Unit - 1

INTRODUCTION

★ **Signals:**

Signal is a function of one or more independent variable along with information. $f(x_1, x_2, x_3, \cdots x_n)$.

A function of one independent variable is called one dimensional signal and a function of more than one independent variable is called multidimensional signal.

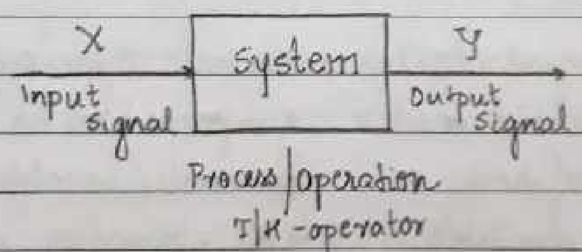
Examples

1. Electrical Signal : One dimensional signal → $f(t)$
   voltage : $v(t)$
   current : $i(t)$
2. Photograph / Image Signal : Two dimensional signal : $f(x, y)$
   image : $I(R, C)$ R: row and C: column
3. Video Signal : Three dimensional signal : $f(x, y, z)$
   video : $V(R, G, B)$ Illuminance, Chrominance, Brightness
4. Biomedical signal :
   ECG, EEG, EMG etc.

★ **Systems:**

A system is any process that produces an output signal in response to an input signal.

Ex : Communication systems, control system, Biomedical systems.

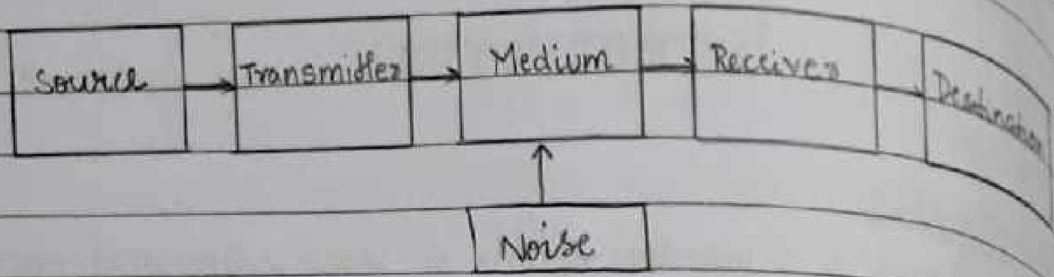


$y = T\{x\}$ or $H\{x\}$

Output is obtained by operation on the input.

Ex: operator: filter
operation: filteration.

Ex: Block diagram of basic communication system



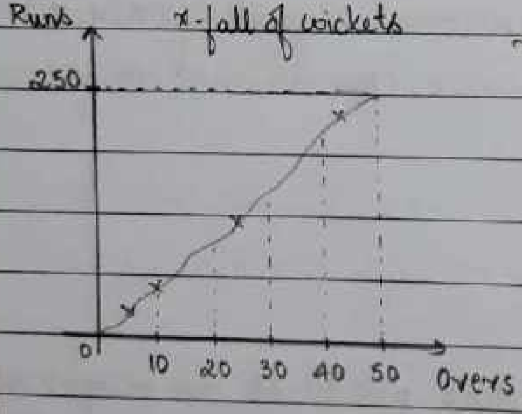
Source: Mic
Transmitter: Antenna | Modulator
Medium: Wired | Wireless
Receiver: Antenna | Demodulator
Destination: Speaker

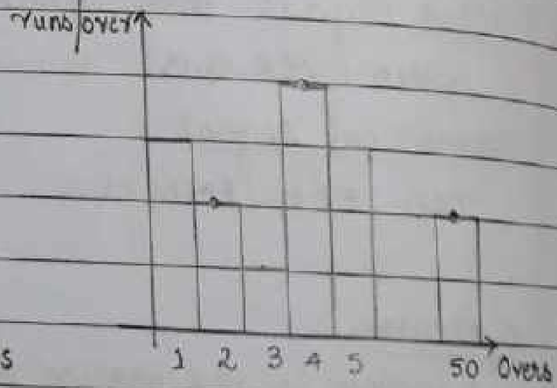
Filter can be used on the output side to remove noise.

* Classification of signals:

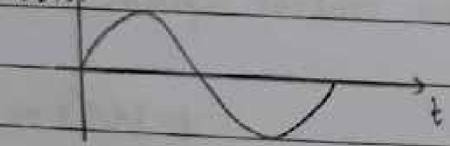
a. Continuous time signal or Discrete Time Signal

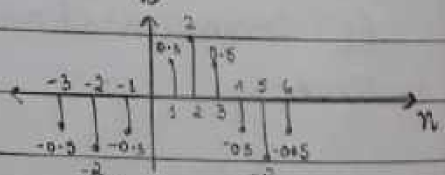
Ex: Cricket Score





Ex: sinusoidal wave





— Continuous Time Signal (CTS)
* Information can be obtained at every instant of time.
* All information is available.

— Discrete Time Signal (DTS)
* Information at integer values can only be obtained
* All information will not be avai

b. <u>Deterministic Signal or Random Signal</u>

- <u>Deterministic Signal</u>: The signal can be represented in a mathematical form is called a deterministic signal.

  <u>Ex</u>: Sinusoidal signal: $x(t) = k + A\sin(\omega t + \phi)$

- Random Signal: Since the signal has random information it cannot be represented in a mathematical form.

  <u>Ex</u>: Runs scored in a cricket match.

<u>NOTE</u>:

Sampled CTS is DTS and sampled DTS is digital signal.

c. <u>Even Signal or Odd signal</u>

$x(t)$	CTS	DTS
even	$x(-t) = x(t)$	$x(-n) = x(n)$
odd	$x(-t) = -x(t)$	$x(-n) = -x(n)$

<u>Ex</u>: $x(t) = \sin\omega t$ $\qquad$ $x(t) = \cos\omega t$

$x(-t) = \sin(-\omega t)$ $\qquad$ $x(-t) = \cos(-\omega t)$

$x(-t) = -\sin\omega t$ $\qquad$ $x(-t) = \cos\omega t$

$x(-t) = -x(t)$ $\qquad$ $x(-t) = x(t)$

$\therefore$ It is an odd signal $\qquad$ $\therefore$ It is an even signal.

Actual signal can be represented as sum of odd and even signal:

CTS: $x(t) = x_e(t) + x_o(t)$ —— (1)

DTS: $x(n) = x_e(n) + x_o(n)$ —— (2)

at $t = -t$

From eq (1) $x(-t) = x_e(-t) + x_o(-t)$

$x(-t) = x_e(t) - x_o(t)$ —— (3)

From eq (2) $x(-n) = x_e(-n) + x_o(-n)$

$x(-n) = x_e(n) - x_o(n)$ —— (4)

Adding eq (1) and (3)

$$\boxed{x_e(t) = \frac{x(t) + x(-t)}{2}}$$

subracting eq ① and ③

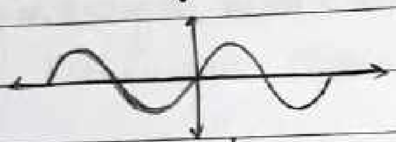
$$\boxed{x_o(t) = \frac{x(t) - x(-t)}{2}}$$

similarly

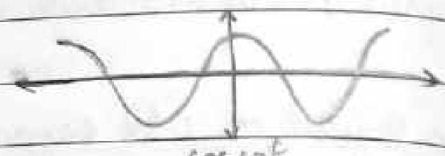
$$\boxed{x_e(n) = \frac{x(n)+x(-n)}{2}} \qquad \boxed{x_o(n) = \frac{x(n) - x(-n)}{2}}$$

NOTE: Even/odd signal helps to find the symmetry of any signal

→ Even signals are symmetric about the vertical axis
Odd signals are antisymmetric about the vertical axis



$\sin\omega t$
(odd signal)
antisymmetric



$\cos\omega t$
(even signal)
symmetric

→ CTS: $x(t)\big|_{t=0} = 0$ and DTS: $x(n)\big|_{n=0} = 0$

→ A signal can be neither odd nor even signal.

Q: check whether the following signals are odd or even:

1. $x(t) = 10\sin\omega t$
   $x(-t) = 10\sin(-\omega t) = -10\sin\omega t$
   $\therefore x(-t) = -x(t) \Rightarrow$ ∴ It is an odd signal.

2. $x(t) = A\cos(t)$
   $x(-t) = A\cos(-t) = A\cos t$
   $\therefore x(-t) = x(t) \Rightarrow$ It is an even signal.

3. $x(n) = n + n^3 + 5n^5$
   $x(-n) = (-n) + (-n)^3 + 5(-n)^5 = -n - n^3 - 5n^5 = -(n+n^3+5n^5)$
   $\therefore x(-n) = -x(n) \Rightarrow$ It is an odd signal.

4. $x(n) = n^2 + 3n^4$

$x(-n) = (-n)^2 + 3(-n)^4 = n^2 + 3n^4$

$\therefore x(-n) = x(n) \Rightarrow$ It is an even signal.

5. $x(t) = 10 + t + 2t^2$

$x(-t) = 10 + (-t) + 2(-t)^2 = 10 - t + 2t^2$

$\therefore$ It is neither even nor odd.

6. $x(t) = e^{jt}$

$x(-t) = e^{-jt}$ $\therefore$ It is neither even nor odd.

Q: Find the amount of even and odd part available in the signal:

1. $x(t) = \cos(t) + \sin(t) + \cos(t)\sin(t)$

$x(-t) = \cos(-t) + \sin(-t) + \cos(-t)\sin(-t)$

$= \cos t - \sin t - \cos t \sin t$

Odd part of the function is

$$x_o(t) = \frac{x(t) - x(-t)}{2}$$

$$x_o(t) = \frac{\cancel{\cos t} + \sin t + \cos t \sin t - \cancel{\cos t} + \sin t + \cos t \sin t}{2}$$

$$x_o(t) = \frac{\cancel{2}(\sin t + \cos t \sin t)}{\cancel{2}} = \underline{\sin t(1 + \cos t)}$$

Even part of the function is

$$x_e(t) = \frac{x(t) + x(-t)}{2}$$

$$x_e(t) = \frac{\cos t + \cancel{\sin t} + \cancel{\cos t \sin t} + \cos t - \cancel{\sin t} - \cancel{\cos t \sin t}}{2}$$

$$x_e(t) = \frac{\cancel{2}\cos t}{\cancel{2}} = \underline{\cos t}$$

2 $x(t) = [1+t^3]\cos^3 t$

$x(-t) = [1+(-t)^3][\cos(-t)]^3$

$= [1-t^3]\cos^3 t$

Even part of the signal is

$$x_e(t) = \frac{x(t)+x(-t)}{2}$$

$$x_e(t) = \frac{[1+t^3]\cos^3 t + [1-t^3]\cos^3 t}{2}$$

$$x_e(t) = \frac{\cos^3 t + \cancel{t^3\cos^3 t} + \cos^3 t - \cancel{t^3\cos^3 t}}{2}$$

$$x_e(t) = \frac{2\cos^3 t}{2} = \cos^3 t$$

odd part of the signal is

$x_o(t) = [1+t^3]\cos^3 t - \cos^3 t$ (i.e., $x(t) - x_e(t)$)

$x_o(t) = \cancel{\cos^3 t} + t^3\cos^3 t - \cancel{\cos^3 t}$

$x_o(t) = t^3\cos^3 t$

Q: Show that the product of two even signals results into an even signal.

Ex: Let the two even signals be

$x(t) = \cos\omega t$

$y(t) = t^2$

$x(t)y(t) = t^2\cos\omega t$

at $t = -t$

$x(-t)y(-t) = (-t)^2\cos(-\omega t)$

$x(-t)y(-t) = t^2\cos\omega t$

Since $x(t)y(t) = x(-t)y(-t)$

$\therefore$ It is an even signals

$\therefore$ Product of two even signal results into even signal.

Let $x_1(t)$ and $x_2(t)$ are two even signal.

Let the resulting signal be $x(t)$

$\therefore x(t) = x_1(t)\,x_2(t)$

at $t = -t$

$x(-t) = x_1(-t)\,x_2(-t)$

since x_1 and x_2 are even

$x_1(t) = x_1(-t)$

$x_2(t) = x_2(-t)$

$\therefore x(-t) = x_1(t)\,x_2(t)$

Hence It is an even signal ($x(t)$)

Q: Show that the product of two odd signals result into an even signal.

Let $x_1(t)$ and $x_2(t)$ be two odd signals

Let the resulting signal be $x(t)$

$\therefore x(t) = x_1(t)\, x_2(t)$

at $t = -t$

$x(-t) = x_1(-t)\, x_2(-t)$

Since x_1 and x_2 are odd

$x_1(t) = -x_1(t)$

$x_2(-t) = -x_2(t)$

$\therefore x(-t) = [-x_1(t)][-x_2(t)] = x_1(t)\, x_2(t)$

$\therefore$ The resulting signal is an even signal.

Q: Show that the product of an odd signal and an even signal results into an odd signal.

Let $x_1(t)$ be odd signal and $x_2(t)$ be even signal.

Let the resulting signal be $x(t)$

$\therefore x(t) = x_1(t)\, x_2(t)$

at $t = -t$

$x(-t) = x_1(-t)\, x_2(t)$

Since x_1 is odd and x_2 is even

$x_1(-t) = -x_1(t)$

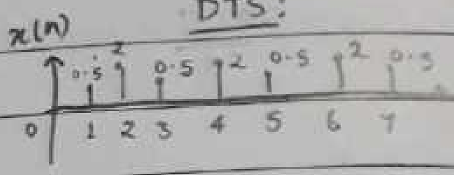
$x_2(-t) = x_2(t)$

$\therefore x(-t) = -x_1(t)\, x_2(t)$

$\therefore$ The resulting signal is an odd signal.

d Periodic Signal or Aperiodic Signal

Periodic signal: The signal repeats after its fundamental time period.



$x(n) = x(n+N)$

$x(t) = x(t+T)$

Aperiodic signal: Any signal that does not repeat after any ~~its fundamental~~ time period.

(random signal)

Periodic signal

CTS signal

CTS $x(t) = k + A\sin(\omega t + \phi)$ sine signal $\omega = 2\pi f = \frac{2\pi}{T}$

$x(t) = k + A\cos(\omega t + \phi)$ cosine signal

$x(t) = k + Ae^{j\omega t + \phi}$ complex signal

DTS $x(n) = k + A\sin(\Omega n + \phi)$ DTS signal

$x(n) = k + A\cos(\Omega n + \phi)$ $\Omega = 2\pi \frac{m}{N}$

NOTE:

- Two signals (periodic) with same frequency are additive in nature and the result is also periodic.
- Two ~~diff~~ same signals with different frequency then $\frac{T_1}{T_2}$ if rational then $T = LCM(T_1, T_2)$

Ex: $T_1 = 5$ and $T_2 = 10$

$\frac{T_1}{T_2} = \frac{5}{10} = 0.5$ It is rational

$\therefore T = LCM(T_1, T_2) = 10$ Hence is periodic.

Ex: $T_1 = 2\pi/3$ and $T_2 = 2\pi/7$

$T = \frac{LCM(2\pi, 2\pi)}{HCF(3,7)} = \frac{2\pi}{21}$

Q: Check whether the following signals are periodic or aperiodic. If periodic find its fundamental time period.

1. $x(t) = 10\cos(3t + \pi/4)$

$\omega = 3 \Rightarrow \frac{2\pi}{T} = 3$

$x(t)$ is periodic signal with $T = 2\pi/3$ sec.

$\therefore \underline{T = 2\pi/3}$

2. $x(t) = \cos^2(4\pi t)$

$x(t) = \frac{1}{2} + \frac{1}{2}\cos(8\pi t)$

$x(t)$ is periodic signal with $T = 1/4$ sec.

$\omega = 8\pi \Rightarrow \frac{2\pi}{T} = 8\pi$

$\therefore T = \frac{1}{4}$ //

3. $x(t) = \cos(\pi t/3) + \sin(\pi t/4)$

$\omega_1 = \frac{\pi}{3} \Rightarrow \frac{2\pi}{T_1} = \frac{\pi}{3}$ $\therefore T_1 = 6$

$\omega_2 = \frac{\pi}{4} \Rightarrow \frac{2\pi}{T_2} = \frac{\pi}{4}$ $\therefore T_2 = 8$

$\therefore x(t)$ is periodic signal with $T = 24$ sec.

$T = LCM(T_1, T_2) = LCM(6, 8)$

$\underline{T = 24}$

4. $x(n) = \sin\left(\frac{\pi n}{4}\right)$

$\Omega = \frac{\pi}{4} = 2\pi\left(\frac{1}{8}\right)$

$2\pi\left(\frac{n}{N}\right) = 2\pi\left(\frac{1}{8}\right)$ $\underline{N = 8 \text{ samples}}$

$1/8$ is rational

$\therefore$ the signal is periodic

5. $x(n) = \sin^2\left(\frac{\pi n}{4}\right)$

$x(n) = \frac{1}{2} - \frac{1}{2}\cos\left(\frac{\pi n}{2}\right)$

$\Omega = \frac{\pi}{2} = 2\pi\left(\frac{1}{4}\right)$

since $\frac{n}{N} = \frac{1}{4}$ is rational

$2\pi\left(\frac{m}{N}\right) = 2\pi\left(\frac{1}{4}\right)$

It is periodic signal.

$\therefore$ $\underline{N = 4 \text{ samples}}$

6. $x(n) = \cos\left(\frac{\pi n}{3}\right) + \sin\left(\frac{\pi n}{5}\right)$

$\Omega_1 = \frac{\pi}{3} = 2\pi\left(\frac{1}{6}\right)$ & $\Omega_2 = \frac{\pi}{5} = 2\pi\left(\frac{1}{10}\right)$

$2\pi\left(\frac{m}{N_1}\right) = 2\pi\left(\frac{1}{6}\right)$ $\quad$ $2\pi\left(\frac{m}{N_2}\right) = 2\pi\left(\frac{1}{10}\right)$

$N_1 = \underline{6 \text{ samples}}$ $\quad$ $N_2 = \underline{10 \text{ samples}}$

$N = \frac{N_1}{N_2} = \frac{6}{10} = 0.6$ $\therefore$ rational

Hence it is periodic signal.

$N = \text{LCM}(N_1, N_2)$

$N = \text{LCM}(6, 10) = \underline{30 \text{ samples}}$

7. $x(n) = \pi + \cos(2n)$

$\Omega = 2 = 2\pi\left(\frac{1}{\pi}\right)$

$2\pi\left(\frac{m}{N}\right) = 2\pi\left(\frac{1}{\pi}\right)$ $\Rightarrow \therefore$ It is irrational

Hence not a periodic signal.

8. $x(n) = 15\cos(0.2\pi n)$

$$\Omega = 0.2\pi = 2\pi\left(\frac{1}{10}\right)$$

$$\therefore 2\pi\left(\frac{1}{10}\right) = 2\pi\left(\frac{m}{N}\right)$$

$\therefore x(n)$ is a periodic signal

$N = 10$ samples

9. $x(t) = \cos t + \sin(\sqrt{2}t)$

$$\omega_1 = 1 \Rightarrow \frac{2\pi}{T_1} = 1$$

$\therefore x(t)$ is an aperiodic signal

$$\therefore T_1 = 2\pi$$

$$\omega_2 = \sqrt{2} \Rightarrow \frac{2\pi}{T_2} = \sqrt{2}$$

$$\therefore T_2 = \sqrt{2}\pi$$

$\dfrac{T_1}{T_2} = \dfrac{2\pi}{\sqrt{2}\pi} = \sqrt{2}$ it is irrational

10. $x(n) = R_e\left[e^{jn\pi/4}\right] + I_m\left[e^{jn\pi/3}\right]$

$$x(n) = \cos\frac{n\pi}{4} + \sin\frac{n\pi}{3}$$

$\cancel{\Omega_1 = \frac{\pi}{4} \Rightarrow \frac{2\pi}{T_1} = \frac{\pi}{4}}$

$\cancel{\therefore T_1 = 8}$

$\cancel{\omega_2 = \frac{\pi}{3} \Rightarrow \frac{2\pi}{T_2} = \frac{\pi}{3}}$

$\cancel{\therefore T_2 = 6}$

$\cancel{\frac{T_1}{T_2} = \frac{8}{6}}$ ~~It is rational~~

$\cancel{T = LCM(T_1, T_2)}$

$\cancel{T = LCM(8, 6)}$

$\cancel{T = 24}$

$$\Omega_1 = \frac{\pi}{4} = 2\pi\left(\frac{1}{8}\right)$$

$$\therefore 2\pi\left(\frac{m}{N_1}\right) = 2\pi\left(\frac{1}{8}\right)$$

$\therefore N_1 = 8$ samples

$$\Omega_2 = \frac{\pi}{3} = 2\pi\left(\frac{1}{6}\right)$$

$$\therefore 2\pi\left(\frac{m}{N_2}\right) = 2\pi\left(\frac{1}{6}\right)$$

$\therefore$

$\therefore N_2 = 6$ samples

$\dfrac{N_1}{N_2} = \dfrac{8}{6}$ is rational hence periodic

$N = LCM(N_1, N_2) = LCM(8, 6) = 24$ samples

11. $x(n) = \cos\left(\frac{\pi n}{5}\right)\sin\left(\frac{\pi n}{3}\right)$

$$= \frac{1}{2}\left(\sin\left(\frac{\pi n}{5} + \frac{\pi n}{3}\right) - \sin\left(\frac{n\pi}{5} - \frac{n\pi}{3}\right)\right)$$

$$= \frac{1}{2}\left(\sin\left(\frac{8\pi n}{15}\right) + \sin\left(\frac{2\pi n}{15}\right)\right)$$

$$\Omega_1 = \frac{8\pi}{15} = 2\pi\left(\frac{4}{15}\right)$$

$\therefore 2\pi\left(\frac{m}{N_1}\right) = 2\pi\left(\frac{4}{15}\right)$ $\therefore N_1 = 15$ samples

$$\Omega_2 = \frac{2\pi}{15} = 2\pi\left(\frac{1}{15}\right)$$

$\therefore 2\pi\left(\frac{m}{N_2}\right) = 2\pi\left(\frac{1}{15}\right)$ $\therefore N_2 = 15$ samples

$\frac{N_1}{N_2} = \frac{15}{15} = 1$ is rational

$\therefore x(n)$ is periodic signal

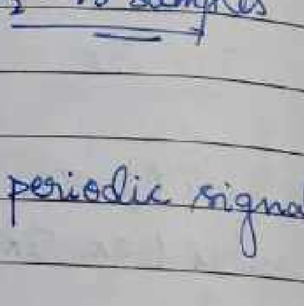
$\therefore N = LCM(N_1, N_2)$

$N = LCM(15, 15) = 15$ samples

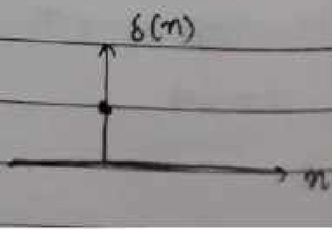
★ Basic Elementary Signals:

1. **Unit Impulse Signal:** It is a signal whose area is equal to 1 and information is available only at $t=0$. It is denoted by $\delta(t)$

CTS: $\delta(t) = \begin{cases} 1 & \text{at } t=0 \\ 0 & \text{Else} \end{cases}$



DTS: $\delta(n) = \begin{cases} 1 & \text{at } n=0 \\ 0 & \text{Else} \end{cases}$



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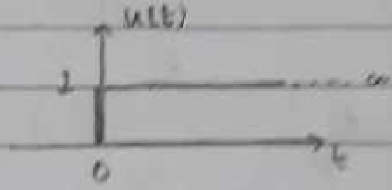
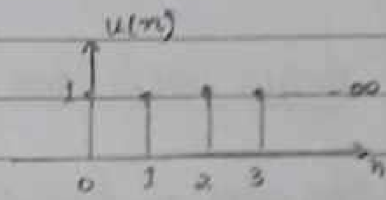
2. Unit Step Signal:

It is denoted by $u(t)$

CTS: $u(t) = \begin{cases} 1 & t \geq 0 \\ 0 & \text{Else} \end{cases}$



$u(t) = \int_0^{\infty} \delta(t)\, dt$



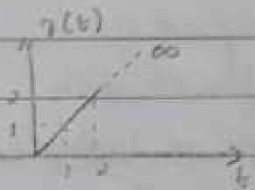
DTS: $u(n) = \begin{cases} 1 & ; n \geq 0 \\ 0 & ; \text{Else} \end{cases}$

$u(n) = \sum_0^{\infty} \delta(n-k)$

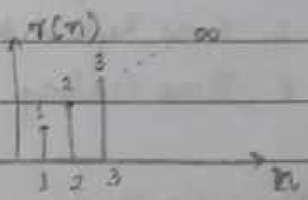
3. Ramp Signal:

It is denoted by $r(t)$

CTS: $r(t) = \begin{cases} t & t \geq 0 \\ 0 & \text{Else} \end{cases}$



$\frac{d}{dt}[r(t)] = u(t)$



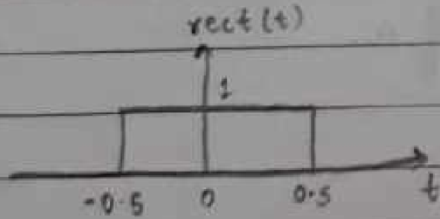
DTS: $r(n) = \begin{cases} n & n \geq 0 \\ 0 & \text{Else} \end{cases}$

4. Pulse Signal:

Rectangular Pulse

CTS:

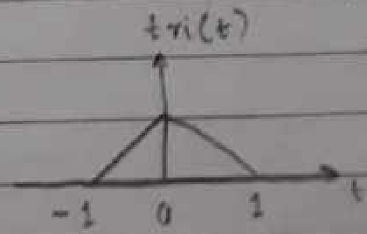
$rect(t) = \begin{cases} 1 & ; -0.5 \leq t \leq 0.5 \\ 0 & ; \text{Else} \end{cases}$



Triangular Pulse

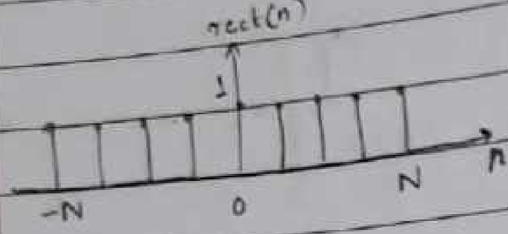
CTS:

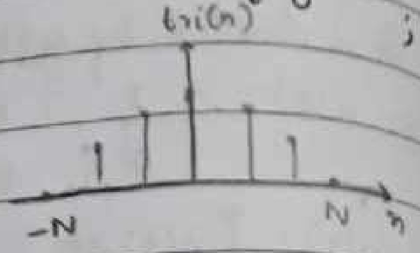
$tri(t) = \begin{cases} t+1 & ; -1 \leq t \leq 0 \\ -t+1 & ; 0 \leq t \leq 1 \\ 0 & ; \text{else} \end{cases}$



DTS: $rect(n) = \begin{cases} 1 & ; -N \le n \le N \\ 0 & ; \text{Else} \end{cases}$

DTS: $Tri(n) = \begin{cases} 1 - \frac{|n|}{N} & ; |n| \le N \\ 0 & ; \text{Else} \end{cases}$





* **Basic Operations on Signals:**

- Operation on dependent variable: $(x(t) / x(n))$
  1. Time scaling
  2. Addition
  3. Multiplication
  4. Integration
  5. Differentiation
- Operation on Independent variable: (t/n)
  1. Time Shift
  2. Time Reflection
  3. Time Scaling

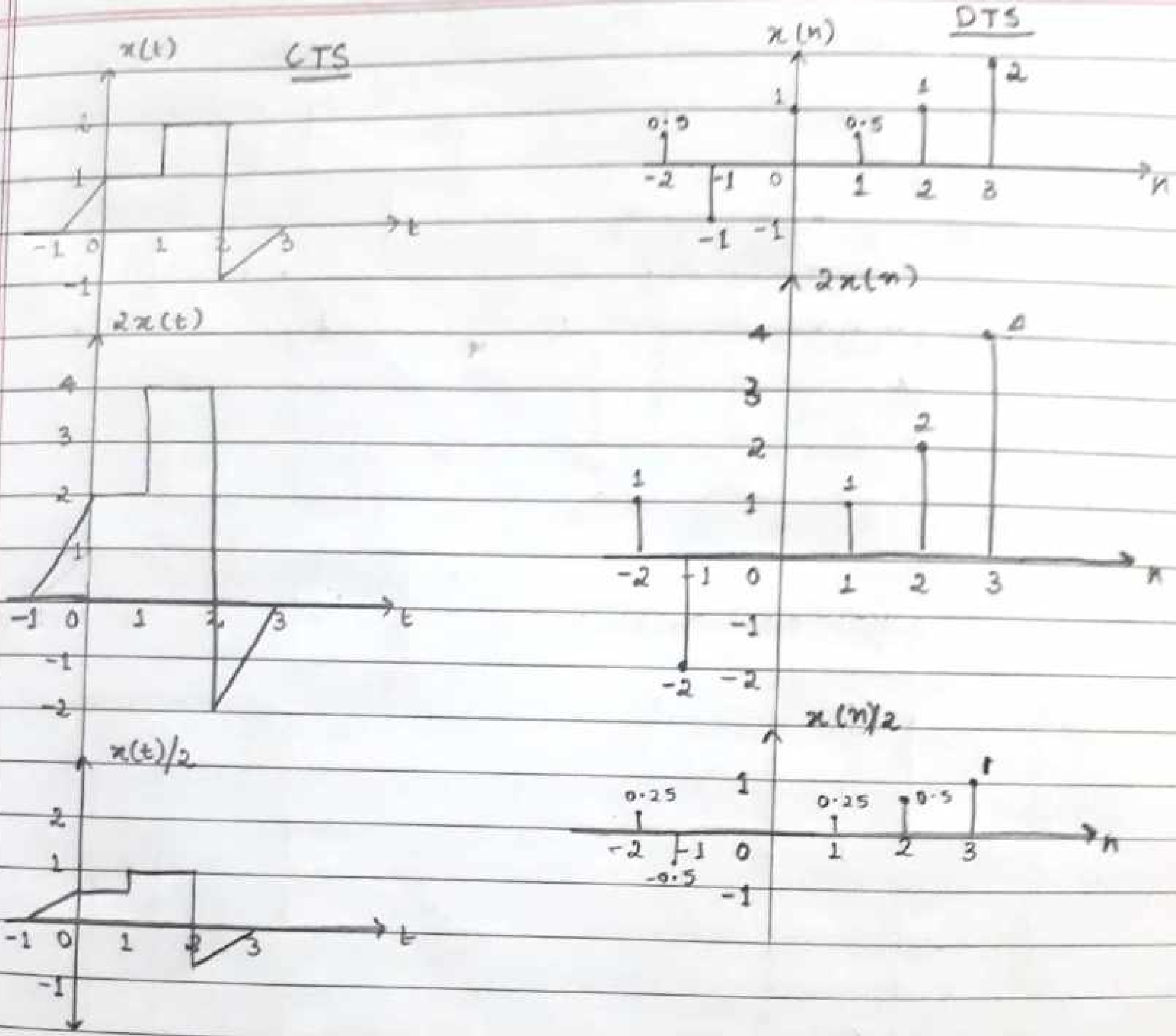
* **Time Scaling:**

$y(t) = a x(t)$ $\quad$ $a > 1$: amplification

$y(n) = a x(n)$ $\quad$ $a < 1$: attenuation.

Here a is the scaling factor.

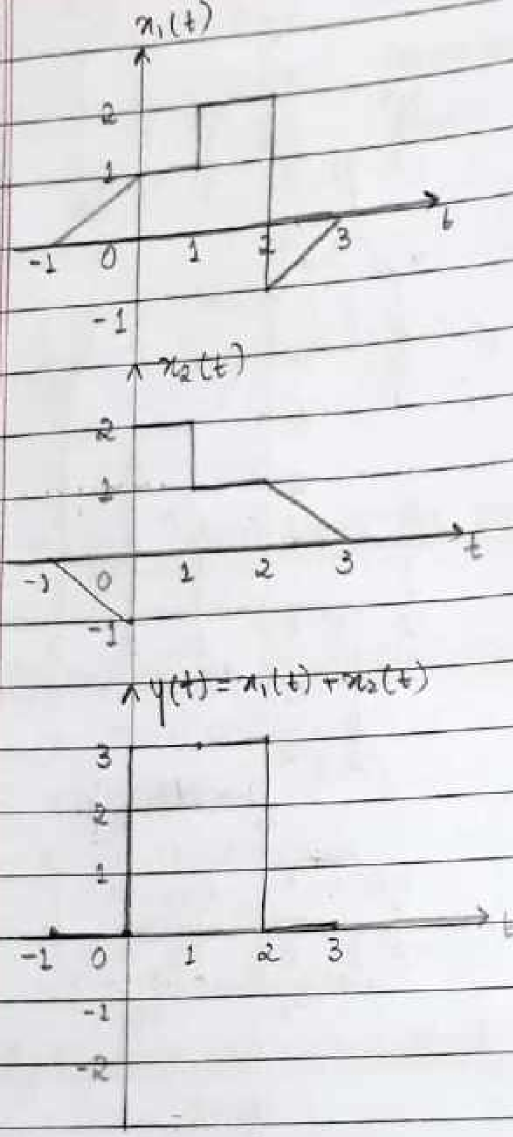
A physical example of a device that performs amplitude scaling is an electronic amplifier or an electronic attenuat which depends on the value of a.



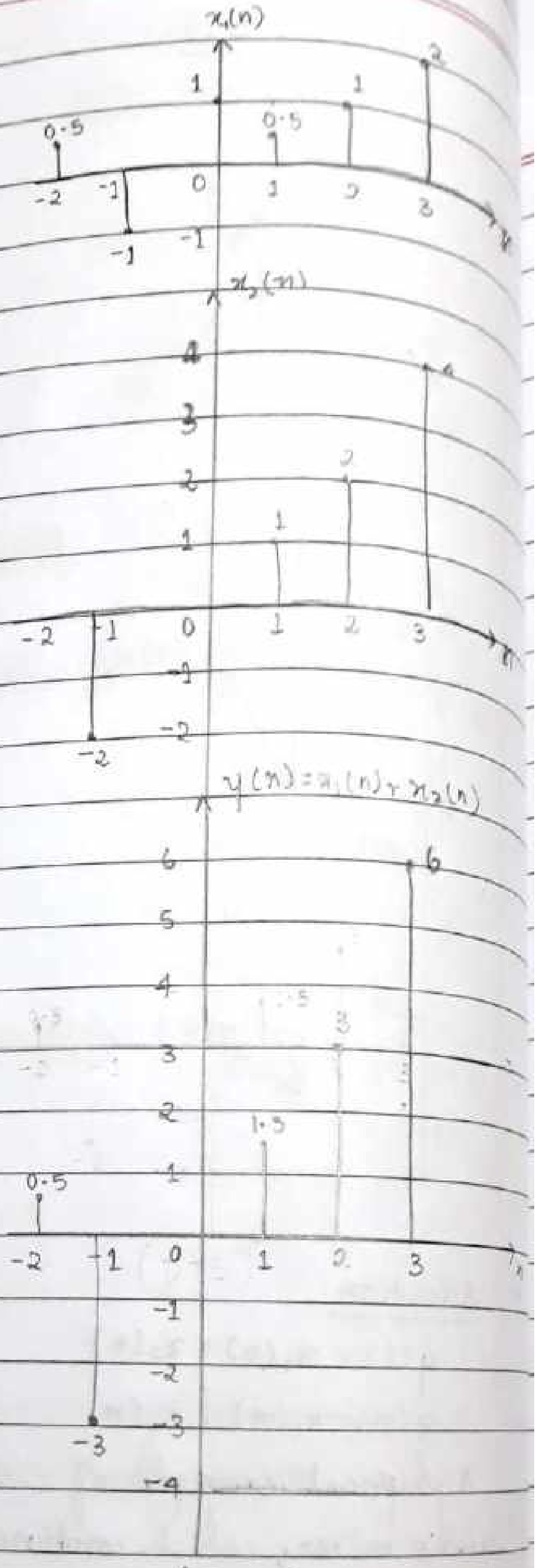
* **Addition:**

$$y(t) = x_1(t) + x_2(t)$$
$$y(n) = x_1(n) + x_2(n)$$

A physical example of a device that adds signals is an audio mixer, which combines music and voice signals.
Addition of signals is the addition of their corresponding amplitudes.



(CTS)

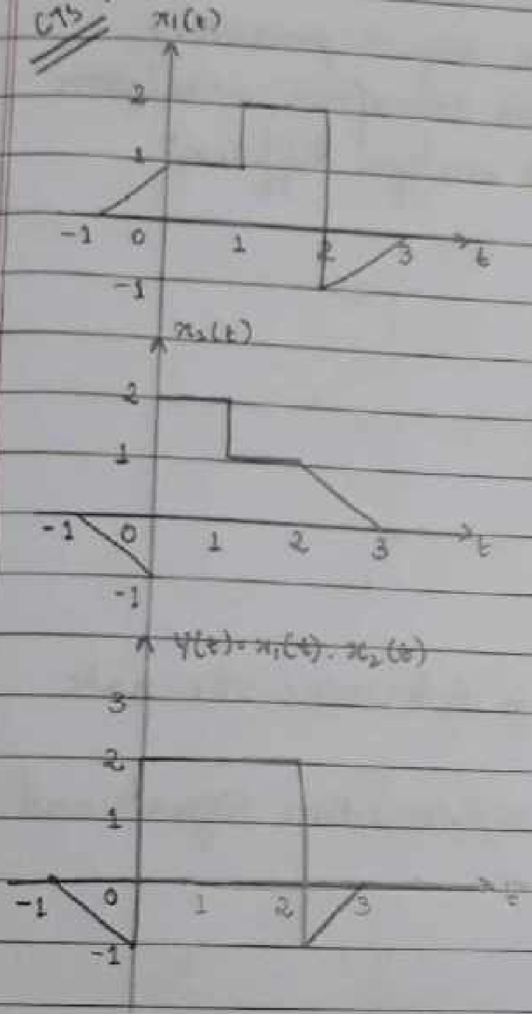


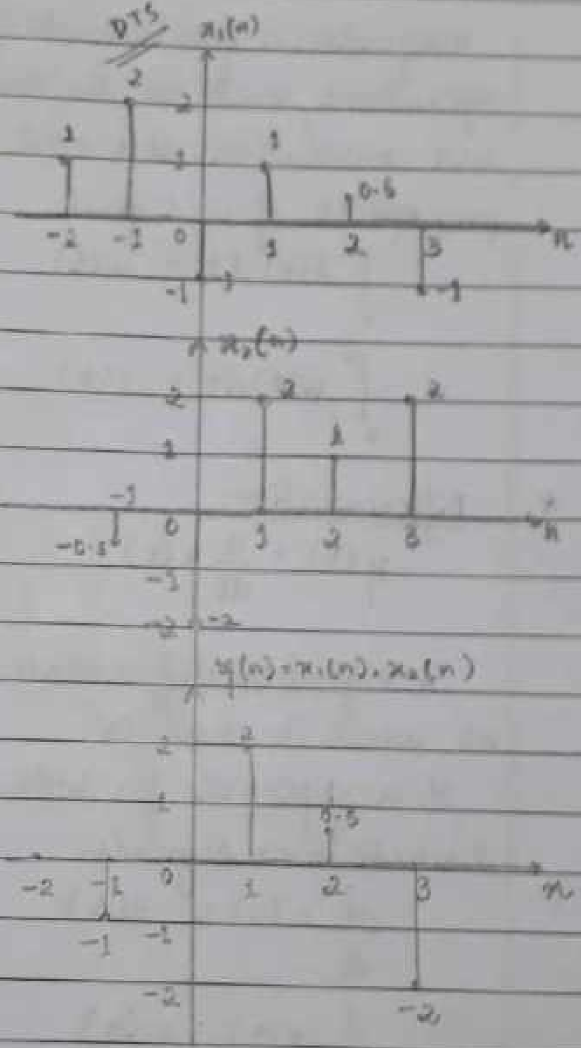
(DTS)

* **Multiplication**:

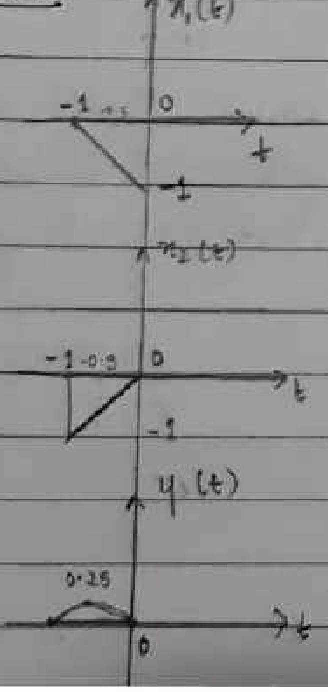
$y(t) = x_1(t) \times x_2(t)$

$y(n) = x_1(n) \times x_2(n)$





NOTE:



Here multiplication of ramp with a ramp gives zero signal. But multiplication of any two signal we expect an output. In such condition we have to consider the intermediate points.

* **Integration**

$$y(t) = \int_{-\infty}^{t} x(\tau)\,d\tau$$

Integration is fundamental in signal processing operations such as the Fourier transform, correlation and convolution. It is used to analyse different properties of a signal.

$$\int_{0}^{\infty} \delta(t)\,dt = u(t)$$

$$\int_{0}^{\infty} u(t)\,dt = r(t)$$

* **Differentiation**

$$y(t) = \frac{d}{dt}x(t)$$

A signal is differentiated to determine the rate at which it changes.

It is applicable for both continuous time signals and discrete time signals.

$$\frac{d}{dt}u(t) = \delta(t)$$

$$\frac{d}{dt}r(t) = u(t)$$

* <u>Time Shift</u>:

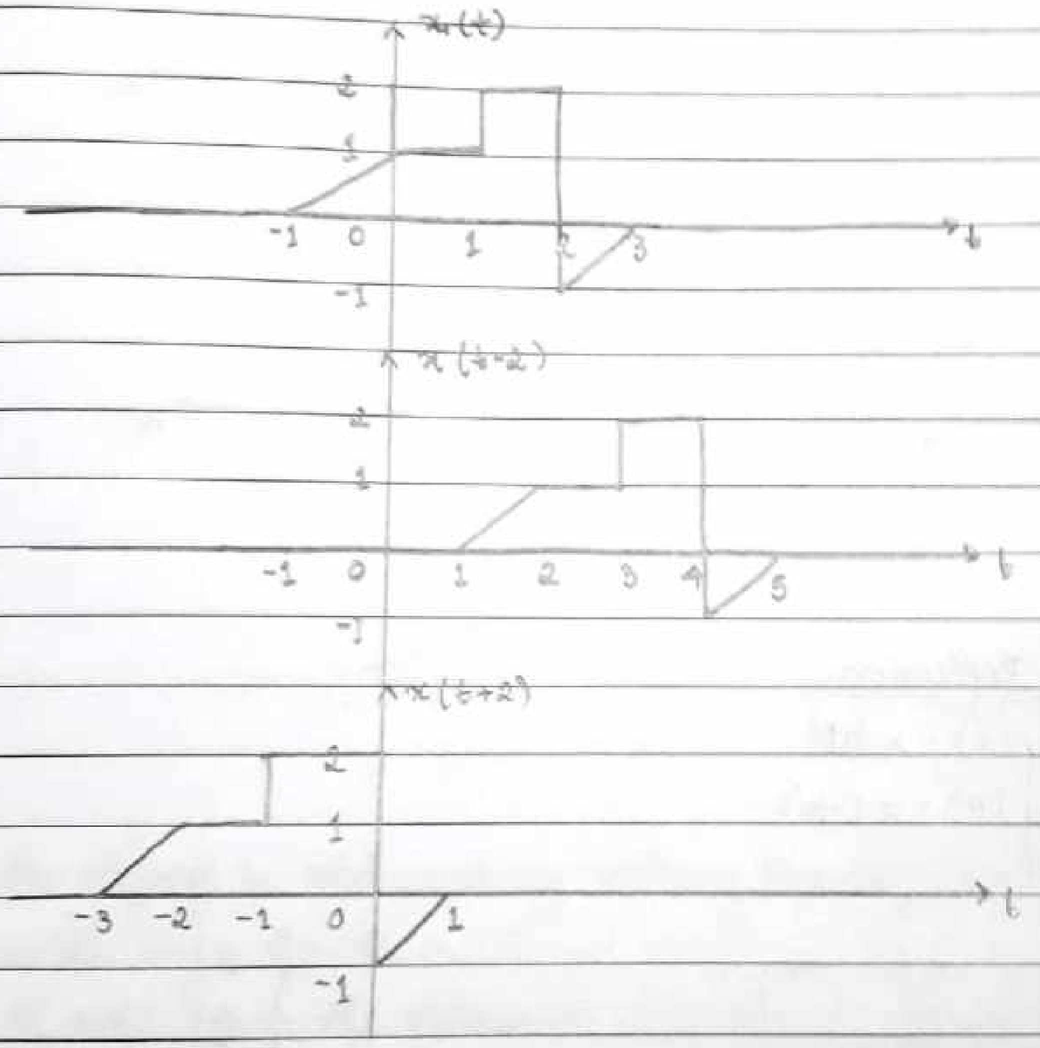
$y(t) = x(t-k)$ (right shift by k units)

$y(t) = x(t+k)$ (left shift by k units)

$y(n) = x(n-k)$ (right shift by k units)

$y(n) = x(n+k)$ (left shift by k units)

<u>CTS</u>



k - denotes the time shift.

The value of k ~~dep~~ determines towards which side the signal is shifted. and by how many units.

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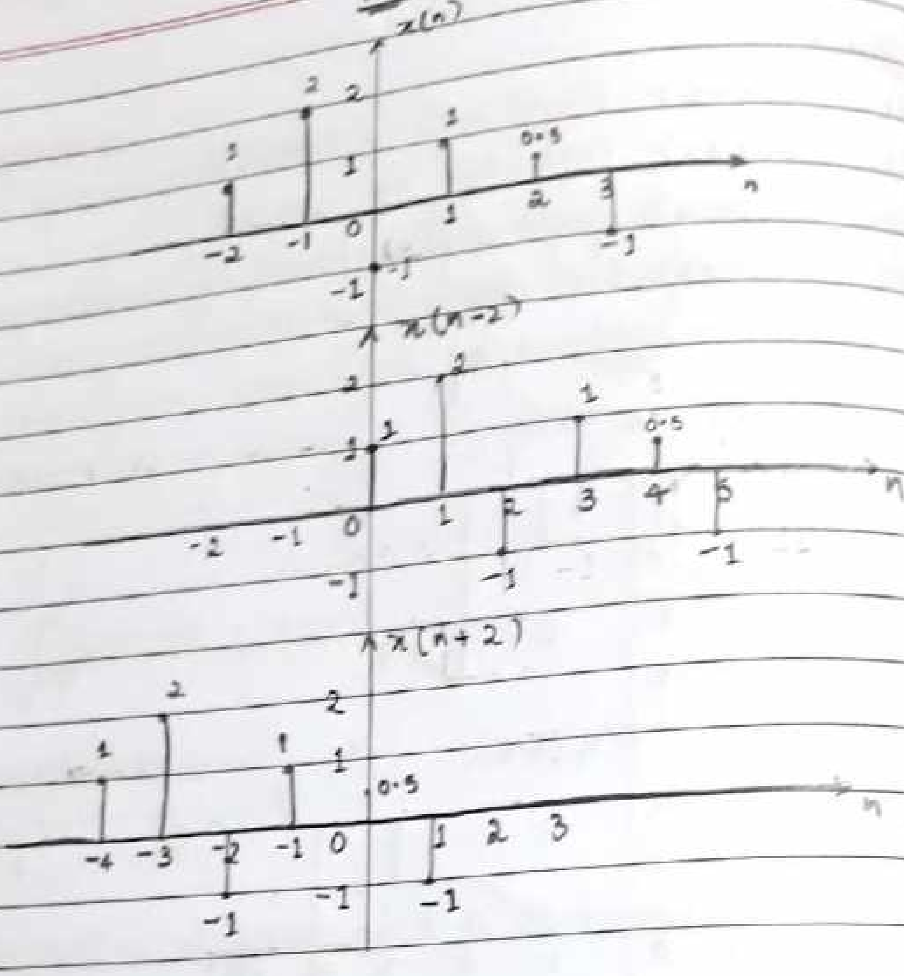
DTS

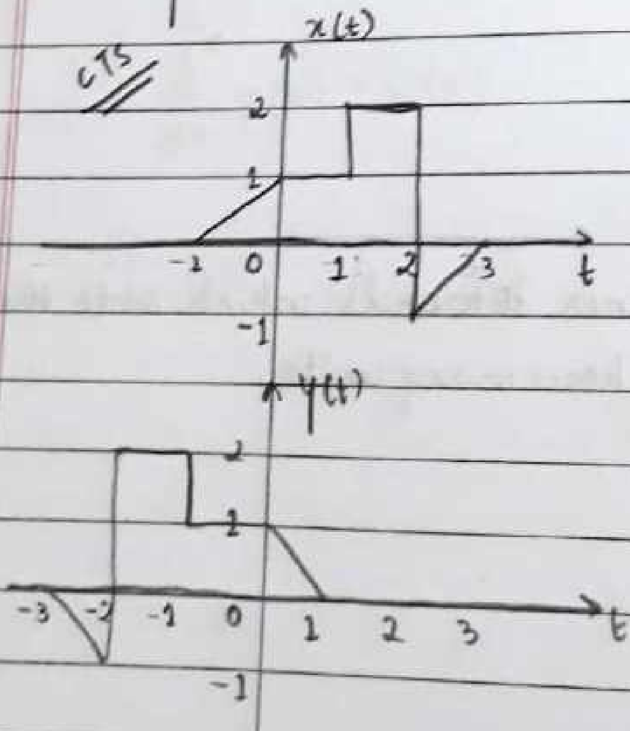


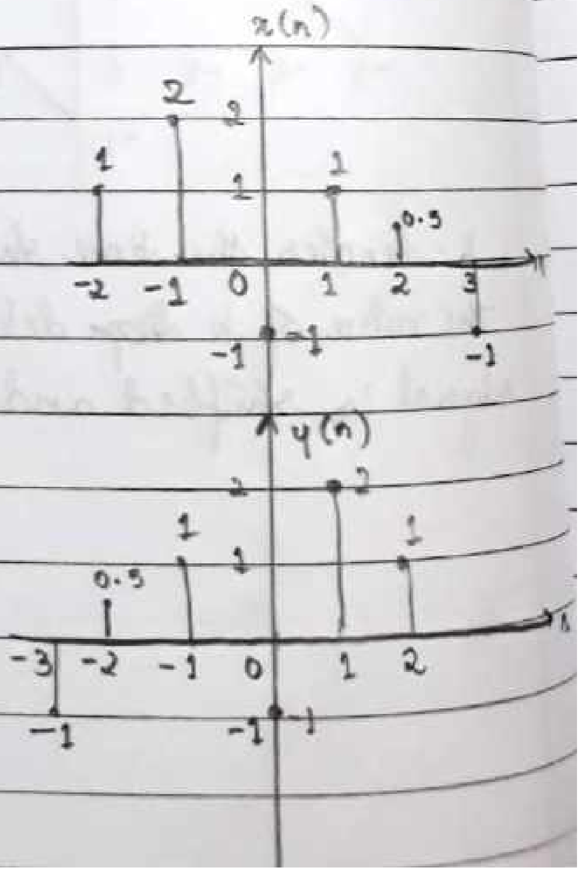
★ **Time Reflection:**

$y(t) = x(-t)$

$y(n) = x(-n)$

CTS



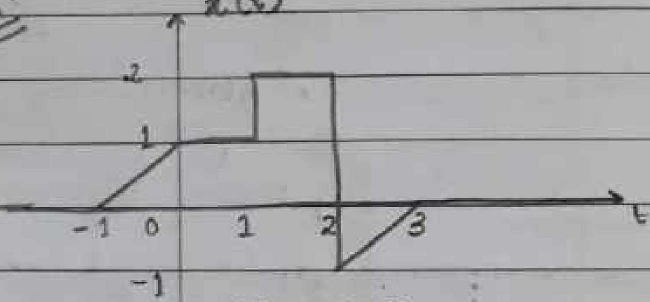


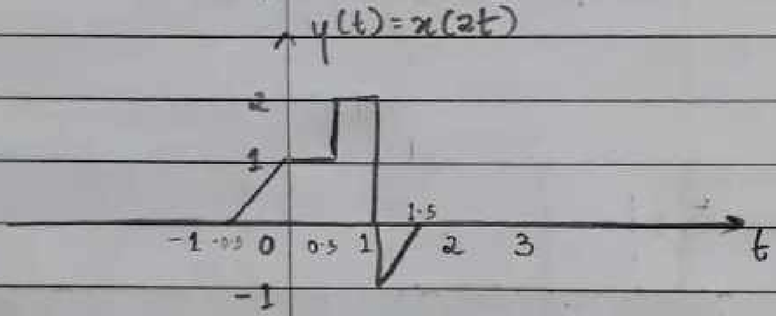
* Time Scaling:

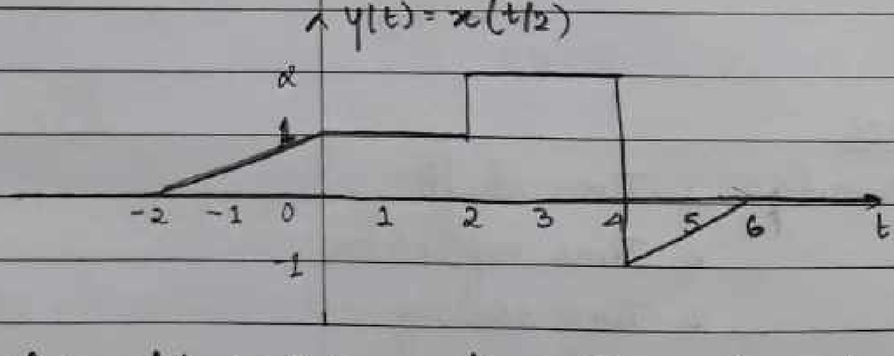
$y(t) = x(at)$ $\qquad$ $a < 1$: Expansion

$y(n) = x(an)$ $\qquad$ $a > 1$: compression

CTS:







The signal is obtained by scaling the independent variable by a factor a.

If $a > 1$, then the obtained signal is a compressed version

If $a < 1$, then the obtained signal is an expanded version.

In DTS signal, on time scaling there is loss in information for non integer values.

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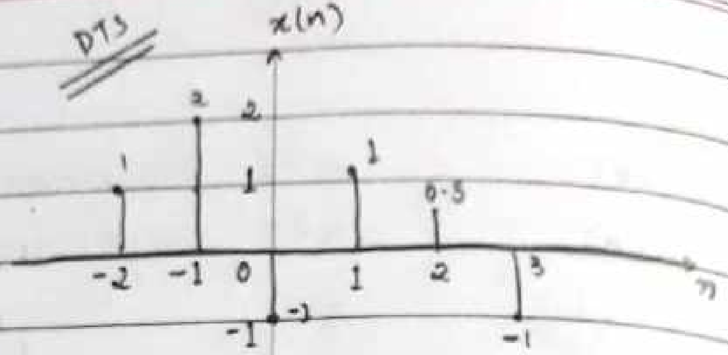
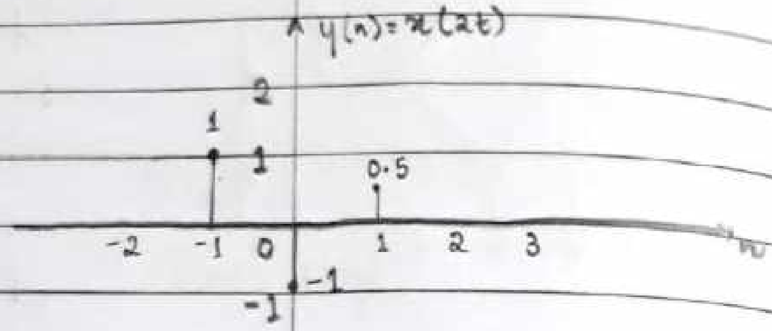
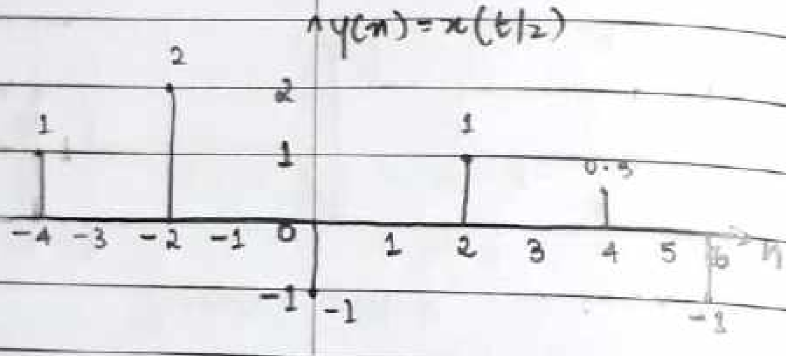
DTS







<u>NOTE:</u>

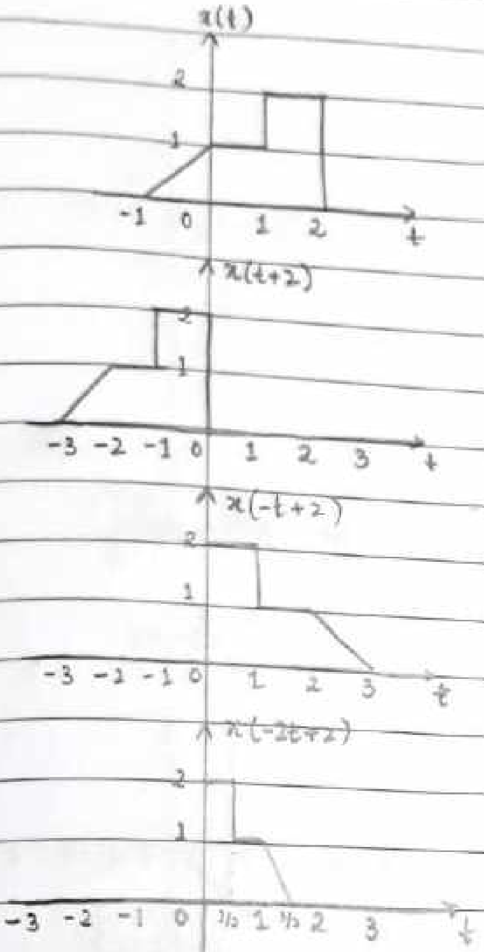
Priority : 1. Time shift
2. Time reflection
3. Time scaling

<u>Ex</u>: For $x(-2t+2)$

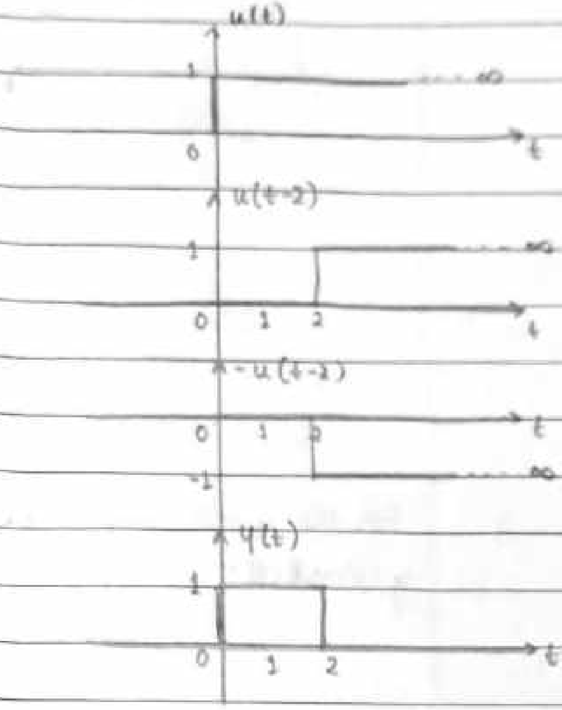
1. $x(t+2)$ time shift
2. $x(-t+2)$ time reflection
3. $x(-2t+2)$ time scaling

It is very necessary to perform the operation of time shifting, time reflection and time scaling in the right order to obtain the output correctly.

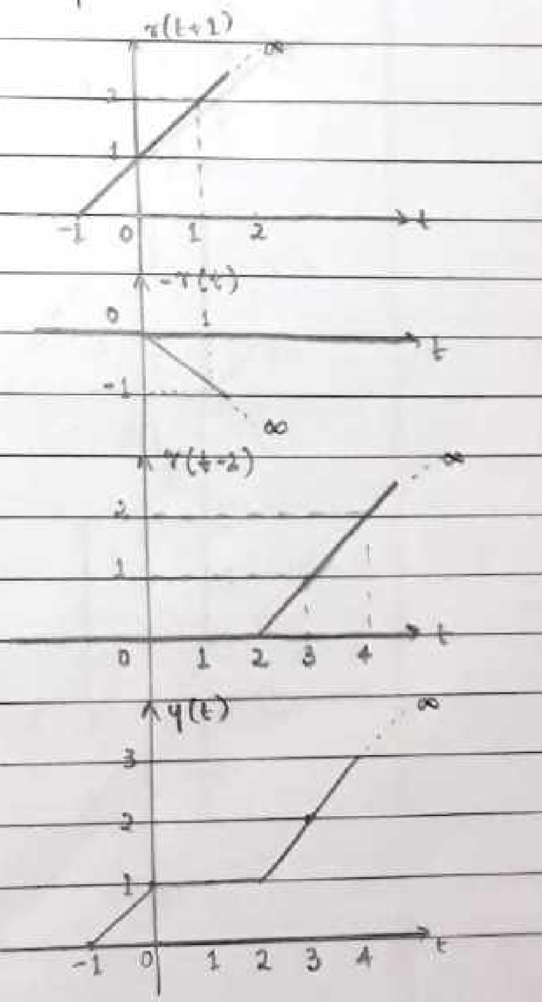
Q1: $x(-2t+2)$



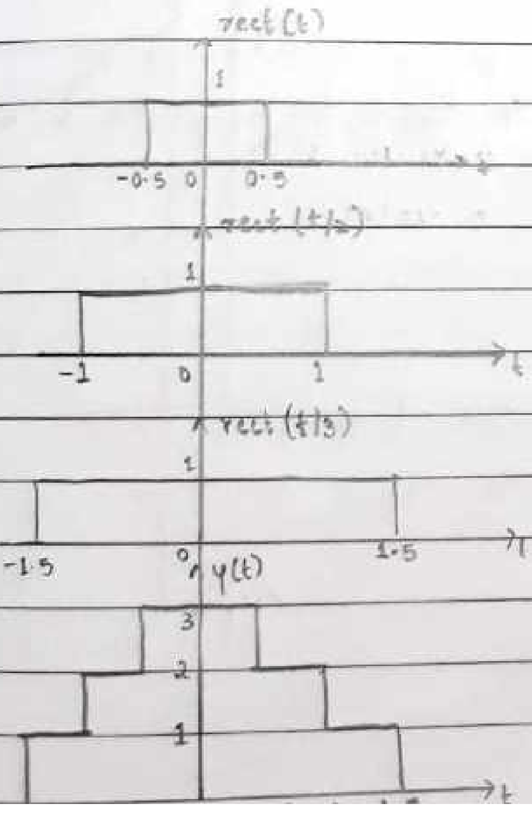
Q2: $y(t) = u(t) - u(t-2)$



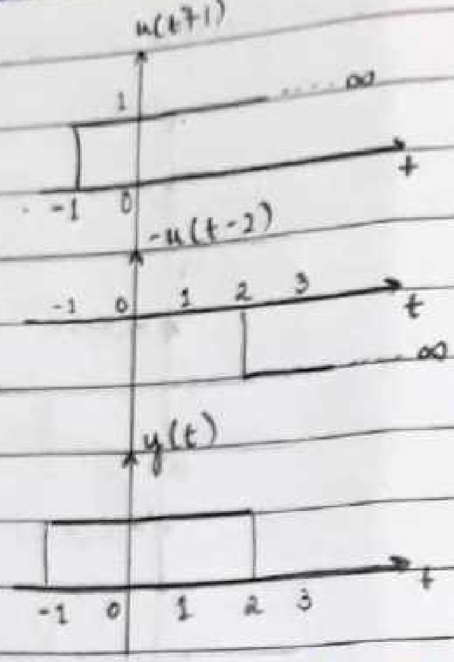
Q4: $y(t) = r(t+1) - r(t) + r(t-2)$



Q3: $y(t) = Rect(t) + Rect(t/2) + Rect(t/3)$

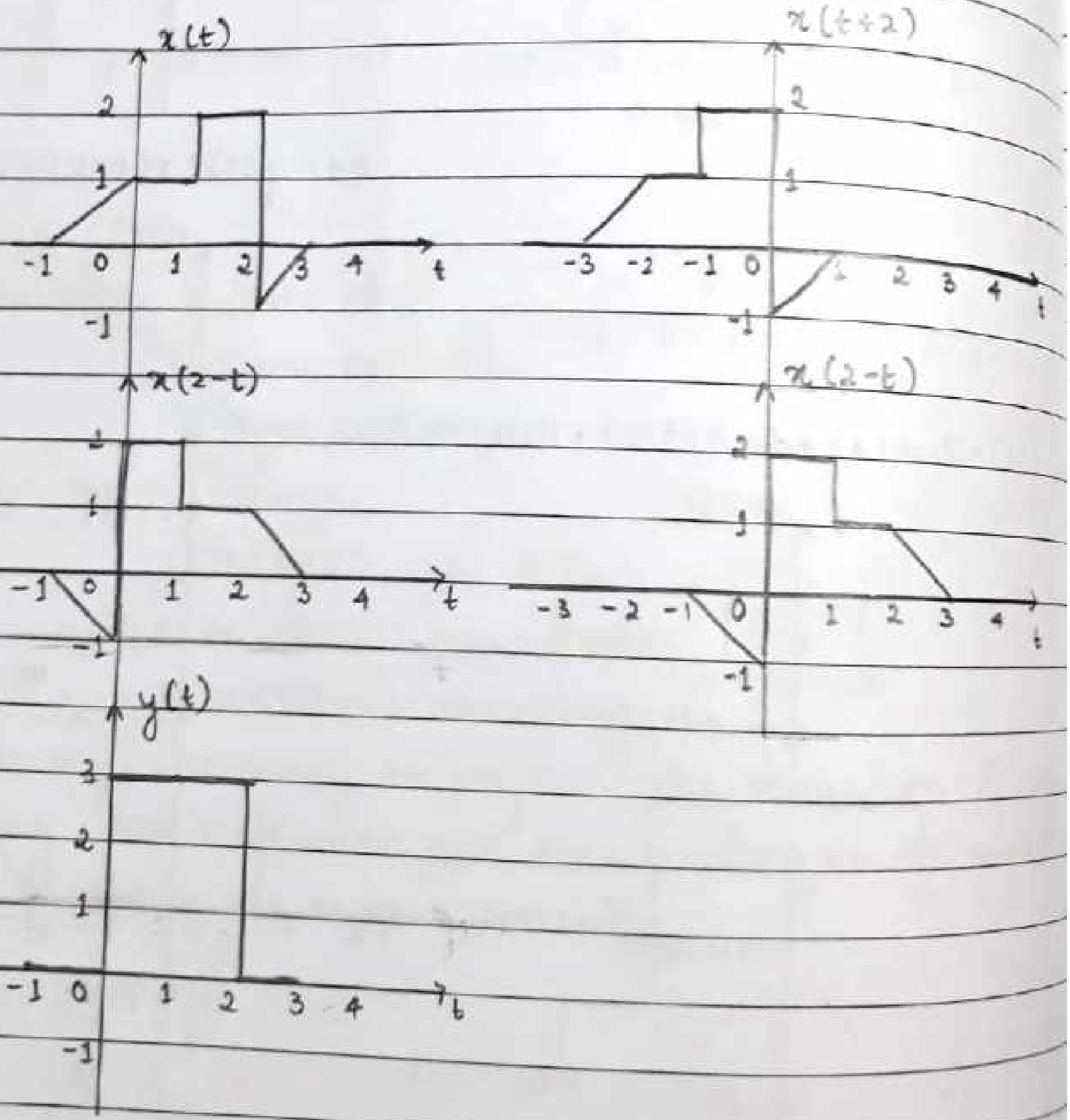


Q5: $y(t) = u(t+1) - u(t-2)$



Q: For the given signal $x(t)$ sketch the following:

Q1. $y(t) = x(t) + x(2-t)$



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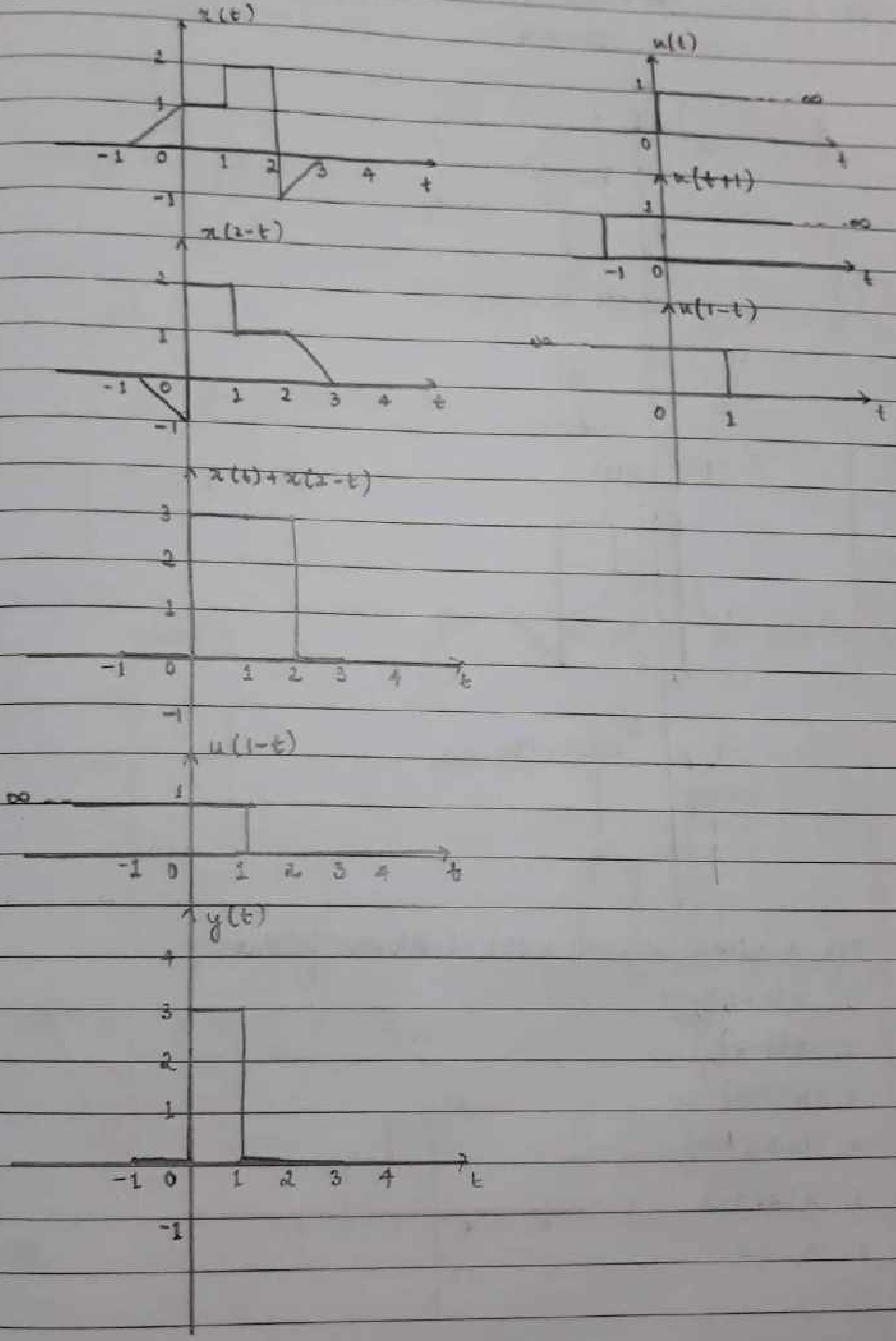
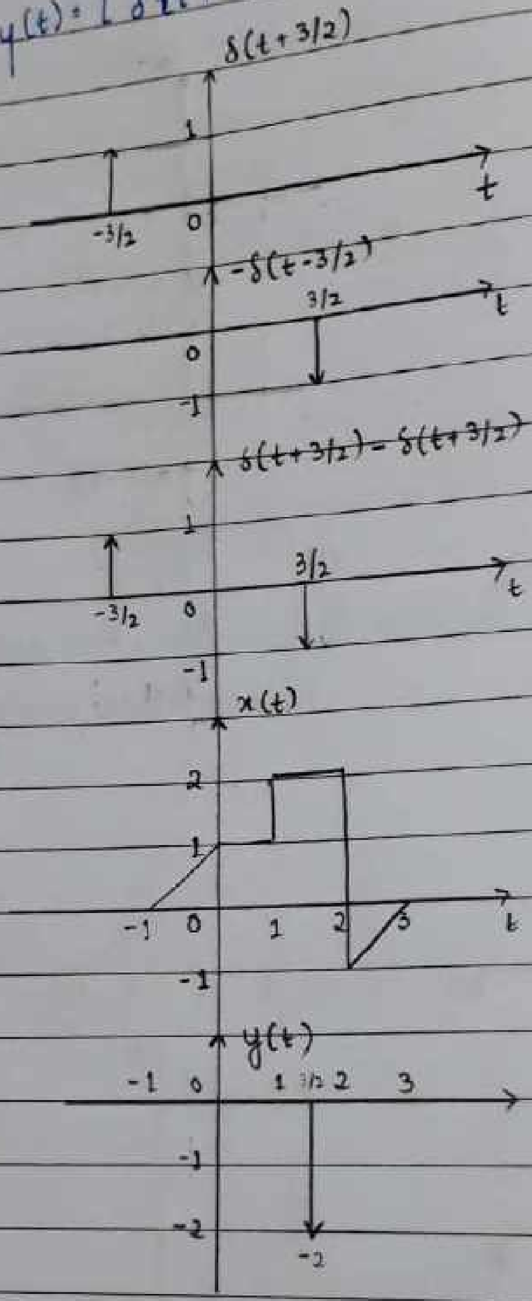
Q2. $y(t) = [x(t) + x(2-t)]\,u(1-t)$

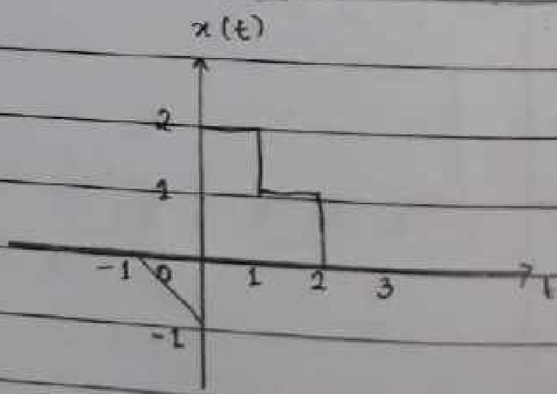


Q3: $y(t) = [\delta(t+3/2) - \delta(t-3/2)]\, x(t)$

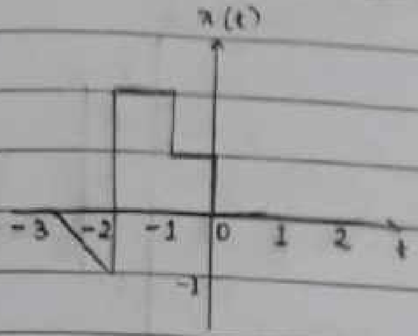


Q: For a given signal $x(t)$ sketch the following

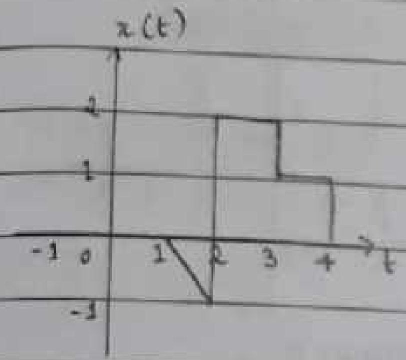
1. $x(t+2)$
2. $x(t-2)$
3. $3x(t/2)$
4. $\frac{1}{2}x(-t/2)$
5. $x(2t+1)$
6. $x(2-3t)$



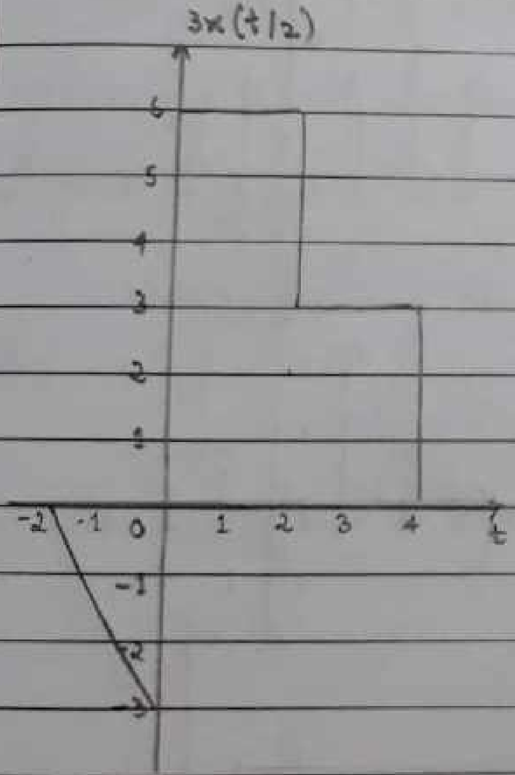
1. $x(t+2)$



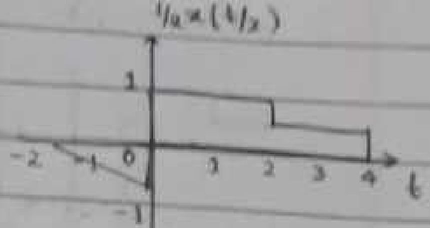
2. $x(t-2)$

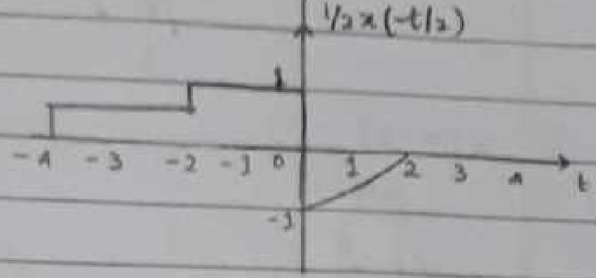


3. $3x(t/2)$

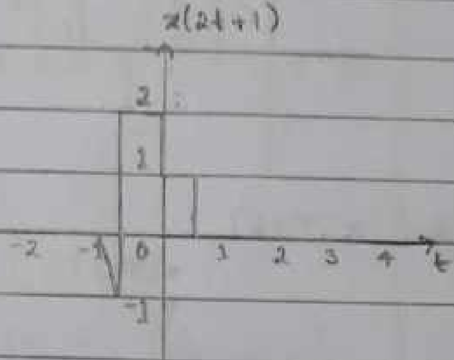


4. $\frac{1}{2}x(-t/2)$

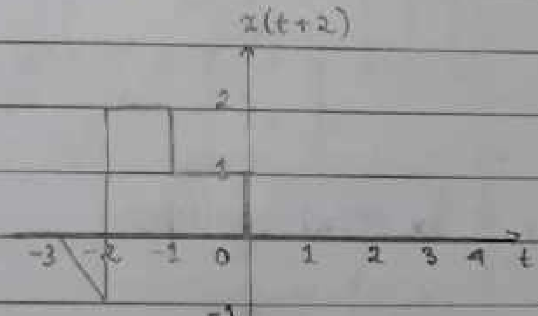


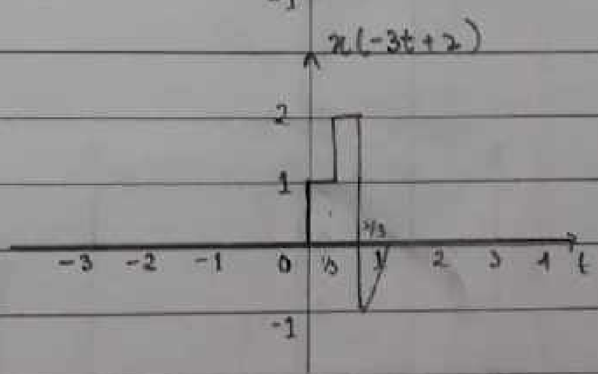


5. $x(2t+1)$

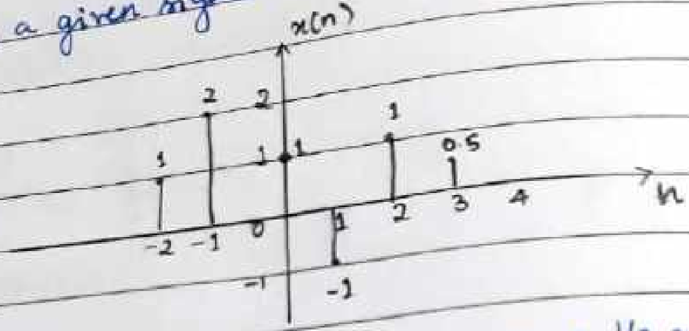


6. $x(2-3t)$

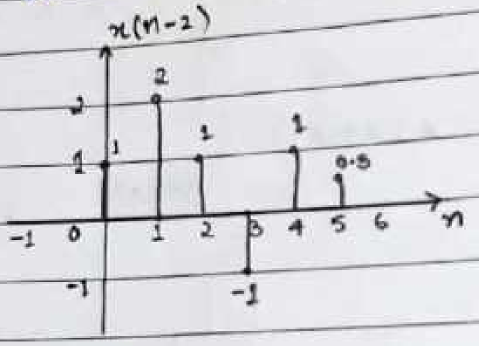




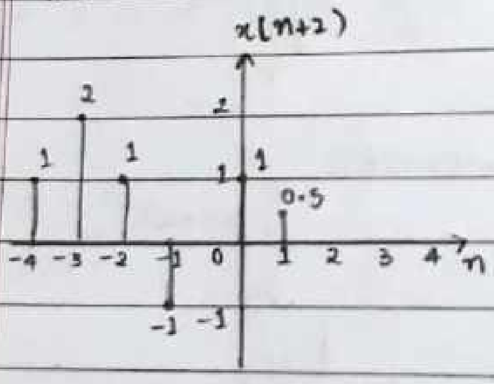
Q: For a given signal $x(n)$ sketch the following



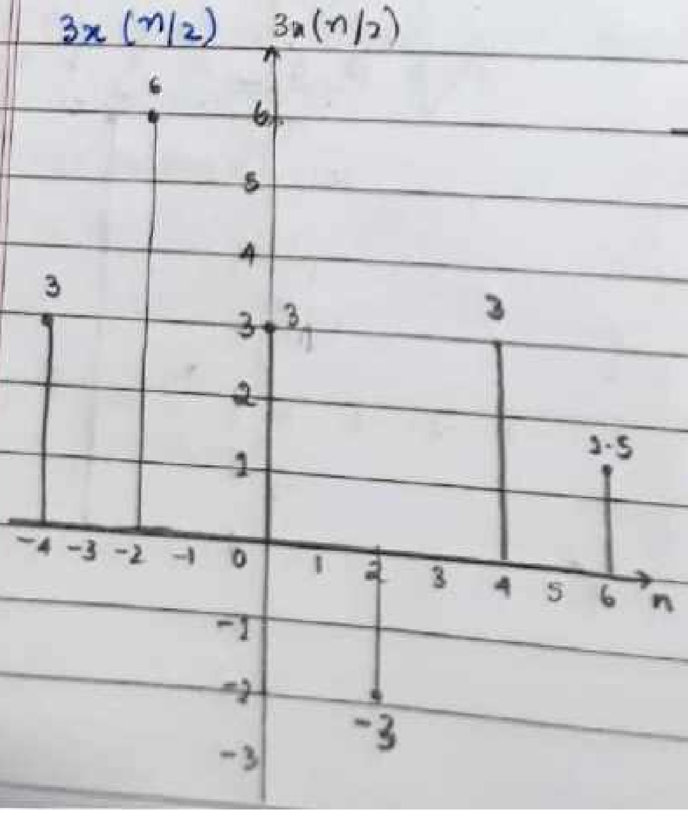
1. $x(n-2)$



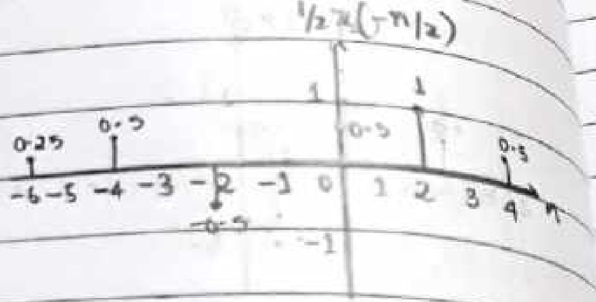
2. $x(n+2)$



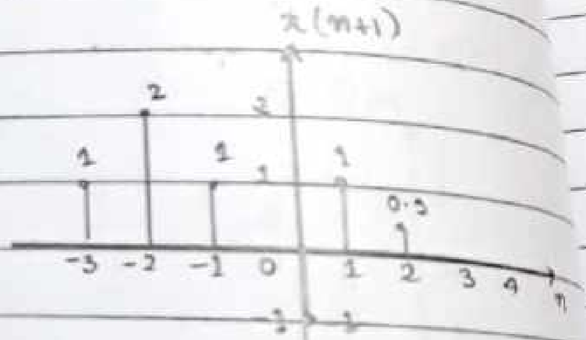
3. $3x(n/2)$

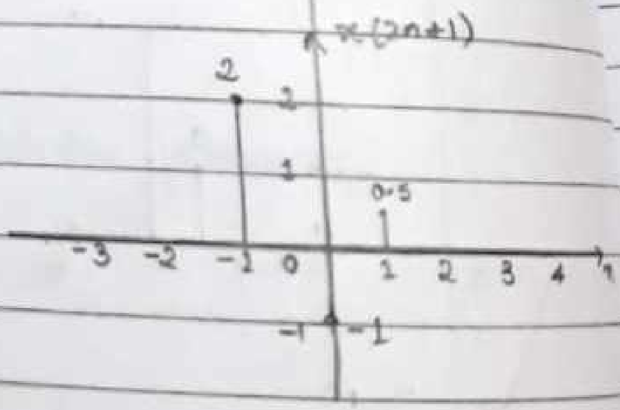


4. $1/2\, x(-n/2)$



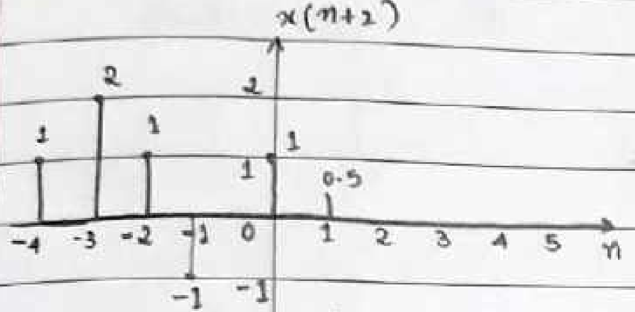
5. $x(2n+1)$

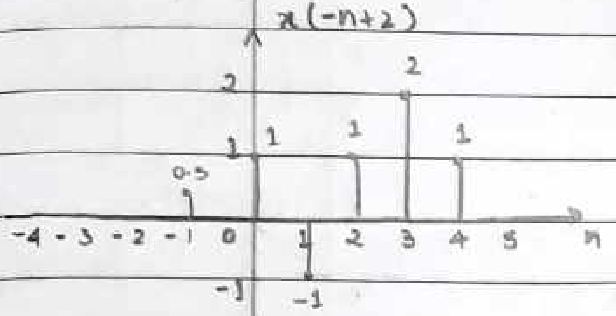


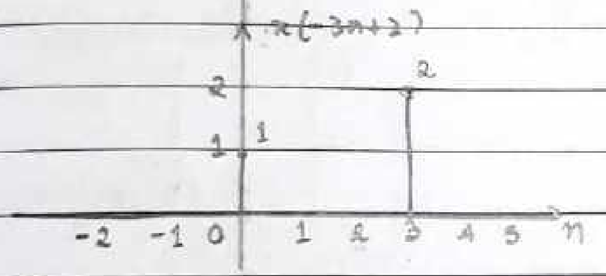


6. x(

6. $x(2-3n)$

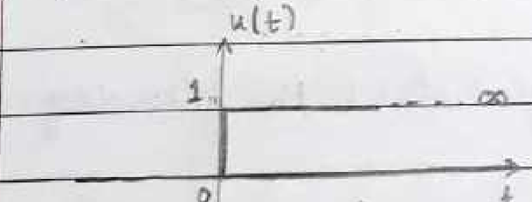


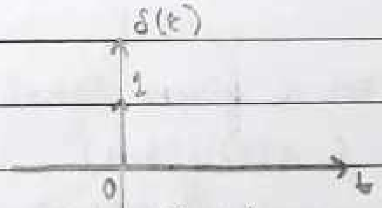


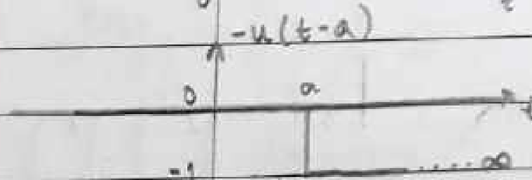


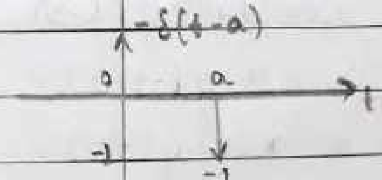
Q: sketch the signal and its derivative for the following:

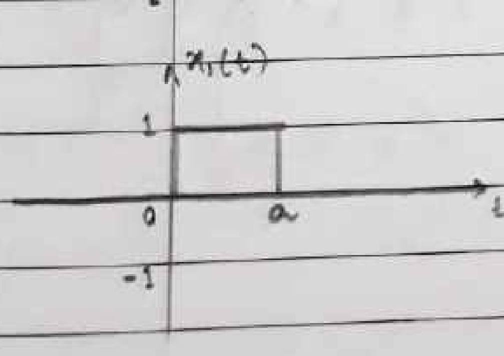
1. $x_1(t) = u(t) - u(t-a)$ $\qquad \frac{d}{dx} x_1(t) = \delta(t) - \delta(t-a)$

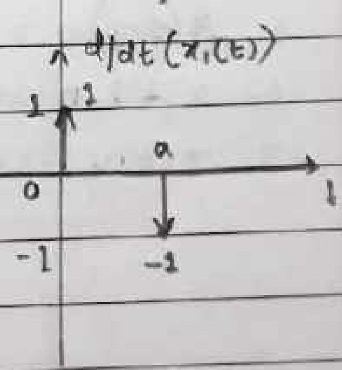






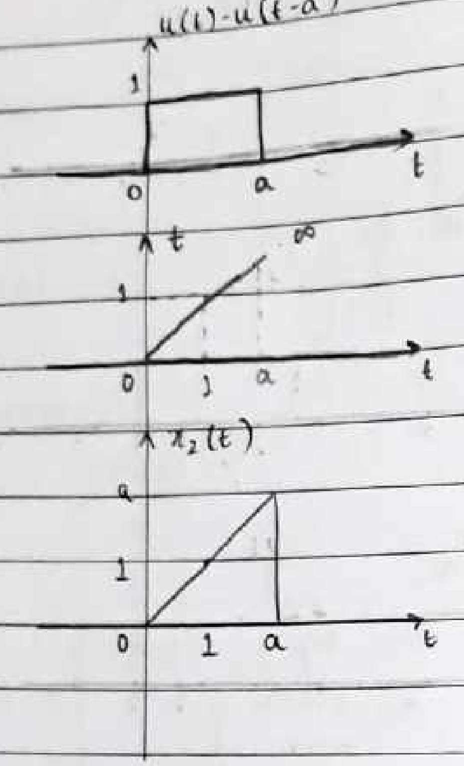


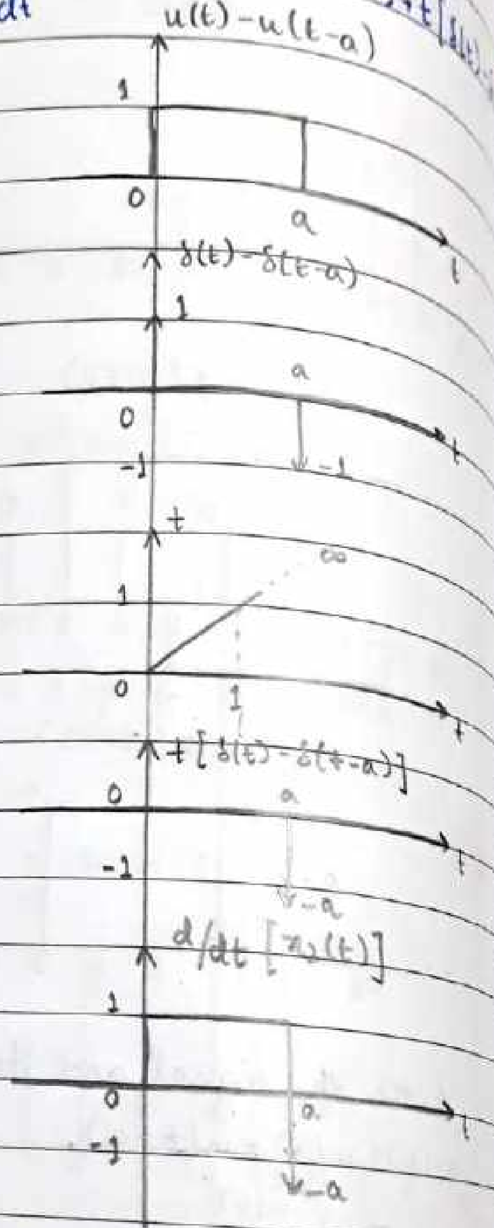




2. $x_2(t) = t\,[u(t)-u(t-a)]$

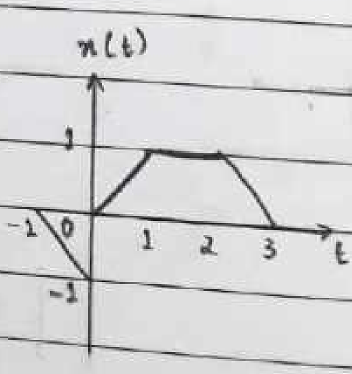
$\frac{d}{dt} x_2(t) = u(t) - u(t-a) + t\,[\delta(t) \ldots$

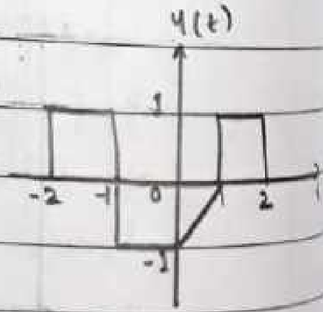




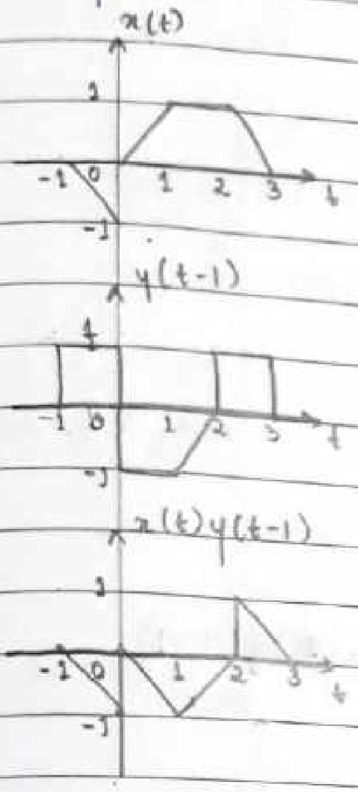
Q: For a given signal $x(t)$ and $y(t)$ sketch the following:

1. $x(t)\,y(t-1)$
2. $x(t+1)\,y(t-2)$
3. $x(t)\,y(-t-1)$
4. $x(t).y(-t)$
5. $x(4-t)\,y(t)$
6. $x(2t).\,y(2t+1)$
7. $x(t)\,y(-t+2)$

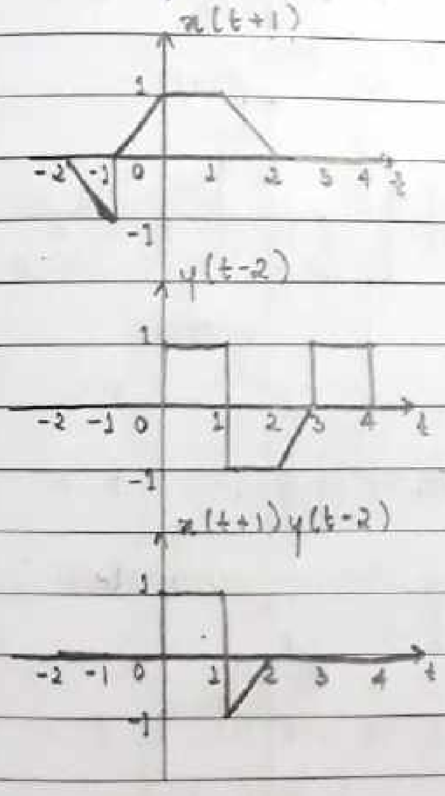




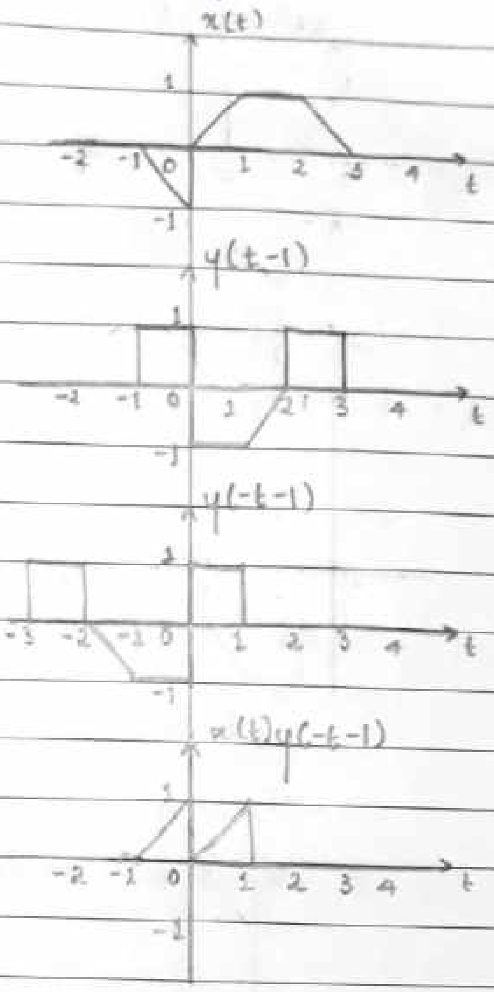
1. $x(t)\,y(t-1)$



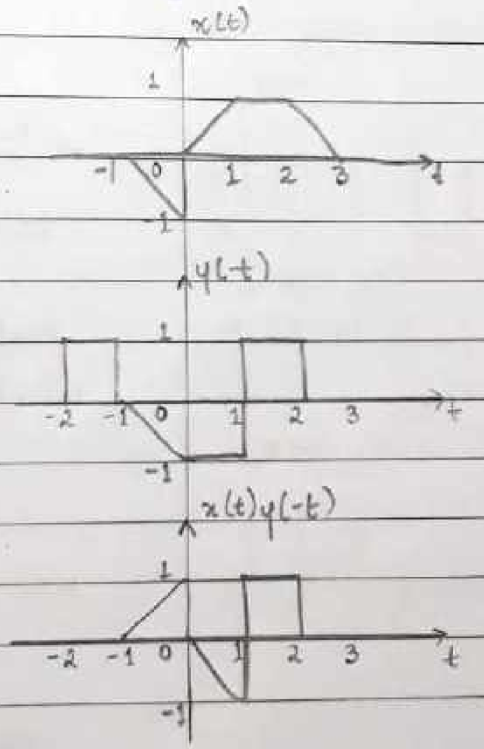
2. $x(t+1)\,y(t-2)$



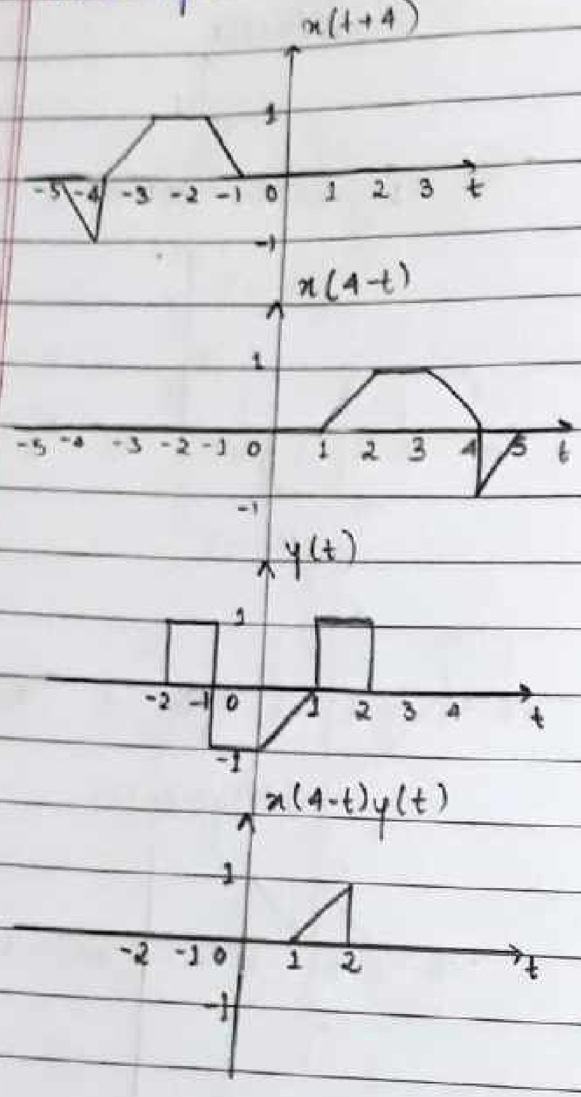
3. $x(t)\,y(-t-1)$



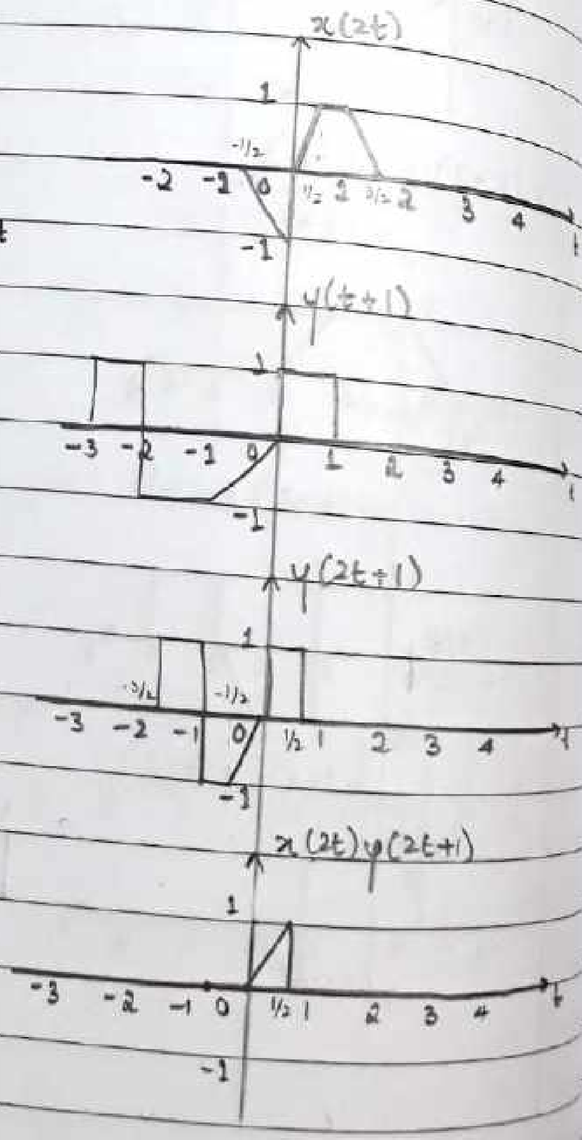
4. $x(t)\,y(-t)$



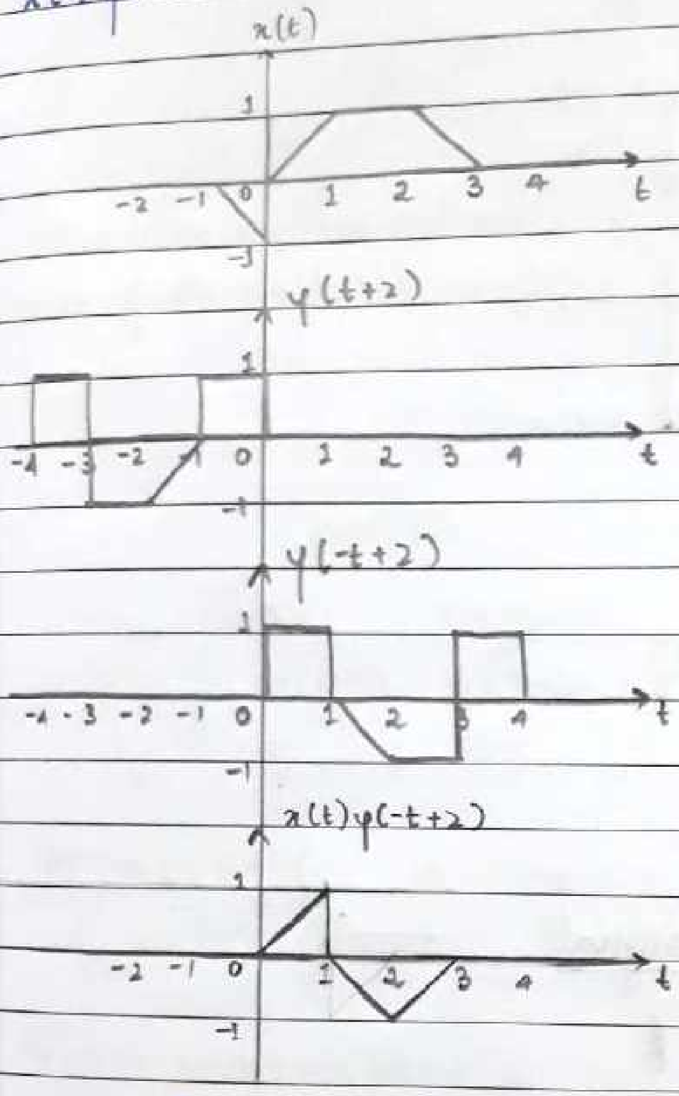
5. $x(4-t)\,y(t)$



6. $x(2t)\,y(2t+1)$

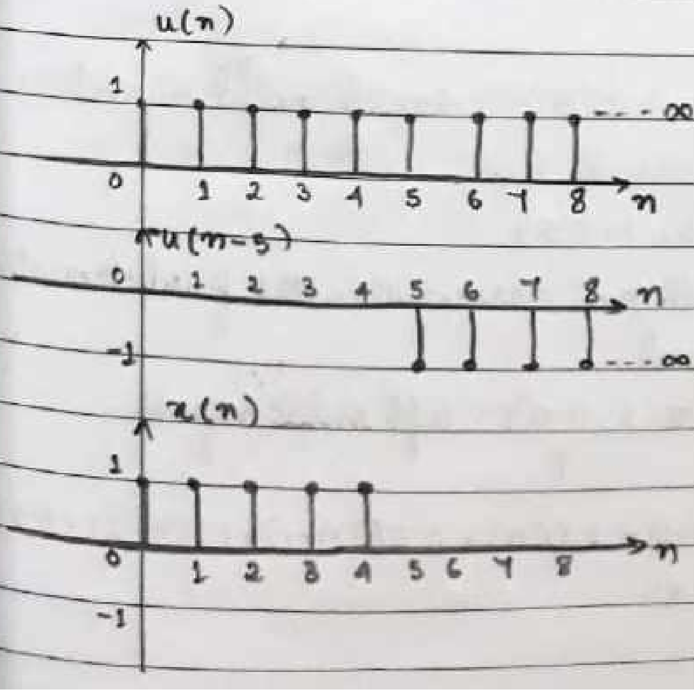


7. $x(t)\, y(-t+2)$

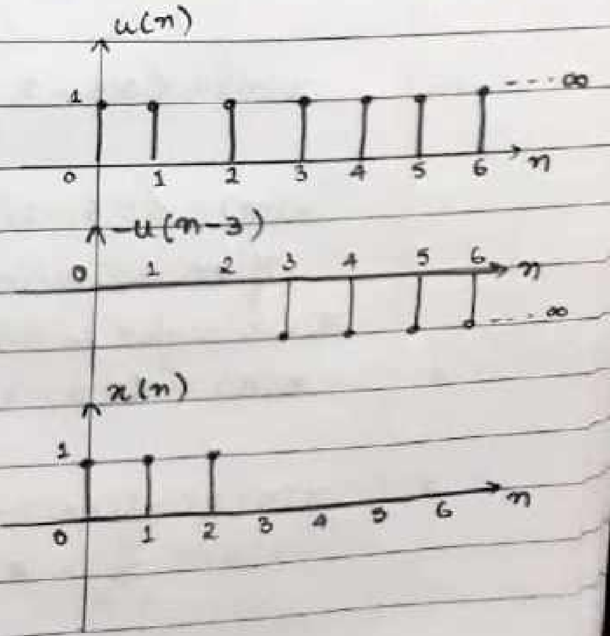


Q: For the following functions, sketch the DTS signal.

1. $x(n) = u(n) - u(n-5)$



2. $x(n) = u(n) - u(n-3)$



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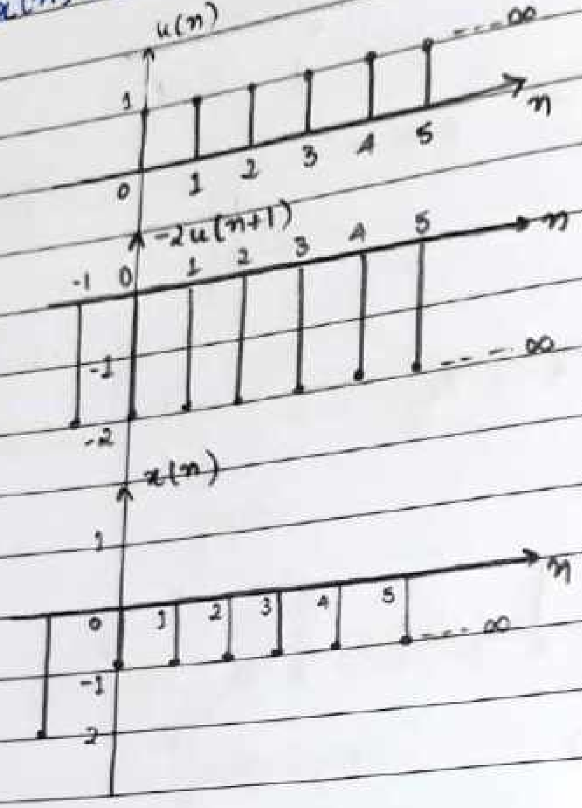
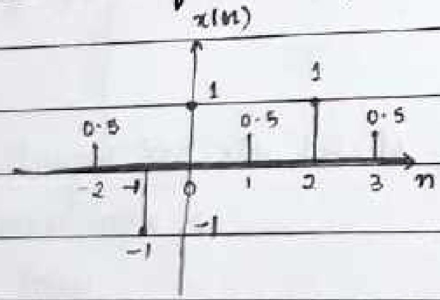
3. $x(n) = u(n) - 2u(n+1)$



★ **Representation of DTS signal**

1.



2. $x(n) = \{0.5, -1, \underset{\uparrow}{1}, 0.5, 1, 0.5\}$: double sided signal

It represents $t = 0$

3. $x(n) = \{0.5, -1, 1, 0.5, 1, 0.5\}$

If no indication with an arrow then the first element is at this a right sided signal.

4. $x(n) = \{0.5, -1, 1, 0.5, 1, \underset{\uparrow}{0.5}\}$: left sided signal.

5. $x(n) = 0.5\delta(n+2) - 1\delta(n+1) + 1\delta(n) + 0.5\delta(n-1) + 1\delta(n-2) + 0.5(n-3)$

$x(n) = \sum_{k=-\infty}^{\infty} x(k)\delta(n-k)$ $x(k)$ = amplitude

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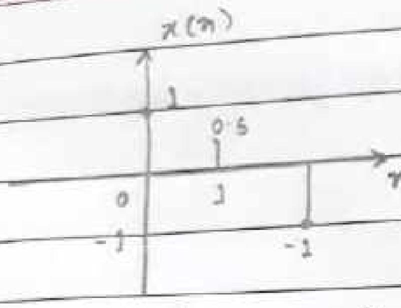
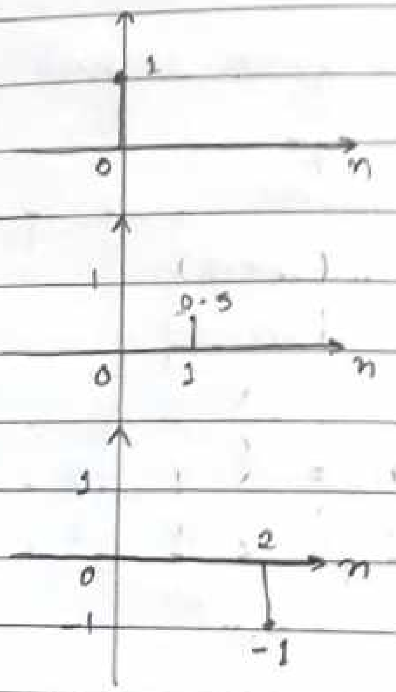


$\Rightarrow$



$\delta(n)$

$0.5\delta(n-1)$

$-1\delta(n-2)$

It can be represented as sum of the impulse functions.

6.

$$x(n) = \begin{cases} n+1 & ; \ 0 \le n \le 2 \\ \cos(n\pi) & ; \ 3 \le n \le 5 \\ 0 & ; \ \text{Else} \end{cases}$$

$$\Rightarrow x(n) = \{\underset{\uparrow}{1}, 2, -1, 1, -1\}$$

Amplitude

$$\begin{array}{lccccc} n\rightarrow & 0 & 1 & 3 & 4 & 5 \end{array}$$

7. $x(n) = u(n) - u(n-4)$

$$\Rightarrow x(n) = \{1, 1, 1, 1\}$$

$$\begin{array}{lcccc} n\rightarrow & 0 & 1 & 2 & 3 \end{array}$$

8. $x(n) = (n+1)[u(n) - u(n-5)\}$

Amplitude

$$\Rightarrow x(n) = \{1, 2, 3, 4, 5\}$$

$$\begin{array}{lccccc} n\rightarrow & 0 & 1 & 2 & 3 & 4 \end{array}$$

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Q: For the given signal $x(n) = (6-n)[u(n) - u(n-6)]$ sketch:

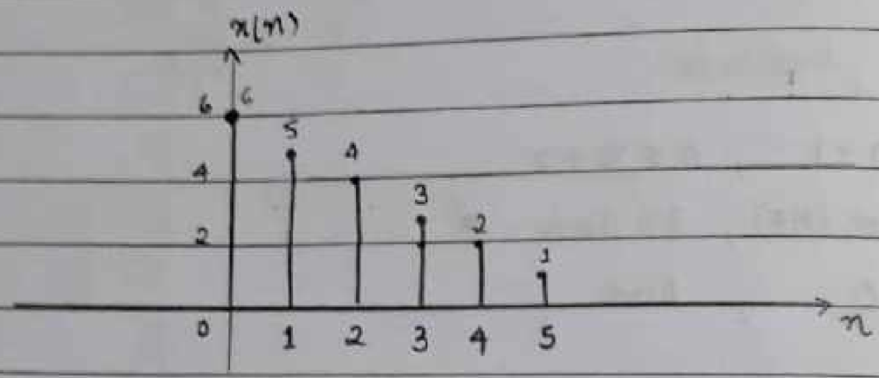
i. $x(n)$

ii. $x(4-n)$

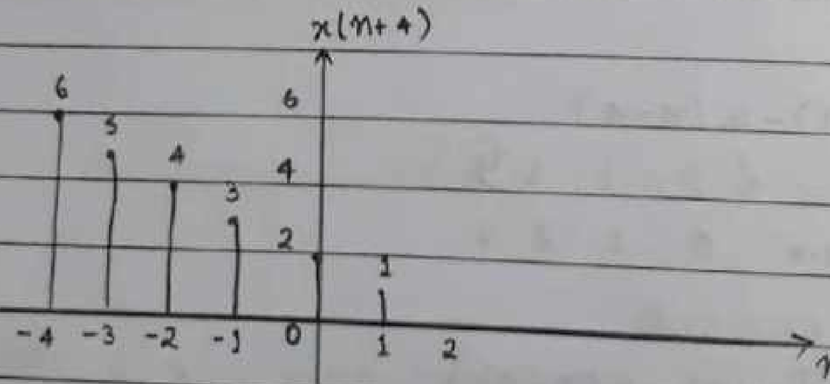
iii. $x(2n-3)$

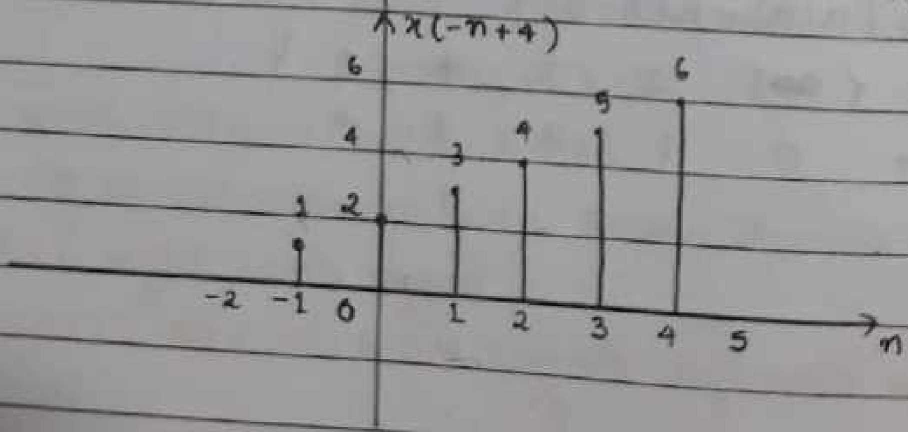
i. $x(n) = \{6, 5, 4, 3, 2, 1\}$

$n \to$	0	1	2	3	4	5

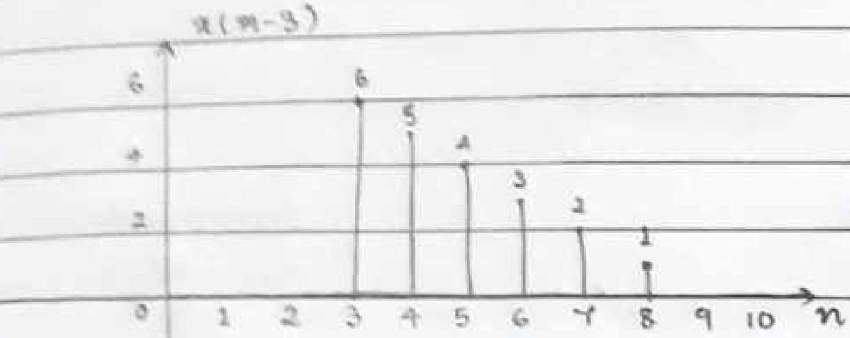


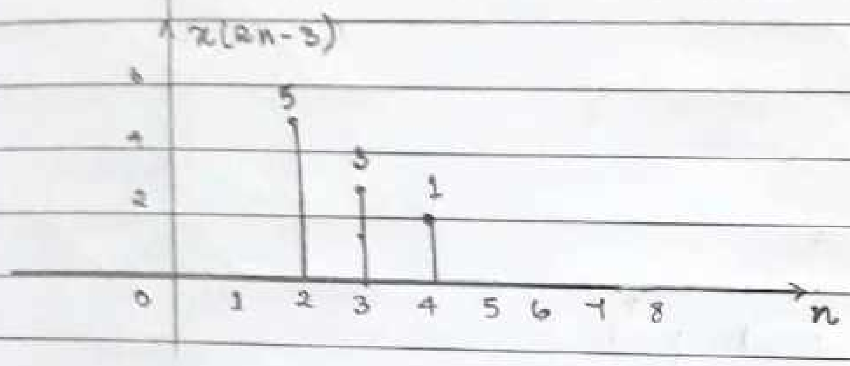
ii. $x(-n+4)$





iii. $x(2n-3)$





★ **Odd and even of a signal:**

$$x(t) = x_e(t) + x_o(t)$$

Even part

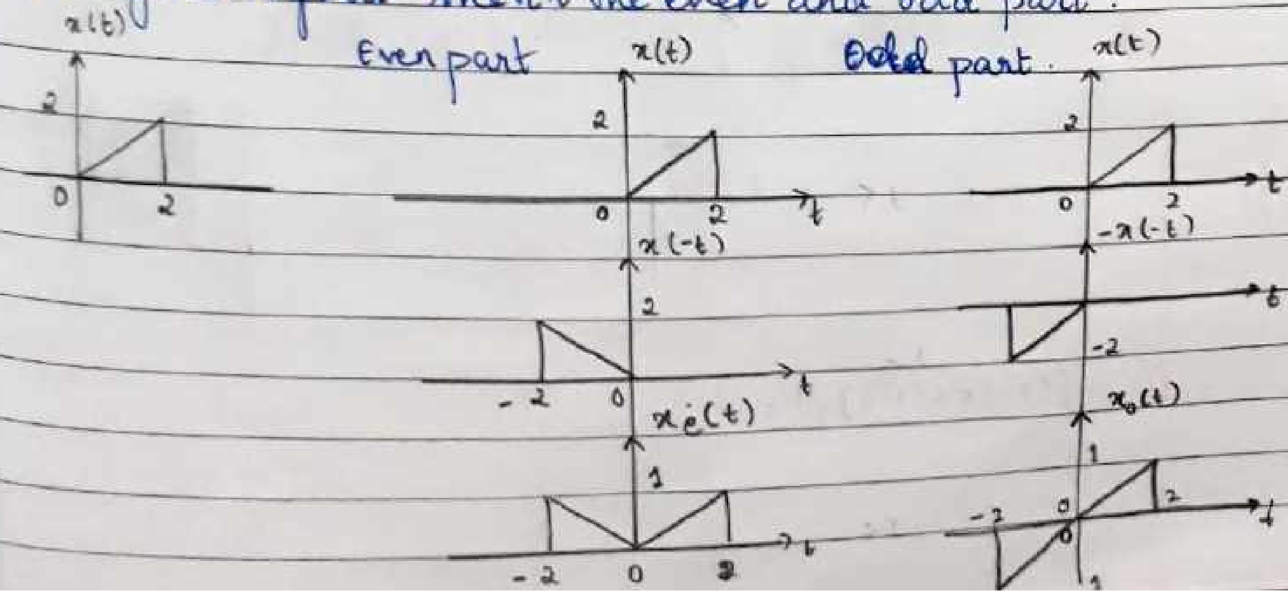
$$x_e(t) = \frac{x(t) + x(-t)}{2}$$

Odd part

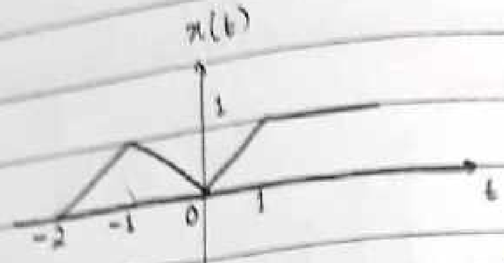
$$x_o(t) = \frac{x(t) - x(-t)}{2}$$

Q: For the given signal sketch the even and odd part:

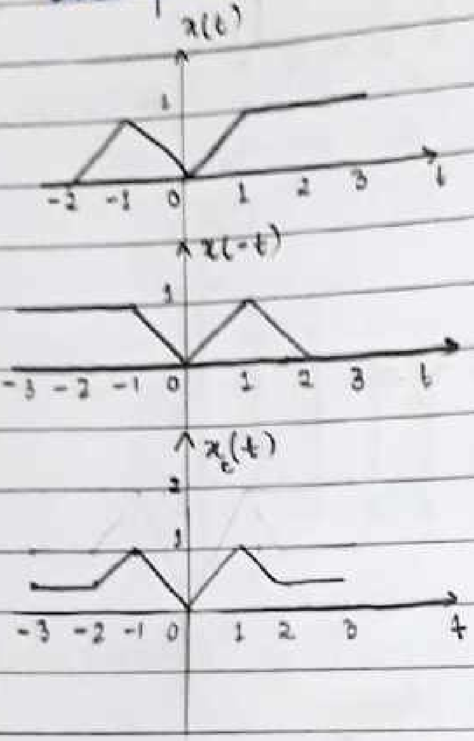
1.



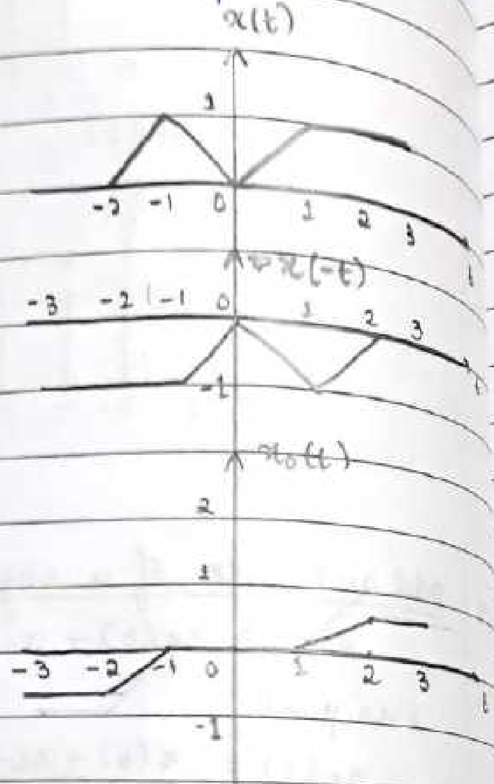
2.



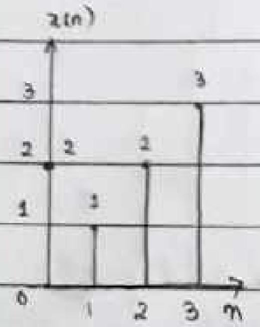
Even part.



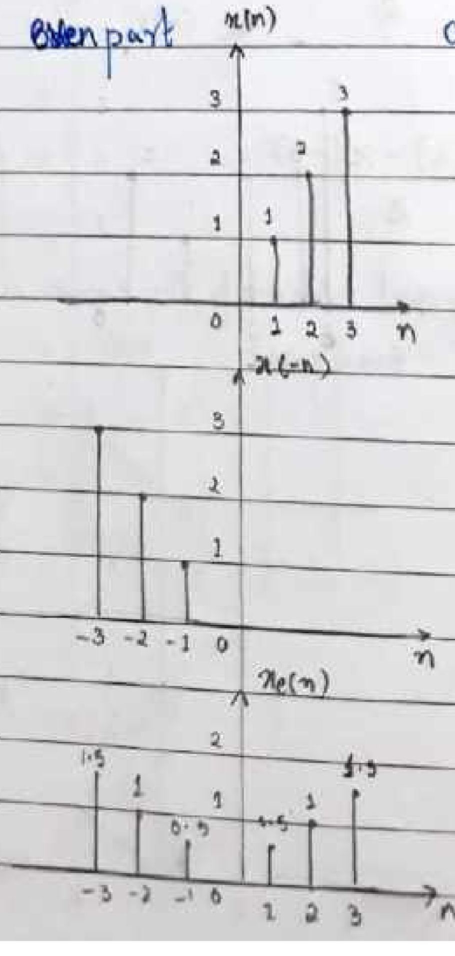
Odd part



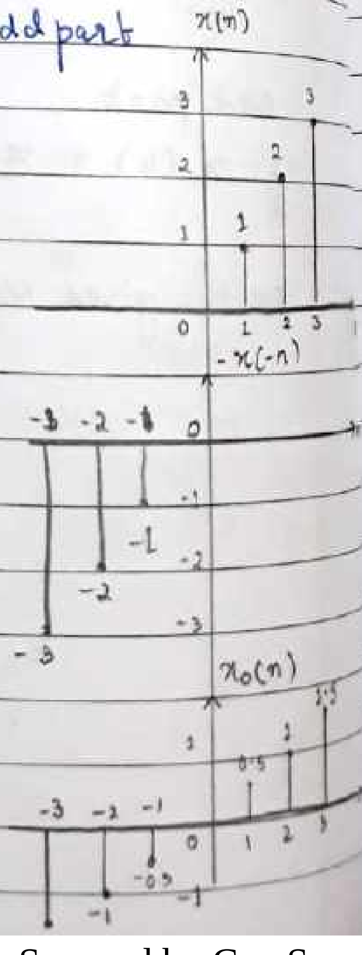
3.



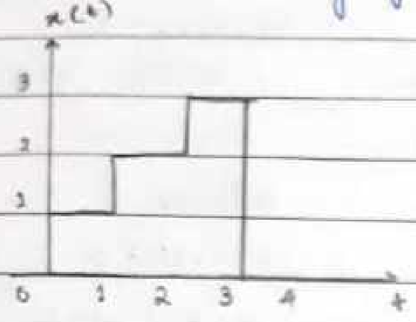
Even part

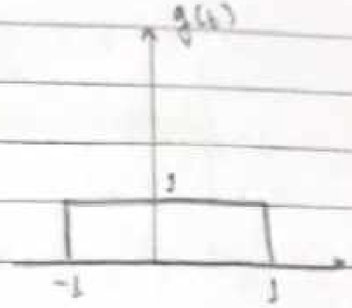


Odd part

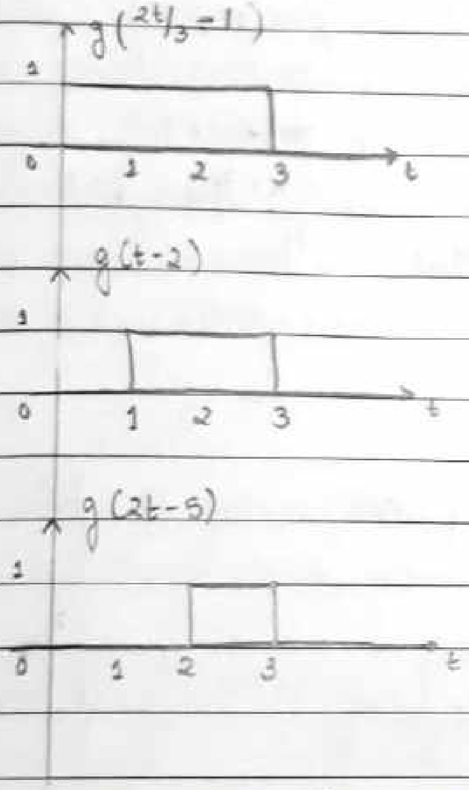


Q1: construct $x(t)$ using $g(t)$





Sol:



$mx + c = y$

$\rightarrow m(0) + 1 = -1$

$m(3) + 1 = 1$

$\therefore x = 2/3 \quad y = -1$

$\rightarrow m(1) + 1 = -1$

$m(3) + 1 = 1$

$\therefore x = 1 \quad y = -2$

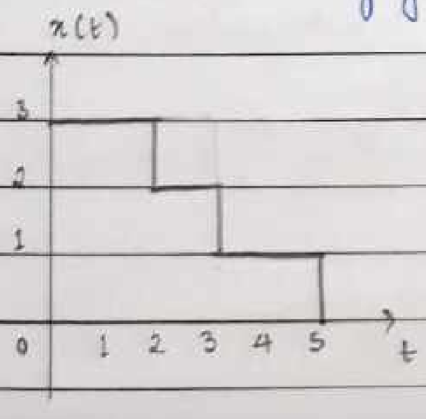
$\rightarrow m(2) + 1 = -1$

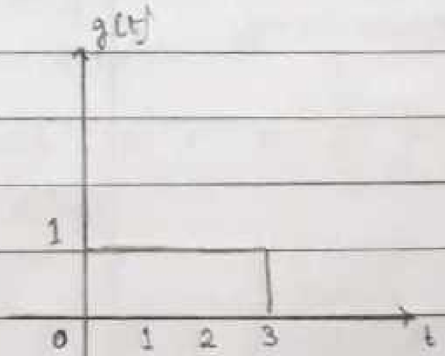
$m(3) + 1 = 1$

$\therefore x = 2 \quad y = -5$

$\therefore x(t) = g(2t/3 - 1) + g(t-2) + g(2t-5)$

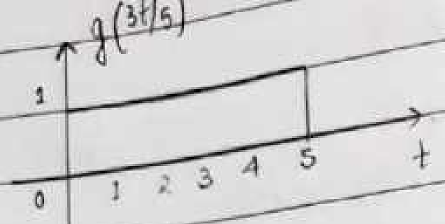
Q2: construct $x(t)$ using $g(t)$



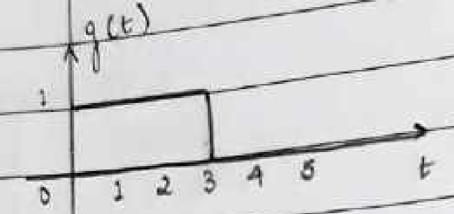


$$mx + c = y$$

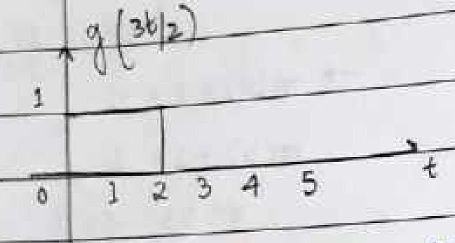
sol:



$$m(0)+1=0$$
$$m(5)+1=3$$
$$\therefore x = 3/5 \quad y=0$$



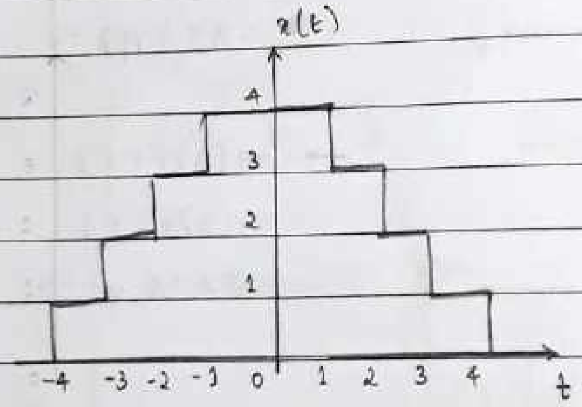
$$m(0)+1=0$$
$$m(3)+1=3$$
$$\therefore x=1 \quad y=0$$

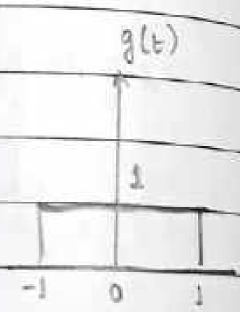


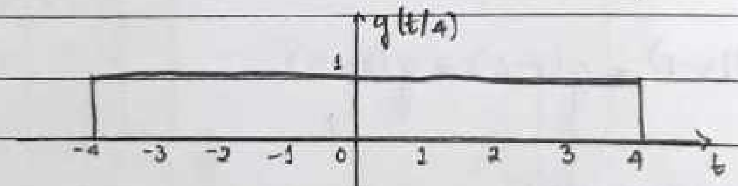
$$m(0)+1=0$$
$$m(2)+1=3$$
$$\therefore x=3/2 \quad y=0$$

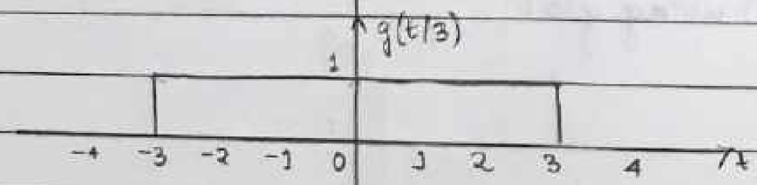
$$\therefore x(t) = g(3t/5) + g(t) + g(3t/2)$$

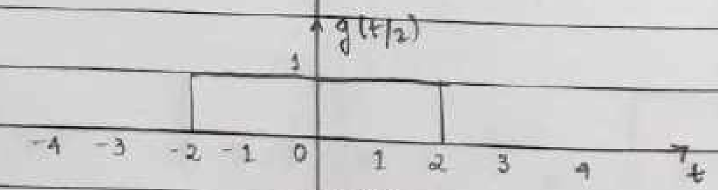
Q 3:

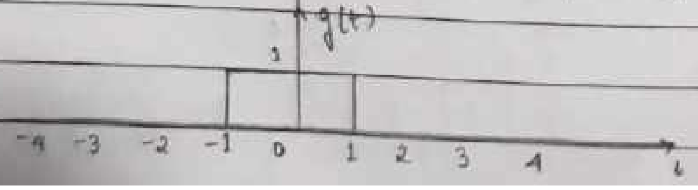




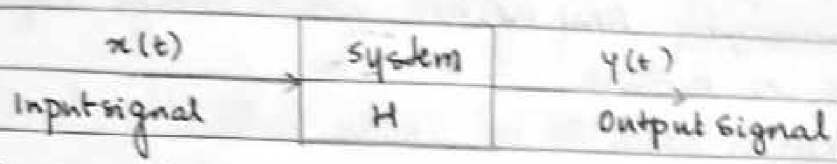








* **System:**



H or T: operator.

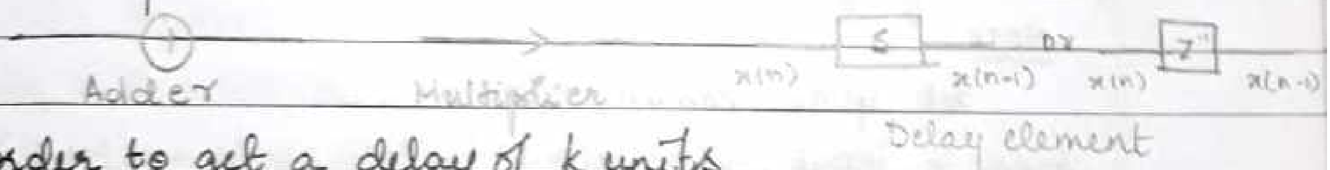
Output is obtained by performing operation on the input.

CTS: $y(t) = H\{x(t)\}$

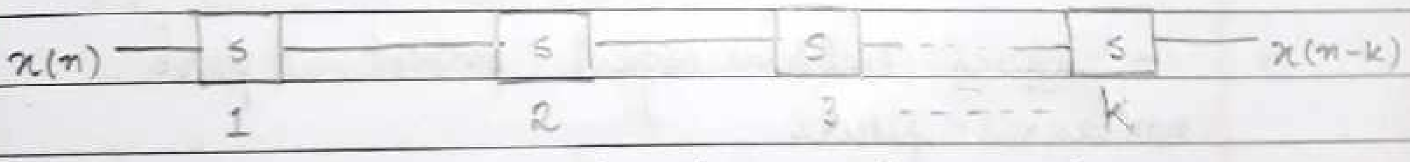
DTS: $y(n) = H\{x(n)\}$

For continuous time system given a CTS as input the output will be CTS similarly for Discrete Time System or Digital system given a DTS as input the output will be DTS.

A digital system usually consists of an adder, a multiplier and delay elements.

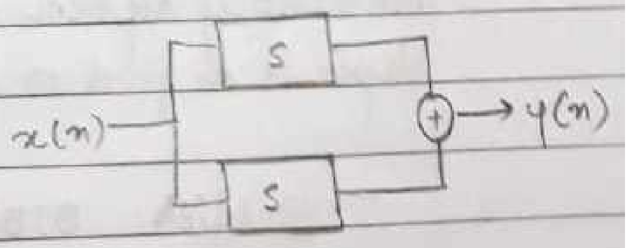


In order to get a delay of k units we can have k number of delay elements.



This can be done by different types of connections.

1. Cascade connection (end to end connection of subsystems)
2. Parallel connection (common input terminal for all subsystems)



* Properties of system:

1. – Linearity: Any system which obeys superposition is linear in nature.

A system is said to be linear when

$\underbrace{a x_1(n) + b x_2(n)}_{x_3(n)} = \underbrace{a y_1(n) + b y_2(n)}_{y_3(n)}$

Any modification in input should lead to same modification in the output.

2. – Time Invariant: If there is any delay in the input then the same delay is expected at the output. That is there should be same time shifting even after any processing of input.

NOTE:

let $y(n) = \sin 6n + x(n)$

change in input $x(n) - x(n-k)$; $y(n) = \sin 6n + x(n-k)$

change in output $y(n) - y(n-k)$; $y(n\ \ ) = \sin 6(n-k) + x(n-k)$

3. – Causal: Real time signal: present and past

noncausal: future.

4. – Memory: When the output is dependent on past or future or both then it is called memory system.

It can be dynamic and static.

nonmemory system or memoryless system where the output does depends on past or future.

5. – Stable: BIBO: Bounded Input Bounded Output.

i.e., a finite input gives a finite output.

Ex: $y(n) = \frac{1}{n}$; $n = 0$ $\quad y(n) = \infty$ $\quad$ Hence not stable

input: finite $\quad$ output: infinite

6. - Invertible :

$$x(t) \rightarrow \boxed{H} \xrightarrow{y(t)} \boxed{H^{-1}} \rightarrow f(t) = x(t)$$

$$y(t) = H\{x(t)\}$$
$$f(t) = H^{-1}\{y(t)\}$$
$$= H^{-1}\{H\{x(t)\}\}$$
$$\underline{\underline{f(t) = x(t)}}$$

Q. check whether the following systems are, linear, time invariant, causal, memory and stable.

1. $y(t) = x(t/2)$

sol: Linear: here $y_1(t) = x_1(t/2)$

$y_2(t) = x_2(t/2)$

$y_3(t) = x_3(t/2)$

Let $x_3(t) = ax_1(t) + bx_2(t)$

$y_3(t) = ax_1(t/2) + bx_2(t/2) \quad (\because y_3(t) = x_3(t/2))$

$\therefore y_3(t) = ay_1(t) + by_2(t)$

Hence $y_3(t) = x_3(t)$

Therefore the system is linear.

Time invariant

change in input: $x(t) \to x(t-t_0)$; $y(t) = x(t/2 - t_0)$

Change in output: $y(t) \to y(t-t_0)$; $y(t) = x\left(\frac{t-t_0}{2}\right)$

Therefore the system is time variant

Causal

$t = -1 \quad y(-1) = x(-1/2)$: future

$t = 0 \quad y(0) = x(0)$: present

$t = +1 \quad y(1) = x(1/2)$: past

Therefore the system is non causal.

Memory

$t=-1$ $y(-1) = x(-1/2)$: future

$t=0$ $y(0) = x(0)$: present

$t=1$ $y(1) = x(1/2)$: past

$\therefore$ Therefore the system is a memory system.

Stability

$x(t) \leq N_x < \infty$

$y(t) \leq N_y < \infty$

Therefore it is a stable system.

2. $y(t) = \sin 6t + x(t)$

Sol: linear $y_1(t) = \sin 6t + x_1(t)$

$y_2(t) = \sin 6t + x_2(t)$

$y_3(t) = \sin 6t + x_3(t)$

let $x_3(t) = a x_1(t) + b x_2(t)$

$y_3(t) = a x_1(t) + b x_2(t) + \sin 6t$

$\therefore y_3(t) \neq a y_1(t) + b y_2(t)$

Hence it is a nonlinear system.

Time invarient

Change in input $x(t) \to x(t-t_0)$; $y(t) = \sin 6t + x(t-t_0)$

Change in output $y(t) \to y(t-t_0)$; $y(t) = \sin 6(t-t_0) + x(t-t_0)$

Hence it is a time varient system.

Causal

$t=-1$ $y(-1) = \sin 6t + x(-1)$: Present

$t=0$ $y(0) = \sin 6t + x(0)$: Present

$t=1$ $y(1) = \sin 6t + x(1)$: Present

Therefore it is a causal system.

Memory

$t=-1$ $y(-1) = \sin 6t + x(-1)$: Present

$t=0$ $y(0) = \sin 6t + x(0)$: Present

$t=1$ $y(1) = \sin 6t + x(1)$: Present

Therefore the system is a memoryless system.

Stability

$x(t) \leq M_x < \infty$

$y(t) \leq M_y < \infty$

Therefore it is a stable system.

3 $y(t) = \sqrt{x(t)}$

Sol: Linearity

$y_1(t) = \sqrt{x_1(t)}$

$y_2(t) = \sqrt{x_2(t)}$

$y_3(t) = \sqrt{x_3(t)}$

Let $x_3(t) = a x_1(t) + b x_2(t)$

$y_3(t) = \sqrt{a x_1(t) + b x_2(t)}$

$\therefore y_3(t) \neq a y_1(t) + b y_2(t)$

Hence it is a non linear system.

Time invarient

change in input $x(t) \to x(t-t_0)$; $y(t) = \sqrt{x(t-t_0)}$

change in output $y(t) \to y(t-t_0)$; $y(t) = \sqrt{x(t-t_0)}$

Therefore it is a time invarient system.

Causal

$t=-1$ $\quad y(-1) = \sqrt{x(-1)}$: Present

$t=0$ $\quad y(0) = \sqrt{x(0)}$: Present

$t=+1$ $\quad y(1) = \sqrt{x(1)}$: Present

Therefore it is a causal system.

Memory

$t=-1$ $\quad y(-1) = \sqrt{x(-1)}$: Present

$t=0$ $\quad y(0) = \sqrt{x(0)}$: Present

$t=1$ $\quad y(1) = \sqrt{x(1)}$: Present

Therefore it is a memoryless system (static memory)

Stability

$x(t) \leq M_x < \infty$ Therefore it is a stable system.

$y(t) \leq M_y < \infty$

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4. $y(n) = n\,x(n+2)$

Sol: Linearity

$y_1(n) = n x_1(n+2)$

$y_2(n) = n x_2(n+2)$

$y_3(n) = n x_3(n+2)$

Let $x_3(n) = a x_1(n) + b x_2(n)$

$y_3(n) = n[a x_1(n+2) + b x_2(n+2)]$

$= n a x_1(n+2) + n b x_2(n+2)$

$y_3(n) = a y_1(n) + b y_2(n)$

Therefore it is a linear system

Time invariant

change in input $x(n) \to x(n-n_0)$; $y(n) = n x_1(n+2-n_0)$

change in output $y(n) \to y(n-n_0)$; $y(n) = (n-n_0) x_1(n-n_0+2)$

Therefore it is a time variant system.

Causal

$n=-1$ $y(-1) = n x(1)$: future

$n=0$ $y(0) = n x(2)$: future

$n=1$ $y(1) = n x(3)$: future

Therefore it is a causal system.

Memory

$n=-1$ $y(-1) = n x(1)$: future

$n=0$ $y(0) = n x(2)$: future

$n=1$ $y(1) = n x(3)$: future

Therefore it is a memory system (dynamic memory)

Stability

$x(n) \le M_x < \infty$

$y(n) \le M_y < \infty$

It does not obey BIBO Hence it is an unstable system

5. $y(n) = \sum_{k=-\infty}^{n} x(k+2)$

Sol: Linearity

$y_1(n) = \sum_{k=-\infty}^{n} x_1(k+2)$

$y_2(n) = \sum_{k=-\infty}^{n} x_2(k+2)$

$y_3(n) = \sum_{k=-\infty}^{n} x_3(k+2)$

Let $x_3(n) = ax_1(n) + bx_2(n)$

$y_3(n) = \sum_{k=-\infty}^{n} [ax_1(k+2) + bx_2(k+2)]$

$y_3(n) = a\sum_{k=-\infty}^{n} x_1(k+2) + b\sum_{k=-\infty}^{n} x_2(k+2)$

$y_3(n) = ay_1(n) + by_2(n)$

Therefore it is a linear system

Time invarient

change in input $x(n) \to x(n-n_0)$; $y(n) = \sum_{k=-\infty}^{n} x_1(k+2-n_0)$

Change in output $y(n) \to y(n-n_0)$; $y(n) = \sum_{k=-\infty}^{n-n_0} x(k+2)$

Therefore the system is time varient system.

Causal

$n = -k$ $\quad y(-k) = \sum_{k=-\infty}^{-k} x(k+2)$: future

$n = 0$ $\quad y(0) = \sum_{k=-\infty}^{0} x(k+2)$: future

$n = k$ $\quad y(k) = \sum_{k=-\infty}^{k} x(k+2)$: future

Therefore the system is noncausal system

Memory

$n = -k \quad y(-k) = \sum_{k=-\infty}^{-k} x(k+2)$: future

$n = 0 \quad y(0) = \sum_{k=-\infty}^{0} x(k+2)$: future

$n = k \quad y(k) = \sum_{k=-\infty}^{k} x(k+2)$: future

Therefore it is a memory system.

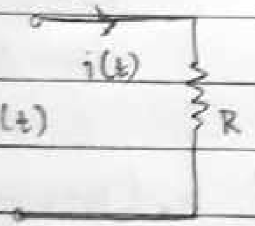
Stability

$x(t) \le M_x < \infty$

$y(t) \le M_y < \infty$

Therefore it is a stable system.

★ **Energy and Power Signal:**



$P = v(t)\, i(t)$

$P = i^2(t) \cdot R \quad \text{or} \quad P = v^2(t)/R$

If $R = 1\Omega$

$P = i^2(t) \quad \text{or} \quad P = v^2(t)$

(Hence $P = x^2(t)$)

Power is rate of work done

wkt $P = W/T$

$\therefore$ Energy: $E = W = PT$

$$\therefore E = \int_{-\infty}^{\infty} |x(t)|^2\, dt$$

Average power

$$P_{av} = \lim_{T \to \infty} \frac{1}{T} \int_{-T/2}^{T/2} |x(t)|^2\, dt$$

Similarly of DTS: we have

$$E = \sum_{n=-\infty}^{\infty} |x(n)|^2$$

Given signal

$$P = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} |x(n)|^2$$

E signal ⇒ $E = F$; $P = \infty$

P signal ⇒ $E = \infty$; $P = F$

$$P = \frac{1}{N} \sum_{n=0}^{N-1} |x(n)|^2$$

Periodic signal for wide range then it is a power signal

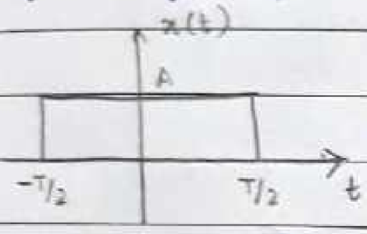
If $E = \infty$ and $P = \infty$, then the signal is neither power nor energy

Finite range : Energy

Wide range : Power $(-\infty$ to $\infty)$

Q1. For a given signal check whether it is energy or power signal, if so find the same.

1.



~~Energy signal (because of finite range)~~

sol:

$$x(t) = \begin{cases} A \; ; & -T/2 \le t \le T/2 \\ 0 \; ; & \text{else} \end{cases}$$

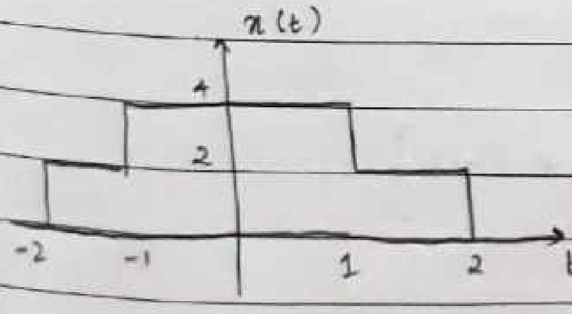
$$E = \int_{-\infty}^{\infty} |x(t)|^2 \, dt$$

$$E = \int_{-T/2}^{T/2} A^2 dt = At \Big|_{-T/2}^{T/2}$$

$$E = A^2 \left[\frac{T}{2} + \frac{T}{2} \right]$$

$\therefore$ $\underline{E = A^2 T \;\; J}$ $\therefore$ It is an energy signal.

2.



sol:

$$x(t) = \begin{cases} 2 \; ; & -2 \le t \le -1 \\ 4 \; ; & -1 \le t \le 1 \\ 2 \; ; & 1 \le t \le 2 \\ 0 \; ; & \text{else} \end{cases}$$

$$E = \int_{-\infty}^{\infty} |x(t)|^2 \, dt$$

$$E = \int_{-2}^{-1} 2^2 dt + \int_{-1}^{1} 4^2 dt + \int_{1}^{2} 2^2 dt$$

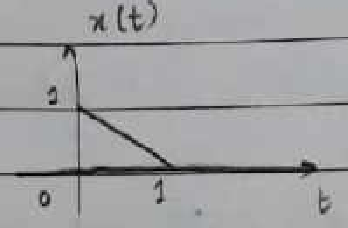
$$E = 4t \Big|_{-2}^{-1} + 16t \Big|_{-1}^{1} + 4t \Big|_{1}^{2}$$

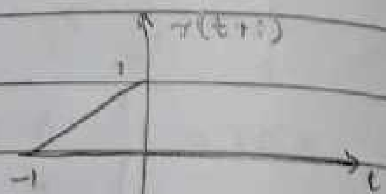
$$E = 4[-1+2] + 16[1+1] + 4[2-1]$$

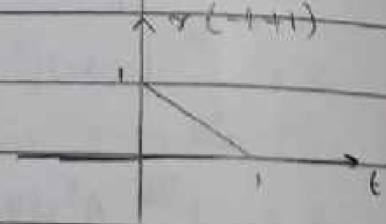
$$E = 4 + 32 + 4$$

$$\underline{E = 40J}$$ $\therefore$ It is an energy signal.

3.







<u>Sol:</u> $x(t) = \begin{cases} -t+1 ; & 0 \le t \le 1 \\ 0 ; & \text{else} \end{cases}$

$$E = \int_{-\infty}^{\infty} |x(t)|^2 dt$$

$$E = \int_0^1 (-t+1)^2 dt \; \cancel{= \frac{(-t+1)^3}{3} \Big|_0^1}$$

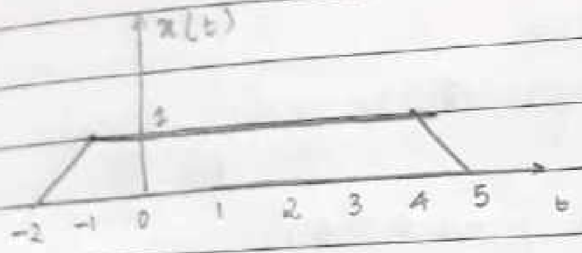
$$\cancel{E = \frac{1}{3} = 0.33J}$$ $\therefore$ It is an energy signal.

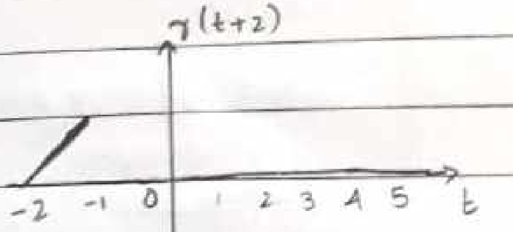
$$E = \int_0^1 (t^2 + 1 - 2t) \, dt$$

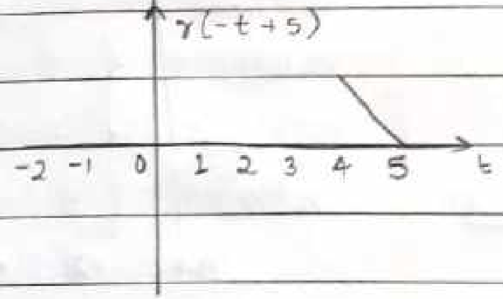
$$E = \left[\frac{t^3}{3} + t - t^2\right]_0^1 = \left[\frac{1}{3} + 1 - 1\right]$$

$$\underline{E = 0.33J}$$

4.







sol:
$$x(t) = \begin{cases} t+2 & ; -2 \le t \le -1 \\ 1 & ; -1 \le t \le 4 \\ -t+5 & ; 4 \le t \le 5 \\ 0 & ; \text{else} \end{cases}$$

$$E = \int_{-\infty}^{\infty} |x(t)|^2 dt$$

$$E = \int_{-2}^{-1} (t+2)^2 dt + \int_{-1}^{4} 1\, dt + \int_{4}^{5} (-t+5)^2 dt$$

$$E = \int_{-2}^{-1} (t^2+4-4t) dt + \int_{-1}^{4} dt + \int_{4}^{5} (t^2+25-10t)\, dt$$

$$E = \left[\frac{t^3}{3} + 4t - 2t^2\right|_{-2}^{-1} + t\Big|_{-1}^{4} + \left[\frac{t^3}{3} + 25t - 5t^2\right|_{4}^{5}$$

$$E = \left[\frac{-1}{3} - 4 + 2 + \frac{8}{3} + 8 + 8 + 4 - 1 + \frac{125}{3} + \cancel{125} - \cancel{125} - \frac{64}{3} - 100 + 80\right]$$

$$\underline{E = 5.67\text{J}}$$

6. For the given function is power or energy signal, if so find the same

1. $x(n) = \{1, 2, -3, \underset{\uparrow}{2}, 4\}$

sol:
$$E = \sum_{n=-\infty}^{\infty} |x(n)|^2$$

$$E = \sum_{n=-3}^{1} |x(n)|^2$$

$$E = |x(-3)|^2 + |x(-2)|^2 + |x(-1)|^2 + |x(0)|^2 + |x(1)|^2$$

$E = 1 + 4 + 9 + 4 + 16$

$E = 34J$ therefore it is an energy signal.

2. $x(n) = \begin{cases} 5\cos(n\pi) & ; -2 \le n \le 1 \\ 0 & ; \text{else} \end{cases}$

Sol: $x(n) = \{5, -5, 5, -5\}$

$n \rightarrow \quad -2 \quad -1 \quad 0 \quad 1$

$E = \sum_{n=-\infty}^{\infty} |x(n)|^2$

$E = \sum_{n=-2}^{1} |x(n)|^2$

$E = |x(-2)|^2 + |x(-1)|^2 + |x(0)|^2 + |x(1)|^2$

$E = 25 + 25 + 25 + 25 = 100J$ ∴ It is an energy signal.

3. $x(n) = (0.5)^n u(n)$

Sol: $E = \sum_{n=-\infty}^{\infty} |x(n)|^2$

$E = \sum_{n=-\infty}^{\infty} |(0.5)^n u(n)|^2$

$u(n) = \begin{cases} 1 & ; 0 \le n \le \infty \\ 0 & ; \text{Else} \end{cases}$

$(0.5)^n : -\infty$ to ∞

$E = \sum_{n=0}^{\infty} |(0.5)^{2n}|$

$E = \sum_{n=0}^{\infty} (0.25)^n$

$E = \frac{1}{1 - 0.25}$

$\sum_{n=k}^{\infty} a^k = \frac{a^k}{1-a} \; ; \; a < 1$

$E = 1.33J$ ∴ It is an energy signal.

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4. $x(n) = 8\left(\frac{1}{2}\right)^n u(n-1)$

sol: $E = \sum_{n=-\infty}^{\infty} |x(n)|^2$

$E = \sum_{n=-\infty}^{\infty} \left|8\left(\frac{1}{2}\right)^n u(n-1)\right|^2$

$E = \sum_{n=1}^{\infty} \left|8\left(\frac{1}{2}\right)^n\right|^2$

$E = \sum_{n=1}^{\infty} \left|64\left(\frac{1}{2}\right)^{2n}\right|$

$E = 64\sum_{n=1}^{\infty}\left[\frac{1}{4}\right]^n$

$E = 64\left[\frac{1}{1-1/4}\right]$

$E = 64\,(1.33)$

$\underline{E = 85.12\,J}$ ∴ It is an energy signal.

5. $x(t) = e^{2t}\,u(-t)$

sol: $E = \int_{-\infty}^{\infty} |x(t)|^2\,dt$

$E = \int_{-\infty}^{\infty} |e^{2t}u(-t)|^2\,dt$

$E = \int_{-\infty}^{0} |e^{4t}|\,dt$

$E = \left.\frac{e^{4t}}{4}\right|_{-\infty}^{0}$

$E = \frac{1}{4}$

$\underline{E = 0.25\,J}$ ∴ It is an energy signal

IMP

6. $x(n) = 2^n u(-n-1)$

$u(n-1) \Rightarrow 1$ to ...
$u(-n-1) \Rightarrow -\infty$ to ...

Sol: $E = \sum_{n=-\infty}^{\infty} |x(n)|^2$

$E = \sum_{n=-\infty}^{\infty} |2^n u(-n-1)|^2$

$E = \sum_{n=-\infty}^{-1} |2^{2n}| = \sum_{n=-\infty}^{-1} 4^n$

$E = \frac{1}{4} + \frac{1}{4^2} + \frac{1}{4^3} + \cdots\cdots$

$E = \frac{1/4}{1-1/4} = \frac{1/4}{3/4} = \frac{1}{3} = 0.33\,J$ $\therefore$ It is an energy signal.

7. $x(t) = e^{-at}u(t)$

Sol: $E = \int_{-\infty}^{\infty} |x(t)|^2 dt$

$E = \int_{0}^{\infty} |e^{-at}|^2 dt$

$E = \int_{0}^{\infty} e^{-2at} dt$

$E = \left.\frac{e^{-2at}}{-2a}\right|_0^{\infty}$

$E = \frac{[0-1]}{-2a} = \frac{1}{2a}\,J$ $\therefore$ It is an energy signal

8. $x(n) = \cos(n\pi/2)$

Since the given function is periodic we consider the power eq

Sol: $P = \frac{1}{N}\sum_{n=0}^{N-1} |x(n)|^2$

$\Omega = \frac{\pi}{2} \Rightarrow 2\pi\left(\frac{m}{N}\right) = 2\pi\left(\frac{1}{4}\right)$ $\therefore N = 4$ samples

$$\therefore P = \frac{1}{4} \sum_{n=0}^{3} |\cos^2(n\pi/2)|$$

$$P = \frac{1}{4}\left[\cos^2(0) + \cancel{\cos^2(\pi/2)} + \cos^2(\pi) + \cancel{\cos^2(3\pi/2)}\right]$$

$$P = \frac{1}{4}(2) = \frac{1}{2}$$

$P = 0.5\,W$ $\therefore$ It is a power signal

9. $$x(n) = e^{j(4\pi n/3)}$$

$$x(n) = \cos\frac{4\pi n}{3} + j\sin\frac{4\pi n}{3}$$

$$\Omega = \frac{4\pi}{3} \Rightarrow 2\pi\left(\frac{m}{N}\right) = 2\pi\left(\frac{2}{3}\right)$$

$\therefore$ N = 3 samples

$$P = \frac{1}{N} \sum_{n=0}^{N-1} |x(n)|^2$$

$$P = \frac{1}{3} \sum_{n=0}^{2} \left|e^{j(4\pi n/3)}\right|^2$$

$$\left|e^{j(4\pi n/3)}\right| = \sqrt{\cos^2\frac{4\pi n}{3} + \sin^2\frac{4\pi n}{3}} = 1$$

$$P = \frac{1}{3} \sum_{n=0}^{2} 1^2 = \frac{1}{3}[2-0+1]$$

$P = \frac{1}{3} \times 3 = 0.33\,W$ $\therefore$ It is a power signal.

(or) $$P = \frac{1}{3}\left[|e^0| + |e^{8\pi j/3}| + |e^{j16\pi/3}|\right]$$

$$\sum_{n=n_1}^{n_2} 1 = n_2 - n_1 + 1$$

10. $$x(t) = A\cos(\omega t)$$

sol: $$P = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} |x(t)|^2\,dt$$

$$\cancel{\omega = \frac{2\pi}{T}} \Rightarrow \cancel{T = 2\pi}$$

$$P = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} A^2\cos^2\omega t \, dt$$

$$P = \lim_{T\to\infty} \frac{1}{T} A^2 \int_{-T/2}^{T/2} \frac{\cos 2\omega t + 1}{2} \, dt$$

$$P = \lim_{T\to\infty} \frac{A^2}{T} \frac{1}{2} \left[\frac{\sin 2\omega t}{2\omega} + t \right]_{-T/2}^{T/2}$$

$$P = \lim_{T\to\infty} \frac{A^2}{2\cancel{T}} \left[0 + \frac{\cancel{T}}{\cancel{2}} - 0 + \frac{\cancel{T}}{\cancel{2}} \right]$$

$P = \dfrac{A^2}{2}$ W $\quad\therefore$ It is a power signal.

Unit - 2

TIME - DOMAIN REPRESENTATION FOR LTI SYSTEMS

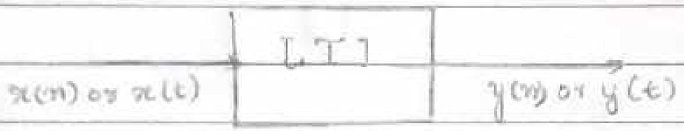
★ <u>Frequency domain:</u>

Spectral Analysis : ECG

It gives more information when compared to time domain.

★ <u>Time domain:</u>

LTI systems: linear time invariant systems.



1. $x(n) = \delta(n)$ $\qquad$ $y(n)$ = Impulse response = $h(n)$

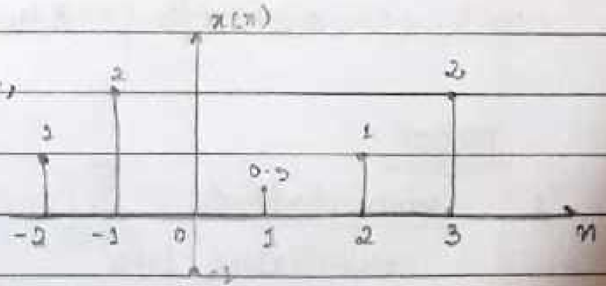
   $x(t) = \delta(t)$ $\qquad$ $y(t)$ = Impulse response = $h(t)$

2. $x(n) = u(n)$ $\qquad$ $y(n)$ = step response

   $x(t) = u(t)$ $\qquad$ $y(t)$ = step response

★ <u>Convolution Sum:</u>

The response of a system when the input is an impulse function, is defined as the impulse response or unit sample response of the system.



$x(n) = \{1, 2, -1, 0.5, 1, 2\}$

$x(n) = 1\,\delta(n+2) + 2\,\delta(n+1) + (-1)\,\delta(n) + 0.5\,\delta(n-1) + 1\,\delta(n-2) + 2\,\delta(n-3)$

$x(n) = x(-2)\,\delta(n+2) + x(-1)\,\delta(n+1) + x(0)\,\delta(n) + x(1)\,\delta(n-1) + x(2)\,\delta(n-2) + x(3)\,\delta(n-3)$

$\therefore\ x(n) = \sum_{k=-2}^{3} x(k)\,\delta(n-k)$

In general

$$\boxed{x(n) = \sum_{k=-\infty}^{\infty} x(k)\,\delta(n-k)}$$

Let $x(n)$ be input and $y(n)$ be output for LTI system.

$$x(n) \longrightarrow \boxed{\text{LTI}} \longrightarrow y(n)$$

$$y(n) = T\{x(n)\}$$

$$y(n) = T\left\{\sum_{k=-\infty}^{\infty} x(k)\,\delta(n-k)\right\}$$

$$y(n) = \sum_{k=-\infty}^{\infty} \underbrace{x(k)}_{\text{Amplitude}}\, T\{\delta(n-k)\}$$

$$y(n) = \sum_{k=-\infty}^{\infty} x(k)\,h(n-k)$$

$$\boxed{y(n) = x(n) * h(n)}$$

knowing past and present we can predict the future, this one of the application of convolution.

* **Properties of convolution sum:**

1. commutative

$$x(n) * h(n) = h(n) * x(n)$$

2. Distributive

$$x(n) * [h_1(n) + h_2(n)] = x(n) * h_1(n) + x(n) * h_2(n)$$

3. Associative

$$x(n) * [h_1(n) * h_2(n)] = [x(n) * h_1(n)] * h_2(n)$$

PROOF:

1. Commutative

considering LHS

$$x(n) * h(n) = \sum_{k=-\infty}^{\infty} x(k)\,h(n-k)$$

Let $n-k = r \Rightarrow k = n-r$

$$\therefore x(n) * h(n) = \sum_{r=-\infty}^{\infty} x(n-r)\,h(r)$$

$$= \sum_{r=-\infty}^{\infty} h(r)\,x(n-r)$$

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since r is just any variable

$$\therefore x(n) * h(n) = \sum_{k=-\infty}^{\infty} h(k)\, x(n-k)$$

$$\boxed{x(n) * h(n) = h(n) * x(n)}$$

2. Distributive

considering LHS

$$x(n) * [h_1(n) + h_2(n)] = \sum_{k=-\infty}^{\infty} x(k)\left[h_1(n-k) + h_2(n-k)\right]$$

$$= \sum_{k=-\infty}^{\infty} x(k) h_1(n-k) + \sum_{k=-\infty}^{\infty} x(k)\, h_2(n-k)$$

$$\boxed{x(n) * [h_1(n) + h_2(n)] = x(n) * h_1(n) + x(n) * h_2(n)}$$

3. Associative

considering RHS

$$[x(n) * h_1(n)] * h_2(n) = \sum_{k=-\infty}^{\infty} [x(k) * h_1(k)]\, h_2(n-k)$$

$$= \sum_{k=-\infty}^{\infty}\left[\sum_{m=-\infty}^{\infty} x(m)\, h_1(k-m)\right] h_2(n-k)$$

changing the order of summation, we get

$$[x(n) * h_1(n)] * h_2(n) = \sum_{m=-\infty}^{\infty} x(m)\left[\sum_{k=-\infty}^{\infty} h_1(k-m)\, h_2(n-k)\right]$$

let $r = k - m$

$$\therefore [x(n) * h_1(n)] * h_2(n) = \sum_{m=-\infty}^{\infty} x(m)\left[\sum_{r=-\infty}^{\infty} h_1(r)\, h_2(n-m-r)\right]$$

$$= \sum_{m=-\infty}^{\infty} x(m)\left[h_1(n-m) * h_2(n-m)\right]$$

$$\cancel{= \sum_{k=-\infty}^{\infty} x(k)\left[h_1(n-k) * h_2(n-k)\right]}$$

$$\boxed{[x(n) * h_1(n)] * h_2(n) = x(n) * [h_1(n) * h_2(n)]}$$

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properties:

Q: Prove the following:

1. $x(n) * \delta(n) = x(n)$

sol: considering LHS

$$x(n) * \delta(n) = \sum_{k=-\infty}^{\infty} x(k)\,\delta(n-k) \qquad \delta(n-k) = \begin{cases} 1 \,; & \text{at } n=k \\ 0 \,; & \text{else} \end{cases}$$

$$= x(k).1 \Big|_{n=k}$$

$$\therefore x(n) * \delta(n) = x(n)$$

2. $x(n) * \delta(n-n_0) = x(n-n_0)$

sol: considering LHS

$$x(n) * \delta(n-n_0) = \sum_{k=-\infty}^{\infty} x(k)\,\delta(n-n_0-k)$$

$$= x(k).1 \Big|_{k=n-n_0}$$

$$\therefore x(n) * \delta(n-n_0) = x(n-n_0)$$

3. $x(n) * u(n) = \sum_{k=-\infty}^{\infty} x(k)$

sol:

considering LHS

$$x(n) * u(n) = \sum_{k=-\infty}^{\infty} x(k)\,u(n-k)$$

$$= \sum_{k=-\infty}^{\infty} x(k)$$

$$\therefore x(n) * u(n) = \sum_{k=-\infty}^{\infty} x(k)$$

4. $\delta(n-m) * \delta(n-k) = \delta(n-(m+k))$

Ex: A LTI system characterised by impulse response $h(n)$

$$h(n) = \{1, \underset{\uparrow}{2}, 3, 4\}$$

and input $x(n) = \{1, \underset{\uparrow}{5}, 2\}$. find the convolution between the two.

Sol: $h(n) = \{1, \underset{\uparrow}{2}, 3, 4\}$: -1 to 2 : $L_1 = 4$

$n \rightarrow \;\; -1 \;\; 0 \;\; 1 \;\; 2$

$x(n) = \{1, \underset{\uparrow}{5}, 2\}$: -1 to 1 : $L_2 = 3$

$n \rightarrow \;\; -1 \;\; 0 \;\; 1$

$y(n) = -2$ to 3 : $L = L_1 + L_2 - 1$

$L = 4 + 3 - 1 = 6.$

$$y(n) = x(n) * h(n)$$

$$= [1\delta(n+1) + 5\delta(n) + 2\delta(n-1)] * [1\delta(n+1) + 2\delta(n) + 3\delta(n-1) + 4\delta(n-2)]$$

$$= 1\delta(n+2) + 2\delta(n+1) + 3\delta(n) + 4\delta(n-1)$$

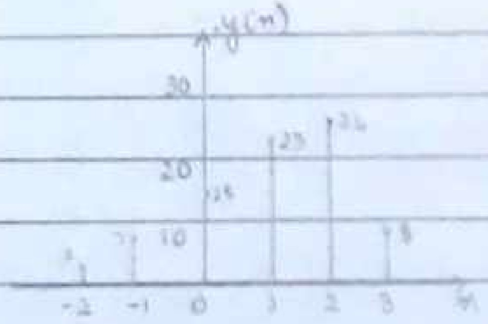
$$+ 5\delta(n+1) + 10\delta(n) + 15\delta(n-1) + 20\delta(n-2)$$

$$+ 2\delta(n) + 4\delta(n-1) + 6\delta(n-2) + 8\delta(n-3)$$

$$= 1\delta(n+2) + 7\delta(n+1) + 15\delta(n) + 23\delta(n-1) + 26\delta(n-2) + 8\delta(n-3)$$

$$\therefore y(n) = \{1, 7, \underset{\uparrow}{15}, 23, 26, 8\}$$

$n \rightarrow \;\; -2 \;\; -1 \;\; 0 \;\; 1 \;\; 2 \;\; 3$



verification

	1	2	3	4
1	1	2	3	4
5	5	10	15	20
2	2	4	6	8

Graphical method

A LTI system characterised by impulse function
$$h(n) = \{1, \underset{\uparrow}{2}, 3, 4\}$$

with input $x(n) = \{1, \underset{\uparrow}{5}, 2\}$

find the convolution using graphical method.

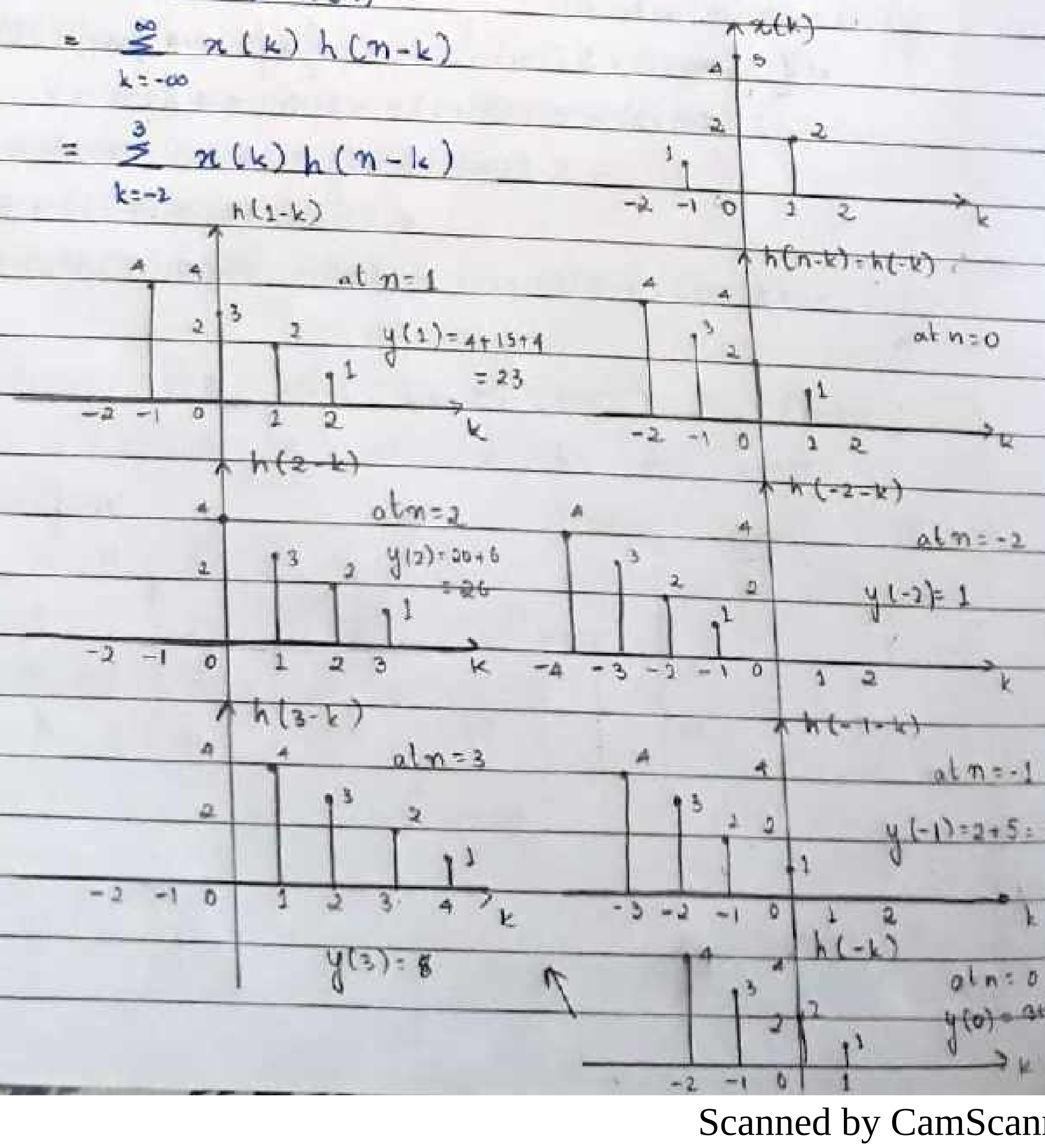
Sol:

$h(n) = \{1, 2, 3, 4\} \rightarrow -1$ to 2 : $L_1 = 4$
$\quad\quad -1 \;\; 0 \;\; 1 \;\; 2$

$x(n) = \{1, 5, 2\} \rightarrow -1$ to 1 : $L_2 = 3$
$\quad\quad -1 \;\; 0 \;\; 1$

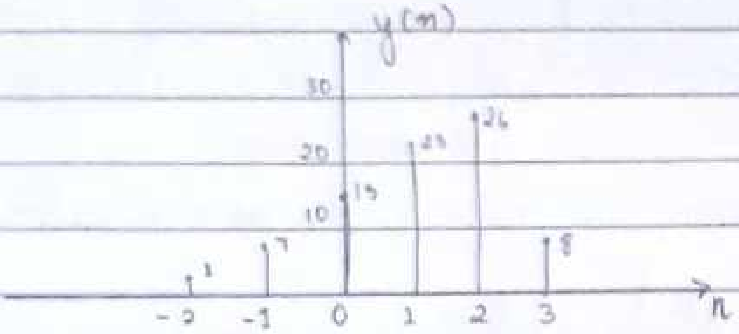
$y(n) = -2$ to 3 : $L = L_1 + L_2 - 1$
$\underline{L = 6}$

$$y(n) = x(n) * h(n)$$
$$= \sum_{k=-\infty}^{\infty} x(k)\, h(n-k)$$
$$= \sum_{k=-2}^{3} x(k)\, h(n-k)$$



$y(n) = \{1, 7, 15, 23, 26, 8\}$

$n \rightarrow$ -2 -1 0 1 2 3



Q1: $x(n) = \{3, 5, -2, 4\}$
↑

$h(n) = \{3, 1, 2\}$
↑

Perform convolution using graphical method.

Sol: $x(n) = \{3, 5, -2, 4\} \rightarrow -2$ to 1 : $L_1 = 4$

$n \rightarrow$ -2 -1 0 1

$h(n) = \{3, 1, 2\} \rightarrow 0$ to 2 : $L_2 = 3$

$n \rightarrow$ 0 1 2

$y(n)$: ~~-4 to 1~~ $L = 6$

-2 to 3

$$y(n) = x(n) * h(n) = \sum_{k=-\infty}^{\infty} x(k)h(n-k)$$

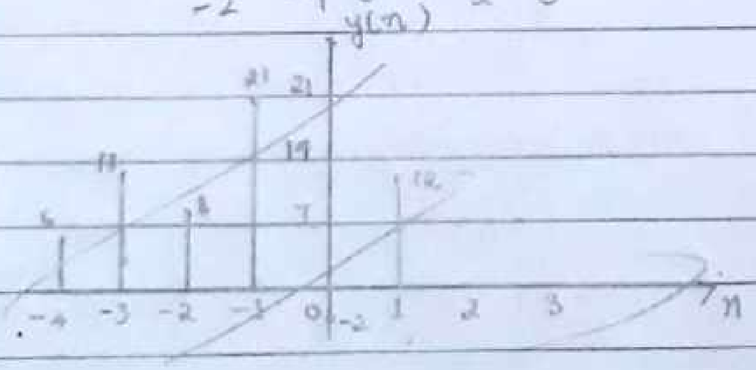
$$= \sum_{k=-4}^{1} x(k)h(n-k)$$

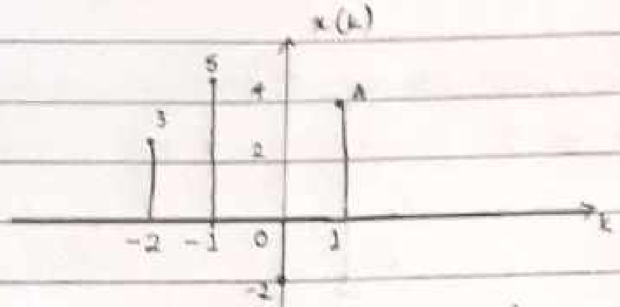
9 18 5 20 0 8

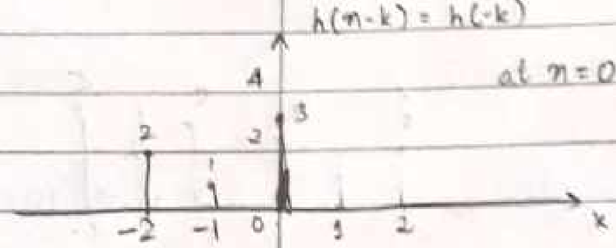
$y(n) = \{\cancel{6}, \cancel{13}, \cancel{8}, \cancel{21}, \cancel{-2}, \cancel{12}\}$

$n \rightarrow$ ~~-4 -3 -2 -1 0 1~~

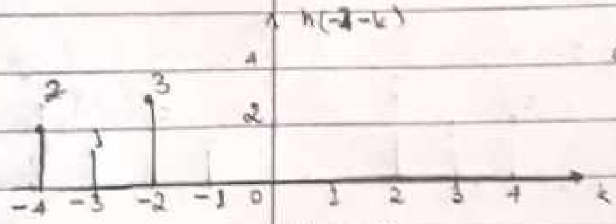
-2 -1 0 1 2 3





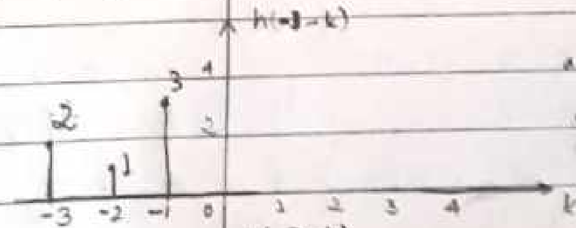


at $n=0$



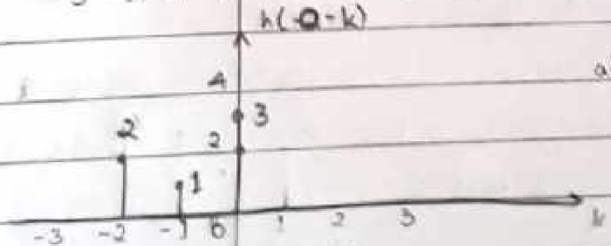
at $n=-2$

$y(-2) = 9$



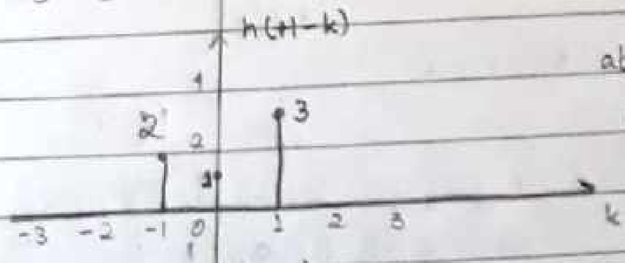
at $n=-1$

$y(-1) = 3+15 = 18$



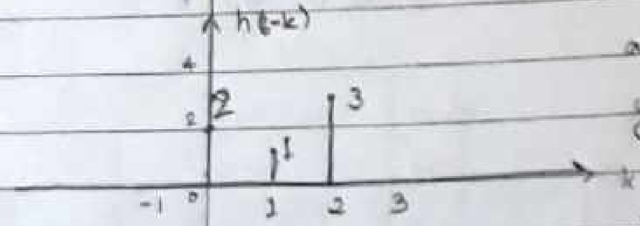
at $n=0$

$y(0) = 6+3-6 = 3$



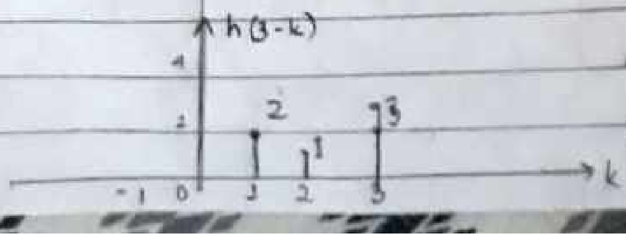
at $n=+1$

$y(+1) = 10-2+12 = 20$



at $n=2$

$y(2) = -4+4 = 0$



at $n=3$

$y(1) = 8$

* **NOTE:**

1. $\sum_{n=k}^{\infty} a^n = \frac{a^k}{1-a}$; $a<1$

2. $\sum_{n=n_1}^{n_2} a^n = \frac{a^{n_1} - a^{n_2+1}}{1-a}$

3. $\sum_{n=n_1}^{n_2} 1 = n_2 - n_1 + 1$

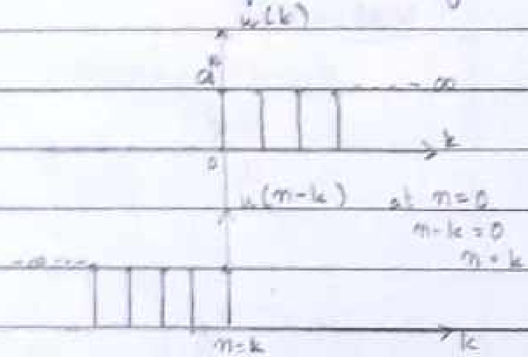
Q1: An LTI system charcterised by impulse function $h(n) = u(n)$ with input $x(n) = a^n u(n)$. Find the output of the LTI system

Sol:

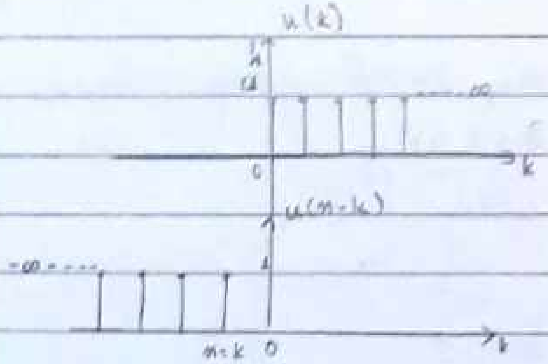
$y(n) = x(n) * h(n)$

$y(n) = \sum_{k=-\infty}^{\infty} x(k)\, h(n-k)$

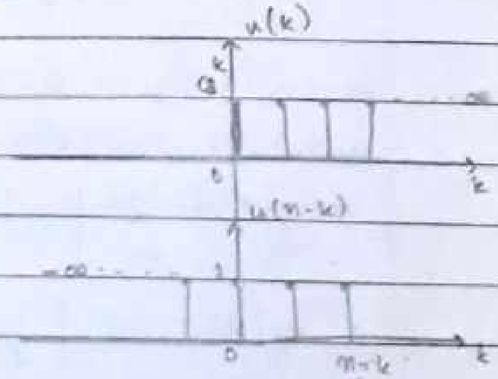
$y(n) = \sum_{k=-\infty}^{\infty} a^k u(k) u(n-k)$



CASE 1: $n<0$



CASE 2: $n>0$

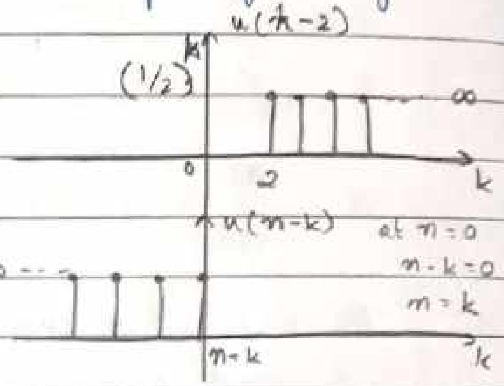


$u(k) \cdot u(n-k) = 1$, $0 \le k \le n$

$\therefore y(n) = \sum_{k=0}^{n} a^k \cdot 1 = \sum_{k=0}^{n} a^k = \frac{a^0 - a^{n+1}}{1-a} = \frac{1-a^{n+1}}{1-a}$

$$y(n) = \begin{cases} \left[\frac{1-a^{n+1}}{1-a}\right] u(n) & ; n>0 \\ 0 & ; n<0 \end{cases}$$

Q2: An LTI function characterised by impulse function $h(n) = u(n)$ with input $x(n) = \left[\frac{1}{2}\right]^n u(n-2)$. Find the output of LTI system.



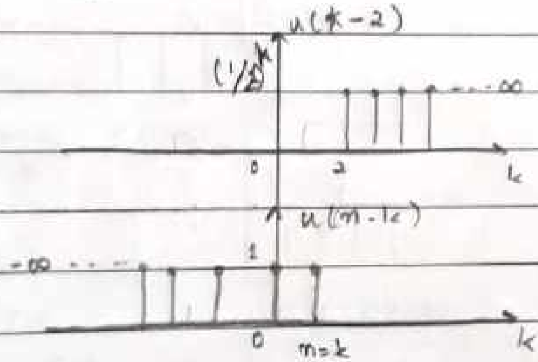
Sol: $y(n) = x(n) * h(n)$

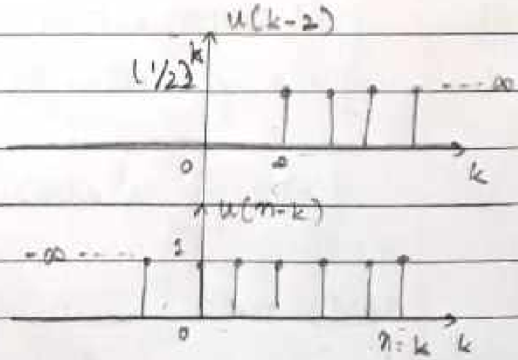
$$y(n) = \sum_{k=-\infty}^{\infty} x(k)\, h(n-k)$$

$$y(n) = \sum_{k=2}^{n} \left[\frac{1}{2}\right]^n u(k-2)\, u(n-k)$$

CASE 1: $n<2$

CASE 2: $n>2$





case 1: $u(k-2)\, u(n-k) = 0$ as they do not overlap.

case 2: $u(k-2)\, u(n-k) = 1$; $2 \le k \le n$

$$y(n) = \sum_{k=2}^{n} \left[\frac{1}{2}\right]^k = \frac{[1/2]^2 - [1/2]^{n+1}}{1 - 1/2}$$

$$y(n) = 2\left[\frac{1}{4} - \left(\frac{1}{2}\right)^{n+1}\right]$$

$$\therefore\ y(n) = \begin{cases} 2\left[\frac{1}{4} - \left(\frac{1}{2}\right)^{n+1}\right] u(n-2) & ; n>0 \\ 0 & ; n<0 \end{cases}$$

Q3: In an LTI system impulse response $h(n) = 3^n u(-n)$. Find the step response of the LTI system.

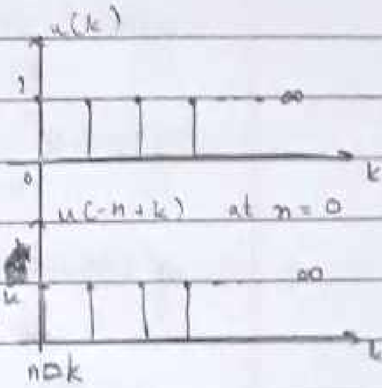
Sol: $h(n) = 3^n u(-n)$

$h(n-k) = 3^{n-k} u[-(n-k)]$

$x(n) \longrightarrow \boxed{LTI} \longrightarrow y(n)$

If $x(n) = u(n)$, then $y(n)$ is step response

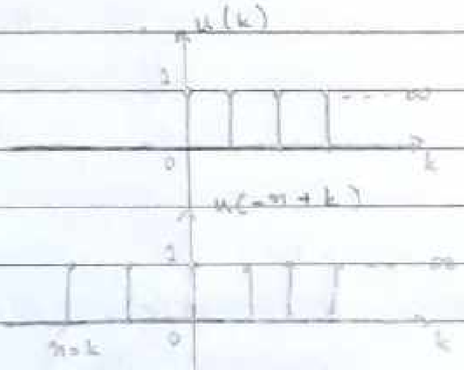
$$y(n) = \sum_{k=-\infty}^{\infty} x(k)\, h(n-k)$$



$$y(n) = \sum_{k=-\infty}^{\infty} u(k)\, 3^{n-k} u[-(n-k)]$$

$$y(n) = 3^n \sum_{k=-\infty}^{\infty} \left(\frac{1}{3}\right)^k u(k)\, u(-n+k)$$

CASE 1: $n < 0$



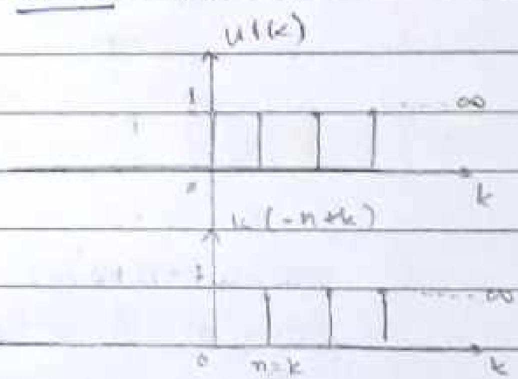
$u(k) \cdot u(-n+k) = 1 \;;\; 0 \le k \le \infty$

$$y(n) = 3^n \sum_{k=0}^{\infty} \left[\frac{1}{3}\right]^k$$

$$y(n) = 3^n \left[\frac{1}{1 - 1/3}\right]$$

$$y(n) = \frac{3^{n+1}}{2}$$

CASE 2: $n > 0$



$u(k) \cdot u(-n+k) = 1 \;;\; n \le k \le \infty$

$$y(n) = 3^n \sum_{k=n}^{\infty} \left[\frac{1}{3}\right]^k$$

$$y(n) = 3^n \left[\frac{(1/3)^n}{1 - 1/3}\right] = 3^n \frac{(3^{-n})}{2/3} = \frac{3}{2}$$

$$y(n) = \begin{cases} 3^{n+1}/2 & ; n < 0 \\ 3/2 & ; n > 0 \end{cases}$$

Q4: Find the convolution of

$x(n) = \alpha^n u(n)$

and $h(n) = \alpha^{-n} u(-n)$

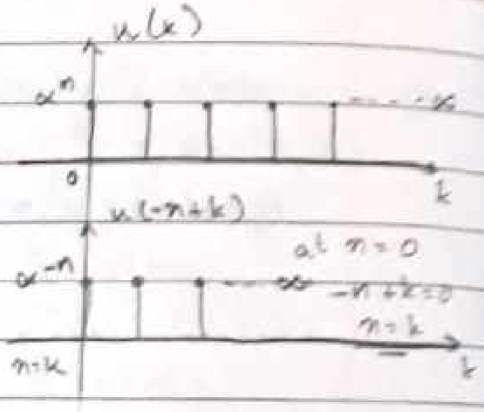
sol:

$$y(n) = x(n) * h(n)$$

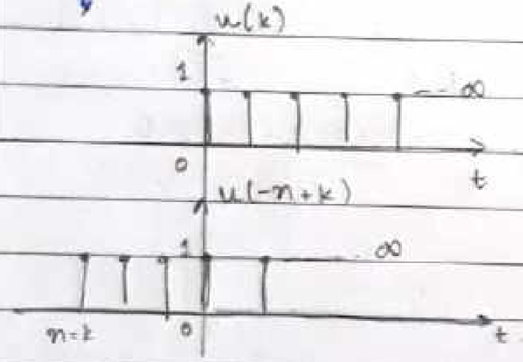
$$y(n) = \sum_{k=-\infty}^{\infty} x(k) \cdot h(n-k)$$

$$y(n) = \sum_{k=-\infty}^{\infty} \alpha^{k} u(k)\, \alpha^{-(n-k)} u(-(n-k))$$

$$y(n) = \alpha^{-n} \sum_{k=-\infty}^{\infty} \alpha^{2k} u(k)\, u(-n+k)$$



CASE 1: $n < 0$



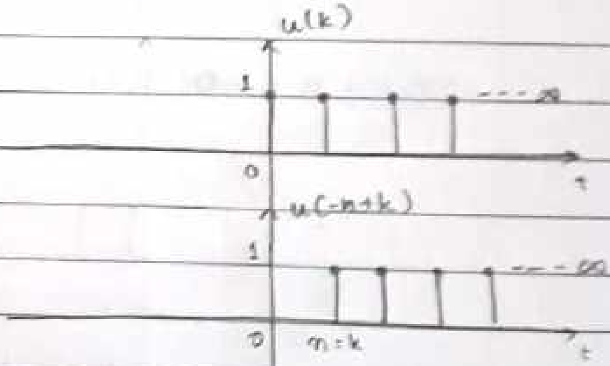
$u(k)\,u(-n+k) = 1 \quad 0 \le k \le \infty$

$$y(n) = \alpha^{-n} \sum_{k=0}^{\infty} \alpha^{2k}$$

$$y(n) = \alpha^{-n} \left[\frac{\alpha^{0}}{1-\alpha^{2}} \right]$$

$$y(n) = \alpha^{-n} \left[\frac{1}{1-\alpha^{2}} \right]$$

CASE 2: $n > 0$



$u(k)\,u(-n+k) = 1 \quad n \le k \le \infty$

$$y(n) = \alpha^{-n} \sum_{k=n}^{\infty} \alpha^{2k}$$

$$y(n) = \alpha^{-n} \left[\frac{\alpha^{2n}}{1-\alpha^{2}} \right]$$

$$y(n) = \frac{\alpha^{n}}{1-\alpha^{2}}$$

Q5: A LTI system characterised by impulse response $h(n) = [1/2]^n u(n)$ with input $x(n) = 2\delta(n) - 3\delta(n-1) + 4\delta(n-3)$. Find the output of the system.

sol: Given an impulse respose with input

$$h(n) = \left[\frac{1}{2}\right]^{n} u(n) \quad \text{and} \quad x(n) = 2\delta(n) - 3\delta(n-1) + 4\delta(n-3)$$

sol: $x(n) = \{2, -3, 0, 4\}$
$n \rightarrow \quad 0 \quad 1 \quad 2 \quad 3$

note: $x(n) * \delta(n) = x(n)$

$y(n) = x(n) * h(n)$

$y(n) = [2\delta(n) - 3\delta(n-1) + 4\delta(n-3)] * h(n)$

$y(n) = 2(\delta(n) * h(n)) - 3(\delta(n-1) * h(n)) + 4(\delta(n-3) * h(n))$

$y(n) = 2h(n) - 3h(n-1) + 4h(n-3)$

$y(n) = 2\left[\frac{1}{2}\right]^{n} u(n) - 3\left[\frac{1}{2}\right]^{n-1} u(n-1) + 4\left[\frac{1}{2}\right]^{n-3} u(n-3)$ //

* Convolution integral:

The continuous time counterpart of convolution sum is the convolution integral.

Let $x(t)$ be the input and $y(t)$ be the output for a LTI system with T operator.

$x(t) \rightarrow \boxed{\text{LTI}} \rightarrow y(t) = T\{x(t)\}$

The response of the system

$y(t) = T\{x(t)\}$

$y(t) = T\left\{\int_{\tau=-\infty}^{\infty} x(\tau)\delta(t-\tau)\,d\tau\right\}$

$y(t) = \int_{\tau=-\infty}^{\infty} x(\tau)\,T\{\delta(t-\tau)\}\,d\tau$

$y(t) = \int_{\tau=-\infty}^{\infty} x(\tau)\,h(t-\tau)\,d\tau$

$\therefore \boxed{y(t) = x(t) * h(t)}$

The output of any continuous time LTI system is the convolution of the input $x(t)$ with the impulse response $h(t)$ of the system.

* Properties of convolution integral

1. commutative

$$x(t) * h(t) = h(t) * x(t)$$

considering LHS

$$x(t) * h(t) = \int_{\tau=-\infty}^{\infty} x(\tau) h(t-\tau) d\tau$$

let $t-\tau = \gamma$

$$\therefore x(t) * h(t) = \int_{\gamma=-\infty}^{\infty} x(t-\gamma) h(\gamma) d\gamma$$

$$= \int_{\gamma=-\infty}^{\infty} h(\gamma) x(t-\gamma) d\gamma$$

since γ is just any variable

$$x(t) * h(t) = \int_{\tau=-\infty}^{\infty} h(\tau) x(t-\tau) d\tau$$

$$\therefore \boxed{x(t) * h(t) = h(t) * x(t)}$$

2. Distributive

$$x(t) * [h_1(t) + h_2(t)] = x(t) * h_1(t) + x(t) * h_2(t)$$

considering LHS

$$x(t) * [h_1(t) + h_2(t)] = \int_{\tau=-\infty}^{\infty} x(\tau) [h_1(t-\tau) + h_2(t-\tau)] d\tau$$

$$= \int_{\tau=-\infty}^{\infty} x(\tau) h_1(t-\tau) d\tau + \int_{\tau=-\infty}^{\infty} x(\tau) h_2(t-\tau) d\tau$$

$$\therefore \boxed{x(t) * [h_1(t) + h_2(t)] = x(t) * h_1(t) + x(t) * h_2(t)}$$

3. Associative.

$$[x(t) * h_1(t)] * h_2(t) = x(t) * [h_1(t) * h_2(t)]$$

Considering LHS

$$[x(t) * h_1(t)] * h_2(t) = \int_{\tau=-\infty}^{\infty} [x(\tau) * h_1(\tau)] h_2(t-\tau) d\tau$$

$$= \int_{\tau=-\infty}^{\infty} \left[\int_{\gamma=-\infty}^{\infty} x(\gamma) h_1(\tau-\gamma) d\gamma \right] h_2(t-\tau) d\tau$$

let $\lambda = \tau - \gamma$ and interchanging the order of integration.

$$[x(t) * h_1(t)] * h_2(t) = \int_{\gamma=-\infty}^{\infty} x(\gamma) \left[\int_{\lambda=-\infty}^{\infty} h_1(\lambda)\, h_2(t-\gamma-\lambda)\, d\lambda \right] d\gamma$$

$$= \int_{\gamma=-\infty}^{\infty} x(\gamma) \left[h_1(t-\gamma) * h_2(t-\gamma) \right] d\gamma$$

$$\boxed{[x(t) * h_1(t)] * h_2(t) = x(t) * [h_1(t) * h_2(t)]}$$

properties

1. $x(t) * \delta(t) = x(t)$

Proof: $x(t) * \delta(t) = \int_{\tau=-\infty}^{\infty} x(\tau)\, \delta(t-\tau)\, d\tau$

$$= x(\tau) \cdot 1 \Big|_{\tau = t}$$

$\therefore x(t) * \delta(t) = x(t)$ //

2. $x(t) * \delta(t-t_0) = x(t-t_0)$

Proof: $x(t) * \delta(t-t_0) = \int_{\tau=-\infty}^{\infty} x(\tau)\, \delta(t-t_0-\tau)\, d\tau$

$$= x(\tau) \cdot 1 \Big|_{\tau = t-t_0}$$

$\therefore x(t) * \delta(t-t_0) = x(t-t_0)$ //

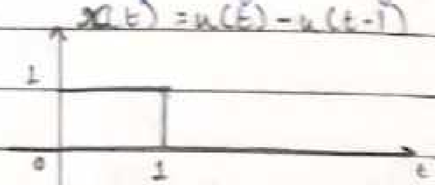
3. $x(t) * u(t) = \int_{-\infty}^{t} x(\tau)\, d\tau$

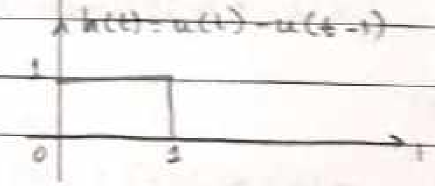
Proof: $x(t) * u(t) = \int_{\tau=-\infty}^{\infty} x(\tau)\, u(t-\tau)\, d\tau$

$$= \int_{\tau=-\infty}^{\infty} x(\tau)\, d\tau$$

$\therefore x(t) * u(t) = \int_{\tau=-\infty}^{\infty} x(\tau)\, d\tau$ //

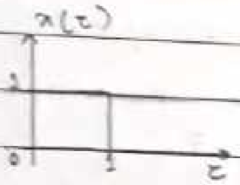
Q1: A LTI system characterised by impulse response $h(t) = u(t) - u(t-1)$ with input $x(t) = u(t) - u(t-1)$. Find the output of the system.

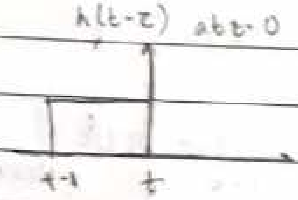




Sol: $y(t) = x(t) * h(t)$

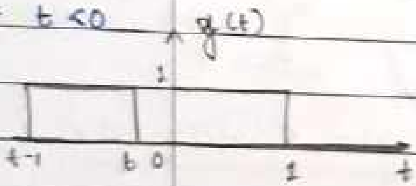
$$y(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau$$





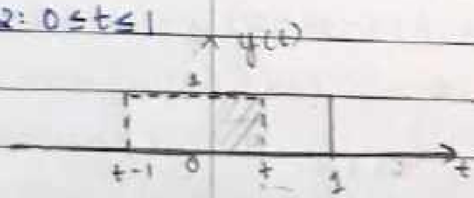
The sliding signal of square pulse is easy.

case 1: $t < 0$



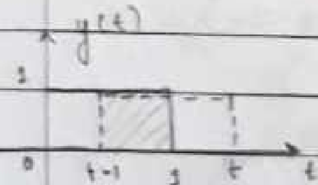
$y(t) = 0$; no overlapping

case 2: $0 \le t \le 1$



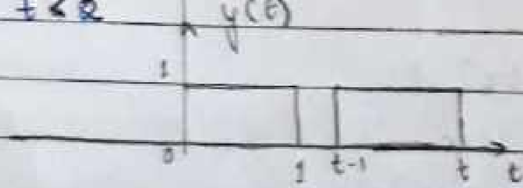
$$y(t) = \int_0^t (1)(1)\, d\tau = \tau \Big|_0^t = t$$

case 3: $1 \le t \le 2$



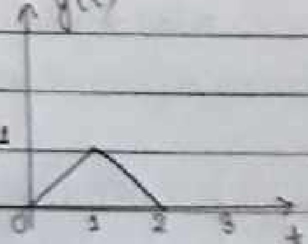
$$y(t) = \int_{t-1}^{1} d\tau = \tau \Big|_{t-1}^{1} = 2 - t$$

case 4: $t \geq 2$



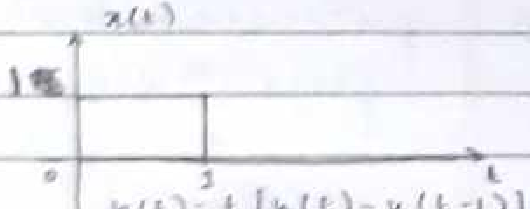
$y(t) = 0$; no overlapping

$$y(t) = \begin{cases} \cancel{0 \quad ; \quad t < 0} \\ t \quad ; \quad 0 \le t \le 1 \\ 2-t \quad ; \quad 1 \le t \le 2 \\ 0 \quad ; \quad \text{else} \end{cases}$$



Q2: A LTI system characterised by impulse response $h(t) = t\,[u(t) - u(t-1)]$ with input $x(t) = \begin{cases} 1 & ; 0 \le t \le 1 \\ 0 & ; \text{else.} \end{cases}$

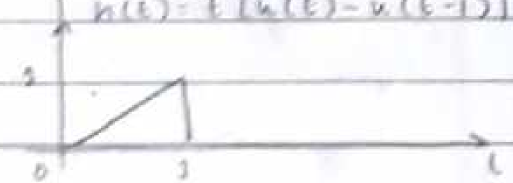
Find the output of the system.

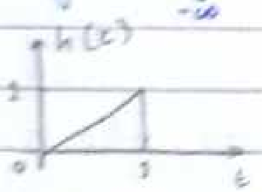


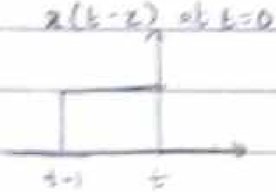
Sol: $y(t) = x(t) * h(t)$

$y(t) = h(t) * x(t)$

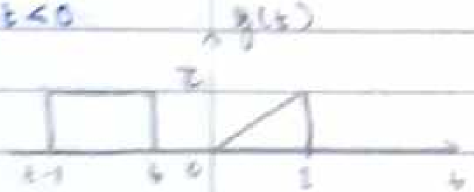
$$y(t) = \int_{-\infty}^{\infty} h(\tau)\,x(t-\tau)\,d\tau$$





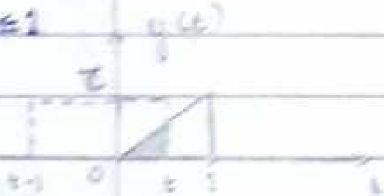


case 1: $t < 0$



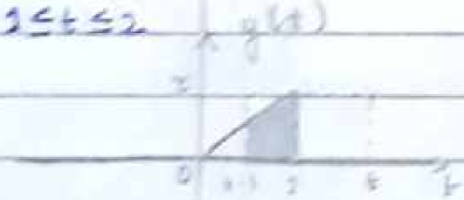
$y(t) = 0$; no overlapping

case 2: $0 \le t \le 1$



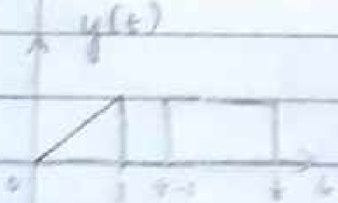
$$y(t) = \int_0^t \tau\,d\tau = \frac{\tau^2}{2}\Big|_0^t = \frac{t^2}{2}$$

case 3: $1 \le t \le 2$



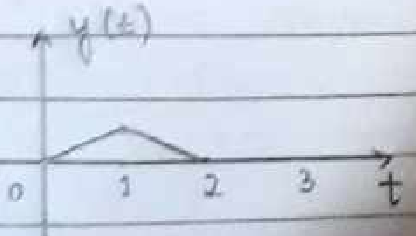
$$y(t) = \int_{t-1}^{1} \tau\,d\tau = \frac{\tau^2}{2}\Big|_{t-1}^{1} = \frac{-(t-1)^2+1}{2} = \frac{2t-t^2}{2}$$

case 4: $t < 2$



$y(t) = 0$; no overlapping

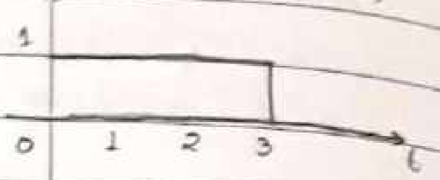
$$\therefore y(t) = \begin{cases} 0 & ; t < 0 \\ t^2/2 & ; 0 \le t \le 1 \\ (2t-t^2)/2 & ; 1 \le t \le 2 \\ 0 & ; \text{else} \end{cases}$$



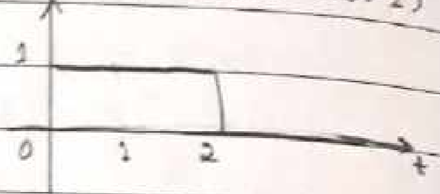
Q3: $x(t) = u(t) - u(t-3)$

$h(t) = u(t) - u(t-2)$

$x(t) = u(t) - u(t-3)$

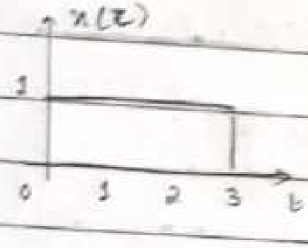


$h(t) = u(t) - u(t-2)$

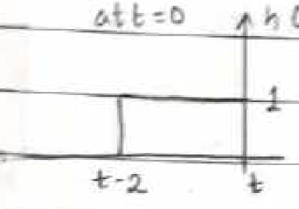


sol: $y(t) = x(t) * h(t)$

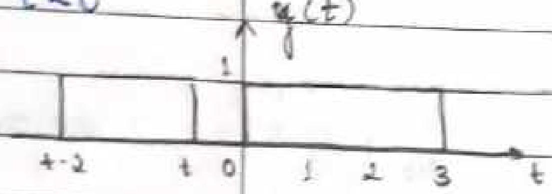
$$y(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau$$



at t=0

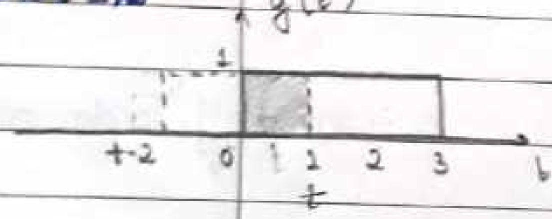


case 1: $t<0$



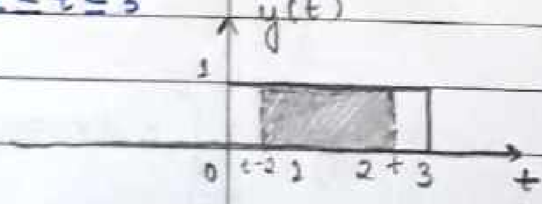
$y(t) = 0$; no overlapping

case 2: $0 \le t \le 2$



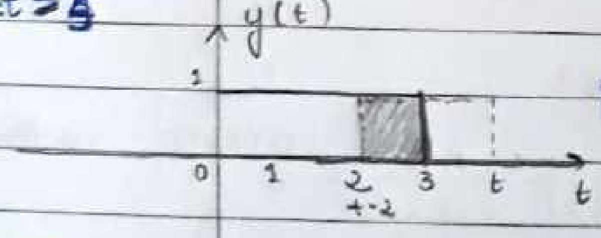
$$y(t) = \int_0^t 1 \cdot 1\, d\tau = \tau\Big|_0^t = t$$

case 3: $2 \le t \le 3$



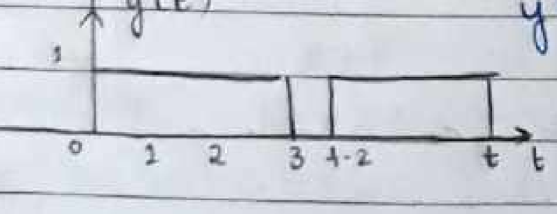
$$y(t) = \int_{t-2}^{t} 1 \cdot 1\, d\tau = \tau\Big|_{t-2}^{t} = t-(t-2) = 2$$

case 4: $3 \le t \le 5$



$$y(t) = \int_{t-2}^{3} 1 \cdot 1\, d\tau = \tau\Big|_{t-2}^{3} = 3-t+2 = 5-t$$

case 5: $t > 5$



$y(t) = 0$; no overlapping

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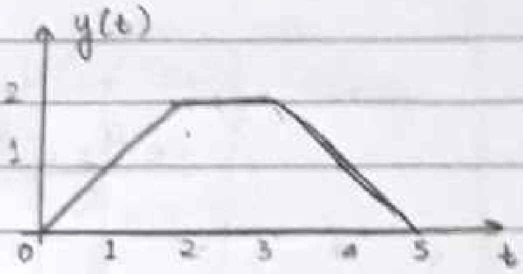
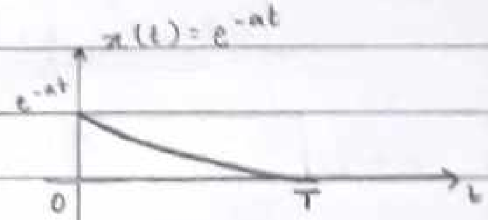
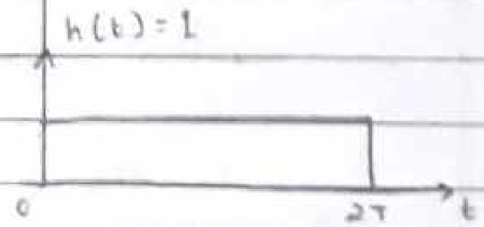
$$\therefore\ y(t) = \begin{cases} t & ; \ 0 \le t \le 2 \\ 2 & ; \ 2 \le t \le 3 \\ 5-t & ; \ 3 \le t \le 5 \\ 0 & ; \ \text{otherwise} \end{cases}$$



Q9: $x(t) = e^{-at} \quad ; \ 0 \le t \le T$

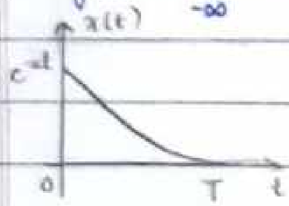
$h(t) = 1 \quad ; \ 0 \le t \le 2T$

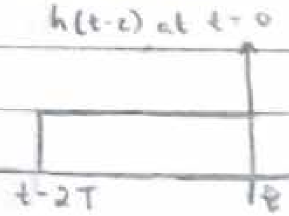




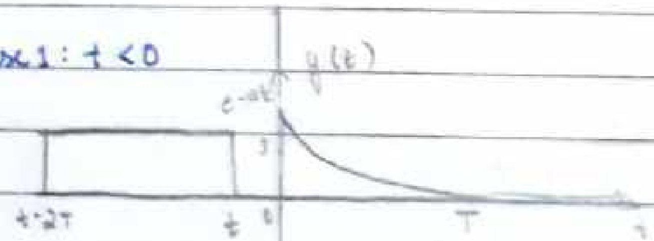
Sol: $y(t) = x(t) * h(t)$

$y(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau$



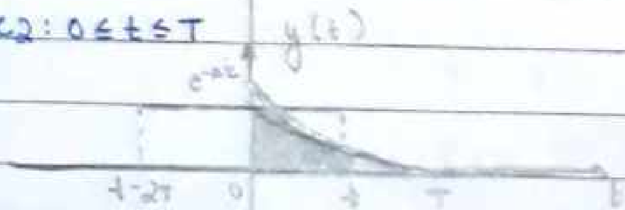


case 1: $t < 0$



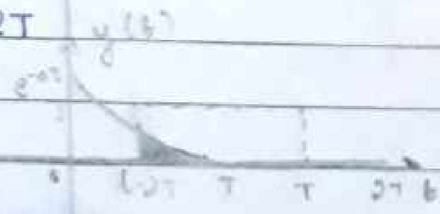
$y(t) = 0$; no overlapping

case 2: $0 \le t \le T$



$$y(t) = \int_0^t e^{-a\tau}\, d\tau = \frac{e^{-a\tau}}{-a}\Bigg|_0^t = \frac{-e^{-at}+1}{a}$$

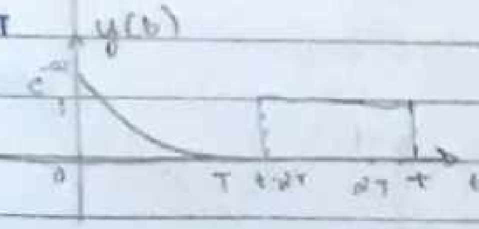
case 3: $T \le t \le 2T$



$$y(t) = \int_{t-2T}^{T} e^{-a\tau}\, d\tau = \frac{e^{-a\tau}}{-a}\Bigg|_{t-2T}^{T}$$

$$= \frac{-e^{-aT} + e^{-at+2T}}{a}$$

case 4: $t > 2T$

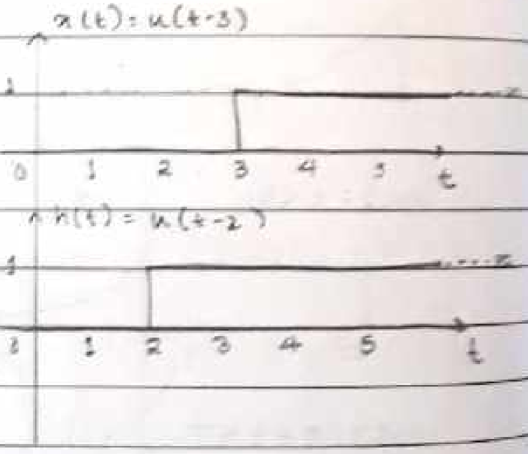


$y(t) = 0$; no overlapping

$$y(t) = \begin{cases} (-e^{-at}+1)/a\; u(t) ; & 0 \le t \le T \\ \dfrac{-e^{-aT} + e^{-a(t+2T)}}{a} u(t) ; & T \le t \le 2T \\ 0 & ; \text{elsewhere.} \end{cases}$$

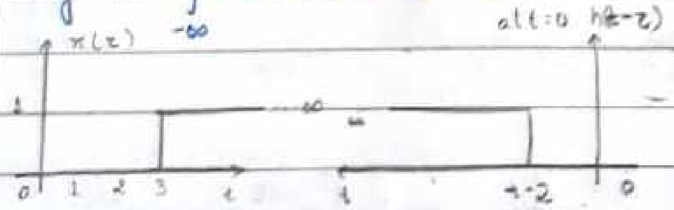
Q5: $x(t) = u(t-3)$

$h(t) = u(t-2)$



sol: $y(t) = x(t) * h(t)$

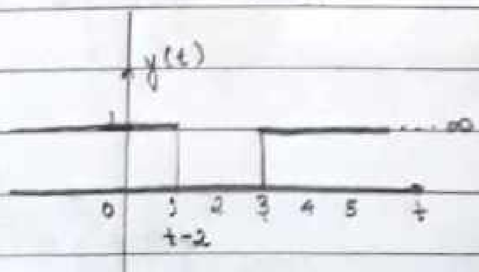
$y(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau$



case 1:

$t-2<3$

$t<5$

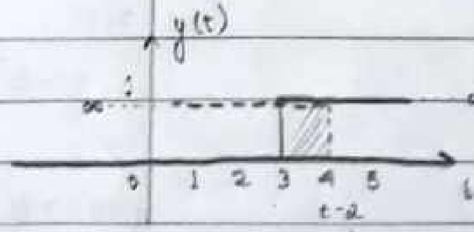


$y(t) = 0$; no overlapping

case 2:

$t-2>3$

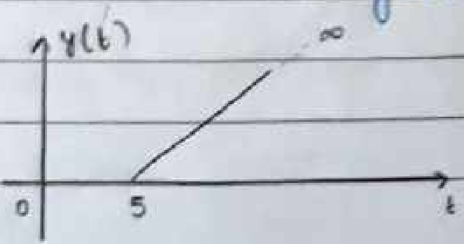
$t>5$



$y(t) = \int_{3}^{t-2} 1\cdot 1\, d\tau = \tau \Big|_{3}^{t-2}$

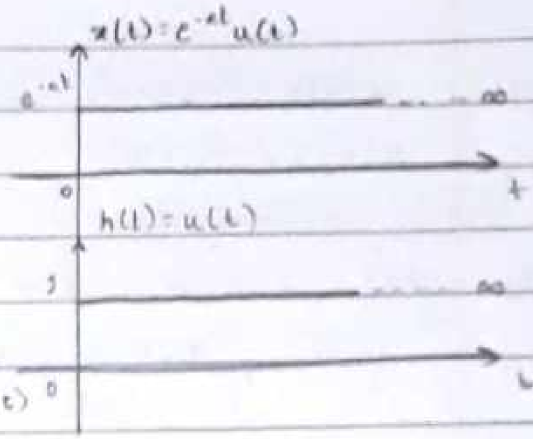
$y(t) = t-2-3 = t-5$

$$y(t) = \begin{cases} (t-5)u(t-5) ; t-2>3 & t>5 \\ 0 \quad ; \text{else} \end{cases}$$



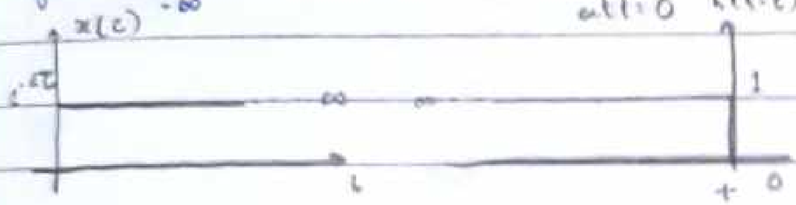
Q6: $x(t) = e^{-at} u(t)$

$h(t) = u(t)$

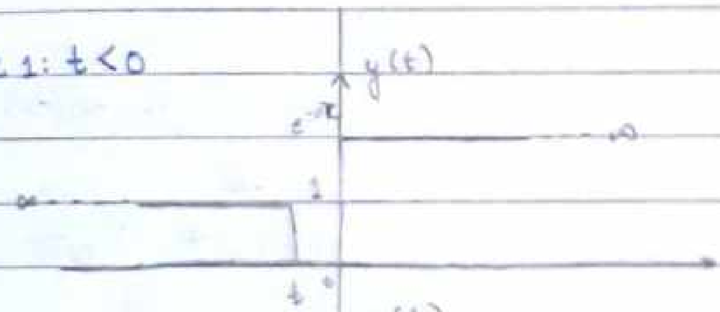


Sol: $y(t) = x(t) * h(t)$

$$y(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau$$

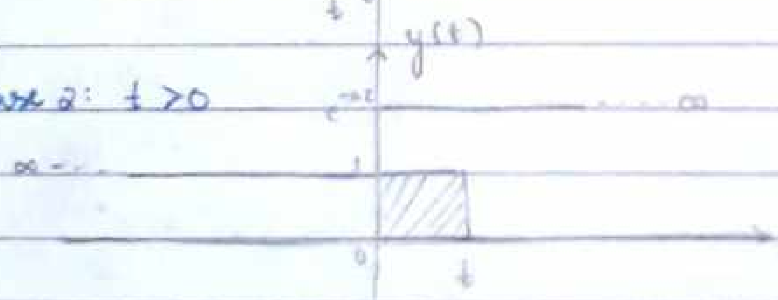


case 1: $t<0$



$y(t) = 0$

case 2: $t>0$



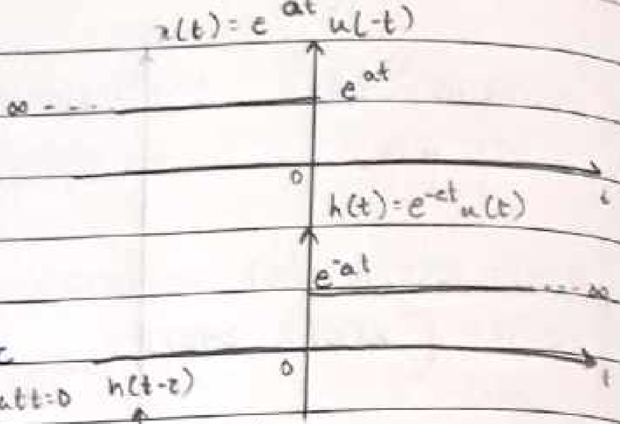
$$y(t) = \int_{0}^{t} e^{-a\tau} \cdot 1 \cdot d\tau$$

$$y(t) = \frac{e^{-a\tau}}{-a}\Big|_0^t = -\frac{e^{-at}+1}{a}$$

$$\therefore\ y(t) = \begin{cases} \dfrac{1-e^{-at}}{a} u(t); & t>0 \\ 0 ; & t<0 \end{cases}$$

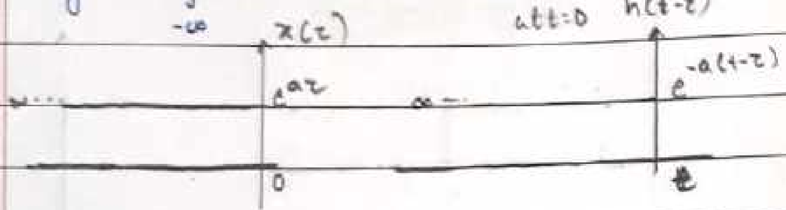
Q7: $x(t) = e^{at}u(-t)$

$h(t) = e^{-at}u(t)$

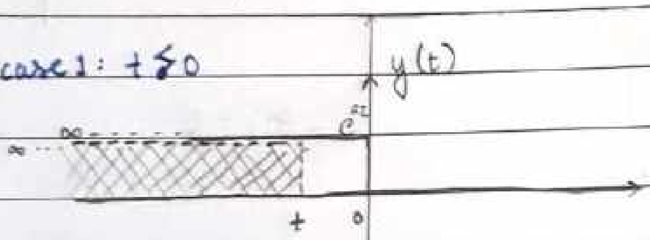


sol: $y(t) = x(t) * h(t)$

$y(t) = \int_{-\infty}^{\infty} x(\tau) * h(t-\tau)\,d\tau$



case 1: $t \leq 0$



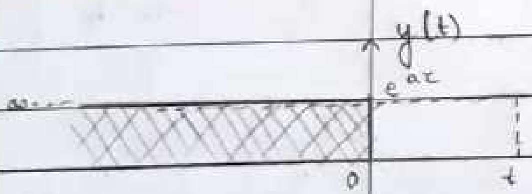
$y(t) = \int_{-\infty}^{t} e^{a\tau} e^{-a(t-\tau)}\,d\tau$

$y(t) = e^{-at} \int_{-\infty}^{t} e^{2a\tau}\,d\tau$

$y(t) = e^{-at} \frac{e^{2a\tau}}{2a}\Big|_{-\infty}^{t} = \frac{e^{-at}e^{2at}}{2a}$

$y(t) = \frac{e^{+at}}{2a}$ //

case 2: $t > 0$



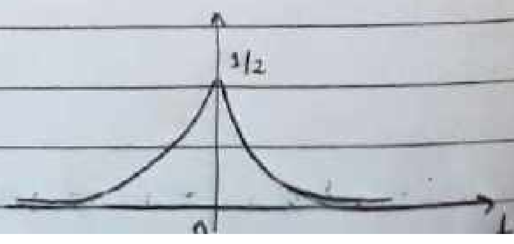
$y(t) = \int_{-\infty}^{0} e^{a\tau} e^{-a(t-\tau)}\,d\tau$

$y(t) = e^{-at} \int_{-\infty}^{0} e^{2a\tau}\,d\tau$

$y(t) = e^{-at} \frac{e^{2a\tau}}{2a}\Big|_{-\infty}^{0} = \frac{e^{-at}}{2a}$

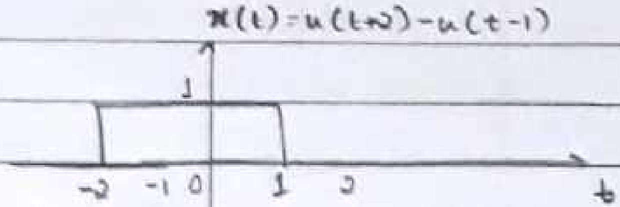
$y(t) = \frac{e^{-at}}{2a}$ //

$$y(t) = \begin{cases} \frac{e^{at}}{2a} u(-t) \ ; & t < 0 \\ \frac{e^{-at}}{2a} u(t) \ ; & t > 0 \end{cases}$$



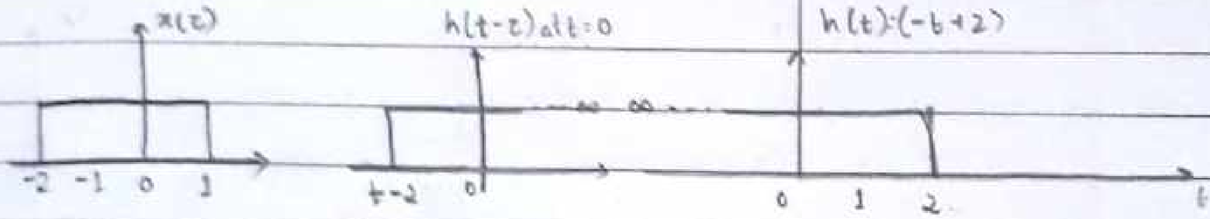
Q8: $h(t) = u(-t+2)$ Find the output of LTI system.

$x(t) = u(t+2) - u(t-1)$



Sol: $y(t) = x(t) * h(t)$

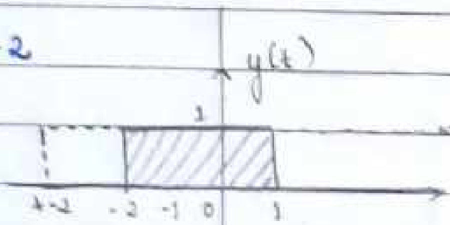
$y(t) = \int_{-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau$



case 1

$t-2 < -2$

$t < 0$

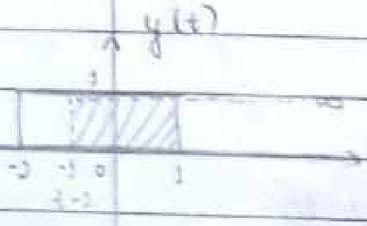


$y(t) = \int_{-2}^{1} 1 \cdot 1\, d\tau = \tau \Big|_{-2}^{1} = \underline{\underline{3}}$

case 2

$t-2 > -2$

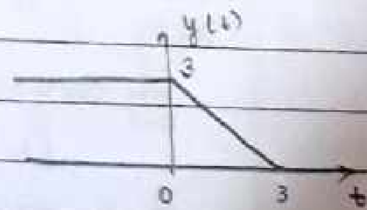
$t > 0$



$y(t) = \int_{t-2}^{1} 1 \cdot 1\, d\tau = \tau \Big|_{t-2}^{1} = 1 - t + 2$

$= \underline{\underline{3-t}}$

$$y(t) = \begin{cases} 3 & ; \quad t < 0 \\ 3-t & ; \quad t > 0 \end{cases}$$



Q9: $x(t) = \delta(t) + 2\delta(t-1) + \delta(t-2)$

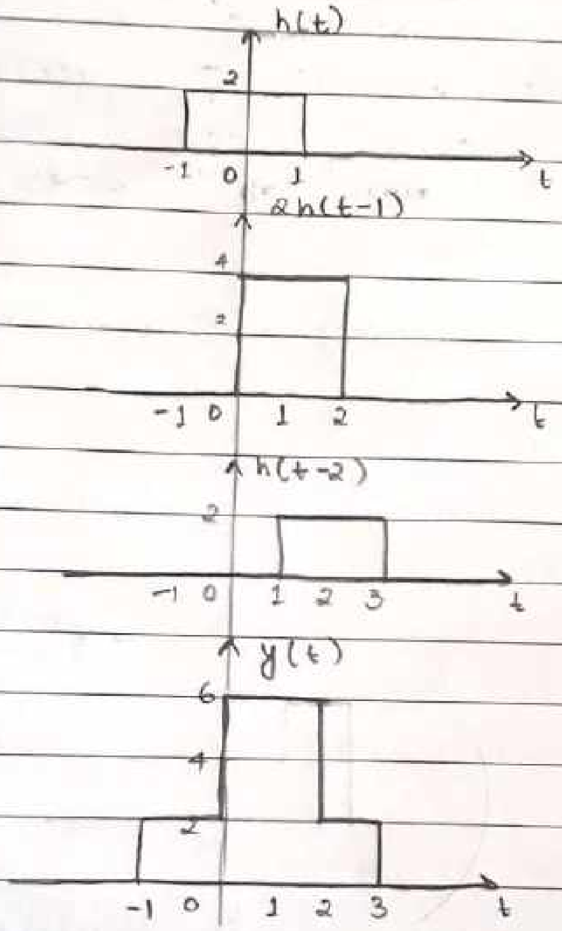
$$h(t) = \begin{cases} 2 & ; \quad -1 \le t \le 1 \\ 0 & ; \quad \text{else.} \end{cases}$$

Sol: $y(t) = x(t) * h(t)$

$y(t) = [\delta(t) + 2\delta(t-1) + \delta(t-2)] * h(t)$

$y(t) = \delta(t) * h(t) + 2\delta(t-1) * h(t) + \delta(t-2) * h(t)$

$y(t) = h(t) + 2h(t-1) + h(t-2)$



* **Block diagram representation:**

DTS:

$$\sum_{k=0}^{M} a_k y(n-k) = \sum_{k=0}^{N} b_k x(n-k)$$

let $M = N = 2$

$$\sum_{k=0}^{2} a_k y(n-k) = \sum_{k=0}^{2} b_k x(n-k)$$

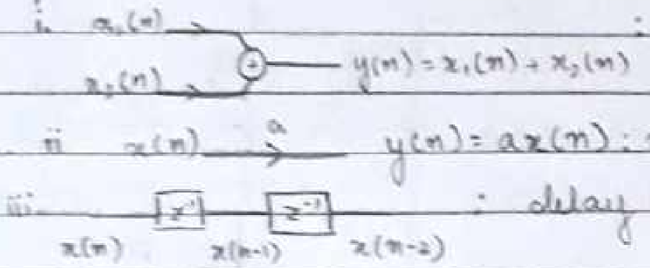
$$a_0 y(n) + a_1 y(n-1) + a_2 y(n-2) = b_0 x(n) + b_1 x(n-1) + b_2 x(n-2)$$

$$y(n) = \frac{b_0}{a_0} x(n) + \frac{b_1}{a_0} x(n-1) + \frac{b_2}{a_0} x(n-2)$$

$a_0 = 1$

$$- \frac{a_1}{a_0} y(n-1) - \frac{a_2}{a_0} y(n-2)$$

The major blocks required are
- adder
- multiplier
- delay element



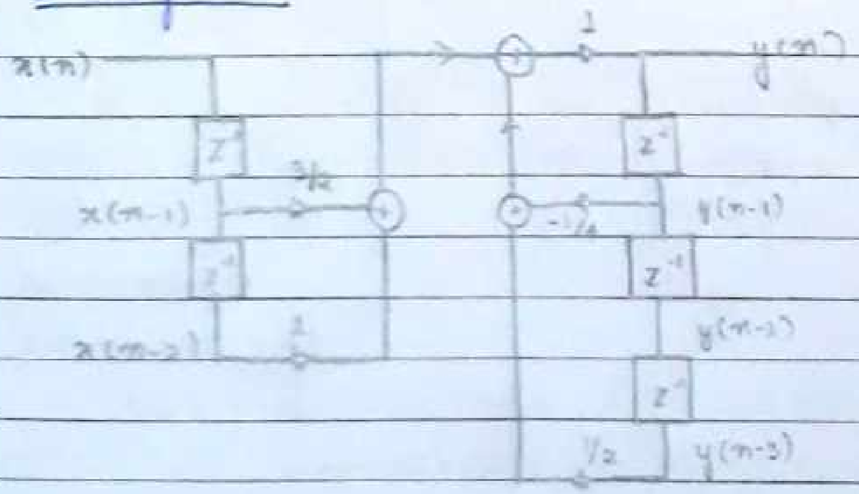
Q: Draw direct form-I and direct form II implementations for the difference equation: $2y(n)+\frac{1}{2}y(n-1)-y(n-3) = 3x(n-1)+2x(n-2)$.

Sol: Given: $2y(n)+\frac{1}{2}y(n-1)-y(n-3) = 3x(n-1)+2x(n-2)$

$$2y(n) = 3x(n-1)+2x(n-2)-\frac{1}{2}y(n-1)+y(n-3)$$

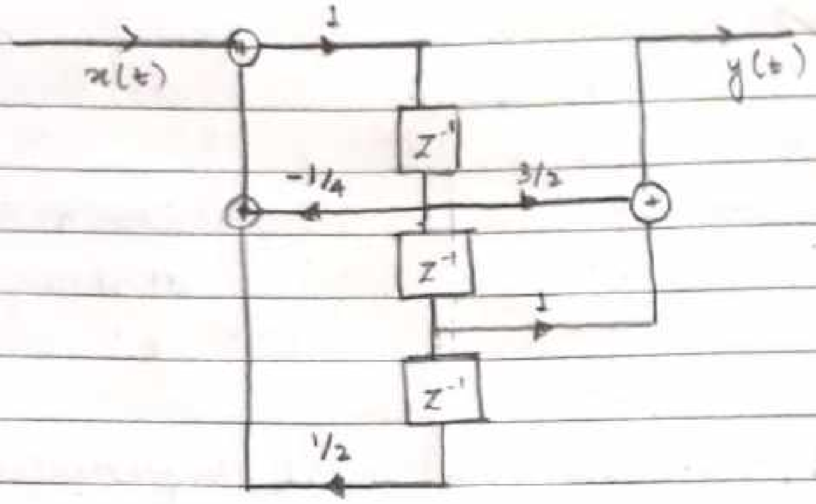
$$\therefore y(n) = \frac{3}{2}x(n-1)+x(n-2)-\frac{1}{4}y(n-1)+\frac{1}{2}y(n-3)$$

Direct form I:



Adder - 3
Delay element - 5
Multiplier - 5

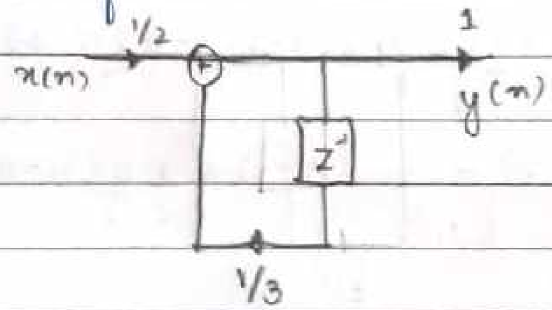
Direct form II:
Same delay element is used for input as well as output.



adders - 3
delay element - 3
multiplier - 5.

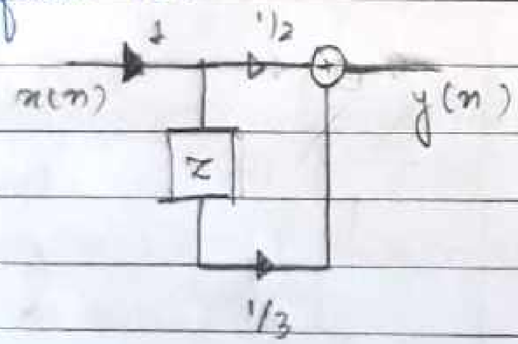
Q2: $y(n) = \frac{1}{3} y(n-1) + \frac{1}{2} x(n)$

sol: Direct form - I



Adders - 1
multipliers - 3
delay element - 1

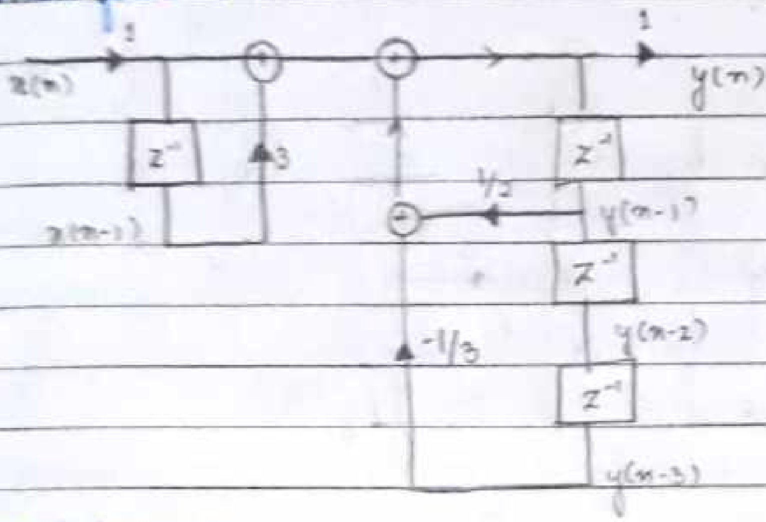
Direct form - II.



Q3: $y(n) - \frac{1}{2} y(n-1) + \frac{1}{3} y(n-3) = x(n) + 3x(n-1)$

sol: $y(n) = x(n) + 3x(n-1) + \frac{1}{2} y(n-1) - \frac{1}{3} y(n-3)$

Direct form - I

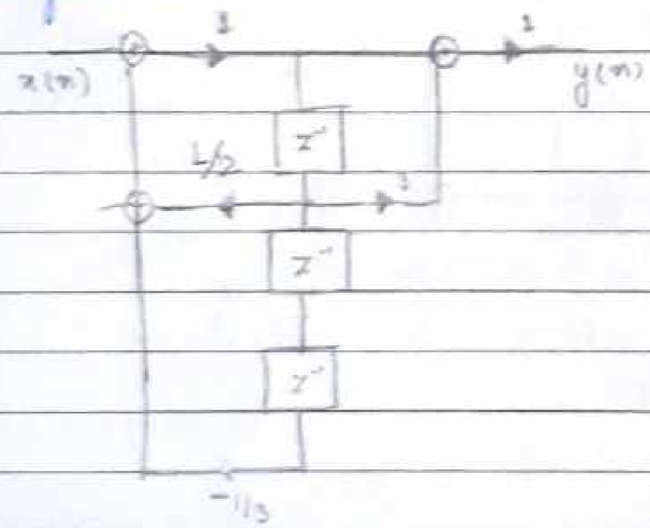


Adders - 3

delay element - 4

multiplier - 5

Direct form - II



Q4: $\frac{d^2y(t)}{dt^2} + 5\frac{d}{dt}y(t) + 4y(t) = x(t) + 3\frac{dx(t)}{d}$

sol: ~~Direct form~~

Integrating w.r.t t

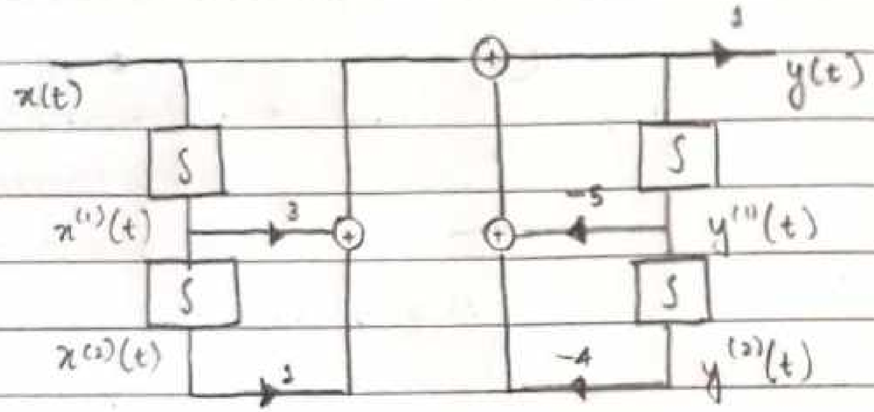
$\frac{d}{dt}y(t) + 5y(t) + 4y^{(1)}(t) = x^{(1)}(t) + 3x(t)$

Integrating w.r.t t

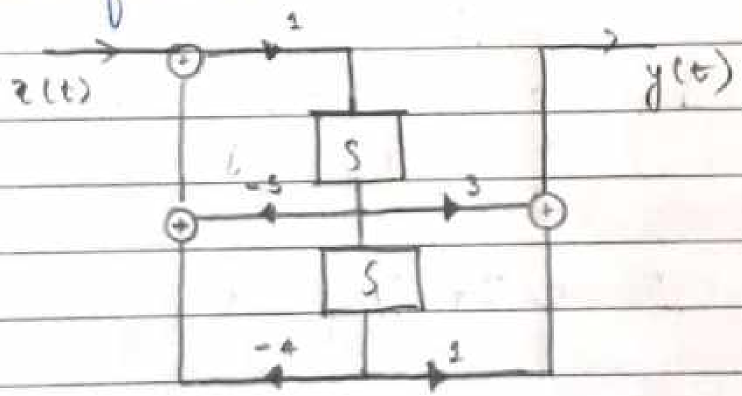
$y(t) + 5y^{(1)}(t) + 4y^{(2)}(t) = x^{(2)}(t) + 3x^{(1)}(t)$

$y(t) = x^{(2)}(t) + 3x^{(1)}(t) - 5y^{(1)}(t) - 4y^{(2)}(t)$

Direct form - 1



Direct form - II



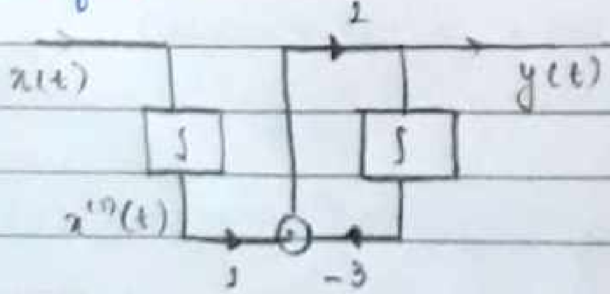
Q5: $\frac{d}{dt} y(t) + 3y(t) = x(t)$

sol: Integrating wrt t

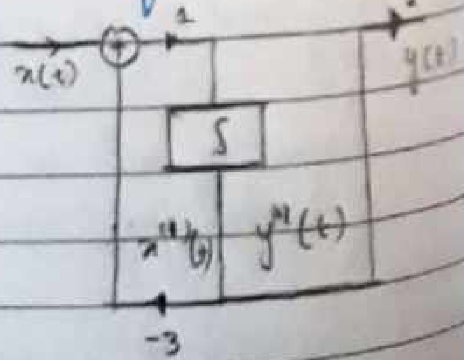
$y(t) + 3y^{(1)}(t) = x^{(1)}(t)$

$y(t) = x^{(1)}(t) - 3y^{(1)}(t)$

Direct form - 1



Direct form - II



Q6: $\frac{d^2}{dt^2}y(t) + 8\frac{d}{dt}y(t) + 4y(t) = \frac{d}{dt}x(t) + \frac{1}{4}\frac{d^2}{dt^2}x(t)$

sol: Integrating wrt t

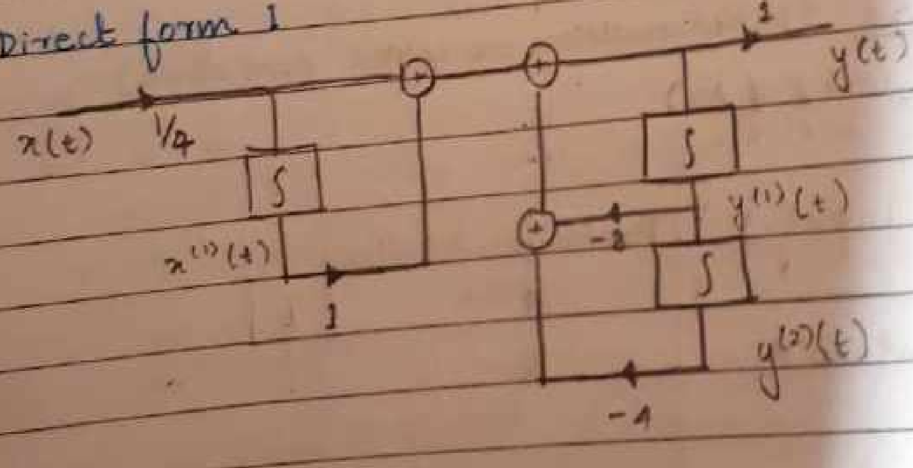
$\frac{d}{dt}y(t) + 8y(t) + 4y^{(1)}(t) = x(t) + \frac{1}{4}\frac{d}{dt}x(t)$

Integrating wrt t

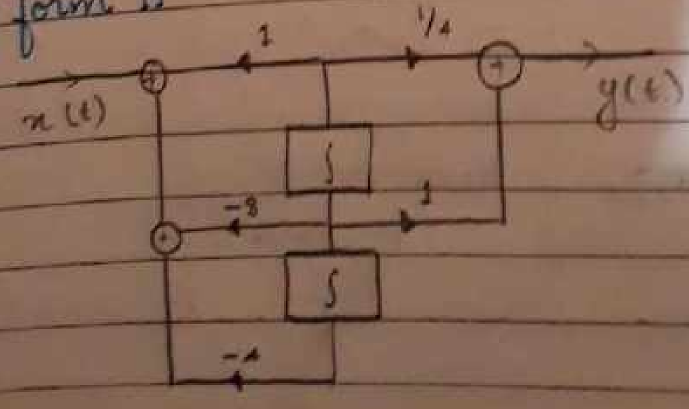
$y(t) + 8y^{(1)}(t) + 4y^{(2)}(t) = x^{(1)}(t) + \frac{1}{4}x(t)$

$y(t) = x^{(1)}(t) + \frac{1}{4}x(t) - 8y^{(1)}(t) - 4y^{(2)}(t).$

Direct form I



Direct form II



* Difference equation:

— Natural response (zero input response)

1. If the order of the equation is 1.

$y_N(n) = A(q)^n$

2. If the order of the equation is 2

a. If roots are distinct

$y_N(n) = A_1(q_1)^n + A_2(q_2)^n$

b. If roots are equal

$y_N(n) = [A_1 + A_2 n]\, q^n$

— Forced response (Initial conditions are zero : zero state)

1. $y_F(n) = y_N(n) + y_p(n)$

	Input	Particular Solution $y_p(n)$
1.	constant A	k
2.	$A\alpha^n$	i. $k\alpha^n$ ii. $kn^m\alpha^n$
3.	$A\cos[\Omega n + \phi]$	$k_1\cos(\Omega n) + k_2\sin(\Omega n)$

— Complete Response (total response) (initial conditions are considered along with the input)

1. $y_T(n) = y_N(n) + y_p(n)$

Q1: $y(n) - 1/2\, y(n-1) = x(n)$ with $y(0) = 2$.
Find the natural response.

Sol: For natural response, input is zero.

$\therefore y(n) - 1/2\, y(n-1) = 0$.

Here the order of the equation is 1

$\therefore y_N(n) = A(q)^n$

$y(n) - 1/2\, y(n-1) = 0$

$q^0 - \frac{1}{2} q^{-1} = 0 \Rightarrow q = \frac{1}{2}$

$\therefore y(n) = A\left[\frac{1}{2}\right]^n$ ——— (1)

initial conditions.

at $n = 0$

$y(0) = A\left[\frac{1}{2}\right]^0$ From eq (1)

$2 = A$ $(\because y(0) = 2$: given)

$\therefore y_N(n) = 2\left[\frac{1}{2}\right]^n$

Q2: For the given difference equation

$y(n) - \frac{1}{4} y(n-1) - \frac{1}{8} y(n-2) = x(n) + x(n-1)$

with $y(-1) = 0$ and $y(-2) = 1$

Find zero input response of the system.

Sol: Here the order of the filter is 2

$y(n) - \frac{1}{4} y(n-1) - \frac{1}{8} y(n-2) = 0$

$q^0 - \frac{1}{4} q^{-1} - \frac{1}{8} q^{-2} = 0$

$q^2 - \frac{1}{4} q - \frac{1}{8} = 0$

$\therefore q_1 = 0.5 \qquad q_2 = -0.25$

$\Rightarrow q_1 = 1/2 \qquad q_2 = -1/4$ //

As the roots are distinct

$y_N(n) = A_1 q_1^n + A_2 q_2^n$ —— (1)

~~Taking the initial conditions for eq (1)~~

at $n=0$

$y(0) = \frac{1}{4} y(-1) + \frac{1}{8} y(-2)$

$y(0) = \frac{1}{4}(0) + \frac{1}{8}(1)$

$y(0) = 1/8$

at $n=1$

$y(1) = \frac{1}{4} y(0) + \frac{1}{8} y(-1)$

$y(1) = \frac{1}{4}\left(\frac{1}{8}\right) + \frac{1}{8}(0)$

$y(1) = \frac{1}{32}$

From eq (2)

$y(0) = A_1\left(\frac{1}{2}\right)^0 + A_2\left(\frac{-1}{4}\right)^0$

$A_1 + A_2 = 1/8$

$y(1) = A_1\left(\frac{1}{2}\right)^1 + A_2\left(\frac{-1}{4}\right)^1 \qquad \Rightarrow A_1 = \frac{1}{12}$ and $A_2 = \frac{1}{24}$

$\frac{A_1}{2} - \frac{A_2}{4} = \frac{1}{32}$

$\therefore y_N(n) = \frac{1}{12}\left(\frac{1}{2}\right)^n + \frac{1}{24}\left(\frac{-1}{4}\right)^n$ //

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Q3: $y(n) - \frac{1}{4}y(n-1) - \frac{1}{8}y(n-2) = x(n) + x(n-1)$ —— ①

with $x(n) = (1/8)^n u(n)$

Find zero state or forced response of the system.

Sol: $y_p(n) = y_N(n) + y_p(n)$

For $y_N(n)$

$y(n) - \frac{1}{4}y(n-1) - \frac{1}{8}(n-2) = 0$

$q^0 - \frac{1}{4}q^{-1} - \frac{1}{8}q^{-2} = 0$

$q^2 - \frac{1}{4}q - \frac{1}{8} = 0$

$q_1 = \frac{1}{2}$ and $q_2 = \frac{-1}{4}$ //

$\therefore y_N(n) = A_1\left[\frac{1}{2}\right]^n + A_2\left[\frac{-1}{4}\right]^n$ —— ②

For $y_p(n)$

$x(n) = (1/8)^n u(n)$

$x(n) = (1/8)^n$ for $n \geq 0$.

$\therefore y_p(n) = k\left[\frac{1}{8}\right]^n$

For k

$k(1/8)^n - \frac{1}{4}k(1/8)^{n-1} - \frac{1}{8}k(1/8)^{n-2} = (1/8)^n + (1/8)^{n-1}$

$k - \frac{1}{\cancel{4}}k(8) - \frac{1}{\cancel{8}}k(8)^{\cancel{2}} = 1 + (8)$

$k - 2k - 8k = 9$

$-9k = 9$

$\underline{k = -1}$

$\therefore \underline{y_p(n) = -\left[\frac{1}{8}\right]^n}$

$$y(n) = \frac{1}{4} y(n-1) + \frac{1}{8} y(n-2) + x(n) + x(n-1)$$

at $n=0$ Forced response: the initial conditions is zero.

$$y(0) = \frac{1}{4} \cancel{y(-1)}^{0} + \frac{1}{8} \cancel{y(-2)}^{0} + x(0) + x(-1)$$

$$y(0) = \left[\frac{1}{8}\right]^0 u(0) + \left[\frac{1}{8}\right]^{-1} \cancel{u(-1)}^{0}$$

$$y(0) = 1$$

at $n=1$

$$y(1) = \frac{1}{4} y(0) + \frac{1}{8} \cancel{y(-1)}^{0} + x(1) + x(0)$$

$$y(1) = \frac{1}{4}(1) + \left[\frac{1}{8}\right]^1 u(1) + \left[\frac{1}{8}\right]^0 u(0)$$

$$y(1) = \frac{1}{4} + \frac{1}{8} + 1$$

$$y(1) = 11/8$$

To find A_1 and A_2

$$y_F(n) = y_N(n) + y_p(n)$$

$$y_F(n) = A_1\left[\frac{1}{2}\right]^n + A_2\left[\frac{-1}{4}\right]^n - \left[\frac{1}{8}\right]^n$$

at $n=0$

$$y(0) = A_1 + A_2 - 1$$

$$2 = A_1 + A_2 \quad \text{—— } (3)$$

at $n=1$

$$y(1) = A_1(1/2) + A_2(-1/4) - (1/8)$$

$$\frac{11}{8} = 4A_1 - 2A_2 - 1$$

$$4A_1 - 2A_2 = 12$$

$$2A_1 - A_2 = 6 \quad \text{—— } (4)$$

classmate

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solving eq (3) and (4)

$$A_1 = \frac{-2}{3} \text{ and } A_2 = \frac{8}{3}$$

$$\therefore y_c(n) = \frac{-2}{3}\left[\frac{1}{2}\right]^n + \frac{8}{3}\left[\frac{-1}{4}\right]^n - \left[\frac{1}{8}\right]^n$$

Ex: $y(n) - 1/4\, y(n-1) - 1/8\, y(n-2) = x(n) + x(n-1)$ —— (1)

with $x(n) = 2u(n)$ and $y(-1) = 0$ and $y(-2) = 1$.

Find the complete response.

sol:

$$y_c(n) = y_h(n) + y_p(n)$$

To find $y_N(n)$

$$y(n) - 1/4\, y(n-1) - 1/8\, y(n-2) = 0$$

$$q^0 - 1/4\, q^{-1} - 1/8\, q^{-2} = 0$$

$$q^2 - \frac{1}{4}q - \frac{1}{8} = 0$$

$$\Rightarrow q_1 = \frac{1}{2} \text{ and } q_2 = \frac{-1}{4}$$

$$\therefore y_N(n) = A_1\left[\frac{1}{2}\right]^n + A_2\left[\frac{-1}{4}\right]^n \quad \text{—— (2)}$$

To find $y_p(n)$

$$x(n) = 2u(n)$$

$$x(n) = 2 \quad ; \quad n \geq 0$$

$$\therefore y_p(n) = k$$

substituting in eq (1)

$$k - 1/4\, k - 1/8\, k = 2 + 2$$

$$8k - 2k - k = 32$$

$$5k = 32$$

$$k = \frac{32}{5}$$

$$\therefore y_p(n) = \frac{32}{5}$$

$$y(n) = \frac{1}{4}y(n-1) + \frac{1}{8}y(n-2) + x(n) + x(n-1)$$

at $n=0$

$$y(0) = \frac{1}{4}\cancel{y(-1)}^{\,0} + \frac{1}{8}y(-2) + x(0) + \cancel{x(-1)}^{\,0}$$

$$y(0) = 2 + \frac{1}{8}(1)$$

$$\underline{y(0) = 17/8}$$

at $n=1$.

$$y(1) = \frac{1}{4}y(0) + \frac{1}{8}y(-1) + x(1) + x(0)$$

$$y(1) = \frac{1}{4}\cdot\frac{17}{8} + \frac{1}{8}(0) + 2 + 2$$

$$\underline{y(1) = 145/32}$$

To find A_1 and A_2

$$y_c(n) = y_h(n) + y_p(n)$$

$$y_c(n) = A_1\left[\frac{1}{2}\right]^n + A_2\left[\frac{-1}{4}\right]^n + \frac{32}{5}$$

at $n=0$

$$y(0) = A_1(1/2)^0 + A_2(-1/4)^0 + 32/5$$

$$17/8 = A_1 + A_2 + 32/5$$

$$A_1 + A_2 = 171/40$$

at $n=1$

$$y(1) = A_1(1/2)^1 + A_2(-1/4)^1 + 32/5$$

$$145/32 = A_1/2 - A_2/4 + 32/5$$

$$2A_1 - A_2 = -299/40$$

$$\therefore A_1 = -16/15 \quad A_2 = 641/120$$

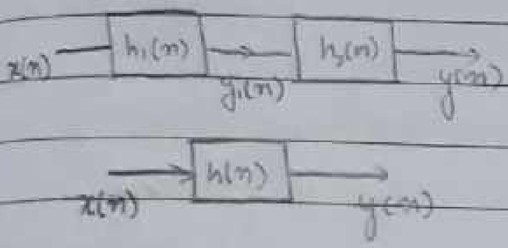
$$\therefore y_c(n) = \left[\frac{-16}{15}\right]\left[\frac{1}{2}\right]^n + \left[\frac{641}{120}\right]\left[\frac{-1}{4}\right]^n + \frac{32}{5}$$

UNIT - 3

LTI System and Fourier Representation for Signals

- Interconnection of LTI system:

Series



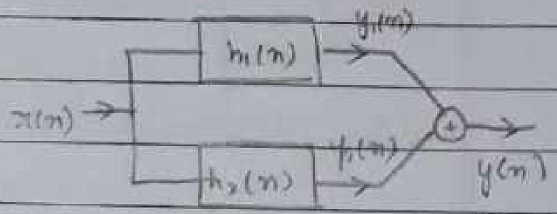
$y_1(n) = x(n) * h_1(n)$

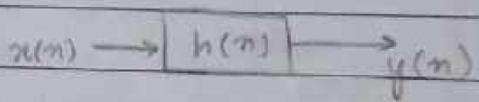
$y(n) = y_1(n) * h_2(n)$

$y(n) = x(n) * h_1(n) * h_2(n)$

here $\boxed{h_1(n) * h_2(n) = h(n)}$

Parallel





$y_1(n) = x(n) * h_1(n)$

$y_2(n) = x(n) * h_2(n)$

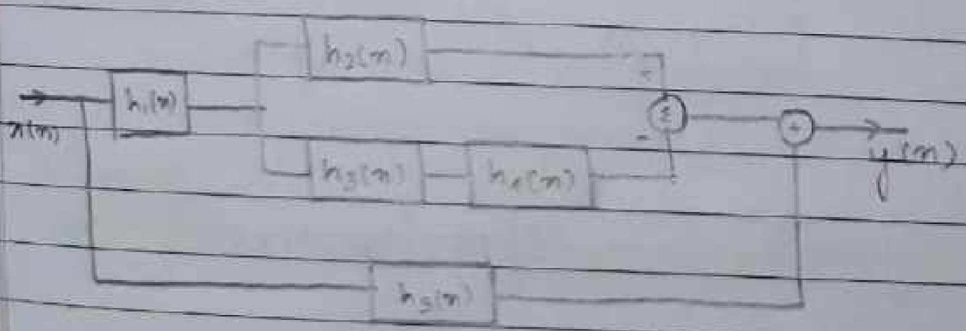
$y(n) = y_1(n) + y_2(n)$

$y(n) = x(n) * h_1(n) + x(n) * h_2(n)$

$y(n) = x(n) * [h_1(n) + h_2(n)]$

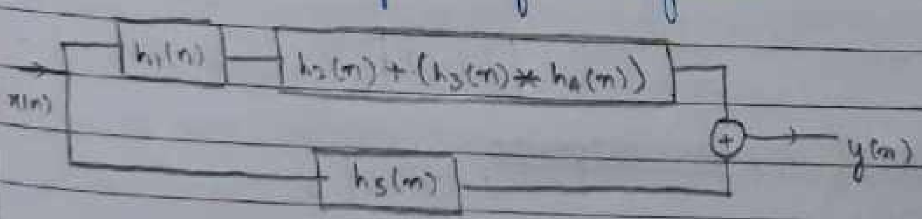
Here $\boxed{h(n) = h_1(n) + h_2(n)}$

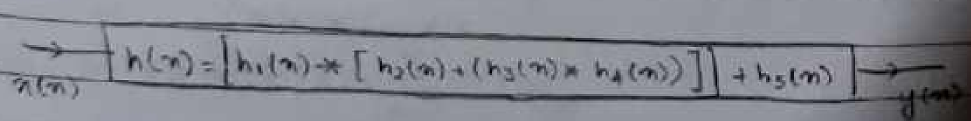
Q1:



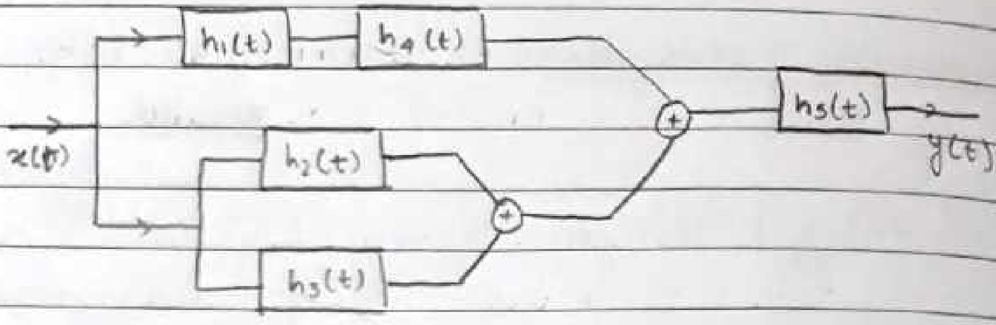
Find the overall response of the system.

Sol:

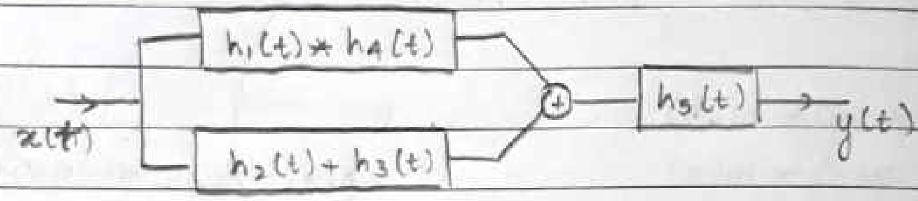


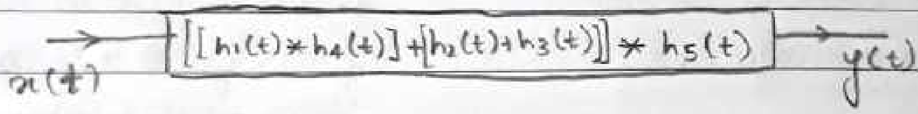


Q2:

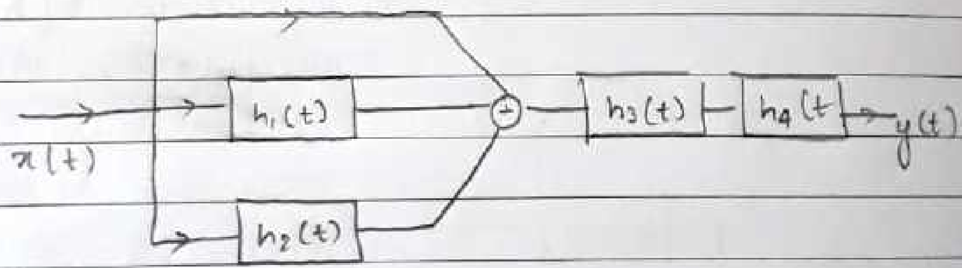


Sol:

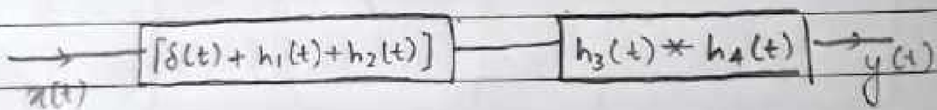


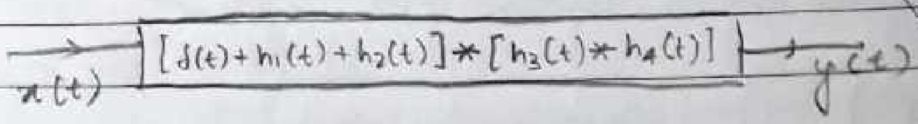


Q3:



Sol:





* **Properties of Impulse response**:

$$y(n) = x(n) * h(n)$$
$$= h(n) * x(n)$$

$$y(n) = \sum_{k=-\infty}^{\infty} h(k)\, x(n-k)$$

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$\therefore y(n) = h(-\infty)x(-\infty) + \cdots + h(-1)h(n+1) + h(0)x(n)$
$+ h(1)x(n-1) + \cdots + h(\infty)x(\infty)$

DTS

1. **Memory less**

$h(n) = 0$ for $n \neq 0$: memory less

If $h(n) \neq 0$ for $n \neq 0$ then it is memory based system.

For memory less: $y(n) = h(0)x(n)$

2. **causal**

$h(n) = 0$ for $n < 0$: causal

For causal system: $y(n) = h(0)x(n) + h(1)x(n-1) + \cdots + h(\infty)x(\infty)$

3. **stable**:

$$\sum_{k=-\infty}^{\infty} |h(k)| < \infty$$

CTS

1. **Memoryless**

$h(t) = 0$ for $t \neq 0$

$y(t) = h(0)x(t)$

2. **causal**

$h(t) = 0$ for $t < 0$

$y(n) = h(0)x(t) + h(1)x(t-1) + \cdots + h(\infty)x(\infty)$

3. **stable**:

$$\int_{-\infty}^{\infty} |h(z)|\,dz < \infty$$

Q1: $h(n) = (1/2)^n u(n)$

- Memory: $h(n) = 0$ for $n \neq 0$

$n = 1$; $h(1) = (1/2)^1 u(1) = 1/2 \neq 0$

$\therefore$ It is a memory based system.

- causal: $h(n) = 0$ for $n < 0$

$n = -1$ $h(-1) = (1/2)^{-1} u(-1) = 2(0) = 0 \Rightarrow h(-n) = 0$.

$\therefore$ It is a causal system.

- stability: $\sum_{k=-\infty}^{\infty} |h(k)| < \infty$

$$= \sum_{k=0}^{\infty} (1/2)^k u(k) = \sum_{k=0}^{\infty} (1/2)^k = \frac{(1/2)^0 - \cancel{(1/2)^{\infty}}}{1 - 1/2} = 2 < \infty$$

$\therefore$ It is a stable system

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Q2: $h(n) = (1/2)^n u(n-10)$

- Memoryless: $h(n)=0$ for $n \neq 0$.

  $n=10$: $h(10) = (1/2)^{10} u(0) = (1/2)^{10} \neq 0$

  $\therefore$ It is memory based system.

- Causal: $h(n)=0$ for $n<0$

  $n=-1$: $h(-1) = (1/2)^{-1} u(-11) = 0$

  $n=-2$: $h(-2) = (1/2)^{-2} u(-12) = 0$

  $\therefore$ $h(n) = 0$ $\therefore$ It is a causal system.

- Stability: $\sum_{k=-\infty}^{\infty} |h(k)| < \infty$

$$= \sum_{k=-\infty}^{\infty} (1/2)^k u(k-10) = \sum_{k=10}^{\infty} (1/2)^k = \frac{(1/2)^{10}}{1-1/2} = \frac{2^{-10}}{1/2} = 2^{-9} < \infty$$

$\therefore$ Stable system.

imp

Q3: $h(n) = 4^{-n} u(2-n)$

- Memoryless: $h(n)=0$ for $n \neq 0$

  $n=1$: $h(1) = 4^{-1} u(1) = 1/4 \neq 0$

  $\therefore$ It is a memory based system.

- Causal: $h(n)=0$ for $n<0$

  $n=-1$: $h(-1) = 4^{1} u(3) = 4 \neq 0$

  $\therefore$ It is a non causal system.

- Stability: $\sum_{k=-\infty}^{\infty} |h(k)| < \infty$

$$= \sum_{k=-\infty}^{\infty} 4^{-k} u(2-k)$$

$$= \sum_{k=-\infty}^{2} 4^{-k}$$

$$= \sum_{k=2}^{-\infty} 4^{-k} = \sum_{k=-2}^{\infty} 4^{k}$$

$$= 4^{-2} + 4^{-1} + 4^{0} + \cdots + 4^{\infty} = \infty$$

$\therefore$ It is a unstable system.

Q: $h(n) = (0.5)^{|n|} = (0.5)^{-n} u(-n-1) + (0.5)^n u(n)$

- Memoryless: $h(n) = 0$ for $n \neq 0$

$n = 1$: $h(1) = (0.5)^1 = 0.5 \neq 0$

$\therefore$ It is memory based system.

- causal: $h(n) = 0$ for $n < 0$

$n = -1$: $h(-1) = (0.5)^{1} u(-2) + (0.5)^{1} u(-1) \neq 0.$

$\therefore$ It is a non causal system.

- stability: $\sum_{k=-\infty}^{\infty} |h(k)| < \infty$

$$= \sum_{k=-\infty}^{\infty} \left[(0.5)^{-k} u(-k-1) + (0.5)^{k} u(k) \right]$$

$$= \sum_{k=-\infty}^{-1} (0.5)^{-k} u(-k-1) + \sum_{k=0}^{\infty} (0.5)^{k} u(k)$$

$$= \sum_{k=1}^{\infty} (0.5)^{k} u(k-1) + \sum_{k=0}^{\infty} (0.5)^{k} u(k)$$

$$= \sum_{k=1}^{\infty} (0.5)^{k} + \sum_{k=0}^{\infty} (0.5)^{k}$$

$$= \frac{0.5}{1-0.5} + \frac{1}{1-0.5} = 1 + 2 = 3 \neq \infty$$

$\therefore$ It is a stable system.

Q: $h(t) = e^{-3t} u(t-1)$

- Memoryless: $h(t) = 0$ for $t \neq 0$

$t = 1$: $h(1) = e^{-3} u(0) = e^{-3} \neq 0$

$\therefore$ It is memory based system.

- causal: $h(t) = 0$ for $t < 0$

$t = -1$: $h(-1) = e^{3} u(-2) = 0$

$t = -2$: $h(-2) = e^{2} u(-3) = 0$

$\therefore h(t) = 0$ $\therefore$ It is a causal system.

- stability: $\int_{-\infty}^{\infty} |h(\tau)| < \infty$

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$$= \int_{-\infty}^{\infty} e^{-3t} u(t-1)\, dt$$

$$= \int_{1}^{\infty} e^{-3t} u(t-1)\, dt$$

$$= \int_{1}^{\infty} e^{-3t} dt = \left.\frac{e^{-3t}}{-3}\right|_{1}^{\infty} = \frac{-1}{3}\left[0 - e^{-3}\right] = \frac{e^{-3}}{3} < \infty$$

$\therefore$ It is a stable system.

Q6: $h(t) = e^{-2|t|}$

$h(t) = e^{2t} u(-t) + e^{-2t} u(t)$

— Memoryless: $h(t) = 0$ for $t \neq 0$

$t = 1 : h(1) = e^{2} \cancel{u(-1)}^{0} + e^{-2} u(1) \neq 0$

$\therefore$ It is a memory based system.

— causal: $h(t) = 0$ for $t < 0$

$t = -1 : h(-1) = e^{-2} \cancel{u(-2)}^{0} + e^{2} u(1) \neq 0$

$\therefore$ It is a non causal system.

— stability: $\int_{-\infty}^{\infty} |h(\tau)|\, d\tau < \infty$

$$= \int_{-\infty}^{\infty} \left[e^{2\tau} u(-\tau) + e^{-2\tau} u(\tau) \right] d\tau$$

$$= \int_{-\infty}^{0} e^{2\tau} d\tau + \int_{0}^{\infty} e^{-2\tau} d\tau$$

$$= \left.\frac{e^{2\tau}}{2}\right|_{-\infty}^{0} + \left.\frac{e^{-2\tau}}{-2}\right|_{0}^{\infty}$$

$$= 1 < \infty$$

$\therefore$ It is a stable system.

* <u>step response</u>:

DTS:

$$x(n) \rightarrow \boxed{\begin{array}{c} LTI \\ h(n) \end{array}} \rightarrow y(n)$$

$$y(n) = x(n) * h(n)$$
$$= h(n) * x(n)$$

$$y(n) = \sum_{k=-\infty}^{\infty} h(k)\, x(n-k)$$

if $x(n) = u(n)$

$y(n)$: step response $= s(n)$

$$\therefore s(n) = \sum_{k=-\infty}^{\infty} h(k)\, u(n-k)$$

$$u(n-k) = \begin{cases} 1 & ; (n-k) \geq 0 \\ 0 & ; \text{else} \end{cases}$$

$$\therefore \boxed{s(n) = \sum_{k=-\infty}^{n} h(k)}$$

similarly for CTS

$$\boxed{s(t) = \int_{-\infty}^{t} h(\tau)\, d\tau}$$

Q: Find the step response for the following:

1. $h(n) = (1/2)^n u(n)$

step response for DTS is given by

$$s(n) = \sum_{k=-\infty}^{n} h(k)$$

$$s(n) = \sum_{k=-\infty}^{n} \left[\frac{1}{2}\right]^k u(k)$$

$$s(n) = \sum_{k=0}^{n} \left[\frac{1}{2}\right]^k = \frac{(1/2)^0 - (1/2)^{n+1}}{1 - 1/2} = \frac{1 - (1/2)^{n+1}}{1/2} = \underline{2 - (1/2)^n}$$

2. $h(n) = u(n-1)$

step response

$$s(n) = \sum_{k=-\infty}^{n} u(k-1)$$

$$s(n) = \sum_{k=1}^{n} 1 = n-1+1 = n$$

3. $h(n) = u(n+2)$

step response

$$s(n) = \sum_{k=-\infty}^{n} u(k+2)$$

$$s(n) = \sum_{k=-2}^{n} 1 = n+2+1 = n+3$$

4. $h(t) = t\,u(t)$

step response

$$s(t) = \int_{-\infty}^{t} h(\tau)\,d\tau$$

$$s(t) = \int_{-\infty}^{t} \tau\,u(\tau)\,d\tau$$

$$s(t) = \int_{0}^{t} \tau\,d\tau = \frac{\tau^2}{2}\Big|_0^t = \frac{t^2}{2}$$

IMP

5. $h(t) = e^{-|t|} = e^{t}u(-t) + e^{-t}u(t)$

step response

$$s(t) = \int_{-\infty}^{t} h(\tau)\,d\tau$$

$$s(t) = \int_{-\infty}^{t} \left[e^{\tau}u(-\tau) + e^{-\tau}u(\tau)\right] d\tau$$

$$s(t) = \int_{-\infty}^{0} e^{\tau}\,d\tau + \int_{0}^{t} e^{-\tau}\,d\tau$$

$$s(t) = e^{\tau}\Big|_{-\infty}^{0} + -e^{-\tau}\Big|_{0}^{t}$$

$$s(t) = 1 - 0 - e^{-t} + 1 = \underline{2 - e^{-t}}$$

classmate

Date

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6. $h(t) = u(t+1) - u(t-1)$

step response

$$s(t) = \int_{-\infty}^{t} h(\tau)\, d\tau$$

$$s(t) = \int_{-\infty}^{t} [u(\tau+1) - u(\tau-1)]\, d\tau$$

$$s(t) = \int_{-1}^{t} d\tau - \int_{1}^{t} d\tau$$

$$s(t) = \tau \Big|_{-1}^{t} - \tau \Big|_{1}^{t} = t + 1 - t + 1 = 2$$

UNIT - 06

Z - Transform

NOTE:

- Fourier Transform

  continuous Time signal Fourier Tranform and Discrete Time signal Fourier Transform is possible. Only analysis of Energy or power signal can be done.

- Laplace Transform

  Only applicable for countinuous time signal. Here analysis of Energy, power or neither energy nor power is possible.

- Z - Transform

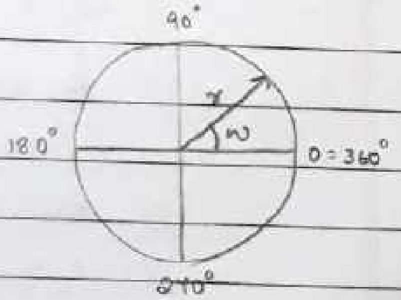
  Applicable for discrete time signal and analysis of energy, power or neither energy nor power signal is possible.

* Z - Transform:

$$x(n) \underset{IZT}{\overset{ZT}{\rightleftharpoons}} X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$: Bilateral Z-T

$-n \rightarrow$ integer

$z \rightarrow$ complex variable



$$x(n) \underset{IUZT}{\overset{UZT}{\rightleftharpoons}} \sum_{n=0}^{\infty} x(n) z^{-n}$$: Unilateral Z - T

ROC $\rightarrow$ $0 < |z| < \infty$

or $r_{min} < |z| < r_{min}$

Ex: Unilateral Z - Transform

$x(n) = u(n+2)$

$$\sum_{n=0}^{\infty} x(n) z^{-n}$$

$x(n) = u(n-2)$

$$\sum_{n=0}^{\infty} x(n) z^{-n}$$

Bilateral Z - Transform

$x(n) = u(n+2)$

$$\sum_{n=-2}^{\infty} x(n) z^{-2}$$

$x(n) = u(n-2)$

$$\sum_{n=2}^{\infty} x(n) z^{-n}$$

$x(n)$

Infinite signal.

- $x(n) = u(n)$
  (right sided signal)
- $x(n) = u(-n-1)$
  (left sided signal)
- $x(n) = u(n) + u(-n-1)$
  (Double sided signal)

Finite signals

- $x(n) = \{\underset{\uparrow}{1}, 2, 4, -5\}$
  (right sided signal)
- $x(n) = \{1, 2, 4, \underset{\uparrow}{-5}\}$
  (left sided signal)
- $x(n) = \{1, 2, \underset{\uparrow}{4}, -5\}$
  (double sided signal)

Q: For the following signal find the z-transform and indicate its ROC:

1. $x(n) = \{1, 2, 4, -5\}$

Sol: $x(n) = \{\underset{\uparrow}{1}, 2, 4, -5\}$

$n \rightarrow \quad 0 \quad 1 \quad 2 \quad 3$

z-transform

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

$$= \sum_{n=0}^{3} x(n) z^{-n}$$

$$= x(0) z^{0} + x(1) z^{-1} + x(2) z^{-2} + x(3) z^{-3}$$

$$= 1 + 2z^{-1} + 4z^{-2} - 5z^{-3}$$

$$\therefore X(z) = 1 + \frac{2}{z} + \frac{4}{z^2} - \frac{5}{z^3}$$

$0 < |z| < \infty$

$|z| > 0 \quad 1$ to ∞

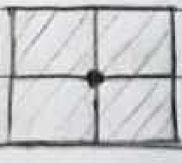
$|z| < \infty \quad 0$ to ∞ x

$z \to 0 \quad X(z) \to \infty$

$z \to \infty \quad X(z) \to 1$

ROC: $|z| > 0$ right sided signal

z_{im}



z_{re}

Entire z plane except $z = 0$

Property 1 of ROC

For a right sided signal ROC is the entire z-plane except $z = 0$

2. $x(n) = \{1, 2, 4, \underset{\uparrow}{-5}\}$

sol: $x(n) = \{1, 2, 4, -5\}$

$n \rightarrow \quad -3 \quad -2 \quad -1 \quad 0$

z - transform

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

$$= \sum_{n=-3}^{0} x(n) z^{-n}$$

$$= x(-3) z^{3} + x(-2) z^{2} + x(-1) z^{1} + x(0) z^{0}$$

$$\therefore X(z) = z^{3} + 2z^{2} + 4z - 5$$

ROC: $|z| < \infty$ left sided signal

Entire z plane except $z = \infty$

$0 < |z| < \infty$: double sided

$|z| > 0 \quad 1$ to ∞)

$|z| < \infty \quad 0$ to ∞

$z \rightarrow 0 \quad X(z) \rightarrow -5$

$z \rightarrow \infty \quad X(z) \rightarrow \infty$

Property 2 of ROC:

For a left sided signal ROC is the entire z plane except $z = \infty$

3. $x(n) = \{1, 2, \underset{\uparrow}{4}, -5\}$

sol: $x(n) = \{1, 2, 4, -5\}$

$n \rightarrow \quad -2 \quad -1 \quad 0 \quad 1$

z transform

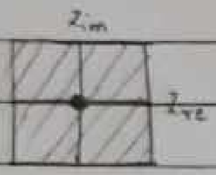
$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

$$= \sum_{n=-2}^{1} x(n) z^{-n}$$

$$= x(-2) z^{2} + x(-1) z^{1} + x(0) z^{0} + x(1) z^{-1}$$

$$\therefore X(z) = z^{2} + 2z + 4 - \frac{5}{z}$$

ROC: $0 < |z| < \infty$ double sided signal



Entire z plane except $z=0$ and $z=\infty$.

$0 < |z| < \infty$

$|z| > 0$ 1 to ∞

$|z| \leq \infty$ 0 to ∞

$z \to 0$ $X(z) \to \infty$

$z \to \infty$ $X(z) \to \infty$

Property 3 for ROC:

For a double sided signal ROC ~~is~~ in the entire z-plane except $z=0$ and $z=\infty$.

Infinite length

1. $x(n) = a^n u(n)$

sol: Z transform

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} a^n u(n) z^{-n}$$

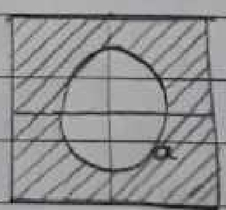
$$= \sum_{n=0}^{\infty} a^n z^{-n} = \sum_{n=0}^{\infty} \left(\frac{a}{z}\right)^n = \sum_{n=0}^{\infty} (az^{-1})^n$$

$$= \frac{(az^{-1})^0}{1-az^{-1}} = \frac{(a/z)^0}{1-a/z} = \frac{z}{z-a}$$

ROC: $|az^{-1}| < 1$

$|a/z| < 1$

$|z| > a$



$\therefore$ for $x(n) = a^n u(n)$

ZT: $X(z) = \dfrac{z}{z-a}$

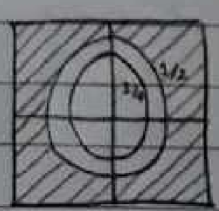
2. $x(n) = \left(\frac{1}{2}\right)^n u(n) + \left(\frac{1}{4}\right)^n u(n)$

sol: Z transform

$$X(z) = \frac{z}{z - 1/2} + \frac{z}{z - 1/4}$$

ROC: $|z| > r_{max}$

$|z| > 1/2$



3. $x(n) = -a^n u(-n-1)$

sol: z transform

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} -a^n u(-n-1) z^{-n}$$

$$= -\sum_{n=-\infty}^{-1} a^n z^{-n} = -\sum_{n=-1}^{-\infty} a^n z^{-n} = -\sum_{n=1}^{\infty} a^{-n} z^n$$

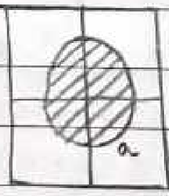
$$= -\sum_{n=1}^{\infty} (a^{-1}z^1)^n = -\frac{(a^{-1}z)^1}{1-(a^{-1}z)} = \frac{-z/a}{(a-z)/a} = \frac{z}{z-a}$$

$\therefore$ $x(n) = -a^n u(-n-1)$

ZT : $X(n) = \dfrac{z}{z-a}$

ROC: $|a^{-1}z| < 1$

$|z| < a$



4. $x(n) = (1/2)^n u(-n-1) + (1/4)^n u(-n-1)$

sol: z transform

$$X(z) = \frac{z}{z-1/2} + \frac{z}{z-1/4}$$

ROC 1

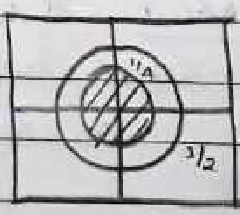
$|z| < r_{max}$

$|z| < 1/2$

ROC 2

$|z| > r_{min}$

$|z| > 1/4$



5. $x(n) = a^n u(n) - b^n u(-n-1)$

sol: z transform.

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} \left(a^n u(n) - b^n u(-n-1)\right) z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} a^n u(n) z^{-n} - \sum_{n=-\infty}^{\infty} b^n u(-n-1) z^{-n}$$

$$= \sum_{n=0}^{\infty} a^n z^{-n} - \sum_{n=-\infty}^{-1} b^n z^{-n}$$

$$= \sum_{n=0}^{\infty} (az^{-1})^n - \sum_{n=1}^{\infty} (b^{-1}z)^n$$

$$= \frac{(az^{-1})^0}{1-(az^{-1})} - \frac{(b^{-1}z)^1}{1-(b^{-1}z)}$$

$$= \frac{1}{(z-a)/z} - \frac{z/b}{(b-z)b}$$

$$= \frac{z}{z-a} + \frac{z}{z-b}$$

$$= \frac{(z-b)z + (z-a)z}{(z-a)(z-b)}$$

$$= \frac{z^2 - zb + z^2 - za}{(z-a)(z-b)}$$

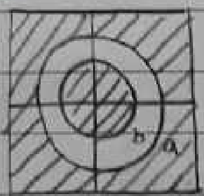
ROC 1: $|z| > a$ ROC 2: $|z| < b$

$$= \frac{2z^2 - z(a+b)}{(z-a)(z-b)}$$

Case 1: If $a > b$

ROC = 0

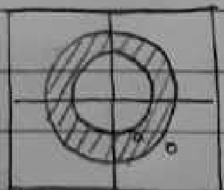
No intersection.



Case 2: If $a < b$

ROC: $a < |z| < b$

$r_{min} < |z| < b$



6. $x(n) = a^n u(n) - b^{-n} u(-n)$

Sol: Z transform

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} \left(a^n u(n) - b^{-n} u(-n)\right) z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} a^n u(n) z^{-n} - \sum_{n=-\infty}^{\infty} b^{-n} u(-n) z^{-n}$$

$$= \sum_{n=0}^{\infty} a^n z^{-n} - \sum_{n=-\infty}^{0} b^{-n} z^{-n}$$

$$= \sum_{n=0}^{\infty} (az^{-1})^n - \sum_{n=0}^{\infty} (bz)^n$$

$$= \frac{(az^{-1})^0}{1-(az^{-1})} - \frac{(bz)^0}{1-bz}$$

$$= \frac{z}{z-a} - \frac{1}{1-bz}$$

$$= \frac{z}{z-a} + \frac{1}{bz-1}$$

$$= \frac{z(bz-1) + z-a}{(z-a)(bz-1)} = \frac{z^2b + z - \cancel{z} - a}{(z-a)(bz-1)} = \frac{z^2b - a}{(z-a)(bz-1)}$$

ROC 1 $\quad$ ROC 2

$|z| > a \quad |z| < 1/b$

Q: Find the z transforms.

1. $x(n) = \delta(n)$

sol: z transform

$$x(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

$$\delta(n) = \begin{cases} 1 & ; \text{ at } n=0 \\ 0 & ; \text{ else} \end{cases}$$

$$= \sum_{n=-\infty}^{\infty} \delta(n) z^{-n}$$

$$= 1 \left| z^{-n} \right|_{n=0} = z^0 = 1$$

2. $x(n) = \delta(n-5)$

sol: $$\delta(n-5) = \begin{cases} 1 & ; \text{ at } n=5 \\ 0 & ; \text{ else} \end{cases}$$

z - transform

$$x(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} \delta(n-5) z^{-n} = 1 \left| z^{-n} \right|_{n=5} = z^{-5}$$

3. $x(n) = 10\,\delta(n+10)$

Sol: $\delta(n+10) = \begin{cases} 1 & ; \text{ at } n=-10 \\ 0 & ; \text{ else} \end{cases}$

Z - transform

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} 10\,\delta(n+10) z^{-n}$$

$$= 10(1) \left| z^{-n} \right|_{n=-10} = 10\, z^{10}$$

★ Linearity Property:

$$a x_1(n) + b x_2(n) = a X_1(z) + b X_2(z)$$

ROC1 ROC2 ROC = ROC1 $\cap$ ROC2

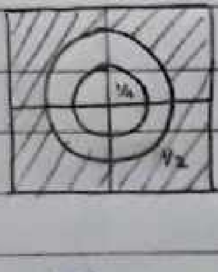
Q1: $x(n) = \left(\frac{1}{2}\right)^n u(n) + \left(\frac{1}{4}\right)^n u(n)$

Sol: Z transform

$$X(z) = \frac{z}{z - 1/2} + \frac{z}{z - 1/4}$$

ROC1: $|1/2\, z^{-1}| < 1$
$|2z| > 1$
$|z| > 1/2$

ROC2: $|1/4\, z^{-1}| < 1$
$|4z| > 1$
$|z| > 1/4$



$\therefore$ $|z| > 1/2$ is the required ROC.

IMP

Q2: $x(n) = \alpha^{|n|} = \alpha^{-n} u(-n-1) + \alpha^n u(n)$

Sol: Z transform

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

$$X(z) = \sum_{n=-\infty}^{\infty} \alpha^{-n} u(-n-1) z^{-n} + \sum_{n=-\infty}^{\infty} \alpha^{n} u(n) z^{-n}$$

$$= \sum_{n=-\infty}^{-1} \alpha^{-n} z^{-n} + \sum_{n=0}^{\infty} \alpha^{n} z^{-n}$$

$$= \sum_{n=1}^{\infty} (\alpha z)^{n} + \sum_{n=0}^{\infty} (\alpha z^{-1})^{n}$$

$$= \frac{(\alpha z)^{1}}{1-\alpha z} + \frac{(\alpha z^{-1})^{0}}{1-\alpha z^{-1}}$$

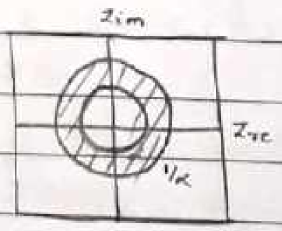
$$= \frac{\alpha z}{1-\alpha z} + \frac{z}{z-\alpha}$$

$$= \frac{\cancel{\alpha z^2} - \alpha^2 z + z \cancel{- \alpha z^2}}{(1-\alpha z)(z-\alpha)}$$

$$X(z) = \frac{z(1-\alpha^2)}{(z-\alpha)(1-\alpha z)}$$

$r_{max} = 1/\alpha \quad r_{min} = \alpha$

$\therefore$ ROC : $\alpha < |z| < 1/\alpha$



* **Differential Property:**

$$x(n) \xrightarrow{ZT} X(z) \longrightarrow \text{ROC } 1$$

$$n x(n) \xrightarrow{ZT} -z \frac{d}{dz} X(z) \longrightarrow \text{ROC } 1$$

Q1: $x(n) = n a^n u(n)$

Sol: Let $x_1(n) = a^n u(n)$

ZT of $x_1(n)$

$$X_1(z) = \frac{z}{z-a} \qquad \text{ROC : } |z| > a$$

ZT of $x(n)$

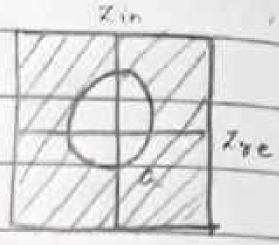
$$X(z) = -z \frac{d}{dz} X_1(z)$$

$$= -z \frac{d}{dz}\left[\frac{z}{z-a}\right] = -z\left[\frac{(z-a)(1) - z(1)}{(z-a)^2}\right] = \frac{az}{(z-a)^2}$$

$x(n) = na^n u(n)$

ZT : $X(z) = \dfrac{az}{(z-a)^2}$

ROC : $|z| > a$



Q2: $x(n) = n^2 a^n u(n) = n \cdot na^n u(n)$

sol: Let $x_1(n) = a^n u(n)$

ZT of $x_1(n)$

$X_1(z) = \dfrac{z}{z-a}$

Let $x_2(n) = na^n u(n)$

ZT of $x_2(n)$

$X_2(z) = -z \dfrac{d}{dz} X_1(z)$

$= -z \dfrac{d}{dz} \left[\dfrac{z}{z-a} \right]$

$= -z \left[\dfrac{z-a-z}{(z-a)^2} \right] = \dfrac{az}{(z-a)^2}$

ZT of $x(n)$

$X(z) = -z \dfrac{d}{dz} X_2(z)$

$= -z \dfrac{d}{dz} \left[\dfrac{az}{(z-a)^2} \right]$

$= -z \left[\dfrac{(z-a)^2(a) - az(2(z-a))}{(z-a)^4} \right]$

$= -z \left[\dfrac{a(z-a) - 2az}{(z-a)^3} \right]$

$= -z \left[\dfrac{za - a^2 - 2za}{(z-a)^3} \right] = -z \left[\dfrac{-a^2 - za}{(z a)^3} \right]$

$= \dfrac{a^2 z + z^2 a}{(z-a)^3}$ ROC : $|z| > a$

Q3: $x(n) = n(n+1)a^n u(n)$

Sol: $n(n+1)a^n u(n)$

$= (n^2+n)a^n u(n)$

$= n^2 a^n u(n) + n\, a^n u(n)$

z transform

$$X(z) = \frac{a^2z + z^2a}{(z-a)^3} + \frac{az}{(z-a)^2} \quad \text{(from previous problem)}$$

$$= \frac{(a^2z + z^2a) + az(z-a)}{(z-a)^3}$$

$$= \frac{\cancel{a^2z} + z^2a + az^2 - \cancel{a^2z}}{(z-a)^3}$$

$$= \frac{2az^2}{(z-a)^3} \quad // \qquad \text{ROC: } |z| > a$$

* convolution Property

$x(n)/X(z)$ →	LTI $h(n)/H(z)$	→ $y(n)/Y(z)$

$h(n)$: impulse response

$H(z)$: system / transfer function.

$y(n) = x(n) * h(n)$

$$H(z) = \frac{Y(z)}{X(z)}$$

$Y(z) = X(z) \cdot H(z)$

$x(n) \xrightarrow{ZT} X(z) \longrightarrow \text{ROC1}$

$h(n) \xrightarrow{ZT} H(z) \longrightarrow \text{ROC2}$

$$\underbrace{x(n) * h(n)}_{y(n)} \xrightarrow{ZT} \underbrace{X(z) \cdot H(z)}_{Y(z)} \longrightarrow \text{ROC : ROC1} \cap \text{ROC2}$$

Q1: Find the convolution between the

$x(n) = n(1/2)^n u(n)$ and $h(n) = (1/4)^n u(n)$

using z transforms.

Sol: $x(n) = n(1/2)^n u(n)$

Let $x_1(n) = (1/2)^n u(n)$

ZT of $x_1(n)$

$$X_1(z) = \frac{z}{z - 1/2}$$

ZT of $x(n)$

$$X(z) = -z \frac{d}{dz} X_1(z)$$

$$= -z \frac{d}{dz}\left[\frac{z}{z-1/2}\right]$$

$$= -z\left[\frac{(z-1/2)-z}{(z-1/2)^2}\right] = \frac{z/2}{(z-1/2)^2}$$

$h(n) = (1/4)^n u(n)$

ZT of $h(n)$

$$H(z) = \frac{z}{z-1/4}$$

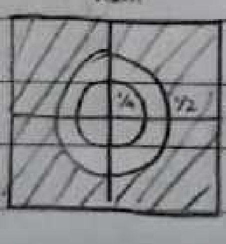
$\therefore Y(z) = X(z) \cdot H(z)$

$$= \frac{z/2}{(z-1/2)^2} \cdot \frac{z}{(z-1/4)}$$

$$= \frac{1}{2} \frac{z^2}{(z-1/2)^2 (z-1/4)}$$

ROC 1: $|z| > 1/2$ ROC 2: $|z| > 1/4$

$\therefore$ ROC $= |z| > 1/2$



Q2: $x(n) = \begin{cases} n+1 & ; 0 \le n \le 2 \\ 0 & ; \text{else} \end{cases}$

$h(n) = \delta(n) + 3\delta(n-2) - 2\delta(n-3)$

Sol: $x(n) = \{1, 2, 3\}$
$n \rightarrow 0 \quad 1 \quad 2$

$h(n) = \{1, 0, 3, -2\}$
$n \rightarrow 0 \quad 1 \quad 2 \quad 3$

z transform

$X(z) = 1 + 2z^{-1} + 3z^{-2}$ ROC 1 : $|z| > 0$

$H(z) = 1 + 0 + 3z^{-2} - 2z^{-3}$ ROC 2 : $|z| > 0$

We know that

$Y(z) = X(z) \cdot H(z)$

$\therefore Y(z) = (1 + 2z^{-1} + 3z^{-2})(1 + 3z^{-2} - 2z^{-3})$

$= 1 + 3z^{-2} - 2z^{-3} + 2z^{-1} + 6z^{-3} - 4z^{-4} + 3z^{-2} + 9z^{-4} - 6z^{-5}$

$= \underline{1 + 2z^{-1} + 6z^{-2} + 4z^{-3} + 5z^{-4} - 6z^{-5}}$

NOTE:

$x(n) = a^n \sin \Omega_0 n \; u(n) \xrightarrow{ZT} \dfrac{a \sin \Omega_0 z}{z^2 - 2a\cos\Omega_0 z + a^2}$

$x(n) = a^n \cos \Omega_0 n \; u(n) \xrightarrow{ZT} \dfrac{z^2 - a\cos\Omega_0 z}{z^2 - 2a\cos\Omega_0 z + a^2}$

* $x(n) \xrightarrow{ZT} X(z)$: ROC   Time reversal property
$x(-n) \xrightarrow{ZT} X(1/z)$: 1/ROC

Q1: $x(n) = (1/3)^{-n} u(-n)$

Sol: Let $x_1(n) = (1/3)^n u(n)$

ZT of $x_1(n)$

$X_1(z) = \dfrac{z}{z - 1/3}$

ROC = $|z| > 1/3$

ZT of $x(n)$

$X(z) = X_1(1/z) = \dfrac{z^{-1}}{z^{-1} - 1/3}$

ROC : $|1/z| > 1/3 = |z| < 3$

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Q2: $x(n) = 2 \cdot 2^{n} u(-n)$

Sol: $x(n) = 2 \cdot \frac{1}{2^{-n}} u(-n) = 2 \left(\frac{1}{2}\right)^{-n} u(-n)$

Let $x_1(n) = \left(\frac{1}{2}\right)^{n} u(n)$

ZT of $x_1(n)$

$X_1(z) = \frac{z}{z - 1/2}$

ZT of $x_1(-n)$

$X_1(1/z) = \frac{z^{-1}}{z^{-1} - 1/2}$

$\therefore$ ZT of $x(n) = X(n) = \frac{2z^{-1}}{z^{-1} - 1/2}$ //

Q3: $x(n) = n \sin(n\pi/2) u(-n)$

Sol: $x(n) = -n \sin\left(\frac{\pi}{2}(-n)\right) u(-n)$

Let $x_1(n) = \sin(n\pi/2) u(n)$

ZT of $x_1(n)$ Here $a = 1$; $\Omega_0 = \pi/2$

$X_1(z) = \frac{\sin \pi/2 \, z}{z^2 - \cos \pi/2 \, z + 1} = \frac{z}{z^2+1}$

Let $x_2(n) = n \sin(n\pi/2) u(n)$

ZT of $x_2(n)$

$X_2(z) = -z \frac{d}{dz}\left[\frac{z}{z^2+1}\right]$

$= -z\left[\frac{z^2+1-2z^2}{(z^2+1)^2}\right] = \frac{z^3 - z}{(z^2+1)^2}$ /

$\therefore$ ZT of $x(n)$

$X(z) = \frac{z^{-3} - z^{-1}}{(z^{-2}+1)^2}$ //

* **Time Shifting Property:**

$x(n) \xrightarrow{ZT} X(z)$: ROC1

$x(n-k) \xrightarrow{ZT} z^{-k}X(z)$: ROC1

$x(n+k) \xrightarrow{ZT} z^{k}X(z)$: ROC1

Q1: $x(n) = (1/3)^{n-3} u(n-3)$

sol: Let $x_1(n) = (1/3)^n u(n)$

ZT of $x_1(n)$

$$X_1(z) = \frac{z}{z-1/3}$$

ZT of $x(n)$

$$X(z) = z^{-3} X_1(z)$$

$$= z^{-3}\frac{z}{z-1/3} = \frac{1}{z^2(z-1/3)}$$

Q2: $x(n) = (n-3)(1/2)^{n-3} \sin\Omega(n-3)\, u(n-3)$

sol: Let $x_1(n) = (1/2)^n \sin\Omega n\, u(n)$

ZT of $x_1(n)$

$$X_1(z) = \frac{1/2 \sin\Omega z}{z^2 - \cos\Omega z + 1/4}$$

Let $x_2(n) = n\,(1/2)^n \sin\Omega n\, u(n)$

ZT of $x_2(n)$

$$X_2(z) = -z\frac{d}{dz}X_1(z)$$

$$= \frac{-z}{2}\frac{d}{dz}\left[\frac{\sin\Omega z}{z^2-\cos\Omega z+1/4}\right]$$

$$= \frac{-z}{2}\left[\frac{\Omega(z^2-\cos\Omega z+1/4)\cos\Omega z - \sin\Omega z(2z+\Omega\sin\Omega z)}{(z^2-\cos\Omega z+1/4)^2}\right]$$

$\therefore$ ZT of $x(n)$

$$X(z) = z^{-3} X_2(z)$$

$$= \frac{-1}{2z^2}\left[\frac{\Omega\cos\Omega z(z^2-\cos\Omega z+1/4) - \sin\Omega z(2z+\Omega\sin\Omega z)}{(z^2-\cos\Omega z+1/4)^2}\right]$$

Q3: $x(n) = n\left(\frac{1}{2}\right)^n \sin\Omega(n-3)\,u(n-3)$

Sol: Let $x_1(n) = \left(\frac{1}{2}\right)^n \sin\Omega n\, u(n)$

ZT of $x_1(n)$

$$X_1(z) = \frac{\frac{1}{2}\sin\Omega \cancel{z}}{z^2 - \cos\Omega z + \frac{1}{4}}$$

Let $x_2(n) = \left(\frac{1}{2}\right)^n \sin\Omega(n-3)\,u(n-3)$

ZT of $x_2(n)$

$$X_2(z) = \frac{z^{-3}}{2}\left[\frac{\sin\Omega z}{z^2 - \cos\Omega z + \frac{1}{4}}\right]$$

$\therefore$ ZT of $x(n)$

$$X(z) = -z\frac{d}{dz}X_2(z)$$

$$= -z\frac{d}{dz}\left[\frac{1}{2z^3}\left(\frac{\sin\Omega z}{z^2 - \cos\Omega z + \frac{1}{4}}\right)\right]$$

$$= \frac{-z}{2}\frac{d}{dz}\left[\frac{\sin\Omega z}{z^5 - z^3\cos\Omega z + \frac{1}{4}}\right]$$

$$= \frac{-z}{2}\left[\frac{(z^5 - z^3\cos\Omega z + \frac{1}{4})\Omega\cos\Omega z - \sin\Omega z(5z^4 - 3z^2\cos\Omega z + z^3\Omega\sin\Omega z)}{(z^5 - z^3\cos\Omega z + \frac{1}{4})^2}\right]$$

$$= \frac{-z}{2}\left[\frac{(z^5 + \frac{1}{4})\Omega\cos\Omega z - z^3\Omega\cos^2\Omega z - (5z^4 - 3z^2\cos\Omega z)\sin\Omega z - z^3\Omega\sin^2\Omega z}{(z^5 - z^3\cos\Omega z + \frac{1}{4})^2}\right]$$

$$= \frac{z}{2}\left[\frac{(z^5 + \frac{1}{4})\Omega\cos\Omega z - (5z^4 - 3z^2\cos\Omega z)\sin\Omega z}{(z^5 - z^3\cos\Omega z + \frac{1}{4})^2}\right]$$

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Q4: $x(n) = (n-2)a^{n-2}u(n-2)$

sol: Let $x_1(n) = a^n u(n)$

ZT of $x_1(n)$

$$X_1(z) = \frac{z}{z-a}$$

Let $x_2(n) = n a^n u(n)$

ZT of $x_2(n)$

$$X_2(z) = -z\frac{d}{dz} X_1(z)$$

$$= -z\frac{d}{dz}\left[\frac{z}{z-a}\right]$$

$$= -z\left[\frac{(z-a)-z}{(z-a)^2}\right] = \frac{az}{(z-a)^2}$$

$\therefore$ ZT of $x(n)$

$$X(z) = z^{-2} X_2(z)$$

$$= z^{-2}\frac{az}{(z-a)^2} = \frac{a}{z(z-a)^2} \quad : |z| > a$$

Q5: $x(n) = n(1/3)^{n-2}u(n-2)$

sol: Let $x_1(n) = (1/3)^n u(n)$

ZT of $x_1(n)$

$$X_1(z) = \frac{z}{z-1/3}$$

Let $x_2(n) = (1/3)^{n-2}u(n-2)$

ZT of $x_2(n)$

$$X_2(z) = z^{-2}\frac{z}{z-1/3} = \frac{1}{z(z-1/3)}$$

ZT of $x(n)$

$$X(n) = -z\frac{d}{dz} X_2(z)$$

$$= -z\frac{d}{dz}\left[\frac{1}{z(z-1/3)}\right]$$

$$= -z \left[\frac{-[z-1/3] - z}{z^2(z-1/3)^2} \right]$$

$$= \frac{2z + 1/3}{z(z-1/3)^2}$$

Q6: $x(n) = 2^n \sin \Omega_0 (n-2) u(n-2)$

$= 2^2 \, 2^{n-2} \sin \Omega_0 (n-2) u(n-2)$

Sol: Let $x_1(n) = 2^n \sin \Omega_0 n \; u(n)$

ZT of $x_1(n)$

$$X_1(z) = \frac{2 \sin \Omega_0 z}{z^2 - 4 \cos \Omega_0 z + 4}$$

ZT of $x(n)$

$$X(z) = 2^2 z^{-2} X_1(z)$$

$$= \frac{8 \sin \Omega_0 z}{z^2(z^2 - 4\cos\Omega_0 z + 4)}$$

Q7: $x(n) = a^n \cos \Omega_0 n \; u(n)$

Sol: Z Transform

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

$$= \sum_{n=-\infty}^{\infty} a^n \cos \Omega_0 n \; u(n) z^{-n}$$

$$= \sum_{n=0}^{\infty} a^n \cos \Omega_0 n \; z^{-n}$$

$$= \sum_{n=0}^{\infty} \left[a^n \left(\frac{e^{j\Omega_0 n} + e^{-j\Omega_0 n}}{2} \right) z^{-n} \right]$$

$$= \frac{1}{2} \left[\sum_{n=0}^{\infty} a^n e^{j\Omega_0 n} z^{-n} + \sum_{n=0}^{\infty} a^n e^{-j\Omega_0 n} z^{-n} \right]$$

$$= \frac{1}{2} \left[\sum_{n=0}^{\infty} (a e^{j\Omega_0} z^{-1})^n + \sum_{n=0}^{\infty} (a e^{-j\Omega_0} z^{-1})^n \right]$$

$$= \frac{1}{2} \left[\frac{z}{z - a e^{j\Omega_0}} + \frac{z}{z - a e^{-j\Omega_0}} \right]$$

$$= \frac{1}{2}\left[\frac{z^2 - az e^{-j\Omega_0} + z^2 - az e^{j\Omega_0}}{(z-ae^{j\Omega_0})(z-ae^{-j\Omega_0})}\right]$$

$$= \frac{1}{2}\left[\frac{2z^2 - az(e^{j\Omega_0}+e^{-j\Omega_0})}{z^2 - aze^{j\Omega_0} - aze^{-j\Omega_0} + a^2}\right]$$

$$= \frac{z^2 - a\cos\Omega_0 z}{z^2 - a\cos\Omega_0 z + a^2} \quad : |z| > a$$

* **Inverse z transform:**

$$H(z) = \frac{\sum_{k=0}^{M} b_k z^{-k}}{1 + \sum_{k=1}^{N} a_k z^{-k}}$$

$$= \frac{b_0 + b_1 z^{-1} + b_2 z^{-2} + \dots + b_M z^{-M}}{1 + a_1 z^{-1} + a_2 z^{-2} + \dots + a_N z^{-N}} \quad \begin{array}{l} : \text{zeroes} \\ : \text{poles} \end{array}$$

if $M < N$: partial fractions

$M \geq N$: long division.

Poles: $(1-x_1 z^{-1})(1-x_2 z^{-1})(1-x_3 z^{-1}) \dots (1-x_N z^{-1})$

· **NOTE:** z transforms of some standard functions

	$x(n)$	$X(z)$	
1.	$\delta(n)$	1	
2.	$k'\delta(n-k)$	$k' z^{-k}$	
3.	$\delta(n+k)$	z^{k}	
4.	$a^n u(n)$	$\frac{z}{z-a}$	Causal: $\lvert z\rvert > r_{max}$
5.	$-a^n u(-n-1)$	$\frac{z}{z-a}$	Noncausal: $\lvert z\rvert < r_{min}$

Stable: $r_{min} < |z| < r_{max}$

$-a^n u(n) \pm b^n u(-n-1)$

6.	$na^n u(n)$	$\frac{za}{(z-a)^2}$

Q1: $X(z) = \dfrac{5z^{-1} - 1}{1 - \frac{3}{2}z^{-1} + \frac{1}{2}z^{-2}}$

Find $x(n)$ for:

i. causal   ii. Non causal   iii. stable

Sol: To find poles.

$1 - \frac{3}{2}z^{-1} + \frac{1}{2}z^{-2}$: $P_1 = 1$ and $P_2 = 1/2$

Here $M < N$: Partial fraction

$$\frac{5z^{-1} - 1}{1 - \frac{3}{2}z^{-1} + \frac{1}{2}z^{-2}} = \frac{A}{1 - z^{-1}} + \frac{B}{1 - z^{-1}/2}$$

$$5z^{-1} - 1 = A(1 - z^{-1}/2) + B(1 - z^{-1})$$

at $z^{-1} = 2$: $5(2) - 1 = B(-1)$

$$B = -9$$

at $z^{-1} = 1$: $5 - 1 = A(1/2)$

$$A = 8$$

$$\therefore \frac{5z^{-1} - 1}{1 - \frac{3}{2}z^{-1} + \frac{1}{2}z^{-2}} = \frac{8}{1 - z^{-1}} - \frac{9}{1 - z^{-1}/2}$$

Taking inverse Z transform

CASE 1: Causal : $|z| > r_{max} \Rightarrow |z| > 1$ right sided signal

$\therefore x(n) = 8(1)^n u(n) - 9(1/2)^n u(n)$

CASE 2: Non causal : $|z| < r_{min} \Rightarrow |z| < 1/2$ left sided signal

$x(n) = -8(1)^n u(-n-1) + 9(1/2)^n u(-n-1)$

CASE 3: Stable : $r_{min} < |z| < r_{max} \Rightarrow 1/2 < |z| < 1$

$x(n) = -8(1)^n u(-n-1) - 9(1/2)^n u(n)$

Q2: $X(z) = \dfrac{1 - z^{-1} + z^{-2}}{(1-z^{-1})(1-\frac{1}{2}z^{-1})(1-2z^{-1})}$

Find ~~for~~ $x(n)$ for :

i. causal   ii. Non causal   iii. stable.

sol: To find poles

$P_1 = 1 \quad P_2 = 1/2 \quad P_3 = 2$

Here $M < N$: Partial fraction

$$\frac{1 - z^{-1} + z^{-2}}{(1-z^{-1})(1-\frac{1}{2}z^{-1})(1-2z^{-1})} = \frac{A}{(1-z^{-1})} + \frac{B}{(1-\frac{1}{2}z^{-1})} + \frac{C}{(1-2z^{-1})}$$

$$1 - z^{-1} + z^{-2} = A(1-\tfrac{1}{2}z^{-1})(1-2z^{-1}) + B(1-z^{-1})(1-2z^{-1}) + C(1-z^{-1})(1-\tfrac{1}{2}z^{-1})$$

at $z^{-1} = 1$: $1 - 1 + 1 = A(\frac{1}{2})(-1)$

$\therefore A = -2$ //

at $z^{-1} = 2$: $1 - 2 + 4 = B(-1)(-3)$

$\therefore B = 1$ //

at $z^{-1} = 1/2$: $1 - \frac{1}{2} + \frac{1}{4} = C(\frac{1}{2})(\frac{3}{4})$

$\therefore C = 2$ //

$$\therefore X(z) = \frac{-2}{1-z^{-1}} + \frac{1}{1-\frac{1}{2}z^{-1}} + \frac{2}{1-2z^{-1}}$$

Taking inverse z transform

CASE 1 : Causal : $|z| > r_{max}$

$|z| > 2$

$\therefore x(n) = -2(1)^n u(n) + 1(\frac{1}{2})^n u(n) + 2(2)^n u(n)$ //

CASE 2 : Noncausal : $|z| < r_{min}$

$|z| < 1$

$\therefore x(n) = 2(1)^n u(-n-1) - 1(\frac{1}{2})^n u(-n-1) - 2(2)^n u(-n-1)$ //

CASE 3 : Stable : $r_{min} < |z| < r_{max}$

$1 < |z| < 2$

$\therefore x(n) = 2(1)^n u(-n-1) + 1(\frac{1}{2})^n u(n) + 2(2)^n u(n)$ //

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Q3: $X(z) = \dfrac{4-3z^{-1}+3z^{-2}}{(1+2z^{-1})(1-3z^{-1})^2}$

Find $x(n)$ for

i. causal ii. Noncausal iii. stable.

Sol: Here $M < N$: partial fraction

$$\frac{4-3z^{-1}+3z^{-2}}{(1+2z^{-1})(1-3z^{-1})^2} = \frac{A}{(1+2z^{-1})} + \frac{B}{(1-3z^{-1})} + \frac{C}{(1-3z^{-1})^2}$$

$$4-3z^{-1}+3z^{-2} = A(1-3z^{-1})^2 + B(1+2z^{-1})(1-3z^{-1}) + C(1+2z^{-1})$$

at $z^{-1} = -1/2$: $4+3/2+3/4 = A(1+3/2)^2$

$A = 25/4 / 25/4 = 1$

at $z^{-1} = 1/3$: $4-1+1/3 = C(1+2/3)$

$C = 10/3 / 5/3 = 2$

at $z^{-1} = 0$: $4 = A+B+C$

$4 = 1+B+2 \Rightarrow B = 1$

$$\therefore X(z) = \frac{1}{(1+2z^{-1})} + \frac{1}{(1-3z^{-1})} + \frac{2z\,z^{-1}}{3(1-3z^{-1})^2}$$

Taking inverse z transform

CASE 1: causal: $|z| > r_{max}$

$|z| > 3$

$$\therefore x(n) = (-2)^n u(n) + (3)^n u(n) + \frac{2}{3}(3)^{n+1}(n+1)u(n+1)$$

CASE 2: Non causal: $|z| < r_{min}$

$|z| < -2$

$$\therefore x(n) = -(-2)^n u(-n-1) - (3)^n u(n-1) - \frac{2}{3}(3)^{n+1}(-n+1)u(n+1)$$

CASE 3: Stable: $r_{min} < |z| < r_{max}$

$-2 < |z| < 3$

$$\therefore x(n) = (-2)^n u(n) - (3)^n u(-n-1) - \frac{2}{3}(3)^{n+1}(-n+1)u(-n+1)$$

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Q4: $X(z) = \dfrac{16z^2 - 4z + 1}{8z^2 + 2z - 1}$ $|z| > 1/2$

Find $x(n)$

sol: $X(z) = \dfrac{16 - 4z^{-1} + z^{-2}}{8 + 2z^{-1} - z^{-2}}$

Here M = N : long division is employed.

$X(z) = \dfrac{2 - 1/2\, z^{-1} + 1/8\, z^{-2}}{1 + 1/4\, z^{-1} - 1/8\, z^{-2}}$

$$\begin{array}{r|l} & \quad -1 \\ -1/8z^{-2} + 1/4z^{-1} + 1 & \cancel{1/8z^{-2}} - 1/8z^{-1} + 2 \\ & \cancel{1/8z^{-2}} - 1/4z^{-1} - 1 \\ & - \quad\quad + \quad\quad + \\ \hline & \quad 1/4z^{-1} + 3 \end{array}$$

$X(z) = -1 + \dfrac{1/4\, z^{-1} + 3}{-1/8z^{-2} + 1/4z^{-1} + 1}$

Let $X_1(z) = \dfrac{1/4\, z^{-1} + 3}{-1/8z^{-2} + 1/4z^{-1} + 1}$

Poles are $P_1 = -1/2$ $P_2 = 1/4$

$X_1(z) = \dfrac{1/4\, z^{-1} + 3}{(1 + 1/2\, z^{-1})(1 - 1/4\, z^{-1})} = \dfrac{A}{(1 + 1/2\, z^{-1})} + \dfrac{B}{(1 - 1/4\, z^{-1})}$

$1/4\, z^{-1} + 3 = A(1 - 1/4\, z^{-1}) + B(1 + 1/2\, z^{-1})$

at $z^{-1} = 4$: $1 + 3 = B(1 + 2)$

$B = 4/3$

at $z^{-1} = -2$: $-1/2 + 3 = A(1 + 1/2)$

$A = 5/2 \,/\, 3/2 = 5/3$ //

$X_1(z) = \dfrac{5}{3(1 + 1/2\, z^{-1})} + \dfrac{4}{3(1 - 1/4\, z^{-1})}$

$\therefore X(z) = -1 + \dfrac{5}{3(1 + 1/2\, z^{-1})} + \dfrac{4}{3(1 - \frac{1}{4} z^{-1})}$

By taking inverse Z transform :

$$x(n) = -\delta(n) + (5/3)(-1/2)^n u(n) + (4/3)(1/4)^n u(n)$$

Q: An LTI system characterised by system function

$$H(z) = \frac{3-4z^{-1}}{1-7/2z^{-1}+3/2z^{-2}} \quad ; \text{ Find :}$$

i. the impulse response of the system for
   a. causal   b. non-causal   c. stable

ii. Check whether the system is stable or not.

iii. the frequency response of the system.

Sol: i. $$H(z) = \frac{3-4z^{-1}}{1-7/2z^{-1}+3/2z^{-2}} = \frac{A}{1-3z^{-1}} + \frac{B}{1-1/2z^{-1}}$$

$$3-4z^{-1} = A(1-1/2z^{-1}) + B(1-3z^{-1})$$

at $z=2$: $3-8 = B(1-6)$ $\therefore B = 1$

at $z=1/3$: $3-4/3 = A(1-1/6)$ $\therefore A = 2$

$$\therefore H(z) = \frac{2}{1-3z^{-1}} + \frac{1}{1-1/2z^{-1}}$$

By taking inverse Z transform

a. CASE 1 : For causal : $|z| > r_{max}$: $|z| > 3$

$$h(n) = 2(3)^n u(n) + (1/2)^n u(n)$$

b. CASE 2 : Non causal : $|z| < r_{min}$: $|z| < 1/2$

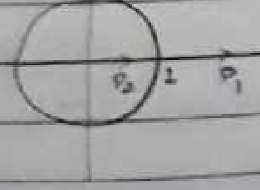
$$h(n) = -2(3)^n u(-n-1) - (1/2)^n u(-n-1)$$

c. CASE 3 : Stable : $r_{min} < |z| < r_{max}$: $1/2 < |z| < 3$

$$h(n) = -2(3)^n u(-n-1) + (1/2)^n u(n)$$

ii. System stability

$$H(z) = \frac{3-4z^{-1}}{1-7/2z^{-1}+3/2z^{-2}} \quad \begin{matrix} : \text{zeros} \\ : \text{poles} \end{matrix}$$



$$1-7/2z^{-1}+3/2z^{-2} = 0$$

$$P_1 = 3 \quad P_2 = 1/2$$

ii. $H(z) = \dfrac{3-4z^{-1}}{1-\frac{7}{2}z^{-1}+\frac{3}{2}z^{-2}} = \dfrac{z^{-1}(3z-4)}{z^{-2}(z^2-7/2z+3/2)}$

$H(z) = \dfrac{3z^2-4z}{z^2-7/2z+3/2} = H(e^{j\Omega}) = \dfrac{-4e^{j\Omega}+3e^{j2\Omega}}{3/2-7/2e^{j\Omega}+e^{j2\Omega}}$ //

Q. A causal system has an input $x(n) = \delta(n)+1/4\,\delta(n-1)-1/8\,\delta(n-2)$ and output $y(n) = \delta(n)-3/4\,\delta(n-1)$. Find:

i. impulse response

ii. output of the system if $x(n) = (1/2)^n u(n)$ using ZT.

Sol. $x(n) = \delta(n)+1/4\,\delta(n-1)-1/8\,\delta(n-2)$

By z transform: $X(z) = 1+1/4z^{-1}-1/8z^{-2}$

and $Y(z) = 1-3/4z^{-1}$

$H(z) = \dfrac{Y(z)}{X(z)} = \dfrac{1-3/4z^{-1}}{1+1/4z^{-1}-1/8z^{-2}} = \dfrac{A}{1-1/4z^{-1}} + \dfrac{B}{1+1/2z^{-1}}$

$1-3/4z^{-1} = A(1+1/2z^{-1})+B(1-1/4z^{-1})$

at $z=4$: $1-3 = A(1+2) \Rightarrow A = -2/3$ //

at $z=-2$: $1+3/2 = B(1+1/2) \Rightarrow B = 5/3$ //

$\therefore H(z) = \dfrac{-2}{3(1-1/4z^{-1})} + \dfrac{5}{3(1+1/2z^{-1})}$ //

i. Impulse response

causal system: $|z| > r_{max}$: $|z| > 1/2$

$h(n) = -2/3\,(1/4)^n u(n) + 5/3\,(-1/2)^n u(n)$ //

ii. Output of system when $x(n) = (1/2)^n u(n)$

$X(z) = \dfrac{z}{z-1/2}$

$\therefore Y(z) = H(z)\cdot X(z) = \dfrac{1-3/4z^{-1}}{1+1/4z^{-1}-1/8z^{-2}} \cdot \dfrac{1}{1-1/2z^{-1}}$

$= \dfrac{A}{1-1/4z^{-1}} + \dfrac{B}{1+1/2z^{-1}} + \dfrac{C}{1-1/2z^{-1}}$

$1-3/4z^{-1} = A(1+1/2z^{-1})(1-1/2z^{-1}) + B(1-1/4z^{-1})(1-1/2z^{-1}) + C(1-1/4z^{-1})(1+1/2z^{-1})$

at $z=-2$: $1+3/2 = B(1+1/2)(2) \Rightarrow B = 5/6$ //

at $z=4$: $1-3 = A(1+2)(1-2) \Rightarrow A = 2/3$ //

at $z=2$: $1-3/2 = C(1-1/2)(1+1) \Rightarrow C = -1/2$ //

$$\therefore Y(z) = \frac{2}{3(1-1/4z^{-1})} + \frac{5}{6(1+1/2z^{-1})} - \frac{1}{2(1-1/2z^{-1})}$$

$$\therefore y(n) = 2/3(1/4)^n u(n) + 5/6(-1/2)^n u(n) - 1/2(1/2)^n u(n)$$ //

Q3. consider the difference equation $y(n) - \frac{3}{4}y(n-1) + \frac{1}{8}y(n-2) = x(n)$

Find i. impulse response

ii. step response of the system using z transforms.

sol: i. $y(n) - 3/4\,y(n-1) + 1/8\,y(n-2) = x(n)$

By ZT: $X(z) = Y(z) - 3/4\,Y(z)z^{-1} + 1/8\,Y(z)z^{-2}$

$X(z) = Y(z)[1 - 3/4z^{-1} + 1/8z^{-2}]$

$$\therefore H(z) = \frac{Y(z)}{X(z)} = \frac{1}{1-3/4z^{-1}+1/8z^{-2}} = \frac{A}{1-1/4z^{-1}} + \frac{B}{1-1/2z^{-1}}$$

$1 = A(1-1/2z^{-1}) + B(1-1/4z^{-1})$

at $z^{-1}=2$: $1 = B(1-1/2) \Rightarrow B = 2$ //

at $z^{-1}=4$: $1 = A(1-2) \Rightarrow A = -1$ //

$$\therefore H(z) = \frac{-1}{1-1/4z^{-1}} + \frac{2}{1-1/2z^{-1}}$$

By inverse z transform : causal system : $|z| > 1/2$

$h(n) = -(1/4)^n u(n) + 2(1/2)^n u(n)$ //

ii. step response

If $x(n) = u(n)$ then $y(n)$: step response

$Y(z) = H(z) \cdot X(z)$

	LTI	
$X(z)$	$H(z)$	$Y(z)$

$$Y(z) = \frac{1}{1-3/4z^{-1}+1/8z^{-2}} \cdot \frac{1}{1-z^{-1}}$$

$$Y(z) = \frac{A}{(1-1/2z^{-1})} + \frac{B}{(1-1/4z^{-1})} + \frac{C}{1-z^{-1}}$$

$1 = A(1-1/4z^{-1})(1-z^{-1}) + B(1-1/2z^{-1})(1-z^{-1}) + C(1-1/2z^{-1})(1-1/4z^{-1})$

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at $z^{-1}=2$: $1 = A(1-1/2)(1-2) \Rightarrow A = -2$
at $z^{-1}=4$: $1 = B(1-2)(1-4) \Rightarrow B = 1/3$
at $z^{-1}=1$: $1 = C(1-1/2)(1-1/4) \Rightarrow C = 3/8$

$\therefore Y(z) = \dfrac{-2}{(1-1/2 z^{-1})} + \dfrac{1}{3(1-1/4 z^{-1})} + \dfrac{3}{8(1-z^{-1})}$

$\therefore$ By Inverse ZT: $y(n) = -2(1/2)^n u(n) + 1/3(1/4)^n u(n) + 3/8(1)^n u(n)$

Q4: For the difference equation $y(n) - (5/6)y(n-1) - (1/6)y(n-2) = x(n) - 2x(n-1)$. Find:
i. system function
ii. Does a stable and causal system exist?
iii. Does a stable and inverse system exist?

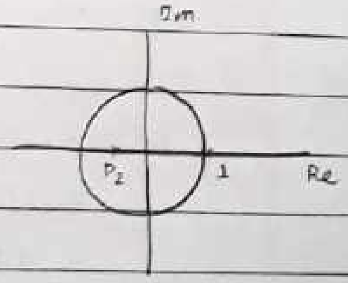
Sol: i. $y(n) - 5/6\, y(n-1) - 1/6\, y(n-2) = x(n) - 2x(n-1)$

ZT: $Y(z) - 5/6\, Y(z) z^{-1} - 1/6\, Y(z) z^{-2} = X(z) - 2X(z) z^{-1}$

$Y(z)\left[1 - 5/6 z^{-1} - 1/6 z^{-2}\right] = X(z)\left[1 - 2z^{-1}\right]$

$H(z) = \dfrac{Y(z)}{X(z)} = \dfrac{1-2z^{-1}}{1 - 5/6 z^{-1} - 1/6 z^{-2}}$

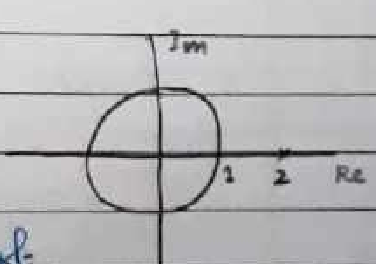
ii. $H(z) = \dfrac{1-2z^{-1}}{1-5/6 z^{-1} - 1/6 z^{-2}}$



Poles: $P_1 = 1 \quad P_2 = -1/6$

$\therefore$ The system is stable and will exist.

iii. $H^{-1}(z) = \dfrac{1}{H(z)} = \dfrac{1 - 5/6 z^{-1} - 1/6 z^{-2}}{1 - 2z^{-1}}$



Poles: $p = 2$

The system is not stable and will not exist.

Q5: If $h(n) = 3(1/4)^n u(n) - 2(1/2)^n u(n)$. Find:
i. Transfer function
ii. Its difference equation.

Sol: $h(n) = 3(1/4)^n u(n) - 2(1/2)^n u(n)$

$$ZT: \frac{3}{1-1/4z^{-1}} - \frac{2}{1-1/2z^{-1}} = \frac{3 - 3/2z^{-1} - 2 + 1/2z^{-1}}{1 - 1/2z^{-1} - 1/4z^{-1} + 1/8z^{-2}}$$

i. $$H(z) = \frac{1 - z^{-1}}{1 - 3/4z^{-1} + 1/8z^{2}}$$

ii. $$H(z) = \frac{1 - z^{-1}}{1 - 3/4z^{-1} + 1/8z^{-2}} = \frac{Y(z)}{X(z)}$$

$$Y(z)[1 - 3/4z^{-1} + 1/8z^{-2}] = X(z)[1 - z^{-1}]$$

$$Y(z) - 3/4Y(z)z^{-1} + 1/8Y(z)z^{-2} = X(z) - X(z)z^{-1}$$

Taking Inverse ZT:

$$y(n) - 3/4y(n-1) - 1/8y(n-2) = x(n) - x(n-1)$$

Q6: A system has a transfer function.

$$H(z) = \frac{2}{1-0.9e^{j\pi/4}z^{-1}} + \frac{2}{1-0.9e^{-j\pi/4}z^{-1}} + \frac{3}{1+2z^{-1}}$$

Find the impulse response for: i. causal ii. stable.

$(P_1 = 0.9 \quad P_2 = 0.9 \quad P_3 = -2)$

Sol: i. Causal: $|z| > r_{max}$: $|z| > 2$

$$h(n) = 2(0.9)^n e^{j\pi/4n}u(n) + 2(0.9)^n e^{-j\pi/4n}u(n) + 3(-2)^n u(n)$$

$$= 2(0.9)^n u(n)[e^{j\pi/4n} + e^{-j\pi/4n}] + 3(-2)^n u(n)$$

$$= 4(0.9)^n u(n) \cos\frac{n\pi}{4} + 3(-2)^n u(n)$$

ii. Stable: $r_{min} < |z| < r_{max}$: $0.9 < |z| < 2$

$$h(n) = 2(0.9)^n e^{j\pi/4n}u(n) + 2(0.9)e^{-j\pi/4n} - 3(-2)^n u(-n-1)$$

$$= 2(0.9)^n u(n)[e^{j\pi/4n} + e^{-j\pi/4n}] - 3(-2)^n u(-n-1)$$

$$= 4(0.9)^n u(n) \cos\frac{n\pi}{4} - 3(-2)^n u(-n-1)$$

UNIT - 04

Fourier Representation

* Discrete Time Fourier series : DTFS

$$X(k) = \frac{1}{N} \sum_{n=0}^{N-1} x(n) e^{-jk\Omega_0 n}$$

$$x(n) = \sum_{k=0}^{N-1} X(k) e^{jk\Omega_0 n}$$

DTFS is applicable only for periodic signals
N: Number of samples
$\Omega_0 = \frac{2\pi}{N}$ rad

* Properties :

1. Linearity :

$$x(n) \rightarrow X(k)$$
$$y(n) \rightarrow Y(k)$$
$$z(n) = ax(n) + by(n) \rightarrow Z(k) = aX(k) + bY(k)$$

Proof : Considering

$$Z(k) = \frac{1}{N} \sum_{n=0}^{N-1} z(n) e^{-jk\Omega_0 n}$$

$$Z(k) = \frac{1}{N} \sum_{n=0}^{N-1} \left(aX(k) + bY(k) \right) e^{-jk\Omega_0 n}$$

$$Z(k) = \frac{1}{N} \left[\sum_{n=0}^{N-1} aX(k) e^{-jk\Omega_0 n} + \sum_{n=0}^{N-1} bY(k) e^{-jk\Omega_0 n} \right]$$

$$Z(k) = aX(k) + bY(k) //$$

2. Time - Shift :

$$x(n) \longrightarrow X(k)$$
$$z(n) = x(n - n_0) \longrightarrow Z(k) = e^{-jk\Omega_0 n_0} X(k)$$

Proof : Considering

$$Z(k) = \frac{1}{N} \sum_{n=0}^{N-1} z(n) e^{-jk\Omega_0 n}$$

$$z(k) = \frac{1}{N} \sum_{n=0}^{N-1} x(n-n_0)\, e^{-jk\Omega_0 n}$$

let $n - n_0 = m$

$n = m + n_0$

$$z(k) = \frac{1}{N} \sum_{m=0}^{N-1} x(m)\, e^{-jk\Omega_0(m+n_0)}$$

$$z(k) = e^{-jk\Omega_0 n_0} \frac{1}{N} \sum_{m=0}^{N-1} x(m)\, e^{-jk\Omega_0 m}$$

$$z(k) = e^{-jk\Omega_0 n_0} X(k) \quad //$$

3. Frequency shift:

$$x(n) \longrightarrow X(k)$$

$$w(n) = e^{jk_0\Omega_0 n} x(n) \longrightarrow W(k) = X(k-k_0)$$

Proof: considering

$$W(k) = \frac{1}{N} \sum_{n=0}^{N-1} w(n)\, e^{-jk\Omega_0 n}$$

$$W(k) = \frac{1}{N} \sum_{n=0}^{N-1} e^{jk_0\Omega_0 n} x(n)\, e^{-jk\Omega_0 n}$$

$$W(k) = \frac{1}{N} \sum_{n=0}^{N-1} x(n)\, e^{-j\Omega_0 n(k-k_0)}$$

$$W(k) = X(k-k_0) \quad //$$

4. Convolution:

$$x(n) \longrightarrow X(k)$$

$$y(n) \longrightarrow Y(k)$$

$$w(n) = x(n) * y(n) \longrightarrow W(k) = N \cdot X(k) Y(k)$$

Proof: considering

$$W(k) = \frac{1}{N} \sum_{n=0}^{N-1} w(n)\, e^{-jk\Omega_0 n}$$

$$W(k) = \frac{1}{N} \sum_{n=0}^{N-1} [x(n) * y(n)]\, e^{-jk\Omega_0 n}$$

$$w(k) = \frac{1}{N} \sum_{n=0}^{N-1} \left[\sum_{l=0}^{N-1} x(l)\, y(n-l) \right] e^{-jk\Omega_0 n}$$

$$w(k) = \frac{1}{N} \left[\sum_{l=0}^{N-1} x(l) \sum_{n=0}^{N-1} y(n-l) \right] e^{-jk\Omega_0 n}$$

let $n-l=m$

$n = l+m$

$$w(k) = \frac{1}{N} \left[\sum_{l=0}^{N-1} x(l) \sum_{m=0}^{N-1} y(m) \right] e^{-jk\Omega_0 (l+m)}$$

$$w(k) = \frac{1}{N} \left[\sum_{l=0}^{N-1} x(l)\, e^{-jk\Omega_0 l} \sum_{m=0}^{N-1} y(m)\, e^{-jk\Omega_0 m} \right]$$

$$w(k) = \frac{1}{N} \left[N X(k) \cdot N Y(k) \right]$$

$$w(k) = N \cdot X(k)\, Y(k)$$

5. <u>Modulation</u>:

$x(n) \rightarrow X(k)$

$y(n) \rightarrow Y(k)$

<u>$w(n) = x(n) \cdot y(n) \rightarrow W(k) = X(k) * Y(k)$</u>

Proof: considering

$$W(k) = \frac{1}{N} \sum_{n=0}^{N-1} w(n)\, e^{-jk\Omega_0 n}$$

$$W(k) = \frac{1}{N} \sum_{n=0}^{N-1} [x(n)\, y(n)]\, e^{-jk\Omega_0 n}$$

we have $x(n) = \sum_{l=0}^{N-1} X(l)\, e^{jl\Omega_0 n}$

$$\therefore W(k) = \frac{1}{N} \sum_{n=0}^{N-1} \left[\sum_{l=0}^{N-1} X(l)\, e^{jl\Omega_0 n} \right] y(n)\, e^{-jk\Omega_0 n}$$

$$W(k) = \frac{1}{N} \sum_{l=0}^{N-1} X(l) \sum_{n=0}^{N-1} y(n)\, e^{-j(k-l)\Omega_0 n}$$

$$W(k) = \sum_{l=0}^{N-1} X(l)\, Y(k-l)$$

$$W(k) = X(k) * Y(k)$$

6. **Parseval's Theorem:**

If $x(n) \xrightarrow{\text{DTFS } \Omega_0} X(k) : \Omega_0 = 2\pi/N$

then $\frac{1}{N} \sum_{n=0}^{N-1} |x(n)|^2 = \sum_{k=0}^{N-1} |X(k)|^2$

Proof: LHS of the equation is the average power of a periodic discrete-time signal with fundamental period N, i.e., $P = \frac{1}{N} \sum_{n=\langle N\rangle} |x(n)|^2$

$$P = \frac{1}{N} \sum_{n=\langle N\rangle} x(n) \cdot x^*(n)$$ complex signal

$$P = \frac{1}{N} \sum_{n=\langle N\rangle} x(n) \left[\sum_{k=\langle N\rangle} X^*(k) e^{-jk\Omega_0 n} \right]$$

Changing the order of summation.

$$P = \sum_{k=\langle N\rangle} X^*(k) \left[\frac{1}{N} \sum_{n=\langle N\rangle} x(n) e^{-jk\Omega_0 n} \right]$$

$$P = \sum_{k=\langle N\rangle} X^*(k) X(n)$$

$$P = \sum_{k=\langle N\rangle} |X(k)|^2$$

Therefore $\frac{1}{N} \sum_{n=\langle N\rangle} |x(n)|^2 = \sum_{k=\langle N\rangle} |X(k)|^2$ //

Q1: Determine DTFS of the signal $x(n) = \cos(\pi n/3)$. Plot its magnitude and phase spectrum.

Sol: $x(n) = \cos(\pi n/3)$

$\Omega_0 = \pi/3 = 2\pi(1/6) \Rightarrow N = 6$ samples //

$$x(n) = \cos(\pi n/3) = \frac{e^{j(n\pi/3)} + e^{-j(n\pi/3)}}{2}$$

$$x(n) = \frac{1}{2} e^{j(1)(\pi/3)n} + \frac{1}{2} e^{j(-1)(\pi/3)n}$$

~~x(n) = x~~

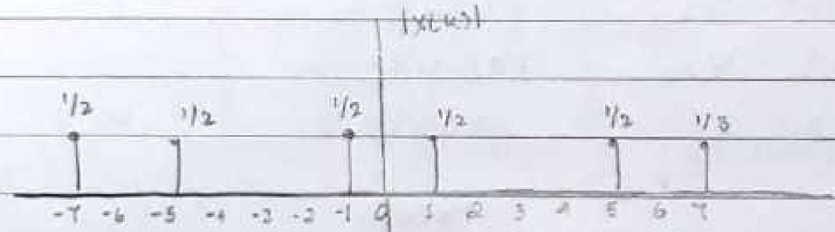
Therefore $X(1) = 1/2$ and $X(-1) = 1/2$

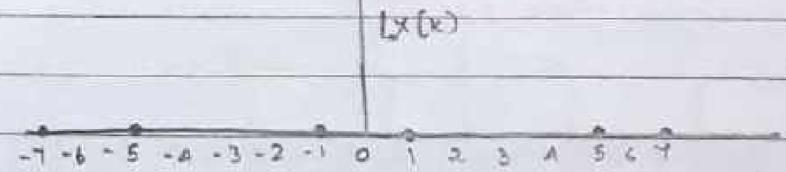
$$x(n) = \sum_{k=0}^{N-1} X(k) e^{jk\Omega_0 n}$$

$$x(n) = X(0) + X(1) e^{j(1)\Omega_0 n} + \cdots\cdots$$

$\therefore X(1) = X(7) = X(13) = \cdots = 1/2$

$X(-1) = X(5) = X(11) = \cdots = 1/2$





Q2: $x(n) = \sin\left(\frac{4\pi n}{21}\right) + \cos\left(\frac{10\pi n}{21}\right) + 1$

Sol: $\Omega_{0_1} = \frac{4\pi}{21} = 2\pi\left(\frac{2}{21}\right) \Rightarrow \underline{N_1 = 21}$

$\Omega_{0_2} = \frac{10\pi}{21} = 2\pi\left(\frac{5}{21}\right) \Rightarrow \underline{N_2 = 21}$

$\Omega_0 = GCD[\Omega_{01}, \Omega_{02}] = GCD\left[\frac{4\pi}{21}, \frac{10\pi}{21}\right] = \frac{2\pi}{21}$ //

$N = LCM[N_1, N_2] = LCM[21, 21] = 21$ //

$$x(n) = \sin\left(\frac{4\pi n}{21}\right) + \cos\left(\frac{10\pi n}{21}\right) + 1$$

$$x(n) = \frac{e^{j(4\pi n/21)} - e^{-j(4\pi n/21)}}{2j} + \frac{e^{j(10\pi n/21)} + e^{-j(4\pi n/21)}}{2} + 1$$

$$x(n) = \frac{1}{2j} e^{j(2)(2\pi/21)n} - \frac{1}{2j} e^{j(-2)(2\pi/21)n} + \frac{1}{2} e^{j(5)(2\pi/21)n}$$

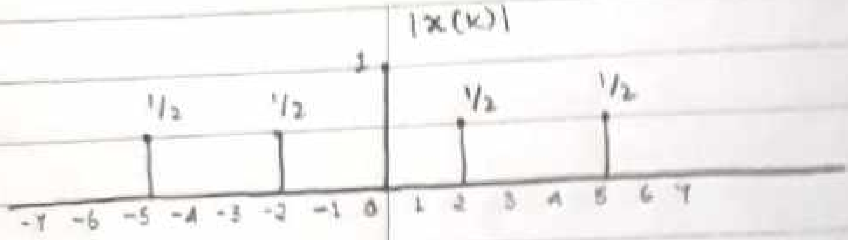
$$+\frac{1}{2}e^{j(5)(2\pi/21)n}+1$$

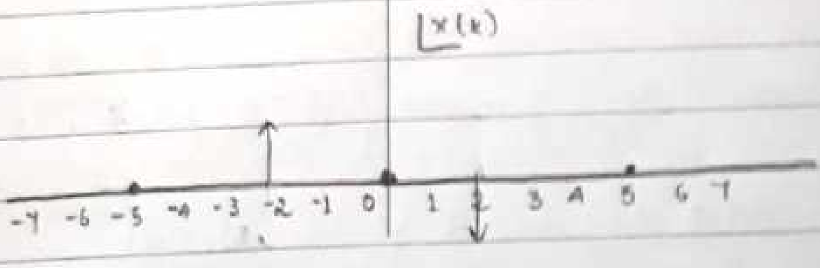
~~x(n)=~~ Therefore

$x(2)=1/2j$; $x(-2)=-1/2j$; $x(5)=1/2$; $x(-5)=1/2$

and $x(0)=1$.

	$\lvert x(k)\rvert$	$\angle x(k)$
$x(2) = 1/2j$	$\lvert x(2)\rvert = 1/2$	$0.5\ \angle 90°$
$x(-2) = -1/2j$	$\lvert x(-2)\rvert = 1/2$	$0.5\ \angle 90°$
$x(5) = 1/2$	$\lvert x(5)\rvert = 1/2$	0
$x(-5) = 1/2$	$\lvert x(-5)\rvert = 1/2$	0
$x(0) = 1$	$\lvert x(0)\rvert = 1$	0





Q3: $x(n)=2+2\cos\left(\frac{\pi n}{4}\right)+\cos\left(\frac{\pi n}{2}\right)+\frac{1}{2}\cos\left(\frac{3\pi n}{4}\right)$. Evaluate the power of the signal

Sketch the power density spectrum.

Sol: $\Omega_{01}=\frac{\pi}{4}=2\pi\left(\frac{1}{8}\right) \Rightarrow \underline{N_1=8}$

$\Omega_{02}=\frac{\pi}{2}=2\pi\left(\frac{1}{4}\right) \Rightarrow \underline{N_2=4}$

$\Omega_{03}=\frac{3\pi}{4}=2\pi\left(\frac{3}{8}\right) \Rightarrow \underline{N_3=8}$

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$$\Omega_0 = GCD[\Omega_{01}, \Omega_{02}, \Omega_{03}] = GCD\left[\frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}\right] = \frac{\pi}{4}$$

$$N = LCM[N_1, N_2, N_3] = LCM[8, 4, 8] = 8$$

$$x(n) = 2 + 2\cos\left(\frac{\pi n}{4}\right) + \cos\left(\frac{\pi n}{2}\right) + \frac{1}{2}\cos\left(\frac{3\pi n}{4}\right)$$

$$x(n) = 2 + 2\,\frac{e^{j(\pi n/4)} + e^{-j(\pi n/4)}}{2} + \frac{e^{j(\pi n/2)} + e^{-j(\pi n/2)}}{2} + \frac{1}{2}\,\frac{e^{j(3\pi n/4)} + e^{-j(3\pi n/4)}}{2}$$

$$x(n) = 2 + e^{j(1)(\pi/4)n} + e^{j(-1)(\pi/4)n} + \frac{1}{2}e^{j(2)(\pi/4)n} + \frac{1}{2}e^{j(-2)(\pi/4)n} + \frac{1}{4}e^{j(3)(\pi/4)n} + \frac{1}{4}e^{j(-3)(\pi/4)n}$$

$\therefore\ X(0) = 2 \qquad |X(0)|^2 = 4$

$X(1) = 1 \qquad |X(1)|^2 = 1$

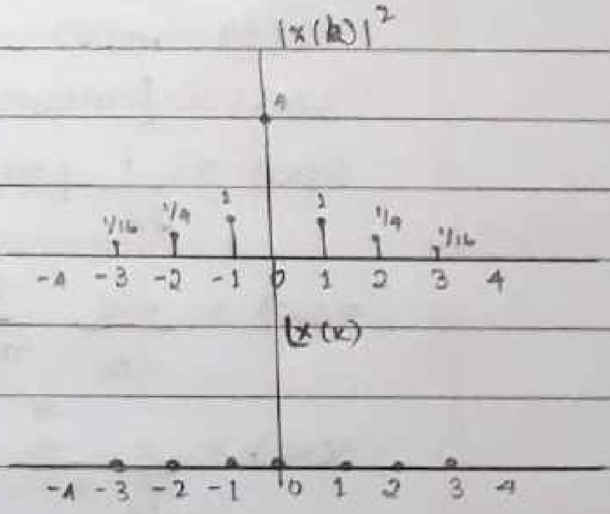
$X(-1) = 1 \qquad |X(-1)|^2 = 1$

$X(2) = 1/2 \qquad |X(2)|^2 = 1/4$

$X(-2) = 1/2 \qquad |X(-2)|^2 = 1/4$

$X(3) = 1/4 \qquad |X(3)|^2 = 1/16$

$X(-3) = 1/4 \qquad |X(-3)|^2 = 1/16$



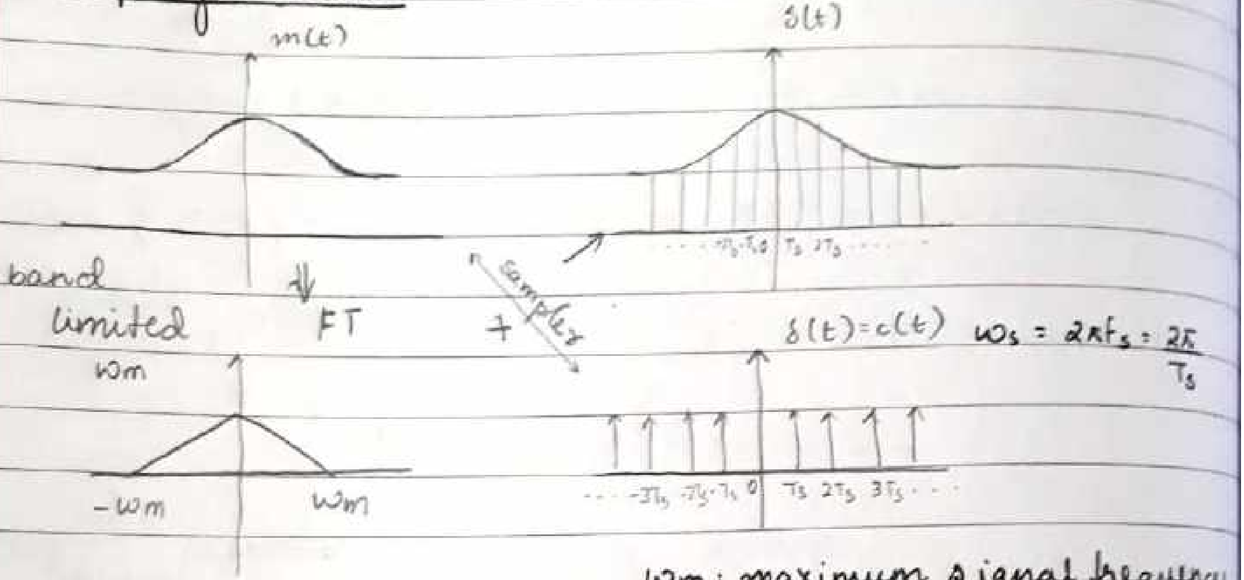
Power of the signal

$$P = \sum_{k=\langle N\rangle} |X(k)|^2 = 4 + 1 + 1 + 1/4 + 1/4 + 1/16 + 1/16 = \underline{\underline{53/8\ W}}$$

UNIT - 5

Applications of Fourier Transform

★ **Sampling Theorem:**



ωm : maximum signal frequency of message signal.
ωs : sampled frequency.

$$\delta(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT_s)$$

$$s(t) = m(t) \cdot c(t)$$

$$s(\omega) = [m(\omega) * c(\omega)] \qquad \left(\because c(\omega) = \sum_{n=-\infty}^{\infty} \delta(\omega - n\omega_s)\right)$$

$$s(\omega) = \frac{1}{2\pi}\left[m(\omega) * \omega_s \sum_{n=-\infty}^{\infty} \delta(\omega - n\omega_s)\right]$$

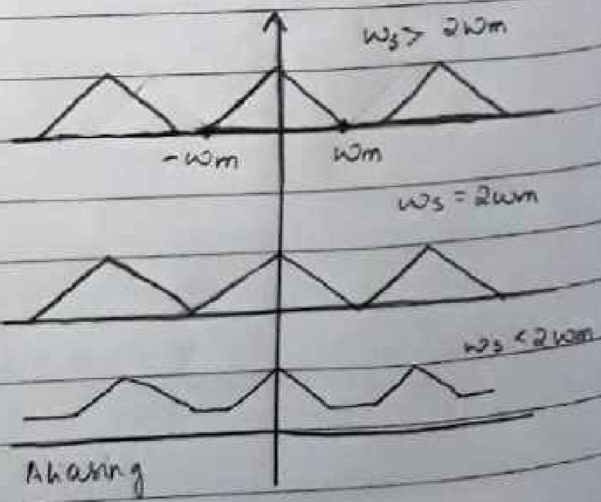
$$s(\omega) = \frac{\omega_s}{2\pi} \sum_{n=-\infty}^{\infty} [m(\omega) * \delta(\omega - n\omega_s)]$$

$$s(\omega) = \frac{1}{T_s} \sum_{n=-\infty}^{\infty} m(\omega - n\omega_s)$$

$$s(\omega) = \frac{1}{T_s}[\dots m(\omega + \omega_s) + m(\omega) + m(\omega - \omega_s) \dots]$$

condition

$\omega_s \geq 2\omega_m$; $f_s \geq 2 f_m$



classmate
Date
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Q: $x(t) = \cos(5\pi t) + 0.5\cos(10\pi t)$ is ideally sampled with sampling period Ts. Find the sampling frequency and nyquist rate.

sol: $\omega_1 = 5\pi$ and $\omega_2 = 10\pi$

$\omega_s \geq 2\omega_m$

$\omega_s = 20\pi$

Frequency

$$f_s = \frac{\omega_s}{2\pi} = \frac{20\pi}{2\pi} = 10\,Hz$$

Q. specify the nyquist rate

i. $x_1(t) = \text{sinc}(200t)$

ii. $x_2(t) = \text{sinc}^2(200t)$

NOTE:

$$\text{sinc}\,\theta = \frac{\sin(\pi\theta)}{\pi\theta}$$

sol: i. $x_1(t) = \text{sinc}(200t)$

$$x_1(t) = \frac{\sin(200\pi t)}{\pi t}$$

$\omega_m = 200\pi$

$\omega_s \geq 200\pi(2)$

$\omega_s = 400\pi$

$$f_s = \frac{\omega_s}{2\pi} = \frac{400\pi}{2\pi} = 200\,Hz$$

ii. $x_2(t) = \text{sinc}^2(200t)$

$$x_2(t) = \frac{\sin^2(200\pi t)}{\pi t} = \frac{1-\cos(400\pi t)}{2\pi t}$$

$\omega_m = 400\pi$

$\omega_s \geq 2\omega_m = 2(400\pi) = 800\pi$

$$f_s = \frac{\omega_s}{2\pi} = \frac{800\pi}{2\pi} = 400\,Hz$$

Scanned by CamScanner

Unit 5

— Frequency response problems (textbook)

SLE

Matlab

★ **COMMANDS :**

input : input data
display : print
x label : x axis
y label : y axis
title :

To plot a wave
plot : CTS
stem : DTS

Q: Write a matlab code to perform convolution between the two signals.

x = {1, 2, 3}
h = {-1, 2, 4}

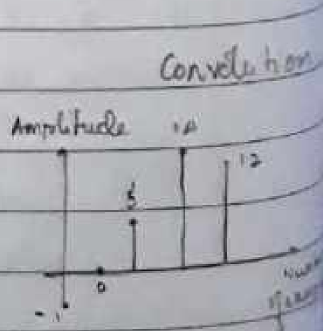
	1	2	3
-1	-1	-2	-3
2	2	4	6
4	4	8	12

y = {-1, 0, 5, 14, 12}

Editor window =

```
clc; //clear command window
x = input ('Enter the 1st sequence x:');
h = input ('Enter the 2nd sequence h:');
y = conv (x,h);
disp ('The convolution output is:');
disp (y);
stem (y);
x label ('Number of samples');
y label ('Amplitude');
title ('convolution');
```

Convolution



command window =

```
Enter the 1st sequence x: [1 2 3]
Enter the 2nd sequence h: [-1 2 4]
The convolution output is: [-1 0 5 14 12]
```

Date ____
Page ____

Q: Write a Matlab code to generate sinusoidal signal.

```
= clc;
  t = -π : 0.001 : π      (interval)
  y = sin(t);
  plot(y);
  xlabel('Time');
  ylabel('Amplitude');
  title('Sine wave');
```

NOTE:
δ(n) , δ(t)
u(n) , u(t)
r(n) , r(t)
default function is not present.

Q: Write a Matlab code to plot discrete time unit step signal:

```
= clc;
  n = input('Enter the range for unit step DTS:');
  (or) u = [1 1 1 1 1];
  stem(u);
  xlabel('number of samples');
  ylabel('amplitude');
  title('Unit step DTS');
```

Unit Step DTS