2(k_2'+k_3') + k_4'\right] \quad \text{—— ②}$$

$k_1 = hf(x_0, y_0, z_0)$

$k_1 = 0.1 f(0, 1, 0)$

$k_1 = 0.1\,[0]$

$\boxed{k_1 = 0}$

$k_1' = hg(x_0, y_0, z_0)$

$k_1' = 0.1\, g(0, 1, 0)$

$k_1 = 0.1\,[-0(0) - 1]$

$\boxed{k_1' = -0.1}$

$k_2 = hf(x_0 + h/2, y_0 + k_1/2, z_0 + k_1'/2)$

$k_2 = 0.1 f(0 + 0.05, 1 + 0, 0 - 0.05)$

$k_2 = 0.1\,[-0.05]$

$\boxed{k_2 = -0.005}$

$k_2' = hg(x_0 + h/2, y_0 + k_1/2, z_0 + k_1'/2)$

$k_2' = 0.1\, g(0.05, 1, -0.05)$

$k_2' = 0.1\,[-0.05(-0.05) - 1]$

$\boxed{k_2' = -0.09975}$

$k_3 = hf(x_0 + h/2,\ y_0 + k_2/2,\ z_0 + k_2'/2)$

$k_3 = 0.1 f(0.05,\ 1 - 0.0025,\ -0.049875)$

$k_3 = 0.1\,[-0.049875]$

$\boxed{k_3 = -0.004987}$

$k_3' = hg(x_0 + h/2,\ y_0 + k_2/2,\ z_0 + \ldots$

$k_3' = 0.1\, g(0.05,\ 1 - 0.0025,\ - \ldots$

$k_3' = 0.1\,[-0.05(-0.049875) - \ldots$

$\boxed{k_3' = -0.09975}$

$k_4 = hf(x_0 + h,\ y_0 + k_3,\ z_0 + k_3')$

$k_4 = 0.1\, f(0.1,\ 1 - 0.004987,\ -0.09975)$

$k_4 = 0.1\,[-0.09975]$

$\boxed{k_4 = -0.00997}$

$k_4' = hg(x_0 + h,\ y_0 + k_3,\ z_0 + k_3' \ldots$

$k_4' = 0.1\, g(0.1,\ 1 - 0.004987,\ - \ldots$

$k_4' = 0.1\,[-0.1(-0.09975) - 0.995 \ldots$

$\boxed{k_4' = -0.09850}$

Substituting in eq ①, we get

$$y(0.1) = 1 + \frac{1}{6}\left[0 + 2(-0.005 - 0.004987) - 0.00997\right]$$

$$y(0.1) = 1 + \frac{1}{6}(-0.02994)$$

$$\underline{y(0.1) = 0.99500}$$

Substituting in eq ①, we get

$$z(0.1) = 1 + \frac{1}{6}\left[-0.1 + 2(-0.09975 - 0.099750) - 0.09850\right]$$

$$z(0.1) = 1 + \frac{1}{6}\left[\overset{-0.5974}{\cancel{-0.05\ldots}}\right]$$

$$\underline{z(0.1) = 0.0995}$$

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Milne's Predictor and Corrector Formula:

consider a differential equation of the form $\frac{dy}{dx} = f(x,y)$ with the set of initial conditions $y(x_0) = y_0$, $y(x_0+h) = y_1$, $y(x_0+2h) = y_2$, $y(x_0+3h) = y_3$.

To compute $y_4 = y(x_0+4h)$ we use the Milne's predictor formula.

$$y_4^{(P)} = y_0 + \frac{4h}{3}\left[2y_1' - y_2' + 2y_3'\right]$$

The value thus predicted is improved by using the Milne's corrector formula

$$y_4^{(c)} = y_2 + \frac{h}{3}\left[y_2' + 4y_3' + y_4'\right]$$

Q1: Use Milne's formula to find $y(0.8)$. given $y' = x - y^2$ and $y(0) = 0$, $y(0.2) = 0.02$, $y(0.4) = 0.0796$ and $y(0.6) = 0.1762$.

Sol:

x	y	$y' = x - y^2$
$x_0 = 0$	$y_0 = 0$	$y_0' = x_0 - y_0^2 = 0 - 0 = 0$
$x_1 = 0.2$	$y_1 = 0.02$	$y_1' = x_1 - y_1^2 = 0.2 - 0.0004 = 0.1996$
$x_2 = 0.4$	$y_2 = 0.0796$	$y_2' = x_2 - y_2^2 = 0.4 - 0.0063 = 0.3937$
$x_3 = 0.6$	$y_3 = 0.1762$	$y_3' = x_3 - y_3^2 = 0.6 - 0.0310 = 0.569$

Milne's predictor formula is

$$y_4^{(P)} = y_0 + \frac{4h}{3}\left[2y_1' - y_2' + 2y_3'\right]$$

$$y_4^{(P)} = 0 + \frac{4(0.2)}{3}\left[2(0.1996) - 0.3937 + 2(0.569)\right]$$

$$y_4^{(P)} = 0.3049$$

This value is improved by using the Milne's corrector formula as follows.

1st iteration

$$y_4^{(c)} = y_2 + \frac{h}{3}\left[y_2' + 4y_3' + y_4'\right]$$

$$y_4' = x_4 - (y_4^{P})^2 = 0.8 - (0.3049)^2$$

$$\underline{y_4' = 0.4070}$$

$$\therefore y_4^{(c)} = 0.0796 + \frac{0.2}{3}\left[0.3937 + 4(0.569) + 0.4070\right]$$

$$\underline{y_4^{(c)} = 0.3047}$$

2nd iteration

$$y_4' = x_4 - (y_4^{c})^2 = 0.8 - (0.3047)^2$$

$$\underline{y_4' = 0.4071}$$

$$\therefore y_4^{(c)} = 0.0796 + \frac{0.2}{3}\left[0.3937 + 4(0.569) + 0.4071\right]$$

$$\boxed{y_4^{(c)} = 0.3047}$$

NOTE: If no initial conditions are given, we can find the initial conditions by Taylor's method and then continue with the Milne's method.

Q2: $\frac{dy}{dx} = 2e^x - y$ and $y(0) = 2$, $y(0.1) = 2.010$, $y(0.2) = 2.040$, $y(0.3) = 2.090$ find $y(0.4)$.

Sol:

x	y	$y' = 2e^x - y$
$x_0 = 0$	$y_0 = 2$	$y_0' = 2e^{x_0} - y_0 = 2e^0 - 2 = 0$
$x_1 = 0.1$	$y_1 = 2.010$	$y_1' = 2e^{x_1} - y_1 = 2e^{0.1} - 2.010 = 0.2003$
$x_2 = 0.2$	$y_2 = 2.040$	$y_2' = 2e^{x_2} - y_2 = 2e^{0.2} - 2.040 = 0.4003$
$x_3 = 0.3$	$y_3 = 2.090$	$y_3' = 2e^{x_3} - y_3 = 2e^{0.3} - 2.090 = 0.6097$

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Milne's predictor formula is given by

$$y_4^{(P)} = y_0 + \frac{4h}{3}\left[2y_1' - y_2' + 2y_3'\right]$$

$$y_4^{(P)} = 2 + \frac{4(0.1)}{3}\left[2(0.2003) - 0.4003 + 2(0.6097)\right]$$

$$\underline{y_4^{(P)} = 2.1626}$$

This value is improved by using the Milne's corrector formula

$$y_4^{(c)} = y_2 + \frac{h}{3}\left[y_2' + 4y_3' + y_4'\right]$$

1st iteration

$$y_4' = 2e^{x_4} - y_4^{P} = 2e^{0.4} - 2.1626$$

$$\underline{y_4' = 0.8210}$$

$$y_4^{(c)} = 2.040 + \frac{0.1}{3}\left[0.4003 + 4(0.6097) + 0.8210\right]$$

$$\underline{y_4^{(c)} = 2.1620}$$

2nd iteration

$$y_4' = 2e^{x_4} - y_4^{c} = 2e^{0.4} - 2.1620 = \underline{0.8216}$$

$$y_4^{(c)} = 2.040 + \frac{0.1}{3}\left[0.4003 + 4(0.6097) + 0.8216\right]$$

$$\underline{y_4^{(c)} = 2.1620}$$

Q 3: Using Milne's predictor and collector formula find $y(1.4)$ given that $\frac{dy}{dx} = x^2 + \frac{y}{2}$ with $y(1) = 2$.

Obtain the initial values of y at $x = (1.1, 1.2, 1.3)$ by Taylor series method.

Sol: We have $y' = x^2 + y/2$

$y_0 = 2$ at $x_0 = 1$

By taylor series

$$y(x_0+h) = y(x_0) + hy'(x_0) + \frac{h^2}{2!} y''(x_0) + \frac{h^3}{3!} y'''(x_0) + \dots \quad \text{--- (1)}$$

To find y at $x = 1.1$ we take $h = 0.1$

$y' = x^2 + y/2$

$y'(x_0=1) = x_0^2 + y_0/2 = 1 + 1 = 2$

$y'(x_0=1) = 2$

$y'' = 2x + y'/2$

$y''(x_0=1) = 2x_0 + y_0'/2 = 2(1) + 2/2$

$y''(x_0=1) = 3$

$y''' = 2 + y''/2$

$y'''(x_0=1) = 2 + y_0''/2 = 2 + 3/2$

$y'''(x_0=1) = 3.5$

$y'''' = y'''/2$

$y''''(x_0=1) = y_0'''/2 = 3.5/2$

$y''''(x_0=1) = 1.75$

Substituting in eq (1)

$$y(1+0.1) = 2 + 0.1(2) + \frac{(0.1)^2}{2}(3) + \frac{(0.1)^3}{6}(3.5) + \frac{(0.1)^4}{24}(1.75)$$

$y(1.1) = 2 + 0.2 + 0.015 + 0.00058 + 0.000007$

$y(1.1) = 2.2156$

To find y at $x = 1.2$ we take $h = 0.2$

Substituting in eq ①

$$y(1+0.2) = 2 + 0.2(2) + \frac{(0.2)^2}{2}(3) + \frac{(0.2)^3}{6}(3.5) + \frac{(0.2)^4}{24}(1.75)$$

$$y(1.2) = 2 + 0.4 + 0.06 + 0.00467 + 0.0001167$$

$$\underline{\underline{y(1.2) = 2.4647}}$$

To find y at $x = 1.3$ we take $h = 0.3$

Substituting in eq ①

$$y(1+0.3) = 2 + (0.3)2 + \frac{(0.3)^2}{2}3 + \frac{(0.3)^3}{6}(3.5) + \frac{(0.3)^4}{24}(1.75)$$

$$y(1.3) = 2 + 0.6 + 0.135 + 0.01575 + 0.00059$$

$$\underline{\underline{y(1.3) = 2.75134}}$$

x	y	$y' = x^2 + y/2$
$x_0 = 1$	$y_0 = 2$	$y_0' = x_0^2 + y_0/2 = 1^2 + 2/2 = 2$
$x_1 = 1.1$	$y_1 = 2.2156$	$y_1' = x_1^2 + y_1/2 = 1.1^2 + 2.2156/2 = 2.3178$
$x_2 = 1.2$	$y_2 = 2.4647$	$y_2' = x_2^2 + y_2/2 = 1.2^2 + 2.4647/2 = 2.67235$
$x_3 = 1.3$	$y_3 = 2.7513$	$y_3' = x_3^2 + y_3/2 = 1.3^2 + 2.7513/2 = 3.0656$

Milne's Predictor formula is given by

$$y_4^P = y_0 + \frac{4h}{3}\left[2y_1' - y_2' + 2y_3'\right]$$

$$y_4^P = 2 + \frac{4(0.1)}{3}\left[2(2.3178) - 2.6723 + 2(3.0656)\right]$$

$$\underline{\underline{y_4^P = 3.0792}}$$

This value is improved by using the Milne's corrector formula

$$y_4^{(c)} = y_2 + \frac{h}{3}\left[y_2' + 4y_3' + y_4'\right]$$

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1st iteration

$y_4' = x_4^2 + y_4^{(p)}/2 = 1.4^2 + 3.0792/2$

$y_4' = 3.4996$

$\therefore y_4^c = 2.4647 + \frac{0.1}{3}\left[2.6723 + 4(3.0656) + 3.4996\right]$

$y_4^c = 3.07917$

2nd iteration

$y_4' = x_4^2 + y_4^{(c)}/2 = 1.4^2 + 3.07917/2$

$y_4' = 3.49958$

$\therefore y_4^c = 2.4647 + \frac{0.1}{3}\left[2.6723 + 4(3.0656) + 3.49958\right]$

$y_4^c = 3.07917$

UNIT - 2

Complex Variables

A complex number is of the form $z = a+ib$ where $\{a, b\}$ are real numbers and $i = \sqrt{-1}$ is the imaginary unit.
The complex number $a+ib$ can be represented as a point in a x-y plane. There is a one to one correspondance between the pair of real numbers (a, b) and complex number of the type $a+ib$. In such a case, the xy plane is called the plane of a complex variable or simply complex plane.

★ <u>Function of a complex variable:</u>

Let $z = x+iy$ and $w = u+iv$ be two complex numbers, If for each value of z in a certain portion R of a complex plane, there corresponds one or more values of w, then w is said to be a function of z and is denoted as

$$\boxed{w = f(z) = f(x+iy) = u(x,y) + iv(x,y)}$$

Here $u(x,y)$ and $v(x,y)$ are the real functions of the real variables x and y.

If for each value of z in R, there is corresponding only one value of w, then w is called a <u>single valued function</u> of z otherwise it is called multiple valued function.

★ <u>Analytic Function (Regular Function) (Holomorphic Function):</u>

If a single valued function $w = f(z)$ posses a unique derivative at $z = z_0$ and at every point in the neighbourhood of z_0 then $f(z)$ is said to be analytic at z_0 and the point z_0 is called the <u>regular point</u> of the function.

If $f(z)$ is analytic at every point of the region R, then $f(z)$ is said to be analytic in R.

A point at which an analytic function ceases to have a derivative is called a singular point.

Conditions for a function $f(z)$ to be analytic:

- Necessary condition

$w = f(z) = u + iv$ must satisfy the Cauchy-Reimann equations: $\boxed{\begin{array}{l} u_x = v_y \\ u_y = -v_x \end{array}}$ $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$; $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

- Sufficient condition

The four partial derivatives u_x, u_y, v_x, v_y of $w = f(z)$ exists and are continuous on a region R.

NOTE:

If u and v are real single valued functions of x and y with which the four partial derivatives are continuous throughout the region R, then the Cauchy-Reimann equations are necessary and sufficient conditions for the function $f(z)$ to be analytic.

* Derivation of cauchy-Reimann Equations in cartesian form

Let $w = f(z)$ be an analytic function of a complex variable in a region R, the $f'(z)$ exists.

By defination

$$f'(z) = \lim_{\delta z \to 0} \frac{f(z+\delta z) - f(z)}{\delta z}$$

we have $w = f(z) = f(x+iy) = u(x,y) + iv(x,y)$

$$f'(z) = \lim_{\substack{\delta x \to 0 \\ \delta y \to 0}} \frac{[u(x+\delta x, y+\delta y) + iv(x+\delta x, y+\delta y)] - [u(x,y)+iv(x,y)]}{\delta x + i\delta y} \quad \text{---} (1)$$

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CASE 1: Let y be a constant, $\delta y = 0$ and let $\delta x \to 0$

Eq I reduces to

$$f'(z) = \lim_{\delta x \to 0} \frac{[u(x+\delta x, y) + iv(x+\delta x, y)] - [u(x,y) + iv(x,y)]}{\delta x}$$

$$f'(z) = \lim_{\delta x \to 0} \frac{u(x+\delta x, y) - u(x,y)}{\delta x} + i \lim_{\delta x \to 0} \frac{v(x+\delta x, y) - v(x,y)}{\delta x}$$

$$\boxed{f'(z) = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}} \quad \text{——— } (1)$$

CASE 2: Let x be a constant, $\delta x = 0$ and let $\delta y \to 0$

Eq I reduces to

$$f'(z) = \lim_{\delta y \to 0} \frac{[u(x, y+\delta y) + iv(x, y+\delta y)] - [u(x,y) + iv(x,y)]}{i\,\delta y}$$

$$f'(z) = \lim_{\delta y \to 0} \frac{u(x, y+\delta y) - u(x,y)}{i\,\delta y} + \cancel{i} \lim_{\delta y \to 0} \frac{v(x, y+\delta y) - v(x,y)}{\cancel{i}\,\delta y}$$

$$f'(z) = \frac{1}{i}\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}$$

$$\boxed{f'(z) = -i\frac{\partial u}{\partial y} + \frac{\partial v}{\partial y}} \quad \text{——— } (2)$$

From eq (1) and eq (2) we have

$$\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}$$

Equating the real and the imaginary parts on both sides of the above equation we get.

$$\boxed{\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y} \qquad \text{ie, } u_x = v_y,\ u_y = -v_x}$$

are the required Cauchy - Reimann equations in the cartesian form.

★ **Derivation of Cauchy-Reimann equations in Polar form:**

Consider a complex valued analytic function $w = f(z)$.

then, $f'(z)$ exists.

By defination

$$f'(z) = \lim_{\delta z \to 0} \frac{f(z+\delta z) - f(z)}{\delta z}$$

NOTE:

$z = x + iy$

$z = r\cos\theta + i r\sin\theta$

$z = r(\cos\theta + i\sin\theta)$

$\therefore z = re^{i\theta}$

we have $w = f(z) = f(re^{i\theta}) = u(r,\theta) + iv(r,\theta)$

$$f'(z) = \lim_{\substack{\delta r \to 0 \\ \delta\theta \to 0}} \frac{[u(r+\delta r, \theta+\delta\theta) + iv(r+\delta r, \theta+\delta\theta)] - [u(r,\theta) + iv(r,\theta)]}{re^{i\theta} i\,\delta\theta + e^{i\theta}\delta r} \quad \text{--- I}$$

$\delta z = \delta(re^{i\theta})$

$\delta z = re^{i\theta} i\,\delta\theta + e^{i\theta}\delta r$

CASE 1: Let θ be constant, $\delta\theta = 0$ and let $\delta r \to 0$

Eq I reduces to

$$f'(z) = \lim_{\delta r \to 0} \frac{[u(r+\delta r, \theta) + iv(r+\delta r, \theta)] - [u(r,\theta) + iv(r,\theta)]}{e^{i\theta}\delta r}$$

$$f'(z) = e^{-i\theta}\left[\lim_{\delta r \to 0} \frac{u(r+\delta r,\theta) - u(r,\theta)}{\delta r} + i\lim_{\delta r \to 0} \frac{v(r+\delta r,\theta) - v(r,\theta)}{\delta r}\right]$$

$$\boxed{f'(z) = e^{-i\theta}\left[\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}\right]} \quad \text{--- (1)}$$

CASE 2: Let r be constant, $\delta r = 0$ and let $\delta\theta \to 0$

Eq I reduces to

$$f'(z) = \lim_{\delta\theta \to 0} \frac{[u(r, \theta+\delta\theta) + iv(r, \theta+\delta\theta)] - [u(r,\theta) + iv(r,\theta)]}{ire^{i\theta}\delta\theta}$$

$$f'(z) = \frac{e^{-i\theta}}{r}\left[\lim_{\delta\theta \to 0} \frac{u(r,\theta+\delta\theta) - u(r,\theta)}{i\,\delta\theta} + \cancel{i}\lim_{\delta\theta \to 0} \frac{v(r,\theta+\delta\theta) - v(r,\theta)}{\cancel{i}\,\delta\theta}\right]$$

$$f'(z) = \frac{e^{-i\theta}}{r}\left[\frac{1}{i}\frac{\partial u}{\partial\theta} + \frac{\partial v}{\partial\theta}\right]$$

$$\boxed{f'(z) = \frac{e^{-i\theta}}{r}\left[-i\frac{\partial u}{\partial\theta} + \frac{\partial v}{\partial\theta}\right]} \quad \text{--- (2)}$$

From eq (1) and eq (2) we have

$$e^{-i\theta}\left[\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r}\right] = \frac{e^{-i\theta}}{r}\left[-i\frac{\partial u}{\partial\theta} + \frac{\partial v}{\partial\theta}\right]$$

$$\frac{\partial u}{\partial r} + i\frac{\partial v}{\partial r} = \frac{1}{r}\left[\frac{\partial v}{\partial \theta} - i\frac{\partial u}{\partial \theta}\right]$$

Equating the real and imaginary parts on both sides of the above equation we get.

$$\boxed{\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \text{ and } \frac{\partial v}{\partial r} = \frac{-1}{r}\frac{\partial u}{\partial \theta} \quad \text{ie., } u_r = v_\theta / r \text{ and } v_r = -u_\theta / r}$$

are the required Cauchy-Reimann equations in the polar form.

Q1: Show that $w = z^2$ is analytic and hence finds its derivative.

Sol: We have $w = z^2$

$w = (x+iy)^2$

$w = x^2 - y^2 + 2ixy$

$w = (x^2-y^2) + i(2xy)$

this is of the form $u(x,y) + i\, v(x,y)$

$\therefore u(x,y) = x^2 - y^2$

$v(x,y) = 2xy$

we check the CR equations

$u_x = 2x \quad u_y = -2y$

$v_x = 2y \quad v_y = 2x$

We observe that

$u_x = v_y$ and $u_y = -v_x$

Hence CR equations are satisfied

Therefore it is an analytical function.

$f'(z) = u_x + iv_x$

$= 2x + i(2y)$

$= 2[x+iy]$

$\therefore \underline{f'(z) = 2z}$

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Q2: Show that $f(z) = \sin z$ is analytic. If so find its derivative.

We have
$\sin i\theta = i\sinh\theta$
$\cos i\theta = \cosh\theta$
$\tan i\theta = i\tanh\theta$

Sol: Given $f(z) = \sin z$

$f(z) = \sin(x+iy)$

$f(z) = \sin x \cos iy + \cos x \sin iy$

$f(z) = \sin x \cosh y + \cos x \, i\sinh y$

$f(z) = u(x,y) + v(x,y)$

$\therefore u(x,y) = \sin x\cosh y$ and $v(x,y) = \cos x \sinh y$

We check the CR equations

$u_x = \cos x \cosh y \qquad u_y = \sin x \sinh y$

$v_x = -\sin x \sinh y \qquad v_y = \cos x \cosh y$

We observe that

$u_x = v_y$ and $u_y = -v_x$

Hence the CR equations are satisfied

Therefore $f(z) = \sin z$ is analytical function.

$f'(z) = u_x + iv_x$

$= \cos x \cosh y - i\sin x \sinh y$

$= \cos x \cos iy - \sin x \sin iy$

$= \cos(x+iy)$

$\underline{f'(z) = \cos z}$

Q 3: $f(z) = e^z$ show that it is analytic. If so find its derivative

Sol: We have $f(z) = e^z$

$f(z) = e^{(x+iy)}$

$f(z) = e^x e^{iy}$

$f(z) = e^x(\cos y + i\sin y)$

$f(z) = e^x\cos y + ie^x\sin y$

it is of the form $u(x,y) + iv(x,y)$

$\therefore u(x,y) = e^x \cos y$ and $v(x,y) = e^x \sin y$

We now check the CR equations

$u_x = e^x \cos y \qquad u_y = -e^x \sin y$

$v_x = e^x \sin y \qquad v_y = e^x \cos y$

We observe that

$u_x = v_y$ and $u_y = -v_x$

Hence the CR equations are satisfied

Therefore $f(z) = e^z$ is an analytical function.

$f'(z) = u_x + i v_x$

$= e^x \cos y + i e^x \sin y$

$= e^x (\cos y + i \sin y)$

$= e^x e^{iy}$

$= e^{x+iy}$

$\therefore \underline{\underline{f'(z) = e^z}}$

Q4: $w = z + e^z$: show that it is analytic and find its derivative.

Sol: We have: $w = z + e^z$

$w = x + iy + e^{x+iy}$

$w = x + iy + e^x e^{iy}$

$w = x + iy + e^x (\cos y + i \sin y)$

$w = (x + e^x \cos y) + i(y + e^x \sin y)$

It is of the form $w = u(x,y) + iv(x,y)$

$\therefore u(x,y) = x + e^x \cos y$

$v(x,y) = y + e^x \sin y$

We now check the CR equations

$u_x = 1 + e^x \cos y \qquad u_y = -e^x \sin y$

$v_x = e^x \sin y \qquad v_y = 1 + e^x \cos y$

We observe that

$u_x = v_y$ and $u_y = -v_x$

Hence the CR equations are satisfied

Therefore $f(z) = z + e^z$ is an analytic function

$$f'(z) = u_x + i v_x$$
$$= 1 + e^x \cos y + i(e^x \sin y)$$
$$= 1 + e^x(\cos y + i \sin y)$$
$$= 1 + e^x e^{iy}$$
$$= 1 + e^{x+iy}$$
$$\therefore \underline{f'(z) = 1 + e^z}$$

Q 5: Determine whether the function $w = \bar{z} = x - iy$ is analytic

Sol: We have $w = \bar{z} = x - iy$

$w = x - iy$

It is of the form $w = u(x,y) + iv(x,y)$

$\therefore u(x,y) = x$

and $v(x,y) = -y$

Now we check the CR equations

$u_x = 1 \qquad u_y = 0$

$v_x = 0 \qquad v_y = -1$

We observe that

$u_x \neq v_y$ and $u_y \neq -v_x$

Hence it does not satisfy the CR equations

Therefore function $w = \bar{z} = x - iy$ is not analytic.

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Q: Show that $w = f(z) = \frac{1}{z^3}$ $(z \neq 0)$ is analytic, if so find its derivative

Sol: We have $w = f(z) = \frac{1}{z^3}$

$e^{in\theta} = \cos n\theta + i\sin n\theta$

$e^{-in\theta} = \cos n\theta - i\sin n\theta$

$$w = \frac{1}{(re^{i\theta})^3}$$

$$w = \frac{e^{-3i\theta}}{r^3}$$

$$w = \frac{1}{r^3}\left[\cos 3\theta - i\sin 3\theta\right]$$

It is of the form $w = u(r,\theta) + iv(r,\theta)$

$\therefore$ $u = \frac{\cos 3\theta}{r^3}$ and $v = \frac{-\sin 3\theta}{r^3}$

We now check for CR equations

$u_r = \frac{1}{r} v_\theta$ and $v_r = -\frac{1}{r} u_\theta$

$$u_r = \frac{1}{r}\left(\frac{-3\cos 3\theta}{r^3}\right) = \frac{-3\cos 3\theta}{r^4}$$

$$v_r = \frac{-1}{r}\left(\frac{-3\sin 3\theta}{r^3}\right) = \frac{3\sin 3\theta}{r^4}$$

Hence the CR equations are satisfied

Therefore $w = f(z) = \frac{1}{z^3}$ is analytic.

Now $f'(z) = e^{-i\theta}\left[u_r + iv_r\right]$

$$= e^{-i\theta}\left[\frac{-3\cos 3\theta}{r^4} + i\frac{3\sin 3\theta}{r^4}\right]$$

$$= \frac{-3e^{-i\theta}}{r^4}\left[\cos 3\theta - i\sin 3\theta\right]$$

$$= \frac{-3e^{-i\theta}}{r^4} e^{-i3\theta}$$

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$$f'(z) = \frac{-3e^{-4i\theta}}{r^4}$$

$$= \frac{-3}{r^4 e^{4i\theta}}$$

$$= \frac{-3}{(re^{i\theta})^4}$$

$$\therefore f'(z) = \frac{-3}{z^4}$$

Q7: Show that $w = z^n$ is analytic for $n \in z^+$. If so find its derivative.

Sol: We have $w = z^n$

$w = (re^{i\theta})^n$

$w = r^n e^{in\theta}$

$w = r^n[\cos n\theta + i \sin n\theta]$

$w = r^n \cos n\theta + i r^n \sin n\theta$

It is of the form $w = f(z) = f(re^{i\theta}) = u(r,\theta) + iv(r,\theta)$

$\therefore u(r,\theta) = r^n \cos n\theta$

and $v(r,\theta) = r^n \sin n\theta$

We now check the CR equations

$u_r = \frac{1}{r} v_\theta$ and $v_r = \frac{-1}{r} u_\theta$

$u_r = nr^{n-1}\cos n\theta$ $\quad u_\theta = -r^n n \sin n\theta$

$v_r = nr^{n-1}\sin n\theta$ $\quad v_\theta = r^n n \cos n\theta$

$u_r = \frac{1}{r}(r^n n \cos n\theta) = r^{n-1} n \cos n\theta$

$v_r = \frac{-1}{r}(-r^n n \sin n\theta) = r^{n-1} n \sin n\theta$

Hence the CR equations are satisfied

Therefore $w = z^n$ is analytic.

$$f'(z) = e^{-i\theta}[u_r + iv_r]$$
$$= e^{-i\theta}[nr^{n-1}\cos n\theta + inr^{n-1}\sin n\theta]$$
$$= e^{-i\theta} nr^{n-1}[\cos n\theta + i\sin n\theta]$$
$$= e^{-i\theta} nr^{n-1}[e^{in\theta}]$$
$$= nr^{n-1} e^{i\theta(n-1)}$$
$$= n(re^{i\theta})^{n-1}$$
$$\therefore f'(z) = nz^{n-1}$$

Q8: Show that $\log z$ is analytic for $z \neq 0$. If so find its derivative.

Sol: We have $w = \log z$

$$w = \log(re^{i\theta})$$
$$w = \log r + \log e^{i\theta}$$
$$w = \log r + i\theta \log e$$
$$w = \log r + i\theta$$

It is of the form $u(r,\theta) + iv(r,\theta)$

where $u(r,\theta) = \log r$ and $v(r,\theta) = \theta$

We now check for CR equations

$$u_r = \frac{1}{r} v_\theta \qquad v_r = \frac{-1}{r} u_\theta$$

$$u_r = \frac{1}{r} \; ; \; u_\theta = 0 \; ; \; v_r = 0 \; ; \; v_\theta = 1$$

$$u_r = \frac{1}{r}(1) = \frac{1}{r} \; ; \quad v_r = \frac{-1}{r}(0) = 0$$

Hence CR equations are satisfied, therefore $\log z$ is analytic

$$f'(z) = e^{-i\theta}[u_r + iv_r]$$
$$= e^{-i\theta}\left[\frac{1}{r} + i(0)\right]$$
$$= e^{-i\theta} r^{-1}$$
$$= (re^{i\theta})^{-1}$$
$$f'(z) = z^{-1} = 1/z$$

★ Harmonic Functions:

Any function possesing continuous second order partial derivatives and which satisfies the Laplace equation is called a Harmonic Function.

Two harmonic functions u and v which are such that $u+iv$ is an analytic function are called conjugate Harmonic functions.

Property of Analytic functions ① NOTE: If $f(z) = u + iv$ is an analytic function then both u and v are harmonic functions, i.e., both the real and the imaginary part of any analytic function satisfy the laplace equation: $\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0$.

Proof: Since $w = f(z) = u(x,y) + iv(x,y)$ is an analytic function, the CR equations are satisfied.

i.e, $u_x = v_y$ —— ①

$u_y = -v_x$ —— ②

The laplace equation in two dimension is $\nabla^2 \phi = 0$ and is defined as $\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0$.

To show that $\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$, i.e, $u_{xx} + u_{yy} = 0$

Differentiating eq ① wrt x on both sides, we get

$u_{xx} = v_{xy}$

Differentiating eq ② wrt y on both sides, we get

$u_{yy} = -v_{yx}$

Adding the above two equations we get

$u_{xx} + u_{yy} = v_{xy} - v_{yx}$

$\therefore u_{xx} + u_{yy} = 0$ because $v_{xy} = v_{yx}$

because v is continuous and posses continous second order partial derivatives.

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The real part $u(x,y)$ of $f(z)$ satisfies laplace equation.

To show that, $v_{xx} + v_{yy} = 0$

we have $v_x = -u_y$ and $v_y = u_x$

Differentiating wrt x on both sides we get

$$v_{xx} = -u_{yx}$$

Differentiating wrt y on both sides we get

$$v_{yy} = u_{xy}$$

Adding the above two equations we get.

$$v_{xx} + v_{yy} = -u_{yx} + u_{xy}$$

$v_{xx} + v_{yy} = 0$ because $u_{yx} = u_{xy}$ since u is continuous and possess continuous second order partial derivatives.

The real part $v(x,y)$ of $f(z)$ satisfies laplace equations.

<u>NOTE</u>: converse need not be true.

- In Polar form

If $w = f(z) = u(r,\theta) + iv(r,\theta)$ is an analytic function, then show that u and v satisfy laplace equation.

The laplace equation in polar form is given by

$$\frac{\partial^2\phi}{\partial r^2} + \frac{1}{r}\frac{\partial\phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2\phi}{\partial\theta^2} = 0$$

Since $w = f(z) = u(r,\theta) + iv(r,\theta)$ is an analytic function, it satisfies the CR equations.

$$u_r = \frac{1}{r} v_\theta \quad \text{—— (1)}$$

$$v_r = \frac{-1}{r} u_\theta \quad \text{—— (2)}$$

To show that $u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\theta\theta} = 0$.

Differentiating eq (1) wrt r on both sides.

$$u_{rr} = \frac{1}{r}v_{r\theta} - \frac{1}{r^2}v_\theta$$

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We have $u_\theta = -r v_r$

Differentiate wrt θ on both sides

$$u_{\theta\theta} = -r v_{\theta r}$$

$$LHS = \frac{1}{r} v_{r\theta} - \frac{1}{r^2} v_\theta + \frac{1}{r}\left(\frac{1}{r} v_\theta\right) + \frac{1}{r^2}\left(-r v_{\theta r}\right) = 0$$

The real part $u(r,\theta)$ of $f(z)$ satisfies laplace equation.

Now, TST: $v_{rr} + \frac{1}{r} v_r + \frac{1}{r^2} v_{\theta\theta} = 0$

Differentiating eq ② wrt r on both sides.

$$v_{rr} = \frac{-1}{r} u_{r\theta} - \frac{1}{r^2} u_\theta$$

We have $v_\theta = r u_r$

Differentiating wrt θ on both sides

$$v_{\theta\theta} = r u_{\theta r}$$

$$LHS = \frac{-1}{r} u_{r\theta} - \frac{1}{r^2} u_\theta + \frac{1}{r} v_r + \frac{1}{r^2} r u_{\theta r} = 0$$

The real part $v(r,\theta)$ of $f(z)$ satisfies the laplace equation.

② PROPERTY

If $w = f(z) = u(x,y) + iv(x,y)$ is an analytic function then the family of curves $u(x,y) = c_1$ and $v(x,y) = c_2$ cut each other orthogonally (c_1 and c_2 are constants)

Consider the family of curves $u(x,y) = c_1$, by total differentiation we have

$$du = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy = 0$$

$$\Rightarrow \frac{dy}{dx} = \frac{-\partial u/\partial x}{\partial u/\partial y}$$

The slope of the general curve from the family of curves $u(x,y) = c_1$ is $m_1 = \frac{-u_x}{u_y}$

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Similarly the slope of the general curve from the family of curves $v(x,y)=c_2$ is $m_2 = \frac{-v_x}{v_y}$

$$\therefore m_1 \cdot m_2 = \frac{-u_x}{u_y} \cdot \frac{-v_x}{v_y}$$

since $f(z)$ is an analytic function, CR equations are satisfied.

$$u_x = v_y$$
$$u_y = -v_x$$
$$\Rightarrow m_1 \cdot m_2 = -\frac{u_x}{u_y} \cdot \frac{u_y}{u_x} = -1$$

therefore the two family of curves intersect each other orthogonally.

NOTE: The converse of the above theorem need not be true.

EX: Property 2 : converse need not be true.

Q: Show that $u = \frac{x^2}{y}$ and $v = x^2 + 2y^2$ intersect orthogonally

but $f(z) = u + iv$ is not analytic.

Sol: The slope to $u(x,y)$ family of curves is $m_1 = \frac{-u_x}{u_y}$

$$\therefore m_1 = \frac{-u_x}{u_y} = \frac{-2x/y}{-x^2/y^2} = \frac{2y}{x}$$

The slope to $v(x,y)$ family of curves is $m_2 = \frac{-v_x}{v_y}$

$$\therefore m_2 = \frac{-v_x}{v_y} = \frac{-2x}{4y} = \frac{-x}{2y}$$

$$\Rightarrow m_1 m_2 = \left(\frac{2y}{x}\right)\left(\frac{-x}{2y}\right) = -1$$

$\therefore$ The given curves intersect orthogonally.

Now consider $f(z) = u + iv = \frac{x^2}{y} + i(x^2 + 2y^2)$

$u_x = \frac{2x}{y}$ $v_y = 4y$

$\therefore u_x \neq v_y$

Therefore the CR equations are not satisfied
Hence $f(z) = u + iv$ is not analytic.

Ex: Property 1: converse need not be true
i.e., ~~if~~ u and v are harmonic function it need not imply that $f(z) = u + iv$ is analytic

Q1: consider $f(z) = x - iy$ ~~$v(x,y)$~~

Sol: Here $u(x,y) = x$
$v(x,y) = -y$

Thus

$u_x = 1$ $u_y = 0$
$u_{xx} = 0$ $u_{yy} = 0$ $\therefore u_{xx} + u_{yy} = 0$

Also

$v_x = 0$ $v_y = -1$
$v_{xx} = 0$ $v_{yy} = 0$ $\therefore v_{xx} + v_{yy} = 0$

$\therefore$ u and v are harmonic functions.
We check for CR equations for analyticity.

$u_x = 1$
$v_y = -1$ $\therefore u_x \neq v_y$

Therefore CR equation is not satisfied
Hence $f(z) = x - iy$ is not an analytic function

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Q1: If u and v are harmonic functions show that $\left(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}\right) + i\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}\right)$ is an analytic function.

Sol: Let $p = \frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}$ and $q = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y}$

To show that $p+iq$ is analytic we need to show that $p_x = q_y$ and $p_y = -q_x$

$$p_x = \frac{\partial^2 u}{\partial x \partial y} - \frac{\partial^2 v}{\partial x^2} = u_{xy} - v_{xx}$$

$$q_y = \frac{\partial^2 u}{\partial x \partial y} + \frac{\partial^2 v}{\partial y^2} = u_{xy} + v_{yy}$$

Since u and v are harmonic we have

$$u_{xx} + u_{yy} = 0 \Rightarrow u_{xx} = -u_{yy}$$
$$v_{xx} + v_{yy} = 0 \Rightarrow v_{yy} = -v_{xx}$$

We have

$$p_x = u_{xy} - v_{xx} \qquad \therefore p_x = q_y$$
$$q_y = u_{xy} + v_{yy}$$

similarly

$$p_y = \frac{\partial^2 u}{\partial y^2} - \frac{\partial^2 v}{\partial x \partial y} = u_{yy} - v_{xy}$$

$$q_x = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 v}{\partial x \partial y} = u_{xx} + v_{xy} = -(u_{yy} - v_{xy})$$

$$\therefore -p_y = q_x$$

Hence CR equations are satisfied

$\therefore$ It is an analytic function.

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Q 2: If $f(z)$ is a regular (analytic) function. show that
$$\left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)|f(z)|^2 = 4|f'(z)|^2$$

Sol:
$f(z) = u + iv$
$f(\bar{z}) = u - iv$
$|f(z)| = \sqrt{u^2+v^2}$

$z = x+iy$
$\bar{z} = x-iy$
$|z|^2 = z\bar{z}$
$|z| = \sqrt{z\bar{z}}$
$|z| = \sqrt{(x+iy)(x-iy)}$
$|z| = \sqrt{x^2+y^2}$

$|f(z)|^2 = u^2+v^2 = \text{let } \phi$

We have
$f'(z) = u_x + iv_x$
$\Rightarrow |f'(z)|^2 = u_x^2 + v_x^2$

$\therefore$ We need to prove that
$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 4(u_x^2+v_x^2)$$

Here $\phi = u^2+v^2$
$$\frac{\partial\phi}{\partial x} = 2uu_x + 2vv_x$$
$$\frac{\partial^2\phi}{\partial x^2} = 2[u_xu_x + uu_{xx} + v_xv_x + vv_{xx}]$$
$$= 2[uu_{xx} + vv_{xx} + u_x^2 + v_x^2]$$

Similarly
$$\frac{\partial^2\phi}{\partial y^2} = 2[uu_{yy} + vv_{yy} + u_y^2 + v_y^2]$$

Adding the above two equations we get
$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 2[u(u_{xx}+u_{yy}) + v(v_{xx}+v_{yy}) + u_x^2 + v_x^2 + u_y^2 + v_y^2]$$

Since $f(z)$ is an analytic function, both u and v satisfy Laplace equation hence they are harmonic functions.

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Therefore $u_{xx} + u_{yy} = 0$
and $v_{xx} + v_{yy} = 0$

Hence
$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 2\left[u_x^2 + v_x^2 + u_y^2 + v_y^2\right]$$

Since $f(z)$ is an analytic function, CR equations are satisfied $\Rightarrow$ $u_x = v_y$ and $v_x = -u_y$

$$\therefore \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 2\left[u_x^2 + v_x^2 + v_x^2 + u_x^2\right]$$
$$= 2\left[2u_x^2 + 2v_x^2\right]$$

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 4\left[u_x^2 + v_x^2\right]$$ // Hence proved.

Q3: If $f(z)$ is a regular function then show that
$$\left(\frac{\partial |f(z)|}{\partial x}\right)^2 + \left(\frac{\partial |f(z)|}{\partial y}\right)^2 = |f'(z)|^2$$

sol: $f(z) = u + iv$

$|f(z)| = \sqrt{u^2+v^2}$ Let us consider it as ϕ.

We have $f'(z) = u_x + iv_x$

$\therefore |f'(z)|^2 = u_x^2 + v_x^2$

We need to prove that
$$\left(\frac{\partial \phi}{\partial x}\right)^2 + \left(\frac{\partial \phi}{\partial y}\right)^2 = u_x^2 + v_x^2$$

Here $\phi = \sqrt{u^2+v^2}$
$$\frac{\partial \phi}{\partial x} = \frac{2uu_x + 2vv_x}{2\sqrt{u^2+v^2}} = \frac{uu_x + vv_x}{\sqrt{u^2+v^2}}$$
$$\Rightarrow \left(\frac{\partial \phi}{\partial x}\right)^2 = \frac{(uu_x + vv_x)^2}{u^2+v^2}$$

$$\frac{\partial\phi}{\partial y} = \frac{2uu_y + 2vv_y}{2\sqrt{u^2+v^2}} = \frac{uu_y + vv_y}{\sqrt{u^2+v^2}}$$

$$\therefore \left(\frac{\partial\phi}{\partial y}\right)^2 = \frac{(uu_y + vv_y)^2}{u^2+v^2}$$

Adding the above two equations we get

$$\left(\frac{\partial\phi}{\partial x}\right)^2 + \left(\frac{\partial\phi}{\partial y}\right)^2 = \frac{(uu_x + vv_x)^2 + (uu_y + vv_y)^2}{u^2+v^2}$$

$$= \frac{u^2u_x^2 + v^2v_x^2 + 2uvu_xv_x + u^2u_y^2 + v^2v_y^2 + 2uvu_yv_y}{u^2+v^2}$$

Since $f(z)$ is analytic function, CR equations are satisfied

$\therefore u_x = v_y$ and $v_x = -u_y$

$$\Rightarrow \left(\frac{\partial\phi}{\partial x}\right)^2 + \left(\frac{\partial\phi}{\partial y}\right)^2 = \frac{u^2u_x^2 + v^2v_x^2 + \cancel{2uvu_xv_x} + u^2v_x^2 + v^2u_x^2 - \cancel{2uvv_xu_x}}{u^2+v^2}$$

$$= \frac{u_x^2(u^2+v^2) + v_x^2(v^2+u^2)}{u^2+v^2}$$

$$= u_x^2 + v_x^2$$

$$\therefore \left(\frac{\partial\phi}{\partial x}\right)^2 + \left(\frac{\partial\phi}{\partial y}\right)^2 = u_x^2 + v_x^2 \quad //$$ Hence proved.

★ **Applications of analytic functions:**

Analytic functions $f(z) = u + iv$ have several applications in various fields of engineering.

– 1. Fluid flow problems

$u(x,y)$: velocity potential of the fluid

$v(x,y)$: stream function.

$f(z)$: complex potential.

– 2. Electrostatics and gravitational fluid flow problems

$u(x,y)$: equipotential lines

$v(x,y)$: lines of force

$f(z)$:

3. Heat conduction problems

$u(x,y)$: Isothermals

$v(x,y)$: heat flow lines.

★ TYPE 1 : When u or v is given to find $f(z)$:

Q1: In a two dimensional fluid flow, the velocity potential of the fluid is given by x^3-3xy^2+3x+1. Find the complex potential.

Sol: Given: $u(x,y): x^3-3xy^2+3x+1$

By defination we have

$$f'(z) = u_x + iv_x$$

Since $f(z) = u+iv$ is an analytic function, CR equations are satisfied, i.e., $u_x = v_y$ and $v_x = -u_y$

$$\therefore f'(z) = u_x - iu_y$$

$$= (3x^2-3y^2+3) - i(-6xy)$$

put $x=z$ and $y=0$

$$f'(z) = 3z^2+3-i(0)$$

$$f'(z) = 3z^2+3$$

integrating wrt z

$$f(z) = z^3+3z+C$$

Q2: In a two dimensional fluid flow, the stream function is given by $v = \dfrac{\sinh 2y}{\cos 2x - \cosh 2y}$. Compute the complex potential $f(z)$.

Sol: By defination we have

$$f'(z) = u_x + iv_x$$

Since $f(z)=u+iv$ is an analytic function, CR equations are satisfied, i.e., $u_x = v_y$ and $v_x = -u_y$

$$\therefore f'(z) = v_y + iv_x$$

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$$f'(z) = \frac{(\cos 2x - \cosh 2y)\,2(\cosh 2y) - (\sinh 2y)(-2\sinh 2y)}{(\cos 2x - \cosh 2y)^2} + i\left[\frac{-(\sinh 2y)(-2\sin 2x)}{(\cos 2x - \cosh 2y)^2}\right]$$

put $x = z$ and $y = 0$

$$f'(z) = \frac{(\cos 2z - 1)2 - 0 + i(0)}{(\cos 2z - 1)^2}$$

$$f'(z) = \frac{2\cancel{(\cos 2z - 1)}}{(\cos 2z - 1)^{\cancel{2}}}$$

$$f'(z) = \frac{2}{\cos 2z - 1} = \frac{2}{-2\sin^2 z} = -\operatorname{cosec}^2 z$$

integrating w.r.t z

$$f(z) = \cot z + c$$

Q3: Given: $u = e^x(x\cos y - y\sin y)$. Find $f(z)$

Sol: By defination we know that

$$f'(z) = u_x + iv_x$$

since $f(z) = u + iv$ is an analytic function, CR equations are satisfied, ie, $u_x = v_y$ and $v_x = -u_y$

$$\therefore f'(z) = u_x - iu_y$$

$$= [e^x(\cos y) + e^x(x\cos y - y\sin y)] - i[e^x(-x\sin y - \sin y - y\cos y)]$$

$$= e^x[(\cos y + x\cos y - y\sin y) + i(x\sin y + \sin y + y\cos y)]$$

put $x = z$ and $y = 0$

$$f'(z) = e^z[1 + z(1) - 0 + i(0 + 0 + 0)]$$

$$f'(z) = e^z[1 + z] = e^z + ze^z$$

integrating w.r.t z.

$$f(z) = e^z + [ze^z - e^z] + c$$

$$f(z) = ze^z + c$$

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Q4: Given: $v = r\sin\theta + \frac{\cos\theta}{r}$ $(r \neq 0)$. Find $f(z)$

Sol: By defination we know that

$$f'(z) = e^{-i\theta}[u_r + iv_r]$$

since $f(z) = u + iv$ is an analytic function, CR equations are satisfied i.e, $u_r = \frac{1}{r}v_\theta$ and $v_r = -\frac{1}{r}u_\theta$

$$\therefore f'(z) = e^{-i\theta}\left[\frac{1}{r}v_\theta + iv_r\right]$$

$$f'(z) = e^{-i\theta}\left[\frac{1}{r}\left(r\cos\theta - \frac{\sin\theta}{r}\right) + i\left(\sin\theta - \frac{\cos\theta}{r^2}\right)\right]$$

put $r = z$ and $\theta = 0$

$$f'(z) = e^0\left[\frac{1}{z}(z(1) - 0) + i\left(0 - \frac{1}{z^2}\right)\right]$$

$$f'(z) = 1 - \frac{i}{z^2}$$

integrating wrt z

$$f(z) = z + \frac{i}{z} + c$$

★ **TYPE 2**: When u or v is given, to find $f(z)$ and then v or u:

Q1: Given: $v = e^x(x\sin y + y\cos y)$. find $f(z)$ and hence u.

Sol: By defination we know that

$$f'(z) = u_x + iv_x$$

since $f(z) = u + iv$ is an analytic function, CR equations are satisfied, i.e., $u_x = v_y$ and $v_x = -u_y$.

$$\therefore f'(z) = v_y + iv_x$$

$$= [e^x(x\cos y + \cos y - y\sin y)] + i[e^x(x\sin y + y\cos y) + e^x(\sin y)]$$

$$f'(z) = e^x[x\cos y + \cos y - y\sin y + i(x\sin y + y\cos y + \sin y)]$$

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put $x = z$ and $y = 0$

$f'(z) = e^z[z(1) + 1 - 0 + i(0+0+0)]$

$f'(z) = e^z z + e^z$

integrating wrt z

$f(z) = ze^z - e^z + e^z + c$

$f(z) = ze^z + c$

put $z = x + iy$, to find u.

$f(z) = (x+iy)e^{x+iy}$

$= (x+iy)e^x e^{iy}$

$= e^x(\cos y + i\sin y)(x+iy)$

$= e^x[x\cos y + iy\cos y + ix\sin y - y\sin y]$

$f(z) = e^x[(x\cos y - y\sin y) + i(x\sin y + y\cos y)]$

$\therefore u(x,y) = e^x[x\cos y - y\sin y]$

Q2: In a two dimensional fluid flow, the stream function is $\frac{-y}{x^2+y^2}$. Find the complex potential $f(z)$ and also the velocity potential $u(x,y)$.

Sol: given $v(x,y) = \frac{-y}{x^2+y^2}$

By defination we know that

$f'(z) = u_x + iv_x$

Since $f(z) = u + iv$ is an analytic function, CR equations are satisfied, i.e., $u_x = v_y$ and $v_x = -u_y$.

$\therefore f'(z) = v_y + iv_x$

$= \frac{(x^2+y^2)(-1) - (-y)(2y)}{(x^2+y^2)^2} + i\,\frac{(x^2+y^2)(0) - (-y)(2x)}{(x^2+y^2)^2}$

$f'(z) = \frac{-(x^2+y^2) + 2y^2 + 2ixy}{(x^2+y^2)^2} = \frac{y^2 - x^2 + 2ixy}{(x^2+y^2)^2}$

put $z = z$ and $y = 0$

$$f'(z) = \frac{-z^2 + 0}{z^4} = \frac{-1}{z^2}$$

Integrating w.r.t z

$$f(z) = \frac{1}{z} + c$$ // is the complex potential.

put $z = x + iy$, to find u

$$f(z) = \frac{1}{x+iy} \times \frac{x-iy}{x-iy}$$

$$f(z) = \frac{x-iy}{x^2+y^2}$$

$$f(z) = \frac{x}{x^2+y^2} + i\left(\frac{-y}{x^2+y^2}\right)$$

$\therefore u(x,y) = \frac{x}{x^2+y^2}$ // is the required velocity potential

Q3: Given: $v = \left(r - \frac{k^2}{r}\right)\sin\theta$, $(r \neq 0)$, find $f(z)$ and hence u.

Sol: By defination

$$f'(z) = e^{-i\theta}[u_r + i\, v_r]$$

since $f(z) = u + iv$ is an analytic function, CR equations are satisfied, ie, $u_r = \frac{1}{r} v_\theta$ and $v_r = \frac{-1}{r} u_\theta$.

$$\therefore f'(z) = e^{-i\theta}\left[\frac{1}{r} v_\theta + i\, v_r\right]$$

$$f'(z) = e^{-i\theta}\left[\frac{1}{r}\left(r - \frac{k^2}{r}\right)\cos\theta + i\left(1 + \frac{k^2}{r^2}\right)\sin\theta\right]$$

put $r = z$ and $\theta = 0$

$$f'(z) = e^0\left[\frac{1}{z}\left(z - \frac{k^2}{z}\right)(1) + i\left(1 + \frac{k^2}{z^2}\right)(0)\right]$$

$$f'(z) = 1 - \frac{k^2}{z^2}$$

Integrating wrt z

$$f(z) = z + \frac{k^2}{z} + c$$

Put $z = re^{i\theta}$, to find u

$$f(z) = re^{i\theta} + \frac{k^2}{re^{i\theta}}\left(\frac{e^{-i\theta}}{e^{-i\theta}}\right)$$

$$f(z) = re^{i\theta} + \frac{k^2 e^{-i\theta}}{r}$$

$$f(z) = r(\cos\theta + i\sin\theta) + \frac{k^2}{r}(\cos\theta - i\sin\theta)$$

$$f(z) = \left[r + \frac{k^2}{r}\right]\cos\theta + i\left[r - \frac{k^2}{r}\right]\sin\theta$$

It is of the form $f(z) = u + iv$

$$\therefore u = \left[r + \frac{k^2}{r}\right]\cos\theta$$

★ **TYPE 3: Given u or v to find v or u and then f(z):**

Q1: Show that $u = \cos x \cosh y$ is harmonic. Find its harmonic conjugate v and hence the analytic function $f(z) = u + iv$.

Sol: To show that it is harmonic

$u_x = -\sin x \cosh y$ $\qquad$ $u_y = \cos x \sinh y$

$u_{xx} = -\cos x \cosh y$ $\qquad$ $u_{yy} = \cos x \cosh y$

$\therefore u_{xx} + u_{yy} = -\cos x \cosh y + \cos x \cosh y = 0$

Hence it satisfies the Laplace equation.

Therefore it is harmonic.

Hence by CR equations

$u_x = v_y = -\sin x \cosh y$

$v = -\int \sin x \cosh y \, dy$

$v = -\sin x \sinh y + f(x)$ — ①

Similarly $v_x = -u_y = -\cos x \sinh y$

$v = -\int \cos x \sinh y \, dx$

$v = -\sin x \sinh y + g(y)$ — ②

From eq ① and eq ②

$f(x) = 0$

$g(x) = 0$

Therefore $v = -\sin x \sinh y$

The analytic function is

$f(z) = \cos x \cosh y - i \sin x \sinh y$

Q2: An electrostatic field in the x-y planes, the potential function is given by $u = 3x^2y - y^3$. Find the stream function.

Sol: $u = 3x^2y - y^3$

$\therefore u_x = 6xy \qquad u_y = 3x^2 - 3y^2$

$u_{xx} = 6y \qquad u_{yy} = -6y$

$\therefore u_{xx} + u_{yy} = 6y - 6y = 0$

Therefore it satisfies the laplace equation, hence it is an harmonic function.

By CR equations

$u_x = v_y = 6xy$

$\therefore v = \int 6xy \, dy = \frac{6xy^2}{2} + f(x) = 3xy^2 + f(x)$ — ①

III^{ly} $v_x = -u_y = -3x^2 + 3y^2$

$\therefore v = \int (-3x^2 + 3y^2) dx = -x^3 + 3xy^2 + g(y)$ — ②

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From eq ① and eq ②

$f(x) = -x^3$

$g(y) = 0$

Hence the required stream function is

$v = 3xy^2 - x^3$

Q3: In a two dimensional fluid flow, if the velocity potential $u = e^{-x}\cos y + xy$, find the stream function.

Sol:

$u = e^{-x}\cos y + xy$

$u_x = -e^{-x}\cos y + y \qquad u_y = -e^{-x}\sin y + x$

$\therefore$ By CR equations: $u_x = v_y$ and $v_x = -u_y$

$\therefore v_y = -e^{-x}\cos y + y$

$v = \int -e^{-x}\cos y + y \, dy$

$v = -e^{-x}\sin y + \frac{y^2}{2} + f(x)$ —— ①

Similarly $v_x = e^{-x}\sin y - x$

$\therefore v = \int e^{-x}\sin y - x \, dx$

$v = -e^{-x}\sin y - \frac{x^2}{2} + g(y)$ —— ②

From eq ① and eq ②

$f(x) = -\frac{x^2}{2}$ and $g(y) = \frac{y^2}{2}$

$\therefore$ The stream function is given by

$v = -e^{-x}\sin y - \frac{x^2}{2} + \frac{y^2}{2}$

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Q4: Show that $u = \left[r + \frac{1}{r}\right]\cos\theta$ is harmonic. Find its harmonic conjugate and corresponding analytic function.

Sol:
$$u = \left[r + \frac{1}{r}\right]\cos\theta$$

$$u_r = \left[1 - \frac{1}{r^2}\right]\cos\theta \qquad u_\theta = -\left[r + \frac{1}{r}\right]\sin\theta$$

$$u_{rr} = \left[\frac{2}{r^3}\right]\cos\theta \qquad u_{\theta\theta} = -\left[r + \frac{1}{r}\right]\cos\theta$$

$$u_{rr} + \frac{u_r}{r} + \frac{u_{\theta\theta}}{r^2} = \frac{2}{r^3}\cos\theta + \left[\cancel{\frac{1}{r}} - \cancel{\frac{1}{r^3}}\right]\cos\theta - \left[\cancel{\frac{1}{r}} + \cancel{\frac{1}{r^3}}\right]\cos\theta = 0$$

∴ It is a harmonic function.

By CR equations: $u_r = v_\theta / r$ and $v_r = -u_\theta / r$.

∴ $v_\theta = r u_r = \left[r - \frac{1}{r}\right]\cos\theta$

$$v = \int \left[r - \frac{1}{r}\right]\cos\theta \, d\theta = r\sin\theta - \frac{\sin\theta}{r} + f(r) \quad \text{--- (1)}$$

Similarly
$$v_r = \frac{-u_\theta}{r} = \left[1 + \frac{1}{r^2}\right]\sin\theta$$

$$v = \int \left[1 + \frac{1}{r^2}\right]\sin\theta \, dr = r\sin\theta - \frac{\sin\theta}{r} + g(\theta) \quad \text{--- (2)}$$

From eq (1) and eq (2)
$f(r) = 0$ and $g(\theta) = 0$

∴ The harmonic conjugate of u is
$$v = r\sin\theta - \frac{\sin\theta}{r}$$

$$f(z) = \left[r\cos\theta + \frac{\cos\theta}{r}\right] + i\left[r\sin\theta - \frac{\sin\theta}{r}\right]$$

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★ **TYPE 4: Relation between u and v is given, to find f(z)**

Q1: given: $u+v = e^{-x}(\cos y + \sin y)$. Find $f(z)$

sol: differentiating wrt x

$$u_x + v_x = -e^{-x}(\cos y + \sin y) \quad \text{——— (1)}$$

differentiating wrt y

$$u_y + v_y = e^{-x}(-\sin y + \cos y)$$

By CR equations

$$u_x = v_y \text{ and } v_x = -u_y$$

$$\therefore u_x - v_x = e^{-x}(-\sin y + \cos y) \quad \text{——— (2)}$$

Adding eq (1) and (2)

$$u_x + \cancel{v_x} = -e^{-x}(\cos y + \sin y)$$
$$u_x - \cancel{v_x} = e^{-x}(-\sin y + \cos y)$$
$$2u_x = -2e^{-x}\sin y$$
$$\therefore u_x = -e^{-x}\sin y$$

subtracting eq (1) and eq (2)

$$\cancel{u_x} + v_x = -e^{-x}(\cos y + \sin y)$$
$$-\cancel{u_x} + v_x = -e^{-x}(-\sin y + \cos y)$$
$$2v_x = -2e^{-x}\cos y$$
$$\therefore v_x = -e^{-x}\cos y$$

wkt $f'(z) = u_x + iv_x$

$$f'(z) = -e^{-x}\sin y - ie^{-x}\cos y$$

Put $x = z$ and $y = 0$

$$f'(z) = -e^{-z}(0) - ie^{-z}(1)$$
$$f'(z) = -ie^{-z}$$

Integrating wrt z

$$f(z) = ie^{-z} + C$$

Q2: Given $u + v = x^3 - y^3 + 3x^2y - 3xy^2$. Find $f(z)$.

Sol: differentiating wrt x

$u_x + v_x = 3x^2 + 6xy - 3y^2$ —— ①

differentiating wrt y

$u_y + v_y = -3y^2 + 3x^2 - 6xy$

By CR equations

$u_x = v_y$ and $v_x = -u_y$

$u_x - v_x = -3y^2 + 3x^2 - 6xy$ —— ②

Adding eq ① and eq ②

$u_x + v_x = 3x^2 + 6xy - 3y^2$

$u_x - v_x = -3y^2 + 3x^2 - 6xy$

$2u_x = 6x^2 - 6y^2$

$u_x = 3x^2 - 3y^2$

Subtracting eq ① and eq ②

$u_x + v_x = 3x^2 + 6xy - 3y^2$

$-u_x + v_x = 3y^2 - 3x^2 + 6xy$

$2v_x = 12xy$

$v_x = 6xy$

wkt $f'(z) = u_x + iv_x$

$f'(z) = 3x^2 - 3y^2 + 6ixy$

Put $x = z$ and $y = 0$

$f'(z) = 3z^2$

Integrating wrt z

$f(z) = z^3 + C$

★ **Graphical representation of a complex valued function:**

To represent the complex valued function

$$w = f(z) = f(x+iy) = u(x,y) + iv(x,y)$$

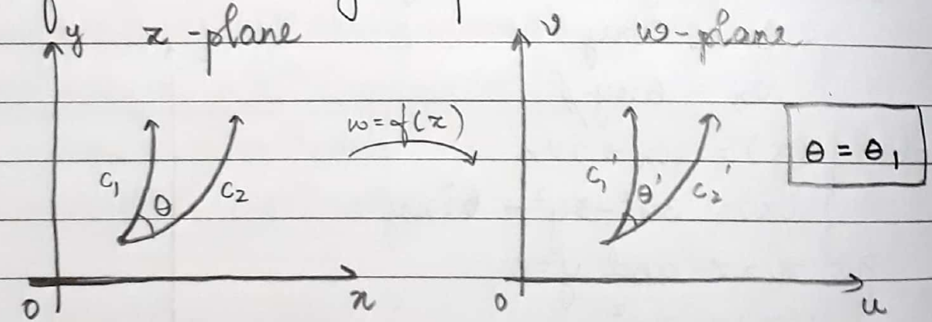
graphically, we need a 4 dimensional coordinate system which is not possible.

Instead we make use of two different planes: one for x and y called the z-plane and ~~another~~ another for u and v called the w-plane.

For each point in the z-plane there is corresponding point in the w-plane, in such a case we say that the point on the z plane is mapped on to the point on the w plane under the mapping $w = f(z)$.

— **Conformal Mapping:**

A function $w = f(z)$ is said to be conformal if under this transformation angles are preserved in magnitude and in sense of rotation between every pair of curves through a point.



Q: Discuss the transformation $w = f(z) = z^2$

Sol: $w = f(z) = z^2 = (x+iy)^2$

$$= x^2 - y^2 + 2ixy$$

$$= u(x,y) + i\,v(x,y)$$

that is: $u(x,y) = x^2 - y^2$ and $v(x,y) = 2xy$ $\Big\}$ —— 1

CASE 1: consider lines parallel to x axis in z plane.

i.e., $y = k$ where k is a constant.

Substituting in eq ①, we get.

$$u = x^2 - k^2$$

$$v = 2xk \Rightarrow x = \frac{v}{2k}$$

$$\Rightarrow u = \frac{v^2}{4k^2} - k^2$$

$$\frac{v^2}{4k^2} = u + k^2$$

$$v^2 = 4k^2(u+k^2) //$$

This represents family of parabolas having the origin of the w-plane as focus, $v=0$ as the axis and vertex at $(-k^2, 0)$.

CASE 2: consider lines parallel to y axis in z plane.

i.e., $x = c$ where c is a constant.

Substituting in eq ①, we get

$$u = c^2 - y^2$$

$$v = 2cy \Rightarrow y = \frac{v}{2c}$$

$$\Rightarrow u = c^2 - \frac{v^2}{4c^2}$$

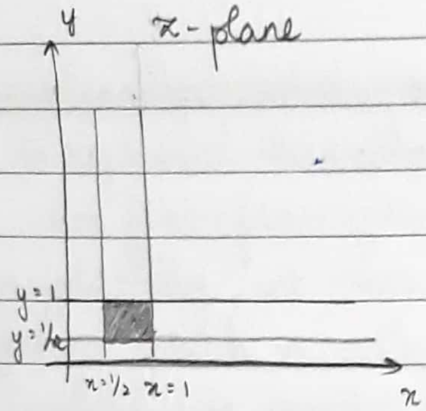
$$\frac{v^2}{4c^2} = c^2 - u$$

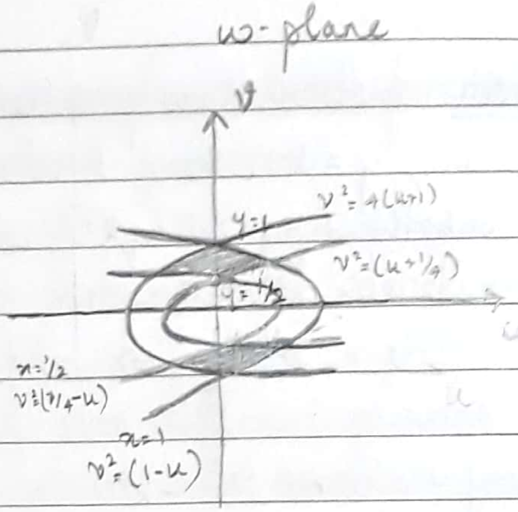
$$v^2 = 4c^2(c^2 - u) //$$

This represents family of parabolas having the origin of the w-plane as the focus, $v=0$ as the axis and vertex at $(c^2, 0)$.

For Ex: $x = 1/2, x = 1, y = 1/2, y = 1$

$\{x = c_1, x = c_2, y = k_1, y = k_2\}$





The rectangular region in the z-plane is mapped on to shaded region shown in the w plane.

Q2: Discuss the transformation : $w = z + \frac{a^2}{z}$

Sol :

$$w = f(z) = z + \frac{a^2}{z}$$

$$w = re^{i\theta} + \frac{a^2 e^{-i\theta}}{r}$$

$$w = r(\cos\theta + i\sin\theta) + \frac{a^2}{r}(\cos\theta - i\sin\theta)$$

$$w = \left(r + \frac{a^2}{r}\right)\cos\theta + i\left(r - \frac{a^2}{r}\right)\sin\theta$$

$$w = u + iv$$

i.e.,
$$\left.\begin{aligned} u &= \left(r + \frac{a^2}{r}\right)\cos\theta \\ v &= \left(r - \frac{a^2}{r}\right)\sin\theta \end{aligned}\right\} \text{ --- } (1)$$

From these equations we eliminate r and θ.

$$\frac{u}{(r + a^2/r)} = \cos\theta \quad \text{and} \quad \frac{v}{(r - a^2/r)} \quad \sin\theta$$

By squaring and adding we get

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$$\frac{u^2}{(r+a^2/r)^2} + \frac{v^2}{(r-a^2/r)} = 1 \quad \text{where } a \neq r. \quad \text{——— (2)}$$

similarly

$$\frac{u}{\cos\theta} = r + \frac{a^2}{r} \quad \text{and} \quad \frac{v}{\sin\theta} = r - \frac{a^2}{r}$$

By squaring and subtracting we get

$$\frac{u^2}{\cos^2\theta} - \frac{v^2}{\sin^2\theta} = \left(r+\frac{a^2}{r}\right)^2 - \left(r-\frac{a^2}{r}\right)$$

$$\frac{u^2}{\cos^2\theta} - \frac{v^2}{\sin^2\theta} = \cancel{r^2} + \cancel{\frac{a^4}{r^2}} + 2a^2 - \cancel{r^2} - \cancel{\frac{a^4}{r^2}} + 2a^2$$

$$\frac{u^2}{\cos^2\theta} - \frac{v^2}{\sin^2\theta} = 4a^2$$

$$\frac{u^2}{(2a\cos\theta)^2} - \frac{v^2}{(2a\sin\theta)^2} = 1 \quad \text{——— (3)}$$

when r is a constant

we have $z = re^{i\theta}$

$$|z| = r$$

$$\sqrt{x^2+y^2} = r$$

$\therefore x^2+y^2 = r^2$ represents a circle in the z plane with center as the origin and radius as r.

When θ is a constant

we have $\tan\theta = y/x$

$$\text{constant} = \frac{y}{x}$$

$\Rightarrow y = \text{constant}.x$, this represents straight line passing through the origin in the z plane.

We shall discuss the image in the w plane corresponding to r which is equal to a constant (r = constant (circle)) and θ which is equal to a constant (θ = constant (straight line)).

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CASE 1 : when $r =$ constant

From eq ②, we have

$\frac{u^2}{A^2} + \frac{v^2}{B^2} = 1$ which represents an ellipse in the w plane with foci $(\pm\sqrt{A^2-B^2}, 0)$.

where $A = r + \frac{a^2}{r}$ and $B = r - \frac{a^2}{r}$.

$$\therefore \text{foci} = \left(\pm\sqrt{r^2 + \frac{a^4}{r^2} + 2a^2 - r^2 - \frac{a^4}{r^2} + 2a^2}\,,\ 0 \right)$$

$$\text{foci} = (\pm 2a, 0)$$

CASE 2 : When $\theta =$ constant

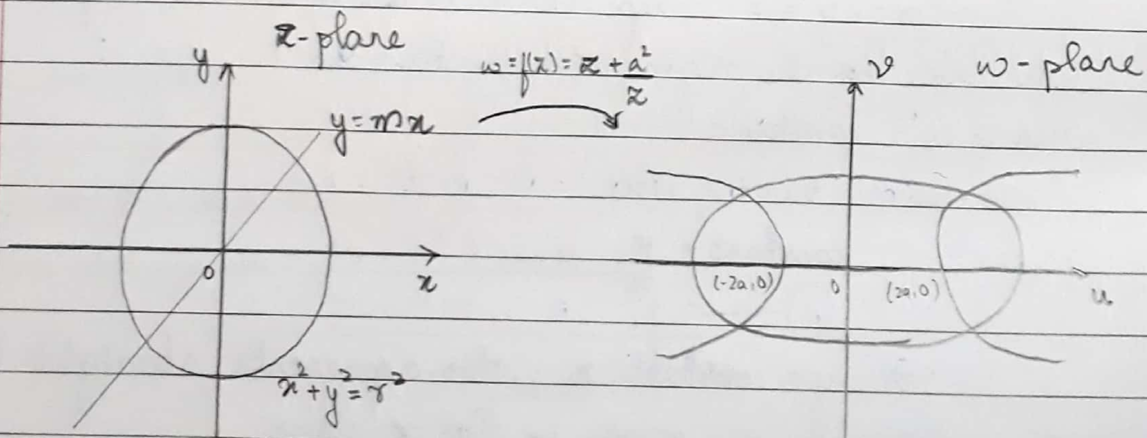
From eq ③, we have

$\frac{u^2}{A^2} - \frac{v^2}{B^2} = 1$ where $A = 2a\cos\theta$ and $B = 2a\sin\theta$

which represents an hyperbola in the w plane with foci $(\pm\sqrt{A^2+B^2}, 0)$

$$\text{foci} = (\pm\sqrt{4a^2\cos^2\theta + 4a^2\sin^2\theta}\,, 0)$$

$$\text{foci} = (\pm 2a, 0)$$



Q3: Show that the transformation $w = \frac{1}{z}$ transforms all circles and straight line in the z plane into circles or straight lines in the w plane.

Sol: consider the equation

$$a(x^2+y^2) + bx + cy + d = 0 \quad \text{——} ①$$

If $a = 0$ then eq ① is

$bx + cy + d = 0$; it represents a straight line

If $a \neq 0$ then eq ① is

$a(x^2+y^2) + bx + cy + d = 0$; it represents a circle.

consider $w = \frac{1}{z} \Rightarrow z = \frac{1}{w} = \frac{1}{u+iv}$

$$z = \frac{u-iv}{u^2+v^2}$$

$$x+iy = \frac{u}{u^2+v^2} + i\left(\frac{-v}{u^2+v^2}\right)$$

$$\therefore x = \frac{u}{u^2+v^2} \text{ and } y = \frac{-v}{u^2+v^2}$$

Substituting x and y in eq ①, we get

$$a\left[\frac{u^2}{(u^2+v^2)^2} + \frac{v^2}{(u^2+v^2)^2}\right] + b\frac{u}{u^2+v^2} - c\frac{v}{u^2+v^2} + d = 0$$

$$a\left[\frac{\cancel{u^2+v^2}}{(u^2+v^2)^{\cancel{2}}}\right] + \frac{bu-cv}{u^2+v^2} + d = 0$$

$$\frac{a}{u^2+v^2} + \frac{bu-cv}{u^2+v^2} + d = 0$$

$$\therefore d(u^2+v^2) + bu - cv + a = 0 \quad \text{——} ②$$

CASE 1: when $a = 0$ and $d = 0$

From eq ①, it represents a straight line passing through the origin in the z plane.

Eq ② represents straight line passing through the origin in the w plane.

$\therefore$ straight lines passing through the origin the z plane are mapped on to again straight lines passing through the origin in the w plane.

CASE 2: When $a = 0$ and $d \neq 0$.

Eq ① represents straight lines not passing through the origin in the z plane.

Eq ② represents circles passing through the origin in the w plane.

Therefore straight lines which are not passing through the origin in the z plane are mapped on to circles which pass through the origin in the w plane.

CASE 3: When $a \neq 0$ and $d = 0$

Eq ① represents circle passing through the origin in z plane

Eq ② represents straight line not passing through the origin in w plane.

Therefore circle passing through the origin in z plane are mapped on to straight lines not passing through the origin in w plane.

CASE 4: When $a \neq 0$ and $d \neq 0$

Eq ① represents circles not passing through the origin in z plane.

Eq ② represents circles not passing through the origin in w plane

Therefore circles not passing through the origin in z plane are mapped on to circles not passing through the origin in w plane.

UNIT - 03

Complex Variables - II

* **Bilinear Transformation : BLT**

The transformation of the type $w = \frac{az+b}{cz+d}$ where a, b, c and d are complex constants, such that $ad - bc \neq 0$ is called a bilinear transformation.

* **Invariant points : (fixed points)**

Invariant points are points such that the image of z is z itself, i.e., $f(z_0) = z_0$.

Q1: Find the BLT which maps the set of points $z = -1, i, 1$ into $w = 1, i, -1$. Also find the invariant points.

Sol: Let the required BLT be $w = \frac{az+b}{cz+d}$ —— ①

Substituting the given points in eq ① we get.

- $z = -1$; $w = 1$

$$1 = \frac{-a+b}{-c+d} \Rightarrow -a+b = -c+d \quad \text{—— ②}$$

- $z = i$; $w = i$

$$i = \frac{ai+b}{ci+d} \Rightarrow ai+b = -c+di \quad \text{—— ③}$$

- $z = 1$; $w = -1$

$$-1 = \frac{a+b}{c+d} \Rightarrow a+b = -c-d \quad \text{—— ④}$$

Solving the above equations

eq ② + eq ④

$$2b = -2c \Rightarrow \underline{\underline{b = -c}}$$

Substituting in eq ③

$ai - c = -c + id$

$\Rightarrow \underline{a = d}$

Substituting in eq ④

$a - c = -c - a$

$2a = 0 \Rightarrow \underline{a = 0} \quad \therefore \underline{d = 0}$

Substituting in eq ①

$$w = \frac{-c}{cz} = \frac{-1}{z} \quad \text{—— ⑤}$$

To find the invariant points.

we put $w = z$ in eq ⑤, we get.

$$z = \frac{-1}{z}$$

$$z^2 = -1$$

$\therefore \underline{z = \pm i}$ are the invariant points.

Q2: Find the BLT which maps the set of points $z = \infty, i, 0$ into $w = -1, -i, 1$. Also find the invariant points.

Sol: Let the required BLT be $w = \dfrac{az+b}{cz+d}$ —— ①

Substituting the given points in eq ①, we get

- $z = \infty$; $w = -1$

$$w = \frac{z(a + b/z)}{z(c + d/z)} = \frac{a + b/z}{c + d/z}.$$

$$-1 = \frac{a}{c} \Rightarrow \underline{a = -c} \quad \text{—— ②}$$

- $z = i$; $w = -i$

$$-i = \frac{ai+b}{ci+d} \Rightarrow ai + b = c - di \quad \text{—— ③}$$

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$z = 0$; $w = 1$

$1 = \frac{b}{d} \Rightarrow b = d$ — (4)

Substituting eq (2) and eq (4) in eq (3)

$ai + b = -a - bi$

$ai + a + b + bi = 0$

$i(a+b) + (a+b) = 0$

$(i+1)(a+b) = 0$

$a + b = 0$

$a = -b$ $\therefore b = c = d = -a$

Substituting in eq (1)

$w = \frac{az - a}{-az - a}$

$w = \frac{1 - z}{1 + z}$ // — (5)

to find the invariant points we put $w = z$ in eq (5)

$z = \frac{1-z}{1+z}$

$z^2 + z = 1 - z$

$z^2 + 2z - 1 = 0$ — (6)

solving eq (6)

$z = \frac{-2 \pm \sqrt{4+4}}{2} = \frac{-2 \pm 2\sqrt{2}}{2} = -1 \pm \sqrt{2}$

Q3: Find the BLT which maps the set of points $z = 1, 0, -1$ into $w = i, 1, \infty$. Also find the invariant points.

Sol: Let the required BLT be $w = \frac{az+b}{cz+d}$ — (1)

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Substituting the given points in eq ①, we get

— $z=1$; $w=-i$

$$-i = \frac{a+b}{c+d} \Rightarrow a+b = ic+id \quad — ②$$

— $z=0$; $w=1$

$$1 = \frac{b}{d} \Rightarrow b=d \quad — ③$$

— $z=-1$; $w=\infty$

$$\frac{1}{w} = \frac{cz+d}{az+b}$$

$$0 = \frac{-c+d}{-a+b} \Rightarrow c=d \quad — ④$$

Substituting eq ③ and eq ④ in eq ②

$$a+d = id+id$$

$$a+d = 2id$$

$$a = d(2i-1)$$

Substituting in eq ① we get

$$w = \frac{(2i-1)dz+d}{dz+d}$$

$$w = \frac{(2i-1)z+1}{z+1}$$ is the required BLT

— ⑤

To find the invariant points we put $w=z$ in eq ⑤

$$z = \frac{(2i-1)z+1}{z+1}$$

$$z^2+z = (2i-1)z+1$$

$$z^2+z[1-(2i-1)]-1 = 0$$

$$z^2+2z(1-i)-1 = 0 \quad — ⑥$$

solving eq ⑥, we get

$$z = \frac{-2(1-i) \pm \sqrt{4(1-i)^2+4}}{2}$$

$$z = (i-1) \pm \sqrt{(1-i)^2+1}$$

$z = (i-1) \pm \sqrt{1-2i}$ are the invariant points.

Q4: Find the BLT which maps the set of points $z = 0, i, \infty$ into $w = 0, 1/2, \infty$. Also find the invariant points.

Sol: Let the required BLT be $w = \frac{az+b}{cz+d}$ — (1)

Substituting the given points in eq (1), we get

- $z=0$; $w=0$

$$0 = \frac{b}{d} \Rightarrow b=0 \quad \text{— (2)}$$

- $z = i$; $w = 1/2$

$$\frac{1}{2} = \frac{ai+b}{ci+d} \Rightarrow 2ai+2b = ci+d \quad \text{— (3)}$$

- $z = \infty$; $w = \infty$

$$\frac{1}{w} = \frac{cz+d}{az+b} = \frac{c+d/z}{a+b/z}$$

$$0 = \frac{c}{a} \Rightarrow c=0 \quad \text{— (4)}$$

Substituting eq (2) and eq (4) in eq (3)

$$2ai = d$$

$$a = d/2i$$

Substituting in eq (1), we get

$$w = \frac{dz/2i}{d} = \frac{-iz}{2} \quad \text{— (5)}$$

To find the invariant points we put $w = z$ in eq (5)

$$z = \frac{-iz}{2} \Rightarrow z + \frac{iz}{2} = 0 \Rightarrow z = 0$$

Keerthan

$z = 1, i, -1$ into $w = i, 0, -i$

Q5: Find the image of $|z| < 1$ under this transformation.

Sol: Let the required BLT be $w = \frac{az+b}{cz+d}$ — (1)

Substituting the given values in eq (1)

$z = 1$; $w = i$

$i = \frac{a+b}{c+d} \Rightarrow a+b = ci+di$ — (2)

$z = i$; $w = 0$

$0 = \frac{ai+b}{ci+d} \Rightarrow ai+b = 0 \Rightarrow b = -ai$ — (3)

$\Rightarrow a = bi$

$z = -1$; $w = -i$

$-i = \frac{-a+b}{-c+d} \Rightarrow -a+b = ci-di$ — (4)

Adding eq (2) and eq (4)

$a+b = ci+di$

$-a+b = ci-di$

$2b = 2ci$

$b = ci$

Substituting in eq (2)

$bi+b = b+di$

$bi = di$

$b = d$

Substituting in eq (1) we get

$w = \frac{biz+b}{-ibz+b} = \frac{1+iz}{1-iz}$ — (5)

Image of $|z| < 1$ under the transformation $w = \frac{1+iz}{1-iz}$

we have $w = \frac{1+iz}{1-iz}$

$w - iwz = 1+iz$

$iz(1+w) = w-1$

$z = \frac{w-1}{i(1+w)} = \frac{-i(w-1)}{(1+w)} = \frac{i(1-w)}{1+w}$

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Now considering $|z| < 1$

$\therefore \left| \dfrac{i(1-w)}{1+w} \right| < 1$

$|i| \, |(1-w)| < |(1+w)|$

Squaring on both sides

$|(1-w)|^2 < |(1+w)|^2$

$|(1-(u+iv))|^2 < |(1+(u+iv))|^2$

$|(1-u)-iv|^2 < |(1+u)+iv|^2$

$(1-u)^2 + (-v)^2 < (1+u)^2 + (v)^2$

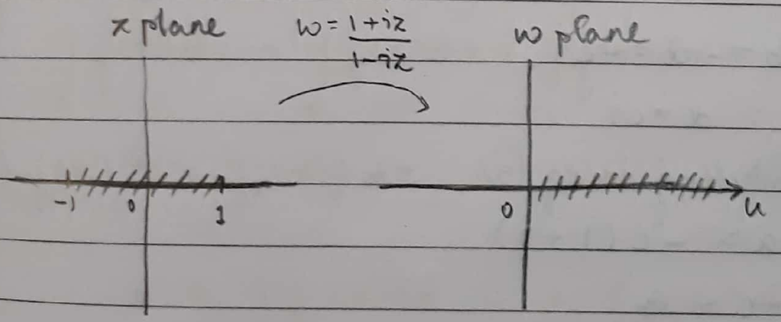
$\cancel{1} - 2u + \cancel{u^2} + \cancel{v^2} < \cancel{1} + 2u + \cancel{u^2} + \cancel{v^2}$

$-2u < 2u$

$0 < 4u \Rightarrow \underline{\underline{u > 0}}$

$z = x+iy$
$|z| = \sqrt{x^2+y^2}$
$|z|^2 = x^2 + y^2$

Therefore $|z| < 1$ in the z plane is mapped on to the positive u axis of the w plane under the transformation $w = \dfrac{1+iz}{1-iz}$



Q: Find the BLT whose fixed points are 1 and i. and which maps 0 to -1.

Sol: Let the required BLT be $w = \dfrac{az+b}{cz+d}$ —— (1)

The BLT has fixed points, hence we have (put $w = z$)

$z = \dfrac{az+b}{cz+d}$ —— (2)

$cz^2 + dz = az + b$

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$$cz^2 + z(d-a) - b = 0$$

$$z^2 + \frac{d-a}{c} z^2 - \frac{b}{c} = 0 \quad \text{——— (3)}$$

Since 1 and i are the fixed points we have.

$$(z-1)(z-i) = 0$$

$$z^2 - iz - z + i = 0$$

$$z^2 - z(1+i) + i = 0 \quad \text{——— (4)}$$

comparing eq (3) and eq (4), we have.

$$\frac{d-a}{c} = -(1+i) \qquad \frac{-b}{c} = i$$

$$d - a = -c(1+i) \quad \text{——— (5)} \qquad \Rightarrow b = -ci \Rightarrow c = ib$$

Since 0 is mapped on to -1, we have from eq (1)

$$-1 = \frac{b}{d} \Rightarrow b = -d$$

$$\therefore b = -d = -ci$$

$$\therefore d = ci$$

substituting in eq (5)

$$ci - a = -c(1+i)$$

$$2ci + c = a$$

$$c(2i+1) = a$$

substituting in eq (1)

$$w = \frac{(2i+1)cz - ci}{cz + ci}$$

$$w = \frac{(2i+1)z - i}{z + i} \quad \text{is the required BLT.}$$

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★ Complex line integrals:

Consider a complex valued function $w = f(z)$ where $z = x + iy$ defined at all points of a curve 'c', then

$\int_c f(z)\,dz$ is called the complex line integral of $f(z)$ along the path c.

$$\int_c f(z)\,dz = \int_c (u+iv)(dx+idy) = \int_c (u\,dx - v\,dy) + i\int_c (v\,dx + u\,dy)$$

Therefore evaluation of a line integral of a complex valued function is nothing but the evaluation of line integrals of real valued functions.

Properties:

1. $-\int_c f(z)\,dz = \int_{-c} f(z)\,dz$ where $-c$ is the curve traversed in the opposite direction.

2. If the curve c is composed of finite number of curves say $c_1, c_2, c_3, \ldots, c_n$, then
$$\int_c f(z)\,dz = \int_{c_1} f(z)\,dz + \int_{c_2} f(z)\,dz + \cdots + \int_{c_n} f(z)\,dz.$$

3. $\int_c (\lambda_1 f_1(z) \pm \lambda_2 f_2(z))\,dz$ where λ_1 and λ_2 are constants
$$= \lambda_1 \int_c f_1(z)\,dz \pm \lambda_2 \int_c f_2(z)\,dz$$

Q1: Evaluate $\int_c z^2\,dz$ where c is the line joining the points $z = 0$ and $z = 3 + i$.

Sol: Given: $z^2\,dz$

$(x, y) \longleftrightarrow x + iy$

$$z^2\,dz = (x+iy)^2(dx+idy)$$
$$= (x^2 - y^2 + 2ixy)(dx + idy) \quad \text{—— (1)}$$

Here

$z = 0 : (0,0)$

$z = 3+i : (3,1)$

Here curve c is the straight line joining the point $\underset{x_1,\ y_1}{(0,0)}$ and $\underset{x_2\ y_2}{(3,1)}$

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Equation of the straight line joining two points (x_1, y_1) and (x_2, y_2) is given by

$$\frac{y-y_1}{x-x_1} = \frac{y_2-y_1}{x_2-x_1}$$

i.e., $\frac{y-0}{x-0} = \frac{1-0}{3-0}$

$\therefore \underline{x = 3y} \Rightarrow \underline{dx = 3dy}$

Substituting in eq ①, we get

$\therefore z^2dz = [(3y)^2 - y^2 + 2i(3y)y][3dy + idy]$

$z^2dz = [9y^2 - y^2 + 6iy^2][3dy + idy]$

$= (8y^2 + 6iy^2)(3+i)dy$

$\therefore \int_c z^2dz = \int_{y=0}^{1} [(8y^2+6iy^2)(3+i)]\,dy$

$= (3+i)\int_{y=0}^{1} (8y^2+6iy^2)\,dy$

$= (3+i)\left[\frac{8y^3}{3} + \frac{\cancel{6}^{2}iy^3}{\cancel{3}}\right]_0^1$

$= (3+i)\left[\frac{8}{3} + 2i - 0 - 0\right]$

$= (3+i)\left(\frac{8}{3} + 2i\right)$

$= 8 + 6i + \frac{8i}{3} - 2$

$\therefore \int_c z^2dz = 6 + \frac{26i}{3}$

Q2: Evaluate $\int_c (\bar{z})^2 dz$ where c is the circle

i. $c: |z| = 1$ ii. $c: |z-1| = 1$

sol: CASE 1: consider $c : |z| = 1$.

This is a circle with the centre at origin and radius is 1.

$z = re^{i\theta} \Rightarrow \bar{z} = - e^{-i\theta}$

$z = e^{i\theta}$ because $r = 1$

$dz = i e^{i\theta} d\theta$

$(\bar{z})^2 = e^{-2i\theta}$

$z^2 dz = e^{-2i\theta} (i e^{i\theta} d\theta)$

$$\int_c (\bar{z})^2 dz = i \int_{\theta=0}^{2\pi} e^{-i\theta} d\theta$$

$$= i \left. \frac{e^{-i\theta}}{-i} \right|_0^{2\pi}$$

$$= -(e^{-2\pi i} - 1)$$

$$= -(\cos 2\pi - i \sin 2\pi - 1)$$

$$= -(1 - i(0) - 1)$$

$$\int_c (\bar{z})^2 dz = 0$$

CASE 2: consider $c : |z-1| = 1$

This is a circle with centre at 1 and radius 1.

$z - 1 = re^{i\theta}$ $\qquad \bar{z} = 1 + e^{-i\theta}$

$z = 1 + re^{i\theta}$ $\qquad (\bar{z})^2 dz = (1+e^{-i\theta})^2 (ie^{i\theta}) d\theta$

$dz = ie^{i\theta} d\theta$ $\qquad = (1 + e^{-2i\theta} + 2e^{-i\theta})(ie^{i\theta}) d\theta$

$\qquad = (ie^{i\theta} + ie^{-i\theta} + 2i) d\theta$

$$\therefore \int_c (\bar{z})^2 dz = i \int_{\theta=0}^{2\pi} (e^{i\theta} + e^{-i\theta} + 2) d\theta$$

$$= i \int_{\theta=0}^{2\pi} (\cos\theta + \cancel{i\sin\theta} + \cos\theta - \cancel{i\sin\theta} + 2) d\theta$$

$$= 2i \left[\sin\theta + \theta \right|_0^{2\pi}$$

$$= 2i [0 + 2\pi - 0 - 0]$$

$$\int_c (\bar{z})^2 dz = 4\pi i$$

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Q3: Show that

$$\int_c (z-a)^n dz = \begin{cases} 0 & ; n \neq -1 \\ 2\pi i & ; n = -1 \end{cases}$$

where $c = |z-a| = r$

<u>Sol:</u> Here $c: |z-a| = r$

which is a circle with center at a and radius is r

$$z - a = re^{i\theta}$$

$$z = re^{i\theta} + a$$

$$dz = rie^{i\theta} d\theta$$

— considering the case when $n \neq -1$.

$$\int_c (z-a)^n dz = \int_{\theta=0}^{2\pi} (re^{i\theta})^n r i e^{i\theta} d\theta$$

$$= r^{n+1} i \int_0^{2\pi} e^{(n+1)i\theta} d\theta$$

$$= \cancel{i} r^{n+1} \left. \frac{e^{(n+1)i\theta}}{(n+1)\cancel{i}} \right|_0^{2\pi}$$

$$= \frac{r^{n+1}}{n+1} \left[e^{(n+1)i2\pi} - e^0 \right]$$

$$= \frac{r^{n+1}}{n+1} \left[\cos(2n\pi + 2\pi) + i\sin(2n\pi + 2\pi) - 1 \right]$$

$$= \frac{r^{n+1}}{n+1} \left[\cos 2n\pi \cos 2\pi - \sin 2n\pi \sin 2\pi + i \sin 2n\pi \cos 2\pi + i \cos 2n\pi \sin 2\pi - 1 \right]$$

$$\int_c (z-a)^n dz = \frac{r^{n+1}}{n+1} [1 - 1] = 0$$

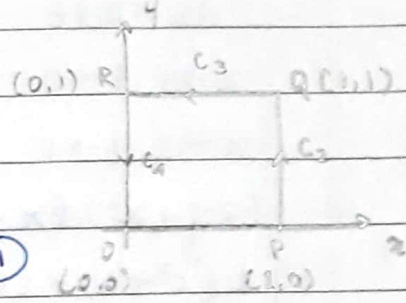
— considering the case when $n = -1$.

$$\int_c (z-a)^{-1} dz = \int \frac{dz}{z-a} = \int_{\theta=0}^{2\pi} \frac{\cancel{r} i e^{i\theta} d\theta}{\cancel{r} e^{i\theta}}$$

$$= i\theta \Big|_0^{2\pi} = i(2\pi - 0)$$

$$\int_c (z-a)^{-1} dz = 2\pi i$$

Q4: Evaluate $\int |z|^2 dz$ where c is a square with radius (0,0), (1,0), (1,1), (0,1).



Sol: $\int_c |z|^2 dz = \int_{c_1} |z|^2 dz + \int_{c_2} |z|^2 dz$

$+ \int_{c_3} |z|^2 dz + \int_{c_4} |z|^2 dz$ —— ①

$z = x + iy \qquad dz = dx + idy$

$|z|^2 = x^2 + y^2$

Along OP : C_1 : $y = 0 \Rightarrow dy = 0$

$\Rightarrow |z|^2 dz = x^2 dx$ where $0 \le x \le 1$

Along PQ : C_2 : $x = 1 \Rightarrow dx = 0$

$\Rightarrow |z|^2 dz = (1+y^2) i dy$ where $0 \le y \le 1$

Along QR : C_3 : $y = 1 \Rightarrow dy = 0$

$\Rightarrow |z^2| dz = (x^2+1) dx$ where $1 \le x \le 0$

Along RO : C_4 : $x = 0 \Rightarrow dx = 0$

$\Rightarrow |z|^2 dz = y^2 i dy$ where $1 \le y \le 0$

Substituting in eq ①

$\int_c |z|^2 dz = \int_0^1 x^2 dx + \int_0^1 (1+y^2) i dy + \int_1^0 (x^2+1) dx + \int_1^0 y^2 i dy$

$= \frac{x^3}{3}\Big|_0^1 + i\left(y + \frac{y^3}{3}\Big|_0^1\right) + \left(\frac{x^3}{3} + x\Big|_1^0\right) + i\frac{y^3}{3}\Big|_1^0$

$= \cancel{\frac{1}{3}} + i + \cancel{\frac{i}{3}} - \cancel{\frac{1}{3}} - 1 - \cancel{\frac{i}{3}}$

$\therefore \int_c |z|^2 dz = -1 + i$

Q5: Evaluate $\int_{(0,3)}^{(2,4)} [(2y + x^2)dx + (3x - y)dy]$

i. along parabola $x = 2t$, $y = t^2 + 3$

ii. along straight line from (0,3) to (2,4)

Sol: i) c : parabola

$x = 2t$ and $y = t^2 + 3$

$dx = 2dt \qquad dy = 2t\,dt$

$x \to 0 \quad t \to 0 \qquad y \to 3 \quad t \to 0$

$x \to 2 \quad t \to 1 \qquad y \to 4 \quad t \to 1$

$(2y + x^2)dx + (3x - y)dy$

$= \left[2(t^2+3) + (2t)^2\right]2\,dt + \left[3(2t) - (t^2+3)\right]2t\,dt$

$= \left[4t^2 + 12 + 8t^2\right]dt + \left[12t^2 - 2t^3 - 6t\right]dt$

$= \left[24t^2 - 2t^3 - 6t + 12\right]dt$

$$\therefore \int_{(0,3)}^{(2,4)} \left[(2y + x^2)dx + (3x - y)dy\right]$$

$$= \int_0^1 (24t^2 - 2t^3 - 6t + 12)\,dt$$

$$= \left(\frac{\cancel{24}^{8}t^3}{\cancel{3}} - \frac{\cancel{2}t^4}{\cancel{4}_2} - \frac{\cancel{6}^{3}t^2}{\cancel{2}} + 12t\right)\Bigg|_0^1$$

$$= 8 - \frac{1}{2} - 3 + 12 = \frac{16 - 1 - 6 + 24}{2} = \frac{33}{2}$$

ii) Equation of straight line joining points $(0,3)$ to $(2,4)$

$$y - y_1 = \left(\frac{y_2 - y_1}{x_2 - x_1}\right)(x - x_1)$$

$$y - 3 = \frac{4-3}{2-0}(x - 0)$$

$$2y - 6 = x \Rightarrow y = \frac{x}{2} + 3$$

$$\therefore dy = dx/2$$

$(2y + x^2)dx + (3x - y)dy$

$= \left[2\left(\frac{x}{2} + 3\right) + x^2\right]dx + \left[3x - \left(\frac{x}{2} + 3\right)\right]dy$

$= \left[x^2 + x + 6\right]dx + \left[\frac{5x}{2} - 3\right]\frac{dx}{2}$

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$$= \left[x^2 + x + 6 + \frac{5x}{4} - \frac{3}{2}\right] dx$$

$$= \left[x^2 + \frac{9x}{4} + \frac{9}{2}\right] dx$$

$$\therefore \int_{(0,3)}^{(2,4)} \left[(2y + x^2)dx + (3x - y)dy\right]$$

$$= \int_0^2 \left(x^2 + \frac{9x}{4} + \frac{9}{2}\right) dx$$

$$= \left(\frac{x^3}{3} + \frac{9x^2}{8} + \frac{9x}{2}\right)\Big|_0^2 = \frac{8}{3} + \frac{\cancel{36}^{9}}{\cancel{8}_{2}} + \frac{\cancel{18}^{9}}{\cancel{2}}$$

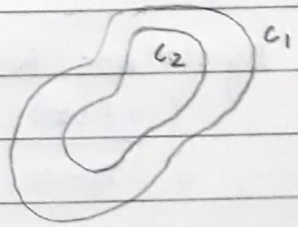
$$= \frac{16 + 27 + 54}{6} = \frac{97}{6}$$

* **Cauchy's theorem:**

If $f(z)$ is analytic inside and along the boundary of a simple closed curve c, then $\int_c f(z)\,dz = 0$.

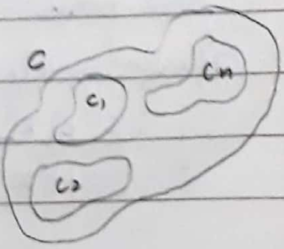
- **Consequences of Cauchy's theorem:**

1. If c_1, c_2 are two simple closed curves such that c_2 lies entirely within c_1 and if $f(z)$ is analytic on c_1, c_2 and in the region bounded by c_1, c_2 then



$$\int_{c_1} f(z)\,dz = \int_{c_2} f(z)\,dz$$

2. If c is a simple closed curve enclosing non overlapping simple closed curves $c_1, c_2, c_3, \dots c_n$ and if $f(z)$ is analytic in the annular region between c and these curves, then



$$\int_c f(z)\,dz = \int_{c_1} f(z)\,dz + \int_{c_2} f(z)\,dz + \dots + \int_{c_n} f(z)\,dz$$

★ Cauchy's Integral Formula: (CIF)

If $f(z)$ is analytic inside and on a simple closed curve c and if 'a' is any point within c, then

$$\boxed{f(a) = \frac{1}{2\pi i}\int_c \frac{f(z)}{z-a}\,dz}$$

Generalised Cauchy's Integral Formula

If $f(z)$ is analytic inside and on a simple closed curve c and if 'a' is any point within c, then

$$\boxed{f^{n}(a) = \frac{n!}{2\pi i}\int_c \frac{f(z)}{(z-a)^{n+1}}\,dz}$$

Problems:

Here we evaluate integrals of the type

$$\int_c \frac{f(z)}{z-a}\,dz \text{ and } \int_c \frac{f(z)}{(z-a)^{n+1}}\,dz$$

Using CIF evaluate the following:

Q1. $\int_c \frac{e^z dz}{(z-i\pi)}$ where c is

i. $|z| = 2\pi$ ii. $|z| = \pi/2$

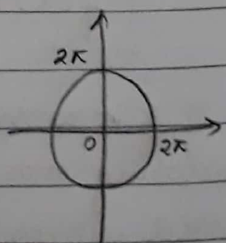
Sol: By CIF we have

$$\int_c \frac{f(z)}{z-a}\,dz = 2\pi i f(a)$$

Here $f(z) = e^z$ and $a = i\pi \Rightarrow (0, \pi)$

i. considering $|z| = 2\pi$

Here $a = i\pi$, which corresponds to the point $(0,\pi)$ which lies within c.



Therefore by CIF

$$\int_c \frac{e^z dz}{(z-i\pi)} = 2\pi i f(a = i\pi)$$

$$= 2\pi i\, e^{i\pi} = 2\pi i(\cos\pi + i\sin\pi)$$

$$= -2\pi i$$

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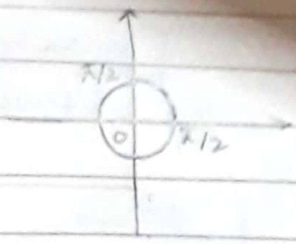
ii. considering $|z| = \pi/2$



here $a = i\pi$ which corresponds to the point $(0,\pi)$ which does not lie within c, i.e., the point $a = i\pi$ lies outside the given curve c.

Hence by Cauchy's theorem

$$\int_c \frac{e^z}{(z-i\pi)}\,dz = 0$$

Q2: $$\int_c \frac{\sin\pi z^2 + \cos\pi z^2}{(z-1)(z-2)}\,dz \quad \text{where } c: |z| = 3$$

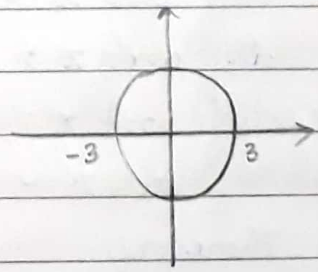
sol: consider

$$\frac{1}{(z-1)(z-2)} = \frac{A}{(z-1)} + \frac{B}{(z-2)}$$

$$1 = A(z-2) + B(z-1)$$

at $z = 2$ $\quad \underline{B = 1}$

at $z = 1$ $\quad \underline{A = -1}$



Hence

$$\therefore \int_c \frac{\sin\pi z^2 + \cos\pi z^2}{(z-1)(z-2)}\,dz$$

$$= -\int_c \frac{\sin\pi z^2 + \cos\pi z^2}{(z-1)}\,dz + \int_c \frac{\sin\pi z^2 + \cos\pi z^2}{(z-2)}\,dz \quad \text{--- (1)}$$

Here $a = 1$ lies within curve c $\qquad$ Here $a = 2$ lies within curve c.

Therefore by CIF, from eq (1), we have

$$\int_c \frac{\sin\pi z^2 + \cos\pi z^2}{(z-1)(z-2)}\,dz = -2\pi i f(1) + 2\pi i f(2)$$

Here $f(z) = \sin\pi z^2 + \cos\pi z^2$

$\Rightarrow f(1) = \sin\pi + \cos\pi = -1$

$\Rightarrow f(2) = \sin 4\pi + \cos 4\pi = 1$

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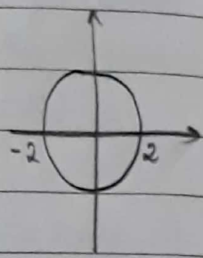
$$\therefore \int_c \frac{\sin \pi z^2 + \cos \pi z^2}{(z-1)(z-2)} dz = -2\pi i(-1) + 2\pi i(1) = 4\pi i$$

Q3: $\displaystyle\int_c \frac{e^{2z} dz}{(z+1)^4}$ where $c: |z| = 2$

Sol: By generalised CIF we have

$$\int_c \frac{f(z)}{(z-a)^{n+1}} dz = \frac{2\pi i}{n!} f^n(a)$$



Here $f(z) = e^{2z}$ and $a = -1$

Therefore $a = -1$ lies within c.

$$\therefore \int_c \frac{e^{2z}}{(z+1)^4} dz = \frac{2\pi i}{3!} f'''(-1)$$

Here $f(z) = e^{2z}$

diff wrt z : $f'(z) = 2e^{2z}$

diff wrt z : $f''(z) = 4e^{2z}$

diff wrt z : $f'''(z) = 8e^{2z}$

at $z = -1$: $f'''(-1) = 8e^{-2}$

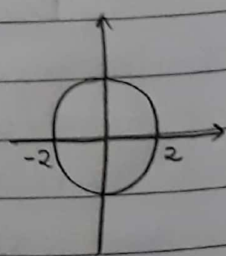
Therefore

$$\int_c \frac{e^{2z} dz}{(z+1)^4} = \frac{2\pi i}{3!}(8e^{-2}) = \frac{8\pi i e^{-2}}{3}$$

Q4: $\displaystyle\int_c \frac{z^3 - 2z + 1}{(z-i)^2} dz$ $\quad c: |z| = 2$

Sol: By generalised CIF we have

$$\int_c \frac{f(z)}{(z-a)^{n+1}} dz = \frac{2\pi i}{n!} f^n(a)$$



Here $f(z) = z^3 - 2z + 1$ and $a = i$

Therefore $a = i$ lies within c

$$\therefore \int_c \frac{z^3 - 2z + 1}{(z-i)^2} = \frac{2\pi i}{1!} f'(i)$$

Here $f(z) = z^3 - 2z + 1$

diff wrt z : $f'(z) = 3z^2 - 2$

at $z = i$: $f'(i) = -3 - 2 = -5$

Therefore

$$\int_c \frac{z^3 - 2z + 1}{(z-i)^2} \, dz = 2\pi i(-5) = -10\pi i$$

* **Poles and Residues:**

- Poles:

A point $z = a$ where $f(z)$ seizes to be analytic is called a singular point of $f(z)$ or pole of $f(z)$.

Singular points or poles of $f(z)$ are those points which reduce the denominator factors to zero or they are the roots of $\frac{1}{f(z)} = 0$.

Ex: $f(z) = \frac{z}{(z-1)(z+2)^2}$

Here $z = 1$ is a pole of order 1 i.e., we say the pole $z = 1$ is a single pole. Also $z = -2$ is a pole of order 2.

- Residues:

Consider the Lorentz series of $f(z)$

$$f(z) = \underbrace{\sum_{n=0}^{\infty} a_n (z-a)^n}_{\text{analytic part}} + \underbrace{\sum_{n=1}^{\infty} a_{-n} (z-a)^{-n}}_{\text{principle part}} \quad \text{(SLF)}$$

The coefficient of $\frac{1}{(z-a)}$ (i.e., a_{-1}) in the Lorentz series expansion of $f(z)$ is called the residue of $f(z)$ at the pole $z = a$

* **Cauchy's residue theorem (CRT):**

If $f(z)$ is analytic inside and on the boundary of a simple closed curve c except for a finite number of poles $a, b, c \dots$ then we have:

$$\int_c f(z) \, dz = 2\pi i (a_{-1} + b_{-1} + c_{-1} + \dots\dots)$$

Formula for the residue at the pole

If $z=a$ is a pole of order m of $f(z)$, then the residue of $f(z)$ at $z=a$ is denoted by $R[m,a]$ and is given by

$$\boxed{R[m,a] = \frac{1}{(m-1)!} \lim_{z\to a} \frac{d^{m-1}}{dz^{m-1}}\left[(z-a)^m f(z)\right]}$$

In particular when $m=1$, i.e., in case of a simple pole

$$\boxed{R[1,a] = a_{-1} = \lim_{z\to a}\left[(z-a)f(z)\right]}$$

is the residue of $f(z)$ at $z=a$ when a is a simple pole.

Determine the poles and residues at the poles for the following functions.

Q1: $f(z) = \frac{2z+1}{z^2-z-2}$

Sol: $f(z) = \frac{2z+1}{z^2-z-2} = \frac{2z+1}{(z-2)(z+1)}$

$z=2$ and $z=-1$ are simple poles $(m=1)$ of $f(z)$

- residue at the simple pole $z=a=2$.

$$R[1,2] = \lim_{z\to 2}\left[(z-2)f(z)\right]$$

$$= \lim_{z\to 2}\left[\cancel{(z-2)}\left(\frac{2z+1}{\cancel{(z-2)}(z+1)}\right)\right]$$

$$= \lim_{z\to 2}\frac{2z+1}{z+1} = \frac{2(2)+1}{2+1} = \frac{5}{3}$$

- residue at the simple pole $z=a=-1$

$$R[1,-1] = \lim_{z\to -1}\left[(z+1)f(z)\right]$$

$$= \lim_{z\to -1}\left[\cancel{(z+1)}\frac{2z+1}{(z-2)\cancel{(z+1)}}\right]$$

$$= \lim_{z\to -1}\frac{2z+1}{z-2} = \frac{2(-1)+1}{-1-2} = \frac{-1}{-3} = \frac{1}{3}$$

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$$f(z) = \frac{z^2}{z^4-1}$$

$$f(z) = \frac{z^2}{z^4-1} = \frac{z^2}{(z^2-1)(z^2+1)} = \frac{z^2}{(z-1)(z+1)(z-i)(z+i)}$$

$z = 1, -1, i$ and $-i$ are simple poles of $f(z)$

- residue at the simple pole $z = a = 1$

$$R[1,1] = \lim_{z\to 1}\left[(z-1)f(z)\right]$$

$$= \lim_{z\to 1}\left[\cancel{(z-1)}\frac{z^2}{\cancel{(z-1)}(z+1)(z-i)(z+i)}\right]$$

$$= \lim_{z\to 1}\left[\frac{z^2}{(z+1)(z-i)(z+i)}\right] = \frac{1}{(1+1)(1-i)(1+i)}$$

$$= \frac{1}{2(1+1)} = \frac{1}{4}$$

- residue at the simple pole $z = a = -1$

$$R[1,-1] = \lim_{z\to -1}\left[(z+1)f(z)\right]$$

$$= \lim_{z\to -1}\left[\cancel{(z+1)}\frac{z^2}{(z-1)\cancel{(z+1)}(z-i)(z+i)}\right]$$

$$= \lim_{z\to -1}\left[\frac{z^2}{(z-1)(z-i)(z+i)}\right] = \frac{1}{(-1-1)(-1-i)(-1+i)}$$

$$= \frac{1}{(-2)(1+1)} = \frac{-1}{4}$$

- residue at the simple pole $z = a = i$

$$R[1,i] = \lim_{z\to i}\left[(z-i)f(z)\right]$$

$$= \lim_{z\to i}\left[\cancel{(z-i)}\frac{z^2}{(z-1)(z+1)\cancel{(z-i)}(z+i)}\right]$$

$$= \lim_{z\to i}\left[\frac{z^2}{(z-1)(z+1)(z+i)}\right] = \frac{-1}{(i-1)(i+1)(i+i)}$$

$$= \frac{-1}{(-1-1)(2i)} = \frac{1}{4i} = \frac{-i}{4}$$

- residue at the simple pole $z=a=-i$

$$R[1,-i] = \lim_{z\to -i}\left[(z+i)\,f(z)\right]$$

$$= \lim_{z\to -i}\left[\cancel{(z+i)}\frac{z^2}{(z-1)(z+1)(z-i)\cancel{(z+i)}}\right]$$

$$= \lim_{z\to -i}\left[\frac{z^2}{(z-1)(z+1)(z-i)}\right] = \frac{-1}{(-i-1)(-i+1)(-i-i)}$$

$$= \frac{-1}{(-1-1)(-2i)} = \frac{-1}{4i} = \frac{i}{4}$$

Q3: $f(z) = \dfrac{z+4}{(z-1)^2(z-2)^3}$

Sol: The poles of $f(z)$ are 1 and 2.

$z=1$ is a pole of order 2.

$z=2$ is a pole of order 3.

- Residue at the pole $z=a=1$ of order 2.

$$R[2,1] = \frac{1}{(2-1)!}\lim_{z\to 1}\frac{d^{2-1}}{dz^{2-1}}\left[(z-1)^{2}f(z)\right]$$

$$= \lim_{z\to 1}\frac{d}{dz}\left[\cancel{(z-1)^2}\frac{z+4}{\cancel{(z-1)^2}(z-2)^3}\right]$$

$$= \lim_{z\to 1}\frac{d}{dz}\left[\frac{z+4}{(z-2)^3}\right]$$

$$= \lim_{z\to 1}\left[\frac{(z-2)^3(1)-(z+4)3(z-2)^2}{(z-2)^6}\right]$$

$$= \lim_{z\to 1}\left[\frac{z-2-3z-12}{(z-2)^4}\right]$$

$$= \lim_{z\to 1}\left[\frac{-2z-14}{(z-2)^4}\right] = \frac{-2-14}{(1-2)^4} = -16$$

- Residue at the pole $z = a = 2$ of order 3

$$R[3,2] = \frac{1}{(3-1)!} \lim_{z\to 2} \frac{d^{3-1}}{dz^{3-1}}\left[(z-2)^3 f(z)\right]$$

$$= \frac{1}{2!} \lim_{z\to 2} \frac{d^2}{dz^2}\left[\cancel{(z-2)^3}\, \frac{z+4}{(z-1)^2\cancel{(z-2)^3}}\right]$$

$$= \frac{1}{2} \lim_{z\to 2} \frac{d^2}{dz^2}\left[\frac{z+4}{(z-1)^2}\right]$$

$$= \frac{1}{2} \lim_{z\to 2} \frac{d}{dz}\left[\frac{(z-1)^2(1)-(z+4)2(z-1)}{(z-1)^4}\right]$$

$$= \frac{1}{2} \lim_{z\to 2} \frac{d}{dz}\left[\frac{z-1-2z-8}{(z-1)^3}\right]$$

$$= \frac{1}{2} \lim_{z\to 2} \frac{d}{dz}\left[\frac{-z-9}{(z-1)^3}\right]$$

$$= \frac{1}{2} \lim_{z\to 2} \left[\frac{(z-1)^3(-1)-(-z-9)3(z-1)^2}{(z-1)^6}\right]$$

$$= \frac{1}{2} \lim_{z\to 2} \left[\frac{-z+1+3z+27}{(z-1)^4}\right]$$

$$= \frac{1}{2} \lim_{z\to 2} \left[\frac{2z+28}{(z-1)^4}\right] = \frac{1}{2}\left[\frac{4+28}{(2-1)^4}\right] = \frac{32}{2} = 16$$

Use Cauchy's residue theorem to evaluate the following:

Q: $\int_c \frac{z-1}{(z+1)^2(z-2)}$ where $c : |z-i| = 2$

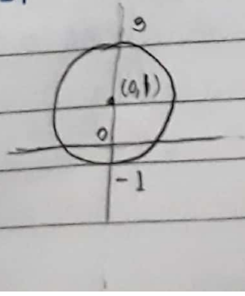
Sol: Here, $f(z) = \frac{z-1}{(z+1)^2(z-2)}$ the poles of $f(z)$ are $z = -1, 2$ of order 2 and 1 respectively.

By Cauchy's residue theorem, we have:

$$\int f(z)\,dz = 2\pi i\,(a_{-1} + b_{-1} + c_{-1} + \dots)$$

- Residues at the poles

$|z-i| = 2$ is a circle with $a = i$ centre ~~i~~ and radius 2.
$(0,1)$



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The equation of the circle is

$$x^2+(y-1)^2=4$$ ~~[illegible]~~

when $y=0$: $x^2+1=4$

$x^2=3$

$x=\pm\sqrt{3}$

The pole $z=a=-1$ lies within c

$$\therefore R[2,-1]=\frac{1}{(2-1)!}\lim_{z\to-1}\frac{d}{dz}\left[\cancel{(z+1)^2}\frac{z-1}{\cancel{(z+1)^2}(z-2)}\right]$$

$$=\lim_{z\to-1}\left[\frac{(z-2)(1)-(z-1)(1)}{(z-2)^2}\right]$$

$$=\lim_{z\to-1}\left[\frac{\cancel{z}-2-\cancel{z}+1}{(z-2)^2}\right]$$

$$=\lim_{z\to-1}\left[\frac{-1}{(z-2)^2}\right]=\frac{-1}{(-1-2)^2}=\frac{-1}{9}$$

The pole $z=a=2$ lies outside $c:|z-i|=2$

$\therefore R[1,2]=0$.

$$\therefore\int_c\frac{z-1}{(z+1)^2(z-2)}=2\pi i\left(\frac{-1}{9}+0\right)=\frac{-2\pi i}{9}$$

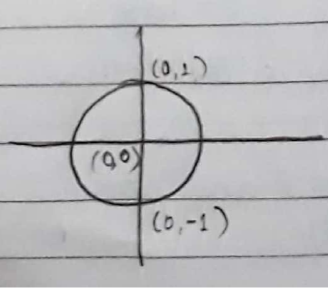
Q2: $\displaystyle\int_c\frac{\sin^6 z}{(z-\pi/6)^3}dz$ where $c:|z|=1$

Sol: By CRT, we have

$$\int_c f(z)dz=2\pi i(a_{-1}+b_{-1}+c_{-1}+\dots) \quad \text{—— (1)}$$

Here $f(z)=\dfrac{\sin^6 z}{(z-\pi/6)^3}$. the pole of $f(z)$ is $z=\pi/6$ of order 3.

Here $|z|=1$ is a circle with centre $(0,0)$ and of radius 1.



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Residue at the pole

The pole $z = a = \pi/6$ lies within the circle: $|z| = 1$

Hence $R[3, \pi/6] = \frac{1}{(3-1)!} \lim_{z \to \pi/6} \frac{d^2}{dz^2}\left[\cancel{(z-\pi/6)^3} \frac{\sin^6 z}{\cancel{(z-\pi/6)^3}}\right]$

$= \frac{1}{2!} \lim_{z\to\pi/6} \frac{d^2}{dz^2}\left[\sin^6 z\right]$

$= \frac{1}{2} \lim_{z\to\pi/6} \frac{d}{dz}\left[6\sin^5 z \cos z\right]$

$= \frac{3}{1} \lim_{z\to\pi/6}\left[5\sin^4 z \cos^2 z + \sin^5 z(-\sin z)\right]$

$= 3 \lim_{z\to\pi/6}\left[5\sin^4 z\cos^2 z - \sin^6 z\right]$

$= 3\left[5\sin^4\left(\frac{\pi}{6}\right)\cos^2\left(\frac{\pi}{6}\right) - \sin^6\left(\frac{\pi}{6}\right)\right]$

$= 3\left[5\left(\frac{1}{2}\right)^4\left(\frac{\sqrt{3}}{2}\right)^2 - \left(\frac{1}{2}\right)^6\right]$

$= 3\left[\frac{15}{64} - \frac{1}{64}\right]$

$= 3\left[\frac{\cancel{14}^{7}}{\cancel{64}_{32}}\right] = \frac{21}{32}$

Substituting in ① we get

$\int_c f(z)\,dz = \int_c \frac{\sin^6 z}{(z-\pi/6)^3}\,dz = 2\pi i\left(\frac{21}{32}\right) = \frac{21\pi i}{16}$

Q3: $\int_c \frac{e^{2z}}{(z-1)^3}\,dz$ where $c: |z| = 2$

Sol: By CRT, we have

$\int_c f(z)\,dz = 2\pi i\,(a_{-1} + b_{-1} + c_{-1} + \cdots)$ ——— ①

Here $f(z) = \frac{e^{2z}}{(z-1)^3}$ the pole of $f(z)$ is $z = a = 1$ of order 3.

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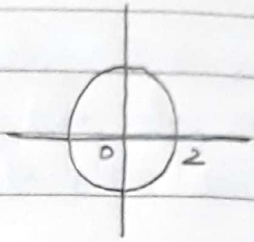
- Residue at the pole

Here c is a circle with center at origin and of radius 2.



The pole $z = a = 1$ lies within given c.

$$\therefore R[3,1] = \frac{1}{(3-1)!} \lim_{z\to1} \frac{d^2}{dz^2}\left[\cancel{(z-1)^3}\frac{e^{2z}}{\cancel{(z-1)^3}}\right]$$

$$= \frac{1}{2!} \lim_{z\to1} \frac{d^2}{dz^2}\left[e^{2z}\right]$$

$$= \frac{1}{\cancel{2}} \lim_{z\to1} \frac{d}{dz}\left[2e^{2z}\right]$$

$$= \lim_{z\to1}\left[2e^{2z}\right]$$

$$= e^{2(1)} = 2e^2$$

Substituting in eq (1)

$$\int_c f(z)\,dz = \int_c \frac{e^{2z}}{(z-1)^3}\,dz = 2\pi i\left(2e^2\right) = 4e^2\pi i$$

Q4 : $\displaystyle\int_c \frac{z+1}{z^2(1-4z^2)}\,dz$ where $c: |z|=1$

Sol: $z^2(1-4z^2) = (z-0)^2(-4z^2-2z+2z+1)$

$= (z-0)^2[-2z(2z+1)+1(2z+1)]$

$= (z-0)^2(2z+1)(1-2z)$

By CRT we have

$$\int_c f(z)\,dz = 2\pi i\,(a_{-1}+b_{-1}+c_{-1}+\dots) \quad (1)$$

Here $f(z) = \dfrac{z+1}{z^2(1+2z)(1-2z)}$

Here $z = a = -1/2,\ 1/2$ are the simple poles of $f(z)$ and $z = a = 0$ is a pole of $f(z)$ of order 2.

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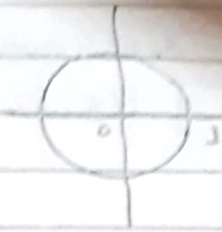


residues at the poles
Here the curve c is a circle with center at the origin of radius 1.
We observe that all the three poles are within c.
Hence we calculate the residues at the poles.

Residue at pole $z = a = 0$.

$$R[2,0] = \frac{1}{(2-1)!} \lim_{z\to 0} \frac{d}{dz}\left[\cancel{(z-0)^2} \frac{z+1}{\cancel{z^2}(1-4z^2)} \right]$$

$$= \lim_{z\to 0} \frac{d}{dz}\left[\frac{z+1}{1-4z^2}\right]$$

$$= \lim_{z\to 0}\left[\frac{(1-4z^2)(1)-(z+1)(-8z)}{(1-4z^2)}\right]$$

$$= \lim_{z\to 0}\left[\frac{1-4z^2+8z^2+8z}{(1-4z^2)}\right] = 1$$

Residue at pole $z = a = 1/2$

$$R[1, 1/2] = \lim_{z\to 1/2}\left[\cancel{\left(z-\frac{1}{2}\right)} \frac{z+1}{z^2\,\cancel{(1-2z)}(1+2z)}\right]$$

$$= \frac{1}{2}\lim_{z\to 1/2}\left[\frac{z+1}{z^2(1+2z)}\right]$$

$$= \frac{1}{2}\left[\frac{1+1/2}{(1/2)^2(1+2(1/2))}\right]$$

$$= \frac{-1}{\cancel{2}}\left[\frac{3/2}{1/\cancel{4}\,(2)}\right] = \frac{-3}{2}$$

Residue at pole $z = a = -1/2$

$$R[1, 1/2] = \lim_{z\to -1/2}\left[\cancel{\left(z+\frac{1}{2}\right)}\frac{z+1}{z^2(1-2z)\cancel{(1+2z)}\,2}\right]$$

$$= \frac{1}{2}\lim_{z\to -1/2}\left[\frac{z+1}{z^2(1-2z)}\right]$$

$$= \frac{1}{2}\left[\frac{-1/2+1}{(-1/2)^2(1-2(-1/2))}\right] = \frac{1}{\cancel{2}}\left[\frac{1/2}{1/4\,(2)}\right] = \frac{1}{2}$$

Substituting in eq ① we get

$$\int_c f(z)\,dz = \int_c \frac{z+1}{z^2(1-4z^2)}\,dz = 2\pi i\left[1-\frac{3}{2}+\frac{1}{2}\right] = 0$$

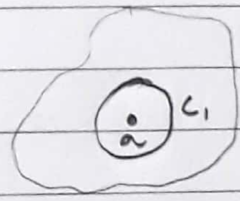
☆ Cauchy's Integral Formula : PROOF

If $f(z)$ is analytic inside and on a closed curve C and if 'a' is any point inside C, then

$$f(a) = \frac{1}{2\pi i}\int_c \frac{f(z)}{z-a}\,dz$$

Let a be any point within C, we shall enclose 'a' by a circle C_1 which lies entirely within such that 'a' is the center and r is the radius.

The function $\frac{f(z)}{z-a}$ is analytic inside and on the boundary of the region between C and C_1.



Therefore as a consequence of Cauchy's theorem we have :

$$\int_c \frac{f(z)}{z-a}\,dz = \int_{c_1}\frac{f(z)}{z-a}\,dz \quad \text{(property)}$$

We know that C_1 is a circle of center a and radius r

$\therefore |z-a| = r \Rightarrow z-a = re^{i\theta}$

$z = a + re^{i\theta}$

diff wrt z

$dz = i re^{i\theta}\,d\theta$

$$\therefore \int_c \frac{f(z)}{z-a}\,dz = \int_{\theta=0}^{2\pi}\frac{f(a+re^{i\theta})}{re^{i\theta}}\,ire^{i\theta}\,d\theta$$

$$\int_c \frac{f(z)}{z-a}\,dz = i\int_{\theta=0}^{2\pi} f(a+re^{i\theta})\,d\theta$$

This is true for any $r>0$

Hence if $r\to 0$, we get

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$$\int_c \frac{f(z)}{z-a}\,dz = i\int_{\theta=0}^{2\pi} f(a)\,d\theta$$

$$\int_c \frac{f(z)}{z-a}\,dz = i f(a)\,\theta\Big|_0^{2\pi}$$

$$\int_c \frac{f(z)}{z-a}\,dz = 2\pi i f(a)$$

$$\therefore \boxed{f(a) = \frac{1}{2\pi i}\int_c \frac{f(z)}{z-a}}$$

* <u>generalised cauchy's integral Formula:</u>

If $f(z)$ is analytic inside and on a closed curve c and if 'a' is any point inside c, then

$$f^n(a) = \frac{n!}{2\pi i}\int_c \frac{f(z)}{(z-a)^{n+1}}\,dz$$

wkt Cauchy's Integral formula is given by:

$$f(a) = \frac{1}{2\pi i}\int_c \frac{f(z)}{z-a}$$

Applying differentiation under integral sign partially with respect to 'a'.

$$f'(a) = \frac{1}{2\pi i}\int_c \frac{\partial}{\partial a}\left[\frac{f(z)}{z-a}\right] dz$$

$$f'(a) = \frac{1}{2\pi i}\int_c f(z)\,\frac{-1}{(z-a)^2}(-1)\,dz$$

$$f'(a) = \frac{1}{2\pi i}\int_c \frac{f(z)}{(z-a)^2}\,dz$$

Again on differentiating partially wrt a.

$$f''(a) = \frac{1}{2\pi i}\int_c \frac{\partial}{\partial a}\left[\frac{f(z)}{(z-a)^2}\right] dz$$

$$f''(a) = \frac{1}{2\pi i}\int_c f(z)\,\frac{-2}{(z-a)^3}(-1)\,dz$$

$$f''(a) = \frac{2}{2\pi i}\int_c \frac{f(z)}{(z-a)^3}\,dz$$

Hence on continuing by differentiating partially wrt a for n times, we get

$$f^n(a) = \frac{n!}{2\pi i} \int_c \frac{f(z)}{(z-a)^{n+1}} dz$$

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UNIT - 4

Special Function

★ The Bessel differential equation of order 'n' is given by

$$x^2 \frac{d^2y}{dx^2} + x \frac{dy}{dx} + (x^2 - n^2) y = 0$$

where n is a non-negative real constant.

The Bessel function is given by

$$J_n(x) = \sum_{r=0}^{\infty} (-1)^r \left(\frac{x}{2}\right)^{n+2r} \frac{1}{\Gamma(n+r+1) \cdot r!}$$

Also the general solution is given by

$$y = a\, J_n(x) + b\, J_n(x)$$

★ **Properties :**

1. $J_n(x) = (-1)^n J_n(x)$

consider

$$J_n(x) = \sum_{r=0}^{\infty} (-1)^r (x/2)^{n+2r} \frac{1}{\Gamma(n+r+1) \cdot r!}$$

$$J_{-n}(x) = \sum_{r=0}^{\infty} (-1)^r (x/2)^{-n+2r} \frac{1}{\Gamma(-n+r+1) \cdot r!}$$

Let $\Gamma(r-(n-1)) = \Gamma(-k)$

$r = 0, 1, 2 \dots n-2$

also $\Gamma(-k) = \Gamma(0)$ for $r = n-1$

$\Gamma(-k) \to \infty, \quad \frac{1}{\Gamma(-k)} \to 0$

$r = 0, 1, 2 \dots n-2$

$$\therefore J_{-n}(x) = \sum_{r=n}^{\infty} (-1)^r (x/2)^{-n+2r} \frac{1}{\Gamma(-n+r+1) \cdot r!}$$

Let $r - n = s$ or $r = s + n$

$$\therefore J_{-n}(x) = \sum_{s=0}^{\infty} (-1)^{n+s} (x/2)^{-n+2(n+s)} \frac{1}{\Gamma(s+1) \cdot (s+n)!}$$

$$J_{-n}(x) = \sum_{s=0}^{\infty} (-1)^n (-1)^s (x/2)^{n+2s} \frac{1}{\Gamma(s+n+1) \cdot s!}$$

$$\therefore \boxed{J_{-n}(x) = (-1)^n J_n(x)}$$

NOTE: $\Gamma(n+1) = n\Gamma(n)$

$\Gamma(n+1) = n!$

2. $J_n(-x) = (-1)^n J_n(x)$

consider

$$J_n(x) = \sum_{r=0}^{\infty} (-1)^r (x/2)^{n+2r} \frac{1}{\Gamma(n+r+1) \cdot r!}$$

$$J_n(-x) = \sum_{r=0}^{\infty} (-1)^r (-x/2)^{n+2r} \frac{1}{\Gamma(n+r+1) \cdot r!}$$

$$J_n(-x) = \sum_{r=0}^{\infty} (-1)^r (-1)^{n+2r} (x/2)^{n+2r} \frac{1}{\Gamma(n+r+1) \cdot r!}$$

$$J_n(-x) = \sum_{r=0}^{\infty} (-1)^{n+3r} (x/2)^{n+2r} \frac{1}{\Gamma(n+r+1) \cdot r!}$$

$$J_n(-x) = \sum_{r=0}^{\infty} (-1)^n (-1)^{3r} (x/2)^{n+2r} \frac{1}{\Gamma(n+r+1) \cdot r!}$$

$$J_n(-x) = \sum_{r=0}^{\infty} (-1)^n (-1)^r (x/2)^{n+2r} \frac{1}{\Gamma(n+r+1) \cdot r!}$$

$$\boxed{J_n(-x) = (-1)^n J_n(x)}$$

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3. $J_n'(x) = \frac{1}{2}[J_{n-1}(x) - J_{n+1}(x)]$

consider

$$J_n(x) = \sum_{r=0}^{\infty} (-1)^r (x/2)^{n+2r} \frac{1}{\Gamma{n+r+1}\cdot r!}$$

$$J_n'(x) = \sum_{r=0}^{\infty} (-1)^r (n+2r)(x/2)^{n+2r-1} (1/2) \frac{1}{\Gamma{n+r+1}\cdot r!}$$

$$2J_n'(x) = \sum_{r=0}^{\infty} \frac{(-1)^r (n+r)(x/2)^{n+2r-1}}{(n+r)\Gamma{n+r}\cdot r!} + \sum_{r=0}^{\infty} \frac{(-1)^r (r)(x/2)^{n+2r-1}}{\Gamma{n+r+1}\, r(r-1)!}$$

$$2J_n'(x) = \sum_{r=0}^{\infty} (-1)^r (x/2)^{(n-1)+2r} \frac{1}{\Gamma{n-1+r+1}\cdot r!} + \sum_{r=0}^{\infty} (-1)^r (x/2)^{n-1+2r} \frac{1}{\Gamma{n+r+1}\,(r-1)!}$$

Let $r-1 = s$ in second summation

$r = s+1$

$$2J_n'(x) = J_{n-1}(x) + \sum_{r=0}^{\infty} (-1)^{s+1} (x/2)^{(n+1)+2s} \frac{1}{\Gamma{(n+1)+s+1}\cdot s!}$$

$$2J_n'(x) = J_{n-1}(x) - J_{n+1}(x)$$

$$\boxed{J_n'(x) = \frac{1}{2}[J_{n-1}(x) - J_{n+1}(x)]}$$

4. $\frac{d}{dx}[x^n . J_n(x)] = x^n J_{n-1}(x)$

consider

$$J_n(x) = \sum_{r=0}^{\infty} (-1)^r (x/2)^{n+2r} \frac{1}{\Gamma{n+r+1}\cdot r!}$$

$$x^n J_n(x) = \sum_{r=0}^{\infty} (-1)^r (x)^{2n+2r} \frac{1}{2^{n+2r}\Gamma{n+r+1}\cdot r!}$$

$$\frac{d}{dx}[x^n J_n(x)] = \sum_{r=0}^{\infty} (-1)^r (2n+2r)\, x^{2n+2r-1} \frac{1}{2^{n+2r}\Gamma{n+r+1}\, r!}$$

$$\frac{d}{dx}[x^n J_n(x)] = \sum_{r=0}^{\infty} (-1)^r \frac{(n+r)\, x^{2n+2r-1}}{2^{n+2r-1}(n+r)\Gamma{n+r}\, r!}$$

$$\frac{d}{dx}[x^n J_n(x)] = \sum_{r=0}^{\infty} (-1)^r (x)^n (x/2)^{(n-1)+2r} \frac{1}{\Gamma{(n-1)+r+1}\, r!}$$

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$$\boxed{\frac{d}{dx}\left[x^n J_n(x)\right] = x^n J_{n-1}(x)}$$

5. $$\frac{d}{dx}\left[x^{-n} J_n(x)\right] = -x^{-n} J_{n+1}(x)$$

Sol: consider

$$J_n(x) = \sum_{r=0}^{\infty} (-1)^r (x/2)^{n+2r} \frac{1}{\Gamma(n+r+1) \cdot r!}$$

$$x^{-n} J_n(x) = \sum_{r=0}^{\infty} (-1)^r x^{2r} \frac{1}{2^{n+2r}\Gamma(n+r+1) \cdot r!}$$

$$\frac{d}{dx}\left[x^{-n} J_n(x)\right] = \sum_{r=0}^{\infty} (-1)^r 2\cancel{r}\, x^{2r-1} \frac{1}{2^{n+2r}\Gamma(n+r+1) \cdot \cancel{r}(r-1)!}$$

Put $r-1 = s \Rightarrow r = s+1$

$$\frac{d}{dx}\left[x^{-n} J_n(x)\right] = \sum_{s=0}^{\infty} (-1)^{s+1} 2x^{2s+1} \frac{1}{2^{n+2s+2}\Gamma(n+s+2) \cdot s!}$$

$$\frac{d}{dx}\left[x^{-n} J_n(x)\right] = -\sum_{s=0}^{\infty} (-1)^s \frac{x^{2s+1}}{2^{n+2s+1}} \frac{1}{\Gamma((n+1)+s+1) \cdot s!}$$

$$\frac{d}{dx}\left[x^{-n} J_n(x)\right] = -\sum_{s=0}^{\infty} x^{-n} \frac{x^{(n+1)+2s}}{2^{n+1+2s}} \frac{1}{\Gamma((n+1)s+1) \cdot s!}$$

$$\boxed{\frac{d}{dx}\left[x^{-n} J_n(x)\right] = -x^{-n} J_{n+1}(x)}$$

6. $x J_n'(x) = n J_n(x) - x J_{n+1}(x)$

consider

$$J_n(x) = \sum_{r=0}^{\infty} (-1)^r (x/2)^{n+2r} \frac{1}{\Gamma(n+r+1) \cdot r!}$$

$$J_n'(x) = \sum_{r=0}^{\infty} (-1)^r (n+2r)(x/2)^{n+2r-1} (1/2) \frac{1}{\Gamma(n+r+1)\, r!}$$

$$J_n'(x) = \sum_{r=0}^{\infty} (-1)^r (n+2r)\, x^{n+2r-1} \frac{1}{2^{n+2r}\Gamma(n+r+1)\, r!}$$

$$x J_n'(x) = \sum_{r=0}^{\infty} n(-1)^r \left(\frac{x}{2}\right)^{n+2r} \frac{1}{\Gamma(n+r+1) \cdot r!} + \sum_{r=0}^{\infty} (-1)^r 2\cancel{r} \left(\frac{x}{2}\right)^{n+2r} \frac{1}{\Gamma(n+r+1) \cdot \cancel{r}(r-1)!}$$

$$x J_n'(x) = n J_n(x) + \sum_{s=0}^{\infty} (-1)^{s+1} \left(\frac{x}{2}\right)^{n+2s+2} \frac{1}{\overline{|n+s+2} \cdot s!}$$

$$x J_n'(x) = n J_n(x) - \sum_{s=0}^{\infty} (-1)^{s} \left(\frac{x}{2}\right)^{n+1+2s+1} \frac{1}{\overline{|(n+1)+s+1} \cdot s!}$$

$$\boxed{x J_n'(x) = n J_n(x) - x J_{n+1}(x)}$$

7. $x J_n'(x) = x J_{n-1}(x) - n J_n(x)$

consider

$$J_n(x) = \sum_{r=0}^{\infty} (-1)^r (x/2)^{n+2r} \frac{1}{\overline{|n+r+1} \cdot r!}$$

$$J_n'(x) = \sum_{r=0}^{\infty} (-1)^r (n+2r)(x/2)^{n+2r-1} \frac{(1/2)\, 1}{\overline{|n+r+1} \cdot r!}$$

$$x J_n'(x) = \sum_{r=0}^{\infty} (-1)^r (n+2r)(x/2)^{n+2r} \frac{1}{\overline{|n+r+1} \cdot r!}$$

considering $n+2r = 2(n+r) - n$

$$\therefore x J_n'(x) = \sum_{r=0}^{\infty} (-1)^r \, 2\cancel{(n+r)} (x/2)^{n+2r} \frac{1}{\cancel{(n+r)}\, \overline{|n+r} \cdot r!}$$

$$+ \sum_{r=0}^{\infty} (-1)^r (-n)(x/2)^{n+2r} \frac{1}{\overline{|n+r+1} \cdot r!}$$

$$x J_n'(x) = \sum_{r=0}^{\infty} (-1)^r \frac{x^{n+2r}}{2^{n+2r-1}} \frac{1}{\overline{|n+r} \cdot r!} - n J_n(x)$$

$$x J_n'(x) = \sum_{r=0}^{\infty} (-1)^r \frac{x \, x^{(n-1)+2r}}{2^{(n-1)+2r}} \frac{1}{\overline{|(n-1)+r+1} \cdot r!} - n J_n(x)$$

$$\boxed{x J_n'(x) = x J_{n-1}(x) - n J_n(x)}$$

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8. $2n J_n(x) = x[J_{n+1}(x) + J_{n-1}(x)]$

consider

$$J_n(x) = \sum_{r=0}^{\infty} (-1)^r (x/2)^{n+2r} \frac{1}{\overline{|n+r+1} \cdot r!}$$

$$2n J_n(x) = \sum_{r=0}^{\infty} (-1)^r (x/2)^{n+2r} \frac{2n}{\overline{|n+r+1} \cdot r!}$$

Let $2n = 2(n+r) - 2r$

$$2n J_n(x) = \sum_{r=0}^{\infty} (-1)^r (x/2)^{n+2r} \frac{2\cancel{(n+r)}}{\cancel{(n+r)}\overline{|n+r} \cdot r!}$$

$$+ \sum_{r=0}^{\infty} (-1)^r (-2\cancel{r}) (x/2)^{n+2r} \frac{1}{\overline{|n+r+1}\, \cancel{r}(r-1)!}$$

$$2n J_n(x) = \sum_{r=0}^{\infty} (-1)^r \frac{x\, x^{n+2r-1}}{2^{n+2r-1}} \frac{1}{\overline{|n+r} \cdot r!}$$

$$- \sum_{r=1}^{\infty} (-1)^r (x/2)^{n+2r} \frac{2}{\overline{|n+r+1}\, (r-1)!}$$

$$2n J_n(x) = x \sum_{r=0}^{\infty} (-1)^r \frac{x^{(n-1)+2r}}{2^{(n-1)+2r}} \frac{1}{\overline{|(n-1)+r+1} \cdot r!}$$

$$- \sum_{r=1}^{\infty} (-1)^r \frac{x^{n+2r}}{2^{n+2r-1}} \frac{1}{\overline{|n+r+1}\, (r-1)!}$$

Let $r - 1 = s$

$$2n J_n(x) = x J_{n-1}(x) - \sum_{s=0}^{\infty} (-1)^{s+1} \frac{x^{n+2s+2}}{2^{n+2s+1}} \frac{1}{\overline{|n+s+2} \cdot s!}$$

$$2n J_n(x) = x J_{n-1}(x) + x \sum_{s=0}^{\infty} (-1)^s \left(\frac{x}{2}\right)^{(n+1)+2s} \frac{1}{\overline{|(n+1)+s+1} \cdot s!}$$

$$2n J_n(x) = x J_{n-1}(x) + x J_{n+1}(x)$$

$$\boxed{2n J_n(x) = x[J_{n+1}(x) + J_{n-1}(x)]}$$

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7. a. $J_{1/2}(x) = \sqrt{\frac{2}{\pi x}} \sin x$

b. $J_{-1/2}(x) = \sqrt{\frac{2}{\pi x}} \cos x$

consider

$$J_n(x) = \sum_{r=0}^{\infty} (-1)^r (x/2)^{n+2r} \frac{1}{\Gamma(n+r+1) \cdot r!}$$

a. Taking $n = 1/2$

$$J_{1/2}(x) = \sum_{r=0}^{\infty} (-1)^r (x/2)^{1/2+2r} \frac{1}{\Gamma(1/2+r+1) \cdot r!}$$

$$J_{1/2}(x) = \sum_{r=0}^{\infty} (-1)^r \sqrt{\frac{x}{2}} \left(\frac{x}{2}\right)^{2r} \frac{1}{\Gamma(r+3/2) \cdot r!}$$

For $r = 0, 1, 2 \ldots$

$$J_{1/2}(x) = \sqrt{\frac{x}{2}} \left[\frac{1}{\Gamma 3/2} - \left(\frac{x}{2}\right)^2 \frac{1}{\Gamma 5/2 \; 1!} + \left(\frac{x}{2}\right)^4 \frac{1}{\Gamma 7/4 \cdot 2!} - \cdots \right]$$

using $\Gamma \frac{1}{2} = \sqrt{\pi}$; $\Gamma n = n-1 \, \Gamma n-1$

$$\Gamma \frac{3}{2} = \frac{1}{2} \Gamma \frac{1}{2} = \frac{\sqrt{\pi}}{2}$$

$$\Gamma \frac{5}{2} = \frac{3}{2} \Gamma \frac{3}{2} = \frac{3}{2} \cdot \frac{1}{2} \Gamma \frac{1}{2} = \frac{3\sqrt{\pi}}{4}$$

$$\Gamma \frac{7}{2} = \frac{5}{2} \cdot \frac{3}{2} \cdot \frac{1}{2} \Gamma \frac{1}{2} = \frac{15\sqrt{\pi}}{8}$$

$$J_{1/2}(x) = \sqrt{\frac{x}{2}} \left[\frac{2}{\sqrt{\pi}} - \left(\frac{x}{\cancel{2}}\right)^2 \frac{\cancel{4}}{3\sqrt{\pi}} + \left(\frac{x}{2}\right)^4 \frac{8}{15\sqrt{\pi}} - \cdots \right]$$

$$J_{1/2}(x) = \sqrt{\frac{x}{2\pi}} \left[2 - \frac{x^2}{3} + \frac{x^4}{60} - \cdots \right]$$

$$J_{1/2}(x) = \sqrt{\frac{x}{2\pi}} \frac{2}{x} \left[x - \frac{x^3}{6} + \frac{x^5}{120} - \cdots \right]$$

$$J_{1/2}(x) = \sqrt{\frac{2}{\pi x}} \left[\frac{x}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} - \cdots \right] = \sqrt{\frac{2}{\pi x}} \sin x \;//$$

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6. Taking $n = -1/2$

$$J_{-1/2}(x) = \sum_{r=0}^{\infty} (-1)^r (x/2)^{-1/2+2r} \cdot \frac{1}{\Gamma{r+1/2} \cdot r!}$$

$$J_{-1/2}(x) = \sum_{r=0}^{\infty} (-1)^r \sqrt{\frac{2}{x}} \left(\frac{x}{2}\right)^{2r} \frac{1}{\Gamma{r+1/2} \cdot r!}$$

For $r = 0, 1, 2 \ldots$

$$J_{-1/2}(x) = \sqrt{\frac{2}{x}} \left[\frac{1}{\Gamma{1/2}} - \left(\frac{x}{2}\right)^2 \frac{1}{\Gamma{3/2}} + \left(\frac{x}{2}\right)^4 \frac{1}{\Gamma{5/2}} - \ldots \right]$$

$$J_{-1/2}(x) = \sqrt{\frac{2}{x}} \left[\frac{1}{\sqrt{\pi}} - \left(\frac{x^2}{2}\right) \frac{2}{\sqrt{\pi}} + \left(\frac{x}{2}\right)^4 \frac{4}{3\sqrt{\pi}} - \ldots \right]$$

$$J_{-1/2}(x) = \sqrt{\frac{2}{x\pi}} \left[1 - \frac{x^2}{2} + \frac{x^4}{12} - \ldots \right]$$

$$J_{-1/2}(x) = \sqrt{\frac{2}{x\pi}} \left[1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots \right] = \sqrt{\frac{2}{\pi x}} \cos x$$

Q: Using the suitable reccurence relation find $J_{3/2}$, $J_{-3/2}$, $J_{5/2}$ and $J_{-5/2}$.

Sol: wkt

$$2n J_n(x) = x\left[J_{n+1}(x) + J_{n-1}(x) \right]$$

$$\Rightarrow J_{n+1}(x) = \frac{2n}{x} J_n(x) - J_{n-1}(x) \quad \text{——— } (1)$$

$$J_{n+1}(x) = \sum_{r=0}^{\infty} (-1)^r (x/2)^{(n+1)+2r} \frac{1}{\Gamma{(n+1)+r+1} \cdot r!}$$

when $n = 1/2$

$$J_{3/2}(x) = \frac{2(1/2)}{x} J_{1/2}(x) - J_{-1/2}(x)$$

$$= \frac{1}{x}\left[\sqrt{\frac{2}{\pi x}} \sin x\right] - \sqrt{\frac{2}{\pi x}} \cos x$$

$$= \sqrt{\frac{2}{\pi x}} \left[\frac{\sin x - x\cos x}{x} \right]$$

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- When $n = -1/2$

$$J_{1/2}(x) = \frac{2(-1/2)}{x} J_{-1/2}(x) - J_{-3/2}(x)$$

$$J_{-3/2}(x) = \frac{-1}{x} J_{-1/2}(x) \quad J_{1/2}(x)$$

$$= \frac{-1}{x}\left[\sqrt{\frac{2}{\pi x}}\cos x\right] - \sin x\sqrt{\frac{2}{\pi x}}$$

$$= -\sqrt{\frac{2}{\pi x}}\left[\frac{\cos x + x\sin x}{x}\right] //$$

- When $n = 3/2$

$$J_{5/2}(x) = \frac{2(3/2)}{x} J_{3/2}(x) - J_{1/2}(x)$$

$$= \frac{3}{x}\left[\frac{\sin x - x\cos x}{x}\right]\sqrt{\frac{2}{\pi x}} - \sqrt{\frac{2}{\pi x}}\sin x$$

$$= \sqrt{\frac{2}{\pi x}}\left[\frac{3\sin x - 3x\cos x - x^2\sin x}{x^2}\right] //$$

- When $= n = -3/2$

$$J_{-1/2}(x) = \frac{2(-3/2)}{x} J_{-3/2}(x) - J_{-5/2}(x)$$

$$J_{-5/2}(x) = \frac{-3}{x} J_{-3/2}(x) - J_{-1/2}(x)$$

$$= \frac{-3}{x}\left[-\sqrt{\frac{2}{\pi x}}\left(\frac{\cos x + x\sin x}{x}\right)\right] - \sqrt{\frac{2}{\pi x}}\cos x$$

$$= \sqrt{\frac{2}{\pi x}}\left[\frac{3\cos x + 3x\sin x - x^2\cos x}{x^2}\right] //$$

* **Legendre's Polynomial**

The Rodrige's formula for Legendre's polynomial is

$$\boxed{P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n} (x^2-1)^n}$$

— $P_0(x) = \frac{1}{2^0 0!} \frac{d^0}{dx^0} (x^2-1)^0$

$$\boxed{P_0(x) = 1}$$

— $P_1(x) = \frac{1}{2^1 1!} \frac{d}{dx} (x^2-1) =$

$P_1(x) = \frac{1}{2}(2x)$

$$\boxed{P_1(x) = x}$$

— $P_2(x) = \frac{1}{2^2 2!} \frac{d^2}{dx^x} (x^2-1)^2$

$P_2(x) = \frac{1}{\cancel{4}(2)} \frac{d}{dx} [\cancel{2}(x^2-1)(\cancel{2}x)]$

$P_2(x) = \frac{1}{2} \frac{d}{dx} (x^3-x)$

$$\boxed{P_2(x) = \frac{1}{2}[3x^2-1]}$$

— $P_3(x) = \frac{1}{2^3 3!} \frac{d^3}{dx^3} (x^2-1)^3$

$P_3(x) = \frac{1}{8(\cancel{6})} \frac{d^2}{dx^2} [\cancel{3}(x^2-1)^2(\cancel{2}x)]$

$P_3(x) = \frac{1}{8} \frac{d^2}{dx^2} [x^5 - 2x^3 + x]$

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$$P_3(x) = \frac{1}{8}\frac{d}{dx}\left[5x^4 - 6x^2 + 1\right]$$

$$P_3(x) = \frac{1}{8}\left[20x^3 - 12x\right]$$

$$\boxed{P_3(x) = \frac{1}{2}\left[5x^3 - 3x\right]}$$

- $$P_4(x) = \frac{1}{2^4 4!}\frac{d^4}{dx^4}(x^2-1)^4$$

$$P_4(x) = \frac{1}{\cancel{16}(24)}\frac{d^3}{dx^3}\left[\cancel{4}(x^2-1)^3 \cancel{2}x\right]$$

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$$P_4(x) = \frac{1}{48}\frac{d^3}{dx^3}\left[x^7 - 3x^5 + 3x^3 - x\right]$$

$$P_4(x) = \frac{1}{48}\frac{d^2}{dx^2}\left[7x^6 - 15x^4 + 9x^2 - 1\right]$$

$$P_4(x) = \frac{1}{\cancel{48}_8}\frac{d}{dx}\left[\overset{7}{\cancel{42}}x^5 - \overset{10}{\cancel{60}}x^3 + \overset{3}{\cancel{18}}x\right]$$

$$P_4(x) = \frac{1}{8}\frac{d}{dx}\left[7x^5 - 10x^3 + 3x\right]$$

$$\boxed{P_4(x) = \frac{1}{8}\left[35x^4 - 30x^2 + 3\right]}$$

* <u>Expressing algebraic terms in terms of Legendre's polynomials :</u>

- $\boxed{P_0(x) = 1}$

- $\boxed{P_1(x) = x}$

- $$P_2(x) = \frac{1}{2}\left[3x^2 - 1\right]$$

$$\boxed{x^2 = \frac{2P_2(x) + P_0(x)}{3}}$$

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— $P_3(x) = \frac{1}{2}[5x^3 - 3x]$

$$x^3 = \frac{2P_3(x) + 3x}{5}$$

$$\boxed{x^3 = \frac{2P_3(x) + 3P_1(x)}{5}}$$

— $P_4(x) = \frac{1}{8}[35x^4 - 30x^2 + 3]$

$$x^4 = \frac{8P_4(x) + 30x^2 - 3}{35}$$

$$x^4 = \frac{8P_4(x)}{35} + \frac{6}{7}\left[\frac{2P_2(x) + P_0(x)}{3}\right] - \frac{3}{35}P_0(x)$$

$$x^4 = \frac{8}{35}P_4(x) + \frac{4P_2(x)}{7} + \frac{2P_0(x)}{7} - \frac{3}{35}P_0(x)$$

$$\boxed{x^4 = \frac{8}{35}P_4(x) + \frac{4}{7}P_2(x) + \frac{1}{5}P_0(x)}$$

Q 1: Express $x^3 + 2x^2 - 4x + 5$ in terms of legendre's polynomial

sol: $f(x) = x^3 + 2x^2 - 4x + 5$

$$f(x) = \left[\frac{2}{5}P_3(x) + \frac{3}{5}P_1(x)\right] + 2\left[\frac{2}{3}P_2(x) + \frac{1}{3}P_0(x)\right] - 4\left[P_1(x)\right] + 5\left[P_0(x)\right]$$

$$f(x) = \frac{2}{5}P_3(x) + \frac{3}{5}P_1(x) + \frac{4}{3}P_2(x) + \frac{2}{3}P_0(x) - 4P_1(x) + 5P_0(x)$$

$$f(x) = \frac{2}{5}P_3(x) + \frac{4}{3}P_2(x) - \frac{17}{5}P_1(x) + \frac{17}{3}P_0(x)$$

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Q2: If $x^3+2x^2-x+1 = aP_0(x) + bP_1(x) + cP_2(x) + dP_3(x)$, find the values of a, b, c and d.

sol: $f(x) = x^3+2x^2-x+1$

$$f(x) = \left[\frac{2}{5}P_3(x) + \frac{3}{5}P_1(x)\right] + 2\left[\frac{2}{3}P_2(x) + \frac{1}{3}P_0(x)\right] - P_1(x) + P_0(x)$$

$$f(x) = \frac{2}{5}P_3(x) + \frac{3}{5}P_1(x) + \frac{4}{3}P_2(x) + \frac{2}{3}P_0(x) - P_1(x) + P_0(x)$$

$$f(x) = \frac{5}{3}P_0(x) - \frac{2}{5}P_1(x) + \frac{4}{3}P_2(x) + \frac{2}{5}P_3(x)$$

On comparing: $a = \frac{5}{3}$; $b = \frac{-2}{5}$; $c = \frac{4}{3}$; $d = \frac{2}{5}$ //

Q3: Show that:

i. $P_2(\cos\theta) = \frac{1}{4}(1+3\cos 2\theta)$

ii. $P_3(\cos\theta) = \frac{1}{8}(3\cos\theta + 5\cos 3\theta)$

sol: i. wkt

$$P_2(x) = \frac{1}{2}[3x^2-1]$$

at $x = \cos\theta$

$$P_2(x) = \frac{1}{2}[3\cos^2\theta - 1]$$

$$= \frac{1}{2}\left[3\left(\frac{1+\cos 2\theta}{2}\right) - 1\right]$$

$$= \frac{1}{4}\left[3+3\cos 2\theta - 2\right]$$

$$= \frac{1}{4}\left[1+3\cos 2\theta\right]$$ //

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ii. wkt

$$P_3(x) = \frac{1}{2}[5x^3 - 3x]$$

at $x = \cos\theta$

$$P_3(\cos\theta) = \frac{1}{2}\left[5\cos^3\theta - 3\cos\theta\right]$$

$$= \frac{1}{2}\left[5\left(\frac{\cos 3\theta + 3\cos\theta}{4}\right) - 3\cos\theta\right]$$

$$= \frac{1}{8}\left[5\cos 3\theta + 15\cos\theta - 12\cos\theta\right]$$

$$= \frac{1}{8}\left[5\cos 3\theta + 3\cos\theta\right]$$

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Unit - 5

STATISTICS AND PROBABILITY - I

Curve Fitting

* **Curve Fitting by the method of least squares:**

Several equations of different types can be obtained to express the given data approximately, but the problem is to find the equation of the curve of best fit, which may be most suitable for predicting the unknown values. The process of finding such an equation of best fit is known as curve fitting.

- **Fitting of a straight line $y = a + bx$:**

For clarity suppose it is required to fit the curve $y = a + bx$ to a given set of observations such as (x_1, y_1), (x_2, y_2), (x_3, y_3), ... (x_n, y_n).

For any x_i, the observed value is y_i and the expected value is η_i, $\eta_i = a + bx_i$ such that the error $e_i = y_i - \eta_i$.

Therefore the sum of the squares of these errors is:

$$E = e_1^2 + e_2^2 + e_3^2 + \dots + e_n^2$$

$$\therefore E = [y_1 - (a + bx_1)]^2 + [y_2 - (a + bx_2)]^2 + [y_3 - (a + bx_3)]^2 + \dots + [y_n - (a + bx_n)]^2$$

$$\therefore E = \sum_{i=1}^{n} [y_i - (a + bx_i)]^2$$

For E to be minimum we must have

$$\frac{\partial E}{\partial a} = 0 \text{ and } \frac{\partial E}{\partial b} = 0$$

$$\frac{\partial E}{\partial a} = \sum_{i=1}^{n} 2[y_i - (a + bx_i)](-1)$$

$$\frac{\partial E}{\partial a} = -2 \sum_{i=1}^{n} [y_i - (a + bx_i)] = 0 \quad \text{——— (1)}$$

$$\frac{\partial E}{\partial b} = \sum_{i=1}^{n} 2\left[y_i - (a+bx_i)\right](-x_i)$$

$$\frac{\partial E}{\partial b} = -2\sum_{i=1}^{n} x_i\left[y_i - (a+bx_i)\right] = 0 \quad \text{——— (2)}$$

From eq (1)

$$\sum_{i=1}^{n}\left[y_i - (a+bx_i)\right] = 0$$

$$\sum_{i=1}^{n} y_i - \Sigma a - b\sum_{i=1}^{n} x_i = 0 \quad \text{——— (3)}$$

From eq (2)

$$\sum_{i=1}^{n} x_i\left[y_i - (a+bx_i)\right] = 0$$

$$\sum_{i=1}^{n} x_i y_i - a\sum_{i=1}^{n} x_i - b\sum_{i=1}^{n} x_i^2 = 0 \quad \text{——— (4)}$$

From eq (3)

$$\sum_{i=1}^{n} y_i - na - b\sum_{i=1}^{n} x_i = 0 \quad \text{——— (5)}$$

From eq (4)

$$\sum_{i=1}^{n} x_i y_i - a\sum_{i=1}^{n} x_i - b\sum_{i=1}^{n} x_i^2 = 0$$

~~In general~~

In general eq (5) and eq (4) can be written as

$$\Sigma y = na + b\Sigma x \quad \text{——— (6)}$$

$$\Sigma xy = a\Sigma x + b\Sigma x^2 \quad \text{——— (7)}$$

Equations (6) and (7) are called normal equations which fits the straight line $y = a+bx$ in the least square sense. By solving these equations we get the values of a and b.

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Q1: Fit a straight line $y = a + bx$ in the least square sense for the following data.

x	1	3	4	6	8	9	11	14
y	1	2	4	4	5	7	8	9

Sol:

x	y	xy	x^2
1	1	1	1
3	2	6	9
4	4	16	16
6	4	24	36
8	5	40	64
9	7	63	81
11	8	88	121
14	9	126	196
$\Sigma x = 56$	$\Sigma y = 40$	$\Sigma xy = 364$	$\Sigma x^2 = 524$

The normal equations are given by

$\Sigma y = na + b\Sigma x$ — (1)

$\Sigma xy = a\Sigma x + b\Sigma x^2$ — (2)

substituting the values in eq (1) an (2)

$40 = 8a + 56b$ — (3)

$364 = 56a + 524b$ — (4)

solving eq (3) and eq (4)

$a = 0.545 \quad b = 0.636$

$\therefore y = 0.54 + 0.63x$ is the required straight line.

Q2: Find the equation of the best fitting straight line $y = a + bx$ for the following data.

x	5	10	15	20	25
y	16	19	23	26	30

Sol:

x	y	xy	x^2
5	16	80	25
10	19	190	100
15	23	345	225
20	26	520	400
25	30	750	625
$\Sigma x = 75$	$\Sigma y = 114$	$\Sigma xy = 1885$	$\Sigma x^2 = 1375$

The normal equations are

$\Sigma y = na + b\Sigma x$ — (1)

$\Sigma xy = a\Sigma x + b\Sigma x^2$ — (2)

Substituting the values

$114 = 5a + 75b$ — (3)

$1885 = 75a + 1375b$ — (4)

Solving eq (3) and (4)

$a = 12.3 \quad b = 0.7$

$\therefore y = 12.3 + 0.7x$
is the required straight line.

Q3: Find the equation of best fitting straight line for the data.

x	0	1	2	3	4	5
y	9	8	24	28	26	20

Sol:

x	y	xy	x^2
0	9	0	0
1	8	8	1
2	24	48	4
3	28	84	9
4	26	104	16
5	20	100	25
$\Sigma x = 15$	$\Sigma y = 115$	$\Sigma xy = 344$	$\Sigma x^2 = 55$

The normal equations are

$\Sigma y = na + b\Sigma x$ — (1)

$\Sigma xy = a\Sigma x + b\Sigma x^2$ — (2)

Substituting in eq (1) and (2)

$115 = 6a + 15b$ — (3)

$344 = 15a + 55b$ — (4)

solving eq (3) and eq (4)

$a = 11.095$ $b = 3.228$

$\therefore y = 11.095 + 3.228x$ is the required straight line.

Q4: A simply supported beam carries a concentrated load at its mid point. corresponding to various values of P, the maximum deflection y is measured and is given by

P	100	120	140	160	180	200
y	0.45	0.55	0.60	0.70	0.80	0.85

Find a law of the form $y = a + bP$ and hence estimate y when $P = 150$.

Sol: The normal equations is given by

$\Sigma y = na + b\Sigma P$ — (1)

$\Sigma Py = a\Sigma P + b\Sigma P^2$ — (2)

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P	y	Py	P^2
100	0.45	45	10000
120	0.55	66	14400
140	0.60	84	19600
160	0.70	112	25600
180	0.80	144	32400
200	0.85	170	40000
$\Sigma P = 900$	$\Sigma y = 3.95$	$\Sigma Py = 621$	$\Sigma P^2 = 142000$

Substituting in eq ① and eq ②

$3.95 = 6a + 900b$ — ③

$621 = 900a + 142000b$ — ④

Solving eq ③ and eq ④

$a = 0.047$ $b = 0.004$

$\therefore y = 0.047 + 0.004P$

is the required form.

at $P = 150$

$y = 0.047 + 0.004(150)$

$y = 0.647$

- Fitting of a second degree parabola $y = a + bx + cx^2$

For clarity suppose it is required to fit the curve $y = a + bx + cx^2$ to a given set of observations such as $(x_1, y_1), (x_2, y_2), (x_3, y_3) \dots (x_n, y_n)$.

For any x_i, the observed value is y_i and the expected value is η_i, $\eta_i = a + bx_i + cx_i^2$ such that the error $e_i = y_i - \eta_i$.

Therefore the sum of squares of these errors is

$$E = e_1^2 + e_2^2 + e_3^2 + \dots + e_n^2$$

$$\therefore E = [y_1 - \eta_1]^2 + [y_2 - \eta_2]^2 + [y_3 - \eta_3]^2 + \dots + [y_n - \eta_n]^2$$

$$E = [y_1 - (a + bx_1 + cx_1^2)]^2 + [y_2 - (a + bx_2 + cx_2^2)]^2 + [y_3 - (a + bx_3 + cx_3^2)]^2 + \dots + [y_n - (a + bx_n + cx_n^2)]^2$$

$$\therefore E = \sum_{i=1}^{n} [y_i - (a + bx_i + cx_i^2)]^2$$

For E to be minimum, we ~~must~~ have

$$\frac{\partial E}{\partial a} = 0, \quad \frac{\partial E}{\partial b} = 0 \text{ and } \frac{\partial E}{\partial c} = 0$$

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$$\frac{\partial E}{\partial a} = \sum_{i=1}^{n} 2\left[y_i - (a + bx_i + cx_i^2)\right]\left[-1\right] = 0$$

$$\Rightarrow \sum_{i=1}^{n}\left[y_i - (a + bx_i + cx_i^2)\right] = 0$$

$$\sum_{i=1}^{n} y_i - \sum_{i=1}^{n} a - b\sum_{i=1}^{n} x_i - c\sum_{i=1}^{n} x_i^2 = 0$$

$$\sum_{i=1}^{n} y_i = \sum_{i=1}^{n} a + b\sum_{i=1}^{n} x_i + c\sum_{i=1}^{n} x_i^2 = 0 \quad \text{—— } 1$$

$$\therefore \cancel{\Sigma y = na + b\Sigma x + c\Sigma x^2 = 0} \quad \text{—— } 1$$

$$\frac{\partial E}{\partial b} = \sum_{i=1}^{n} 2\left[y_i - (a + bx_i + cx_i^2)\right]\left[-x_i\right] = 0$$

$$\Rightarrow \sum_{i=1}^{n} x_i\left[y_i - (a + bx_i + cx_i^2)\right] = 0$$

$$\sum_{i=1}^{n} x_i y_i - a\sum_{i=1}^{n} x_i - b\sum_{i=1}^{n} x_i^2 - c\sum_{i=1}^{n} x_i^3 = 0$$

$$\sum_{i=1}^{n} x_i y_i = a\sum_{i=1}^{n} x_i + b\sum_{i=1}^{n} x_i^2 + c\sum_{i=1}^{n} x_i^3 \quad \text{—— } 2$$

$$\therefore \cancel{\Sigma xy = a\Sigma x + b\Sigma x^2 + c\Sigma x^3} \quad \text{—— } 2$$

$$\frac{\partial E}{\partial c} = \sum_{i=1}^{n} 2\left[y_i - (a + bx_i + cx_i^2)\right]\left[-x_i^2\right] = 0$$

$$\Rightarrow \sum_{i=1}^{n} x_i^2\left[y_i - (a + bx_i + cx_i^2)\right] = 0$$

$$\sum_{i=1}^{n} x_i^2 y_i - a\sum_{i=1}^{n} x_i^2 - b\sum_{i=1}^{n} x_i^3 - c\sum_{i=1}^{n} x_i^4 = 0$$

$$\sum_{i=1}^{n} x_i^2 y_i = a\sum_{i=1}^{n} x_i^2 + b\sum_{i=1}^{n} x_i^3 + c\sum_{i=1}^{n} x_i^4 = 0 \quad \text{—— } 3$$

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In general eq ①, eq ② and eq ③ can be written as

$$\Sigma y = na + b\Sigma x + c\Sigma x^2$$

$$\Sigma xy = a\Sigma x + b\Sigma x^2 + c\Sigma x^3$$

$$\Sigma x^2 y = a\Sigma x^2 + b\Sigma x^3 + c\Sigma x^4$$

Equations ④, ⑤ and ⑥ are called the normal equations which fits the parabola $y = a + bx + cx^2$ in the least square sense.

By solving these equations we get the values of a, b and c.

Q1: Fit the parabola of second degree $y = a + bx + cx^2$ for the following data.

x	0	1	2	3	4	5	6
y	14	18	23	29	36	40	46

Sol: The normal equations for a second degree parabola is given by:

$$\Sigma y = na + b\Sigma x + c\Sigma x^2 \quad \text{—— ①}$$

$$\Sigma xy = a\Sigma x + b\Sigma x^2 + c\Sigma x^3 \quad \text{—— ②}$$

$$\Sigma x^2 y = a\Sigma x^2 + b\Sigma x^3 + c\Sigma x^4 \quad \text{—— ③}$$

x	y	xy	x^2y	x^2	x^3	x^4
0	14	0	0	0	0	0
1	18	18	18	1	1	1
2	23	46	92	4	8	16
3	29	87	261	9	27	81
4	36	144	576	16	64	256
5	40	200	1000	25	125	625
6	46	276	1656	36	216	1296
$\Sigma x = 21$	$\Sigma y = 206$	$\Sigma xy = 441$	$\Sigma x^2y = 3603$	$\Sigma x^2 = 91$	$\Sigma x^3 = 441$	$\Sigma x^4 = 2275$

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Substituting in eq ①, eq ② and eq ③

$206 = 7a + 21b + 91c$ — ④

$771 = 21a + 91b + 441c$ — ⑤

$3603 = 91a + 441b + 2275c$ — ⑥

Solving eq ④, eq ⑤ and eq ⑥

$a = 13.452$ $\quad b = 4.964$ $\quad c = 0.083$

$\therefore$ $y = 13.452 + 4.964x + 0.083x^2$ is the required second degree parabola.

Q2: Fit a parabola of second degree in the least square sense for the following data

x	0	1	2	3	4
y	1	1.8	1.3	2.5	2.3

Sol: The normal equations for a second degree parabola is given by

$\Sigma y = na + b\Sigma x + c\Sigma x^2$ — ①

$\Sigma xy = a\Sigma x + b\Sigma x^2 + c\Sigma x^3$ — ②

$\Sigma x^2 y = a\Sigma x^2 + b\Sigma x^3 + c\Sigma x^4$ — ③

x	y	xy	x^2y	x^2	x^3	x^4
0	1	0	0	0	0	0
1	1.8	1.8	1.8	1	1	1
2	1.3	2.6	5.2	4	8	16
3	2.5	7.5	22.5	9	27	81
4	2.3	9.2	36.8	16	64	256
$\Sigma x = 10$	$\Sigma y = 8.9$	$\Sigma xy = 21.1$	$\Sigma x^2y = 66.3$	$\Sigma x^2 = 30$	$\Sigma x^3 = 100$	$\Sigma x^4 = 354$

Substituting in eq ①, eq ② and eq ③

$8.9 = 5a + 10b + 30c$ —— ④

$21.1 = 10a + 30b + 100c$ —— ⑤

$66.3 = 30a + 100c + 224c$ —— ⑥

solving eq ④, eq ⑤ and eq ⑥

$a = 1.0721$ $\quad b = 0.4157$ $\quad c = -0.0814$

$\therefore y = 1.0721 + 0.4157x - 0.0814x^2$ is the required second degree parabola.

Q3: Fit a parabola of second degree in the least square sense for the following data

x	1	2	3	4	5
y	25	28	33	39	46

Sol: The normal equations for a second degree parabola is given by:

$\Sigma y = na + b\Sigma x + c\Sigma x^2$ —— ①

$\Sigma xy = a\Sigma x + b\Sigma x^2 + c\Sigma x^3$ —— ②

$\Sigma x^2y = a\Sigma x^2 + b\Sigma x^3 + c\Sigma x^4$ —— ③

x	y	xy	x^2y	x^2	x^3	x^4
1	25	25	25	1	1	1
2	28	56	112	4	8	16
3	33	99	297	9	27	81
4	39	156	624	16	64	256
5	46	230	1150	25	125	625
Σx	Σy	Σxy	Σx^2y	Σx^2	Σx^3	Σx^4
$=15$	$=171$	$=566$	$=2208$	$=55$	$=225$	$=979$

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Substituting in eq ①, eq ② and eq ③

$171 = 5a + 15b + 55c$

$566 = 15a + 55b + 225c$

$2208 = 55a + 225b + 979c$

$a = 22.8 \quad b = 1.4428 \quad c = 0.6428$

$\therefore y = 22.8 + 1.4428x + 0.6428x^2$ is the required second degree parabola.

Q4: Fit a parabola of second degree in the least square sense for the following data.

x	-2	-1	0	1	2
y	-3.15	-1.39	0.62	2.88	5.378

sol: The normal equations for a second degree parabola is given

$\Sigma y = na + b\Sigma x + c\Sigma x^2$ — ①

$\Sigma xy = a\Sigma x + b\Sigma x^2 + c\Sigma x^3$ — ②

$\Sigma x^2 y = a\Sigma x^2 + b\Sigma x^3 + c\Sigma x^4$ — ③

x	y	xy	x^2y	x^2	x^3	x^4
-2	-3.15	6.3	-12.6	4	-8	16
-1	-1.39	1.39	-1.39	1	-1	1
0	0.62	0	0	0	0	0
1	2.88	2.88	2.88	1	1	1
2	5.378	10.756	21.512	4	8	16
$\Sigma x = 0$	$\Sigma y = 4.338$	$\Sigma xy = 21.326$	$\Sigma x^2y = 10.402$	$\Sigma x^2 = 10$	$\Sigma x^3 = 0$	$\Sigma x^4 = 34$

Substituting in eq ①, eq ② and eq ③

$4.338 = 5a + 10c$ — ④

$21.326 = 10b$ — ⑤

$10.402 = 10a + 34c$ — ⑥

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solving eq ④, eq ⑤ and eq ⑥

$a = 0.621$ $\quad b = 2.1326$ $\quad c = 0.1232$

$\therefore y = 0.621 + 2.1326x + 0.1232x^2$ is the required second degree parabola.

- Fitting of curves of the form:
  1. $y = ab^x$
  2. $y = ax^b$
  3. $y = ae^{bx}$

- 1. $y = ab^x$

Taking $\log_e$ on both sides

$\log_e y = \log_e(ab^x)$

$\log_e y = \log_e a + x\log_e b$

$Y = A + BX$

where $Y = \log_e y$; $A = \log_e a \Rightarrow a = e^A$

$X = x$; $B = \log_e b \Rightarrow b = e^B$

- 2. $y = ax^b$

Taking $\log_e$ on both sides

$\log_e y = \log_e (ax^b)$

$\log_e y = \log_e a + b\log_e x$

$Y = A + BX$

where $Y = \log_e y$; $A = \log_e a \Rightarrow a = e^A$

$X = \log_e x$; $B = b$

- 3. $y = ae^{bx}$

Taking $\log_e$ on both sides

$\log_e y = \log_e (ae^{bx})$

$\log_e y = \log_e a + bx\log_e e = \log_e a + bx$

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$Y = A + BX$

where $Y = \log_e y$; $A = \log_e a \Rightarrow a = e^A$

$X = x$; $B = b$.

The normal equations for the above three curves is given

$\Sigma Y = nA + B\Sigma X$

$\Sigma XY = A\Sigma X + B\Sigma X^2$

Q1: Fit a curve of the form $y = ab^x$ for the data and hence estimate y when $x = 8$.

x	0	1	2	3	4	5	6
y	32	47	65	92	132	190	275

Sol: We have $y = ab^x$

taking $\log_e$ on both sides

$\log_e y = \log_e(ab^x)$

$\log_e y = \log_e a + x\log_e b$

$Y = A + BX$

where $Y = \log_e y$ $\quad X = x$

$A = \log_e a$ $\quad B = \log_e b$

$\Rightarrow a = e^A$ $\quad \Rightarrow b = e^B$

X	$Y = \log_e y$	XY	X^2
0	3.46573	0	0
1	3.85014	3.85014	1
2	4.17438	8.34876	4
3	4.52178	13.56534	9
4	4.88280	19.53120	16
5	5.24702	26.23510	25
6	5.61677	33.70062	36
$\Sigma X = 21$	$\Sigma Y = 31.75862$	$\Sigma XY = 105.23116$	$\Sigma X^2 = 91$

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The normal equations is given by

$\Sigma Y = nA + B\Sigma X$ — (1)

$\Sigma XY = A\Sigma X + B\Sigma X^2$ — (2)

Substituting in eq (1) and eq (2)

$31.75862 = 7A + 21B$ — (3)

$105.23116 = 21A + 91B$ — (4)

solving eq (3) and eq (4)

$A = 3.4403$ $B = 0.3555$

$\therefore a = e^A = e^{3.4403} = 32.14638$

$b = e^B = e^{0.3555} = 1.42689$

~~y = log_e~~

$\therefore y = (32.14638)(1.42689)^x$ is the required curve.

at $x = 8$

$y = (32.14638)(1.42689)^8$

$y = 552.403$

Q2: At constant temperature, the pressure P and the volume V of a gas are connected by the relation PV^γ = constant. Find the best fitting equation of this form to the following data and estimate V at P = 4.

P(kg sq cm)	0.5	1.0	1.5	2.0	2.5	3.0
V (c.c)	1620	1000	750	620	520	460

Sol: Consider $PV^\gamma = K$

taking $\log_e$ on both sides

$\log_e P + \gamma \log_e V = \log_e k \Rightarrow \log_e P = \log_e k - \gamma \log_e V$

$y = a + bx$

where $a = \log_e k$ $y = \log_e P$ $x = \log_e V$ $b = -\gamma$

The normal equations is given by

$\Sigma y = na + b\Sigma x$ —— ①

$\Sigma xy = a\Sigma x + b\Sigma x^2$ —— ②

$x = \log_e V$	$y = \log_e P$	xy	x^2
7.39018	−0.69314	−5.12242	54.61476
6.90775	0	0	47.71701
6.62007	0.40546	2.68417	43.82532
6.42971	0.69314	4.45668	41.34114
6.25382	0.91629	5.73031	39.11026
6.13122	1.09861	6.73581	37.59185
Σx = 39.73275	Σy = 2.42036	Σxy = 14.48455	Σx^2 = 264.20037

Substituting in eq ① and eq ②

$2.42036 = 6a + 39.73275\,b$ —— ③

$14.48455 = 39.73275a + 264.20037$ —— ④

solving eq ③ and eq ④

$a = 9.82199$ $\quad b = -1.42229$

$a = \log_e k$

$\Rightarrow k = e^a$

$k = 18434.69$

$b = -\gamma$

$\Rightarrow \gamma = 1.42229$

$PV^{1.42229} = 18434.69$ is the required curve

at $P = 4$

$4V^{1.42229} = 18434.69$

$V^{1.42229} = 4608.6725$

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Q3: Fit the curve ax^b for the following data

x	1	2	3	4	5
y	0.5	2	4.5	8	12.5

Sol: we have $y = ax^b$

taking $\log_e$ on both sides

$$\log_e y = \log_e a + b \log_e x$$
$$Y = A + BX$$

where $Y = \log_e y$ $\quad X = \log_e x$

$A = \log_e a \quad B = b.$

x	$X = \log_e x$	y	$Y = \log_e y$	XY	X^2
1	0	0.5	-0.69314	0	0
2	0.69314	2	0.69314	0.48044	0.48042
3	1.09861	4.5	1.50407	1.65238	1.20694
4	1.38629	8	2.07944	2.88240	1.92179
5	1.60943	12.5	2.52572	4.06496	2.59026
	$\Sigma X = 4.78747$		$\Sigma Y = 6.10923$	$\Sigma XY = 9.08048$	$\Sigma X^2 = 6.19943$

The normal equations is given by

$$\Sigma Y = nA + B\Sigma X \quad \text{--- (1)}$$
$$\Sigma XY = A\Sigma X + B\Sigma X^2 \quad \text{--- (2)}$$

Substituting in eq (1) and eq (2)

$$6.10923 = 5A + 4.78747B \quad \text{--- (3)}$$
$$9.08048 = 4.78747A + 6.19943B \quad \text{--- (4)}$$

Solving eq (3) and eq (4)

$A = -0.69315 \qquad B = 2.00001$

$\log_e a = -0.69315 \qquad b = 2.00001$

$a = 0.499$

$\therefore y = 0.5x^2$ is the required equation.

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Q4: Fit the curve $y = ax^b$ for the following data.

x	1	2	3	4	5	6
y	2.98	4.26	5.21	6.1	6.8	7.5

Sol: we have $y = ax^b$

Taking log on both sides

$\log_e y = \log_e a + b \log_e x$

$Y = A + BX$

where $Y = \log_e y$, $X = \log_e x$

$A = \log_e a$, $B = b$

x	$X = \log_e x$	y	$Y = \log_e y$	XY	X^2
1	0	2.98	1.0919	0	0
2	0.6931	4.26	1.4492	1.0045	0.4803
3	1.0986	5.21	1.6505	1.8132	1.2069
4	1.3862	6.1	1.8082	2.5065	1.9215
5	1.6094	6.8	1.9169	3.0850	2.5901
6	1.7917	7.5	2.0149	3.6100	3.2101
	$\Sigma X = 6.579$		$\Sigma Y = 9.9316$	$\Sigma XY = 12.0192$	$\Sigma X^2 = 9.4089$

The normal equations are given by

$\Sigma Y = nA + B\Sigma X$ —— ①

$\Sigma XY = A\Sigma X + B\Sigma X^2$ —— ②

Substituting in eq ① and eq ②

$9.9316 = 6A + 6.579B$ —— ③

$12.0192 = 6.579A + 9.40891B$ —— ④

Solving eq ③ and eq ④

$A = 1.0911$ $B = 0.5144$

$\log_e A = 1.0911$ $b = 0.5144$

$a = 2.977$

$\therefore y = 2.977x^{0.5144}$ is the required curve

Q5: Fit the curve $y = ae^{bx}$ for the following data

x	0	2	4
y	8.12	10	31.82

Sol: we have $y = a e^{bx}$

Taking log on both sides

$\log_e y = \log_e a + bx \log_e e$

$\log_e y = \log_e a + bx$

$Y = A + BX$

where $Y = \log_e y \quad X = x$

$A = \log_e a \quad B = b$

$x = X$	y	$Y = \log_e y$	XY	X^2
0	8.12	2.0943	0	0
2	10	2.3025	4.6050	4
4	31.82	3.4600	13.8400	16
$\Sigma X = 6$		$\Sigma Y = 7.8568$	$\Sigma XY = 18.4450$	$\Sigma X^2 = 20$

The normal equations are given by

$\Sigma Y = nA + B\Sigma X$ —— (1)

$\Sigma XY = A\Sigma X + B\Sigma X^2$ —— (2)

Substituting in eq (1) and eq (2)

$7.8568 = 3A + 6B$ —— (3)

$18.4450 = 6A + 20B$ —— (4)

solving eq (3) and eq (4)

$A = 1.936 \qquad B = 0.3414$

$\log_e a = 1.936 \qquad b = 0.3414$

$a = 6.9309$

$\therefore y = 6.9309 e^{0.3414x}$ is the required curve.

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Q6: Fitt the curve $y = ae^{bx}$ for the following data

x	5	6	7	8	9	10
y	133	55	23	7	2	2

Sol: We have $y = ae^{bx}$

taking log on both sides

$$\log_e y = \log_e a + bx \log_e e$$

$$\log_e y = \log_e a + bx$$

$$Y = A + BX$$

where $Y = \log_e y$ $\quad A = \log_e a$

$B = b$ $\quad X = x$

$x = X$	y	$Y = \log_e y$	XY	X^2
5	133	4.8903	24.4515	25
6	55	4.0073	24.0438	36
7	23	3.1354	21.9478	49
8	7	1.9459	15.5672	64
9	2	0.6931	6.2379	81
10	2	0.6931	6.931	100
$\Sigma X = 45$		$\Sigma Y = 15.3651$	$\Sigma XY = 99.1792$	$\Sigma X^2 = 355$

The normal equations are given by

$$\Sigma Y = nA + B\Sigma X \quad (1)$$

$$\Sigma XY = A\Sigma X + B\Sigma X^2 \quad (2)$$

substituting in eq (1) and eq (2)

$$15.3651 = 6A + 45B \quad (3)$$

$$99.1792 = 45A + 355B \quad (4)$$

solving eq (3) and eq (4)

$A = 9.4433$ $\quad B = -0.91766$

$\log_e a = 9.4433$ $\quad b = -0.91766$

$a = 12623.3$

$$\therefore y = 12623.3e^{(-0.91766)x}$$

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Probability

Random variables
- Discrete random variables
Binomial and Poisson distributions

★ **Random Variables**

Probability: The extent to which an event E is likely to occur, measured by the ratio of the favourable cases to the number of cases possible.

i.e., $P(E) = \dfrac{\text{number of favourable cases}}{\text{Total number of cases}}$

Exhaustive event: An event consisting of all the various possibilities is an exhaustive event

Mutually Exclusive events: Two or more events are said to be mutually exclusive if the happening of one event prevents the simultaneous happening of the other.

Ex: In the tossing of a coin, obtaining a head or tail is mutually exclusive in the view of the fact that if head is the turnout, getting tail is not possible.

Independent events: Two or more events are said to be independent if the happening or not happening of the event does not prevent the happening or not happening of the other.

Ex: If two coins are tossed, getting a head is an independent event as both coins can turn out to be head.

★ **Properties:**

1. $P(E) + P(\bar{E}) = 1$
2. $P(A \cup B) = P(A) + P(B) - P(A \cap B)$

   In mutually exclusive event $P(A \cap B) = 0$.
3. $P(A \cap B) = P(A) \cdot P\left(\frac{B}{A}\right)$.

Random variable (X) is a real valued function whose domain is the sample space of a random experiment.

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★ **Discrete Probability Distribution:**

For each value of x_i of a discrete random variable the mean and variance of a discrete probability distribution is given by

i. **Mean** (μ)

$$\mu = \sum_i x_i P(x_i)$$

ii. **Variance** (V)

$$V = \sum_i (x_i - \mu)^2 P(x_i)$$

iii. **Standard Deviation** (SD)

$$SD = \sqrt{V}$$

Also $P(x_i) \geq 0$

and $\sum_i P(x_i) = 1$.

Q1: A coin is tossed twice. A random variable X represents the number of heads turning up. Find the discrete probability distribution for X. Also find its mean and variance.

Sol: Here $S = \{HH, HT, TH, TT\}$

The association of the elements of S to the random variable X are respectively 0, 1, 1, 2.

$\therefore P(HH) = 1/4$

$P(HT) = 1/4$

$P(TH) = 1/4$

$P(TT) = 1/4$

$P(X=0$, i.e. no head$) = P(TT) = 1/4$

$P(X=1$, i.e. one head$) = P(HT \cup TH) = P(HT) + P(TH) - P(HT \cap TH)$

$P(HT \cup TH) = 1/4 + 1/4 = 1/2$

$P(X=2$, i.e. two heads$)$ $P(HH) = 1/4$

The discrete probability distribution for X is as follows

$X = x_i$	0	1	2
$P(x_i)$	1/4	1/2	1/4

Here $P(x_i) > 0$

and $\sum P(x_i) = 1$

Mean

$\mu = \sum_i x_i P(x_i) = 0(1/4) + 1(1/2) + 2(1/4)$

$\mu = 1$

Variance

$V = \sum_i (x_i - \mu)^2 P(x_i) = (0-1)^2(1/4) + (1-1)^2(1/2) + (2-1)^2(1/4)$

$V = 1/4 + 1/4$

$V = 1/2$

Q2: A random experiment of rolling a die twice is performed. Random variable X on this sample space is defined to be the sum of two numbers turning up on the toss. Find the discrete probability distribution for X. compute the corresponding mean and standard deviation.

Sol: Here $S = \{(x, y)\}$ where $x = 1,2,3,4,5,6$ and $y = 1,2,3,4,5,6$.

Number of elements in S : $n(S) = 36$

The set of values of the random variable X defined as sum of two numbers on the face of the die.

i.e., X is 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12

The association of the elements of S to the sample variable X are as follows.

Elements of S	$X = x_i$	Total number of events
(1,1)	2	1
(1,2)(2,1)	3	2
(1,3)(3,1)(2,2)	4	3
(1,4)(4,1)(2,3)(3,2)	5	4
(1,5)(5,1)(2,4)(4,2)(3,3)	6	5
(1,6)(6,1)(2,5)(5,2)(3,4)(4,3)	7	6
(2,6)(6,2)(3,5)(5,3)(4,4)	8	5

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	9		4
(3,6)(6,3)(4,5)(5,4)	10		3
(4,6)(6,4)(5,5)	11		2
(5,6)(6,5)	12		1
(6,6)			

~~P(x=2, i.e sum=2) = P[(1,1)] = 1/36~~

$\rightarrow$ Discrete probability distribution
$P(x_i) > 0$ and $\Sigma P(x_i) = 1$.

$X = x_i$	$P(x_i)$
2	1/36
3	2/36 = 1/18
4	3/36 = 1/12
5	4/36 = 1/9
6	5/36
7	6/36 = 1/6
8	5/36
9	4/36 = 1/9
10	3/36 = 1/12
11	2/36 = 1/18
12	1/36

Mean:

$$\mu = \sum_i x_i P(x_i)$$
$$= 2(1/36) + 3(1/18) + 4(1/12) + 5(1/9) + 6(5/36) + 7(1/6) + 8(5/36) + 9(1/9) + 10(1/12) + 11(1/18) + 12(1/36)$$

$$\mu = 7$$

Variance

$$V = \sum_i (x_i - \mu)^2 P(x_i)$$
$$= (2-7)^2(1/36) + (3-7)^2(1/18) + (4-7)^2(1/12) + (5-7)^2(1/9) + (6-7)^2(5/36) + (7-7)^2(1/6) + (8-7)^2(5/36) + (9-7)^2(1/9) + (10-7)^2(1/12) + (11-7)^2(1/18) + (12-7)^2(1/36)$$

$$V = 5.8333$$

Standard deviation

$$SD = \sqrt{V} = 2.4152$$

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Q8: A random variable x has the following probability function for the various values of x.

$x = x_i$	0	1	2	3	4	5	6	7
$P(x_i)$	0	k	2k	2k	3k	k^2	$2k^2$	$7k^2+k$

i. Find k.

ii. Evaluate $P(x<6)$

iii. Evaluate $P(3 \le x \le 6)$

sol: i. we know that

$\sum P(x_i) = 1$

$\therefore \; 0+k+2k+2k+3k+k^2+2k^2+7k^2+k = 1$

$9k + 10k^2 = 1$

$10k^2 + 9k - 1 = 0$

$10k^2 + 10k - k - 1 = 0$

$10k(k+1) - 1(k+1) = 0$

$10k - 1 = 0 \qquad k+1 = 0$

$k = 1/10 \qquad k = -1$

but $P(x_i) > 0$

$\therefore \underline{k = 0.1}$

Discrete Probability distribution for X is

$X = x_i$	0	1	2	3	4	5	6	7
$P(x_i)$	0	0.1	0.2	0.2	0.3	0.01	0.02	0.17

ii. Number of elements : $n(E) = 8$

$P(x<6) = 0 + 0.1 + 0.2 + 0.2 + 0.3 + 0.01 = \underline{0.81}$

iii $P(3 \le x \le 6) = 0.2 + 0.3 + 0.01 + 0.02 = \underline{0.53}$

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Q4: A probability distribution of finite random variable, is given by the following:

x_i	-2	-1	0	1	2	3
$P(x_i)$	0.1	k	0.2	2k	0.3	k

Find the value of k and also mean and variance.

Sol: we know that

$P(x_i) > 0$ and $\sum_i P(x_i) = 1$

$\therefore 0.1 + k + 0.2 + 2k + 0.3 + k = 1$

$4k + 0.6 = 1$

$4k = 0.4$

$k = 0.1$

$\therefore$ Discrete Probability Distribution of X is

x_i	-2	-1	0	1	2	3
$P(x_i)$	0.1	0.1	0.2	0.2	0.3	0.1

Mean

$\mu = \sum_i x_i [P(x_i)]$

$= -2(0.1) - 1(0.1) + 0(0.2) + 1(0.2) + 2(0.3) + 3(0.1)$

$= -\cancel{0.2} - \cancel{0.1} + 0.2 + 0.6 + \cancel{0.3}$

$\mu = 0.8$

Variance

$V = \sum_i (x_i - \mu)^2 P(x_i)$

$= (-2-0.8)^2(0.1) + (-1-0.8)^2(0.1) + (0-0.8)^2(0.2)$
$+ (1-0.8)^2(0.2) + (2-0.8)^2(0.3) + (3-0.8)^2(0.1)$

$V = 2.16$

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Q8: If X is a discrete random variable taking the values 1, 2, 3... with $P(x) = \frac{1}{2}\left[\frac{2}{3}\right]^x$. Find P(X, being an odd number).

By first establishing that $p(x)$ is a probability function.

Sol: consider $\sum_i P(x_i)$

$$\sum_i P(x_i) = \sum_{x=1}^{\infty} \frac{1}{2}\left[\frac{2}{3}\right]^x$$

$$\sum_i P(x_i) = \frac{1}{2}\sum_{x=1}^{\infty}\left[\frac{2}{3}\right]^x$$

$$\sum_i P(x_i) = \frac{1}{2}\left[\left(\frac{2}{3}\right)+\left(\frac{2}{3}\right)^2+\left(\frac{2}{3}\right)^3+\cdots\right]$$

$$\sum_i P(x_i) = \frac{1}{2}\left[\frac{2/3}{1-2/3}\right] = \frac{1}{2}\left[\frac{2/3}{1/3}\right]$$

$$\therefore \sum_i P(x_i) = 1$$

Therefore $P(x)$ is a probability function.

$$P(X\text{, being odd number}) = \sum_{x=1,3,5\ldots}^{\infty} P(x_i)$$

$$P(x) = \sum_{x=1,3,5}^{\infty} \frac{1}{2}\left[\frac{2}{3}\right]^x$$

$$P(x) = \frac{1}{2}\left[\frac{2}{3}+\left(\frac{2}{3}\right)^3+\left(\frac{2}{3}\right)^5+\cdots\right]$$

$$P(x) = \frac{1}{2}\left[\frac{2/3}{1-(2/3)^2}\right]$$

$$P(x) = \frac{1}{\cancel{2}}\left[\frac{\cancel{2}/\cancel{3}}{5/\cancel{9}\,3}\right]$$

$$P(x) = \frac{3}{5}$$

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Q6: The range of random variable $x = \{1, 2, 3 \ldots n\}$ and the probabilities of x are kx. Find the value of k and also compute the mean and variance of the probability distribution

NOTE : Alternative formula for variance

$$V = \sum_i x_i^2 P(x_i) - \mu^2$$

Sol:

x_i	1	2	3	4	5 ..... n
$P(x_i)$	k	$2k$	$3k$	$4k$	$5k$ ..... nk

we know that

$$\sum_i P(x_i) = 1$$

$$\therefore k + 2k + 3k + 4k + 5k + \cdots + nk = 1$$

$$k(1 + 2 + 3 + 4 + 5 + \cdots + n) = 1$$

$$k\left[\frac{n(n+1)}{2}\right] = 1$$

$$k = \frac{2}{n(n+1)}$$

Mean

$$\mu = \sum_i x_i P(x_i)$$

$$= k + 4k + 9k + 16k + 25k + \cdots + n^2 k$$

$$= k(1 + 4 + 9 + 16 + 25 + \cdots + n^2)$$

$$= k\left[\frac{n(n+1)(2n+1)}{6}\right]$$

$$= \frac{2}{\cancel{n(n+1)}}\left[\frac{\cancel{n(n+1)}(2n+1)}{\cancel{6}\ 3}\right]$$

$$\mu = \frac{2n+1}{3}$$

Variance

$$V = \sum_i x_i^2 P(x_i) - \mu^2$$

$$V = k - \frac{(2n+1)^2}{9} + 8k - \frac{(2n+1)^2}{9} + 27k - \frac{(2n+1)^2}{9} + \cdots$$

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$$+ n^3 k - \frac{(2n+1)^2}{9}$$

$$v = k(1+8+27+64+125+\cdots+n^3) - \frac{(2n+1)^2}{9}$$

$$v = \frac{2}{n(n+1)}\left[\frac{n(n+1)}{2}\right]^2 - \frac{(2n+1)^2}{9}$$

$$v = \frac{n(n+1)}{2} - \frac{(2n+1)^2}{9}$$

$$v = \frac{9n(n+1) - 2(2n+1)^2}{18}$$

$$v = \frac{n^2 + 5n - 2}{18}$$

Q7. A random variable X has $P(x) = 2^{-x}$ where $x = 1, 2, 3 \ldots$ Show that $P(x)$ is a probability function. Also find

i. $P(X, \text{even})$

ii. $P(X, \text{divisible by } 3)$

iii. $P(X \geq 5)$

Sol:

x_i	1	2	3	4	5	n
$P(x_i)$	0.5	0.25	0.125	0.0625	0.03125	2^{-n}

we know that

$$\sum_i P(x_i) = 1$$

$$\therefore 0.5 + 0.25 + 0.125 + 0.0625 + 0.03125 + \cdots + 2^{-n}$$

$$= (1/2 + 1/4 + 1/8 + 1/16 + 1/32 + \cdots + 2^{-n})$$

$$= \left(\frac{1/2}{1 - 1/2}\right) = \frac{1/2}{1/2} = 1$$

$\therefore \sum_i P(x_i) = 1$ Hence $P(x)$ is a probability function.

$$\therefore P(x, \text{even}) = 1/4 + 1/16 + 1/64 + \dots$$

$$P(x, \text{even}) = \frac{1/4}{1-1/4} = \frac{1/4}{3/4}$$

$$P(x, \text{even}) = \frac{1}{3}$$

$$\text{ii. } P(x, \text{divisible by } 3) = 1/8 + 1/64 + 1/512 + \dots$$

$$P(x, \text{divisible by } 3) = \frac{1/8}{1-1/8} = \frac{1/8}{7/8}$$

$$P(x, \text{divisible by } 3) = \frac{1}{7}$$

$$\text{iii. } P(x \geq 5) = 1/32 + 1/64 + 1/128 + \dots$$

$$P(x \geq 5) = \frac{1/32}{1-1/2} = \frac{1/32}{1/2}$$

$$P(x \geq 5) = \frac{1}{16}$$

Q8: From a sealed box containing a dozen apples. It was found that 3 apples where perished. Optain the probability of perished apple when two apples are drawn at random. Also find the mean and variance of this distribution.

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* **Binomial Trial:**

A random experiment with only two possible outcomes categorised as success and failure is called as Bernoulli trial, where the probability of success is same for each trial.

* **Bernoulli's Theorem:**

The probability of 'x' successes in 'n' trials is given as ${}^nC_x p^x q^{n-x}$, where p is the probability of success and q is the probability of failure. Since p is the probability of success the probability of x successes is p.p.p... x times i.e., p^x.

Also 'x' successes implies n−x failures and since q is the probability of failures the probability of n−x failures is q^{n-x}.

By multiplication rule, the probability of simultaneous happening is $p^x q^{n-x}$

x successes in n trials can occur in ${}^nC_x$ ways and all these cases are favourable to the event. Hence by the addition rule the probability of x successes out of n trials is given by $p^x q^{n-x} + p^x q^{n-x} + \dots + p^x q^{n-x}$ for ${}^nC_x$ times

Therefore ${}^nC_x p^x q^{n-x}$ represents a binomial distribution.

* **Mean and Standard Deviation of the Binomial Distribution:**

1. **MEAN:**

$$\mu = \sum_{x=0}^{n} x P(x)$$

$$\mu = \sum_{x=0}^{n} x \, {}^nC_x p^x q^{n-x}$$

$$\mu = \sum_{x=0}^{n} \frac{x n!}{(n-x)! x!} p^x q^{n-x}$$

$$\mu = \sum_{x=0}^{n} \frac{x n!}{(n-x)!\, x(x-1)!} p^x q^{n-x}$$

$$\mu = \sum_{x=1}^{n} \frac{n!}{(n-x)!\,(x-1)!} p^x q^{n-x}$$

$$\mu = \sum_{x=1}^{n} \frac{n(n-1)!}{(n-x)!\,(x-1)!} p\, p^{x-1} q^{n-x}$$

$$\mu = np \sum_{x=1}^{n} \frac{(n-1)!}{(n-x)!\,(x-1)!} p^{x-1} q^{n-x}$$

$$\mu = np \sum_{x=1}^{n} \frac{(n-1)!}{(x-1)!\,((n-1)-(x-1))!} p^{x-1} q^{(n-1)-(x-1)}$$

Let $n-1 = m$ and $x-1 = y$

$$\mu = np \sum_{y=0}^{m} \frac{m!}{y!\,(m-y)!} p^y q^{m-y}$$

$$\mu = np \sum_{y=0}^{m} {}^{m}C_y\, p^y q^{m-y}$$

$$\mu = np\,(p+q)^m$$

$$\boxed{\mu = np}$$

2. VARIANCE:

$$V = \sum_{x=0}^{n} x^2 P(x) - \mu^2$$

consider $\sum_{x=0}^{n} x^2 P(x) = \sum_{x=0}^{n} [x(x-1)+x]\, P(x)$

$$= \sum_{x=0}^{n} x(x-1)P(x) + \sum_{x=0}^{n} x P(x)$$

$$= \sum_{x=0}^{n} x(x-1)\, {}^{n}C_x\, p^x q^{n-x} + np$$

$$= \sum_{x=0}^{n} x(x-1) \frac{n!}{(n-x)!\, x!} p^x q^{n-x} + np$$

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$$= \sum_{x=0}^{n} \frac{\cancel{x(x-1)}\, n!}{(n-x)!\, \cancel{x(x-1)}(x-2)!} p^x q^{n-x} + np$$

$$= \sum_{x=2}^{n} \frac{n!}{(n-x)!(x-2)!} p^x q^{n-x} + np$$

$$= \sum_{x=2}^{n} \frac{n(n-1)(n-2)!}{(n-x)!(x-2)!} p^2 p^{x-2} q^{n-x} + np$$

$$= n(n-1)p^2 \sum_{x=2}^{n} \frac{(n-2)!}{(x-2)!(n-x)!} p^{x-2} q^{n-x} + np$$

$$= n(n-1)p^2 \sum_{x=2}^{n} \frac{(n-2)!\, p^{x-2} q^{n-x}}{(x-2)!(n-x)!} + np$$

$$= n(n-1)p^2 \sum_{x=2}^{n} \frac{(n-2)!}{(x-2)![(n-2)-(x-2)]!} p^{x-2} q^{(n-2)-(x-2)} + np$$

$$= n(n-1)p^2 \sum_{x=2}^{n} {}^{(n-2)}C_{(x-2)}\, p^{x-2} q^{(n-2)-(x-2)} + np$$

$$= n(n-1)p^2 (p+q)^{n-2} + np$$

$$\therefore \sum_{x=0}^{n} x^2 p(x) = n(n-1)p^2 + np$$

Substituting in eq ①

$$V = \cancel{\sum_{x}} n(n-1)p^2 + np - (np)^2 \qquad \because \mu = np$$

$$= \cancel{n^2p^2} - np^2 + np - \cancel{n^2p^2}$$

$$= np(1-p)$$

$$= npq$$

Hence variance $\boxed{V = npq}$

3. STANDARD DEVIATION:

SD : $\sigma = \sqrt{V}$

$$\therefore \boxed{\sigma = \sqrt{npq}}$$

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Q1: Find the binomial probability distribution which has mean 2 and variance 4/3.

sol: given $np = 2$ and $npq = 4/3$

$npq = 4/3$

$2q = 4/3$

$q = 2/3$

$p + q = 1$

$p + \frac{2}{3} = 1$

$p = \frac{1}{3}$ $\Rightarrow n\left(\frac{1}{3}\right) = 2 \quad \therefore n = 6$

$P(x) = {}^nC_x\, p^x q^{n-x}$

$P(x) = {}^6C_x \left(\frac{1}{3}\right)^x \left(\frac{2}{3}\right)^{6-x}$

$x=0$: $P(0) = {}^6C_0 \left(\frac{1}{3}\right)^0 \left(\frac{2}{3}\right)^{6-0} = 1\left(\frac{1}{3}\right)\left(\frac{2}{6}\right)^6 = \left(\frac{2}{6}\right)^6$

$x=1$: $P(1) = {}^6C_1 \left(\frac{1}{3}\right)^1 \left(\frac{2}{3}\right)^5 = 2\left(\frac{2}{3}\right)^5$

$x=2$: $P(2) = {}^6C_2 \left(\frac{1}{3}\right)^2 \left(\frac{2}{3}\right)^4 = 15\left(\frac{1}{3}\right)^2\left(\frac{2}{3}\right)^4$

$x=3$: $P(3) = {}^6C_3 \left(\frac{1}{3}\right)^3 \left(\frac{2}{3}\right)^3 = 20\left(\frac{1}{3}\right)^3\left(\frac{2}{3}\right)^3$

$x=4$: $P(4) = {}^6C_2 \left(\frac{1}{3}\right)^4 \left(\frac{2}{3}\right)^2 = 15\left(\frac{1}{3}\right)^4\left(\frac{2}{3}\right)^2$

$x=5$: $P(5) = {}^6C_1 \left(\frac{1}{3}\right)^5 \left(\frac{2}{3}\right)^1 = 2\left(\frac{1}{3}\right)^5\left(\frac{2}{3}\right)$

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Q 2: When a coin is tossed 4 times find the probability of getting
i. exactly one head.
ii. atmost three heads
iii. atleast two heads

sol: $n = 4$

i. Success: $p = 1/2 \Rightarrow q = 1/2$

We have $P(x) = {}^nC_x p^x q^{n-x}$

$P(x) = 4C_x p^x q^{4-x}$

$P(x) = 4C_x \left(\frac{1}{2}\right)^x \left(\frac{1}{2}\right)^{4-x}$

exactly one head $\Rightarrow x = 1$

$P(1) = 4C_1 \left(\frac{1}{2}\right)\left(\frac{1}{2}\right)^3 = 4\left(\frac{1}{2}\right)^4$

$P(1) = 0.25$

ii. atmost three heads : ≤ 3

$P(0) = 4C_0 \left(\frac{1}{2}\right)^0 \left(\frac{1}{2}\right)^4 = \frac{1}{16} = 0.0625$

$P(1) = 4C_1 \left(\frac{1}{2}\right)^1 \left(\frac{1}{2}\right)^3 = 4\left[\frac{1}{2}\right]^4 = 0.25$

$P(2) = 4C_2 \left(\frac{1}{2}\right)^2 \left(\frac{1}{2}\right)^2 = 6\left(\frac{1}{2}\right)^4 = 0.375$

$P(3) = 4C_3 \left(\frac{1}{2}\right)^3 \left(\frac{1}{2}\right) = 4\left(\frac{1}{2}\right)^4 = 0.25$

$P(x) = 0.0625 + 0.25 + 0.375 + 0.25$

$P(x \leq 3) = 0.9375$

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iii. atleast 2 heads.

$P(x \geq 2) = 1 - [P(x=0) + P(x=1)]$

$P(x \geq 2) = 1 - \left[{}^4C_0 \left(\frac{1}{2}\right)^0 \left(\frac{1}{2}\right)^4 + {}^4C_1 \left(\frac{1}{2}\right)^1 \left(\frac{1}{2}\right)^3 \right]$

$P(x \geq 2) = 1 - 0.3125$

$P(x \geq 2) = 0.6875$

Q3: An airline knows that 5% of people making reservations on a certain flight will not turnup. consequently air policy is to sell 52 tickets for a flight that can only hold 50 passangers. What is the probability that there will be a seat for every passengers who turn up?

Sol: $n = 52$

$p = 5\% = 0.05$

$\therefore q = 0.95$

Let x denotes the number of passangers who will not turnup

We have $P(x) = {}^nC_x p^x q^{n-x}$

$P(x) = {}^{52}C_x p^x q^{52-x} = {}^{52}C_x (0.05)^x (0.95)^{52-x}$

A seat is assured for every passanger if the number of passanger who fail to turn up is more than or equal to 2

$\therefore P(x \geq 2) = [1 - P(x<2)]$

$P(x \geq 2) = [1 - P(x=0) - P(x=1)]$

$P(x \geq 2) = [1 - {}^{52}C_0 (0.05)^0 (0.95)^{52} - {}^{52}C_1 (0.05)^1 (0.95)^{51}]$

$P(x \geq 2) = [1 - 0.069 - 0.190]$

$P(x \geq 2) = 0.7405$

Q4: In 800 families with 5 children each how many families would be expected to have

i. 3 boys

ii. 5 girls

iii. either 2 or 3 boys

iv. atmost 2 girls by assuming probability of boy and girls are equal.

sol: $f(x) = 800\,P(x)$

$n = 5$ - five children. $p = 1/2$ (boy) $q = 1/2$ (girl)

Let x be the number of boys.

We have $P(x) = {}^nC_x\, p^x q^{n-x}$

$P(x) = {}^5C_x\, p^x q^{n-x}$

i. 3 boys

Here $x = 3$

$P(3) = {}^5C_3 \left(\frac{1}{2}\right)^3 \left(\frac{1}{2}\right)^{5-3} = 10\left(\frac{1}{2}\right)^3\left(\frac{1}{2}\right)^2$

$\therefore P(3) = 10\left(\frac{1}{2}\right)^5$ // for each family

For 800 families : $800P(3) = 250 = f(3)$ //

ii. 5 girls

Here $x = 0$

$P(0) = {}^5C_0 \left(\frac{1}{2}\right)^0 \left(\frac{1}{2}\right)^5 = 1\left(\frac{1}{2}\right)^5$

$P(0) = \left(\frac{1}{2}\right)^5$ // for each family

For 800 families : $f(0) = 800P(0) = 800(1/2)^5 = 25$ //

iii. either 2 or 3 boys.

$P(2) + P(3) = {}^5C_2 \left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)^3 + {}^5C_3 \left(\frac{1}{2}\right)^3\left(\frac{1}{2}\right)^2$

$= [10 + 10]\left(\frac{1}{2}\right)^5 = 20\left(\frac{1}{2}\right)^5$ // for each family

For 800 families : $f(2) + f(3) = 800[P(2) + P(3)] = 500$ //

iv. atmost 2 girls (less than or equal to 2)

0 girls $\Rightarrow x = 5$

1 girl $\Rightarrow x = 4$

2 girls $\Rightarrow x = 3$

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$$P(5)+P(4)+P(3) = {}^5C_5\left(\frac{1}{2}\right)^5\left(\frac{1}{2}\right)^0 + {}^5C_4\left(\frac{1}{2}\right)^4\left(\frac{1}{2}\right)^1 + {}^5C_3\left(\frac{1}{2}\right)^3\left(\frac{1}{2}\right)^2$$

$$= 1\left(\frac{1}{2}\right)^5 + 5\left(\frac{1}{2}\right)^5 + 10\left(\frac{1}{2}\right)^5$$

$$= 16\left(\frac{1}{2}\right)^5 \text{ // each family}$$

For 800 families: $f(5)+f(4)+f(3) = 800\left[P(5)+P(4)+P(3)\right]$

$$= 800\left[16\left(\frac{1}{2}\right)^5\right] = 400 \text{ //}$$

Q5: Show that for a binomial distribution:

$$P(x+1) = \frac{n-x}{x+1}\,\frac{p}{q}\,P(x)$$

Sol: We know that

$$P(x) = {}^nC_x\, p^x q^{n-x}$$

$$P(x+1) = {}^nC_{x+1}\, p^{x+1} q^{n-(x+1)}$$

$$P(x+1) = \frac{n!}{(n-(x+1))!\,(x+1)!}\, p^{x+1} q^{n-(x+1)}$$

$$P(x+1) = \frac{n!}{[n-(x+1)]!\,(x+1)x!}\, p^{x+1} q^{n-(x+1)}$$

$$P(x+1) = \frac{n!\,(n-x)}{(x+1)x!\,[n-(x+1)]!\,(n-x)}\, p^{x+1} q^{n-(x+1)}$$

$$P(x+1) = \frac{n!\,(n-x)}{(x+1)x!\,(n-x)!}\,\frac{p}{q}\,p^x q^{n-x}$$

$$P(x+1) = \frac{n!}{(n-x)!\,x!}\,\frac{n-x}{x+1}\,\frac{p}{q}\,p^x q^{n-x}$$

$$P(x+1) = {}^nC_x\,\frac{n-x}{x+1}\,\frac{p}{q}\,p^x q^{n-x}$$

$$P(x+1) = \frac{n-x}{x+1}\,\frac{p}{q}\,P(x)$$

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Q6: In sampling a large number of parts manufactured by company, the mean number of defectives in samples of 20 is 2. Out of 1000 such samples, how many would be expected to contain at least 3 defective parts?

Sol: Given: $np = 2 \quad n = 20$

$\Rightarrow p = 1/10 = 0.1$

wkt $p+q = 1 \Rightarrow q = 0.9$

Let x denotes the number of defective parts

$\therefore P(x) = {}^nC_x\, p^x q^{n-x}$

$P(x) = {}^{20}C_x\, (0.1)^x (0.9)^{20-x}$

atleast 3 defective parts $\Rightarrow x \geq 3$

$P(x \geq 3) = P(3) + P(4) + \cdots + P(20)$

$\therefore P(x \geq 3) = 1 - [P(0) + P(1) + P(2)]$

$= 1 - \left[{}^{20}C_0 (0.1)^0 (0.9)^{20} + {}^{20}C_1 (0.1)^1 (0.9)^{19} + {}^{20}C_2 (0.1)^2 (0.9)^{18}\right]$

$= 1 - [0.121 + 0.27 + 0.285]$

$= 0.324$ //

$\therefore$ The number of defective samples in 1000 samples is

$1000\, P(x \geq 3) = 1000 (0.324) = 324$ //

Q7: The probability of a shooter hitting a target is 1/3. How many times he should shoot the target, so that the probability of hitting the target at least once is more than 3/4?

Let p be probability of hitting a target be $1/3$

$\therefore q = 1 - 1/3 = 2/3$.

we have $P(x) = {}^nC_x\, p^x q^{n-x}$

$P(x) = {}^nC_x \left(\frac{1}{3}\right)^x \left(\frac{2}{3}\right)^{n-x}$

$P(x \geq 1) > \frac{3}{4}$ given condition

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$P(x \geq 1) = 1 - P(x=0)$

$\therefore 1 - P(x=0) > 3/4$

$\Rightarrow 1 - {}^nC_0 \left(\frac{1}{3}\right)^0 \left(\frac{2}{3}\right)^n > \frac{3}{4}$

$${}^nC_0 \left(\frac{2}{3}\right)^n < \frac{1}{4}$$

$$\left(\frac{2}{3}\right)^n < \frac{1}{4}$$

$$\left(\frac{2}{3}\right)^n < 0.25$$

By inspection

$n = 1$ $\quad 0.66 > 0.25$

$n = 2$ $\quad 0.44 > 0.25$

$n = 3$ $\quad 0.29 > 0.25$ $\qquad \therefore n = 4$

$n = 4$ $\quad 0.19 < 0.25$

Q8: Four coins are tossed 100 times and the following results were obtained. Fit a binomial distribution for the data and calculate the theoretical frequencies.

Data given:

Number of heads:	0	1	2	3	4
Frequency :	5	29	36	25	5

Sol: Let x denotes the number of heads and f denotes the corresponding frequency.

$$\text{Mean} = \frac{\Sigma f \cdot x}{\Sigma f}$$

Since the data is in the form of frequency distribution we shall first calculate the mean.

$$\text{Mean} = \frac{0+29+72+75+20}{5+29+36+25+5}$$

$\text{Mean} = \mu = 1.96$

From the binomial distribution we have $\mu = np$

$\therefore 1.96 = 4(p)$

$\underline{p = 0.49} \Rightarrow \underline{q = 0.51}$

we know that

$P(x) = {}^nC_x p^x q^{n-x}$

$P(x) = {}^4C_x (0.49)^x (0.51)^{4-x}$

now

$P(0) = {}^4C_0 (0.49)^0 (0.51)^4 = 0.0676$

$P(1) = {}^4C_1 (0.49)^1 (0.51)^3 = 0.259$

$P(2) = {}^4C_2 (0.49)^2 (0.51)^2 = 0.375$

$P(3) = {}^4C_3 (0.49)^3 (0.51)^1 = 0.24$

$P(4) = {}^4C_4 (0.49)^4 (0.51)^0 = 0.0576$

But $F(x) = 100P(x)$

$\therefore F(0) = 100P(0) = 100(0.0676) \approx 7$

$F(1) = 100P(1) = 100(0.259) \approx 26$

$F(2) = 100(P(2)) = 100(0.375) \approx 37$

$F(3) = 100P(3) = 100(0.24) \approx 24$

$F(4) = 100P(4) = 100(0.0576) \approx 6$

Q9: A manufacturer of solar heaters claims that 60% of his heaters work satisfactorily for 10 years. Assuming that his claim is legitimate compute the probability that

i. 4 of 5 heaters work satisfactorily

ii. atmost 2 of 5 heaters work satisfactorily for 10 years.

Sol: 60% work satisfactory for 10 years

$\Rightarrow p = 0.6 \quad \therefore q = 0.4$

Here $n = 5$ we have $P(x) = {}^nC_x p^x q^{n-x}$

i. 4 of 5 heaters work satisfactory

$P(x=4) = {}^5C_4 (0.6)^4 (0.4)^1 = \underline{0.2592}$

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ii. atmost 2 of 5 heaters work satisfactorily

$P(x \leq 2) = P(0) + P(1) + P(2)$

$P(x \leq 2) = 5C_0 (0.6)^0 (0.4)^5 + 5C_1 (0.6)^1 (0.4)^4 + 5C_2 (0.6)^2 (0.4)^3$

$P(x \leq 2) = 0.01024 + 0.0768 + 0.2304$

$P(x \leq 2) = 0.31744$

★ **Poisson Distribution:**

It is regarded as the limiting form of the binomial distribution where 'N' is very large ($N \to \infty$) and the probability of success 'p' is very small ($p \to 0$) so that np (i.e., mean) tends to a fixed finite constant say 'm'.

From binomial distribution we have,

$$P(x) = {}^{n}C_{x}\, p^x q^{n-x}$$

$$P(x) = \frac{n!}{(n-x)!\,x!}\, p^x q^{n-x}$$

$$P(x) = \frac{n(n-1)(n-2)\cdots(3)(2)(1)}{x!\,(n-x)(n-x-1)(n-x-2)\cdots}\, p^x q^{n-x}$$

$$P(x) = \frac{n(n-1)(n-2)\cdots(n-x+1)}{x!}\, p^x q^{n-x}$$

$$P(x) = \frac{n\, n^{x-1}}{x!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{x-1}{n}\right) p^x q^{n-x}$$

$$P(x) = \frac{(np)^x}{x!\, q^x}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{x-1}{n}\right) q^{n}$$

mean = m

$\therefore (np)^x = m^x$

wkt $q + p = 1$

$$\therefore q^n = (1-p)^n = \left(1-\frac{m}{n}\right)^n = \left[\left(1-\frac{m}{n}\right)^{-\frac{n}{m}}\right]^{-m}$$

let $-m/n = k$

then $q^n = \left[\left[1+k\right]^{1/k}\right]^{-m}$

wkt $\lim_{k \to 0} \left[(1+k)^{1/k} \right] = e$

$\therefore \quad q^n = e^{-m}$

since $q^x = (1-p^x)$

hence $q^x \to 1$ as $p \to 0$

Also

$$\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right) \cdots \cdots \left(1-\frac{x-1}{n}\right) \to 1 \text{ as } n \to \infty$$

$$\therefore \quad \boxed{P(x) = \frac{m^x e^{-m}}{x!}}$$

This is called poisson distribution.

$P(x)$ is such that e^{-m}; $m e^{-m}$; $\frac{m^2 e^{-m}}{2!}$; $\frac{m^3 e^{-m}}{3!}$ ......

Therefore $\sum_{x=0}^{\infty} P(x) = e^{-m} + me^{-m} + \frac{m^2 e^{-m}}{2!} + \frac{m^3 e^{-m}}{3!} + \ldots\ldots$

$$\sum_{x=0}^{\infty} P(x) = e^{-m}\left[1 + m + \frac{m^2}{2!} + \frac{m^3}{3!} + \cdots\right]$$

$$\sum_{x=0}^{\infty} P(x) = e^{-m} e^{m} = e^{0}$$

$$\therefore \quad \boxed{\sum_{x=0}^{\infty} P(x) = 1}$$

Hence $P(x)$ is a probability distribution function.

★ <u>Mean and variance for Poisson Distribution</u>

– <u>Mean</u>: $\mu = \sum_{x=0}^{\infty} x (P(x))$

$$\mu = \sum_{x=0}^{\infty} \frac{x\, m^x e^{-m}}{x!}$$

$$\mu = \sum_{x=0}^{\infty} \frac{\cancel{x}\, m\, m^{x-1} e^{-m}}{\cancel{x}(x-1)!}$$

$$\mu = m e^{-m} \sum_{x=1}^{\infty} \frac{m^{x-1}}{(x-1)!}$$

$$\mu = me^{-m}\left[1 + \frac{m}{1!} + \frac{m^2}{2!} + \frac{m^3}{3!} + \cdots\cdots\right]$$

$$\mu = me^{-m}e^{m}$$

$$\boxed{\mu = m = np}$$

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Unit - 6

PROBABILITY - II

★ **Continuous Probability Distribution:**

If for every x belonging to the range of a continuous random variable X, we assign a real number $f(x)$ satisfying the conditions:

1. $f(x) \geq 0$
2. $\int_{-\infty}^{\infty} f(x)\,dx = 1$

then $f(x)$ is called a continuous probability function or probability density function.

If (a,b) is a subinterval range space x, then the probability that x lies in (a,b) is defined to be

$$P(a \leq x \leq b) = \int_a^b f(x)\,dx$$

★ **Mean and Variance:**

If x is a continuous random variable with probability density function $f(x)$ where x lies between $-\infty$ to ∞, that is $-\infty < x < \infty$, then the mean μ and variance v is given by.

MEAN:

$$\mu = \int_{-\infty}^{\infty} x f(x)\,dx$$

VARIANCE:

$$\sigma^2 = \int_{-\infty}^{\infty} (x-\mu)^2 f(x)\,dx$$

or $\sigma^2 = \int_{-\infty}^{\infty} x^2 f(x)\,dx - \mu^2$

Q1: Find which of the following functions is a probability density function.

a. $f_1(x) = \begin{cases} 2x & ; 0 < x < 1 \\ 0 & ; \text{Otherwise} \end{cases}$

Sol: 1. $f_1(x) = 2x \; ; \; 0 < x < 1$

Hence $f_1(x) \geq 0$

2. Also

$$\int_0^1 2x \, dx = x^2 \Big|_0^1 = 1$$

$$\therefore \int_{-\infty}^{\infty} f_1(x) dx = 1$$

Therefore $f_1(x) \geq 0$ and $\int_{-\infty}^{\infty} f_1(x) dx = 1$. Hence $f_1(x)$ is a probability density function.

b. $f_2(x) = \begin{cases} 2x \; ; \; -1 < x < 1 \\ 0 \; ; \; \text{otherwise} \end{cases}$

Sol: 1. $f_2(x) = 2x \; ; \; -1 < x < 1$

$$f_2(x) = \begin{cases} 2x & -1 < x < 0 \\ 2x & 0 < x < 1 \\ 0 & \text{otherwise} \end{cases}$$

Clearly for $f_2(x) = 2x \; ; \; -1 < x < 0$ $\quad f_2(x) < 0$

2. $\int_{-\infty}^{\infty} f_2(x) dx = \int_{-1}^{1} 2x \, dx$

$$\int_{-\infty}^{\infty} f_2(x) dx = x^2 \Big|_{-1}^{1} = [1 - 1] = 0$$

Therefore $f_2(x) < 0$ and $\int_{-\infty}^{\infty} f_2(x) dx \neq 1$

Hence $f_2(x)$ is not a probability density function.

c. $f_3(x) = \begin{cases} |x| \; ; \; |x| \leq 1 \\ 0 \; ; \; \text{otherwise} \end{cases}$

Sol: 1. $f_3(x) = |x| \; ; \; |x| \leq 1 \Rightarrow f_3(x) \geq 0$

2. $\int_{-\infty}^{\infty} f_3(x)\,dx = \int_{-1}^{1} |x|\,dx$

$= \int_{-1}^{0} -x\,dx + \int_{0}^{1} x\,dx$

$= \frac{x^2}{2}\Big|_{0}^{-1} + \frac{x^2}{2}\Big|_{0}^{1}$

$\int_{-\infty}^{\infty} f_3(x)\,dx = \frac{1}{2} + \frac{1}{2} = 1$ //

Therefore $f_3(x)$ is a probability density function.

Q2: Find c such that

$$f(x) = \begin{cases} x/6 + c & ; 0 \le x \le 3 \\ 0 & ; \text{elsewhere} \end{cases}$$

is a probability density function. Also find $f(1 \le x \le 2)$

sol: Clearly $f(x) \geq 0$ if $c \geq 0$.

Also $\int_{-\infty}^{\infty} f(x)\,dx = 1$

i.e., $\int_{0}^{3} \left(\frac{x}{6} + c\right) dx = 1$

$\left[\frac{x^2}{12} + cx\right|_{0}^{3} = 1$

$\frac{3}{4} + 3c = 1$

$3c = \frac{1}{4}$

$c = \frac{1}{12}$ //

$f(1 \le x \le 2) = \int_{1}^{2} f(x)\,dx = \int_{1}^{2} \left(\frac{x}{6} + \frac{1}{12}\right) dx$

$= \left[\frac{x^2}{12} + \frac{x}{12}\right|_{1}^{2} = \frac{1}{12}\left[4 + 2 - 1 - 1\right] = \frac{1}{3}$ //

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Q2: Find k such that $f(x) = \begin{cases} kx^2 & ; 0 < x < 3 \\ 0 & ; \text{otherwise} \end{cases}$ is a PDF.

Also compute

a. $P(1 < x < 2)$

b. $P(x \leq 1)$

c. $P(x > 1)$

d. mean

e. variance.

Sol: clearly $f(x) \geq 0$ if $k > 0$

Also $\int_{-\infty}^{\infty} f(x)\,dx = 1$

$$\int_0^3 kx^2\,dx = 1$$

$$\left.\frac{kx^3}{3}\right|_0^3 = 1$$

$$\frac{27k}{3} = 1$$

$$k = 1/9$$

a. $P(1 < x < 2) = \int_1^2 \frac{x^2}{9}\,dx$

$P(1 < x < 2) = \left.\frac{x^3}{27}\right|_1^2 = \frac{8}{27} - \frac{1}{27} = \frac{7}{27}$ //

b. $P(x \leq 1) = \int_0^1 \frac{x^2}{9}\,dx$

$P(x \leq 1) = \left.\frac{x^3}{27}\right|_0^1 = \frac{1}{27}$ //

c. $P(x > 1) = \int_1^3 \frac{x^2}{9}\,dx$

$P(x > 1) = \left.\frac{x^3}{27}\right|_1^3 = \frac{27}{27} - \frac{1}{27} = \frac{26}{27}$ //

d. mean

$$\mu = \int_{-\infty}^{\infty} x f(x)\,dx$$

$$\mu = \int_0^3 x\left(\frac{x^2}{9}\right)dx$$

$$\mu = \frac{x^4}{36}\Big|_0^3 = \frac{81}{36}$$

$$\underline{\mu = 2.25}$$

e. variance

$$\sigma^2 = \int_{-\infty}^{\infty} x^2 f(x)\,dx - \mu^2$$

$$\sigma^2 = \int_0^3 x^2\left(\frac{x^2}{9}\right)dx - (2.25)^2$$

$$\sigma^2 = \frac{x^5}{45}\Big|_0^3 - 5.0625$$

$$\sigma^2 = \left[\frac{243}{45}\right] - 5.0625$$

$$\underline{\sigma^2 = 0.3375}$$

Q4: Find k such that $f(x) = \begin{cases} kxe^{-x} & ; 0 < x < 1 \\ 0 & ; \text{otherwise} \end{cases}$ is a PDF. Also find mean.

Sol: It is evident that $f(x) \geq 0$ if $k > 0$

and $\int_{-\infty}^{\infty} f(x)\,dx = 1$ since it is a PDF

$$\therefore \int_0^1 kxe^{-x}\,dx = 1$$

$$k\int_0^1 xe^{-x}\,dx = 1$$

$$k\left[x\frac{e^{-x}}{-1} - 1\,e^{-x}\Big|_0^1 = 1\right.$$

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$$k\left[-xe^{-x} - e^{-x}\right]_0^1 = 1$$

$$k\left[-1e^{-1} - e^{-1} + 0 + 1\right] = 1$$

$$k\left[-2e^{-1} + 1\right] = 1$$

$$k = \frac{e}{e-2}$$

Mean

$$\mu = \int_{-\infty}^{\infty} x f(x)\, dx$$

$$\mu = \int_0^1 x\left[\frac{e}{e-2}\right] xe^{-x}\, dx$$

$$\mu = \frac{e}{e-2}\int_0^1 x^2 e^{-x}\, dx$$

$$\mu = \frac{e}{e-2}\left[x^2\frac{e^{-x}}{-1} - 2xe^{-x} + 2\frac{e^{-x}}{-1}\right]_0^1$$

$$\mu = \frac{e}{e-2}\left[-x^2e^{-x} - 2xe^{-x} - 2e^{-x}\right]_0^1$$

$$\mu = \frac{e}{e-2}\left[-e^{-1} - 2e^{-1} - 2e^{-1} + 0 + 0 + 2\right]$$

$$\mu = \frac{e}{e-2}\left[\frac{-5}{e} + 2\right]$$

$$\mu = \frac{-5+2e}{e-2}$$

★ **Cummulative Distribution Function:**

If X is a continuous random variable with probability density function $f(x)$ then the function $F(x)$ is given by

$$F(x) = P(X \le x) = \int_{-\infty}^{x} f(x)\, dx$$

NOTE: If r is any real number then

1. $P(x \geq r) = \int_r^{\infty} f(x)\,dx$

2. $P(x < r) = 1 - \int_r^{\infty} f(x)\,dx$

* **Exponential Distribution:**

The continuous probability distribution having the probability density function $f(x)$ is given by

$$f(x) = \begin{cases} \alpha e^{-\alpha x} & \text{for } x > 0 \\ 0, & \text{otherwise} \end{cases} \Rightarrow \alpha > 0$$

is known as exponential distribution.

proof:

It is clear that $f(x) \geq 0$ and

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_0^{\infty} \alpha e^{-\alpha x}\,dx$$

$$= \frac{\alpha e^{-\alpha x}}{-\alpha}\Big|_0^{\infty}$$

$$= -e^{-\alpha x}\Big|_0^{\infty}$$

$$= 0 + 1$$

$$\int_{-\infty}^{\infty} f(x)\,dx = 1$$

Therefore $f(x)$ is a probability density function.

– **Mean and Variance of Exponential Distribution Function:**

Mean

$$\mu = \int_{-\infty}^{\infty} x f(x)\,dx$$

$$\mu = \int_0^{\infty} x\,\alpha e^{-\alpha x}\,dx$$

$$\mu = \alpha \int_0^{\infty} x e^{-\alpha x}\,dx$$

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$$\mu = \alpha \left[x \frac{e^{-\alpha x}}{-\alpha} - \frac{e^{-\alpha x}}{\alpha^2} \right]_0^\infty$$

$$\mu = \cancel{\alpha} \left[\frac{-x\alpha e^{-\alpha x} - e^{-\alpha x}}{\alpha^{\cancel{2}}} \right]_0^\infty$$

$$\mu = \left[\frac{0 - 0 + 0 + 1}{\alpha} \right]$$

$$\boxed{\mu = \frac{1}{\alpha}}$$

Variance

$$\sigma^2 = \int_{-\infty}^{\infty} (x-\mu)^2 f(x)\, dx$$

$$\sigma^2 = \int_0^\infty \left(x - \frac{1}{\alpha} \right)^2 \alpha e^{-\alpha x}\, dx$$

$$\sigma^2 = \alpha \int_0^\infty \left(x - \frac{1}{\alpha} \right)^2 e^{-\alpha x}\, dx$$

$$\sigma^2 = \alpha \left[\left(x - \frac{1}{\alpha} \right)^2 \frac{e^{-\alpha x}}{-\alpha} - 2\left(x - \frac{1}{\alpha} \right) \frac{e^{-\alpha x}}{\alpha^2} + 2 \frac{e^{-\alpha x}}{-\alpha^3} \right]_0^\infty$$

$$\sigma^2 = \left[-\left(x - \frac{1}{\alpha} \right)^2 e^{-\alpha x} - \frac{2}{\alpha}\left(x - \frac{1}{\alpha} \right) e^{-\alpha x} - \frac{2}{\alpha^2} e^{-\alpha x} \right]_0^\infty$$

$$\sigma^2 = \left[\left(\frac{-1}{\alpha} \right)^2 + \frac{2}{\alpha}\left(\frac{-1}{\alpha} \right) + \frac{2}{\alpha^2} \right]$$

$$\sigma^2 = \frac{1 - 2 + 2}{\alpha^2}$$

$$\boxed{\sigma^2 = \frac{1}{\alpha^2}}$$

Standard deviation

$$\boxed{\sigma = \frac{1}{\alpha}}$$

$\therefore$ Mean and SD for an exponential distribution are equal

* Normal Distribution:

The continuous probability distribution having the probability density function $f(x)$ is given by

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-(x-\mu)^2}{2\sigma^2}}$$

where x varies from $-\infty$ to ∞ i.e., $-\infty < x < \infty$ and μ varies from $-\infty$ to ∞ ie, $-\infty < \mu < \infty$ and $\sigma > 0$.

Proof:

Evidently $f(x) \geq 0$ and

$$\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{-(x-\mu)^2}{2\sigma^2}}\,dx$$

Let $t^2 = \frac{(x-\mu)^2}{2\sigma^2}$ as $x \to \infty$: $t \to \infty$

as $x \to -\infty$: $t \to -\infty$

~~then~~ $\cancel{\int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} e^{-t}\,dt}$

$$\Rightarrow t = \frac{x-\mu}{\sqrt{2}\sigma}$$

$$x = \mu + \sqrt{2}\sigma t$$

$$dx = \sqrt{2}\sigma\,dt$$

$$\therefore \int_{-\infty}^{\infty} f(x)\,dx = \int_{-\infty}^{\infty} \frac{1}{\sigma\sqrt{2\pi}} e^{-t^2} \sqrt{2}\sigma\,dt$$

$$= \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-t^2}\,dt$$

$$= \frac{2}{\sqrt{\pi}} \int_{0}^{\infty} e^{-t^2}\,dt \longrightarrow \text{gamma function.}$$

$$\int_{-\infty}^{\infty} f(x)\,dx = \frac{2}{\sqrt{\pi}} \frac{\sqrt{\pi}}{2} = 1$$

Therefore $f(x)$ is a probability density function.

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– Mean and variance of Normal Distribution

Mean

$$\mu = \int_{-\infty}^{\infty} x f(x)\, dx$$

$$\mu = \int_{-\infty}^{\infty} x \frac{e^{\frac{-(x-\mu)^2}{2\sigma^2}}}{\sigma\sqrt{2\pi}}\, dx$$

Put $\frac{(x-\mu)^2}{2\sigma^2} = t^2$

$$t = \frac{x-\mu}{\sqrt{2}\sigma}$$

$$x = \mu + \sqrt{2}\sigma t$$

$$dx = \sqrt{2}\sigma\, dt$$

$$\mu = \int_{-\infty}^{\infty} (\mu + \sqrt{2}\sigma t) \frac{e^{-t^2}}{\cancel{\sigma\sqrt{2}}\sqrt{\pi}} \cancel{\sqrt{2}\sigma}\, dt$$

$$\mu = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} (\mu + \sqrt{2}\sigma t)\, e^{-t^2}\, dt$$

$$\mu = \frac{1}{\sqrt{\pi}} \left[\mu 2 \int_{0}^{\infty} e^{-t^2} dt + \sqrt{2}\sigma \int_{-\infty}^{\infty} t e^{-t^2} dt \right]$$

$$\mu = \frac{1}{\cancel{\sqrt{\pi}}} \left[\mu \cancel{2} \frac{\cancel{\sqrt{\pi}}}{\cancel{2}} + \sqrt{2}\sigma (0) \right]$$

gamma function   odd function

$$\boxed{\mu = \text{Mean}}$$

Variance

$$\sigma^2 = \int_{-\infty}^{\infty} (x-\mu)^2 f(x)\, dx$$

$$\sigma^2 = \int_{-\infty}^{\infty} (x-\mu)^2 \frac{e^{\frac{-(x-\mu)^2}{2\sigma^2}}}{\sigma\sqrt{2\pi}}\, dx$$

Put $\frac{(x-\mu)^2}{2\sigma^2} = t^2$

$$t = \frac{x-\mu}{\sqrt{2}\sigma}$$

$$x = \mu + \sqrt{2}\sigma t$$

$$dx = \sqrt{2}\sigma\, dt$$

$$\sigma^2 = \int_{-\infty}^{\infty} (\sqrt{2}\sigma t)^2 \frac{e^{-t^2}}{\sigma\sqrt{2\pi}} \sqrt{2}\sigma\, dt$$

$$\sigma^2 = \frac{\sigma^2}{\sqrt{\pi}} \int_{-\infty}^{\infty} 2t^2 e^{-t^2} dt$$

$$\sigma^2 = \frac{1}{\sqrt{\pi}}\sigma^2 \left[\int_{-\infty}^{\infty} \underbrace{e^{-t^2}(-2t)}_{v} \underbrace{(-t)}_{u} dt \right]$$

$$\sigma^2 = \frac{2\sigma^2}{\sqrt{\pi}} \left[\int_{0}^{\infty} e^{-t^2}(-2t)(-t)\, dt \right]$$

$$\sigma^2 = \frac{2\sigma^2}{\sqrt{\pi}} \left[-t e^{-t^2} + \int_0^{\infty} e^{-t^2} dt \right]_0^{\infty}$$

gamma function

$$\sigma^2 = \frac{\cancel{2}\sigma^2}{\cancel{\sqrt{\pi}}} \left[0 + 0 + \frac{\cancel{\sqrt{\pi}}}{\cancel{2}} \right]$$

$$\boxed{\text{variance} = \sigma^2}$$

Q1: Find the CDF for the following PDF of a random variable X.

i. $f(x) = \begin{cases} 6x - 6x^2 & ; 0 < x \leq 1 \\ 0 & ; \text{otherwise} \end{cases}$

ii. $f(x) = \begin{cases} xe^{-x/2}/4 & ; 0 < x < \infty \\ 0 & ; \text{otherwise} \end{cases}$

iii. Exponential distribution.

Sol: We have CDF given by

$$F(x) = \int_{-\infty}^{x} f(x)\, dx$$

$$F(x) = \int_{-\infty}^{0} f(x)\, dx + \int_{0}^{x} f(x)\, dx$$

i. $F(x) = \int_{-\infty}^{0} 0\,dx + \int_{0}^{x} (6x-6x^2)\,dx$

$= (3x^2 - 2x^3)\Big|_0^x = 3x^2 - 2x^3$ //

ii. $F(x) = \int_{-\infty}^{0} 0\,dx + \int_{0}^{x} \frac{x e^{-x/2}}{4}\,dx$

$= \frac{1}{4}\int_0^x x e^{-x/2}\,dx$

$= \frac{1}{4}\left[\frac{x e^{-x/2}}{-1/2} - \frac{e^{-x/2}}{1/4} \right]_0^x$

$= \frac{-x e^{-x/2}}{2} - e^{-x/2} + 1$ //

iii. $f(x) = \begin{cases} \alpha e^{-\alpha x} & ; \quad x > 0 \\ 0 & ; \quad \text{otherwise} \end{cases}$

$F(x) = \int_{-\infty}^{0} 0\,dx + \int_0^x \alpha e^{-\alpha x}\,dx$

$= \frac{\cancel{\alpha} e^{-\alpha x}}{-\cancel{\alpha}}\Big|_0^x$

$= -e^{-\alpha x} + 1$ //

Q2: A continuous random variable has the distribution function $F(x) = \begin{cases} 0 & ; \quad x \le 1 \\ c(x-1)^4 & ; \quad 1 \le x \le 3 \\ 1 & ; \quad x > 3 \end{cases}$

Find c and also the PDF.

Sol: wkt PDF of $f(x) = \frac{d}{dx} F(x)$

$\therefore f(x) = \begin{cases} 0 & ; \quad x \le 1 \\ 4c(x-1)^3 & ; \quad 1 \le x \le 3 \\ 0 & ; \quad x > 3 \end{cases}$

$f(x) \geq 0$ and we must have $\int_{-\infty}^{\infty} f(x)dx = 1$.

$\therefore \int_{-\infty}^{1} \cancel{f(x)dx}^{0} + \int_{1}^{3} f(x)\,dx + \int_{3}^{-\infty} \cancel{f(x)dx}^{0} = 1$

$\int_{1}^{3} 4c(x-1)^3\,dx = 1$

$\cancel{4}c \dfrac{(x-1)^4}{\cancel{4}} \Big|_{1}^{3} = 1$

$c[(3-1)^4 - (1-1)^4] = 1$

$16c = 1$

$\underline{\underline{c = 1/16}}$

Q3: The life of a compressor manufactured by a company is known to be 200 months on an average following an exponential distribution. Find the probability that the life of the compressor is.

i. < 200 months.

ii. between 100 months and 25 years.

Sol: Exponential distribution

$$f(x) = \begin{cases} \alpha e^{-\alpha x} & x > 0 \\ 0 & \text{otherwise} \end{cases}$$

given : Mean: $\mu = 1/\alpha = 200$

$\underline{\alpha = 0.005}$

i. $\int_{0}^{200} f(x)dx = \int_{0}^{200} \alpha e^{-\alpha x}\,dx$

$= 0.005 \int_{0}^{200} e^{-0.005x}\,dx$

$= \cancel{0.005} \left[\dfrac{e^{-0.005x}}{-\cancel{0.005}} \right|_{0}^{200}$

$= -e^{-1} + 1 = 0.632$ //

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ii. $\int_{100}^{300} f(x)dx = \int_{100}^{300} \alpha e^{-\alpha x} dx$

$= 0.005 \int_{100}^{300} e^{-0.005x} dx$

$= \cancel{0.005} \left[\frac{e^{-0.005x}}{-\cancel{0.005}} \right]_{100}^{300}$

$= -[e^{-1.5} - e^{-0.5}] = 0.383$

Q4: The mikage which car owners get with a certain kind of radial tyre is a random variable is exponential distribution with mean 40000km. Find the probabilities that one of this tyre will last:

i. atleast 20000 km

ii. atmost 30000 km.

<u>sol</u>: We have mean given by

$\frac{1}{\alpha} = 40000 km \Rightarrow \alpha = \frac{1}{40000}$

1. Atleast 20000 km

$f(x) = \begin{cases} \alpha e^{-\alpha x} & x>0 \\ 0 & \text{otherwise} \end{cases}$

$1 - \int_{0}^{20000} f(x)dx = 1 - \int_{0}^{20000} \alpha e^{-\alpha x} dx$

$= 1 - \int_{0}^{20000} \frac{1}{40000} e^{-\frac{x}{40000}} dx$

$= 1 - \frac{1}{40000} \left[\frac{e^{-\frac{x}{40000}}}{-1/40000} \right]_{0}^{20000}$

$= 1 + \left[e^{-x/40000} \right]_{0}^{20000}$

$= \cancel{1} + e^{-1/2} - \cancel{e^{0}}$

$= 0.606$

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ii. Atmost 30000 km

$$\int_0^{30000} f(x)\,dx$$

$$= \int_0^{30000} \alpha e^{-\alpha x}\, dx$$

$$= \int_0^{30000} \frac{1}{40000} e^{\frac{-x}{40000}}\, dx$$

$$= \frac{1}{40000}\left[\frac{e^{\frac{-x}{40000}}}{-1/40000}\right]_0^{30000}$$

$$= -e^{-0.75} + e^0$$

$$= \underline{\underline{0.5276}}$$

* Standard Normal Distribution:

We have

$$P(a \le x \le b) = \int_a^b f(x)\,dx$$

In case of normal distribution

$$P(a \le x \le b) = \frac{1}{\sigma\sqrt{2\pi}} \int_a^b e^{\frac{-(x-\mu)^2}{2\sigma^2}}\, dx$$

Therefore put $\frac{(x-\mu)^2}{\sigma^2} = z^2$

$$\Rightarrow z = \frac{x-\mu}{\sigma}$$

When $x = a$; $z = \frac{a-\mu}{\sigma} = z_1$

$$x = z\sigma + \mu$$

$$dx = dz\,\sigma$$

When $x = b$; $z = \frac{b-\mu}{\sigma} = z_2$

$$\therefore f(a \le x \le b) = \frac{1}{\sqrt{2\pi}} \int_{z_1}^{z_2} e^{-\frac{z^2}{2}}\, dz$$

Therefore $\boxed{f(z_1 \le z \le z_2) = \frac{1}{\sqrt{2\pi}} \int_{z_1}^{z_2} e^{-z^2/2}\, dz}$

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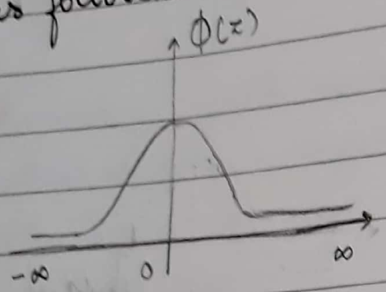
If $F(x) = \frac{1}{\sqrt{2\pi}} e^{\frac{-x^2}{2}}$

it is clear that $F(x)$ is same as normal distribution with $\mu = 0$ and $\sigma = 1$

$F(x)$ is a standard normal curve which is symmetric about the line $x = 0$. The curve of standard normal distribution is as follows:



Therefore

$$\phi(x) = \frac{1}{\sqrt{2\pi}} \int_{z_1=0}^{z_2} e^{-z^2/2} dz$$

In particular if $z_1 = 0$ then we have

$$\boxed{\phi(z) = \frac{1}{\sqrt{2\pi}} \int_0^z e^{-z^2/2} dz}$$

NOTE:

wkt $\phi(x) = \frac{1}{\sqrt{2\pi}} \int_0^x e^{-z^2/2} dz$

Also $\int_{-\infty}^{\infty} \phi(x) dz = 1$.

$$\int_{-\infty}^{0} \phi(x) dx + \int_0^{\infty} \phi(x) dx = 1$$

Since $\int_{-\infty}^{0} \phi(x) dx = \int_0^{\infty} \phi(x) dz = \frac{1}{2}$

i.e., $P(-\infty \le z \le \infty) = P(-\infty \le z \le 0) + P(0 \le z \le \infty) = 1$

consider $P(-\infty \le z \le z_1) = P(-\infty \le z \le 0) + P(0 \le z \le z_1)$

$$\boxed{\begin{aligned} P(-\infty \le z \le z_1) &= \frac{1}{2} + \phi(z_1) \\ P(z > z_1) &= \frac{1}{2} - \phi(z_1) \end{aligned}}$$

Standard Normal variant is given by $z = \frac{x - \mu}{\sigma}$

Q1: Evaluate the following with the help of normal probability table.

i. $P(z \geq 0.85)$

ii. $P(-1.64 \leq z \leq -0.88)$

iii. $P(z \leq -2.43)$

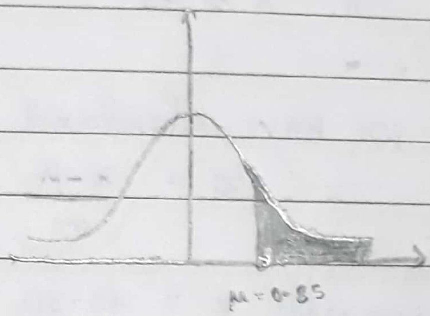
iv. $P(|z| \leq 1.94)$

Sol:- i. $P(Z \geq 0.85)$

$= 0.5 - \phi(0.85)$

$= 0.5 - 0.3023$

$\therefore P(Z \geq 0.85) = 0.1977$



ii. $P(-1.64 \leq z \leq -0.88)$

By symmetry

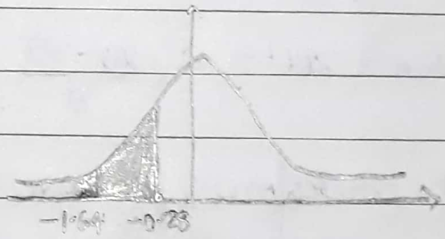
$P(-1.64 \leq z \leq -0.88)$

$= P(0 \leq z \leq 1.64) - P(0 \leq z \leq 0.88)$

$= \phi(1.64) - \phi(0.88)$

$= 0.4495 - 0.3106$

$= 0.1389$



iii. $P(z \leq -2.43)$

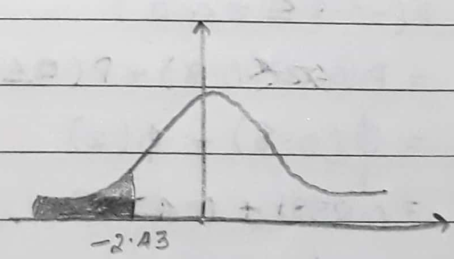
By symmetry

$P(z \leq -2.43) = P(z \geq 2.43)$

$= 0.5 - \phi(2.43)$

$= 0.5 - 0.4925$

$= 0.0075$

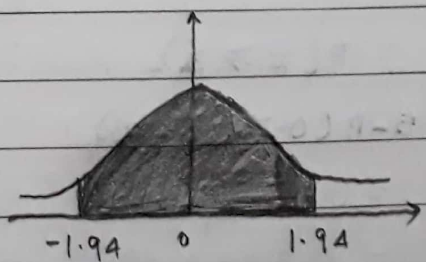


iv. $P(|z| \leq 1.94)$

$= P(-1.94 \leq z \leq 1.94)$

By symmetry

$P(|z| \leq 1.94) = 2P(0 \leq z \leq 1.94)$



$= 2\,\phi(1.94)$

$= 2(0.4738)$

$= 0.9476$

Q2: If x is a normal variant with mean 30 and standard deviation 5. find the probability that

i. $26 \le x \le 40$

ii. $x \geqslant 45$

Sol: We have standard normal variant

$$z = \frac{x-\mu}{\sigma}$$

i.

when $x=40$: $z = \frac{40-30}{5} = 2$

when $x=26$: $z = \frac{26-30}{5} = -0.8$

∴ Therefore we have to find

$P(-0.8 \le z \le 2)$

By symmetry

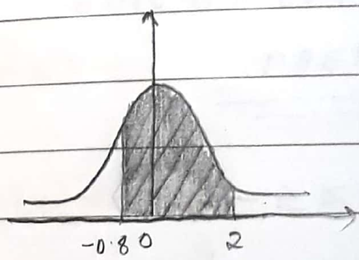
$P(-0.8 \le z \le 2)$

$= P(0 \le z \le 0.8) + P(0 \le z \le 2)$

$= \phi(0.8) + \phi(2)$

$= 0.2881 + 0.4772$

$= 0.7653$



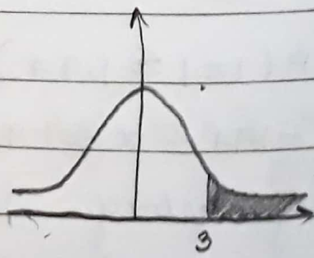
ii. $z = \frac{x-\mu}{\sigma} = \frac{45-30}{5} = 3$

∴ $P(z \geqslant 3)$

$= 0.5 - P(0 \le z \le 3)$

$= 0.5 - \phi(3)$

$= 0.5 - 0.4987 = 0.0013$



Q3: If x is normally distributed with mean 12 and standard deviation 4, then find

i. $P(x \geq 20)$

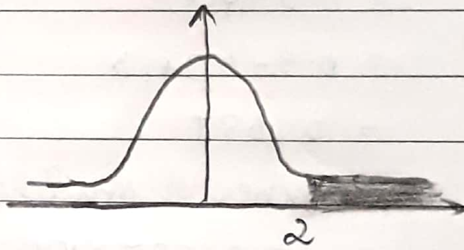
ii. $P(x \leq 20)$

sol: we have standard normal variant

$$z = \frac{x-\mu}{\sigma} = \frac{20-12}{4} = 2$$

i. $P(x \geq 20)$

$P(z \geq 2)$



$\therefore P(z \geq 2) = 0.5 - P(0 \leq z \leq 2)$

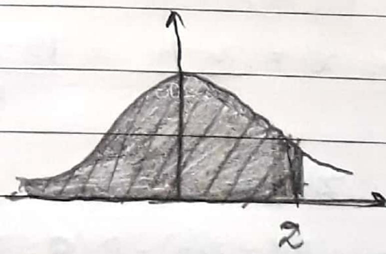
$= 0.5 - \phi(2)$

$= 0.5 - 0.4772$

$= 0.0228$

ii. $P(x \leq 20)$

$P(z \leq 2)$



$= 0.5 + P(0 \leq z \leq 2)$

$= 0.5 + \phi(2)$

$= 0.5 + 0.4772$

$= 0.9772$

Q3: The marks of 1000 students in an examination follows a normal distribution with mean 70 and standard deviation 5. Find the number of students whose marks will be

i. greater than 75.

ii. less than 65

iii. $65 < x < 75$

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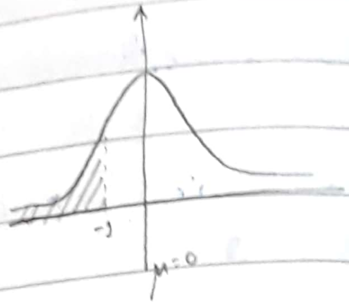
Sol: Given: $\mu = 70$ and $\sigma = 5$

i. less than 65

$P(x<65)$

$z = \frac{x-\mu}{\sigma} = \frac{65-70}{5} = -1$

$\therefore P(z<-1)$



By symmetry

$P(z<-1) = P(z\geq 0) - P(0\leq z\leq 1)$

$= 0.5 - \phi(1)$

$= 0.5 - 0.3413$

$= 0.1587$

$\therefore$ Number of students who got less than 65 ~~ERRS~~

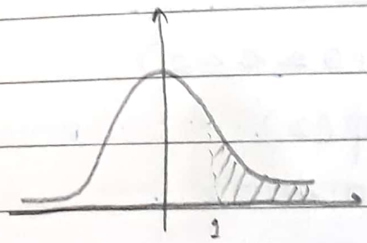
$= 0.1587 \times 1000 \approx$ 159 students

ii. greater than 75

$P(x>75)$

$z = \frac{x-\mu}{\sigma} = \frac{75-70}{5} = 1$

$P(z>1)$



$= P(z\geq 0) - P(0\leq z\leq 1)$

$= 0.5 - \phi(1)$

$= 0.5 - 0.3413$

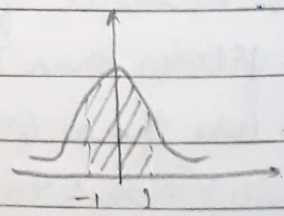
$= 0.1587$

$\therefore$ Number of students who got more than 75 marks

$= 0.1587 \times 1000 \approx$ 159 students

iii. between 65 to 75

$P(65<x<75)$



$= P(-1<z<1)$

$= 2P(0\leq z\leq 1) = 2\phi(1) = 2(0.3413) = 0.6826$

$\therefore 1000 \times 0.6826 =$ 683 students

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Q: In a test on electric bulbs it was found that lifetime of a particular brand was distributed normally with an average lifetime of 2000hrs and standard deviation 60 hrs. If a firm purchases 2500 bulbs, find the number of bulbs that are likely to last for

i. more than 2100hrs

ii. less than 1950 hrs

iii. between 1900 hrs to 2100 hrs.

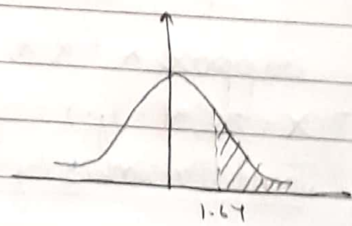
sol: Given: $\mu = 2000$ and $\sigma = 60$.

i. more than 2100 hrs

$P(x > 2100)$

$z = \frac{x-\mu}{\sigma} = \frac{2100 - 2000}{60} = 1.67$

$P(z > 1.67)$



$= P(z \geq 0) - P(0 \leq z \leq 1.67)$

$= 0.5 - \phi(1.67)$

$= 0.5 - 0.4525$

$= 0.0475$

Number of bulbs $= 0.0475 \times 2500 = 118.75 = 119$ bulbs

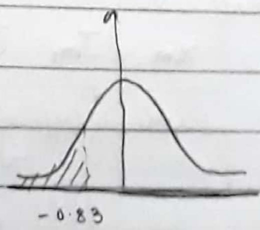
ii. less than 1950 hrs

$P(x < 1950)$

$z = \frac{x-\mu}{\sigma} = \frac{1950 - 2000}{60} = -0.833$

$P(z < -0.833)$

By symmetry



$P(z < -0.83)$

$= P(z \geq 0) - P(0 \leq z \leq 0.83)$

$= 0.5 - \phi(0.83)$

$= 0.5 - 0.2967$

$= 0.2033$

Number of bulbs

$= 0.2033 \times 2500 = 508.25$

≈ 509 bulbs

iii. between 1900 and 2100 hrs

$z = \frac{x-\mu}{\sigma} = \frac{1900-2000}{60} = -1.$

$P(1900 < x < 2100)$

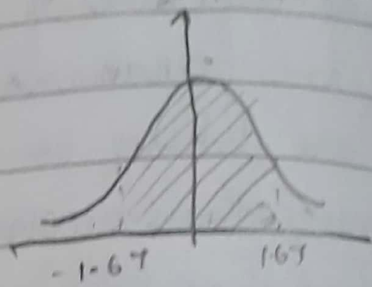
$P(-1.67 < z < 1.67)$

$= 2[P(0 \leq z \leq 1.67)]$

$= 2\phi(1.67)$

$= 2(0.4525)$

$= 0.9050$



$\therefore$ Number of bulbs $= 2500 \times 0.905 = 2262.5 \approx$ 2263 bulbs

★ **Joint Probability Distribution:**

If X and Y are discrete random variables, we define joint probability function of X and Y by $P(X=x, Y=y) = f(x,y)$ where $f(x,y)$ satisfies the conditions $f(x,y) \geq 0$ and $\sum_x \sum_y f(x,y) = 1$

Suppose $X = \{x_1, x_2, x_3 \ldots x_n\}$ and $Y = \{y_1, y_2, y_3, \ldots y_n\}$, then $P(X = x_i, Y = y_j) = f(x_i, y_j)$ denoted by J_{ij}.

Joint Probability Table:

X\Y	y_1	y_2	y_3	...	y_n	sum
x_1	J_{11}	J_{12}	J_{13}	.....	J_{1n}	$f(x_1)$
x_2	J_{21}	J_{22}	J_{23}	....	J_{2n}	$f(x_2)$
x_3	J_{31}	J_{32}	J_{33}	...	J_{3n}	$f(x_3)$
⋮	⋮	⋮	⋮	...	⋮	⋮
x_m	J_{m1}	J_{m2}	J_{m3}	......	J_{mn}	$f(x_m)$
sum	$g(y_1)$	$g(y_2)$	$g(y_3)$	....	$g(y_n)$	1

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Marginal Probability Distribution:

* In the joint probability table $f(x_1), f(x_2), \dots f(x_m)$ respectively represents the sum of all the entries in the 1st row, 2nd row upto to n^{th} row. Similarly $g(y_1), g(y_2), \dots g(y_n)$ respectively represents the sum of all the entries in the 1st, 2nd ... and upto n^{th} column.

where
$$f(x_1) = J_{11} + J_{12} + \dots + J_{1n}$$
$$f(x_2) = J_{21} + J_{22} + \dots + J_{2n}$$
$$\vdots$$
$$f(x_m) = J_{m1} + J_{m2} + \dots + J_{mn}$$

similarly
$$g(y_1) = J_{11} + J_{21} + J_{31} + \dots + J_{m1}$$
$$g(y_2) = J_{12} + J_{22} + J_{32} + \dots + J_{m2}$$
$$\vdots$$
$$g(y_n) = J_{1n} + J_{2n} + J_{3n} + \dots + J_{mn}$$

$\{f(x_1), f(x_2), \dots f(x_m)\}$ and $\{g(y_1), g(y_2), \dots g(y_m)\}$ represents the marginal probability distribution of X and Y respectively.

* Expectation, variance, covariance and correlation:

– If X a discrete random variable taking the values $x_1, x_2, \dots, x_n$, then the expecion of X is denoted by $E(X) = \mu_x$ is defined by

$$\boxed{E(X) = \mu_x = \sum_{i=1}^{n} x_i f(x_i)}$$ Expecion of X

similarly $$\boxed{E(Y) = \mu_Y = \sum_{j=1}^{n} y_j g(y_j)}$$ Expecion of Y

If X and Y are two discrete random variables having the joint probability function $f(x,y)$, then the expecions of X and Y is defined as

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$$\mu_x = E(X) = \sum_x \sum_y x f(x,y) = \sum_i x_i f(x_i)$$

similarly

$$\mu_Y = E(Y) = \sum_x \sum_y y g(x,y) = \sum_j y_j g(y_j)$$

Therefore

$$\boxed{E(XY) = \sum_{i,j} x_i y_j J_{ij}}$$

expectation of X Y

- The variance of X denoted by $V(X)$ is defined by

$$\boxed{V(X) = \sum_{i=1}^{n} (x_i - \mu)^2 f(x_i)}$$

Also $\boxed{\sigma_x = \sqrt{V(X)}}$ is called standard deviation of X.

- If X and Y are random variables having mean μ_x and μ_y respectively then the covariance of X and Y denoted by $\mathrm{cov}(X,Y)$ is defined as

$$\boxed{\mathrm{cov}(X,Y) = \sum_i \sum_j (x_i - \mu_x)(y_j - \mu_Y) J_{ij}}$$

- correlation of X and Y denoted by $\rho(x,y)$ is defined by

$$\boxed{\rho(X,Y) = \frac{\mathrm{cov}(X,Y)}{\sigma_x \sigma_Y}}$$

Q1: The joint distribution of two random variable X and Y is as follows.

X|Y	-4	2	7
1	1/8	1/4	1/8
5	1/4	1/8	1/8

Find:

a. $E(X)$ and $E(Y)$

b. $E(XY)$

c. σ_x and σ_Y

d. $\mathrm{cov}(X,Y)$

e. $\rho(X,Y)$ $(y_j - \mu_Y) J_{ij}$

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Sol: The marginal distribution of μ_x and μ_y is as follows

Distribution of X

x_i	1	5
$f(x_i)$	1/2	1/2

Distribution of Y

y_j	-4	2	7
$g(y_j)$	3/8	3/8	1/4

a. $E(X) = \sum_{i=1}^{n} x_i f(x_i)$

$E(X) = \sum_{i=1}^{2} x_i f(x_i) = 1\left(\frac{1}{2}\right) + 5\left(\frac{1}{2}\right) = 3$ //

$E(Y) = \sum_{j=1}^{n} y_j g(y_j)$

$E(Y) = \sum_{j=1}^{3} y_j g(y_j) = -4\left(\frac{3}{8}\right) + 2\left(\frac{3}{8}\right) + 7\left(\frac{1}{4}\right) = 1$ //

b. $E(XY) = \sum_{i,j} x_i y_j J_{ij}$

$E(XY) = 1(-4)\left(\frac{1}{8}\right) + 1(2)\left(\frac{1}{4}\right) + 1(7)\left(\frac{1}{8}\right)$

$+ 5(-4)\left(\frac{1}{4}\right) + 5(2)\left(\frac{1}{8}\right) + 5(7)\left(\frac{1}{8}\right)$

$E(XY) = \cancel{-\frac{1}{2}} + \cancel{\frac{1}{2}} + \frac{7}{8} - 5 + \frac{5}{4} + \frac{35}{8} = \frac{3}{2}$ //

c. $V(X) = \sum_{i=1}^{n} (x_i - \mu)^2 f(x_i)$

$\cancel{= \sum_{i=1}^{n} \left[x_i^2 f(x_i) \right.}$

$= E\left[(X - \mu_x)^2\right]$

$= E\left[X^2 - 2X\mu_x + \mu_x^2\right]$

$= E[X^2] - 2[E(X)]\mu_x + \mu_x^2 E(1)$

$= E[X^2] - 2[E(X)]^2 + \mu_x^2$

$= E[X^2] - 2[E(X)]^2 + [E(X)]^2$

$= E[X^2] - [E(X)]^2$

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NOTE: Important Formulas

1. $E(X) = \sum_i x_i f(x_i)$

• $E(Y) = \sum_j y_j g(y_j)$

2. $E(XY) = \sum_{ij} x_i y_j J_{ij}$

3. $V(X) = \sum (x-\mu)^2 f(x_i) = E[(X-\mu)^2]$

$= E[X^2] - E[X]^2$

4. $\sigma_x = \sqrt{V(X)}$ and $\sigma_y = \sqrt{V(Y)}$

5. $COV(XY) = E(XY) - \mu_x \mu_y$

6. $\rho(XY) = \frac{COV(XY)}{\sigma_x \sigma_y}$

– Variance

$V(X) = E(X^2) - [E(X)]^2 = 3^2$

$E[X^2] = \sum_i x_i^2 f(x_i)$

$= 1 \times \frac{1}{2} + 25 \times \frac{1}{2} = \frac{26}{2} = 13$ //

$V(X) = 13 - 3^2 = 4$ //

$\sigma_x = \sqrt{V(X)} = \sqrt{4} = 2$ //

$V(Y) = E(Y^2) - [E(Y)]^2$

$E(Y^2) = \sum_i y_i^2 g(y_i)$

$= 16 \times \left(\frac{1}{8} + \frac{1}{4}\right) + 4\left(\frac{1}{8} + \frac{1}{4}\right) + 49\left(\frac{2}{8}\right)$

$= \frac{16 \times 3}{8} + \frac{4 \times 3}{8} + \frac{49 \times 2}{8} = \frac{79}{4}$

$V(Y) = \frac{79}{4} - 1^2 = \frac{75}{4}$ //

$\sigma_y = \sqrt{V(Y)} = \sqrt{\frac{75}{4}} = \frac{5\sqrt{3}}{2}$ //

d. Covariance

$$cov(xy) = E(xy) - \mu_x \mu_y$$

$$= \frac{3}{2} - 3(1) = \frac{-3}{2}$$

e. $\rho(xy) = \dfrac{cov(xy)}{\sigma_x \sigma_y}$

$$= \frac{-3/2}{2 \times 5\sqrt{3}/2} = -0.1732$$

* Independent Random Variables:

The discrete random variable X and Y are said to be independent if $P(X=x, Y=y) = P(X=x) \cdot P(Y=y) = J_{ij}$

Q1: The joint probability distribution for two random variables in as follows:

x / y	-2	1	4	5
1	0.1	0.2	0	0.3
2	0.2	0.1	0.1	0

Determine the marginal probability distribution of X and Y. Also compute:

i. Expectations of X and Y.

ii. E(X) and E(Y)

iii. E(XY)

iv. $\sigma_x \sigma_y$

v. Further verify that X and Y are dependent random variables.

Sol: Distribution of X

x_i	1	2
$f(x_i)$	0.6	0.4

Distribution of Y

y_i	-2	-1	4	5
$g(y_i)$	0.3	0.3	0.1	0.3

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$E(x) = \sum_i x_i f(x_i) = 1 \times 0.6 + 2 \times 0.4$

$= \underline{\underline{1.4}}$

$E(y) = \sum_j y_j f(y_j) = -2(0.3) + -1(0.3) + 4(0.3) + 5(0.3)$

$= -0.6 - 0.3 + 1.2 + 1.5$

$= 1$ //

$E(xy) = \sum_{ij} x_i y_j J_{ij}$

$= 1(-2)(0.1) + (-1)(1)(0.2) + 0 + 5(0.3)$

$+ 2(0.2)(-2) + 2(0.1)(1) + 4(0.1)(2) + 0$

$= -0.2 - 0.2 + 1.5 - 0.8 + 0.2 + 0.8$

$= \underline{\underline{0.9}}$

$V(x) = E(x^2) - E(x)^2$

$E(x^2) = \sum_i x_i^2 f(x_i)$

$= 1(0.6) + 4(0.4)$

$= 2.2$

$V(x) = (2.2) - (1.4)^2$

$= 2.2 - 1.96 = 0.24$ //

$\sigma_x = \sqrt{V(x)} = \underline{0.489}$

$V(y) = E(y^2) - E(y)^2$

$E(y^2) = \sum_j y_j^2 g(y_j)$

$= 4(0.3) + 1(0.3) + 16(0.1) + 25(0.3)$

$= \underline{\underline{10.6}}$

$V(y) = 10.6 - 1^2 = \underline{\underline{9.6}}$

$P(xy) = P(x)P(y)$ considering $P(x)P(y) = 0.1$

$P(1,-2) = 0.6 \times 0.3 = 0.18$

But $P(x,y) = P(1,-2) = 0.1$

as $0.1 \neq 0.18$

$\therefore$ X and Y are dependent random variables.

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Q2: The joint probability distribution of two discrete random variables X and Y is given by $f(x, y) = k(2x + y)$ where x and y are integers such that $0 \le x \le 2$ and $0 \le y \le 3$.

a. compute k

b. compute marginal distribution of X and Y.

Sol:

X / Y	0	1	2	3	
0	0	1k	2k	3k	6k
1	2k	3k	4k	5k	14k
2	4k	5k	6k	7k	22k
	6k	9k	12k	15k	42k

$42k = 1 \Rightarrow k = \frac{1}{42}$ //

Marginal Probability

x_i	0	1	2
$f(x_i)$	$6k = 6/42$	$14k = 14/42$	$22k = 22/42$

y_i	0	1	2	3
$f(y_i)$	$6k = 6/42$	$9k = 9/42$	$12k = 12/42$	$15k = 15/42$