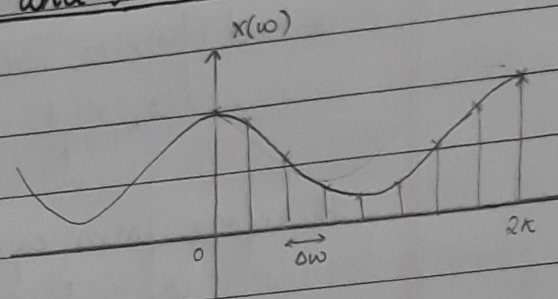
Keerthana Ashok

DIGITAL SIGNAL PROCESSING

UNIT - 01

Discrete Fourier Transform and Properties of DFT.

* Frequency domain sampling and reconstruction of discrete time signals:



Let us consider an aperiodic discrete-time signal $x(n)$ with Fourier transform:

$$X(\omega) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n} \quad \text{—— (1)}$$

Suppose we sample $X(\omega)$ periodically in frequency at a spacing of $\delta\omega$ between successive samples, thus $X(\omega)$ is periodic with period 2π. Let us consider N equidistant samples in the interval $0 \le \omega < 2\pi$ with a spacing

$$\Delta\omega = 2\pi/N$$

$$\Rightarrow \quad \omega = 2\pi k/N$$

Therefore

$$X\left(\frac{2\pi k}{N}\right) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\frac{2\pi}{N}kn} \quad ; \quad k = 0, 1 \ldots N-1 \quad \text{—— (2)}$$

$$X\left(\frac{2\pi k}{N}\right) = \cdots + \sum_{n=-N}^{-1} x(n) e^{-j\frac{2\pi}{N}kn} + \sum_{n=0}^{N-1} x(n) e^{-j\frac{2\pi}{N}kn}$$

$$+ \sum_{n=N}^{2N-1} x(n) e^{-j\frac{2\pi}{N}kn} + \cdots\cdots$$

$$X\left(\frac{2\pi k}{N}\right) = \sum_{l=-\infty}^{\infty} \sum_{n=lN}^{lN+N-1} x(n) e^{-j\frac{2\pi}{N}kn}$$

Changing the index in the inner summation from n to $n-lN$ and interchanging the order of summation.

$$X\left(\frac{2\pi k}{N}\right) = \sum_{n=0}^{N-1} \left[\sum_{l=-\infty}^{\infty} x(n-lN) \right] e^{-j\frac{2\pi}{N}kn}$$

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The signal $x_p(n) = \sum_{l=-\infty}^{\infty} x(n-lN)$ is obtained by the periodic repetition of $x(n)$ every N samples is periodic with fundamental period N.

It can be expanded in fourier series.

$$x_p(n) = \sum_{k=0}^{N-1} a_k e^{j\frac{2\pi}{N}kn} \quad n = 0, 1 \ldots N-1 \quad (3)$$

where fourier coefficients

$$a_k = \frac{1}{N} \sum_{n=0}^{N-1} x_p(n) e^{-j\frac{2\pi}{N}kn} \quad k = 0, 1 \ldots N-1 \quad (4)$$

Therefore form eq (2)

$$a_k = \frac{1}{N} X\left(\frac{2\pi k}{N}\right) \quad k = 0, 1 \ldots N-1$$

substituting in eq (3)

$$x_p(n) = \frac{1}{N} \sum_{k=0}^{N-1} X\left(\frac{2\pi k}{N}\right) e^{j\frac{2\pi}{N}kn} \quad n = 0, 1 \ldots N-1$$

This provides the reconstruction of the periodic signal $x_p(n)$ from the samples of the spectrum $X(\omega)$.

To reconstruct $x(n)$ the number of samples N has to be greater than or equal to the length L of the sequence i.e., $N \geq L$ if not it leads to aliasing effect.

* <u>Discrete Fourier Transform</u> (DFT):

A finite duration sequence $x(n)$ of length L (i.e, $x(n) = 0$ for $n<0$ and $n \geq L$) has a Fourier transform

$$X(\omega) = \sum_{n=0}^{L-1} x(n) e^{-j\omega n}$$

When we sample $X(\omega)$ at equally spaced frequencies $\omega_k = 2\pi k/N$ where $k = 0, 1 \ldots N-1$ where $N \geq L$, then

$$X(k) = X\left(\frac{2\pi k}{N}\right) = \sum_{n=0}^{L-1} x(n) e^{-j\frac{2\pi}{N}kn}$$

$$X(k) = \sum_{n=0}^{N-1} e^{-j\frac{2\pi}{N}kn} x(n); \quad k = 0, 1 \ldots N-1$$

Discrete Fourier Transform

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where $X(k)$ is the N point DFT of the sequence $x(n)$.

To recover the sequence $x(n)$ from the frequency samples: $x(n) = \frac{1}{N}\sum_{k=0}^{N-1} X(k)\, e^{j\frac{2\pi}{N}kn}$; $n = 0, 1 \dots N-1$ Inverse Discrete Fourier Transform

* <u>DFT as a linear transformation</u>:

DFT: $X(k) = \sum_{n=0}^{N-1} x(n) W_N^{kn}$; $k = 0, 1 \dots N-1$

IDFT: $x(n) = \frac{1}{N}\sum_{k=0}^{N-1} X(k)\, W_N^{-kn}$; $n = 0, 1 \dots N-1$

where $W_N = e^{-j\frac{2\pi}{N}}$ which is an Nth root of unity and is called as twiddle factor or phase factor.

Let us define an N-point vector x_N of the signal sequence $x(n)$, $n = 0, 1 \dots N-1$, an N-point vector X_N of frequency samples and an $N \times N$ matrix W_N as

$$x_N = \begin{bmatrix} x(0) \\ x(1) \\ \vdots \\ x(N-1) \end{bmatrix} \qquad X_N = \begin{bmatrix} X(0) \\ X(1) \\ \vdots \\ X(N-1) \end{bmatrix}$$

$$W_N = \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & W_N & W_N^2 & \cdots & W_N^{N-1} \\ 1 & W_N^2 & W_N^4 & \cdots & W_N^{2(N-1)} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ 1 & W_N^{N-1} & W_N^{2(N-1)} & \cdots & W_N^{(N-1)(N-1)} \end{bmatrix}$$

Therefore by defination of DFT as mentioned above

$$X_N = W_N x_N$$

where W_N is the matrix of the linear transformation and it is a symmetric matrix. Assuming the inverse of W_N exists:

$$x_N = W_N^{-1} X_N$$

but this is an expression for IDFT which is given as

$$x_N = \frac{1}{N} W_N^{*} X_N$$

where W_N^* is the conjugate matrix of W_N.

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Therefore

$$W_N^{-1} = \frac{1}{N} W_N^{*}$$

$$W_N^{*} W_N = N I_N$$

where I_N is an $N \times N$ identity matrix

★ Relationship of the DFT to other transforms:

– Relationship to the Fourier series coefficients of a Periodic Sequence:

A periodic sequence $x_p(n)$ with fundamental period N can be represented in a Fourier series of the form:

$$x_p(n) = \sum_{k=0}^{N-1} a_k e^{j\frac{2\pi}{N}kn} \qquad -\infty < n < \infty \quad \text{—(1)}$$

where the Fourier series coefficients are given as

$$a_k = \frac{1}{N}\sum_{n=0}^{N-1} x_p(n) e^{-j\frac{2\pi}{N}kn} \quad ; k = 0, 1 \cdots N-1 \quad \text{—(2)}$$

The N-point DFT of $x(n)$ is given by

$$X(k) = \sum_{n=0}^{N-1} x(n) e^{-j\frac{2\pi}{N}kn} \quad ; k = 0, 1 \cdots N-1 \quad \text{—(3)}$$

The N-point IDFT of $X(k)$ is given by

$$x(n) = \frac{1}{N}\sum_{k=0}^{N-1} X(k) e^{j\frac{2\pi}{N}kn} \quad ; n = 0, 1 \cdots N-1 \quad \text{—(4)}$$

comparing eq (1) and eq (4)

$$x(n) = x_p(n)$$

and eq (2) and eq (3)

$$X(k) = N a_k$$

– Relationship to the Fourier transform of an aperiodic sequence:

consider an aperiodic finite energy sequence $x(n)$ with fourier transform $X(\omega)$ sampled with N samples

$$X(k) = X(\omega)\Big|_{\omega = \frac{2\pi k}{N}} = \sum_{n=-\infty}^{\infty} x(n) e^{-j\frac{2\pi}{N}kn} \quad ; k = 0, 1 \cdots N-1$$

where $X(k)$ are the DFT coefficients of the periodic sequence of period N, given by

$$x_p(n) = \sum_{l=-\infty}^{\infty} x(n-lN)$$

Thus $x_p(n)$ is determined by aliasing $\{x(n)\}$ over the interval $0 \le n \le N-1$. The finite duration sequence

$$\hat{x}(n) = \begin{cases} x_p(n) & ; 0 \le n \le N-1 \\ 0 & ; \text{otherwise} \end{cases}$$

bears no resemblance to the original sequence $\{x(n)\}$ unless $x(n)$ is of finite duration and length $L \le N$

$$x(n) = \hat{x}(n) \qquad 0 \le n \le N-1$$

Only in this case will the IDFT of $X(k)$ yield the original sequence $x(n)$.

- Relationship to the z-transform:

Let us consider a sequence $x(n)$ having the z-transform:

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

If $X(z)$ is sampled at the N equally spaced points on the unit circle $z_k = e^{j\frac{2\pi}{N}kn}$; $z = 0, 1, \cdots N-1$

$$X(k) = X(z)\Big|_{z=e^{j\frac{2\pi}{N}kn}} = \sum_{n=-\infty}^{\infty} x(n) e^{-j\frac{2\pi}{N}kn}$$

$$k = 0, 1 \cdots N-1$$

This is identical to the fourier transform $X(\omega)$ evaluated at the N equally spaced samples.

If the sequence $x(n)$ has a finite duration the sequence can be recovered from its N-point DFT. Hence its z-transform is uniquely determined by its N-point DFT.

$$X(z) = \sum_{n=0}^{N-1} x(n) z^{-n}$$

$$X(z) = \sum_{n=0}^{N-1} \left[\frac{1}{N} \sum_{k=0}^{N-1} X(k) e^{j\frac{2\pi}{N}kn} \right] z^{-n}$$

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$$X(z) = \frac{1}{N}\sum_{k=0}^{N-1} X(k) \sum_{n=0}^{N-1}\left(e^{j\frac{2\pi}{N}k} z^{-1}\right)^n$$

$$X(z) = \frac{1-z^{-N}}{N}\sum_{k=0}^{N-1} \frac{X(k)}{1-e^{j\frac{2\pi}{N}k} z^{-1}}$$

When evaluated on the unit circle, it yields the Fourier Transform of the finite duration sequence in terms of its DFT

$$X(\omega) = \frac{1-e^{-j\omega N}}{N}\sum_{k=0}^{N-1} \frac{X(k)}{1-e^{-j(\omega-2\pi k/N)}}$$

This expression for the Fourier transform is a Lagrange polynomial interpolation formula for $X(\omega)$ expressed in terms of the values $X(k)$ of the polynomial at a set of equally spaced discrete frequencies $\omega_k = 2\pi k/N$; $k = 0, 1 \ldots N-1$.

★ **Properties of the DFT:**

- **Linearity:**

If $x_1(n) \xleftrightarrow[N]{DFT} X_1(k)$

and $x_2(n) \xleftrightarrow[N]{DFT} X_2(k)$

then for any real valued or complex valued constants

$$a_1 x_1(n) + a_2 x_2(n) \xleftrightarrow[N]{DFT} a_1 X_1(k) + a_2 X_2(k)$$

Proof: $y(n) = a_1 x_1(n) + a_2 x_2(n)$

but $Y(k) = \sum_{n=0}^{N-1} y(n) e^{-j\frac{2\pi}{N}kn}$

$$Y(k) = \sum_{n=0}^{N-1} (a_1 x_1(n) + a_2 x_2(n)) e^{-j\frac{2\pi}{N}kn}$$

$$Y(k) = a_1 \sum_{n=0}^{N-1} x_1(n) e^{-j\frac{2\pi}{N}kn} + a_2 \sum_{n=0}^{N-1} x_2(n) e^{-j\frac{2\pi}{N}kn}$$

$$Y(k) = a_1 X_1(k) + a_2 X_2(k)$$

- **Periodicity:**

If $x(n)$ and $X(k)$ are an N-point DFT pair

then $x(n+N) = x(n)$ for all n

$X(k+N) = X(k)$ for all k

Proof $x(k) = \sum_{n=0}^{N-1} x(n) e^{-j\frac{2\pi}{N}kn}$

$$X(k+N) = \sum_{n=0}^{N-1} x(n) e^{-j\frac{2\pi}{N}(k+N)n}$$

$$X(k+N) = \sum_{n=0}^{N-1} x(n) e^{-j\frac{2\pi}{N}kn} e^{-j\frac{2\pi}{N}Nn}$$

$$X(k+N) = \sum_{n=0}^{N-1} x(n) e^{-j\frac{2\pi}{N}kn} = X(k)$$

Similarly

$$x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k) e^{j\frac{2\pi}{N}kn}$$

$$x(n+N) = \frac{1}{N} \sum_{k=0}^{N-1} X(k) e^{j\frac{2\pi}{N}k(n+N)}$$

$$x(n+N) = \frac{1}{N} \sum_{k=0}^{N-1} X(k) e^{j\frac{2\pi}{N}kn} e^{j\frac{2\pi}{N}Nn}$$

$$x(n+N) = \frac{1}{N} \sum_{k=0}^{N-1} X(k) e^{j\frac{2\pi}{N}kn} = x(n)$$

– Multiplication of two DFTs and circular convolution:

Let us consider two finite duration sequences of length N, $x_1(n)$ and $x_2(n)$. Their DFT's are given by:

$$X_1(k) = \sum_{n=0}^{N-1} x_1(n) e^{-j\frac{2\pi}{N}kn} \qquad k = 0, 1, \dots, N-1$$

$$X_2(k) = \sum_{n=0}^{N-1} x_2(n) e^{-j\frac{2\pi}{N}kn} \qquad k = 0, 1, \dots, N-1$$

If we multiply the two DFTs it results to a DFT, $X_3(k)$ of a sequence $x_3(n)$.

$$X_3(k) = X_1(k)\, X_2(k) \qquad k = 0, 1, \dots N-1$$

Taking IDFT

$$x_3(m) = \frac{1}{N} \sum_{k=0}^{N-1} X_3(k) e^{j\frac{2\pi}{N}km}$$

$$x_3(m) = \frac{1}{N} \sum_{k=0}^{N-1} X_1(k)\, X_2(k) e^{j\frac{2\pi}{N}km}$$

substituting for $X_1(k)$ and $X_2(k)$

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$$x_3(m) = \frac{1}{N}\sum_{k=0}^{N-1}\left[\sum_{n=0}^{N-1} x_1(n)e^{-j\frac{2\pi}{N}kn}\right]\left[\sum_{l=0}^{N-1} x_2(l)e^{-j\frac{2\pi}{N}kl}\right]e^{j\frac{2\pi}{N}km}$$

$$= \frac{1}{N}\sum_{n=0}^{N-1} x_1(n)\sum_{l=0}^{N-1} x_2(l)\left[\sum_{k=0}^{N-1} e^{j2\pi k(m-n-l)/N}\right]$$

wkt $\sum_{k=0}^{N-1} a^k = \begin{cases} N & a=1 \\ \frac{1-a^N}{1-a} & a\neq 1 \end{cases}$ here $a = e^{j2\pi k(m-n-l)/N}$

When $m-n-l$ is a multiple of N, $a=1$ and $a^N = 1$ for any value $a \neq 0$.

$$\sum_{k=0}^{N-1} a^k = \begin{cases} N & l = m-n+pN = ((m-n))_N \\ 0 & \text{otherwise} \end{cases}$$

Therefore

$$x_3(m) = \sum_{n=0}^{N-1} x_1(n)\, x_2((m-n))_N \; ; \; m = 0,1,\cdots N-1$$

Circular convolution

Thus multiplication of the DFTs of two sequences is equivalent to the circular convolution of the two sequences in time domain.

Circular convolution

If $x_1(n) \xleftrightarrow[N]{DFT} X_1(n)$

and $x_2(n) \xleftrightarrow[N]{DFT} X_2(n)$

then $x_1(n) \text{ N } x_2(n) \xleftrightarrow[N]{DFT} X_1(k)X_2(k)$

note: similarly

$$x_1(n)x_2(n) \xleftrightarrow[N]{DFT} \frac{1}{N} X_1(k) \text{ N } X_2(k)$$

NOTE: $\sum_{n=0}^{N-1} a^n = \begin{cases} \frac{1-a^N}{1-a} & a\neq 1 \\ N & a = 1 \end{cases}$

$$\therefore \sum_{n=0}^{N-1} W_N^{kn} = \begin{cases} \frac{1-W_N^{Nk}}{1-W_N^k} & W_N^k \neq 1 \\ N & W_N^k = 1 \end{cases}$$

but $W_N^{Nk} = 1$

$$\therefore \sum_{n=0}^{N-1} W_N^{kn} = \begin{cases} N & W_N^k = 1 \\ 0 & W_N^k \neq 1 \end{cases} = N\,\delta[(k)_N]$$

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- Symmetry properties of the DFT:

Let $x(n)$ be a sequence of length N and $X(k)$ be its N-point DFT. Considering $x(n)$ and $X(k)$ to be complex value, we have

$$x(n) = x_R(n) + jx_I(n) \qquad 0 \le n \le N-1$$

$$X(k) = X_R(k) + j\,X_I(k) \qquad 0 \le k \le N-1$$

wkt

$$X(k) = \sum_{n=0}^{N-1} x(n) e^{-j\frac{2\pi}{N}kn}$$

$$\therefore \quad X(k) = \sum_{n=0}^{N-1} \left[x_R(n) + jx_I(n)\right]\left[\cos\frac{2\pi kn}{N} - j\sin\frac{2\pi kn}{N}\right]$$

$$X_R(k) + jX_I(k) = \sum_{n=0}^{N-1}\left[\left\{x_R(n)\cos\frac{2\pi kn}{N} + x_I(n)\sin\frac{2\pi kn}{N}\right\} - j\left\{x_R(n)\sin\frac{2\pi}{N}kn - x_I(n)\cos\frac{2\pi}{N}kn\right\}\right]$$

Hence

$$X_R(k) = \sum_{n=0}^{N-1}\left[x_R(n)\cos\frac{2\pi kn}{N} + x_I(n)\sin\frac{2\pi kn}{N}\right]$$

$$X_I(k) = -\sum_{n=0}^{N-1}\left[x_R(n)\sin\frac{2\pi kn}{N} - x_I(n)\cos\frac{2\pi kn}{N}\right]$$

Similarly

$$x(n) = \frac{1}{N}\sum_{k=0}^{N-1} X(k) e^{j\frac{2\pi}{N}kn}$$

$$\therefore x(n) = \frac{1}{N}\sum_{k=0}^{N-1}\left[X_R(k) + jX_I(k)\right]\left[\cos\frac{2\pi kn}{N} + j\sin\frac{2\pi kn}{N}\right]$$

$$x_R(n) + j\,x_I(n) = \frac{1}{N}\sum_{k=0}^{N-1}\left[\left(X_R(k)\cos\frac{2\pi kn}{N} - X_I(k)\sin\frac{2\pi kn}{N}\right) + j\left(X_R(k)\sin\frac{2\pi kn}{N} + X_I(k)\cos\frac{2\pi kn}{N}\right)\right]$$

Hence

$$x_R(n) = \frac{1}{N}\sum_{k=0}^{N-1}\left[X_R(k)\cos\frac{2\pi kn}{N} - X_I(k)\sin\frac{2\pi kn}{N}\right]$$

$$x_I(n) = \frac{1}{N}\sum_{k=0}^{N-1}\left[X_R(k)\sin\frac{2\pi kn}{N} + X_I(k)\cos\frac{2\pi kn}{N}\right]$$

For real valued sequences

If the sequence $x(n)$ is real:

wkt $X(k) = \sum_{n=0}^{N-1} x(n) W_N^{kn}$

$$X^*(k) = \sum_{n=0}^{N-1} x^*(n) W_N^{-kn}$$

but as $x(n)$ is real valued $x(n) = x^*(n)$

$$X^*(k) = \sum_{n=0}^{N-1} x(n) W_N^{-kn}$$

since $W_N^{kN} = 1$ multiplying on RHS

$$X^*(k) = \sum_{n=0}^{N-1} x(n) W_N^{-kn} W_N^{Nk}$$

$$X^*(k) = \sum_{n=0}^{N-1} x(n) W_N^{(N-k)n}$$

$$\therefore X^*(k) = X(N-k) = X(-k)$$

For real and even sequences

If the sequence $x(n)$ is real and even:

$$x(n) = x(N-n)$$

and $x_I(n) = 0$

$$X(k) = \sum_{n=0}^{N-1} x_R(n) e^{-j\frac{2\pi}{N}kn}$$

odd function

$$X(k) = \sum_{n=0}^{N-1} \Big[x_R(n) \cos \frac{2\pi}{N} kn - j \underbrace{x_R(n)}_{\text{even}} \underbrace{\sin \frac{2\pi}{N} kn}_{\text{odd}} \Big]$$

$$X(k) = \sum_{n=0}^{N-1} x(n) \cos \frac{2\pi}{N} kn$$

similarly

$$x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k) \cos \frac{2\pi kn}{N}$$

For real and odd sequences

If the sequence $x(n)$ is real and odd

$$x(n) = -x(N-n)$$

and $x_I = 0$

$$X(k) = \sum_{n=0}^{N-1} x_R(n) e^{-j\frac{2\pi}{N}kn}$$

odd function

$$x(k) = \sum_{n=0}^{N-1} \Big[\cancel{\underbrace{x_R(n)}_{\text{odd}} \underbrace{\cos \frac{2\pi}{N} kn}_{\text{even}}} - j\, x_R(n) \sin \frac{2\pi}{N} kn \Big]$$

$$x(k) = -j \sum_{n=0}^{N-1} x(n) \sin \frac{2\pi}{N} kn$$

Similarly

$$x(n) = j \frac{1}{N} \sum_{k=0}^{N-1} X(k) \sin \frac{2\pi kn}{N}$$

NOTE:

Properties of W_N

1. $W_N^{k+N} = W_N^{k}$: Periodicity property

$LHS = W_N^{k+N} = e^{-j\frac{2\pi}{N}(k+N)} = e^{-j\frac{2\pi k}{N}} e^{-j2\pi} = e^{-j\frac{2\pi}{N}k} = W_N^{k} = RHS$

2. Symmetry property

$W_N^{k+\frac{N}{2}} = -W_N^{k}$

$LHS = W_N^{k+\frac{N}{2}} = e^{-j\frac{2\pi}{N}(k+\frac{N}{2})} = e^{-j\frac{2\pi k}{N}} e^{-j\pi} = -e^{-j\frac{2\pi}{N}k} = -W_N^{k} = RHS$

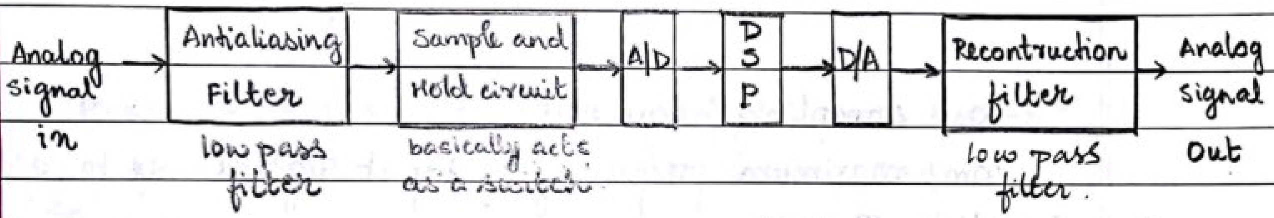
3. $W_N^2 = W_{N/2}$

$LHS = W_N^2 = e^{-j\frac{2\pi}{N}k(2)} = e^{-j\frac{2\pi}{N/2}k} = W_{N/2} = RHS$

UNIT - 01

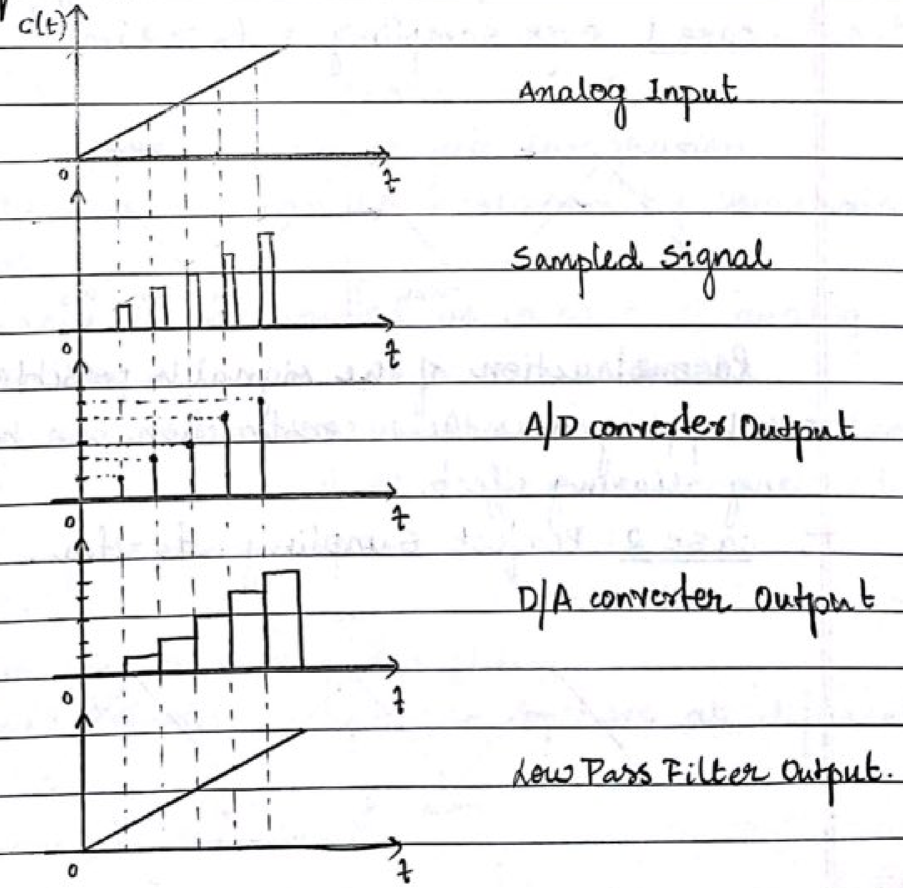
Discrete Fourier Transform and Properties of DFT

★ Digital Signal Processing system:



The analog filters used comply with the sampling theorem. The attialiasing filter is used to remove frequency components above half the sampling rate that would alias during the sampling. The reconstruction filter eliminates frequencies above the Nyquist rate.

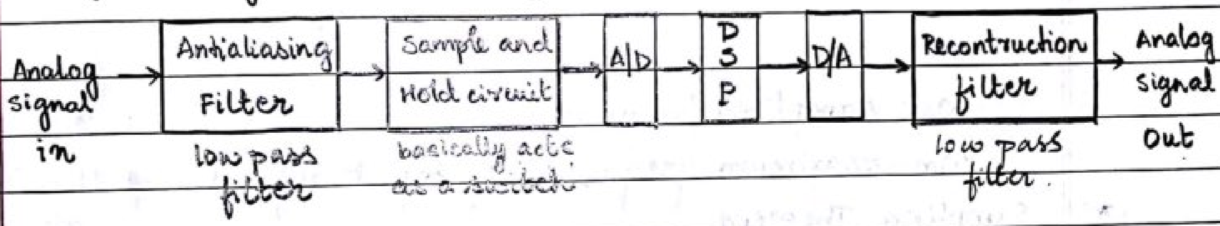
Ex:



UNIT - 01

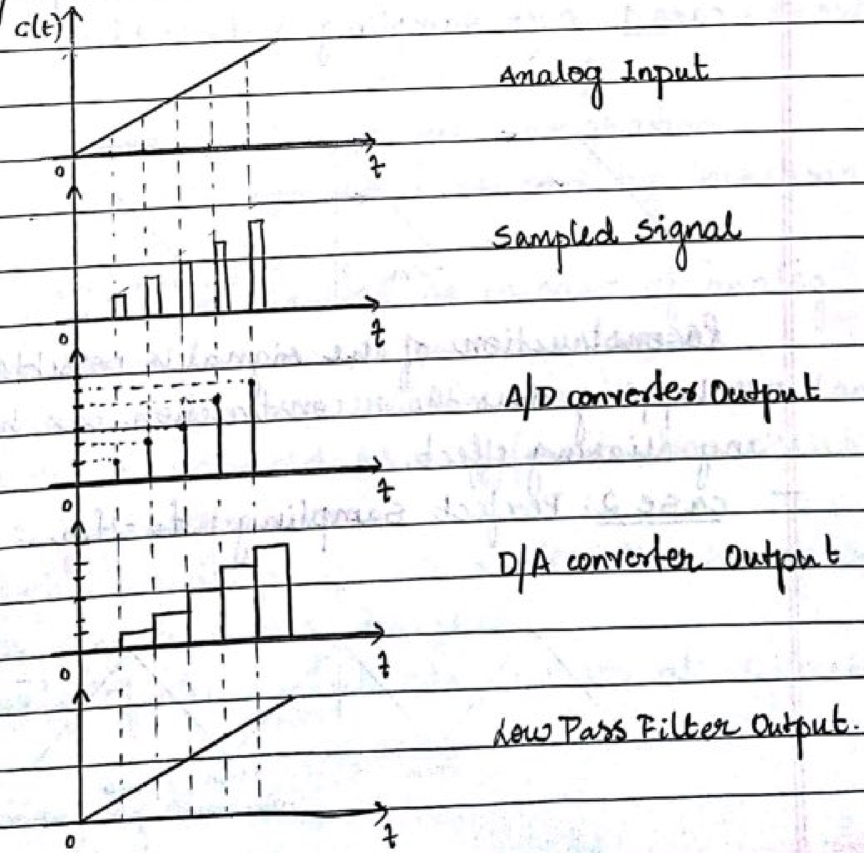
Discrete Fourier Transform and Properties of DFT

★ Digital Signal Processing system:



The analog filters used comply with the sampling theorem. The antialiasing filter is used to remove frequency components above half the sampling rate that would alias during the sampling. The reconstruction filter eliminates frequencies above the Nyquist rate.

Ex:



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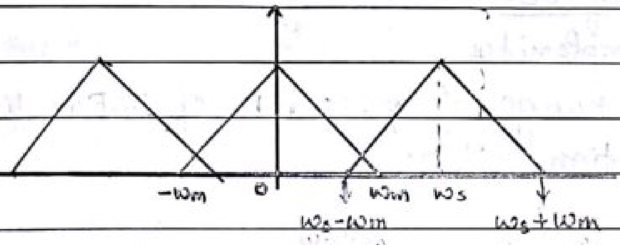
Hence the reconstruction of the signal is possible without any aliasing effect.

case 3: Under sampling : $f_s < 2f_m$



$\omega_s - \omega_m < \omega_m$

$\omega_s < 2\omega_m$

Hence reconstruction of the signal is not possible due to aliasing effect caused due to overlapping.

To avoid this phenomenon we use the antialiasing filter which is a low pass filter.

* Advantages of DSP:

1. Digital Signal Processor is less sensitive to component tolerances and environmental changes.
2. Digital signal allows for volume production without need for adjusting during construction of use.
3. Digital circuits are suitable for full integration.
4. In DSP, the accuracy can be increased by increasing the word length.
5. Dynamic range is not limited as in case of analog signal processing circuits.
6. The same processor can do multiple operations simultaneously.
7. DSP can be easily adjusted for a different characteristic by changing a few coefficients.
8. It shows the linear phase characteristics, hence linear phase systems can be easily designed.
9. Different parts of the circuit can operate at different frequencies.
10. No overloading of the circuit.

11. Easy storage of the digital data.
12. Very low frequency signal processing is possible.

★ **Disadvantages of DSP:**

1. Increased complexity
2. The frequency range of operation of DSP is limited.
3. Power dissipation is high.

★ **Frequency Domain Sampling and Discrete Fourier Transform**

To change from time domain to frequency domain we use Fourier Transform i.e, sampling $X(e^{j\omega}) \rightarrow X(k)$

Let $x(n)$ be a discrete time signal

Let the fourier transform of $x(n)$ be $X(e^{j\omega})$

$X(e^{j\omega})$ is a continous function of frequency ω.

This is periodic in nature with period 2π.

★ **Frequency Domain Sampling and Reconstruction of discrete time signal:**

Consider a non periodic finite energy discrete time signal $x(n)$. Its Fourier Transform is given by

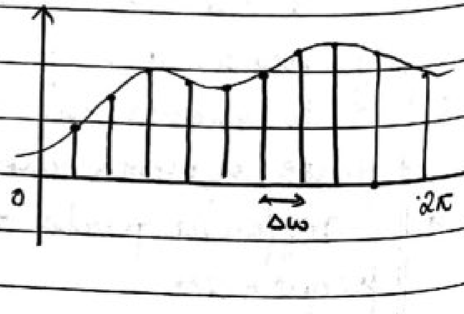
$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\omega n}$$

The Fourier Transform $X(e^{j\omega})$ is periodic in frequency domain with period 2π.

Let the frequency be in the range of 0 to 2π.

Let $\Delta\omega$ be the distance between adjacent samples.

Let N equidistant samples are taken: $0 \le \omega \le 2\pi$



$$\Delta\omega = \frac{2\pi}{N}$$

$$\omega_k = \frac{2\pi}{N} k \quad \text{where} \quad 0 \le k \le N-1$$

$$x(e^{j\omega}) = X\left(\frac{2\pi}{N} k\right)$$

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-\frac{2\pi k n}{\omega}}$$

$$x(e^{j\omega}) = \sum_{n=-N}^{-1} x(n)\, e^{-j\frac{2\pi}{N}kn} + \sum_{n=0}^{N-1} x(n)\, e^{-\frac{2\pi}{N}kn}$$

$$+ \sum_{n=N}^{2N-1} x(n)\, e^{-j\frac{2\pi}{N}kn} + \cdots$$

$$X(e^{j\omega}) = \sum_{l=-\infty}^{\infty} \sum_{n=lN}^{lN+N-1} x(n)\, e^{-j\frac{2\pi}{N}kn}$$

Changing the index of inner term from $n \to n-lN$ and changing the limits

$$X\left(\frac{2\pi}{N}k\right) = \sum_{l=-\infty}^{\infty} \sum_{n=0}^{N-1} x(n-lN)\, e^{-j\frac{2\pi}{N}kn}$$

$$X\left(\frac{2\pi}{N}k\right) = \sum_{n=0}^{N-1} \sum_{l=-\infty}^{\infty} x(n-lN)\, e^{-j\frac{2\pi}{N}kn}$$

where $x_p(n) = \sum_{l=-\infty}^{\infty} x(n-lN)$

the periodic representation of $x(n)$ of fundamental period N.

wkt

$x_p(n)$ can be represented by Fourier series

$$x_p(n) = \sum_{k=0}^{N-1} a_k\, e^{j\frac{2\pi}{N}kn}$$

$$a_k = \frac{1}{N} \sum_{n=0}^{N-1} x_p(n)\, e^{-j\frac{2\pi}{N}kn}$$

$$a_k = \frac{1}{N} X\left(\frac{2\pi}{N}k\right)$$

$$x_p(n) = \frac{1}{N} \sum_{k=0}^{N-1} X\left(\frac{2\pi}{N}k\right) e^{j\frac{2\pi}{N}kn}$$

This equation for $x_p(n)$ gives reconstruction of the periodic signal $x_p(n)$ from the samples of $X(e^{j\omega})$.

To reconstruct $x(n)$ discrete signal the number of samples N that has to be taken from the fourier transform $X(e^{j\omega})$ must be greater than or equal to the length L of the sequence

i.e., $L \leq N$

If not there will be occurance of aliasing effect.

* **Discrete Fourier Transform: DFT**

The relationship between the finite length sequence $x(n)$ which is in the range $0 \leq n \leq N-1$ and its DFT $X(k)$ is obtained by uniformly sampling $X(e^{j\omega})$ on the ω axis between $0 \leq \omega \leq 2\pi$

At $\omega = \omega_k = \frac{2\pi}{N}k \quad ; \quad 0 \leq k \leq N-1$

$$\boxed{X(k) = \sum_{n=0}^{N-1} x(n)\, e^{-j\frac{2\pi}{N}kn}} \quad ; \quad 0 \leq k \leq N-1$$

where $X(k)$ is called the N-point DFT of the sequence $x($

To have $X(k)$ of length greater than the length of the sequence of $x(n)$ it is possible by adding zeroes. This is called as zero padding.

Ex: $x(n) \rightarrow 4 = L$

$X(k) \rightarrow 8 = N$

then $N - L = 8 - 4 = 4$ zeroes must be added.

Here

$W_N = e^{-j\frac{2\pi}{N}}$: Twiddle factor or phase factor

then

$$\boxed{X(k) = \sum_{n=0}^{N-1} x(n)\, W_N^{kn}}$$

$|X(k)|$ and $\angle X(k)$ are known as the magnitude and phase spectrum respectively.

To find magnitude:

$$|X(k)| : x + iy$$
$$\rightarrow \sqrt{x^2+y^2}$$

To find phase value

$$\angle X(k) = \tan^{-1} \frac{y}{x}$$

* Inverse Discrete Fourier Transform: IDFT

Inverse Discrete Fourier Transform is given by

$$\boxed{x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k) e^{j\frac{2\pi}{N}kn}} \quad ; \quad 0 \leq k \leq N-1$$

When computing the N-point of DFT of a sequence $x(n)$ whose length is L, then N must be atleast equal to L to avoid time aliasing of the signal during reconstruction.

Q: Find the DFT of $x(n) = \delta(n)$.

wkt

$$X(k) = \sum_{n=0}^{N-1} x(n) e^{-j\frac{2\pi}{N}kn} \quad ; \quad 0 \leq k \leq N-1$$

$$X(k) = \sum_{n=0}^{N-1} \delta(n) e^{-j\frac{2\pi}{N}kn}$$

Here $\delta(n) = 1$; $n = 0$

and $\delta(n) = 0$; $n \neq 0$ (otherwise)

$$\therefore \underline{\underline{X(k) = 1}}$$

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Q: compute the 4 point DFT of the sequence $x(n)$ where $x(n) = \{\underset{\uparrow}{0}, 1, 2, 3\}$ and sketch the magnitude and phase spectrum.

wkt

$$X(k) = \sum_{n=0}^{N-1} x(n) e^{-j\frac{2\pi}{N}kn} \quad ; \quad 0 \le k \le N-1$$

$$X(k) = \sum_{n=0}^{3} x(n) e^{-j\frac{2\pi}{4}kn}$$

$$X(k) = x(0)e^{-j\frac{2\pi}{4}k(0)} + x(1)e^{-j\frac{2\pi}{4}k(1)} + x(2)e^{-j\frac{2\pi}{4}k(2)} + x(3)e^{-j\frac{2\pi}{4}k(3)}$$

$$X(k) = 0 + 1e^{-j\frac{\pi}{2}k} + 2e^{-j\pi k} + 3e^{-j\frac{3\pi}{2}k}$$

$$X(k) = e^{-j\frac{\pi}{2}k} + 2e^{-j\pi k} + 3e^{-j\frac{3\pi}{2}k} \quad 0 \le k \le N-1$$

$$0 \le k \le 3$$

when $k = 0$

$$X(0) = 1 + 2 + 3 = 6$$

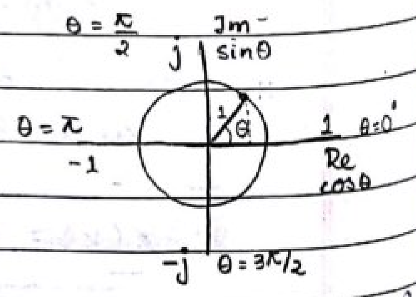
when $k = 1$

$$X(1) = e^{-j\frac{\pi}{2}} + 2e^{-j\pi} + 3e^{-j\frac{3\pi}{2}}$$

$$= \cancel{\cos\frac{\pi}{2}}^{0} - j\sin\frac{\pi}{2} + 2\cos\pi - 2j\cancel{\sin\pi}^{0} + 3\cancel{\cos\frac{3\pi}{2}}^{0} - 3j\sin\frac{3\pi}{2}$$

$e^{i\theta} = \cos\theta + i\sin\theta$

$e^{-i\theta} = \cos\theta - i\sin\theta$



$$= -j - 2 + 3j$$

$$X(1) = -2 + 2j$$

when $k = 2$

$$X(2) = e^{-j\pi} + 2e^{-2j\pi} + 3e^{-3j\pi}$$

$$= \cos\pi - j\cancel{\sin\pi}^{0} + 2\cos 2\pi - 2j\cancel{\sin 2\pi}^{0} + 3\cos 3\pi - 3j\cancel{\sin 3\pi}^{0}$$

$$= -1 + 2 - 3$$

$$X(2) = -2$$

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When $k = 3$

$$X(3) = e^{-j\frac{3\pi}{2}} + 2e^{-j3\pi} + 3e^{-j\frac{9\pi}{2}}$$

$$= \cancel{\cos\frac{3\pi}{2}}^{0} - j\sin\frac{3\pi}{2} + 2\cos 3\pi - \cancel{2\sin 3\pi}^{0} + \cancel{3\cos\frac{9\pi}{2}}^{0} - 3j\sin\frac{9\pi}{2}$$

$$= j - 2 - 3j$$

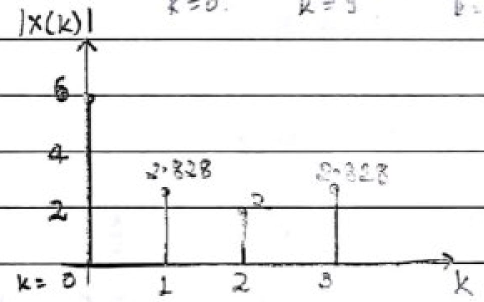
$$X(3) = -2 - 2j$$

$$\therefore X(k) = \{6, -2+2j, -2, -2-2j\}$$

$$|X(k)| = \{6, \sqrt{(-2)^2+(2)^2}, +2, \sqrt{(-2)^2+(-2)^2}\}$$

$$= \{6, 2.828, +2, 2.828\}$$

k=0 k=1 k=2 k=3



Magnitude spectrum.

Phase spectrum

$\tan^{-1}\left(\frac{y}{x}\right) = \tan^{-1}0 \Rightarrow 0$

$\tan^{-1}\left(\frac{2}{-2}\right) = \tan^{-1}(-1) = \frac{-\pi}{4} + \pi = \frac{3\pi}{4} = 135^\circ$

$\tan^{-1}(0) = 0 + \pi = 180^\circ$

$\tan^{-1}\left(\frac{-2}{-2}\right) = \tan^{-1}(1) = \frac{\pi}{4} - \pi = \frac{-3\pi}{4} = -135^\circ$

Phase angle

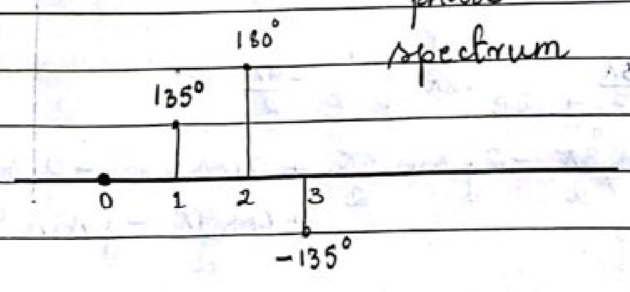
$\tan^{-1}\left(\frac{y}{x}\right)$; $x \geq 0$

$\tan^{-1}\left(\frac{y}{x}\right) + \pi$; $x < 0, y \geq 0$

$\tan^{-1}\left(\frac{y}{x}\right) - \pi$; $x < 0, y < 0$

$\frac{\pi}{2}$; $x = 0$ and $y > 0$

$\frac{-\pi}{2}$; $x = 0$ and $y < 0$



phase spectrum

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Q: Find the 4 point DFT of $x(n) = \{1, 2, 2, 1\}$ and sketch the magnitude and phase spectrum.

wkt

$$X(k) = \sum_{n=0}^{N-1} x(n) e^{-j\frac{2\pi}{N}kn}$$

$$X(k) = \sum_{n=0}^{3} x(n) e^{-\frac{2\pi}{4}kn}$$

$$X(k) = x(0)e^{-\frac{\pi}{2}k(0)} + x(1)e^{-\frac{\pi}{2}k(1)} + x(2)e^{-\frac{\pi}{2}k(2)} + x(3)e^{-\frac{\pi}{2}k(3)}$$

$$X(k) = 1 + 2e^{-\frac{\pi}{2}k} + 2e^{-\pi k} + e^{-\frac{3\pi}{2}k}$$

$$0 \le k \le N-1$$

$$0 \le k \le 3$$

When $k=0$

$$X(0) = 1 + 2 + 2 + 1 = 6$$

When $k=1$

$$X(1) = 1 + 2e^{-\frac{\pi}{2}} + 2e^{-\pi} + e^{-\frac{3\pi}{2}}$$

$$= 1 + 2\cos\frac{\pi}{2} - 2j\sin\frac{\pi}{2} + 2\cos\pi - 2j\sin\pi + \cos\frac{3\pi}{2} - j\sin\frac{3\pi}{2}$$

$$X(1) = 1 - 2j - 2 + j$$

$$X(1) = -1 - j$$

When $k=2$

$$X(2) = 1 + 2e^{-\pi} + 2e^{-2\pi} + e^{-3\pi}$$

$$= 1 + 2\cos\pi - 2j\sin\pi + 2\cos 2\pi - 2j\sin 2\pi + \cos 3\pi - j\sin 3\pi$$

$$= 1 - 2 + 2 - 1$$

$$X(2) = 0$$

When $k=3$

$$X(3) = 1 + 2^{-\frac{3\pi}{2}} + 2e^{-3\pi} + e^{-\frac{9\pi}{2}}$$

$$= 1 + 2\cos\frac{3\pi}{2} - 2j\sin\frac{3\pi}{2} + 2\cos 3\pi - 2j\sin 3\pi + \cos\frac{9\pi}{2} - j\sin\frac{9\pi}{2}$$

$= 1 + 2j - 2 - j$

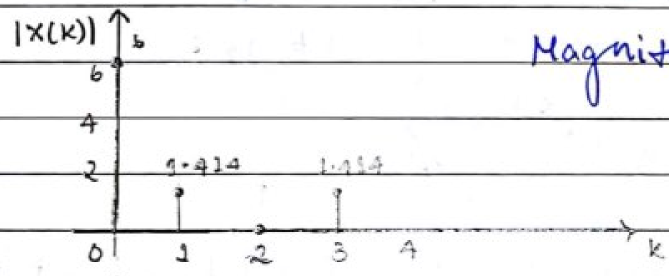
$x(3) = -1 + j$

$\therefore \; X(k) = \{6, -1-j, 0, -1+j\}$

magnitude

~~phase~~

$|X(k)| = \{6, 1.414, 0, 1.414\}$



Magnitude spectrum.

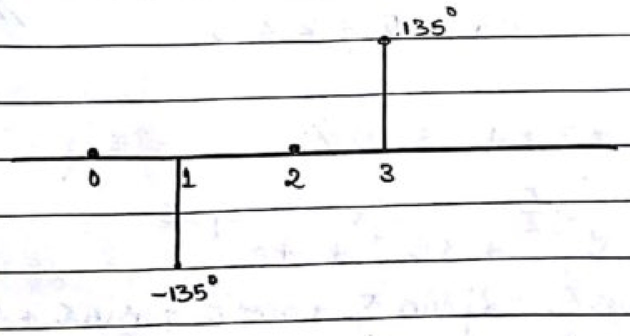
Phase:

$k=0: \tan^{-1}\left(\frac{y}{x}\right) = \tan^{-1}\left(\frac{0}{6}\right) = 0^\circ$

$k=1: \tan^{-1}\left(\frac{y}{x}\right) = \tan^{-1}\left(\frac{-1}{-1}\right) - \pi = \frac{\pi}{4} - \pi = \frac{-3\pi}{4} = -135^\circ$

$k=2: \tan^{-1}\left(\frac{y}{x}\right) = \tan^{-1}(0) = 0^\circ$

$k=3: \tan^{-1}\left(\frac{y}{x}\right) = \tan^{-1}\left(\frac{1}{-1}\right) + \pi = \frac{-\pi}{4} + \pi = \frac{3\pi}{4} = 135^\circ$



phase spectrum

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Q: compute N-point DFT of $x(n) = \delta(n-n_0)$; $0 < n_0 < N$

— wkt.

$$x(k) = \sum_{n=0}^{N-1} x(n) e^{-j\frac{2\pi}{N}kn} \quad ; \quad 0 \leq k \leq N-1$$

$$x(k) = \sum_{n=0}^{N-1} \delta(n-n_0) e^{-j\frac{2\pi}{N}kn}$$

Here $\delta(n-n_0) = 1$ for $n = n_0$
$\delta(n-n_0) = 0$ for $n \neq n_0$

$$\therefore X(k) = e^{-j\frac{2\pi}{N}kn_0}$$ //

Q: Compute 4-point DFT of $x(n) = \{1, 2, 3, 4\}$ sketch the magnitude and phase spectrum.

— wkt

$$x(k) = \sum_{n=0}^{N-1} x(n) e^{-j\frac{2\pi}{N}kn} \quad ; \quad 0 \leq k \leq N-1$$

$$x(k) = \sum_{n=0}^{3} x(n) e^{-j\frac{2\pi}{4}kn}$$

$$x(k) = x(0) e^{-j\frac{\pi}{2}k(0)} + x(1) e^{-j\frac{\pi}{2}k(1)} + x(2) e^{-j\frac{\pi}{2}k(2)} + x(3) e^{-j\frac{\pi}{2}k(3)}$$

$$X(k) = 1 + 2e^{-j\frac{\pi}{2}k} + 3e^{-j\pi k} + 4e^{-j\frac{3\pi}{2}k}$$ //

when $0 \leq k \leq 3$

When $k = 0$

$$X(0) = 1 + 2 + 3 + 4 = 10$$ //

When $k = 1$

$$X(1) = 1 + 2e^{-j\frac{\pi}{2}} + 3e^{-j\pi} + 4e^{-j\frac{3\pi}{2}}$$

$$= 1 + \cancel{2\cos\frac{\pi}{2}}^{0} - 2j\sin\frac{\pi}{2} + 3\cos\pi - \cancel{3j\sin\pi}^{0} + \cancel{4\cos\frac{3\pi}{2}}^{0} - 4j\sin\frac{3\pi}{2}$$

$$= 1 - 2j - 3 + 4j$$

$$X(1) = -2 + 2j$$ //

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When $k = 2$:

$$x(2) = 1 + 2e^{-j\pi} + 3e^{-j2\pi} + 4e^{-j3\pi}$$

$$= 1 + 2\cos\pi - 2j\cancel{\sin\pi} + 3\cos 2\pi - 3j\cancel{\sin 2\pi} + 4\cos 3\pi - 4j\cancel{\sin 3\pi}$$

$$= 1 - 2 + 3 - 4$$

$$x(2) = -2$$

When $k = 3$

$$x(3) = 1 + 2e^{-j\frac{3\pi}{2}} + 3e^{-j3\pi} + 4e^{-j\frac{9\pi}{2}}$$

$$= 1 + 2\cancel{\cos\frac{3\pi}{2}} - 2j\sin\frac{3\pi}{2} + 3\cos 3\pi - 3\cancel{\sin 3\pi} + 4\cancel{\cos\frac{9\pi}{2}} - 4j\sin\frac{9\pi}{2}$$

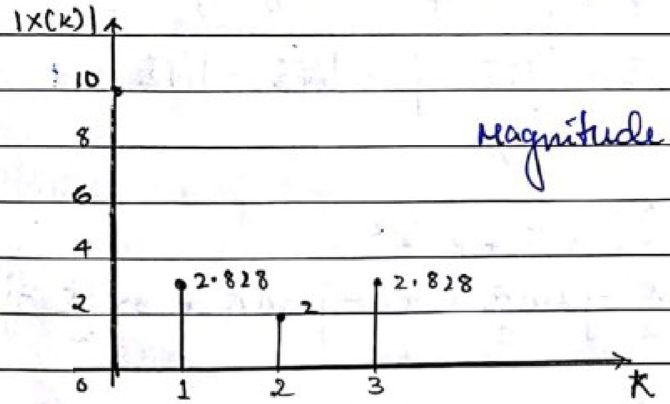
$$= 1 + 2j - 3 - 4j$$

$$= -2 - 2j$$

$$\therefore x(k) = \{10, -2+2j, -2, -2-2j\}$$

Magnitude

$$|x(k)| = \{10, 2.828, 2, 2.828\}$$



Magnitude spectrum.

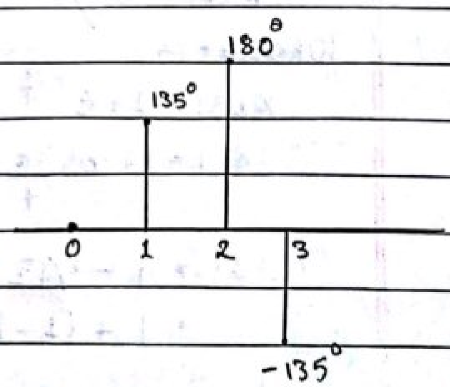
Phase spectrum.

$$\tan^{-1}\left(\frac{y}{x}\right) = \tan^{-1}(0) = 0^\circ$$

$$\tan^{-1}\left(\frac{y}{x}\right) = \tan^{-1}\left(\frac{2}{-2}\right) + \pi = \frac{3\pi}{4}$$

$$\tan^{-1}\left(\frac{y}{x}\right) = \tan^{-1}(0) + \pi = \pi$$

$$\tan^{-1}\left(\frac{y}{x}\right) = \tan^{-1}\left(\frac{-2}{-2}\right) - \pi = -\frac{3\pi}{4}$$



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Q: Compute 8-point DFT of a sequence $x(n)=\{1,1,1,1,0,0,0,0\}$

→ cokt

$$X(k)=\sum_{n=0}^{N-1} x(n)e^{-j\frac{2\pi}{N}kn} \quad ; \quad 0\le k\le N-1$$

$$X(k)=\sum_{n=0}^{7} x(n)e^{-j\frac{2\pi}{8}kn}$$

$$X(k)=x(0)e^{-\frac{\pi}{4}k(0)}+x(1)e^{-\frac{\pi}{4}k(1)}+x(2)e^{-\frac{\pi}{4}k(2)}+x(3)e^{-\frac{\pi}{4}k(3)}$$
$$+x(4)e^{-\frac{\pi}{4}k(4)}+x(5)e^{-\frac{\pi}{4}k(5)}+x(6)e^{-\frac{\pi}{4}k(6)}+x(7)e^{-\frac{\pi}{4}k(7)}$$

$$X(k)=1+e^{-\frac{\pi}{4}k}+e^{-\frac{\pi}{2}k}+e^{-\frac{3\pi}{4}k} - ; \quad 0\le k\le 7$$

When $k=0$

$$X(0)=1+1+1+1=4$$

When $k=1$

$$X(1)=1+e^{-\frac{\pi}{4}}+e^{-\frac{\pi}{2}}+e^{-\frac{3\pi}{4}}$$
$$=1+\cos\frac{\pi}{4}-j\sin\frac{\pi}{4}+\cancel{\cos\frac{\pi}{2}}^{0}-j\sin\frac{\pi}{2}+\cos\frac{3\pi}{4}-j\sin\frac{3\pi}{4}$$
$$=1+\cancel{1/\sqrt{2}}-j/\sqrt{2}-j-\cancel{1/\sqrt{2}}-j/\sqrt{2}$$
$$=1-(1+\sqrt{2})j$$

When $k=2$

$$X(2)=1+e^{-\frac{\pi}{2}}+e^{-\pi}+e^{-\frac{3\pi}{2}}$$
$$=1+\cancel{\cos\frac{\pi}{2}}^{0}-j\sin\frac{\pi}{2}+\cos\pi-j\cancel{\sin\pi}^{0}+\cancel{\cos\frac{3\pi}{2}}^{0}-j\sin\frac{3\pi}{2}$$
$$=1-j-1+j$$
$$X(2)=0$$

When $k=3$

$$X(3)=1+e^{-\frac{3\pi}{4}}+e^{-\frac{3\pi}{2}k}+e^{-\frac{9\pi}{4}k}$$
$$=1+\cos\frac{3\pi}{4}-j\sin\frac{3\pi}{4}+\cancel{\cos\frac{3\pi}{2}}^{0}-j\sin\frac{3\pi}{2}+\cos\frac{9\pi}{4}-j\sin\frac{9\pi}{4}$$
$$\therefore=1-\cancel{1/\sqrt{2}}-j/\sqrt{2}+j+\cancel{1/\sqrt{2}}-j/\sqrt{2}$$
$$=1+(1-\sqrt{2})j$$

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When $k=4$

$x(4) = 1 + e^{-\pi} + e^{-2\pi} + e^{-3\pi}$

$= 1 + \cos\pi - j\cancel{\sin\pi}^{0} + \cos 2\pi - j\cancel{\sin 2\pi}^{0} + \cos 3\pi - j\cancel{\sin 3\pi}^{0}$

$= 1 - 1 + 1 - 1$

$x(4) = 0$ //

When $k=5$

$x(5) = 1 + e^{-\frac{5\pi}{4}} + e^{-\frac{5\pi}{2}} + e^{-\frac{15\pi}{4}}$

$= 1 + \cos\frac{5\pi}{4} - j\sin\frac{5\pi}{4} + \cancel{\cos\frac{5\pi}{2}}^{0} - j\sin\frac{5\pi}{2} + \cos\frac{15\pi}{4} - j\sin\frac{15\pi}{4}$

$= 1 - \cancel{1/\sqrt{2}} + j/\sqrt{2} - j + \cancel{1/\sqrt{2}} + j/\sqrt{2}$

$= 1 - (1-\sqrt{2})j$ //

When $k=6$

$x(6) = 1 + e^{-\frac{3\pi}{2}} + e^{-3\pi} + e^{-\frac{9\pi}{2}}$

$= 1 + \cancel{\cos\frac{3\pi}{2}}^{0} - j\sin\frac{3\pi}{2} + \cos 3\pi - j\cancel{\sin 3\pi}^{0} + \cancel{\cos\frac{9\pi}{2}}^{0} - j\sin\frac{9\pi}{2}$

$= 1 + j - 1 - j$

$x(6) = 0$ //

When $k=7$

$x(7) = 1 + e^{-\frac{7\pi}{4}} + e^{-\frac{7\pi}{2}} + e^{-\frac{21\pi}{4}}$

$= 1 + \cos\frac{7\pi}{4} - j\sin\frac{7\pi}{4} + \cancel{\cos\frac{7\pi}{2}}^{0} - j\sin\frac{7\pi}{2} + \cos\frac{21\pi}{4} - j\sin\frac{21\pi}{4}$

$= 1 + \cancel{1/\sqrt{2}} - j/\sqrt{2} + j - \cancel{1/\sqrt{2}} - j/\sqrt{2}$

$x(7) = 1 + (1-\sqrt{2})j$ //

$\therefore X(k) = \{\underset{k=0}{4}, \underset{k=1}{1-(1+\sqrt{2})j}, \underset{k=2}{0}, \underset{k=3}{1+(1-\sqrt{2})j}, \underset{k=4}{0}, \underset{k=5}{1-(1-\sqrt{2})j}, \underset{k=6}{0}, \underset{k=7}{1+(1-\sqrt{2})j}\}$

Magnitude

$|X(k)| = \{4, 2.6131, 0, 1.0824, 0, 1.0824, 0, 2.6131\}$

Phase

$\angle X(k) = \{0, -1.1781, 0, -0.3927, 0, 0.3927, 0, 1.1781\}$

Q: Find the DFT of the sequence $x(n) = 1; \ 0 \le n \le 2$
$= 0;$ otherwise
For:
a. $N = 4$ sketch the magnitude and phase
b. $N = 8$ spectrum in each case.

— a. $N=4$:

wkt

$$X(k) = \sum_{n=0}^{N-1} x(n) e^{-j\frac{2\pi}{N}kn} \quad ; \quad 0 \le k \le N-1$$

$$X(k) = \sum_{n=0}^{3} x(n) \cdot e^{-j\frac{2\pi}{4}kn}$$

$$X(k) = x(0) e^{-j\frac{\pi}{2}k(0)} + x(1) e^{-j\frac{\pi}{2}k(1)} + x(2) e^{-j\pi k} + x(3) \cdot e^{-j\frac{3\pi}{2}k}$$

$$X(k) = 1 + e^{-j\frac{\pi}{2}k} + e^{-j\pi k}$$

$$X(k) = 1 + \cos\frac{\pi}{2}k - j\sin\frac{\pi}{2}k + \cos\pi k - j\sin\pi k$$

at $k=0$

$X(0) = 3$ //

at $k=1$

$X(1) = 1 + 0 - j - 1 + 0 = -j$ //

at $k=2$

$X(2) = 1 - 1 - 0 + 1 + 0 = 1$ //

at $k=3$

$X(3) = 1 + 0 + j - 1 + 0 = j$ //

$X(k) = \{3, -j, 1, j\}$

Magnitude

$|X(k)| = \{3, 1, 1, 1\}$

Phase

$\angle X(k) = \{0, -1.57, 0, 1.57\}$ //

Keerthana Ashok

b. $N = 8$:

wkt

$$X(k) = \sum_{n=0}^{N-1} x(n) e^{-j\frac{2\pi}{N}kn} \quad ; \quad 0 \le k \le N-1$$

$$X(k) = \sum_{n=0}^{7} x(n)\, e^{-j\frac{2\pi}{8}kn}$$

$$X(k) = x(0) e^{-j\frac{2\pi}{8}k(0)} + x(1) e^{-j\frac{2\pi}{8}k(1)} + x(2) e^{-j\frac{2\pi}{8}k(2)}$$

~~X(k) =~~

$$X(k) = 1 + e^{-j\frac{\pi}{4}k} + e^{-j\frac{\pi}{2}k}$$

$$X(k) = 1 + \cos\frac{\pi}{4}k - j\sin\frac{\pi}{4}k + \cos\frac{\pi}{2}k - j\sin\frac{\pi}{2}k$$

at $k=0$ $X(0) = 3$

at $k=1$

$$X(1) = 1 + 1/\sqrt{2} - j\, 1/\sqrt{2} - j = 1.707 - 1.707j$$

at $k=2$

$$X(2) = 1 + 0 - j - 1 + 0 = -j$$

at $k=3$

$$X(3) = 1 - 1/\sqrt{2} - j\, 1/\sqrt{2} + 0 + j = 0.293 + 0.293j$$

at $k=4$

$$X(4) = 1 - 1 - 0 + 1 + 0 = 1$$

at $k=5$

$$X(5) = 1 - 1/\sqrt{2} + j\, 1/\sqrt{2} + 0 - j = 0.293 - 0.293j$$

at $k=6$

$$X(6) = 1 + 0 + j - 1 + 0 = j$$

at $k=7$

$$X(7) = 1 + 1/\sqrt{2} + j\, 1/\sqrt{2} + 0 + j = 1.707 + 1.707j$$

$$\therefore X(k) = \{3,\ 1.707 - 1.707j,\ -j,\ 0.293 + 0.293j,\ 1,\ 0.293 - 0.293j,\ j,\ 1.707 + 1.707j\}$$

Magnitude

$$|X(k)| = \{3, 2.414, 1, 0.414, 1, 0.414, 1, 2.414\}$$

phase

$$\angle X(k) = \{0, 0.785, -1.57, 0.785, 0, -0.785, 1.57, 0.785\}$$

Keerthana Ashok

Homework : 28th Aug

Q: Find the IDFT of the sequence $X(k) = \{6, -2+j2, -2, -2-2j\}$

— coke

$$x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k) e^{j\frac{2\pi}{N}kn} \quad ; \quad 0 \le n \le N-1$$

$$x(n) = \frac{1}{4} \sum_{k=0}^{3} X(k) e^{j\frac{2\pi}{4}kn}$$

$$x(n) = \frac{1}{4}\left[X(0) e^{j\frac{\pi}{2}n(0)} + X(1) e^{j\frac{\pi}{2}n(1)} + X(2) e^{j\frac{\pi}{2}n(2)} + X(3) e^{j\frac{\pi}{2}k(3)} \right]$$

$$x(n) = \frac{1}{4}\left[6(1) + (-2+2j) e^{j\frac{\pi}{2}n} - 2e^{j\pi n} + (-2-2j) e^{j\frac{3\pi}{2}n} \right]$$

$$x(n) = \frac{1}{4}\left[6 + (-2+2j)(\cos \pi/2 n + j \sin \pi/2 n) - 2(\cos \pi n + j \sin \pi n) + (-2-2j)(\cos 3\pi/2 n + j \sin 3\pi/2 n) \right]$$

at $n = 0$

$$x(0) = \frac{1}{4}\left[6 + (-2+2j)(1) - 2(1) + (-2-2j)(1) \right]$$

$$= \frac{1}{4}\left[\cancel{6} \cancel{-2} + \cancel{2j} \cancel{-2} \cancel{-2} \cancel{-2j} \right] = 0$$

at $n = 1$

$$x(1) = \frac{1}{4}\left[6 + (-2+2j)(j) - 2(-1) + (-2-2j)(-j) \right]$$

$$= \frac{1}{4}\left[6 \cancel{-2j} \cancel{-2} + \cancel{2} + \cancel{2j} - 2 \right] = \frac{4}{4} = 1$$

at $n = 2$

$$x(2) = \frac{1}{4}\left[6 + (-2+2j)(-1) - 2(1) + (-2-2j)(-1) \right]$$

$$= \frac{1}{4}\left[6 + 2 \cancel{-2j} \cancel{-2} + \cancel{2} + \cancel{2j} \right] = \frac{8}{4} = 2$$

at $n = 3$

$$x(3) = \frac{1}{4}\left[6 + (-2+2j)(-j) - 2(-1) + (-2-2j)(j) \right]$$

$$= \frac{1}{4}\left[6 + \cancel{2j} + 2 + 2 \cancel{-2j} + 2 \right] = 3$$

Keerthana Ashok

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Q: Find the IDFT of $X(k) = \{1, 0, 1, 0\}$

— wkt

$$x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k) e^{j\frac{2\pi}{N}kn} \quad ; \quad 0 \le n \le N-1$$

$$x(n) = \frac{1}{4} \sum_{k=0}^{3} X(k) e^{j\frac{2\pi}{4}kn}$$

$$x(n) = \frac{1}{4}\left[X(0)e^{0} + X(1)e^{j\frac{\pi}{2}n(1)} + X(2)e^{j\frac{\pi}{2}n(2)} + X(3)e^{j\frac{\pi}{2}n(3)}\right]$$

$$x(n) = \frac{1}{4}\left[1 + 0 + e^{j\pi n} + 0\right]$$

$$x(n) = \frac{1}{4} + \frac{e^{j\pi n}}{4}$$

$$x(n) = \frac{1}{4} + \frac{1}{4}\cos\pi n - j\frac{1}{4}\sin\pi n$$

at $n = 0$

$$x(0) = \frac{1}{4} + \frac{1}{4}(1) - \frac{j}{4}(0) = \frac{1}{2}$$

at $n = 1$

$$x(1) = \frac{1}{4} + \frac{1}{4}(-1) - \frac{j}{4}(0) = 0$$

at $n = 2$

$$x(2) = \frac{1}{4} + \frac{1}{4}(1) - j\frac{1}{4}(0) = \frac{1}{2}$$

at $n = 3$

$$x(3) = \frac{1}{4} + \frac{1}{4}(-1) - j\frac{1}{4}(0) = 0$$

$$\therefore x(n) = \left\{\frac{1}{2}, 0, \frac{1}{2}, 0\right\}$$

Q: Find the 4 point DFT on the sequence $x(n) = \{1,0,0,1\}$ and verify the answer by taking 4 point IDFT of the result.

— 4 point DFT

wkt

$$X(k) = \sum_{n=0}^{N-1} x(n) e^{-j\frac{2\pi}{N}kn} \quad ; \quad 0 \le k \le N-1$$

$$X(k) = \sum_{n=0}^{3} x(n) e^{-j\frac{\pi}{2}kn}$$

$$X(k) = x(0) e^{-j\frac{\pi}{2}k(0)} + \cancel{x(1) e^{-j\frac{\pi}{2}k(1)}} + \cancel{x(2) e^{-j\frac{\pi}{2}k(2)}} + x(3) e^{-j\frac{\pi}{2}k(3)}$$

$$X(k) = 1 + e^{-j\frac{3\pi}{2}k}$$

$$X(k) = 1 + \cos\frac{3\pi}{2}k - j\sin\frac{3\pi}{2}k$$

at $k=0$

$$X(0) = 1 + \cos 0 - j\sin 0$$
$$= 1 + 1 - j(0)$$
$$= 2$$

at $k=1$

$$X(1) = 1 + \cos 3\pi/2 - j\sin 3\pi/2$$
$$= 1 + 0 + j$$
$$= 1+j$$

at $k=2$

$$X(2) = 1 + \cos 3\pi - j\sin 3\pi$$
$$= 1 - 1$$
$$= 0$$

$$X(k) = \{2,\ 1+j,\ 0,\ 1-j\}$$

at $k=3$

$$X(3) = 1 + \cos 9\pi/2 - j\sin 9\pi/2$$
$$= 1 + 0 - j$$
$$= 1-j$$

verification.

- 4 point IDFT

wkt

$$x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k) e^{j\frac{2\pi}{N}kn} \quad ; 0 \le n \le N-1$$

$$x(n) = \frac{1}{4} \sum_{k=0}^{3} X(k) e^{j\frac{2\pi}{4}kn}$$

$$x(n) = \frac{1}{4}\left[X(0)e^{j\frac{\pi}{2}(0)n} + X(1)e^{j\frac{\pi}{2}(1)n} + X(2)e^{j\frac{\pi}{2}(2)n} + X(3)e^{j\frac{\pi}{2}(3)n}\right]$$

$$x(n) = \frac{1}{4}\left[2(1) + (1+j)e^{j\frac{\pi}{2}n} + 0 + (1-j)e^{j\frac{3\pi}{2}n}\right]$$

$$x(n) = \frac{1}{4}\left[2 + (1+j)\left(\cos\frac{\pi}{2}n + j\sin\frac{\pi}{2}n\right) + (1-j)\left(\cos\frac{3\pi}{2}n + j\sin\frac{3\pi}{2}n\right)\right]$$

at $n=0$

$$x(0) = \frac{1}{4}\left[2 + (1+j)(1) + (1-j)(1)\right] = \frac{1}{4}\left[2+1+\cancel{j}+1-\cancel{j}\right] = 1$$

at $n=1$

$$x(1) = \frac{1}{4}\left[2 + (1+j)(j) + (1-j)(-j)\right] = \frac{1}{4}\left[\cancel{2}+\cancel{j}-\cancel{1}-\cancel{j}-\cancel{1}\right] = 0$$

at $n=2$

$$x(2) = \frac{1}{4}\left[2 + (1+j)(-1) + (1-j)(-1)\right] = \frac{1}{4}\left[\cancel{2}-\cancel{1}-\cancel{j}-\cancel{1}+\cancel{j}\right] = 0$$

at $n=3$

$$x(3) = \frac{1}{4}\left[2 + (1+j)(-j) + (1-j)(j)\right] = \frac{1}{4}\left[2-\cancel{j}+1+\cancel{j}+1\right] = 1$$

Hence proved. by verification.

* **Relation between $X(k)$ and W_N:**

wkt

$$X(k) = \sum_{n=0}^{N-1} x(n)\, e^{-j\frac{2\pi}{N}kn}$$

where $W_N = e^{-j\frac{2\pi}{N}}$: Twiddle Factor

$$\therefore X(k) = \sum_{n=0}^{N-1} x(n)\, W_N^{kn} \quad \text{—— (1)}$$

Eq (1) can be written in matrix form

$$X_N = W_N \cdot x_N$$

where
$$X_N = \begin{array}{c} k=0 \\ k=1 \\ \vdots \\ k=N-1 \end{array} \begin{bmatrix} X(0) \\ X(1) \\ \vdots \\ X(N-1) \end{bmatrix} \quad \text{and} \quad x_N = \begin{array}{c} k=0 \\ k=1 \\ \vdots \\ k=N-1 \end{array} \begin{bmatrix} x(0) \\ x(1) \\ \vdots \\ x(N-1) \end{bmatrix}$$

X_N is the vector containing an N-DFT samples and x_N is the vector containing N input samples and W_N is a symmetric matrix given by $[N \times N]$

$$X_N = W_N\, x_n$$

$$\therefore x_n = W_N^{-1} X_N$$

By IDFT

$$x_n = \frac{1}{N} W_N^{*} X_N$$

$$\therefore \boxed{\frac{1}{N} W_N^{*} = W_N^{-1}}$$

$$\boxed{I = \frac{1}{N} W_N W_N^{*}} \quad \text{where } I = \text{Idenity matric}$$

$$\Rightarrow \boxed{I_N = W_N W_N^{*}}$$

where I_N is an identity matrix of size $[N \times N]$

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$$W_N = \begin{array}{c|cccc} & n=0 & n=1 & \cdots & n=N-1 \\ \hline k=0 & W_N^0 & W_N^0 & \cdots & W_N^0 \\ k=1 & W_N^0 & W_N^1 & \cdots & W_N^{N-1} \\ k=2 & W_N^0 & W_N^2 & \cdots & W_N^{2(N-1)} \\ \vdots & \vdots & \vdots & & \vdots \\ k=N-1 & W_N^0 & W_N^{(N-1)} & \cdots & W_N^{(N-1)(N-1)} \end{array}$$

W_N^{kN} 8 point

$$W_8^0 = e^{-j\frac{2\pi}{8}kn} = e^{-j\frac{\pi}{4}(0)} = 1$$

$$W_8^1 = e^{-j\frac{2\pi}{8}kn} = e^{-j\frac{\pi}{4}(1)} = 1/\sqrt{2} - j/\sqrt{2}$$

$$W_8^2 = e^{-j\frac{2\pi}{8}kn} = e^{-j\frac{\pi}{4}(2)} = e^{-j\frac{\pi}{2}} = -j$$

$$W_8^3 = e^{-j\frac{2\pi}{8}kn} = e^{-j\frac{\pi}{4}(3)} = -1/\sqrt{2} - j/\sqrt{2}$$

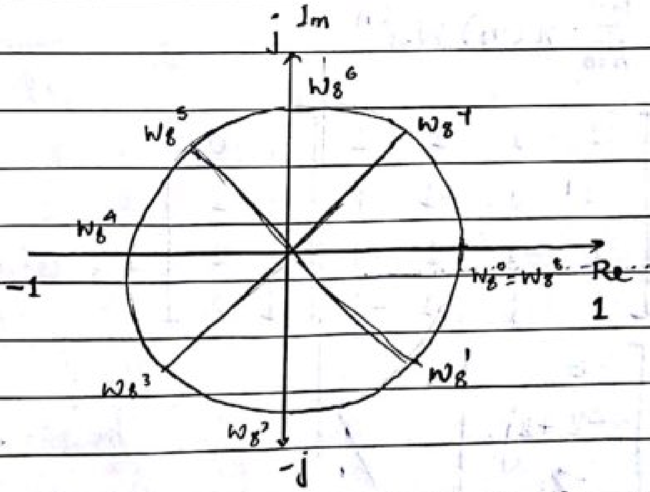
$$W_8^4 = e^{-j\frac{2\pi}{8}kn} = e^{-j\frac{\pi}{4}(4)} = e^{-j\pi} = -1$$

$$W_8^5 = e^{-j\frac{2\pi}{8}kn} = e^{-j\frac{\pi}{4}(5)} = -1/\sqrt{2} + j/\sqrt{2}$$

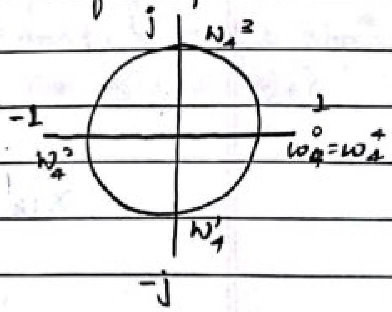
$$W_8^6 = e^{-j\frac{2\pi}{8}kn} = e^{-j\frac{\pi}{4}(6)} = j$$

$$W_8^7 = e^{-j\frac{2\pi}{8}kn} = e^{-j\frac{\pi}{4}(7)} = 1/\sqrt{2} + j/\sqrt{2}$$

$$W_8^8 = e^{-j\frac{2\pi}{8}kn} = e^{-j\frac{\pi}{4}(8)} = e^{-2\pi j} = 1$$



similarly for 4 point



complex plane

Matrix method.

For 4 point

$$W_N = \begin{array}{c c} & \begin{array}{cccc} n=0 & n=1 & n=2 & n=3 \end{array} \\ \begin{array}{c} k=0 \\ k=1 \\ k=2 \\ k=3 \end{array} & \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & W_4^1 & W_4^2 & W_4^3 \\ 1 & W_4^2 & W_4^4 & W_4^6 \\ 1 & W_4^3 & W_4^6 & W_4^9 \end{bmatrix} \end{array}$$

$W_4^1 = e^{-j\frac{\pi}{2}(1)} = -j$

$W_4^2 = e^{-j\pi} = -1$

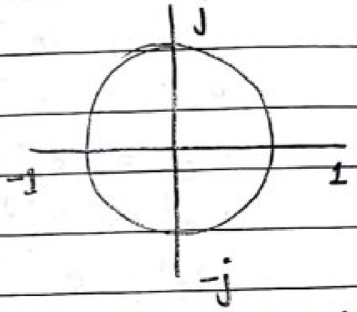
$W_4^3 = e^{-j\frac{3\pi}{2}} = j$

$W_4^4 = e^{-j2\pi} = 1$

$W_4^6 = e^{-j3\pi} = -1$

$W_4^9 = e^{-j\frac{9\pi}{2}} = -j$

$$W_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix}$$



Q: Compute the DFT of the sequence $x(n) = \{0, 1, 2, 3\}$ using linear transformation equation or matrix relation equation.

wkt

$$X(k) = \sum_{n=0}^{N-1} x(n) W_N^{kn}$$

$$\therefore X(k) = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \end{bmatrix}$$

$$X(k) = \begin{bmatrix} 6 \\ -2+2j \\ -2 \\ -2-2j \end{bmatrix}$$

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Q: Find the 4 point DFT of the sequence $x(n) = \{1, 2, 0, 1\}$ using the matrix relation equation:

wkt

$$X(k) = \sum_{n=0}^{N-1} x(n) W_N^{kn}$$

$$X(k) = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix}$$

$$X(k) = \begin{bmatrix} 4 \\ 1-j \\ -2 \\ 1+j \end{bmatrix} //$$

Q: Find the 4 point IDFT of $X(k) = \{2, 1+j, 0, 1-j\}$ using the matrix relation equation

wkt

$$x_n = \frac{1}{N} W_N^* X_N$$

NOTE exchange the sign of imaginary part of the W_N matrix to get W_N^* (conjugate of W_N)

$$\begin{bmatrix} x(0) \\ x(1) \\ x(2) \\ x(3) \end{bmatrix} = \frac{1}{4} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & j & -1 & -j \\ 1 & -1 & 1 & -1 \\ 1 & -j & -1 & j \end{bmatrix} \begin{bmatrix} X(0) \\ X(1) \\ X(2) \\ X(3) \end{bmatrix}$$

$$= \frac{1}{4} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & j & -1 & -j \\ 1 & -1 & 1 & -1 \\ 1 & -j & -1 & j \end{bmatrix} \begin{bmatrix} 2 \\ 1+j \\ 0 \\ 1-j \end{bmatrix}$$

$$\begin{bmatrix} x(0) \\ x(1) \\ x(2) \\ x(3) \end{bmatrix} = \frac{1}{4} \begin{bmatrix} 4 \\ 0 \\ 0 \\ 4 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix} //$$

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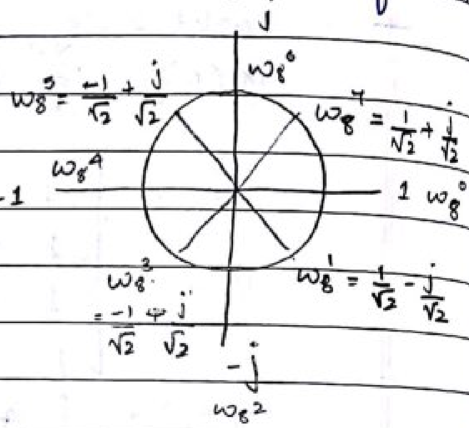
Q: Calculate 8 point DFT of $x(n) = \{1, 1, 1, 1\}$ using W_N matrix. Also calculate magnitude and phase of $X(k)$.

wkt

$$X(k) = \sum_{n=0}^{N-1} x(n) W_N^{kn}$$



$$X(k) = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \end{array} \begin{array}{c} 0 \quad 1 \quad 2 \quad \cdots \quad 7 \\ \begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & W_8^1 & W_8^2 & \cdots & W_8^7 \\ 1 & W_8^2 & W_8^4 & \cdots & W_8^{14} \\ 1 & W_8^3 & W_8^6 & \cdots & W_8^{21} \\ 1 & W_8^4 & W_8^8 & \cdots & W_8^{28} \\ 1 & W_8^5 & W_8^{10} & \cdots & W_8^{35} \\ 1 & W_8^6 & W_8^{12} & \cdots & W_8^{42} \\ 1 & W_8^7 & W_8^{14} & \cdots & W_8^{49} \end{bmatrix} \end{array} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$X(k) = \begin{array}{c} \\ k=0 \\ k=1 \\ k=2 \\ k=3 \\ k=4 \\ k=5 \\ k=6 \\ k=7 \end{array} \begin{array}{c} n=0 \quad n=1 \quad n=2 \quad n=3 \quad n=4 \quad n=5 \quad n=6 \quad n=7 \\ \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1/\sqrt{2} - j/\sqrt{2} & -j & -1/\sqrt{2} - j/\sqrt{2} & -1 & -1/\sqrt{2} + j/\sqrt{2} & j & 1/\sqrt{2} + j/\sqrt{2} \\ 1 & -j & -1 & j & 1 & -j & -1 & j \\ 1 & -1/\sqrt{2} - j/\sqrt{2} & j & 1/\sqrt{2} - j/\sqrt{2} & -1 & 1/\sqrt{2} + j/\sqrt{2} & -j & -1/\sqrt{2} + j/\sqrt{2} \\ 1 & -1 & 1 & -1 & 1 & -1 & 1 & -1 \\ 1 & -1/\sqrt{2} + j/\sqrt{2} & -j & 1/\sqrt{2} + j/\sqrt{2} & -1 & 1/\sqrt{2} - j/\sqrt{2} & j & -1/\sqrt{2} - j/\sqrt{2} \\ 1 & j & -1 & -j & 1 & j & -1 & -j \\ 1 & 1/\sqrt{2} + j/\sqrt{2} & j & -1/\sqrt{2} + j/\sqrt{2} & -1 & -1/\sqrt{2} - j/\sqrt{2} & -j & 1/\sqrt{2} - j/\sqrt{2} \end{bmatrix} \end{array} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$X(k) = \begin{bmatrix} 1 + 1 + 1 + 1 \\ 1 + \cancel{1/\sqrt{2}} - j/\sqrt{2} - j - \cancel{1/\sqrt{2}} - j/\sqrt{2} \\ \cancel{1} - \cancel{j} - \cancel{1} + \cancel{j} \\ 1 - \cancel{1/\sqrt{2}} - j/\sqrt{2} + j + \cancel{1/\sqrt{2}} - j/\sqrt{2} \\ \cancel{1} - \cancel{1} + \cancel{1} - \cancel{1} \\ 1 - \cancel{1/\sqrt{2}} - \cancel{j/\sqrt{2}} - j + \cancel{1/\sqrt{2}} + \cancel{j/\sqrt{2}} \\ \cancel{1} + \cancel{j} - \cancel{1} - \cancel{j} \\ 1 + \cancel{1/\sqrt{2}} + j/\sqrt{2} + j - \cancel{1/\sqrt{2}} + j/\sqrt{2} \end{bmatrix} = \begin{bmatrix} 4 \\ 1 - 2.414j \\ 0 \\ 1 - 0.414j \\ 0 \\ 1 - j \\ 0 \\ 1 + 2.414j \end{bmatrix}$$

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Magnitude

$|X(k)| = \{4, 2.613, 0, 1.082, 0, 1.414, 0, 2.613\}$

Phase

$\angle X(k) = \{0, -1.178, 0, -0.392, 0, -0.785, 0, 1.178\}$ //

HOME WORK: 29th Aug

Q1: Find the 6 point DFT of $x(n)$ using the matrix relation equation where $x(n) = \{1, 1, 0, 0, 0, 2\}$

wkt

$$X(k) = \sum_{n=0}^{N-1} x(n) W_N^{kn}$$

$$X(k) = \begin{array}{c|cccccc|} & n=0 & n=1 & n=2 & n=3 & n=4 & n=5 \\ k=0 & 1 & 1 & 1 & 1 & 1 & 1 \\ k=1 & 1 & W_6^1 & W_6^2 & W_6^3 & W_6^4 & W_6^5 \\ k=2 & 1 & W_6^2 & W_6^4 & W_6^6 & W_6^8 & W_6^{10} \\ k=3 & 1 & W_6^3 & W_6^6 & W_6^9 & W_6^{12} & W_6^{15} \\ k=4 & 1 & W_6^4 & W_6^8 & W_6^{12} & W_6^{16} & W_6^{20} \\ k=5 & 1 & W_6^5 & W_6^{10} & W_6^{15} & W_6^{20} & W_6^{25} \end{array} \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 2 \end{bmatrix}$$

$W_6^0 = e^{-j\frac{2\pi}{6}kn} = e^{-j\frac{\pi}{3}(0)} = 1$

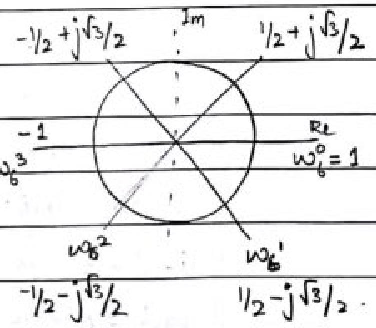
$W_6^1 = e^{-j\frac{\pi}{3}(1)} = \cos\pi/3 - j\sin\pi/3 = 1/2 - j\sqrt{3}/2$

$W_6^2 = e^{-j\frac{\pi}{3}(2)} = \cos 2\pi/3 - j\sin 2\pi/3 = -1/2 - j\sqrt{3}/2$

$W_6^3 = e^{-j\pi/3(3)} = \cos\pi - j\sin\pi = -1$

$W_6^4 = e^{-j\frac{\pi}{3}(4)} = \cos 4\pi/3 - j\sin 4\pi/3 = -1/2 + j\sqrt{3}/2$

$W_6^5 = e^{-j\frac{\pi}{3}(5)} = \cos 5\pi/3 - j\sin 5\pi/3 = 1/2 + j\sqrt{3}/2$



$$X(k) = \begin{array}{c|cccccc|} & n=0 & n=1 & n=2 & n=3 & n=4 & n=5 \\ k=0 & 1 & 1 & 1 & 1 & 1 & 1 \\ k=1 & 1 & 1/2 - j\sqrt{3}/2 & -1/2 - j\sqrt{3}/2 & -1 & -1/2 + j\sqrt{3}/2 & 1/2 + j\sqrt{3}/2 \\ k=2 & 1 & -1/2 - j\sqrt{3}/2 & -1/2 + j\sqrt{3}/2 & 1 & -1/2 - j\sqrt{3}/2 & -1/2 + j\sqrt{3}/2 \\ k=3 & 1 & -1 & 1 & -1 & 1 & -1 \\ k=4 & 1 & -1/2 + j\sqrt{3}/2 & -1/2 - j\sqrt{3}/2 & 1 & -1/2 + j\sqrt{3}/2 & -1/2 - j\sqrt{3}/2 \\ k=5 & 1 & 1/2 + j\sqrt{3}/2 & -1/2 + j\sqrt{3}/2 & -1 & -1/2 - j\sqrt{3}/2 & 1/2 - j\sqrt{3}/2 \end{array} \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 2 \end{bmatrix}$$

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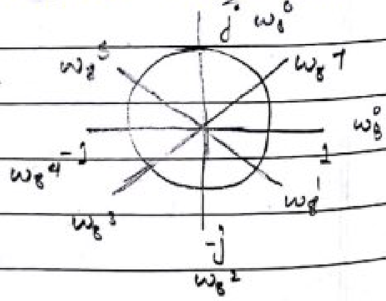
$$x(k) = \begin{bmatrix} 1+1+2 \\ 1+1/\sqrt{2} - j\sqrt{3}/2 + 1 + j\sqrt{3} \\ 1 \mp 1/\sqrt{2} - j\sqrt{3}/2 - 1 + j\sqrt{3} \\ 1 - 1 - 2 \\ 1 - 1/\sqrt{2} + j\sqrt{3}/2 - 1 - j\sqrt{3} \\ 1 + 1/\sqrt{2} + j\sqrt{3}/2 + 1 - j\sqrt{3} \end{bmatrix} = \begin{bmatrix} 4 \\ 2.707 + 0.866j \\ 0.707 + 0.866j \\ -2 \\ -0.707 - 0.866j \\ 2.707 - 0.866j \end{bmatrix}$$

Q: Compute 8 point DFT for the sequence
$x(n) = \{1, 0, 1, 0, 1, 0, 1, 0\}$

wkt

$$X(k) = \sum_{n=0}^{N-1} x(n) W_N^{kn} \quad ; \quad 0 \le k \le N-1$$

$$X(k) = x(0)W_N^0 + \cancel{x(1)}W_N^{1k} + x(2)W_N^{2k} + \cancel{x(3)}W_N^{3k} + x(4)W_N^{4k} + \cancel{x(5)}W_N^{5k} + x(6)W_N^{6k} + \cancel{x(7)}W_N^{7k}$$



$$X(k) = 1 + W_N^{2k} + W_N^{4k} + W_N^{6k}$$

at $k = 0$

$$X(0) = 1 + W_8^0 + W_8^0 + W_8^0 = 1+1+1+1 = 4$$

at $k = 1$

$$X(1) = 1 + W_8^2 + W_8^4 + W_8^6 = 1 \mp j - 1 + j = 0$$

at $k = 2$

$$X(2) = 1 + W_8^4 + W_8^8 + W_8^{12} = 1 - 1 + 1 - 1 = 0$$

at $k = 3$

$$X(3) = 1 + W_8^6 + W_8^{12} + W_8^{18} = 1 + j - 1 - j = 0$$

at $k = 4$

$$X(4) = 1 + W_8^8 + W_8^{16} + W_8^{24} = 1 + 1 + 1 + 1 = 4$$

at $k = 5$

$$X(5) = 1 + W_8^{10} + W_8^{20} + W_8^{30} = 1 - j - 1 + j = 0$$

at $k = 6$

$$X(6) = 1 + W_8^{12} + W_8^{24} + W_8^{36} = 1 \mp 1 + 1 - 1 = 0$$

at $k=7$

$$x(7) = 1 + W_8^{14} + W_8^{28} + W_8^{42} = 1 + j - 1 - j = 0$$

Therefore

$$X(k) = \{4, 0, 0, 0, 4, 0, 0, 0\}$$

Q: **Home Work : 3rd sept**

Find the 4 point DFT of the sequence $x(n) = \cos\left(\frac{n\pi}{4}\right)$

wkt

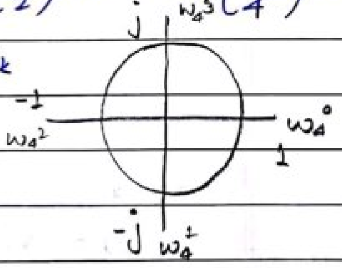
$$X(k) = \sum_{n=0}^{N-1} x(n) W_N^{kn} \; ; \quad 0 \le k \le N-1$$

$$X(k) = \sum_{n=0}^{3} \cos\left(\frac{n\pi}{4}\right) W_N^{kn}$$

$$X(k) = \cos(0) W_N^0 + \cos\left(\frac{\pi}{4}\right) W_N^k + \cos\left(\frac{\pi}{2}\right) W_N^{2k} + \cos\left(\frac{3\pi}{4}\right) W_N^{3k}$$

$$X(k) = 1(1) + (1/\sqrt{2}) W_N^k + (0) W_N^{2k} + (1/\sqrt{2}) W_N^{3k}$$

$$X(k) = 1 + \frac{W_N^k}{\sqrt{2}} + \frac{W_N^{3k}}{\sqrt{2}}$$



at $k=0$

$$X(0) = 1 + \frac{W_4^0}{\sqrt{2}} + \frac{W_4^0}{\sqrt{2}} = 1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} = 1 + \sqrt{2}$$

at $k=1$

$$X(1) = 1 + \frac{W_4^1}{\sqrt{2}} + \frac{W_4^3}{\sqrt{2}} = 1 - \frac{j}{\sqrt{2}} + \frac{j}{\sqrt{2}} = 1$$

at $k=2$

$$X(2) = 1 + \frac{W_4^2}{\sqrt{2}} + \frac{W_4^6}{\sqrt{2}} = 1 - \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}} = 1 - \sqrt{2}$$

at $k=3$

$$X(3) = 1 + \frac{W_4^3}{\sqrt{2}} + \frac{W_4^9}{\sqrt{2}} = 1 + \frac{j}{\sqrt{2}} - \frac{j}{\sqrt{2}} = 1$$

$$\therefore X(k) = (1+\sqrt{2}, 1, 1-\sqrt{2}, 1)$$

Q: compute 4 point DFT of $x(n) = \sin\left(\frac{n\pi}{4}\right)$

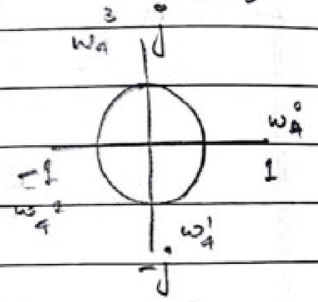
wkt

$$X(k) = \sum_{n=0}^{N-1} x(n) W_N^{kn} \; ; \quad 0 \le k \le N-1$$

$$X(k) = \sum_{n=0}^{3} \sin\left(\frac{n\pi}{4}\right) W_N^{kn}$$

$$X(k) = \cancel{\sin(0)} W_N^0 + \sin\left(\frac{\pi}{4}\right) W_N^k + \sin\left(\frac{\pi}{2}\right) W_N^{2k} + \sin\left(\frac{3\pi}{4}\right) W_N^{3k}$$

$$X(k) = \frac{W_N^k}{\sqrt{2}} + W_N^{2k} + \frac{W_N^{3k}}{\sqrt{2}}$$



at $k=0$

$$X(0) = \frac{W_4^0}{\sqrt{2}} + W_4^0 + \frac{W_4^0}{\sqrt{2}} = \frac{1}{\sqrt{2}} + 1 + \frac{1}{\sqrt{2}} = 1+\sqrt{2}$$

at $k=1$

$$X(1) = \frac{W_4^1}{\sqrt{2}} + W_4^2 + \frac{W_4^3}{\sqrt{2}} = \frac{-j}{\sqrt{2}} - 1 + \frac{j}{\sqrt{2}} = -1$$

at $k=2$

$$X(2) = \frac{W_4^2}{\sqrt{2}} + W_4^4 + \frac{W_4^6}{\sqrt{2}} = \frac{-1}{\sqrt{2}} + 1 - \frac{1}{\sqrt{2}} = 1-\sqrt{2}$$

at $k=3$

$$X(3) = \frac{W_4^3}{\sqrt{2}} + W_4^6 + \frac{W_4^9}{\sqrt{2}} = \frac{j}{\sqrt{2}} - 1 - \frac{j}{\sqrt{2}} = -1$$

Therefore

$$X(k) = \{1+\sqrt{2}, -1, 1-\sqrt{2}, -1\}$$

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Standard Problem

Q: Compute the N point DFT of the sequence $x(n) = an$; $0 \le n \le N-1$

wkt

$$X(k) = \sum_{n=0}^{N-1} x(n) W_N^{kn} \; ; \; 0 \le k \le N-1$$

$$X(k) = \sum_{n=0}^{N-1} an\, W_N^{kn}$$

$$X(k) = a \sum_{n=0}^{N-1} n\, W_N^{kn} \quad \text{—— (1)}$$

at $k=0$: $X(0) = a \sum_{n=0}^{N-1} n = a \frac{N(N-1)}{2}$

If $k \ne 0$

let W_N^k be b.

then $a \sum_{n=0}^{N-1} n b^n = X(k)$

to find $\sum_{n=0}^{N-1} n b^n$

wkt $\sum_{n=0}^{N-1} b^n = \frac{b^N - 1}{b-1}$; $b \ne 1$

differentiating both sides wrt b

$$\sum_{n=0}^{N-1} n b^{n-1} = \frac{(b-1)Nb^{N-1} - (b^N - 1)}{(b-1)^2}$$

multiplying b on both sides.

$$\sum_{n=0}^{N-1} n b^n = \frac{b[Nb^N - Nb^{N-1} - b^N + 1]}{(b-1)^2}$$

$$\sum_{n=0}^{N-1} nb^n = \frac{b[b^N(N-1) - Nb^{N-1} + 1]}{(b-1)^2}$$

but wkt here $b = W_N^k$

$$\therefore \sum_{n=0}^{N-1} n W_N^{kn} = \frac{W_N^k [W_N^{kN}(N-1) - N W_N^{(N-1)k} + 1]}{(W_N^k - 1)^2}$$

Here $W_N^{kN} = e^{-j\frac{2\pi}{N}kN}$ ~~here~~ $n = N$

$\therefore W_N^{kN} = e^{-j2\pi k} = \cos 2\pi = 1$

$$\therefore \sum_{n=0}^{N-1} n W_N^{kn} = \frac{W^k\{(N-1) - N W_N^{-k} + 1\}}{(W_N^k - 1)^2}$$

$$\sum_{n=0}^{N-1} n W_N^{kn} = \frac{W^k [N(1 - W_N^{-k})]}{(W_N^k - 1)^2}$$

$$\sum_{n=0}^{N-1} n W_N^{kn} = \frac{W_N^k N}{(W_N^k - 1)^2}\left[1 - \frac{1}{W_N^k}\right]$$

$$\sum_{n=0}^{N-1} n W_N^{kn} = \frac{\cancel{W_N^k} N}{(W_N^k - 1)^{\cancel{2}}}\left[\frac{\cancel{W_N^k - 1}}{\cancel{W_N^k}}\right]$$

$$\therefore \sum_{n=0}^{N-1} n W_N^{kn} = \frac{N}{W_N^k - 1}$$

substituting in eq (1)

$$X(k) = a \sum_{n=0}^{N-1} n W_N^{kn}$$

$$X(k) = a \frac{N}{W_N^k - 1}$$

Q: compute N-point DFT of $y(n) = e^{-j\frac{2\pi}{N}k_0 n}$; $0 \le n \le N-1$

— wkt

$$Y(k) = \sum_{n=0}^{N-1} y(n) W_N^{kn} \; ; \quad 0 \le k \le N-1$$

$$Y(k) = \sum_{n=0}^{N-1} e^{-j\frac{2\pi}{N}k_0 n} W_N^{kn}$$

Here $W_N^{kn} = e^{-j\frac{2\pi}{N}kn}$

$$Y(k) = \sum_{n=0}^{N-1} e^{-j\frac{2\pi}{N}k_0 n} e^{-j\frac{2\pi}{N}kn}$$

$$Y(k) = \sum_{n=0}^{N-1} e^{-j\frac{2\pi}{N}(k_0+k)n}$$

$$Y(k) = \frac{1 - e^{-j\frac{2\pi}{N}(k_0+k)N}}{1 - e^{-j\frac{2\pi}{N}(k_0+k_0)}}$$

* Properties of DFT :

– Periodicity Property

If $x(n)$ and $X(k)$ are DFT pairs, then the periodicity property is $x(n+N) = x(n)$

and $X(k+N) = X(k)$.

Proof:

wkt $X(k) = \sum_{n=0}^{N-1} x(n) e^{-j\frac{2\pi}{N}kn}$; $0 \le k \le N-1$.

$$X(k+N) = \sum_{n=0}^{N-1} x(n) e^{-j\frac{2\pi}{N}(k+N)n}$$

$$X(k+N) = \sum_{n=0}^{N-1} x(n) e^{-j\frac{2\pi}{N}kn} \cancel{e^{-j\frac{2\pi}{N}Nn}}^{1}$$

$$X(k+N) = \sum_{n=0}^{N-1} x(n) e^{-j\frac{2\pi}{N}kn}$$

$$\boxed{X(k+N) = X(k)}$$

Similarly

$x(n) = \frac{1}{N}\sum_{k=0}^{N-1} X(k) e^{j\frac{2\pi}{N}kn}$; $0 \le n \le N-1$

$$x(n+N) = \frac{1}{N}\sum_{k=0}^{N-1} X(k) e^{j\frac{2\pi}{N}k(n+N)}$$

$$x(n+N) = \frac{1}{N}\sum_{k=0}^{N-1} X(k) e^{j\frac{2\pi}{N}kn} \cancel{e^{j\frac{2\pi}{N}kN}}^{1}$$

$$x(n+N) = \frac{1}{N}\sum_{k=0}^{N-1} X(k) e^{j\frac{2\pi}{N}kn}$$

$$\boxed{x(n+N) = x(n)}$$

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Linearity Property:

If $x_1(n)$ and $X_1(k)$ are DFT pairs and $x_2(n)$ and $X_2(k)$ are DFT pairs then for any real valued or complex valued constants a_1 and a_2 where $y(n)$ and $Y(k)$ are DFT pairs then:

$$y(n) = a_1 x_1(n) + a_2 x_2(n)$$

and $Y(k) = a_1 X_1(k) + a_2 X_2(k)$

such that $a_1 x_1(n) + a_2 x_2(n)$ and $a_1 X_1(k) + a_2 X_2(k)$ are DFT pairs.

Proof:

$$\boxed{y(n) = a_1 x_1(n) + a_2 x_2(n)}$$

wkt $Y(k) = \sum_{n=0}^{N-1} y(n) e^{-j\frac{2\pi}{N}kn}$; $0 \le k \le N-1$

$$Y(k) = \sum_{n=0}^{N-1} (a_1 x_1(n) + a_2 x_2(n)) e^{-j\frac{2\pi}{N}kn}$$

$$Y(k) = a_1 \sum_{n=0}^{N-1} x_1(n) e^{-j\frac{2\pi}{N}kn} + a_2 \sum_{n=0}^{N-1} x_2(n) e^{-j\frac{2\pi}{N}kn}$$

$$\boxed{Y(k) = a_1 X_1(k) + a_2 X_2(k)}$$

Q: Obtain 4 point DFT of $x_1(n), x_2(n)$ where

$x_1(n) = \{1, 2, 0, 3\}$

$x_2(n) = \{0.75, 1.5, 0, 2.25\}$

i. Show that $X_2(k) = 0.75 X_1(k)$

ii. Prove the linearity property of DFT for these sequences given. $x_1(n)$ and $x_2(n)$

therefore

$$X_1(k) = \sum_{n=0}^{N-1} x_1(n) e^{-j\frac{2\pi}{N}kn} \; ; \; 0 \le k \le N-1$$

$$X_1(k) = x_1(0) e^{-j\frac{2\pi}{4}k(0)} + x_1(1) e^{-j\frac{2\pi}{4}k(1)} + x_1(2) e^{-j\frac{2\pi}{4}k(2)} + x_1(3) e^{-j\frac{2\pi}{4}k(3)}$$

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$$X_1(k) = 1 + 2e^{-j\frac{\pi}{2}k} + 3e^{-j\frac{3\pi}{2}k}$$

$$X_1(k) = 1 + 2\cos\frac{\pi}{2}k - j2\sin\frac{\pi}{2}k + 3\cos\frac{3\pi}{2}k - j3\sin\frac{3\pi}{2}k$$

at $k=0$

$X_1(0) = 1+2+3 = 6$ //

at $k=1$

$X_1(1) = 1-2j+3j = 1+j$ //

at $k=2$

$X_1(k) = \{6, 1+j, -4, 1-j\}$

$X_1(2) = 1-2-3 = -4$ //

at $k=3$

$X_1(3) = 1+2j-3j = 1-j$ //

similarly

$$X_2(k) = \sum_{n=0}^{N-1} x_2(n)e^{-j\frac{2\pi}{N}kn}$$

$$X_2(k) = x_2(0)e^{-j\frac{2\pi}{4}k(0)} + x_2(1)e^{-j\frac{2\pi}{4}k(1)} + x_2(2)e^{-j\frac{2\pi}{4}k(2)} + x_2(3)e^{-j\frac{2\pi}{4}k(3)}$$

$$X_2(k) = 0.75 + 1.5e^{-j\frac{\pi}{2}k} + 2.25e^{-j\frac{3\pi}{2}k}$$

$$X_2(k) = 0.75 + 1.5\cos\frac{\pi}{2}k - j1.5\sin\frac{\pi}{2}k + 2.25\cos\frac{3\pi}{2}k - j2.25\sin\frac{3\pi}{2}k$$

at $k=0$

$X_2(0) = 0.75 + 1.5 + 2.25 = 4.5$ //

at $k=1$

$X_2(1) = 0.75 - 1.5j + 2.25j = 0.75 + 0.75j$ //

at $k=2$

$X_2(2) = 0.75 - 1.5 - 2.25 = -3$ //

at $k=3$

$X_2(3) = 0.75 + 1.5j - 2.25j = 0.75 - 0.75j$ //

$\therefore X_2(k) = \{4.5, 0.75+0.75j, -3, 0.75-0.75j\}$ //

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i. To prove

$X_2(k) = 0.75\, X_1(k)$

$X_2(0) = 0.75\, X_1(0)$
$= 0.75(6) = 4.5$

$X_2(1) = 0.75\, X_1(1)$
$= 0.75(1+j) = 0.75 + 0.75j$

$X_2(2) = 0.75\, X_1(2)$
$= 0.75(-4) = -3$

$X_2(3) = 0.75\, X_1(3)$
$= 0.75(1-j) = 0.75 - 0.75j$

Hence proved.

ii. Linearity property.

$y(n) = a_1 x_1(n) + a_2 x_2(n)$

$y(n) = 0.75\, x_1(n) + 1\, x_2(n)$ here $a_1 = 0.75$ $a_2 = 1$

$$y(n) = 0.75\begin{bmatrix}1\\2\\0\\3\end{bmatrix} + 1\begin{bmatrix}0.75\\1.5\\0\\2.25\end{bmatrix} = \begin{bmatrix}1.5\\3\\0\\4.5\end{bmatrix}$$

$$y(k) = \begin{bmatrix}1 & 1 & 1 & 1\\1 & -j & -1 & j\\1 & -1 & 1 & -1\\1 & j & -1 & -j\end{bmatrix}\begin{bmatrix}1.5\\3\\0\\4.5\end{bmatrix} = \begin{bmatrix}9\\1.5+1.5j\\-6\\1.5-1.5j\end{bmatrix}$$

but $Y(k) = a_1 X_1(k) + a_2 X_2(k)$

$Y(k) = 0.75\, X_1(k) + 1\, X_2(k)$

$$Y(k) = 0.75\begin{bmatrix}6\\1+j\\-4\\1-j\end{bmatrix} + \begin{bmatrix}4.5\\0.75+0.75j\\-3\\0.75-0.75j\end{bmatrix} = \begin{bmatrix}9.0\\1.5+1.5j\\-6\\1.5-1.5j\end{bmatrix}$$

Hence proved

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Parseval's theorem

For two complex valued sequences $x(n)$ and $y(n)$, if $x(n)$ and $X(k)$ are DFT pairs and $y(n)$ and $Y(k)$ are DFT pairs then,

$$\left[\sum_{n=0}^{N-1} x(n)\,y^*(n) = \frac{1}{N}\sum_{k=0}^{N-1} X(k)\cdot Y^*(k)\right]$$

∴ considering the LHS

$$LHS = \sum_{n=0}^{N-1} x(n)\cdot y^*(n)$$

$$= \sum_{n=0}^{N-1} x(n)\left[\frac{1}{N}\sum_{k=0}^{N-1} Y^*(k)\, e^{j\frac{2\pi}{N}kn}\right]$$

$$= \frac{1}{N}\sum_{k=0}^{N-1} Y^*(k)\sum_{n=0}^{N-1} x(n)\, e^{j\frac{2\pi}{N}kn}$$

$$= \frac{1}{N}\sum_{k=0}^{N-1} X(k)\cdot Y^*(k) = RHS$$

Circular convolution

If $x_1(n)$ and $X_1(k)$ are DFT pairs and $x_2(n)$ and $X_2(n)$ are DFT pairs then:

$x_1(n)$ Ⓝ $x_2(n)$ and $X_1(k)X_2(k)$ are DFT pairs where Ⓝ denotes circular convolution of the sequences $x_1(n)$ and $x_2(n)$

i.e.,

if $x_1(n) \xleftrightarrow[N]{DFT} X_1(k)$

$x_2(n) \xleftrightarrow[N]{DFT} X_2(k)$

then

$$x_1(n) \, Ⓝ \, x_2(n) \xleftrightarrow[N]{DFT} X_1(k)X_2(k)$$

where $\boxed{x_1(n) \, Ⓝ \, x_2(n) = \sum_{m=0}^{N-1} x(m)\, g(n-m)_N}$

Parseval's theorem

For two complex valued sequences $x(n)$ and $y(n)$, if $x(n)$ and $X(k)$ are DFT pairs and $y(n)$ and $Y(k)$ are DFT pairs then,

$$\left[\sum_{n=0}^{N-1} x(n)\, y^*(n) = \frac{1}{N}\sum_{k=0}^{N-1} X(k).Y^*(k)\right]$$

considering the LHS

$$LHS = \sum_{n=0}^{N-1} x(n)\cdot y^*(n)$$

$$= \sum_{n=0}^{N-1} x(n)\left[\frac{1}{N}\sum_{k=0}^{N-1} Y^*(k)\, e^{j\frac{2\pi}{N}kn}\right]$$

$$= \frac{1}{N}\sum_{k=0}^{N-1} Y^*(k) \sum_{n=0}^{N-1} x(n)\, e^{j\frac{2\pi}{N}kn}$$

$$= \frac{1}{N}\sum_{k=0}^{N-1} X(k)\cdot Y^*(k) = RHS$$

Circular convolution

If $x_1(n)$ and $X_1(k)$ are DFT pairs and $x_2(n)$ and $X_2(n)$ are DFT pairs then:

$x_1(n)$ Ⓝ $x_2(n)$ and $X_1(k)X_2(k)$ are DFT pairs where Ⓝ denotes circular convolution of the sequences $x_1(n)$ and $x_2(n)$

i.e.,

if $x_1(n) \xleftrightarrow[N]{DFT} X_1(k)$

$x_2(n) \xleftrightarrow[N]{DFT} X_2(k)$

then

$$x_1(n) \, Ⓝ \, x_2(n) \xleftrightarrow[N]{DFT} X_1(k)X_2(k)$$

where $\boxed{x_1(n) \, Ⓝ \, x_2(n) = \sum_{m=0}^{N-1} x(m)\, g(n-m)_N}$

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NOTE:

wkt $\sum_{n=0}^{N-1} a^n = \begin{cases} \frac{1-a^N}{1-a} & a \neq 1 \\ N & a = 1 \end{cases}$

If $a = W_N^k$ then

$$\sum_{n=0}^{N-1} W_N^{kn} = \begin{cases} \frac{1-W_N^{Nk}}{1-W_N^k} & W_N^k \neq 1 \\ N & W_N^k = 1 \end{cases}$$

but $W_N^{Nk} = 1$

$$\therefore \sum_{n=0}^{N-1} W_N^{nk} = \begin{cases} 0 & W_N^k \neq 1 \\ N & W_N^k = 1 \end{cases}$$

$W_N^k = 1$ if $k = 0, \pm N, \pm 2N \ldots$

i.e., $W_N^k = 1$ if $(k)_N = 0$

and $W_N^k \neq 1$ if $(k)_N \neq 0$

$$\therefore \sum_{n=0}^{N-1} W_N^{nk} = \begin{cases} 0 & (k)_N \neq 0 \\ N & (k)_N = 0 \end{cases}$$

by using discrete time delta function we can define as.

$$\boxed{\sum_{n=0}^{N-1} W_N^{nk} = N \cdot \delta[(k)_N]}$$

- **Multiplication property**

If $x_1(n)$ and $X_1(k)$ are DFT pairs and $x_2(n)$ and $X_2(k)$ are DFT pairs then.

$$x_1(n)\, x_2(n) \longleftrightarrow \frac{1}{N} X_1(k) \circledast X_2(k)$$

this property is a dual of the circular convolution.

- **Multiplication of two DFTs and circular convolution.**

Let $X_1(k)$ and $X_2(k)$ be N point DFT and $x_1(n)$ and $x_2(n)$ be finite duration sequences of finite length 'N',

Let $X_3(k) = X_1(k)\, X_2(k)$

IFFT: $x_3(m) = \frac{1}{N} \sum_{k=0}^{N-1} X_3(k)\, e^{j\frac{2\pi}{N}kn}$

$$x_3(m) = \frac{1}{N} \sum_{k=0}^{N-1} X_1(k)\, X_2(k)\, e^{j\frac{2\pi}{N}kn}$$

$$x_3(m) = \frac{1}{N} \sum_{k=0}^{N-1} \sum_{n=0}^{N-1} x_1(n) e^{-j\frac{2\pi}{N}kn} \sum_{l=0}^{N-1} x_2(l) e^{-j\frac{2\pi}{N}kn}\, e^{j\frac{2\pi}{N}km}$$

$$x_3(m) = \frac{1}{N} \sum_{n=0}^{N-1} x_1(n)\, e^{-j\frac{2\pi}{N}} \sum_{l=0}^{N-1} x_2(l) e^{-j\frac{2\pi}{N}kl} \sum_{k=0}^{N-1} e^{j\frac{2\pi}{N}km}$$

$$x_3(n) = \frac{1}{N} \sum_{n} x_1(n) \sum_{l} x_2(l) \sum_{k=0}^{N-1} e^{-j\frac{2\pi}{N}k(m-n-l)}$$

but : $\sum_{k=0}^{N-1} a^k = \begin{cases} N & ; a=1 \\ \frac{1-a^N}{1-a} & ; a \neq 1 \end{cases}$

Here

If $m-n-l$ is a multiple of N, i.e. $m-n-l = pN$ then

$$l = m-n-pN$$

$$l = (m-n) \bmod N$$

$$l = (m-n)_N$$

Therefore when $\sum_{k=0}^{N-1} e^{-j\frac{2\pi k}{N}} = \begin{cases} N & \text{for } l=(m-n)_N \\ 0 & \text{for otherwise} \end{cases}$ is a multiple of N.

Therefore

$$x_3(m) = \frac{1}{\cancel{N}} \sum_{n=0}^{N-1} x_1(n) \sum_{l=0}^{N-1} x_2(m-n)_N \; (\cancel{N})$$

$$x_3(m) = [\, x_1(n) \; Ⓝ \; x_2(n) \,]$$

Hence

$$\left[x_1(n) \; Ⓝ \; x_2(n) = X_1(k) X_2(k) \right]$$

Q: compute the circular convolution of the sequences

$x_1(n) = \{1, 3, 5, 3\}$

$x_2(n) = \{2, 3, 1, 1\}$

— Matrix method

$$\begin{bmatrix} 1 & 3 & 5 & 3 \\ 3 & 1 & 3 & 5 \\ 5 & 3 & 1 & 3 \\ 3 & 5 & 3 & 1 \end{bmatrix} \times \begin{bmatrix} 2 \\ 3 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 19 \\ 17 \\ 23 \\ 25 \end{bmatrix}$$

Q: Using matrix method find the 6 point circular convolution of the following sequences.

$x_1(n) = \{1, 2, 3, 4, 5, 6\}$

$x_2(n) = \{0, 0, 1, 0, 0, 0\}$

— Matrix Method

$$\begin{bmatrix} 1 & 6 & 5 & 4 & 3 & 2 \\ 2 & 1 & 6 & 5 & 4 & 3 \\ 3 & 2 & 1 & 6 & 5 & 4 \\ 4 & 3 & 2 & 1 & 6 & 5 \\ 5 & 4 & 3 & 2 & 1 & 6 \\ 6 & 5 & 4 & 3 & 2 & 1 \end{bmatrix} * \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 5 \\ 6 \\ 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$$

Q: Consider the sequences $x_1(n) = \{0,1,2,3,4\}$ and $x_2(n) = \{0,1,0,0,0\}$. Let $X_1(k)$ and $X_2(k)$ be their respective 5 point DFTs, then by using suitable properties of DFT find $y(n)$ so that $Y(k) = X_1(k)X_2(k)$. Also find $x_3(n)$ such that $S(k) = X_1(k)X_3(k)$ and given that $s(n) = \{1,0,0,0\}$.

— Given:

$x_1(n) = \{0,1,2,3,4\}$

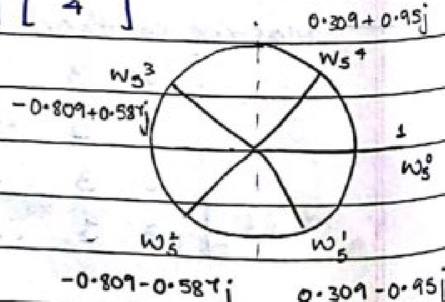
$$X_1(k) = \begin{array}{c} \\ k=0 \\ k=1 \\ k=2 \\ k=3 \\ k=4 \end{array} \begin{array}{c} \begin{array}{ccccc} n=0 & n=1 & n=2 & n=3 & n=4 \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & W_5^1 & W_5^2 & W_5^3 & W_5^4 \\ 1 & W_5^2 & W_5^4 & W_5^6 & W_5^8 \\ 1 & W_5^3 & W_5^6 & W_5^9 & W_5^{12} \\ 1 & W_5^4 & W_5^8 & W_5^{12} & W_5^{16} \end{bmatrix} \end{array} \begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$$

$W_5^1 = e^{-j\frac{2\pi}{5}(1)} = 0.309 - 0.95j$

$W_5^2 = e^{-j\frac{2\pi}{5}(2)} = -0.809 - 0.587j$

$W_5^3 = e^{-j\frac{2\pi}{5}(3)} = -0.809 + 0.587j$

$W_5^4 = e^{-j\frac{2\pi}{5}(4)} = 0.309 + 0.95j$



$$X_1(k) = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 0.309-0.95j & -0.809-0.587j & -0.809+0.587j & 0.309+0.95j \\ 1 & -0.809-0.587j & 0.309+0.95j & 0.309-0.95j & -0.809+0.587j \\ 1 & -0.809+0.587j & 0.309-0.95j & 0.309+0.95j & -0.809-0.587j \\ 1 & 0.309+0.95j & -0.809+0.587j & -0.809-0.587j & 0.309-0.95j \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$$

$$X_1(k) = \begin{bmatrix} 1+2+3+4 \\ 0.309-0.95j-1.618-1.174j-2.427+1.761j+1.236+3.8j \\ -0.809-0.587j+0.618+1.9j+0.927-2.85j-3.236+2.348j \\ -0.809+0.587j+0.618-1.9j+0.927+2.85j-3.236-2.348j \\ 0.309+0.95j-1.618+1.174j-2.427-1.761j+1.236-3.8j \end{bmatrix}$$

$$X_1(k) = \begin{bmatrix} 10 \\ -2.5+3.437j \\ -2.5+0.811j \\ -2.5-0.811j \\ -2.5-3.437j \end{bmatrix} //$$

similarly

$x_2(n) = \{0, 1, 0, 0, 0\}$

$$X_2(k) = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 0.309-0.95j & -0.809-0.587j & -0.809+0.587j & 0.309+0.95j \\ 1 & -0.809-0.587j & 0.309+0.95j & 0.309-0.95j & -0.809+0.587j \\ 1 & -0.809+0.587j & 0.309-0.95j & 0.309+0.95j & -0.809-0.587j \\ 1 & 0.309+0.95j & -0.809+0.587j & -0.809-0.587j & 0.309-0.95j \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$X_2(k) = \begin{bmatrix} 1 \\ 0.309-0.95j \\ -0.809-0.587j \\ -0.809+0.587j \\ 0.309+0.95j \end{bmatrix} //$$

To find $y(n)$:

$Y(k) = X_1(k)\, X_2(k)$

$$Y(k) = \begin{bmatrix} 10 \\ -2.5+3.437j \\ -2.5+0.811j \\ -2.5-0.811j \\ -2.5-3.437j \end{bmatrix} \begin{bmatrix} 1 \\ 0.309-0.95j \\ -0.809-0.587j \\ -0.809+0.587j \\ 0.309+0.95j \end{bmatrix} = \begin{bmatrix} 10 \\ 2.49+3.437j \\ 2.49+0.811j \\ 2.49-0.811j \\ 2.49-3.437j \end{bmatrix} //$$

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Q. $x_1(n) = \{2, 1, 2, 1\}$

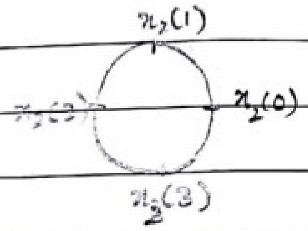
$x_2(n) = \{1, 2, 3, 4\}$

i. Time domain formula

ii. Matrix method

iii. concentric circle method

i. Time domain formula.



$$x_3(m) = \sum_{n=0}^{N-1} x_1(n)\, x_2((m-n))_N$$

→ $x_3(0) = \sum_{n=0}^{3} x_1(n)\, x_2((-n))_4$

$$x_3(0) = x_1(0)x_2((-0))_4 + x_1(1)x_2((-1))_4 + x_1(2)x_2((-2))_4 + x_1(3)x_2((-3))_4$$

$$x_3(0) = x_1(0)x_2(0) + x_1(1)x_2(3) + x_1(2)x_2(2) + x_1(3)x_2(1)$$

$$x_3(0) = 2(1) + 1(4) + 2(3) + 1(2)$$

$$x_3(0) = 2 + 4 + 6 + 2 = 14$$

→ $x_3(1) = \sum_{n=0}^{3} x_1(n)\, x_2((1-n))_4$

$$x_3(1) = x_1(0)x_2((1))_4 + x_1(1)x_2((0))_4 + x_1(2)x_2((-1))_4 + x_1(3)x_2((-2))_4$$

$$x_3(1) = x_1(0)x_2(1) + x_1(1)x_2(0) + x_1(2)x_2(3) + x_1(3)x_2(2)$$

$$x_3(1) = 2(2) + 1(1) + 2(4) + 1(3)$$

$$x_3(1) = 4 + 1 + 8 + 3 = 16$$

→ $x_3(2) = \sum_{n=0}^{3} x_1(n)\, x_2((2-n))_4$

$$x_3(2) = x_1(0)x_2((2))_4 + x_1(1)x_2((1))_4 + x_1(2)x_2((0))_4 + x_1(3)x_2((-1))_4$$

$$x_3(2) = x_1(0)\,x_2(2) + x_1(1)\,x_2(1) + x_1(2)\,x_2(0) + x_1(3)\,x_2(3)$$

$$x_3(2) = 2(3) + 1(2) + 2(1) + 1(4)$$

$$x_3(2) = 6 + 2 + 2 + 4 = 14 \,//$$

$$\rightarrow \; x_3(3) = \sum_{n=0}^{3} x_1(n)\,x_2((3-n))_4$$

$$x_3(3) = x_1(0)\,x_2((3))_4 + x_1(1)\,x_2((2))_4 + x_1(2)\,x_2((1))_4 + x_1(3)\,x_2((0))_4$$

$$x_3(3) = x_1(0)\,x_2(3) + x_1(1)\,x_2(2) + x_1(2)\,x_2(1) + x_1(3)\,x_2(0)$$

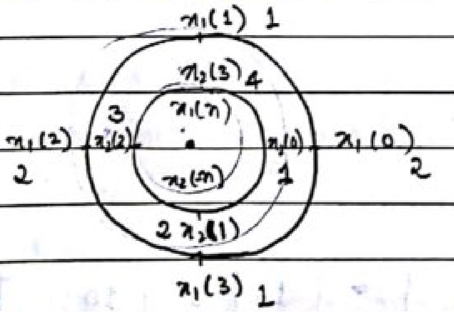
$$x_3(3) = 2(4) + 1(3) + 2(2) + 1(1)$$

$$x_3(3) = 8 + 3 + 4 + 1 = 16 \,//$$

= Matrix method

$$\begin{bmatrix} 2 & 1 & 2 & 1 \\ 1 & 2 & 1 & 2 \\ 2 & 1 & 2 & 1 \\ 1 & 2 & 1 & 2 \end{bmatrix} * \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 14 \\ 16 \\ 14 \\ 16 \end{bmatrix} \,//$$

= Concentric circle method

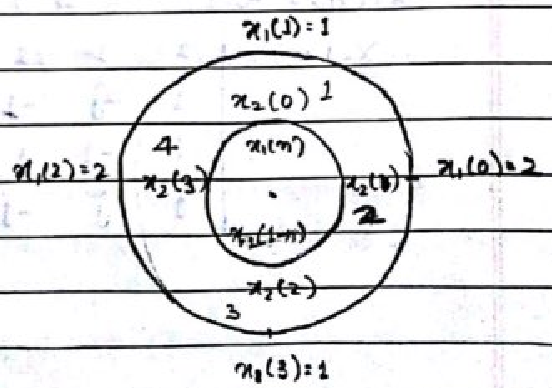


$$x_3(0) = \sum_{n=0}^{3} x_1(n)\,x_2((-n))_4$$

$$x_3(0) = 2 + 4 + 6 + 2 = 14 \,//$$

$$x_3(1) = \sum_{n=0}^{3} x_1(n)\,x_2((-n))_4$$

$$x_3(1) = 4 + 1 + 6 + 3 = 16 \,//$$



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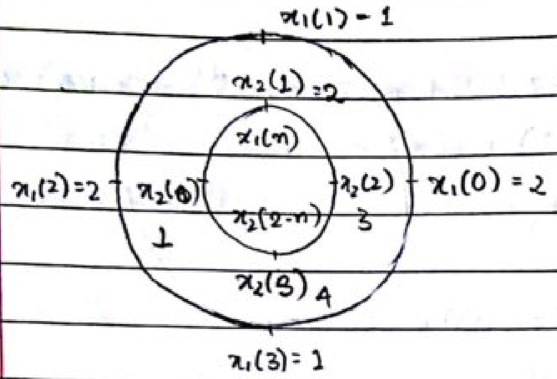
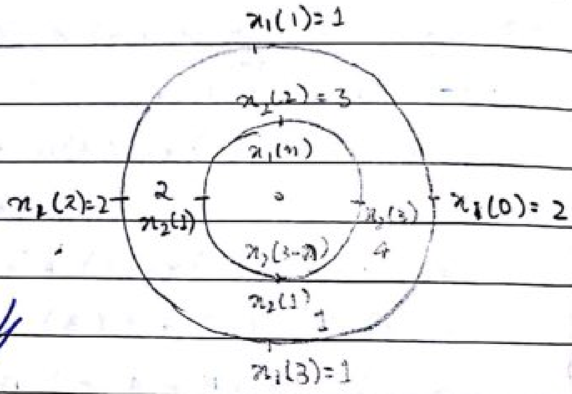


$$x_3(2) = \sum_{n=0}^{3} x_1(n)\, x_2((2-n))_4$$

$$x_3(2) = 6 + 2 + 2 + 4 = 14$$



$$x_3(3) = \sum_{n=0}^{3} x_1(n)\, x_2((3-n))_4$$

$$x_3(3) = 8 + 3 + 4 + 1 = 16$$

Q: compute the circular convolution of the sequences $x_1(n) = \{2, 1, 2, 1\}$ and $x_2(n) = \{1, 2, 3, 4\}$ using DFT and IDFT.

— given $x_1(n) = \{2, 1, 2, 1\}$

$$X_1(k) = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 2+1+2+1 \\ 2-j-2+j \\ 2-1+2-1 \\ 2+j-2-j \end{bmatrix}$$

$$X_1(k) = \begin{bmatrix} 6 \\ 0 \\ 2 \\ 0 \end{bmatrix}$$

$x_2(n) = \{1, 2, 3, 4\}$

$$X_2(k) = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -j & -1 & j \\ 1 & -1 & 1 & -1 \\ 1 & j & -1 & -j \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 10 \\ -2+2j \\ -2 \\ -2-2j \end{bmatrix}$$

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$y = x_1 * x_2$

$$y = \begin{bmatrix} 60 \\ 0 \\ -4 \\ 0 \end{bmatrix}$$

$IDFT(Y(k)) = y(n)$

$$y(n) = \frac{1}{N} \sum_{k=0}^{N-1} Y(k) e^{j\frac{2\pi}{N}nk}$$

$$y(n) = \frac{1}{4} \sum_{k=0}^{3} Y(k)\, e^{j\frac{\pi}{2}nk}$$

$$y(n) = \frac{1}{4}\left[Y(0)e^{j\frac{\pi}{2}n(0)} + Y(1)\, e^{j\frac{\pi}{2}n(1)} + Y(2)e^{j\frac{\pi}{2}n(2)} + Y(3)e^{j\frac{\pi}{2}n(3)} \right]$$

$$y(n) = \frac{1}{4}\left[60 - 4\cos\frac{3\pi}{2}n - 4j\sin\frac{3\pi}{2}n \right]$$

at $n = 0$

$$y(0) = \frac{1}{4}[60 - 4] = \frac{56}{4} = 14$$

at $n = 1$

$$y(1) = \frac{1}{4}[60 + 4j] = 15 + j$$

at $n = 2$

$$y(2) = \frac{1}{4}[60 + 4] = \frac{64}{4} = 16$$

at $n = 3$

$$y(3) = \frac{1}{4}[60 + 4 - 4j] = 16 - j$$

* Relationship of DFT with other transforms:

1. **Discrete Fourier Series coefficients of a periodic sequence (DFS):**

A periodic signal $x_p(n)$ with fundamental period N is represented in fourier series as

$$x_p(n) = \sum_{k=0}^{N-1} a_k e^{j\frac{2\pi}{N}kn} \quad ; -\infty \le n \le \infty \qquad \text{—— (1)}$$

and the fourier coefficients are given by

$$a_k = \frac{1}{N}\sum_{n=0}^{N-1} x_p(n) e^{-j\frac{2\pi}{N}kn} \quad ; \quad 0 \le k \le N-1 \qquad \text{—— (2)}$$

The N-point DFT of $x(n)$ is given by

$$X(k) = \sum_{n=0}^{N-1} x(n) e^{-j\frac{2\pi}{N}kn} \quad ; \; 0 \le k \le N-1 \qquad \text{—— (3)}$$

we also know that

$$x(n) = \frac{1}{N}\sum_{k=0}^{N-1} X(k) e^{j\frac{2\pi}{N}kn} \quad ; \; 0 \le n \le N-1 \qquad \text{—— (4)}$$

comparing eq (2) and eq (3) we know that expression of a_k is similar to that of $X(k)$

Therefore $x(n) = x_p(n)$

where $0 \le n \le N-1$

The DFT of the sequence $x_p(n)$ is given by

$$X(k) = Na_k$$

Also eq (4) resembles the equation for IDFT.

Therefore

$$DFT\{x(n)\} = DFT\{x_p(n)\}$$

2. **Fourier Transform of a non-periodic sequence**

Consider an aperiodic finite energy sequence $x(n)$ with fourier transform

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n}$$

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If $X(e^{j\omega})$ is sampled at N equally spaced samples such that $\omega_k = \frac{2\pi}{N}k$; $0 \le k \le N-1$ then

$$X(k) = X(e^{j\omega})\Big|_{\omega_k = \frac{2\pi}{N}k} = \sum x(n)\, e^{-j\frac{2\pi}{N}kn} \quad ; 0 \le k \le N-1$$
$$; -\infty \le n \le \infty$$

where $X(k)$ as the DFT coefficients of the periodic sequence $x_p(n)$ of period N is given by.

$$x_p(n) = \sum_{l=-\infty}^{\infty} x(n - lN)$$

Therefore $x_p(n)$ is obtained by aliasing $x(n)$ over the interval $0 \le n \le N-1$

Thus the finite duration sequence

$$\hat{x}(n) = \begin{cases} x_p(n) & ; \quad 0 \le n \le N-1 \\ 0 & ; \quad \text{otherwise} \end{cases}$$

which has no resemblance to the original sequence $x(n)$ unless $x(n)$ has a length $L \le N$, in this case

$$x(n) = \hat{x}(n) ; \quad 0 \le n \le N-1$$

Only in this case, the IDFT of $X(k)$ will give the original sequence $x(n)$.

3. <u>Z - Transform:</u>

The Z transform of the sequence $x(n)$ is given by

$$X(z) = \sum_{n=-\infty}^{\infty} x(n)\, z^{-n}$$

If $X(z)$ is sampled at N equally spaced points on the unit circle the $z_k = e^{j\frac{2\pi}{N}k}$

substituting for z we get

$$X(k) = X(z)\Big|_{z = e^{j\frac{2\pi}{N}k}} = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\frac{2\pi}{N}kn}$$

This is identical to the fourier transform $X(e^{j\omega})$ evaluated at N equally spaced samples.

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$$\omega_k = \frac{2\pi k}{N} \quad ; \; 0 \le k \le N-1$$

If $x(n)$ has a finite duration of length $L \le N$, the sequence can be recovered from its N-point DFT.

Therefore the z-transform is also uniquely determined by its N-point DFT and the N-point DFT, $X(k)$ can be obtained as.

$$X(z) = \sum_{n=0}^{N-1} x(n) z^{-n}$$

Also $x(n)$ can be recovered as

$$x(n) = \frac{1}{N} \sum X(z) e^{j\frac{2\pi}{N}kn} \quad \text{—— (1)}$$

we can replace $z = e^{j\frac{2\pi}{N}}$

Therefore

$$X(z) = \sum_{n=0}^{N-1} x(n) e^{-j\frac{2\pi}{N}kn} \quad ; \; 0 \le k \le N-1 \quad \text{—— (2)}$$

Replace $x(n)$ in eq (1) and in eq (2)

$$X(z) = \sum_{n=0}^{N-1} \left[\frac{1}{N} \sum_{k=0}^{N-1} X(k) e^{-j\frac{2\pi}{N}kn} \right] e^{-j\frac{2\pi}{N}kn}$$

$$X(z) = \frac{1}{N} \left[\sum_{n=0}^{N-1} X(k) e^{-j\frac{2\pi}{N}kn} e^{-j\frac{2\pi}{N}kn} \right]$$

$$X(z) = \frac{1}{N} \sum_{k=0}^{N-1} X(k) \frac{1 - z^{-N}}{1 - e^{j\frac{2\pi}{N}k} z^{-1}}$$

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Q: Find the Fourier Transform and 4 point DFT and also z transform for the sequence

$$x(n) = \begin{cases} n ; & 0 \le n \le 3 \\ 0 ; & \text{otherwise} \end{cases}$$

given :

$x(n) = \{0, 1, 2, 3\}$

Fourier transform :

$$X(e^{j\omega}) = \sum_{n=0}^{3} x(n) e^{-j\omega n}$$

$$X(e^{j\omega}) = \cancel{x(0)}^{0} e^{0} + x(1) e^{-j\omega} + x(2) e^{-j\omega 2} + x(3) e^{-3j\omega}$$

$$X(e^{j\omega}) = e^{-j\omega} + 2e^{-2j\omega} + 3e^{-3j\omega}$$

4 point DFT :

$$X(k) = \sum_{n=0}^{3} x(n) e^{-j\frac{2\pi}{4}kn}$$

$$X(k) = \cancel{x(0)}^{0} e^{0} + x(1) e^{-\frac{\pi}{2}k} + x(2) e^{-\pi k} + x(3) e^{-\frac{3\pi k}{2}}$$

$$X(k) = \cos\frac{\pi}{2}k - j\sin\frac{\pi}{2}k + 2\cos\pi k - 2j\sin\pi k + 3\cos\frac{3\pi}{2}k - 3j\sin\frac{3\pi}{2}k$$

at $k = 0$

$X(0) = 1 + 2 + 3 = 6$

at $k = 1$

$X(1) = -j - 2 + 3j = -2 + 2j$

at $k = 2$

$X(2) = -1 + 2 - 3 = -2$

at $k = 3$

$X(3) = j - 2 - 3j = -2 - 2j$

$\therefore X(k) = \{6, -2+2j, -2, -2-2j\}$

z transform :

$$X(z) = \sum_{n=0}^{3} x(n) z^{-n}$$

$$X(z) = x(0) z^{0} + x(1) z^{-1} + x(2) z^{-2} + x(3) z^{-3}$$

$$X(z) = z^{-1} + 2z^{-2} + 3z^{-3}$$

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Q: Find

~~a. Fourier transform~~

a. z transform

b. 4 point DFT

of the given sequence $x(n) = \{0.5, 0, 0.5, 0\}$.

a. z transform

$$X(z) = \sum_{n=0}^{N-1} x(n) z^{-n}$$

$$X(z) = \sum_{n=0}^{3} x(n) z^{-n}$$

$$X(z) = x(0) z^{0} + x(1) z^{-1} + x(2) z^{-2} + x(3) z^{-3}$$

$$X(z) = 0.5 + 0.5 z^{-2}$$

$$X(z) = 0.5 + 0.5 e^{-j\pi k}$$

b. 4 point DFT:

at $k=0$: $X(0) = 0.5 + 0.5 = 1$

at $k=1$: $X(1) = 0.5 - 0.5 = 0$

at $k=2$: $X(2) = 0.5 + 0.5 = 1$

at $k=3$: $X(3) = 0.5 - 0.5 = 0$

Therefore

$$X(k) = \{1, 0, 1, 0\}$$

Q: Consider the sequences $x_1(n) = \{0, 1, 2, 3, 4\}$ and $x_2(n) = \{0, 1, 0, 0, 0\}$ and $\delta(n) = \{1, 0, 0, 0, 0\}$

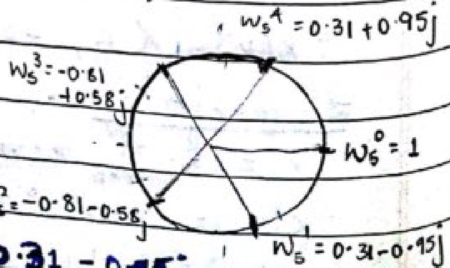
Find i. $X_1(k)$, $X_2(k)$, $\delta(k)$

ii. $y(n)$ where $Y(k) = X_1(k) * X_2(k)$

iii. $x_3(n)$ where $\delta(k) = X_1(k) * X_3(k)$

i. given: $x_1(n) = \{0, 1, 2, 3, 4\}$

wkt $X_1(k) = \sum_{n=0}^{4} x_1(n) W_k^{nk}$



$$W_5^0 = e^{-j\frac{2\pi}{5}(0)} = 1$$

$$W_5^1 = e^{-j\frac{2\pi}{5}(1)} = \cos\frac{2\pi}{5} - j\sin\frac{2\pi}{5} = 0.31 - 0.95j$$

$$W_5^2 = e^{-j\frac{2\pi}{5}(2)} = \cos\frac{4\pi}{5} - j\sin\frac{4\pi}{5} = -0.81 - 0.58j$$

$$W_5^3 = e^{-j\frac{2\pi}{5}(3)} = \cos\frac{6\pi}{5} - j\sin\frac{6\pi}{5} = -0.81 + 0.58j$$

$$W_5^4 = e^{-j\frac{2\pi}{5}(4)} = \cos\frac{8\pi}{5} - j\sin\frac{8\pi}{5} = 0.31 + 0.95j$$

$$x_1(k) = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{array}\begin{array}{c} \begin{array}{ccccc} 0 & 1 & 2 & 3 & 4 \end{array} \\ \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & W_5^1 & W_5^2 & W_5^3 & W_5^4 \\ 1 & W_5^2 & W_5^4 & W_5^6 & W_5^8 \\ 1 & W_5^3 & W_5^6 & W_5^9 & W_5^{12} \\ 1 & W_5^4 & W_5^8 & W_5^{12} & W_5^{16} \end{bmatrix} \end{array} \begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$$

$$x_1(k) = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 0.31-0.95j & -0.81-0.58j & -0.81+0.58j & 0.31+0.95j \\ 1 & -0.81-0.58j & 0.31+0.95j & 0.31-0.95j & -0.81+0.58j \\ 1 & -0.81+0.58j & 0.31-0.95j & 0.31+0.95j & -0.81-0.58j \\ 1 & 0.31+0.95j & -0.81+0.58j & -0.81-0.58j & 0.31-0.95j \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$$

$$x(k) = \begin{bmatrix} 1+2+3+4 \\ 0.31-0.95j - 1.62 - 1.16j - 2.43 + 1.74j + 1.24 + 3.8j \\ -0.81-0.58j + 0.62 + 1.9j + 0.93 - 2.85j - 3.24 + 2.32j \\ -0.81+0.58j + 0.62 - 1.9j + 0.93 + 2.85j - 3.24 - 2.32j \\ 0.31+0.95j - 1.62 + 1.16j - 2.43 - 1.74j + 1.24 - 3.8j \end{bmatrix}$$

$$x(k) = \begin{bmatrix} 10 \\ -2.5+3.43j \\ -2.5+0.49j \\ -2.5-0.49j \\ -2.5-3.43j \end{bmatrix} //$$

similarly

$$x_2(k) = \sum_{n=0}^{4} x_2(n) W_k^{nk}$$

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$$x_2(k)=\begin{bmatrix}1 & 1 & 1 & 1 & 1\\ 1 & 0.31-0.95j & -0.81-0.58j & -0.81+0.58j & 0.31+0.95j\\ 1 & -0.81-0.58j & 0.31+0.95j & 0.31-0.95j & -0.81+0.58j\\ 1 & -0.81+0.58j & 0.31-0.95j & 0.31+0.95j & -0.81-0.58j\\ 1 & 0.31+0.95j & -0.81+0.58j & -0.81-0.58j & 0.31-0.95j\end{bmatrix}\begin{bmatrix}0\\1\\0\\0\\0\end{bmatrix}$$

$$x_2(k)=\begin{bmatrix}1\\0.31-0.95j\\-0.81-0.58j\\-0.81+0.58j\\0.31+0.95j\end{bmatrix}$$ //

similarly

$$S(k)=\sum_{n=0}^{4} \cancel{b}(n)\, \cancel{W}_k^{nk}$$

$$s(k)=\begin{bmatrix}1 & 1 & 1 & 1 & 1\\ 1 & 0.31-0.95j & -0.81-0.58j & -0.81+0.58j & 0.31+0.95j\\ 1 & -0.81-0.58j & 0.31+0.95j & 0.31-0.95j & -0.81+0.58j\\ 1 & -0.81+0.58j & 0.31-0.95j & 0.31+0.95j & -0.81-0.58j\\ 1 & 0.31+0.95j & -0.81+0.58j & -0.81-0.58j & 0.31-0.95j\end{bmatrix}\begin{bmatrix}1\\0\\0\\0\\0\end{bmatrix}$$

$$s(k)=\begin{bmatrix}1\\1\\1\\1\\1\end{bmatrix}$$ //

ii. To find $y(n)$ where $y(k)=x_1(k)*x_2(k)$

Given $Y(k)=X_1(k)*X_2(k)$

then $y(n)=x_1(n) \circledN\, x_2(n)$

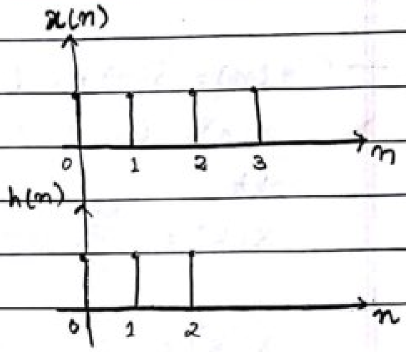
$$\therefore y(n)=\begin{bmatrix}0 & 4 & 3 & 2 & 1\\ 1 & 0 & 4 & 3 & 2\\ 2 & 1 & 0 & 4 & 3\\ 3 & 2 & 1 & 0 & 4\\ 4 & 3 & 2 & 1 & 0\end{bmatrix} * \begin{bmatrix}0\\1\\0\\0\\0\end{bmatrix} = \begin{bmatrix}4\\0\\1\\2\\3\end{bmatrix}$$ //

Q: Find $y(n) = x(n) \, Ⓝ \, h(n)$

$x(n) = u(n) - u(n-4)$

$h(n) = u(n) - u(n-3)$

Assume $N = 8$



given

$x(n) = \{1, 1, 1, 1, 0, 0, 0, 0\}$

$h(n) = \{1, 1, 1, 0, 0, 0, 0, 0\}$

$$y(n) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$y(n) = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 3 \\ 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} //$$

Q: consider a finite length sequence $x(n) = \delta(n) + 2\delta(n-5)$

Find : i. N point DFT of $x(n)$

ii. The 10 point sequence $y(n)$ that has the DFT

$Y(k) = X(k)\, W(k)$

where $X(k)$ and $W(k)$ are N-point DFTs of $x(n)$ & $w(n)$ respectively and $w(n) = \begin{cases} 1 & ; 0 \le n \le 6 \\ 0 & ; \text{otherwise} \end{cases}$

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— $x(n) = \delta(n) + 2\delta(n-5)$

$x(n) = \{1, 0, 0, 0, 0, 2, 0, 0, 0, 0\}$

wkt

$$X(k) = \sum_{n=0}^{N-1} x(n) e^{-j\frac{2\pi}{N}nk}$$

$$X(k) = \sum_{n=0}^{9} x(n) e^{-j\frac{\pi}{5}nk}$$

$$X(k) = x(0) e^{-j\frac{\pi}{5}k(0)} + x(5) e^{-j\frac{\pi}{5}k(5)}$$

$$X(k) = 1 + 2\cos\pi k - 2j\sin\pi k$$

at $k=0$

$X(0) = 1 + 2 = 3$

at $k=1$

$X(1) = 1 - 2 = -1$

at $k=2$

$X(2) = 1 + 2 = 3$

at $k=3$

$X(3) = 1 - 2 = -1$

at $k=4$

$X(4) = 1 + 2 = 3$

at $k=5$

$X(5) = 1 - 2 = -1$

at $k=6$

$X(6) = 1 + 2 = 3$

at $k=7$

$X(7) = 1 - 2 = -1$

at $k=8$

$X(8) = 1 + 2 = 3$

at $k=9$

$X(9) = 1 - 2 = -1$

Therefore

$X(k) = \{3, -1, 3, -1, 3, -1, 3, -1, 3, -1\}$

given $w(n) = \{1, 1, 1, 1, 1, 1, 0, 0, 0, 0\}$

wkt

$$W(k) = \sum_{n=0}^{N-1} w(n) e^{-j\frac{2\pi}{N}nk}$$

$$W(k) = \sum_{n=0}^{9} w(n) e^{-j\frac{\pi}{5}nk}$$

$$W(k) = x(0) e^{-j\frac{\pi}{5}k(0)} + x(1) e^{-j\frac{\pi}{5}k(1)} + x(2) e^{-j\frac{\pi}{5}k(2)} + x(3) e^{-j\frac{\pi}{5}k(3)} + x(4) e^{-j\frac{\pi}{5}k(4)} + x(5) e^{-j\frac{\pi}{5}k(5)}$$

* Symmetry Property:

Let $x(n)$ be a length N sequence whose N-point DFT is given by $X(k)$. Let $x(n)$ and $X(k)$ be complex valued

i.e., $x(n) = x_R(n) + jx_I(n) \quad ; 0 \le n \le N-1$

$X(k) = X_R(k) + jX_I(k) \quad ; 0 \le k \le N-1$

When $x_R(n)$ and $x_I(n)$ are real and imaginary of $x(n)$ and $X_R(k)$ and $X_I(k)$ are real and imaginary of $X(k)$

By defination

$$X(k) = \sum_{n=0}^{N-1} x(n) e^{-j\frac{2\pi}{N}kn}$$

Substituting $x(n)$ in the above equation

$$X(k) = \sum_{n=0}^{N-1} [x_R(n) + jx_I(n)] e^{-j\frac{2\pi}{N}kn}$$

$$X(k) = \sum_{n=0}^{N-1} [x_R(n) + jx_I(n)]\left[\cos\frac{2\pi}{N}kn - j\sin\frac{2\pi}{N}kn\right]$$

Therefore

$$X_R(k) + jX_I(k) = \sum_{n=0}^{N-1}\Big[x_R(n)\cos\frac{2\pi}{N}kn - jx_R(n)\sin\frac{2\pi}{N}kn + jx_I(n)\cos\frac{2\pi}{N}kn + x_I(n)\sin\frac{2\pi}{N}kn\Big]$$

$$X_R(k) + jX_I(k) = \sum_{n=0}^{N-1}\Big[\Big(x_R(n)\cos\frac{2\pi}{N}kn + x_I(n)\sin\frac{2\pi}{N}kn\Big) - j\Big(x_R(n)\sin\frac{2\pi}{N}kn - x_I(n)\cos\frac{2\pi}{N}kn\Big)\Big]$$

Therefore

$$X_R(k) = \sum_{n=0}^{N-1}\Big[x_R(n)\cos\frac{2\pi}{N}kn + x_I(n)\sin\frac{2\pi}{N}kn\Big]$$

$$X_I(k) = -\sum_{n=0}^{N-1}\Big[x_R(n)\sin\frac{2\pi}{N}kn - x_I(n)\cos\frac{2\pi}{N}kn\Big]$$

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By defination

$$x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k) e^{j\frac{2\pi}{N}kn}$$

Substituting $x(n)$ and $X(k)$ in the above equation

$$x_R(n) + j x_I(n) = \frac{1}{N} \sum_{k=0}^{N-1} \left[X_R(k) + j X_I(k) \right] e^{j\frac{2\pi}{N}kn}$$

$$x_R(n) + j x_I(n) = \frac{1}{N} \sum_{k=0}^{N-1} \left[X_R(k) + j X_I(k) \right] \left[\cos \frac{2\pi}{N}kn + j \sin \frac{2\pi}{N}kn \right]$$

$$x_R(n) + j x_I(n) = \frac{1}{N} \sum_{k=0}^{N-1} \left[X_R(k) \cos \frac{2\pi}{N}kn + j X_R \sin \frac{2\pi}{N}kn \right.$$

$$\left. + j X_I(k) \cos \frac{2\pi}{N}kn - X_I(k) \sin \frac{2\pi}{N}kn \right]$$

$$x_R(n) + j x_I(n) = \frac{1}{N} \sum_{k=0}^{N-1} \left[\left(X_R(k) \cos \frac{2\pi}{N}kn - X_I(k) \sin \frac{2\pi}{N}kn \right) \right.$$

$$\left. + j \left(X_R(k) \sin \frac{2\pi}{N}kn + X_I(k) \cos \frac{2\pi}{N}kn \right) \right]$$

Therefore

$$x_R(n) = \frac{1}{N} \sum_{k=0}^{N-1} \left[X_R(k) \cos \frac{2\pi}{N}kn - X_I(k) \sin \frac{2\pi}{N}kn \right]$$

$$x_I(n) = \frac{1}{N} \sum_{k=0}^{N-1} \left[X_R(k) \sin \frac{2\pi}{N}kn + X_I(k) \cos \frac{2\pi}{N}kn \right]$$

Real and even sequence:

Let N be the length of the sequence $x(n)$. Let $x(n)$ be real sequence and even.

i.e., $x(n) = x_R(n)$; $0 \le n \le N-1$

$x(n) = x(N-n) = x((-n))_N$

Let $X(k) = X_R(k) + j X_I(k)$

From the above we substitute for $X(k)$

$$X(k) = \sum_{n=0}^{N-1} x_R(n) \cos \frac{2\pi}{N}kn - j \sum_{n=0}^{N-1} x_R(n) \sin \frac{2\pi}{N}kn$$

here $x_I(n) = 0$

$$X(k) = \sum_{n=0}^{N-1} x_R(n) \cos \frac{2\pi kn}{N}$$

$x(n)$ is real and even, its N-point DFT is also real valued and even.

- Real and odd sequence:

Let N be the length of the sequence $x(n)$. Let $x(n)$ be the real sequence and odd.

i.e, $x(n) = x_R(n)$ $\quad x_I(n) = 0$

$$x(n) = x(N-n) = x((-n))_N$$

Let $X(k) = X_R(k) + jX_I(k)$

Substituting for $X(k)$ (odd function)

$$X(k) = \underbrace{\sum_{n=0}^{N-1} \underbrace{x_R(n)}_{odd} \underbrace{\cos \frac{2\pi}{N} kn}_{even}}_{0} - j \sum_{n=0}^{N-1} \underbrace{x_R(n)}_{odd} \sin \frac{2\pi}{N} kn$$

$$X(k) = -j \sum_{n=0}^{N-1} x_R(n) \sin \frac{2\pi}{N} kn \; ; \; 0 \le k \le N-1$$

Therefore for length N sequence $x(n)$ which is real valued and odd, its N-point DFT $X(k)$ is purely imaginary and odd.

- Symmetry property of DFT for a real valued sequence

Let $x(n)$ be a real valued sequence with $0 \le n \le N-1$.

Then according to symmetry property of real valued sequence

$$X(k) = X^*(N-k) \; ; \; k = 0, \dots N-1$$

wkt

$$X(k) = \sum_{n=0}^{N-1} x(n) W_N^{kn} \; ; \; 0 \le k \le N-1$$

$$X^*(k) = \sum_{n=0}^{N-1} x^*(n) W_N^{-kn}$$

Since $x(n)$ is real valued function : $x(n) = x^*(n)$

Therefore $X^*(k) = \sum_{n=0}^{N-1} x(n) W_N^{-kn}$

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$$X^*(k) = \sum_{n=0}^{N-1} x(n)\, W_N^{-kn}\, W_N^{Nn}$$

$$X^*(k) = \sum_{n=0}^{N-1} x(n)\, W_N^{n(N-k)}$$

$$\therefore \boxed{X^*(k) = X(N-k)}$$

The DFT of a real valued sequence possesses conjugate symmetry about the mid point and if N is Odd, then the index $k = N/2$ is called Folding Index.

Conjugate symmetry implies that one needs to compute only half of the DFT values to find the entire sequence. Similar result holds good for IDFT also.

Q compute the 5 point DFT of the sequence
$x(n) = \{1, 0, 1, 0, 1\}$
and verify its symmetry property.

— wkt

$$X(k) = \sum_{n=0}^{N-1} x(n) e^{-j\frac{2\pi}{N}kn}$$

$$X(k) = \sum_{n=0}^{4} x(n) e^{-j\frac{2\pi}{5}kn}$$

$$X(k) = x(0) e^{-j\frac{2\pi}{5}k(0)} + x(2) e^{-j\frac{2\pi}{5}k(2)} + x(4) e^{-j\frac{2\pi}{5}k(4)}$$

$$X(k) = 1 + e^{-j\frac{4\pi}{5}k} + e^{-j\frac{8\pi}{5}k}$$

$$X(k) = 1 + \cos\frac{4\pi}{5}k - j\sin\frac{4\pi}{5}k + \cos\frac{8\pi}{5}k - j\sin\frac{8\pi}{5}k$$

at $k=0$: $X(0) = 1 + 1 + 1 = 3$

at $k=1$: $X(1) = 1 - 0.81 - j0.58 + 0.31 + j0.95 = 0.5 + 0.36j$

at $k=2$: $X(2) = 1 + 0.31 + j0.95 - 0.81 + j0.58 = 0.5 + 1.538j$

at $k=3$: $X(3) = 1 + 0.31 - j0.95 - 0.81 - j0.58 = 0.5 - 1.538j$

at $k=4$: $X(4) = 1 - 0.81 + j0.58 + 0.31 - j0.95 = 0.5 - 0.36j$

$X^*(k) = X(5-k)$

$\therefore X^*(1) = X(4)$

$X^*(2) = X(3)$ //

Q: The first 5 points of 8 point DFT of a real valued sequence are given by {0.25, 0.5-j0.5, 0, 0.5-j0.86, 0}. Find remaining three points of this DFT.

– By conjugate property

$X^*(k) = X(8-k)$

$\therefore X^*(0) = X(8) = 0.25$ //

$X^*(1) = X(7) = 0.5+j0.5$ //

$X^*(2) = X(6) = 0$ //

$X^*(3) = X(5) = 0.5+j0.86$ //

$X^*(4) = X(4) = 0$ //

– Symmetry property of DFT

DFT of real even and real odd sequences

Let $x(n)$ be a length N real sequence with N-point DFT given by $X(k)$.

If $x(n) = x_e(n) + x_o(n)$

where $x_e(n)$: the even part of $x(n)$

$x_o(n)$: the odd part of $x(n)$

then DFT of $x_e(n)$ is purely real and DFT of $x_o(n)$ is purely imaginary.

$$x_e(n) = \frac{1}{2}\left[x(n) + x((-n))_N\right]$$

$$DFT\{x_e(n)\} = \frac{1}{2}\left[X(k) + X((-k))_N\right]$$

Let $X(k) = A+jB$

then $X^*(k) = A-jB$

$$DFT\{x_e(n)\} = \frac{1}{2}\left[X(k) + X^*(k)\right]$$

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Therefore

$$DFT\{x_e(n)\} = \frac{1}{2}\left[A + jB + A - jB\right]$$

$$DFT\{x_e(n)\} = A$$

$$x_o(n) = \frac{1}{2}\left[x(n) - x((-n))_N\right]$$

$$DFT\{x_o(n)\} = \frac{1}{2}\left[X(k) - X((-k))_N\right]$$

Let $X(k) = A + jB$

then $X^*(k) = A - jB$.

then

$$DFT\{x_o(n)\} = \frac{1}{2}\left[A + jB - (A - jB)\right]$$

$$DFT\{x_o(n)\} = \frac{1}{2}\left[\cancel{A} + jB - \cancel{A} + jB\right]$$

$$DFT\{x_o(n)\} = \frac{1}{2}(2jB)$$

$$\therefore DFT\{x_o(n)\} = jB$$

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* **Properties of W_N**

1. $W_N^{k+N} = W_N^k$

$LHS = W_N^{k+N}$

wkt $W_N = e^{-j\frac{2\pi}{N}}$

$LHS = e^{-j\frac{2\pi}{N}(k+N)}$

$= e^{-j\frac{2\pi}{N}k} \, e^{-j\frac{2\pi}{N}N}$

$= e^{-j\frac{2\pi}{N}k}$

$= W_N^k = RHS$

2. Symmetry property of W_N

$W_N^{k+\frac{N}{2}} = -W_N^k$

$LHS = W_N^{k+\frac{N}{2}}$

wkt $W_N = e^{-j\frac{2\pi}{N}}$

$LHS = e^{-j\frac{2\pi}{N}(k+N/2)}$

$= e^{-j\frac{2\pi}{N}k} \, e^{-j\pi}$

$= -e^{-j\frac{2\pi}{N}k}$

$= -W_N^k = RHS$

3. $W_N^2 = W_{N/2}$

$LHS = W_N^2$

$= e^{-j\frac{2\pi}{N}k(2)}$

$= e^{-j\,2\pi k/(N/2)}$

$= W_{N/2} = RHS$

Applications

- Linear filtering using DFT:
- DFT for linear filtering of long duration sequences.
- Frequency or spectral ananlysis using DFT

UNIT - 02

Fast Fourier Transform Algorithms

* **Direct computations of the DFT:**

$$X(k) = \sum_{n=0}^{N-1} x(n) W_N^{kn}$$

$$X(k) = \sum_{n=0}^{N-1} x(n) e^{-j\frac{2\pi}{N}kn}$$

$$X(k) = \sum_{n=0}^{N-1} \left[x_R(n) + j x_I(n)\right]\left[\cos\frac{2\pi}{N}kn - j\sin\frac{2\pi}{N}kn\right]$$

$$X(k) = \sum_{n=0}^{N-1} \left[x_R(n)\cos\frac{2\pi}{N}kn + x_I(n)\sin\frac{2\pi}{N}kn\right]$$

$$-j\sum_{n=0}^{N-1} \left[x_R(n)\sin\frac{2\pi}{N}kn - x_I(n)\cos\frac{2\pi}{N}kn\right]$$

The direct computation requires:

a. $2N^2$ evaluations of trigonometric functions.
b. $4N^2$ real multiplications
c. $4N(N-1)$ real additions

* **Classifications of FFT algorithms:**

a. divide and conquer approach
 - Radix-2 FFT algorithm
 - Radix-4 FFT algorithm
 - split-Radix FFT algorithm

b. DFT as linear filtering
 - Chirp-z algorithm
 - Goertzel algorithm

* **Radix-2 DIT FFT:**

Let $x(n)$ be split into two $N/2$ point sequences $f_1(n)$ with even indexed values and $f_2(n)$ with odd indexed values.

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considering $x(n) = \{x(0), x(1), x(2), x(3), x(4) \ldots\ldots x(n-1)\}$

where $f_1(n) = \{x(0), x(2), x(4) \ldots\}$

$f_2(n) = \{x(1), x(3), x(5) \ldots\}$

Therefore

$$f_1(n) = x(2n) \quad ; 0 \le n \le (N/2)-1$$

$$f_2(n) = x(2n+1) \quad ; 0 \le n \le (N/2)-1$$

Thus $f_1(n)$ and $f_2(n)$ are obtained by decimating $x(n)$. Hence the resulting FFT algorithm is called a decimation-in-time algorithm.

The N-point DFT is given by

$$X(k) = \sum_{n=0}^{N-1} x(n) W_N^{kn} \quad ; k = 0, 1 \ldots N-1$$

$$X(k) = \sum_{r=0}^{N/2-1} x(2r) W_N^{2kr} + \sum_{r=0}^{N/2-1} x(2r+1) W_N^{k(2r+1)}$$

$$X(k) = \sum_{r=0}^{N/2-1} g(r) W_{N/2}^{kr} + \sum_{r=0}^{N/2-1} h(n) W_{N/2}^{kr} W_N^{k}$$

$$X(k) = G(k) + H(k) W_N^{k} \quad ; k = 0, 1 \ldots N-1$$

where $G(k)$ and $H(k)$ are the $N/2$ point DFTs of the sequences $f_1(n)$ and $f_2(n)$ respectively.

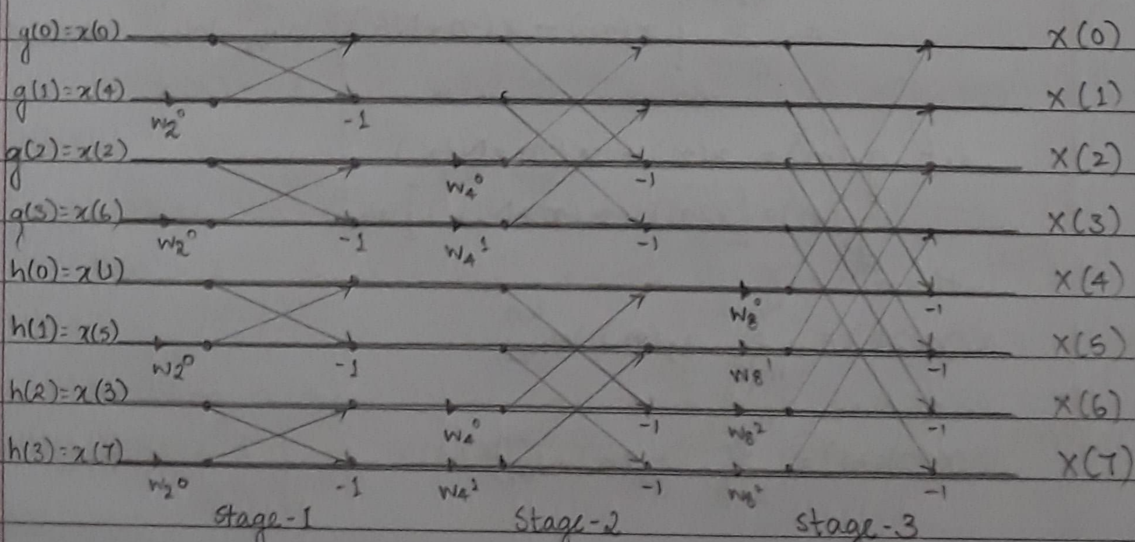
Since $G(k)$ and $H(k)$ are periodic with period $N/2$

$\therefore G(k+N/2) = G(k)$ and $H(k+N/2) = H(k)$

$$X(k) = G(k) + H(k) W_N^{k}$$

$$X(k+N/2) = G(k+N/2) + H(k+N/2) W_N^{k+N/2}$$

$$X(k+N/2) = G(k) - H(k) W_N^{k}$$



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* **Radix-2 DIF FFT:**

Let $x(n)$ be a sequence of length N.
The N-point DFT of the sequence is given by

$$X(k) = \sum_{n=0}^{N-1} x(n) W_N^{kn} \quad ; k = 0, 1, \cdots N-1$$

By splitting the DFT formula into two summation at N/2, we get

$$X(k) = \sum_{n=0}^{N/2-1} x(n) W_N^{kn} + \sum_{n=N/2}^{N-1} x(n) W_N^{kn}$$

$$X(k) = \sum_{n=0}^{N/2-1} x(n) W_N^{kn} + \sum_{n=0}^{N/2-1} x(n+N/2) W_N^{k(n+N/2)}$$

$$X(k) = \sum_{n=0}^{N/2-1} x(n) W_N^{kn} + W_N^{kN/2} \sum_{n=0}^{N/2-1} x(n+N/2) W_N^{kn}$$

but $W_N^{kN/2} = e^{-j\frac{2\pi}{N}k\frac{N}{2}} = e^{-j\pi k} = (-1)^k$

$$X(k) = \sum_{n=0}^{N/2-1} x(n) W_N^{kn} + (-1)^k \sum_{n=0}^{N/2-1} x(n+N/2) W_N^{kn}$$

$$k = 0, 1 \cdots (N/2)-1$$

Dividing or decimating $X(k)$ into the even and odd numbered samples:

$$X(2k) = \sum_{n=0}^{N/2-1} x(n) W_N^{2kn} + (-1)^{2k} \sum_{n=0}^{N/2-1} x(n+N/2) W_N^{2kn}$$

$$= \sum_{n=0}^{N/2-1} \left[x(n) + x(n+N/2)\right] W_{N/2}^{kn} \quad k = 0, 1, \cdots \frac{N}{2}-1$$

$$X(2k+1) = \sum_{n=0}^{N/2-1} x(n) W_N^{(2k+1)n} + (-1)^{2k+1} \sum_{n=0}^{N/2-1} x(n+N/2) W_N^{(2k+1)n}$$

$$= \sum_{n=0}^{N/2-1} \left[x(n) - x(n+N/2)\right] W_{N/2}^{kn} W_N^{n} \quad k = 0, 1, \cdots \frac{N}{2}-1$$

Let $g_1(n) = x(n) + x(n+N/2)$

$g_2(n) = \left[x(n) - x(n+N/2)\right] W_N^{n}$

then

$$X(2k) = \sum_{n=0}^{N/2-1} g_1(n) W_{N/2}^{kn}$$

$$X(2k+1) = \sum_{n=0}^{N/2-1} g_2(n) W_{N/2}^{kn}$$

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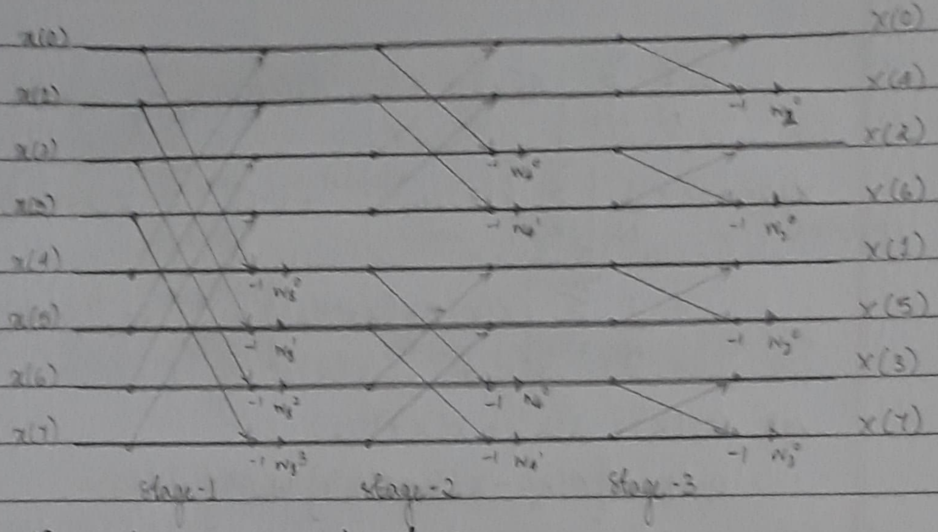


The entire process involves

$v = \log_2 N$ stages of decimation

Here $N=8$: $v = \log_2 8 = \log_2 2^3 = 3$ stages

$(N/2)\log_2 N$ complex multiplications

Here $N=8$: $(8/2)\log_2 8 = 4(3) = 12$ complex multiplications

$N\log_2 N$ complex additions

Here $N=8$: $8\log_2 8 = 8(3) = 24$ complex additions

★ **Radix-2 DIT IDFT:**

The IDFT is given as

$$x(n) = \frac{1}{N}\sum_{k=0}^{N-1} X(k) W_N^{-kn} \quad ; n = 0, 1, \cdots N-1$$

By splitting the IDFT into two summation at $N/2$

$$x(n) = \frac{1}{N}\left[\sum_{k=0}^{N/2-1} X(k) W_N^{-kn} + \sum_{k=N/2}^{N-1} X(k) W_N^{-kn}\right]$$

$$x(n) = \frac{1}{N}\left[\sum_{k=0}^{N/2-1} X(k) W_N^{-kn} + \sum_{k=0}^{N/2-1} X(k+N/2) W_N^{-n(k+N/2)}\right]$$

$$x(n) = \frac{1}{N}\left[\sum_{k=0}^{N/2-1} X(k) W_N^{-kn} + W_N^{-nN/2}\sum_{k=0}^{N/2-1} X(k+N/2) W_N^{-kn}\right]$$

Here $W_N^{-nN/2} = e^{-j\frac{2\pi}{N}\left(\frac{-nN}{2}\right)} = e^{j\pi n} = (-1)^n$

$$x(n) = \frac{1}{N}\left[\sum_{k=0}^{N/2-1} X(k) W_N^{-kn} + (-1)^n\sum_{k=0}^{N/2-1} X(k+N/2) W_N^{-kn}\right]$$

$n = 0, 1, \cdots (N/2)-1$

Decimating $x(n)$ into the even and odd samples

$$x(2n) = \frac{1}{N}\left[\sum_{k=0}^{N/2-1} X(k) W_N^{-2nk} + (-1)^{2n}\sum_{k=0}^{N/2-1} X(k+N/2) W_N^{-2nk}\right]$$

$$= \frac{1}{N} \sum_{k=0}^{N/2-1} \left[X(k) + X(k+N/2) \right] W_{N/2}^{-kn} \quad n = 0, 1 \ldots N/2 - 1$$

$$x(2n+1) = \frac{1}{N} \left[\sum_{k=0}^{N/2-1} X(k) W_N^{-k(2n+1)} + (-1)^{2n+1} \sum_{k=0}^{N/2-1} X(k+N/2) W_N^{-k(2n+1)} \right]$$

$$= \frac{1}{N} \sum_{k=0}^{N/2-1} \left[X(k) - X(k+N/2) \right] W_{N/2}^{-kn} W_N^{-k} \; ; n = 0, 1 \ldots \frac{N}{2} - 1$$

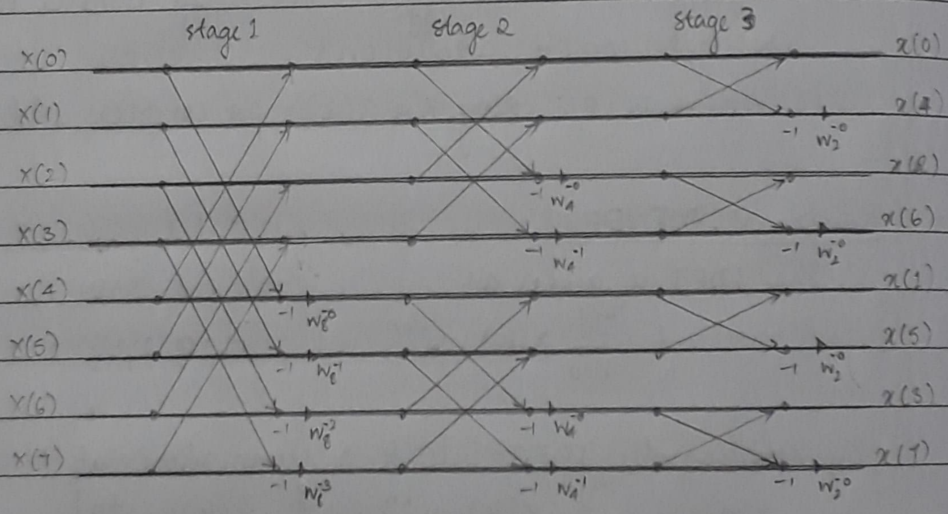
Let $G_1(k) = X(k) + X(k+N/2)$

$G_2(k) = \left[X(k) - X(k+N/2) \right] W_N^{-k}$

then

$$x(2n) = \frac{1}{N} \sum_{k=0}^{N/2-1} G_1(k) W_{N/2}^{-kn}$$

$$x(2n+1) = \frac{1}{N} \sum_{k=0}^{N/2-1} G_2(k) W_{N/2}^{-kn}$$



* Radix-2 DIF IDFT:

Let the DFT $X(k)$ be split into ~~two~~ even numbered sequence $F_1(k)$ and odd numbered sequence $F_2(k)$.

i.e, $F_1(k) = X(2k) \quad ; \quad 0 \le k \le (N/2) - 1$

$F_2(k) = X(2k+1) \quad ; \quad 0 \le k \le (N/2) - 1$

where $X(k) = \{ X(0), X(1), X(2), X(3), \ldots X(N-1) \}$

thus $F_1(k) = \{ X(0), X(2), \ldots \}$

$F_2(k) = \{ X(1), X(3), \ldots \}$

Thus $F_1(k)$ and $F_2(k)$ is obtained by decimating $X(k)$. Hence it is a decimation in frequency algorithm.

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The N-point IDFT is given by

$$x(n) = \frac{1}{N}\sum_{k=0}^{N-1} X(k)\, W_N^{-kn} \qquad n = 0, 1, \dots N-1$$

$$x(n) = \frac{1}{N}\sum_{m=0}^{N/2-1} X(2m)\, W_N^{-2mn} + \frac{1}{N}\sum_{m=0}^{N/2-1} X(2m+1)\, W_N^{-(2m+1)n}$$

$$x(n) = \frac{1}{N}\sum_{k=0}^{N/2-1} G(k)\, W_{N/2}^{-kn} + \frac{1}{N}\sum_{k=0}^{N/2-1} H(k)\, W_{N/2}^{-kn}\, W_N^{-n}$$

$$x(n) = g(n) + h(n)\, W_N^{-n} \quad ; \; n = 0, 1, \dots N-1$$

where $g(n)$ and $h(n)$ are the $N/2$ point IDFTs of the sequences $F_1(k)$ and $F_2(k)$ respectively.

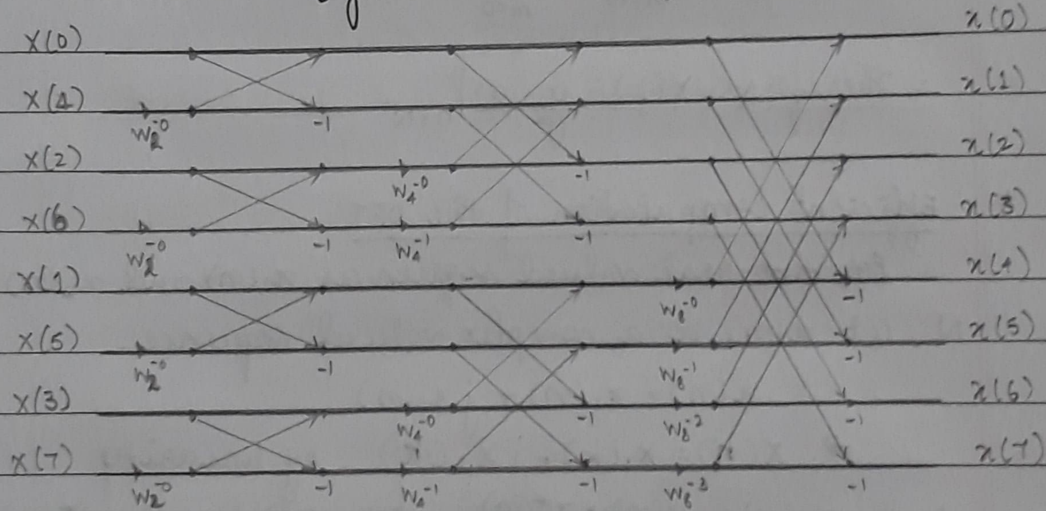
Since $g(n)$ and $h(k)$ are periodic with period $N/2$,

$\therefore g(n+N/2) = g(n)$ and $h(n+N/2) = h(n)$

$$x(n) = [g(n) + h(n)\, W_N^{-n}]/2$$

$$x(n+N/2) = [g(n+N/2) + h(n+N/2)\, W_N^{-(n+N/2)}]/2$$

$$x(n+N/2) = [g(n) - h(n)\, W_N^{-n}]/2$$



★ Goertzel algorithm:

This algorithm is adopted when we want to find the DFT only at certain points.

The N-point DFT is given by:

$$X(k) = \sum_{m=0}^{N-1} x(m)\, W_N^{km} \quad ; \; 0 \le k \le N-1$$

$$X(k) = \sum_{m=0}^{N-1} x(m)\, W_N^{km}\, W_N^{-kN} \qquad \text{as } W_N^{-kN} = 1$$

$$X(k) = \sum_{m=0}^{N-1} x(m)\, W_N^{-k(N-m)}$$

The above equation is in the form of a convolution

ie, $y_k(n) = \sum_{m=0}^{N-1} x(m) W_N^{-k(n-m)}$

$y_k(n)$ is the convolution of the finite-duration input sequence $x(n)$ of length N with a filter that has an impulse response

$$h_k(n) = W_N^{-kn} u(n)$$

$$y_k(n) = \sum_{m=-\infty}^{\infty} x(m) h_k(n-m)$$

$$y_k(n) = \sum_{m=-\infty}^{\infty} x(m) W_N^{-k(n-m)} u(n-m)$$

$$y_k(n) = \sum_{m=0}^{N-1} x(m) W_N^{-k(n-m)}$$

when $n = N$

$$y_k(n)\Big|_{n=N} = \sum_{m=0}^{N-1} x(m) W_N^{-k(N-m)}$$

Therefore $X(k) = y_k(n)\Big|_{n=N}$

* **Efficient computation of the DFT:**

For two real valued sequences $x_1(n)$ and $x_2(n)$ of length N, let $x(n)$ be a complex valued sequence.

$$x(n) = x_1(n) + j x_2(n)$$

$$\Rightarrow X(k) = X_1(k) + j X_2(k) \quad \text{by linearity}$$

but $x_1(n) = \dfrac{x(n) + x^*(n)}{2}$ and $x_2(n) = \dfrac{x(n) - x^*(n)}{2j}$

Therefore

$$X_1(k) = \frac{1}{2}\left[X(k) + X^*(N-k)\right]$$

$$X_2(k) = \frac{1}{2j}\left[X(k) - X^*(N-k)\right]$$

Thus by performing a single DFT on the complex valued sequence $x(n)$, we can obtain the DFT of the two real sequences.

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UNIT - 02

Fast Fourier Transform Algorithms.

* **Direct computation of DFT:**

→ $X(k) = \sum_{n=0}^{N-1} x(n) W_N^{kn}$; $0 \le k \le N-1$

complex addition

$X(k) = \underbrace{x(0) W_N^0}_{\text{complex multiplication CM1}} \overset{CA1}{+} \underbrace{x(1) W_N^k}_{CM2} \overset{CA2}{+} \underbrace{x(2) W_N^{2k}}_{CM3} + \ldots \overset{CA(N-1)}{+} \underbrace{x(N-1) W_N^{k(N-1)}}_{CMN}$

For one value of k there are N complex multiplications. Here k varies from 0 to N-1, i.e., N values of k has $N \times N = N^2$ complex multiplications.

Similary for one value of k there are N-1 complex additions. Here k varies from 0 to N-1 i.e, N values of k. Here there are $N(N-1)$ complex additions.

→ $X(k) = \sum_{n=0}^{N-1} x(n) W_N^{kn}$; $0 \le k \le N-1$

$X(k) = \sum_{n=0}^{N-1} x(n) e^{-j\frac{2\pi}{N}kn}$

$X(k) = \sum_{n=0}^{N-1} x(n) \left[\cos\frac{2\pi}{N}kn - j\sin\frac{2\pi}{N}kn\right]$

For one value of k there are 2N trignometric functions. Here k varies from 0 to N-1, i.e, N values of k. Hence there are $2N^2$ trignometric functions.

→ $X(k) = \sum_{n=0}^{N-1} x(n) W_N^{kn}$; $0 \le k \le N-1$

$x(n) = x_R(n) + j x_I(n)$

$W_N = W_{RN} + j W_{IN}$

$$X(k) = \sum_{n=0}^{N-1} \left[x_R(n) + j x_I(n) \right] \left[W_{RN}^{kn} + j W_{IN}^{kn} \right]$$

Real multiplication

$$X(k) = \sum_{n=0}^{N-1} \left[\left(x_R(n) W_{RN}^{kn} - x_I(n) W_{IN}^{kn} \right) + j \left(x_I(n) W_{RN}^{kn} + x_R(n) W_{IN}^{kn} \right) \right]$$

real addition

For one value of k there are 4N real multiplications and 2N real additions in one complex multiplication.

NOTE: Here subtraction is also considered as addition since it takes almost the same computational time as for addition. The complex numbers are expressed as $a+jb$, in this it is not considered as an addition operation as this is just a way of representing a complex number. The real part and imaginary part of the complex number are basically real numbers.

Hence for N^2 complex multiplications there are $4N^2$ real multiplications and $2N^2$ real addition.

complex additions.

$$(a+jb) + (c+jd)$$

↳ One complex addition

$$(a+c) + j(b+d) \quad : \text{One complex number.}$$

real addition real addition

Hence for $N(N-1)$ complex additions there are $2N(N-1)$ real additions.

Therefore total number of complex additions is

$$2N(N-1) + 2N^2$$

$$2N^2 - 2N + 2N^2$$

$$\therefore 4N^2 - 2N \text{ complex additions}$$

<u>Summary</u>: <u>The computational complexity of direct computation of N-point DFT:</u>

- N number of complex multiplications for each value of k
- N^2 number of complex multiplications for all values of k
- $(N-1)$ complex additions for each value of k
- $N(N-1)$ complex additions for all value of k.

OR

- $4N$ real multiplication for each value of k
- $4N^2$ real multiplications for all values of k
- $4N-2$ real additions for each value of k
- $4N^2-2N$ real additions for all values of k.

* <u>Classification of FFT Algorithms:</u>

For computation of DFT:

a. divide and conquer approach
b. DFT as linear filtering.

a. divide and conquer approach
- Radix - 2 FFT algorithm
- Radix-4 FFT algorithm - DIT FFT and DIF FFT
- split = Radix FFT algorithm - Decimation in time and Frequency respectively

b. DFT as linear filtering.
- chirp-z algorithm
- Goertzel algorithm

* <u>Radix-2 DIT FFT:</u> (Fast Fourier Transform)

Let $x(n)$ be split into two $N/2$ point sequences as $f_1(n)$ and $f_2(n)$.

Let $f_1(n)$ contain ~~samples~~ ~~of~~ $x(n)$ which have even index values or numbers

Let $f_2(n)$ contain samples of $x(n)$ which have odd index values or numbers

$f_2(n)$

Let $x(n) = \{x(0), x(1), x(2), x(3), x(4), \ldots, x(N-1)\}$

$f_1(n)$

Therefore.

$$f_1(n) = x(2n) \; ; \quad 0 \leq n \leq (N/2)-1$$
$$f_2(n) = x(2n+1) \; ; \quad 0 \leq N \leq (N/2)-1$$

The $x(n)$ sequence is divided or split (decimated) into two sequences $f_1(n)$ and $f_2(n)$ and since the division or splitting operation is carried out on the given time domain sequence it is decimation in time (DIT) FFT algorithm.

The N-point DFT of $x(n)$ is given by

$$X(k) = \sum_{n=0}^{N-1} x(n)\, e^{-j\frac{2\pi}{N}kn}$$

$$X(k) = \sum_{r=0}^{N/2-1} x(2r)\, e^{-j\frac{2\pi}{N}k(2r)} + \sum_{r=0}^{N/2-1} x(2r+1)\, e^{-j\frac{2\pi}{N}k(2r+1)}$$

$$X(k) = \sum_{r=0}^{N/2-1} x(2r)\, W_N^{2kr} + \sum_{r=0}^{N/2-1} x(2r+1)\, W_N^{k(2r+1)}$$

$$X(k) = \sum_{r=0}^{N/2-1} g(x)\, W_{N/2}^{kr} + \sum_{r=0}^{N/2-1} h(x)\, W_{N/2}^{kr}\, W_N^{k}$$

$$X(k) = G(k) + H(k)\, W_N^k$$

$G(k)$ and $H(k)$ represent $N/2$ - point DFTs of even and odd indexed sequences. Therefore.

$$X(k) = G(k) + W_N^k H(k).$$

because $G(k)$ and $H(k)$ are $N/2$ point DFTs they are periodic with period $N/2$.

Therefore

$$X(k) = G(k) + W_N^{k} H(k) \quad \text{—} ①$$
$$X(k+N/2) = G(k) - W_N^{k} H(k) \quad \text{—} ②$$

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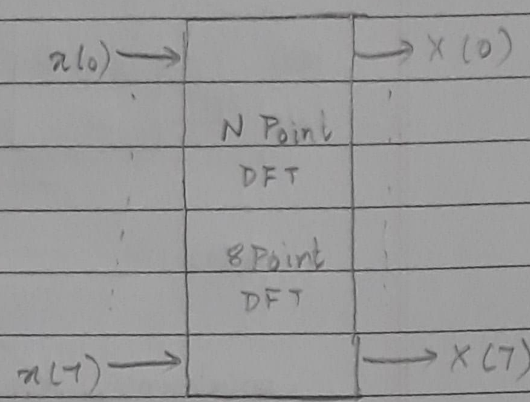
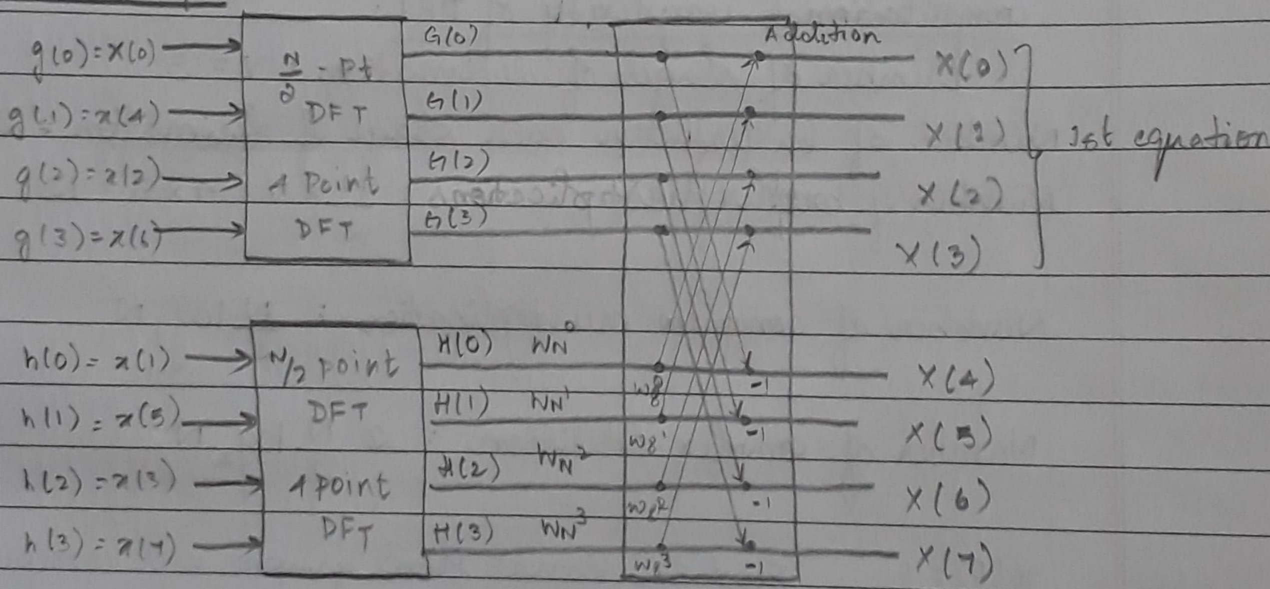
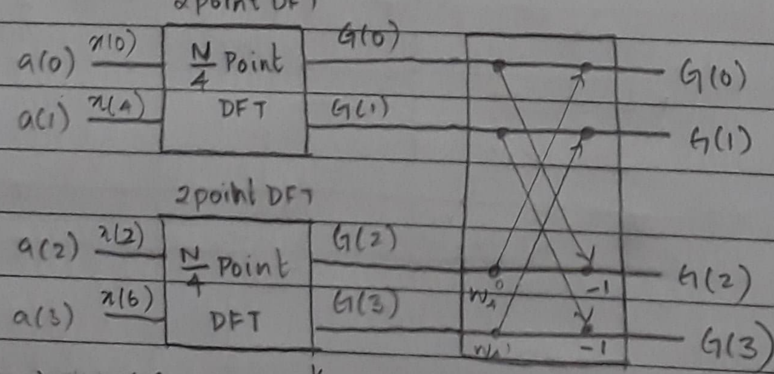
8 point DFT

Natural Order	Bit reversed
000 - (0)	000 - (0)
001 - (1)	100 - (4)
010 - (2)	010 - (2)
011 - (3)	110 - (6)
100 - (4)	001 - (1)
101 - (5)	101 - (5)
110 - (6)	011 - (3)
111 - (7)	111 - (7)



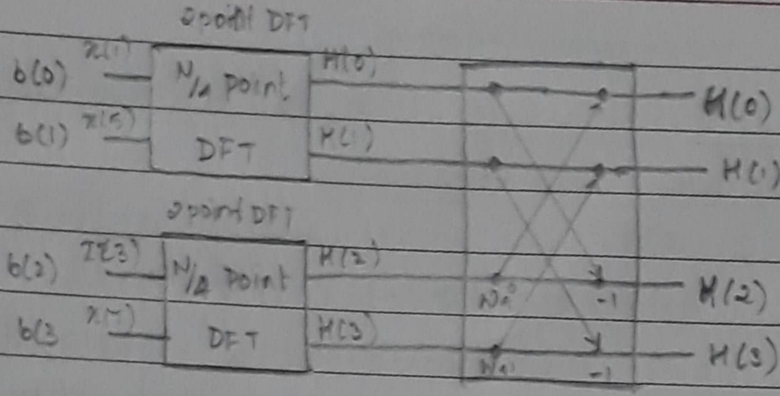
Signal Flow Graph:

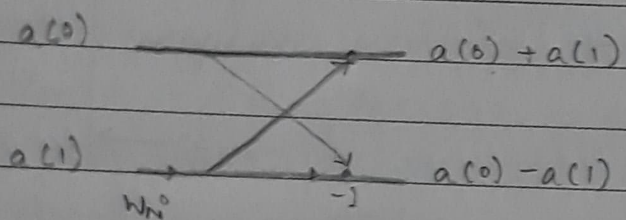




$$G(k) = A(k) + W_{N/2}^{k} B(k)$$

$$G(k+N/4) = A(k) - W_{N/2}^{k} B(k)$$





1 complex multiplication
2 complex addition.

computational complexity of DFT:

Number of stages of decimation : $v = 3$

Number of butterflies in each stage of decimation $= 2^v = 2^3$

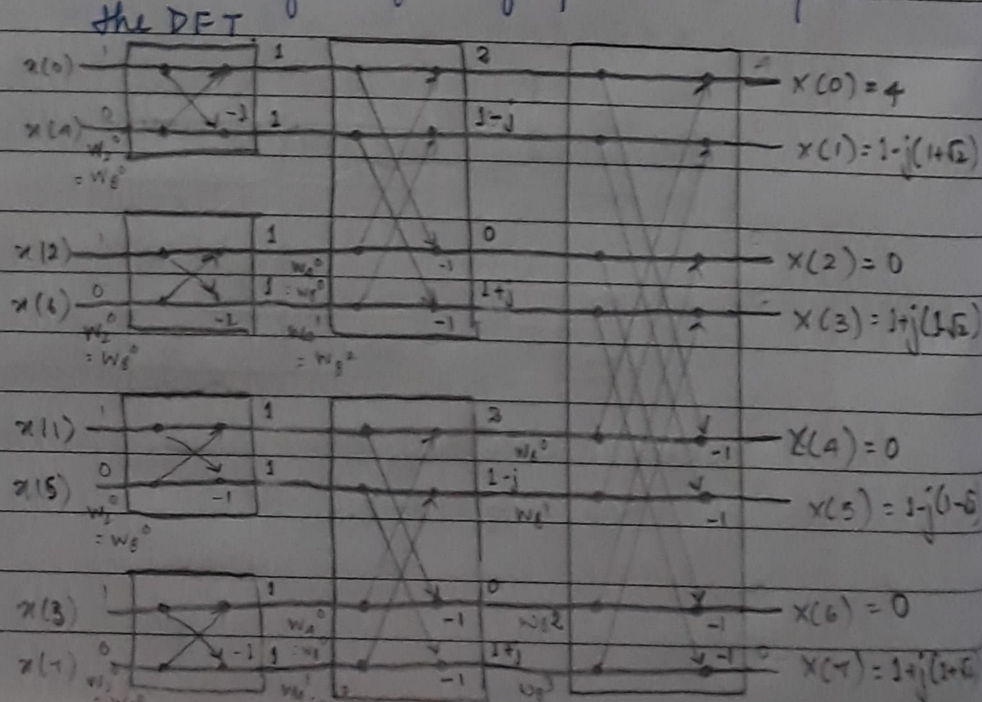
~~Number of butterflies~~ : $\cancel{\frac{N}{2} \times v = 4 \times 3 = 12}$

Number of complex multiplication : $\frac{N}{2} \log_2 N$

Number of complex addition : $2 \frac{N}{2} \log_2 N$

Q: Find the signal flow graph and compute the DFT.

$x(0) = 1$
$x(4) = 0$
$x(2) = 1$
$x(6) = 0$
$x(1) = 1$
$x(5) = 0$
$x(3) = 1$
$x(7) = 0$



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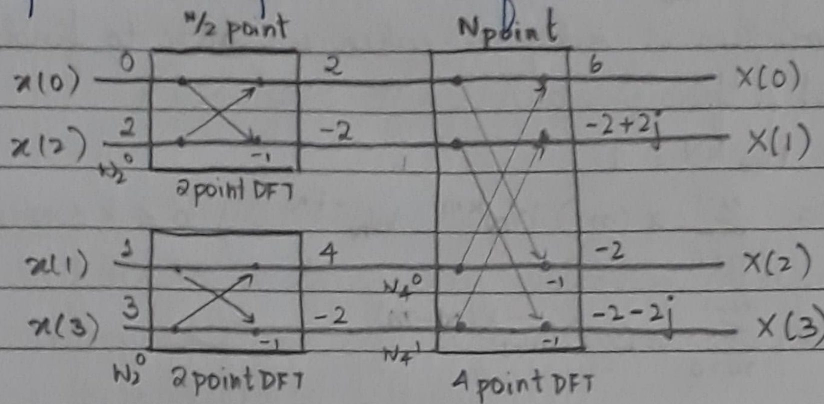
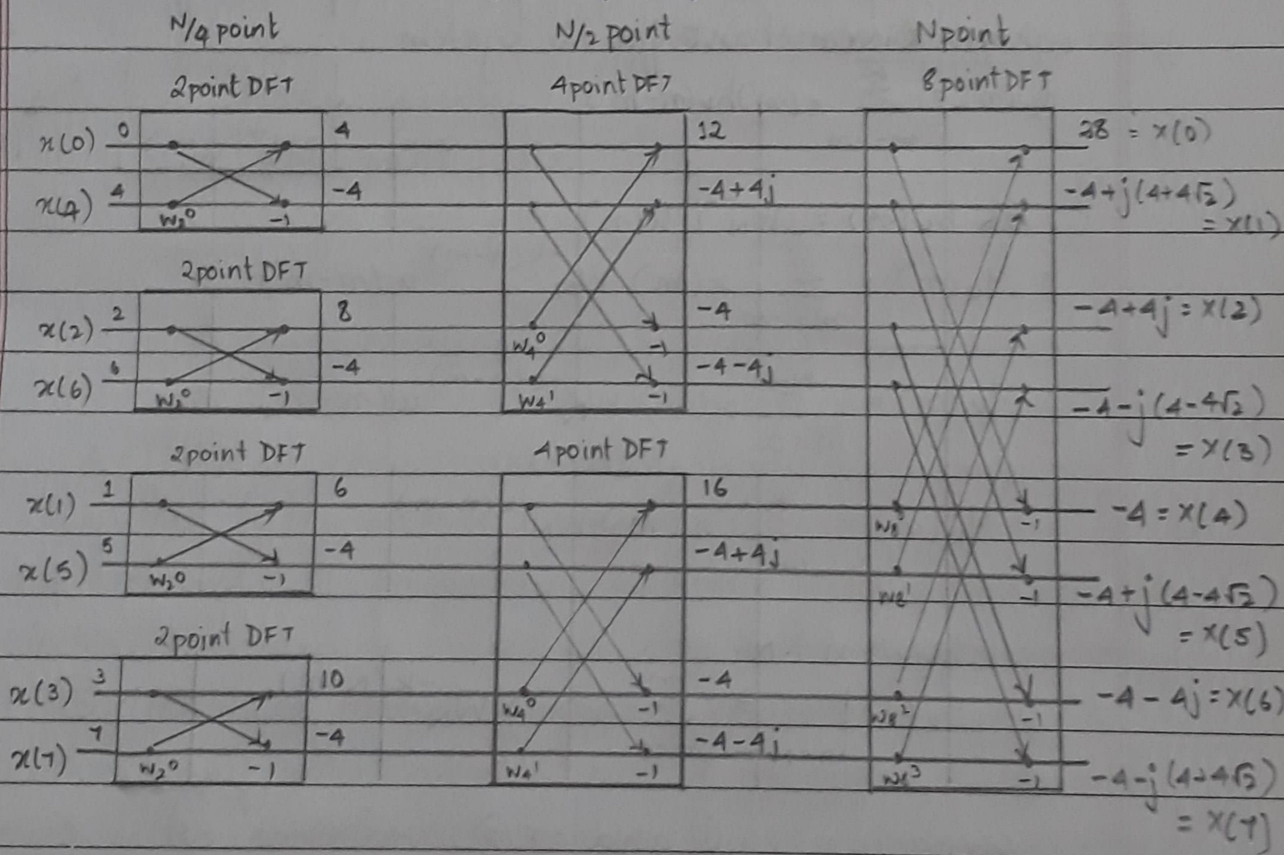
Q: compute the 4 point DFT of $x(n) = \{0, 1, 2, 3\}$



Q: compute the 8 point DFT of the given sequence
$x(n) = \{0, 1, 2, 3, 4, 5, 6, 7\}$



★ **Goertzel Algorithm:**

This algorithm is adopted when we have to find the DFT only at a few points.

wkt $W_N^{-kN} = 1$

$$X(k) = \sum_{m=0}^{N-1} x(m)\, W_N^{km} \cdot W_N^{-kN} \quad ; 0 \leq k \leq N-1$$

$$X(k) = \sum_{m=0}^{N-1} x(m)\, W_N^{-k(N-m)} \quad \text{—— (1)}$$

consider an LTI system of input $x(n)$ and $y_k(n)$ as the output with an impulse response of $h_k(n)$ where k is constant for the system.

$$y_k(n) = \sum_{m=-\infty}^{\infty} x(m) h_k(n-m)$$

$$x(n) \longrightarrow \boxed{\begin{array}{c} h_k(n) \\ = W_N^{-kn} u(n) \end{array}} \longrightarrow y_k(n)$$

as $h_k(n) = W_N^{-kn} u(n)$

$$y_k(n) = \sum_{m=-\infty}^{\infty} x(m)\, W_N^{-k(n-m)}\, u(n-m)$$

$$y_k(n) = \sum_{m=0}^{N-1} x(m)\, W_N^{-k(n-m)}\, u(n-m)$$

$$y_k(n) = \sum_{m=0}^{N-1} x(m)\, W_N^{-k(n-m)}$$

when $n = N$

$$y_k(n)\Big|_{n=N} = \sum_{m=0}^{N-1} x(m)\, W_N^{-k(N-m)} \quad \text{—— (2)}$$

Comparing eq (1) and eq (2)

$$X(k) = y_k(n)\Big|_{n=N}$$

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* DIF - FFT algorithm:

$$X(k) = \sum_{n=0}^{N-1} x(n) W_N^{kn} \quad ; \quad 0 \le k \le N-1$$

$$X(k) = \sum_{n=0}^{\frac{N}{2}-1} x(n) W_N^{kn} + \sum_{m=N/2}^{N-1} x(m) W_N^{km}$$

let $m - N/2 = n$

then if $m = N/2 \quad n = 0$

$m = N-1 \quad n = N-1-N/2 = N/2-1$

$$X(k) = \sum_{n=0}^{N/2-1} x(n) W_N^{kn} + \sum_{n=0}^{N/2-1} x(n+N/2) W_N^{k(n+N/2)}$$

$$X(k) = \sum_{n=0}^{N/2-1} x(n) W_N^{kn} + \sum_{n=0}^{N/2-1} x(n+N/2) W_N^{kn} W_N^{kN/2}$$

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Here $W_N^{kN/2} = e^{-j\frac{2\pi}{N}k\frac{N}{2}} = e^{-\pi jk}$

$= (e^{-j\pi})^k = (-1)^k$

Therefore

$$X(k) = \sum_{n=0}^{N/2-1} x(n) W_N^{kn} + (-1)^k \sum_{n=0}^{N/2-1} x(n+N/2) W_N^{kn}$$

for $0 \le k \le \frac{N}{2}-1$

Dividing $X(k)$ as an odd sequence and an even sequence.

$$X(2k) = \sum_{n=0}^{N/2-1} x(n) W_N^{2kn} + (-1)^{2k} \sum_{n=0}^{N/2-1} x(n+N/2) W_N^{2kn} \quad \text{--- } (1)$$

$$X(2k+1) = \sum_{n=0}^{N/2-1} x(n) W_N^{(2k+1)n} + (-1)^{2k+1} \sum_{n=0}^{N/2-1} x(n+N/2) W_N^{(2k+1)n} \quad \text{--- } (2)$$

From eq (1)

$$X(2k) = \sum_{n=0}^{N/2-1} x(n) W_N^2 W_N^{kn} + \sum_{n=0}^{N/2-1} x(n+N/2) W_N^2 W_N^{kn}$$

wkt $W_N^2 = W_{N/2}$

$$\therefore X(2k) = \sum_{n=0}^{N/2-1} x(n) W_{N/2} W_N^{kn} + \sum_{n=0}^{N/2-1} x(n+N/2) W_{N/2} W_N^{kn}$$

$$X(2k) = \sum_{n=0}^{N/2-1} [x(n) + x(n+N/2)] W_{N/2}^{kn} \quad \text{--- } (3)$$

From eq (2)

$$X(2k+1) = \sum_{n=0}^{N/2-1} x(n) W_N^{2kn} W_N^n + (-1)^{2k}(-1) \sum_{n=0}^{N/2-1} x(n+N/2) W_N^{2kn} W_N^n$$

$$X(2k+1) = \sum_{n=0}^{N/2-1} x(n) W_{N/2}^{kn} W_N^n - \sum_{n=0}^{N/2-1} x(n+N/2) W_{N/2}^{kn} W_N^n$$

$$X(2k+1) = \sum_{n=0}^{N/2-1} [x(n) - x(n+N/2)] W_{N/2}^{kn} W_N^n \quad \text{--- } (4)$$

From eq (3)

let $x(n) + x(n+N/2) = g_1(n)$

From eq (4)

let $[x(n) - x(n+N/2)] W_N^n = g_2(n)$

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Therefore

$$X(2k) = \sum_{n=0}^{N/2-1} g_1(n)\, W_{N/2}^{kn} \quad \text{——— (5)}$$

$$X(2k+1) = \sum_{n=0}^{N/2-1} g_2(n)\, W_{N/2}^{kn} \quad \text{——— (6)}$$

Let $N = 8$ so $N/2 = 4$.

From eq (5)

$g_1(0) = x(0) + x(4)$

$g_1(1) = x(1) + x(5)$

$g_1(2) = x(2) + x(6)$

$g_1(3) = x(3) + x(7)$

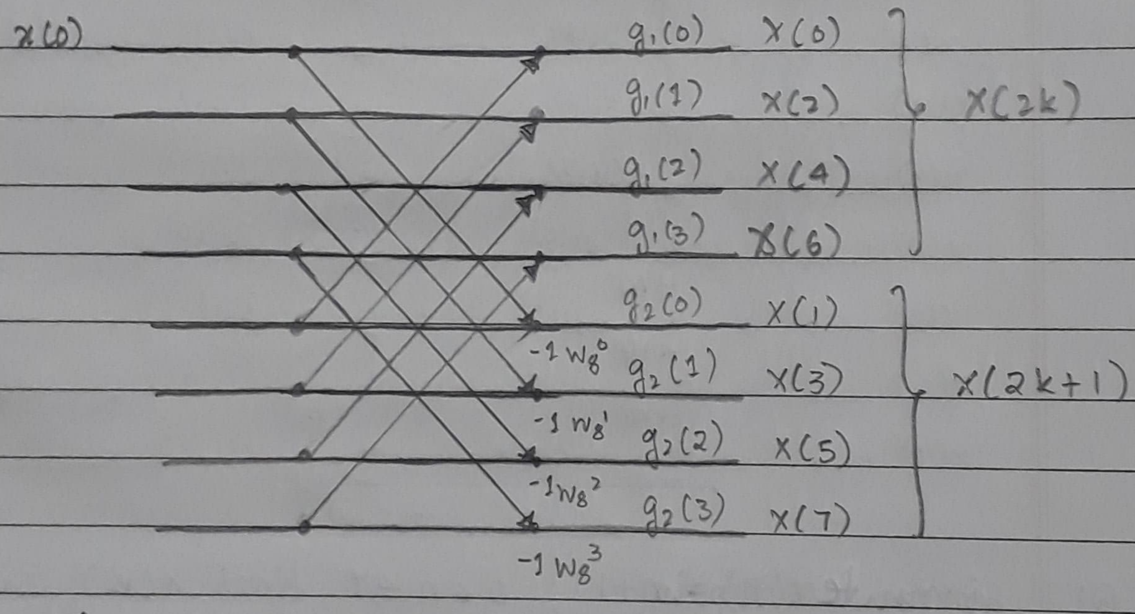
From eq (6)

$g_2(0) = [x(0) - x(4)]\, W_8^0$

$g_2(1) = [x(1) - x(5)]\, W_8^1$

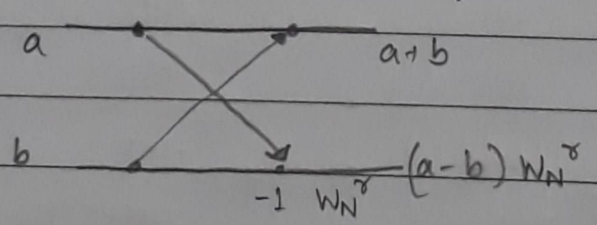
$g_2(2) = [x(2) - x(6)]\, W_8^2$

$g_2(3) = [x(3) - x(7)]\, W_8^3$



The obtained values of $X(2k)$ and $X(2k+1)$ are further decimated into even and odd indexed values.

General butterfly structure for DIF FFT



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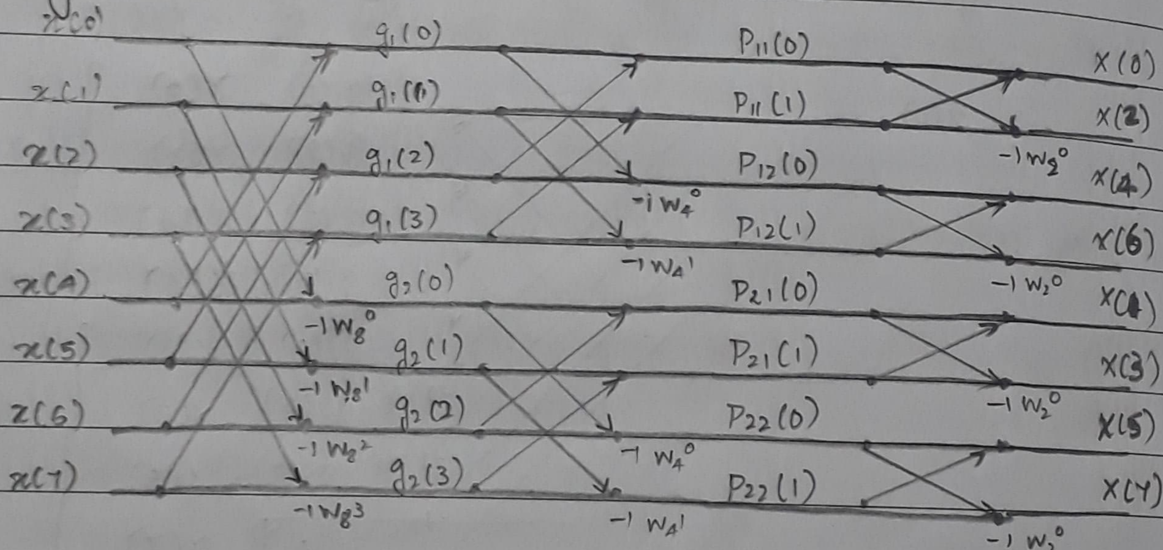
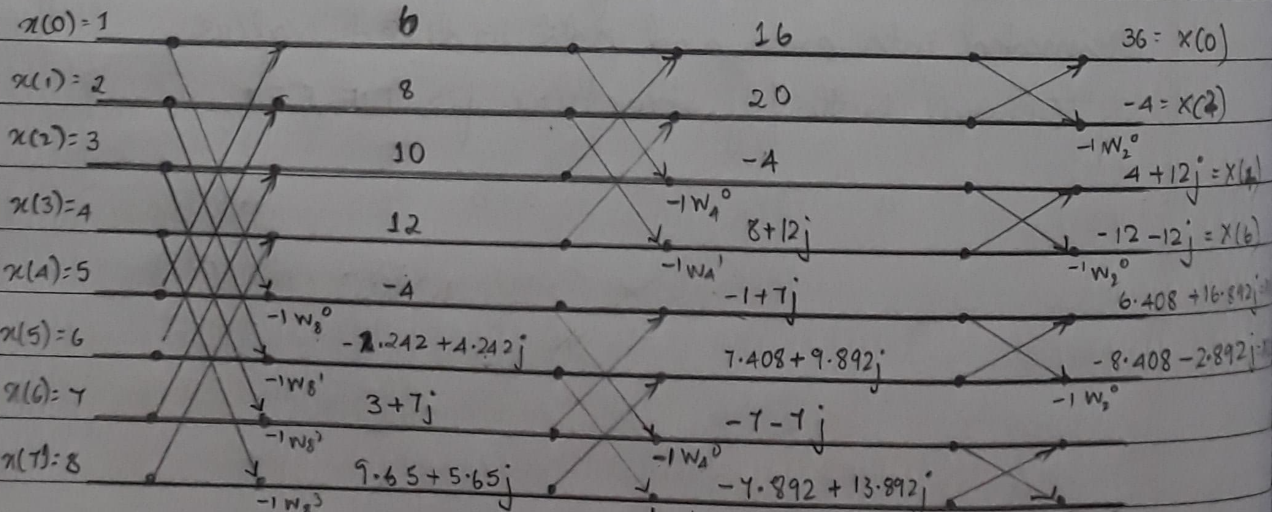
$$P_{11}(0) = g_1(0) + g_1(2)$$
$$P_{11}(1) = g_1(1) + g_1(3)$$
$$P_{12}(0) = [g_1(0) - g_1(2)]\, W_8^0$$
$$P_{12}(1) = [g_1(1) - g(3)]\, W_8^2$$
$$P_{21}(0) = g_2(0) + g_2(2)$$
$$P_{21}(1) = g_2(1) + g_2(3)$$
$$P_{22}(0) = [g_2(0) - g_2(2)]\, W_8^0$$
$$P_{22}(1) = [g_2(1) - g_2(3)]\, W_8^2$$

The next stage is to compute the 2 point DFTs.
The direction of DIF FFT is opposite to that of DIT FFT algorithm.



Q: Given $x(n) = n+1$; $0 \le n \le 7$, find $X(k)$ using DIF FFT.

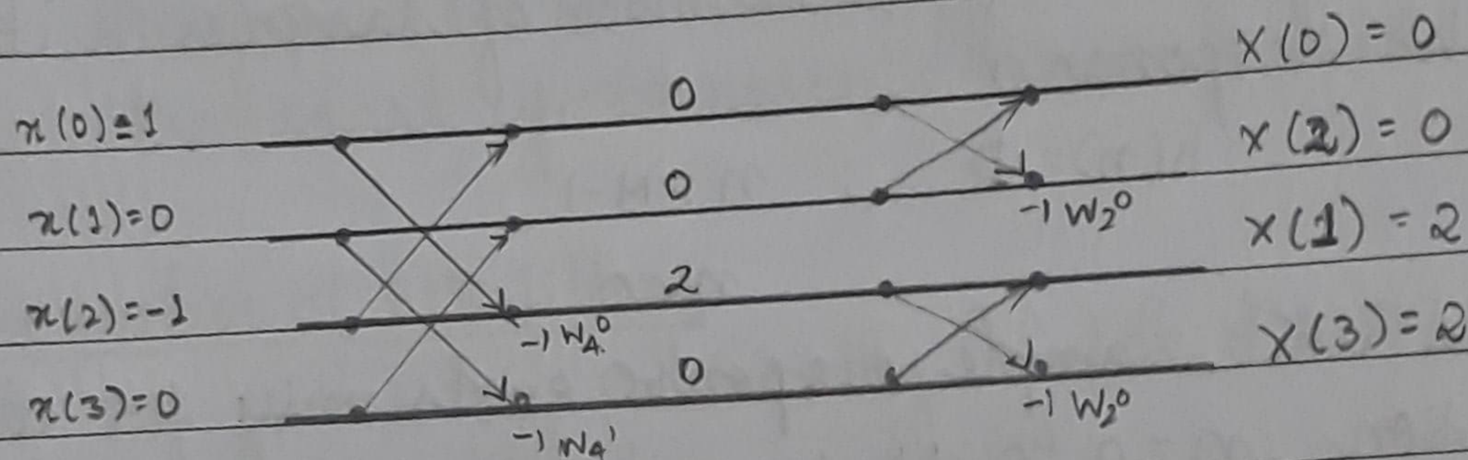
$x(n) = \{1, 2, 3, 4, 5, 6, 7, 8\}$



Q: compute DFT of the sequence $x(n) = \cos(n\pi/2)$ where $N = 4$. using DIF-FFT algorithm.

→ $x(n) = \cos(n\pi/2)$; $N = 4$

$x(n) = \{1, 0, -1, 0\}$



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UNIT - 03

Design of FIR Filters

★ Properties of FIR filters

1. FIR filters are inherently stable

This is because they have no feedback elements thus any bounded input results in a bounded output i.e., it has BIBO stability.

2. FIR filters have linear phase.

A FIR filter is linear phase when its coefficients are symmetrical around the center coefficient (i.e, the first coefficient is same as the last and the second is the same as the next to last and so on).

Symmetric and antisymmetric property of FIR filter

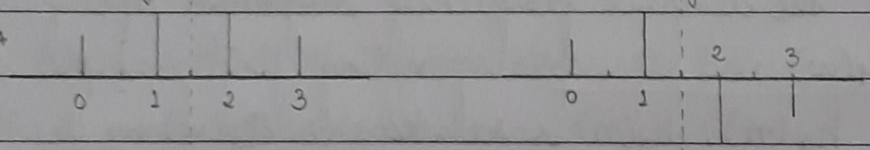
$h(n) = \pm h(M-1-n)$: Linear phase

$h(n) = h(M-1-n)$: symmetric

$h(n) = -h(M-1-n)$: Antisymmetric

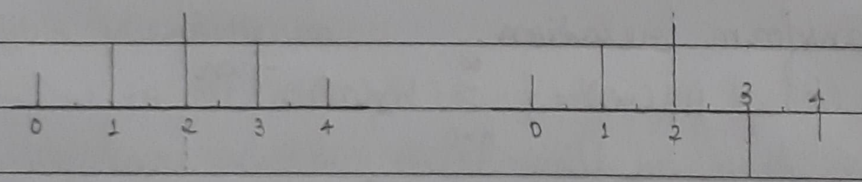
Ex: Symmetric Antisymmetric

M = 4



Ex:

M = 5



Midpoint is $\frac{M-1}{2}$ (center point of $h(n)$)

The phase characteristics of the filter for both M odd and M even is.

$$\theta(\omega) = \begin{cases} -\omega\left(\frac{M-1}{2}\right) & H(\omega) > 0 \\ -\omega\left(\frac{M-1}{2}\right) + \pi & H(\omega) < 0 \end{cases}$$

3. FIR filters require higher filter orders for similar magnitude response when compared to IIR filters.

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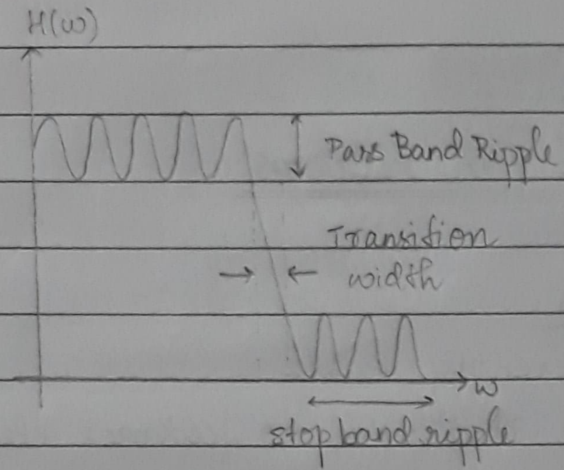

Magnitude characteristic specification for an FIR filter.

★ **Windows:**

A window function is a mathematical function that is zero valued outside of some chosen interval. Mathematically when another function or waveform is multiplied by a window function and the product is zero valued outside the interval and the signal is left only in the part where they overlap (view through the window).

★ **Design of FIR filters using Windows:**

The desired frequency response specification is $H_d(\omega)$ and the corresponding unit sample response is $h_d(n)$. $h_d(n)$ is related to $H_d(\omega)$ by the Fourier transform relation.

$$H_d(\omega) = \sum_{n=0}^{\infty} h_d(n) e^{-j\omega n}$$

$$\text{where } h_d(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(\omega) e^{j\omega n}\, d\omega$$

In general $h_d(n)$ is infinite in duration. Thus to truncate to some point $n = M-1$ we need a FIR filter of length M. Truncation of $h_d(n)$ to length $M-1$ is done by multiplying $h_d(n)$ with a window function $w(n)$. Thus the unit sample response of the FIR filter is:

$$h(n) = h_d(n)\, w(n)$$

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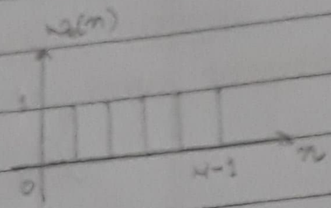


– **Rectangular Window:**

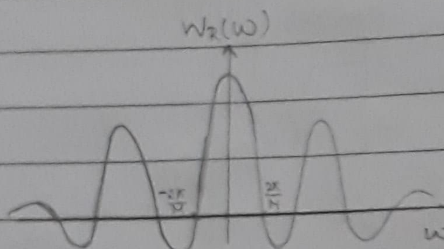
$$w_R(n) = \begin{cases} 1 & ; \quad 0 \le n \le M-1 \\ 0 & ; \quad \text{otherwise} \end{cases}$$

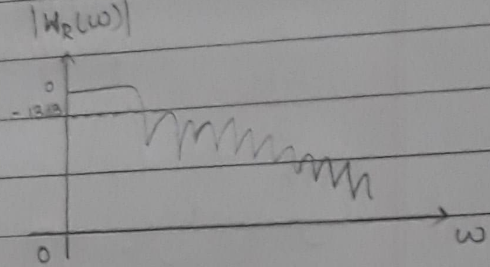
$$W_R(\omega) = \sum_{n=0}^{M-1} w(n) e^{-j\omega n}$$

$$W_R(\omega) = \sum_{n=0}^{M-1} e^{-j\omega n} = \frac{1 - e^{-j\omega M}}{1 - e^{-j\omega}}$$

$$W_R(\omega) = e^{-j\omega(M-1)/2} \cdot \frac{\sin(\omega M/2)}{\sin(M/2)}$$

$$|W_R(\omega)| = \frac{|\sin(\omega M/2)|}{|\sin(M/2)|}$$





Transition width of main lobe : $4\pi/M$

Peak side lobe = -13 dB

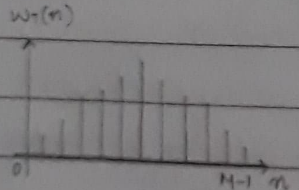
As the length of the filter increases, the transition width of main lobe decreases.

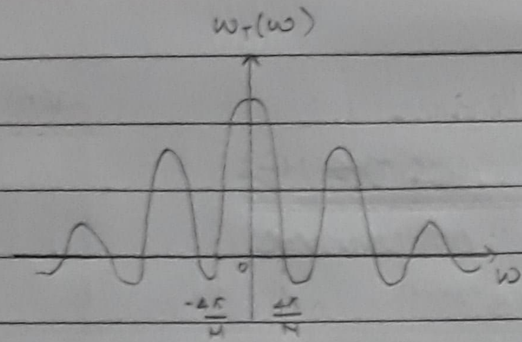
Gibb's phenomenon:

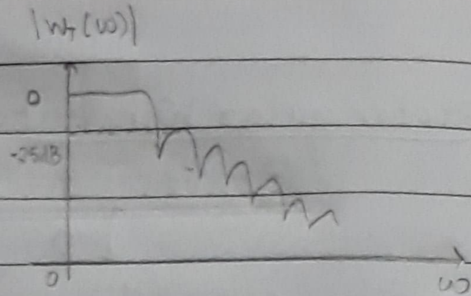
Due to the abrupt truncation of $h_d(n)$ using rectangular window there will be both ripple in pass band and stop band of the low pass filter. This phenomenon is called the Gibb's phenomenon and this effect is higher just before and after the transition band.

– **Bartlett window:** (Triangular window)

$$w_T(n) = \begin{cases} 1 - \dfrac{2\left|n - \dfrac{M-1}{2}\right|}{M-1} & 0 \le n \le M-1 \\ 0 & \text{otherwise} \end{cases}$$





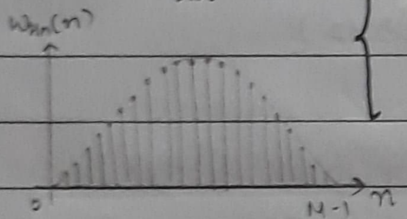


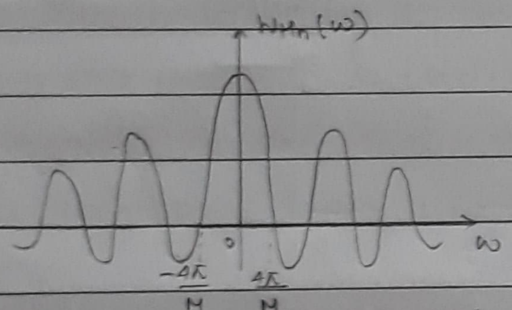
Transition width of main lobe : $8\pi/M$

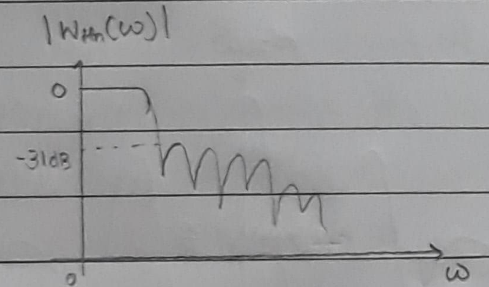
peak sidelobe : -25 dB

- Hanning Window:

$$W_{Hn}(n) = \begin{cases} 0.5 - 0.5 \cos \dfrac{2\pi n}{M-1} & ; 0 \le n \le M-1 \\ 0 & ; \text{otherwise} \end{cases}$$





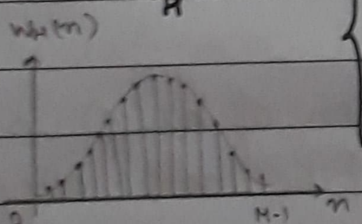


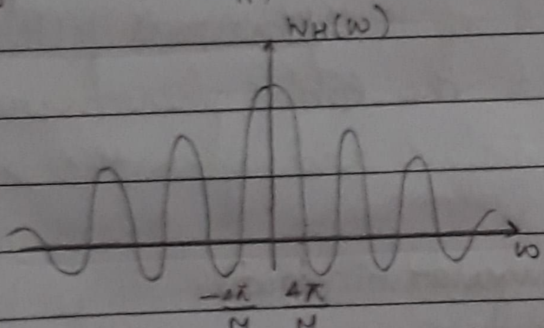
Transition width of main lobe : $8\pi/M$

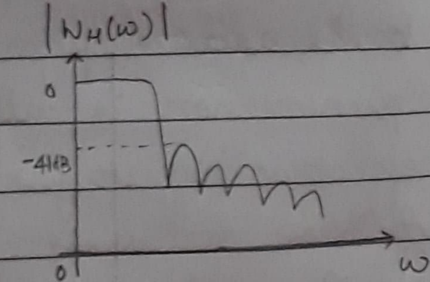
Peak sidelobe : -31 dB

- Hamming Window

$$W_H(n) = \begin{cases} 0.54 - 0.46 \cos \dfrac{2\pi n}{M-1} & ; 0 \le n \le M-1 \\ 0 & ; \text{otherwise} \end{cases}$$







Transition width of main lobe : $8\pi/M$

Peak sidelobe : -41 dB

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Suppose we want to design a symmetric lowpass linear-phase FIR filter having a desired frequency response

$$H_d(\omega) = \begin{cases} e^{-j\omega(M-1)/2} & ; \; 0 \le |\omega| \le \omega_c \\ 0 & ; \; \text{otherwise} \end{cases}$$

A delay of $(M-1)/2$ is incorporated into $H_d(\omega)$ to make the filter of length M. Then the unit response is

$$h_d(n) = \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} e^{j\omega\left(n-\frac{M-1}{2}\right)}\, d\omega$$

$$h_d(n) = \begin{cases} \dfrac{\sin \omega_c\left(n-\frac{M-1}{2}\right)}{\pi\left(n-\frac{M-1}{2}\right)} & ; \; n \ne \frac{M-1}{2} \\ \omega_c & ; \; n = \frac{M-1}{2} \end{cases}$$

$f_c = \dfrac{F_c}{F_s}$ where F_c : cut off frequency

F_s : sampling frequency

$\omega_c = 2\pi f_c$.

* **Difference between FIR and IIR filters:**

FIR Filters	IIR filters
- Always stable	- Can be unstable.
- Unit sample response exists for the duration from 0 to M-1.	- Unit sample response exists for the duration from 0 to ∞.
- Output depends only on present and past inputs. $y(n) = \sum_{k=0}^{M} b_k x(n-k)$	- Output depends on present and past inputs as well as past outputs $y(n) = \sum_{k=1}^{N} a_k y(n-k) + \sum_{k=0}^{M} b_k x(n-k)$
- Limited or finite memory requirements.	- Requires infinite memory.
- Linear phase response	- Non linear phase response

UNIT - 3: FIR FILTERS

FIR : Finite Impulse Response

These are linear time invariant systems which are characterised by unit sample response.

An FIR filter has finite number of samples in its impulse response.

$$h(n) = 0 \quad ; \quad n > M-1, \quad n < 0$$

The unit sample response exists only for the duration $n = 0$ to $M-1$

The FIR and IIR filters are best described by the difference equation. The response of the FIR filter depends on upon present and past inputs.

The response of an IIR input depends on the present input and past input as well as past outputs.

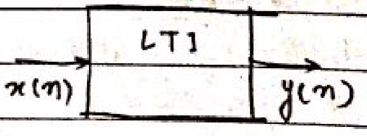
IIR filter

$$y(n) = \underbrace{\sum_{k=1}^{N-1} a_k\, y(n-k)}_{\text{past outputs}} + \underbrace{\sum_{k=0}^{M-1} b_k\, x(n-k)}_{\text{present and past input}} \quad —① $$

FIR filter

$$y(n) = \underbrace{\sum_{k=0}^{M-1} b_k\, x(n-k)}_{\text{present and past input}} = b_0 x(n) + b_1 x(n-1) \ldots\ldots \quad —②$$

In case of FIR filters the first summation is not present.



$$y(n) = \sum_{k=0}^{M-1} h(k)\, x(n-k) \quad —③$$

comparing eq ② and eq ③

The unit sample response of FIR filter can be obtained directly from the coefficients of difference equation

$$h(k) = b_k.$$

* Advantages of FIR filters over IIR filters

- FIR filters are always stable.
- FIR filters can be designed to have exactly linear phase.
- They can be realised efficiently.
- The start up transients have small duration

* Properties of FIR filters

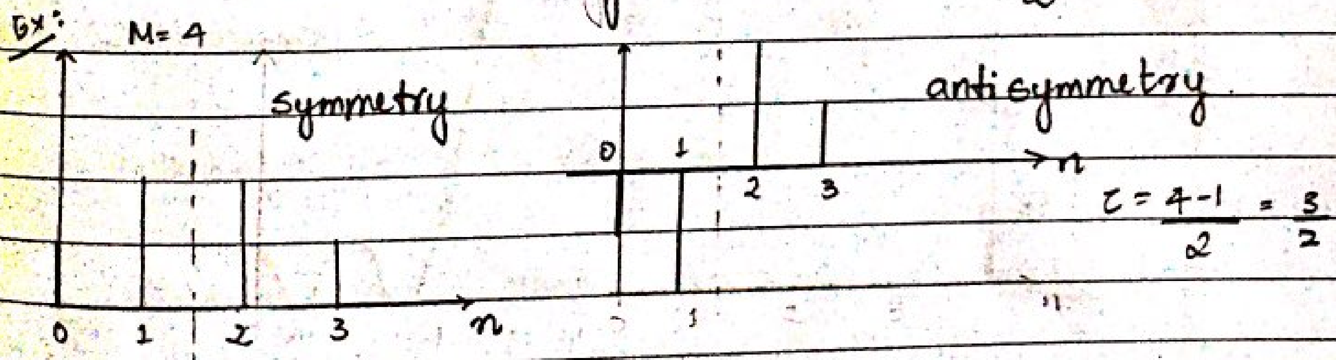
1. FIR filters are inherently stable. (BIBO stability)
2. FIR filters have linear phase.
3. FIR filters require higher filter orders for similar magnitude response compared to IIR filters.

② Symmetric and antisymmetric property of FIR filter

$h(n) = h(M-1-n)$: symmetric

$h(n) = -h(M-1-n)$: Antisymmetric $\qquad \tau = \frac{M-1}{2}$

Ex: $M = 4$



Q: Draw the impulse responses for

i. Symmetric $\quad$ for $\quad$ a) $M = 9$

ii. Antisymmetric $\quad\quad\quad$ b) $M = 8$

For a linear phase FIR filter $h(n)$ is

$$h(n) = \pm h(M-1-n)$$

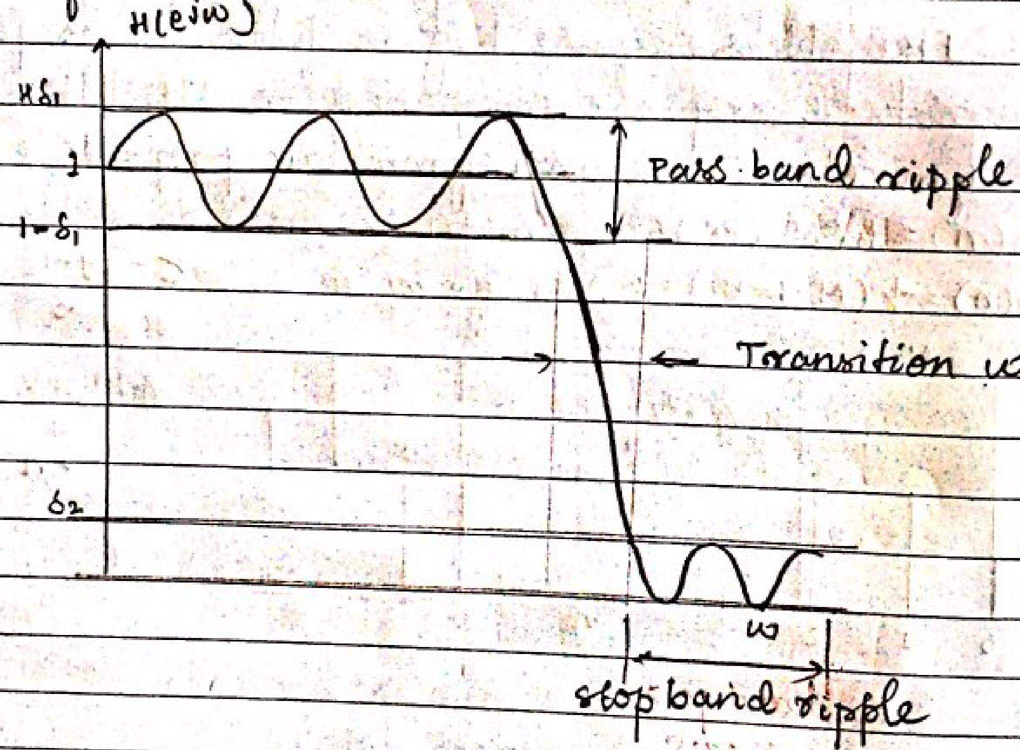
It can be obtained if the unit sample response satisfies this equation.

The phase for angle ω is given by

$$\angle H(\omega) = \begin{cases} -\omega\left(\frac{M-1}{2}\right) & ; \; H(\omega) > 0 \\ -\omega\left(\frac{M-1}{2}\right) + \pi & ; \; |H(\omega)| < 0 \end{cases}$$

because $(M-1)/2$ is constant, the phase $H(\omega)$ is linearly proportional to frequency.

(3) Magnitude characteristic specification for an FIR filter.



* **Windowing:**

Windowing technique is used for designing linear phase FIR filters.

step 1. If $H_d(e^{j\omega})$ is the desired frequency response then $H_d(e^{j\omega})$ is given by

$$H_d(e^{j\omega}) = \sum_{-\infty}^{\infty} h_d(n)\, e^{-j\omega n}$$

where $h_d(n)$: impulse response having a finite length and non causal.

$$h_d(n) = \int H_d(e^{j\omega})\, e^{j\omega n}$$

step2: $h_d(n) \times w(n)$

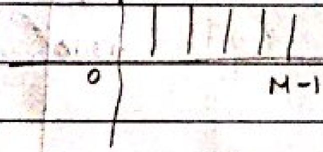
where $w(n)$ is the window function.

step3: shifting $-\frac{M-1}{2}$ to $\frac{M-1}{2}$ to right to make it non causal.

— **Rectangular window**

$$w_R(n) = \begin{cases} 1 & ; \ n \le \left|\frac{M-1}{2}\right| \\ 0 & ; \ \text{otherwise} \end{cases}$$

$$w_R(n) = \begin{cases} 1 & ; \ n = 0 \cdots M-1 \\ 0 & ; \ \text{otherwise} \end{cases}$$



spectrum of rectangular window.

$$W_R(e^{j\omega}) = \int_{-\pi}^{\pi} W_R(n)\, e^{-j\omega n}\, dn$$

$$W_R(e^{j\omega}) = \int_{-(M-1)/2}^{(M-1)/2} e^{-j\omega n}\, dn$$

Let $m = n + \frac{M-1}{2} \Rightarrow n = m - \frac{M-1}{2}$

when $n = -\frac{M-1}{2}$ then $m = 0$

when $n = \frac{M-1}{2}$ then $m = M-1$

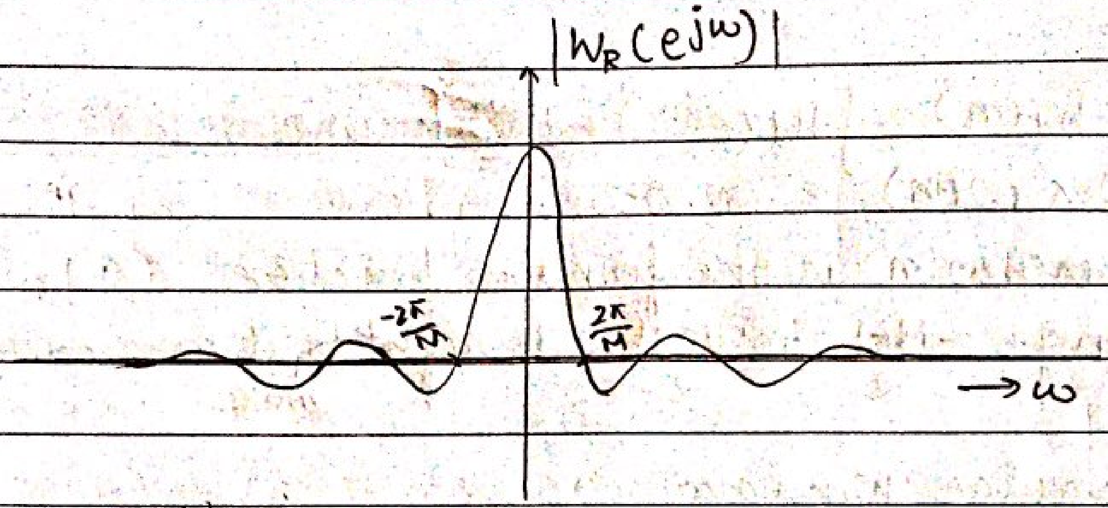
$$W_R(e^{j\omega}) = \int_0^{M-1} e^{-j\omega(m - (M-1)/2)}\, dm$$

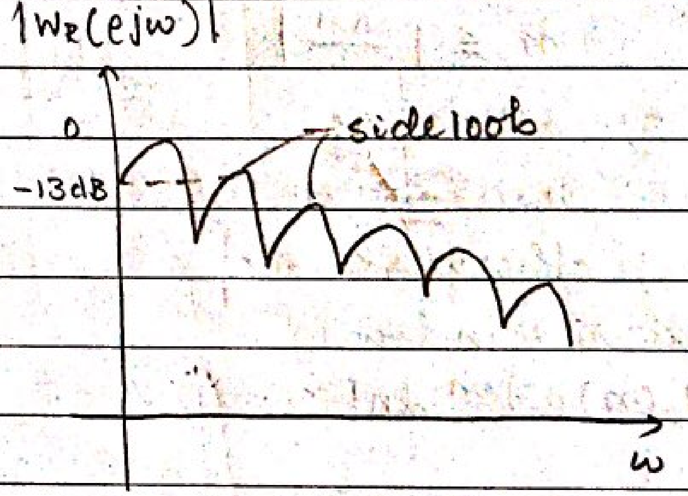
$$W_R(e^{j\omega}) = \int_0^{M-1} e^{-j\omega m}\, e^{j\omega(M-1)/2}\, dm$$

$$W_R(e^{j\omega}) = e^{j\omega\left(\frac{M-1}{2}\right)} \left[\frac{1 - e^{-j\omega M}}{1 - e^{-j\omega}}\right]$$

$$W_R(e^{j\omega}) = \cancel{e^{j\omega\left(\frac{M-1}{2}\right)}} \left[\frac{\cancel{e^{-j\omega\frac{M}{2}}}}{\cancel{e^{-j\omega/2}}} \left(\frac{e^{j\omega\frac{M}{2}} - e^{-j\omega\frac{M}{2}}}{e^{j\omega/2} - e^{-j\omega/2}}\right)\right]$$

$$W_R(e^{j\omega}) = \frac{2j \sin(\omega M/2)}{2j \sin(\omega/2)} = \frac{\sin(\omega M/2)}{\sin(\omega/2)}$$





log magnitude spectrum.

The width of the main lobe is $4\pi/M$ and the highest side lobe is -13dB relative to maximum value at $\omega = 0$.

As the length of the window increases the width of the ~~window~~ main lobe becomes narrower. The transition region width decreases.

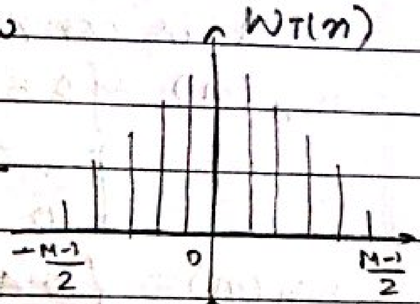
Gibb's Phenomenon

Due to the aprupt trunkation of $h_d(n)$ using rectangular window there will be both ripple in pass band and stop band of the low pass filter. This phenomenon is called the Gibb's phenomenon and this effect is higher just before and after the transition band.

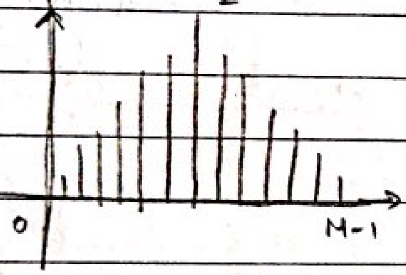
Bartlett Window (triangular window)

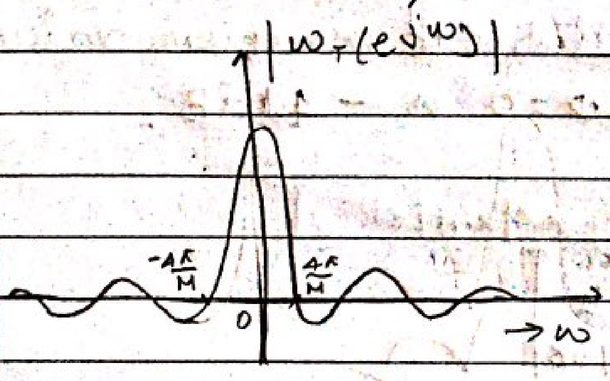
The end point of Bartlett window is given by:

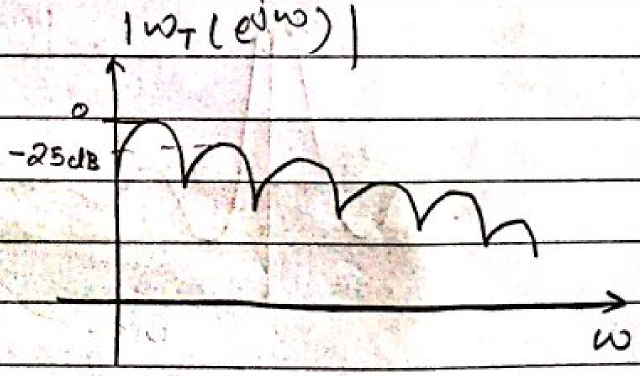
$$w_T(n) = \begin{cases} 1 - \dfrac{2|n|}{M-1} , & |n| \le \dfrac{M-1}{2} \\ 0 ; & \text{otherwise} \end{cases}$$



$$W_T(n) = \begin{cases} 1 - \dfrac{2\left|n - \dfrac{M-1}{2}\right|}{M-1} ; & n = 0, \cdots, M-1 \\ 0 ; & \text{otherwise} \end{cases}$$







The width of the main lobe is $8\pi/M$ and time domain plot has triangular shape.

$$W_T(e^{j\omega}) = \left[\frac{\sin\left(\frac{M-1}{4}\right)\omega}{\sin \frac{\omega}{2}} \right]^2$$

The highest side lobe for Bartlett window is approximately $-25\,dB$ relative to maximum value at $\omega = 0$.

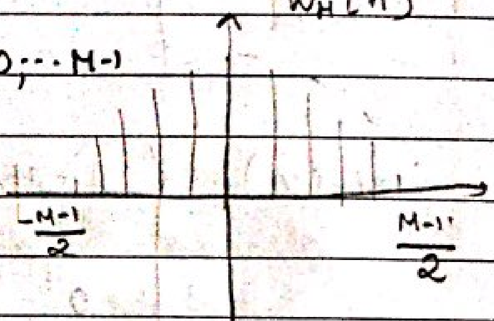
Date
Page

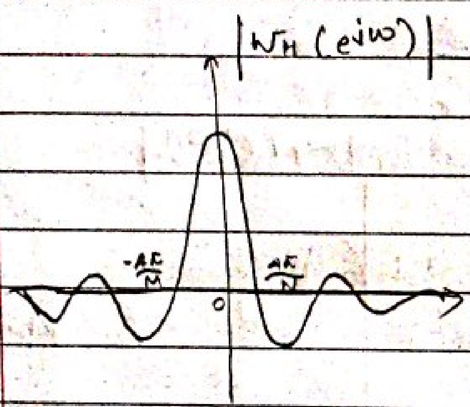
Hamming Window:

It is commonly used in speech processing and

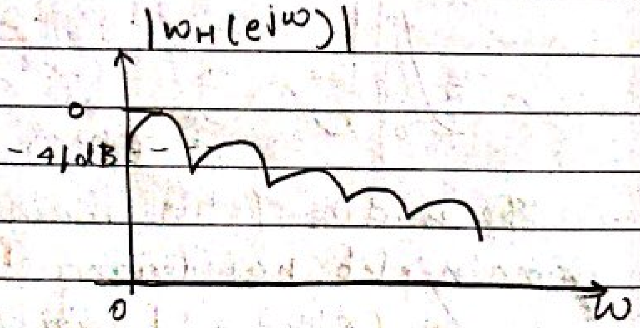
$$w_H(n) = \begin{cases} 0.54 + 0.46 \cos \frac{2\pi n}{M-1} ; & |n| \leq \frac{M-1}{2} \\ 0 ; & \text{otherwise} \end{cases}$$

$$w_H(n) = \begin{cases} 0.46 \cos \frac{2\pi n}{M-1} ; & n = 0; \cdots M-1 \\ 0 ; & \text{otherwise} \end{cases}$$





The highest side lobe relative to maximum value at $\omega = 0$ is -41 dB.



$$w_H(e^{j\omega}) = 0.54 \frac{\sin \frac{\omega N}{2}}{\sin \frac{\omega}{2}} + \frac{0.23 \sin\left(\frac{\omega N}{2} - \frac{\pi M}{M-1}\right)}{\sin\left(\frac{\omega}{2} - \frac{\pi}{M-1}\right)} + \frac{0.23 \sin\left(\frac{\omega N}{2} + \frac{\pi M}{M-1}\right)}{\sin\left(\frac{\omega}{2} + \frac{\pi}{M-1}\right)}$$

width of the main lobe is $8\pi/M$.

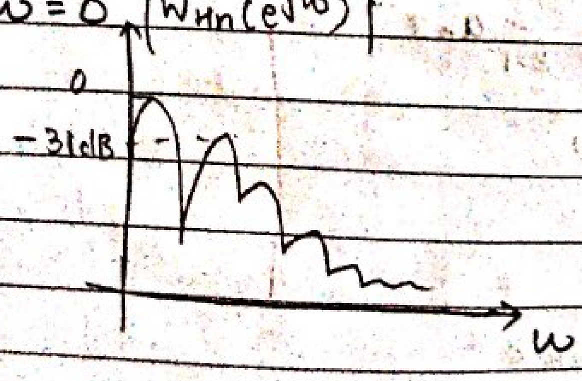
Hanning Window:

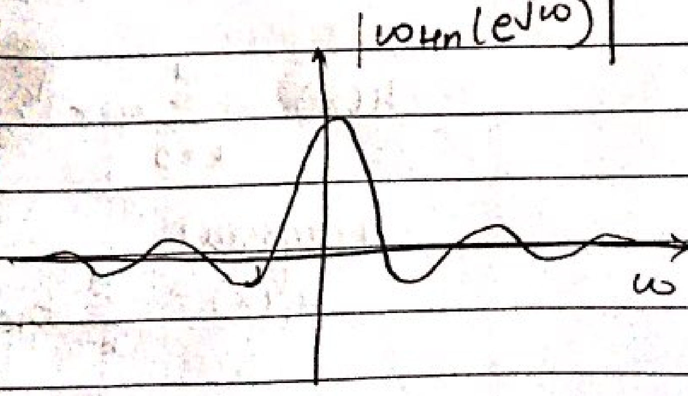
$$W_{Hn}(n) = \begin{cases} 0.5 + 0.5\cos\dfrac{2\pi n}{M-1} & ; |n| \le \dfrac{M-1}{2} \\ 0 ; \text{ otherwise} \end{cases}$$

$$W_{Hn}(n) = \begin{cases} 0.5\left(1 - \cos\dfrac{2\pi n}{M-1}\right) & ; n = 0 \cdots M-1 \\ 0 ; \text{ otherwise} \end{cases}$$

$$W_{Hn}(e^{j\omega}) = \frac{0.5\sin\frac{\omega M}{2}}{\sin\frac{\omega}{2}} + \frac{0.25\sin\left(\frac{\omega M}{2} - \frac{\pi M}{M-1}\right)}{\sin\left(\frac{\omega}{2} - \frac{\pi}{M-1}\right)} + \frac{0.25\sin\left(\frac{\omega M}{2} + \frac{\pi M}{M-1}\right)}{\sin\left(\frac{\omega}{2} + \frac{\pi}{M-1}\right)}$$

The width of the main lobe is $8\pi/M$ and the highest side lobe is $-31\,dB$ relative to maximum value at $\omega = 0$





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UNIT - 04

Design of IIR Filters

* **Frequency Transformations in the analog domain:**

1. **Lowpass to Lowpass**

To convert a lowpass filter with passband edge frequency Ω_p to another lowpass filter with passband edge frequency Ω_p'.

$$s \longrightarrow \frac{\Omega_p}{\Omega_p'} s \qquad H_l(s) \rightarrow H\left(\frac{\Omega_p}{\Omega_p'} s\right)$$

2. **Lowpass to Highpass**

To convert a lowpass filter with passband edge frequency Ω_p to a highpass filter with passband edge frequency Ω_p'.

$$s \longrightarrow \frac{\Omega_p \Omega_p'}{s} \qquad H_h(s) \rightarrow H\left(\frac{\Omega_p \Omega_p'}{s}\right)$$

3. **Lowpass to Bandpass**

To convert a lowpass filter with passband edge frequency Ω_p to a band pass filter with lower band edge frequency Ω_l and upper band edge frequency Ω_u we have to first convert the lowpass filter to another lowpass filter with passband edge frequency $\Omega_p' = 1$ and then perform the transformation.

$$s \rightarrow \Omega_p s \qquad s \rightarrow \frac{\Omega_p}{\Omega_p'} s \quad \text{here } \Omega_p' = 1$$

$$s \rightarrow \Omega_p \frac{s^2 + \Omega_l \Omega_u}{s(\Omega_u - \Omega_l)}$$

4. **Lowpass to Bandstop**

To convert a lowpass filter with passband edge frequency Ω_p to a bandstop filter with lower band edge frequency Ω_l and upper band edge frequency Ω_u

$$s \rightarrow \Omega_p \frac{s(\Omega_u - \Omega_l)}{s^2 + \Omega_l \Omega_u}$$

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* Characteristics of commonly used analog filters:

1. Butterworth filters:

Lowpass butterworth filters are all pole filters characterized by the magnitude-squared frequency response.

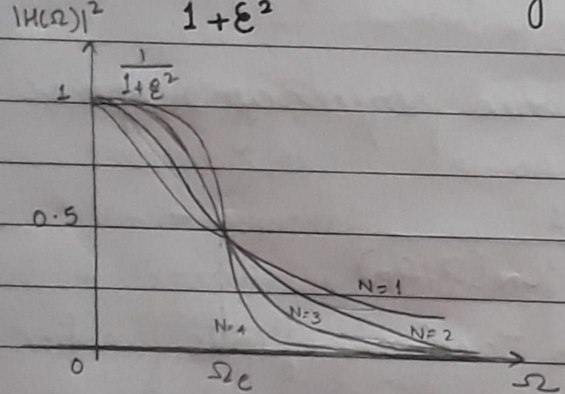
$$|H(\Omega)|^2 = \frac{1}{1+(\Omega/\Omega_c)^{2N}} = \frac{1}{1+\varepsilon^2(\Omega/\Omega_p)^{2N}}$$

where N is the order of the filter

Ω_c: cut off frequency (-3dB frequency)

Ω_p: pass band edge frequency

$\frac{1}{1+\varepsilon^2}$: band edge value of $|H(\Omega)|^2$



Frequency response characteristics of Butterworth filters for several values of N.

2. Chebyshev filters:

Type 1 chebyshev filter

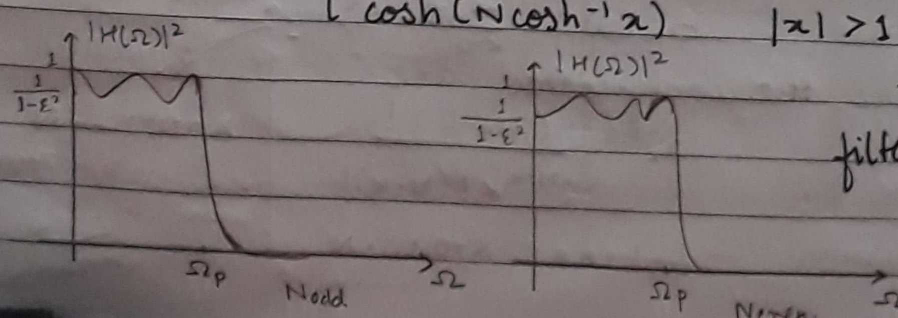
The magnitude squared of the frequency response characteristics of a type 1 chebyshev filter is given as:

$$|H(\Omega)|^2 = \frac{1}{1+\varepsilon^2 T_N^2(\Omega/\Omega_p)}$$

where ε is a parameter of the filter related to the ripple in the passband

and $T_N(x)$ is the Nth order Chebyshev polynomial

$$T_N(x) = \begin{cases} \cos(N\cos^{-1}x) & |x| \le 1 \\ \cosh(N\cosh^{-1}x) & |x| > 1 \end{cases}$$



Type 1 Chebyshev filter characteristic.

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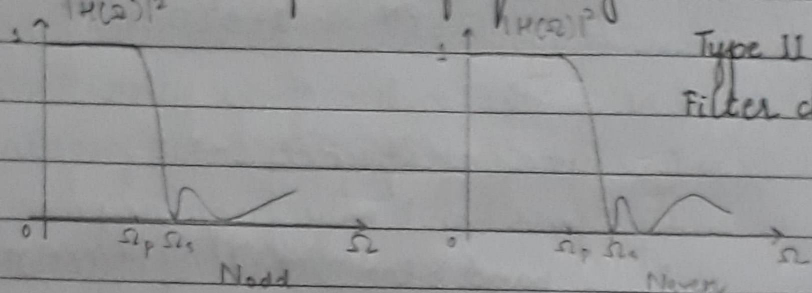
Type II Chebyshev Filter

The magnitude squared frequency response of type II chebyshev filter is given as:

$$|H(\Omega)|^2 = \frac{1}{1+\varepsilon^2\left[T_N^2(\Omega_s/\Omega_p)/T_N^2(\Omega_s/\Omega)\right]}$$

where Ω_s : stopband frequency



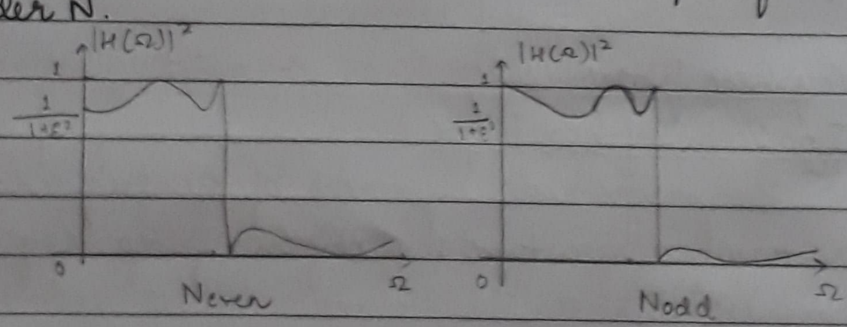
Type II Chebyshev Filter characteristic

3. **Elliptic filters:**

The magnitude squared frequency response of elliptic filters is given as:

$$|H(\Omega)|^2 = \frac{1}{1+\varepsilon^2 U_N(\Omega/\Omega_p)}$$

where $U_N(x)$ is the Jacobian elliptic function of order N.



4. **Bessel filters:**

Bessel filters are characterised by the system function:

$$H(s) = \frac{1}{B_N(s)}$$

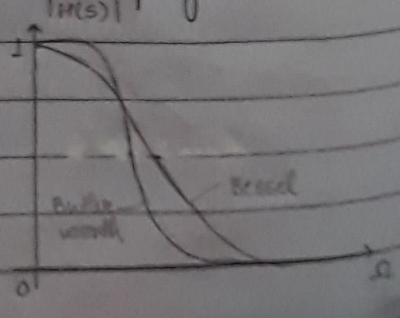
where $B_N(s)$ is the Nth order Bessel polynomial

$$B_N(s) = \sum_{k=0}^{N} a_k s^k$$

where the coefficients are:

$$a_k = \frac{(2N-k)!}{2^{N-k}k!(N-k)!}$$

where $k = 0, 1 \ldots N$



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* **IIR filter design by approximation of derivatives:**

To design digital filters having an infinite duration unit sample response (IIR filters). Here we convert an analog filter into a digital filter.

An analog filter can be described by its system function.

$$H_a(s) = \frac{B(s)}{A(s)} = \frac{\sum_{k=0}^{M} \beta_k s^k}{\sum_{k=0}^{N} \alpha_k s^k}$$

The analog filter having the function $H(s)$ can be described by the linear constant-coefficient differential equation.

$$\sum_{k=0}^{N} \alpha_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} \beta_k \frac{d^k x(t)}{dt^k}$$

where $x(t)$ denotes the input signal and $y(t)$ denotes the output signal

Here to convert an analog filter into a digital filter is to approximate the differential equation by an equivalent difference equation.

For the derivative $\frac{dy(t)}{dt}$ at time $t = nT$, we substitute the back difference.

$$\left.\frac{dy(t)}{dt}\right|_{t=nT} = \frac{y(nT) - y(nT-T)}{T}$$

$$= \frac{y(n) - y(n-1)}{T}$$

where T represents the sampling interval and $y(n) = y(nT)$.

The analog differentiator with output $dy(t)/dt$ has the system function $H(s)$ while the digital system that produces the output $[y(n) - y(n-1)]/T$ has the system function $H(z) = \frac{1 - z^{-1}}{T}$

Therefore

$$s = \frac{1 - z^{-1}}{T}$$ in frequency domain.

$$y(t) \longrightarrow \boxed{H(s) = s} \longrightarrow \frac{dy(t)}{dt}$$

$$y(n) \longrightarrow \boxed{H(z) = \frac{1 - z^{-1}}{T}} \longrightarrow \frac{y(n) - y(n-1)}{T}$$

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Similarly the second difference is substituted for $d^2y(t)/dt^2$

$$\left.\frac{d^2y(t)}{dt^2}\right|_{t=nT} = \frac{d}{dt}\left[\frac{dy(t)}{dt}\right]_{t=nT}$$

$$= \frac{[y(nT)-y(nT-T)]/T - [y(nT-T)-y(nT-2T)]/T}{T}$$

$$= \frac{y(n)-2y(n-1)+y(n-2)}{T^2}$$

In the frequency domain

$$s^2 = \frac{1-2z^{-1}+z^{-2}}{T^2} = \left(\frac{1-z^{-1}}{T}\right)^2$$

Thus for the k^{th} derivative of $y(t)$

$$s^k = \left(\frac{1-z^{-1}}{T}\right)^k$$

Hence the system function for the digital IIR filter obtained as a result of the approximation of the derivatives by finite differences is

$$H(z) = \left.Ha(s)\right|_{s=\frac{1-z^{-1}}{T}}$$

wkt $s = \frac{1-z^{-1}}{T} \Rightarrow z = \frac{1}{1-sT}$

Let $s = j\Omega$

$$z = \frac{1}{1-j\Omega T}$$

Q: Convert the analog bandpass filter with system function $Ha(s) = \frac{1}{(s+0.1)^2+9}$ into a digital IIR filter by using the backward difference for the derivative

$$s = \frac{1-z^{-1}}{T}$$

$$H(z) = \frac{1}{\left(\frac{1-z^{-1}}{T}+0.1\right)^2+9} = \frac{T^2}{1-2(1-0.1T)z^{-1}+0.2T+9.01T^2}$$

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* IIR filter design by Impulse Invariance :

To design an IIR filter having a unit sample response $h(n)$ that is the sampled version of the impulse response of the analog filter. $h(n) \equiv h(nT)$ where T is the sampling interval.

The digital filter with unit sample response $h(n) \equiv h_a(nT)$ has the frequency response

$$H(f) = F_s \sum_{k=-\infty} H_a[(f-k)F_s]$$

$$H(\omega) = F_s \sum_{k=-\infty}^{\infty} H_a[(\omega - 2\pi k)F_s]$$

$$H(\Omega T) = \frac{1}{T} \sum_{k=-\infty}^{\infty} H_a\left(\Omega - \frac{2\pi k}{T}\right)$$

$$\boxed{H_a(s)} \xrightarrow[\text{LT}]{\text{Inverse}} \boxed{h_a(t)} \xrightarrow{\text{sampling}} \boxed{h(nT_s)} \xrightarrow{ZT} \boxed{H(z)}$$

In analog domain $\qquad$ $t = nT_s$ $\qquad$ In digital domain

Let us consider

$$H_a(s) = \sum_{k=1}^{N} \frac{A_k}{s - P_k}$$

A_k : coefficients in partial fraction

P_k : poles of analog filter

Step 1 : Applying inverse laplace transform

$$L^{-1}\{H_a(s)\} = L^{-1}\left\{ \sum_{k=1}^{N} \frac{A_k}{s - P_k} \right\}$$

$$h_a(t) = L^{-1}\left\{ \frac{A_1}{s - P_1} + \frac{A_2}{s - P_2} + \cdots + \frac{A_N}{s - P_N} \right\}$$

$$h_a(t) = L^{-1}\left\{ \frac{A_1}{s - P_1} \right\} + L^{-1}\left\{ \frac{A_2}{s - P_2} \right\} + \cdots + L^{-1}\left\{ \frac{A_N}{s - P_N} \right\}$$

$$h_a(t) = A_1 L^{-1}\left[\frac{1}{s - P_1} \right] + A_2 L^{-1}\left[\frac{1}{s - P_2} \right] + \cdots + A_N L^{-1}\left[\frac{1}{s - P_N} \right]$$

$$h_a(t) = A_1 e^{P_1 t} u(t) + A_2 e^{P_2 t} u(t) + \cdots + A_N e^{P_N t} u(t)$$

$$h_a(t) = \sum_{k=1}^{N} A_k e^{P_k t} u(t)$$

step 2: Sampling at $t = nT$

$$h(nT_s) = \sum_{k=1}^{N} A_k e^{P_k(nT)} u(nT)$$

step 3: Applying z transform

$$H(z) = \sum_{n=-\infty}^{\infty} h(nT) z^{-n}$$

$$H(z) = \sum_{n=0}^{\infty} \left[\sum_{k=1}^{N} A_k e^{P_k(nT)} \right] z^{-n}$$

$$H(z) = \sum_{k=1}^{N} A_k \left[\sum_{n=0}^{\infty} e^{P_k(nT)} z^{-n} \right]$$

$$H(z) = \sum_{k=1}^{N} A_k \left[\sum_{n=0}^{\infty} \left(e^{P_k T} z^{-1} \right)^n \right]$$

wkt $\sum_{n=0}^{\infty} a^n = \frac{1}{1-a}$ here $a = e^{P_k T} z^{-1}$

$$H(z) = \sum_{k=1}^{N} A_k \left[\frac{1}{1 - e^{P_k T} z^{-1}} \right]$$

In analog domain

$$H(s) = \sum_{k=1}^{N} \frac{A_k}{s - P_k}$$

comparing both the equations

Poles: $s = P_k$ and $z = e^{P_k T}$

Therefore $z = e^{sT}$

Criteria for mapping

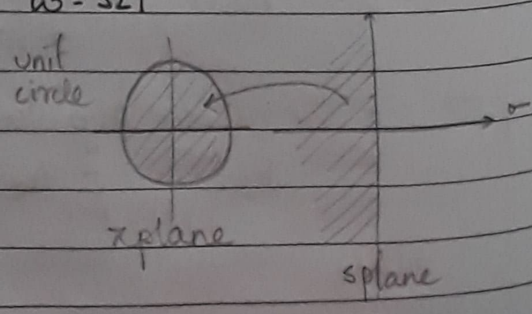
$$z = e^{sT}$$

$s = \sigma + j\Omega$ and $z = re^{j\omega}$

$$re^{j\omega} = e^{(\sigma + j\Omega)T}$$

$$re^{j\omega} = e^{\sigma T} e^{j\Omega T}$$

then $r = e^{\sigma T}$ and $\omega = \Omega T$



Q: convert the analog filter with system function $H_a(s) = \frac{1}{(s+1)(s+2)}$ into a digital IIR filter by means of the impulse invariance method. with $f_s = 5$ samples/sec

— $H_a(s) \xrightarrow[ILT]{} h(t) \xrightarrow[\text{sampling}]{} h(nT_s) \xrightarrow[ZT]{} H(z)$

$$H_a(s) = \frac{1}{(s+1)(s+2)} = \frac{A}{s+1} + \frac{B}{s+2}$$

$$1 = A(s+2) + B(s+1)$$

at $s=-1$: $1 = A$

at $s=-2$: $B = -1$

$$\therefore H_a(s) = \frac{1}{s+1} - \frac{1}{s+2}$$

Taking inverse Laplace transform

$$h_a(t) = e^{-t}u(t) - e^{-2t}u(t)$$

$$h_a(t) = (e^{-t} - e^{-2t})u(t)$$

Sampling : $t = nT$

$$h(nT) = (e^{-nT} - e^{-2nT})u(nT)$$

Applying z transform

$$H(z) = \sum_{n=-\infty}^{\infty} h(nT)z^{-n}$$

$$H(z) = \sum_{n=0}^{\infty} (e^{-nT} - e^{-2nT})z^{-n}$$

$$H(z) = \sum_{n=0}^{\infty} e^{-nT}z^{-n} - \sum e^{-2nT}z^{-n}$$

$$H(z) = \sum_{n=0}^{\infty} (e^{-T}z^{-1})^n - \sum_{n=0}^{\infty} (e^{-2T}z^{-1})^n$$

$$H(z) = \frac{1}{1-e^{-T}z^{-1}} - \frac{1}{1-e^{-2T}z^{-1}}$$

$$H(z) = \frac{1 - e^{-2T}z^{-1} - 1 + e^{-T}z^{-1}}{(1-e^{-T}z^{-1})(1-e^{-2T}z^{-1})}$$

$$H(z) = \frac{(e^{-T} - e^{-2T})z^{-1}}{(1-e^{-T}z^{-1})(1-e^{-2T}z^{-1})}$$

substitute $T = \frac{1}{f_s} = \frac{1}{5}$

★ **IIR filter design by the Bilinear Transform:**

Let us consider an analog linear filter with system function:

$$H(s) = \frac{b}{s+a}$$

The system is also characterised by the differential equation $\frac{dy(t)}{dt} + ay(t) = bx(t)$

Taking Laplace transform

$$sY(s) + aY(s) = bX(s)$$

$$H(s) = \frac{Y(s)}{X(s)} = \frac{b}{s+a}$$

we can write $y(t)$ as (trapezoidal formula)

$$y(t) = \int_{t_0}^{t} y'(\tau)\,d\tau + y(t_0)$$

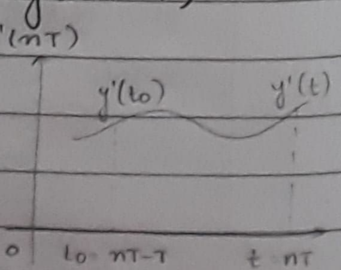
as $y(\tau)\Big|_{t_0}^{t} + y(t_0) = y(t) - y(t_0) + y(t_0) = y(t)$

at $t = nT$ and $t_0 = nT - T$

$$y(nT) = \frac{T}{2}\left[y'(nT) + y'(nT-T)\right] + y(nT-T)$$

Area of trapezium $= \frac{a+b}{2} \times \text{height}$

$$y'(t) = \left[\frac{y'(nT) + y'(nT-T)}{2}\right](nT - nT + T)$$



From the above differential equation

$$y'(nT) = -ay(nT) + bx(nT)$$

Therefore

$$y(nT) = \frac{T}{2}\left[-ay(nT) + bx(nT) - ay(nT-T) + bx(nT-T)\right] + y(nT-T)$$

but $y(nT) = y(n)$ and $x(nT) = x(n)$

$$\therefore y(n) = \frac{T}{2}\left[-ay(n) + bx(n) - ay(n-1) + bx(n-1)\right] + y(n-1)$$

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$$y(n)\left[1+\frac{aT}{2}\right]-y(n-1)\left[1-\frac{aT}{2}\right]$$
$$=\frac{bT}{2}\left[x(n)+x(n-1)\right]$$

Applying z transform

$$Y(z)\left[1+\frac{aT}{2}\right]-Y(z)z^{-1}\left[1-\frac{aT}{2}\right]$$
$$=\frac{bT}{2}\left[X(z)+X(z)z^{-1}\right]$$

$$Y(z)\left[1+\frac{aT}{2}-z^{-1}\left(1-\frac{aT}{2}\right)\right]=X(z)\left[\frac{bT}{2}(1+z^{-1})\right]$$

Therefore

$$H(z)=\frac{Y(z)}{X(z)}=\frac{(bT/2)(1+z^{-1})}{1+aT/2-(1-aT/2)z^{-1}}$$

$$H(z)=\frac{b}{\frac{2}{T}\left[1+\frac{aT}{2}-\left(1-\frac{aT}{2}\right)z^{-1}\right]\frac{1}{1+z^{-1}}}$$

$$H(z)=\frac{b}{\left[\frac{2}{T}+a-\frac{2}{T}z^{-1}+az^{-1}\right]\frac{1}{1+z^{-1}}}$$

$$H(z)=\frac{b}{\left[\frac{2}{T}(1-z^{-1})+a(1+z^{-1})\right]\frac{1}{1+z^{-1}}}$$

$$H(z)=\frac{b}{\frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)+a}$$

Thus comparing with the analog filter function

$$s=\frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)$$

Characteristics of the bilinear transformation

Let $z=re^{j\omega}$ and $s=\sigma+j\Omega$

$$s=\frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)$$

$$s = \frac{2}{T}\left[\frac{z-1}{z+1}\right]$$

$$s = \frac{2}{T}\left[\frac{re^{j\omega}-1}{re^{j\omega}+1}\right]$$

$$s = \frac{2}{T}\left[\frac{r^2-1}{1+r^2+2r\cos\omega} + j\frac{2r\sin\omega}{1+r^2+2r\cos\omega}\right]$$

thus $\sigma = \frac{2}{T}\frac{r^2-1}{1+r^2+2r\cos\omega}$

$$\Omega = \frac{2}{T}\frac{2r\sin\omega}{1+r^2+2r\cos\omega}$$

When $r=1$ then $\sigma = 0$ (because LHP in s maps into the unit circle in z plane)

$$\Omega = \frac{2}{T}\frac{2\sin\omega}{2+2\cos\omega}$$

$$\Omega = \frac{2}{T}\frac{\sin\omega}{1+\cos\omega}$$

$$\Omega = \frac{2}{T}\tan\frac{\omega}{2} \Rightarrow \omega = 2\tan^{-1}\frac{\Omega T}{2}$$

Q: convert the analog filter with system function $H_a(s) = \frac{s+0.1}{(s+0.1)^2+16}$ into a digital IIR filter by means of the bilinear transform. The digital filter is to have a resonant frequency of $\omega_r = \pi/2$.

$$H(s) = \frac{s+0.1}{(s+0.1)^2+16}$$

$$H(z) = H(s)\Big|_{s=\frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)}$$

In general

$$H_a(s) = \frac{s+a}{(s+a)^2+\Omega_c^2}$$

$\therefore$ cut off frequency $\Omega_c = \sqrt{16} = 4$ rad/sec

wkt

$$\Omega = \frac{2}{T}\tan\frac{\omega}{2}$$

$$T = \frac{2}{\Omega}\tan\frac{\omega}{2} = \frac{2}{4}\tan\left(\frac{\pi/2}{2}\right) = 0.5\text{ sec}$$

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Therefore

$$H(z) = H(s)\Big|_{s=4\left(\frac{1-z^{-1}}{1+z^{-1}}\right)}$$

$$H(z) = \frac{4\left(\frac{1-z^{-1}}{1+z^{-1}}\right) + 0.1}{\left[4\left(\frac{1-z^{-1}}{1+z^{-1}}\right) + 0.1\right]^2 + 16}$$

$$H(z) = \frac{\frac{4(1-z^{-1}) + 0.1(1+z^{-1})}{\cancel{(1+z^{-1})}}}{\frac{[4(1-z^{-1}) + 0.1(1+z^{-1})]^2 + 16(1+z^{-1})^2}{(1+z^{-1})^{\cancel{2}}}}$$

$$H(z) = \frac{[4(1-z^{-1}) + 0.1(1+z^{-1})](1+z^{-1})}{[4(1-z^{-1}) + 0.1(1+z^{-1})]^2 + 16(1+z^{-1})^2}$$

$$H(z) = \frac{0.128 + 0.006z^{-1} - 0.122z^{-2}}{1 + 0.0006z^{-1} + 0.975z^{-2}}$$

$$\Omega_p = \frac{\omega_p}{T} \text{ and } \Omega_s = \frac{\omega_s}{T}$$

order of Butterworth filter (3dB) $N = \frac{\log\left[(10^{-\frac{kp}{10}} - 1)/(10^{-\frac{ks}{10}} - 1)\right]}{2\log(\Omega_s/\Omega_p)}$

Cutoff frequency

$$\Omega_c = \frac{1}{2}\left[\frac{\Omega_p}{\left[\frac{1}{A_p^2} - 1\right]^{\frac{1}{2N}}} + \frac{\Omega_s}{\left[\frac{1}{A_s^2} - 1\right]^{1/2N}}\right] = \frac{1}{2}\left[\frac{\Omega_p}{[10^{-0.1A_p} - 1]^{1/2N}} + \frac{\Omega_s}{[10^{-0.1A_s} - 1]^{1/2N}}\right]$$

Normalized butterworth system

$$H_{an}(s) = \frac{1}{s^N + b_{N-1}s^{N-1} + \dots + b_1 s + 1}$$

1st order = $s+1$

2nd order = $s^2 + \sqrt{2}s + 1$

3rd order = $(s^2+s+1)(s+1)$

4th order = $(s^2 + 0.7654s + 1)(s^2 + 1.8478s + 1)$

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UNIT-04

Design of IIR Filters

* Filter:

- A filter is a frequency selective network.
- Depending on the signal processing domain there are two types of filters:
  a. analog filters
  b. digital filters.
- The filters are classified according to the frequency selective characteristics
  a. Low pass filter
  b. High pass filter
  c. Band pass
  d. Band elimination / reject / stop
  e. Comb filter / all pass filter.
- A filter is characterised by a band of frequency which the filter allows to pass through it. Also there is another band of frequency which the filter attenuates or blocks. This band is called as stop band or attenuation.
- A filter that allows frequency components upto Ω_c without any attenuation is called low pass filter and such filter completely attenuates / blocks frequencies above Ω_c. A low pass filter can also be used as denoising filter, as noise is usually of higher frequencies compared to the signal components.

By passing a noisy signal through LPF with the cut off frequency required to pass the signal components, noise can be removed. This is particularly useful when the signal spectrum and noise spectrum do not overlap.

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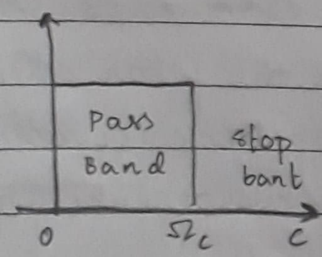
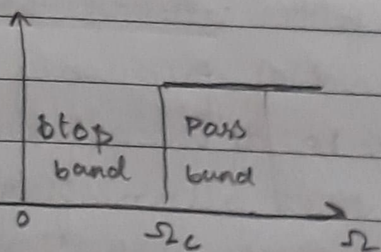
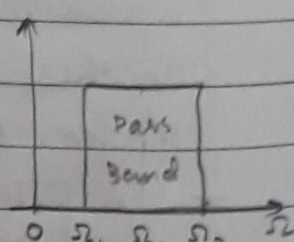
- High pass filter: The high frequency components of an analog signal constitute the finer details of the signal. High pass filter can also be constructed by using a low pass filter.
- Band Pass Filter: They are useful when only selected band of frequencies are to be passed and attenuates all the other frequencies. It is useful in radio and TV receiver circuits.
- Band Stop Filter: It is used in TV receivers to overcome interchannel interference effectively.
- Comb Filter: comb filter is used to denoise when the spectrum of the signal overlaps with the spectrum of noise.
- All pass Filter: It passes different frequency components with different delay. systems that do not have linear phase ~~pamp~~ response can be cascaded with all pass filter to compensate for poor phase response.
  Hence all pass filter is called phase equivalizer filter.



Low pass filter

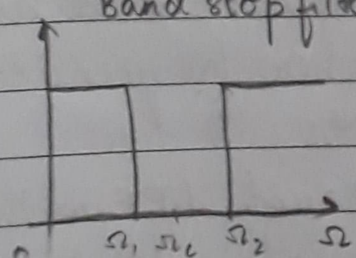


High pass filter

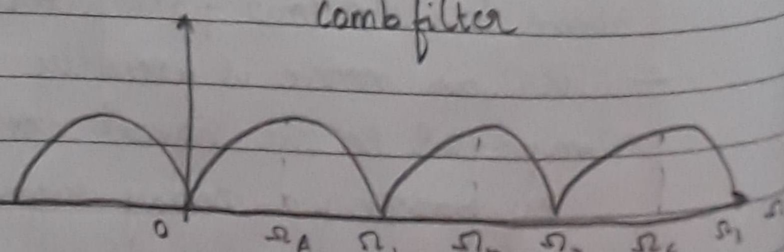


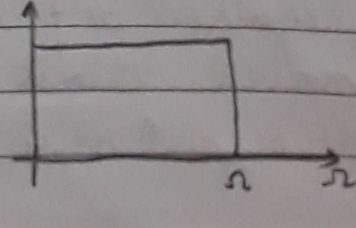
Band pass filter

Band stop filter



Comb filter





All pass filter

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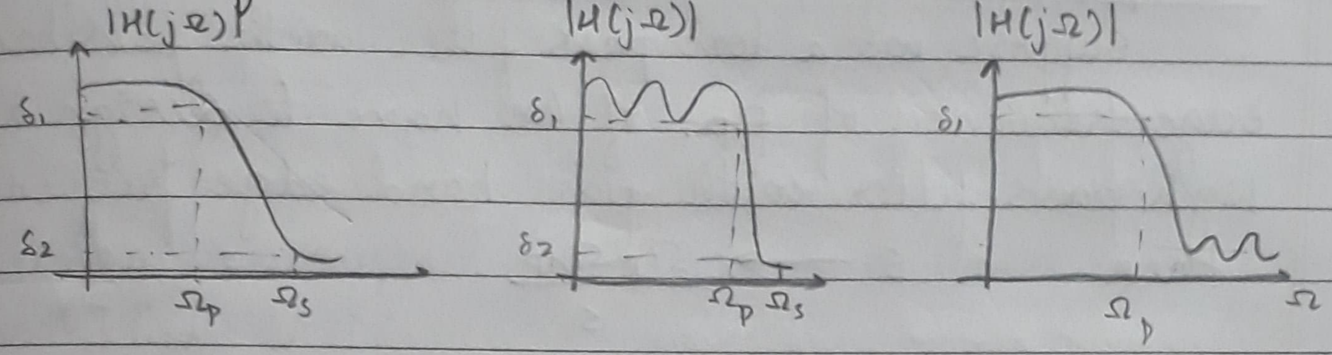
Low pass filter



The width of the pass band, i.e., upto Ω_p is usually reffered to as filter bandwidth.

* **Realisation of analog filter:**

A typical system function of an analog filter is represented as $H(s)$ where s is the laplace complex variable given by

$$s = \sigma + j\Omega.$$

the frequency transfer function $H(j\Omega)$ is given by

$$H(j\Omega) = H(s)\big|_{s=j\omega}$$

The magnitude squared response

$$|H(j\Omega)|^2 = H(j\Omega)\, H^*(j\Omega)$$
$$= H(s)\, H(-s)\big|_{s=j\omega}$$

Design of stable system implies placing the poles and zeros to the left ~~of~~ half of the s plane to satisfy the magnitude-frequency response specifications.

* **Frequency Transformations:**

System function of a normalised low pass filter with a pass band edge frequency of Ω_L.

suppose to construct another low pass filter with pass band edge frequency of Ω_{LP}, then we have to apply the transformation $s \rightarrow \frac{\Omega_P}{\Omega_{LP}} s$.

(Low pass to low pass conversion)

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Low pass to high pass conversion

Suppose we a low pass filter with pass band edge frequency of Ω_p. If we have to design a high pass filter with pass band edge frequency of Ω_{HP} then

$$s \longrightarrow \frac{\Omega_p \Omega_{HP}}{s}$$

Low pass to band pass conversion

Suppose we a low pass filter with pass band edge frequency of Ω_p. To design a band pass filter with a higher band edge frequency Ω_u and a lower band edge frequency Ω_L, then

$$s \longrightarrow \frac{s^2 + \Omega_L \Omega_U}{s(\Omega_u - \Omega_L)}$$

Low pass to band stop/notch conversion

Suppose we have a low pass filter with pass band edge frequency of Ω_p. To convert to a band stop filter with a higher band edge frequency Ω_u and a lower band edge frequency Ω_L, then

$$s \rightarrow \frac{\Omega_p\, s(\Omega_u - \Omega_L)}{s^2 + \Omega_u \Omega_L}$$

Q: The system function of the prototyped (normalised) low pass filter is given as :

$$H_{an}(s) = \frac{0.245}{s^4 + 0.95s^3 + 1.4s^2 + 0.72s + 0.27}$$

Obtain the system function of a low pass filter having a pass band edge frequency of 10 rad/sec.

$\Omega_p = 1$ rad/sec. $\qquad s \rightarrow 1/10\, s = 0.1s$

$$H_a(s) = \frac{0.245}{(0.1s)^4 + 0.95(0.1s)^3 + 1.4(0.1s)^2 + 0.72(0.1s) + 0.27}$$

$$H_a(s) = \frac{0.245}{0.0001s^4 + 0.00095s^3 + 0.014s^2 + 0.072s + 0.27}$$

Q: system function of first order normalised low pass filter is given as:

$$H_{an}(s) = \frac{1}{s+1}$$

obtain the system function of a band pass filter with $F_L = 1\,kHz$ to $f_u = 2k\,Hz$.

$\Omega_u = 2\pi f_u = 4\pi k$

$\Omega_L = 2\pi f_L = 2\pi k$

For band pas

$$s \longrightarrow \frac{s^2 + \Omega_L \Omega_u}{s(\Omega_u - \Omega_L)} = \frac{s^2 + 8\pi^2 M}{s(2\pi k)}$$

$$H_{an}(s) = \frac{1}{\frac{s^2 + \Omega_L \Omega_u}{s(\Omega_u - \Omega_L)} + 1}$$

$$H_{an}(s) = \frac{1}{\frac{s^2 + 8\pi^2 M}{s(2\pi k)} + 1}$$

$$H_{an}(s) = \frac{s(2\pi k)}{s^2 + 8\pi^2 M + 2s\pi k}$$

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UNIT - 05

Digital Filter Structures

★ **Basic IIR filter structures:**

1. **Direct form I and II realizations**

The difference equation representing an IIR filter

$$y(n) = -\sum_{k=1}^{N} a_k y(n-k) + \sum_{k=0}^{M} b_k x(n-k)$$

Taking z transform on both sides

$$Y(z) = -\sum_{k=1}^{N} a_k Y(z) z^{-k} + \sum_{k=0}^{M} b_k X(z) z^{-k}$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{M} b_k z^{-k}}{1 + \sum_{k=1}^{N} a_k z^{-k}}$$

The IIR system can be viewed as two systems in cascade. $H(z) = H_1(z) H_2(z)$

where $H_1(z)$ consists of the zeros of $H(z)$ and $H_2(z)$ consists of the poles of $H(z)$.

i.e., $H_1(z) = \sum_{k=0}^{M} b_k z^{-k}$

$$H_2(z) = \frac{1}{1 + \sum_{k=1}^{N} a_k z^{-k}}$$

Two different direct form realizations are characterized by whether $H_1(z)$ precedes $H_2(z)$ or $H_2(z)$ precedes $H_1(z)$.

- **Direct form I**

$H_1(z)$ precedes $H_2(z)$

$H_1(z) = \frac{W(z)}{X(z)}$ where $W(z)$ is the input to the all zero system

$$\frac{W(z)}{X(z)} = \sum_{k=0}^{M} b_k z^{-k}$$

Taking inverse Z transform

$$w(n) = \sum_{k=0}^{M} b_k \, x(n-k)$$

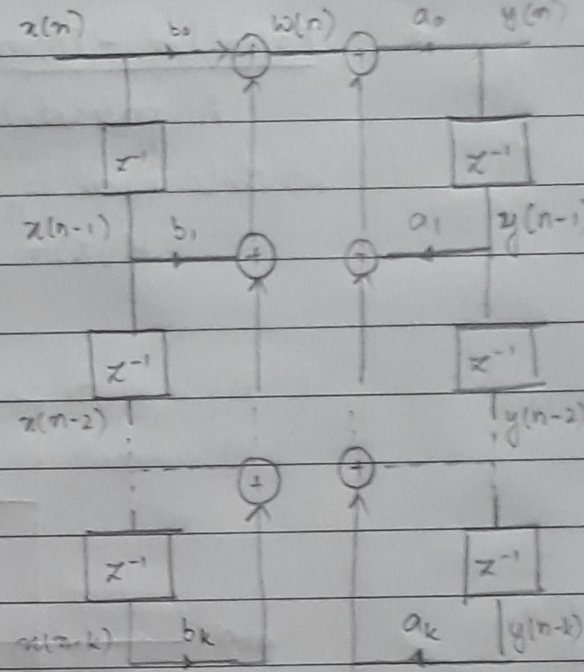
$$H_2(z) = \frac{Y(z)}{W(z)} = \frac{1}{1 + \sum_{k=1}^{N} a_k z^{-k}}$$

$$W(z) = Y(z) + \sum_{k=1}^{N} a_k z^{-k} Y(z)$$

Taking inverse Z transform

$$w(n) = y(n) + \sum_{k=1}^{N} a_k \, y(n-k)$$

$$y(n) = w(n) - \sum_{k=1}^{N} a_k \, y(n-k)$$



Direct form I

- <u>Direct form II</u>

$H_2(z)$ precedes $H_1(z)$

$$H_2(z) = \frac{W(z)}{X(z)} = \frac{1}{1 + \sum_{k=1}^{N} a_k z^{-k}}$$

$$X(z) = W(z) + \sum_{k=1}^{N} a_k z^{-k} W(z)$$

Taking inverse Z transform

$$x(n) = w(n) + \sum_{k=1}^{N} a_k \, w(n-k)$$

$$w(n) = x(n) - \sum_{k=1}^{N} a_k \, w(n-k)$$

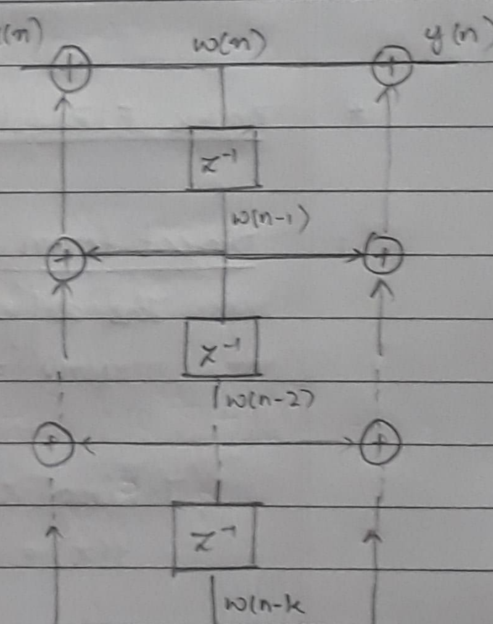
Similarly

$$H_1(z) = \frac{Y(z)}{W(z)} = \sum_{k=0}^{M} b_k z^{-k}$$

$$Y(z) = \sum_{k=0}^{M} b_k z^{-k} W(z)$$

Taking inverse Z transform

$$y(n) = \sum_{k=0}^{M} b_k \, w(n-k)$$

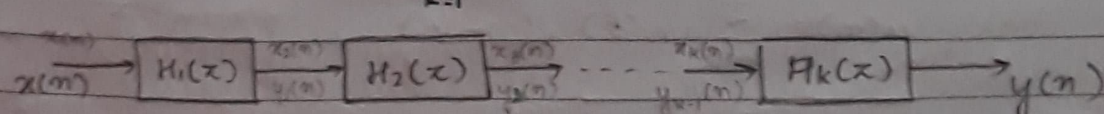


Direct form II

2. <u>Cascade Realization:</u>

The system can be factored into a cascade of second order systems such that

$$H(z) = \prod_{k=1}^{K} H_k(z) = H_1(z) H_2(z) \cdots H_K(z)$$



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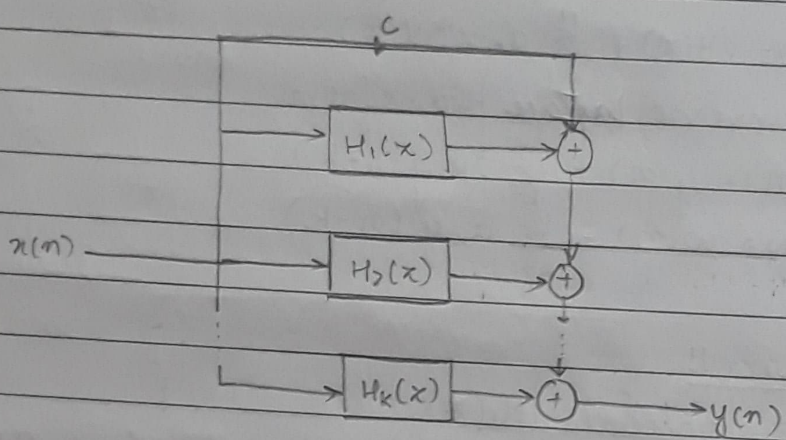
3. **Parallel - Form Realization**

A parallel form realization of an IIR system can be obtained by performing a partial fraction expansion.

$$H(z) = C + \sum_{k=1}^{N} \frac{A_k}{1 - p_k z^{-1}}$$



★ **Basic FIR filter structures:**

1. **Direct form structure:**

The difference equation representing a FIR filter is

$$y(n) = \sum_{k=0}^{M-1} b_k x(n-k)$$ (Feedback is absent in FIR filters)

Taking z transform on both sides

$$Y(z) = \sum_{k=0}^{M-1} b_k X(z) z^{-k}$$

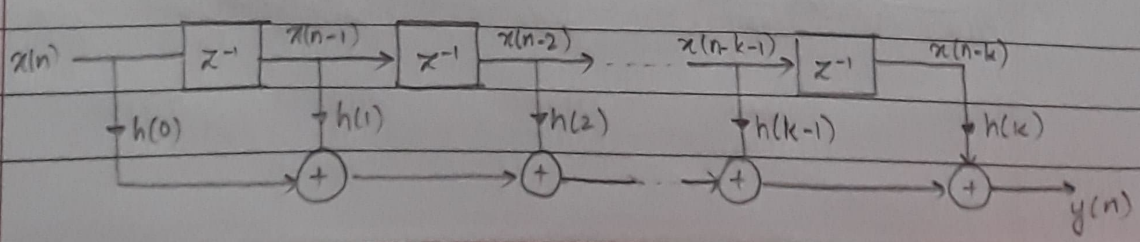
$$H(z) = \frac{Y(z)}{X(z)} = \sum_{k=0}^{M-1} b_k z^{-k}$$

Taking inverse z transform, we obtain the unit sample response of the FIR system.

$$h(n) = \begin{cases} b_n & 0 \le n \le M-1 \\ 0 & \text{otherwise} \end{cases}$$

Thus the direct form structure can be obtained by

$$y(n) = \sum_{k=0}^{M-1} h(k) x(n-k)$$



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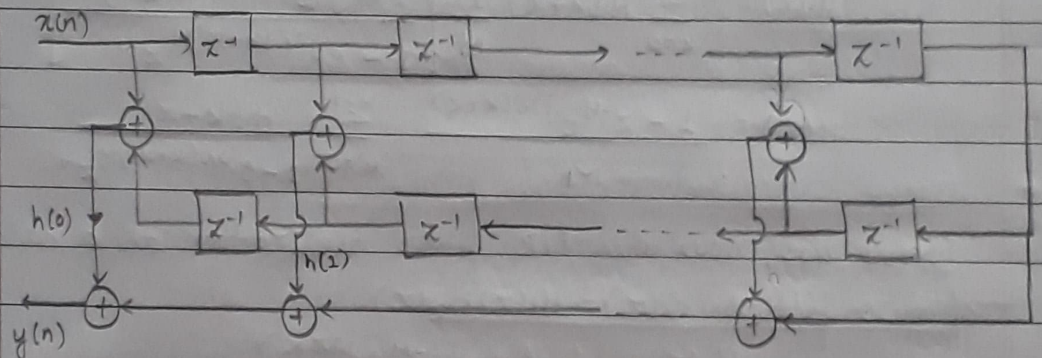
2. Linear Phase FIR structure:

When the FIR filter has linear phase, the unit sample response of the system satisfies either the symmetry or assymmetry condition.

$$h(n) = \pm h(M-1-n)$$

For such a system the number of multiplications is reduced from M to M/2 for M even and M to (M-1)/2 for M odd.



3. Frequency sampling structure:

In frequency sampling realization for an FIR the parameters that characterise the filter are the values of the desired frequency response instead of the impulse response.

The desired frequency response is:

$$H(\omega) = \sum_{n=0}^{M-1} h(n) e^{-j\omega n}$$

at a set of equally spaced frequencies

$$\omega_k = \frac{2\pi}{M}(k+\alpha) \qquad \text{where } k = 0,1 \ldots (M-1)/2 : M \text{ odd}$$

$$k = 0,1 \ldots M/2 - 1 : M \text{ even}$$

$$\alpha = 0 \text{ or } 1/2$$

Therefore

$$H\left[\frac{2\pi}{M}(k+\alpha)\right] = H(k+\alpha)$$

$$= \sum_{n=0}^{M-1} h(n) e^{-j2\pi(k+\alpha)n/M} \qquad k = 0,1, \ldots M-1$$

The set of values $H(k+\alpha)$ are called the frequency samples of $H(\omega)$. In this case $\alpha = 0$, $H(k)$ is the M-point DFT of $h(n)$.

$$\therefore h(n) = \frac{1}{M} \sum_{k=0}^{M-1} H(k+\alpha) e^{j2\pi(k+\alpha)n/M}$$

Taking z transform

$$H(z) = \sum_{n=0}^{M-1} h(n) z^{-n}$$

$$H(z) = \sum_{n=0}^{M-1} \left[\frac{1}{M} \sum_{k=0}^{M-1} H(k+\alpha) e^{j2\pi(k+\alpha)n/M} \right] z^{-n}$$

By interchanging the order of two summations

$$H(z) = \sum_{k=0}^{M-1} H(k+\alpha) \left[\frac{1}{M} \sum_{n=0}^{M-1} \left(e^{j2\pi(k+\alpha)/M} z^{-1} \right)^n \right]$$

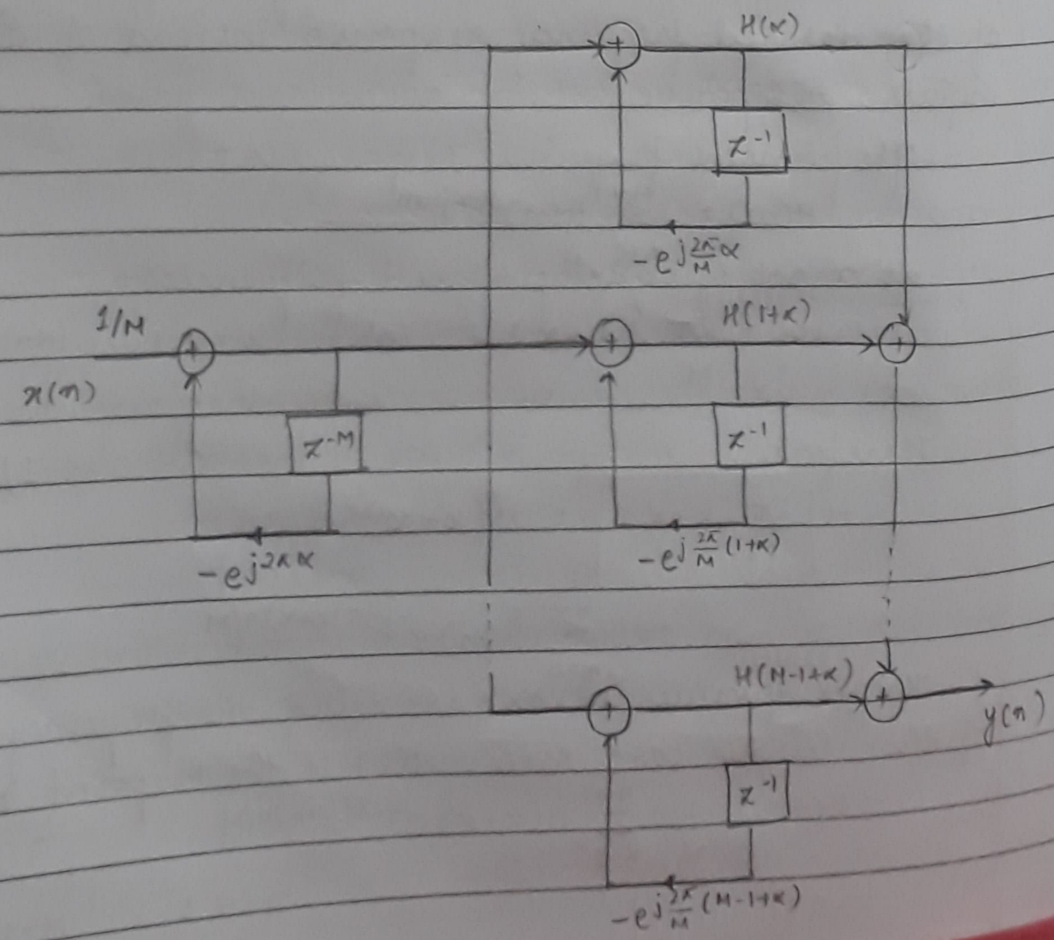
$$H(z) = \sum_{k=0}^{M-1} H(k+\alpha) \frac{1}{M} \left[\frac{1 - \left(e^{j2\pi(k+\alpha)/M} z^{-1} \right)^M}{1 - e^{j2\pi(k+\alpha)/M} z^{-1}} \right]$$

$$H(z) = \frac{1 - z^{-M} e^{j2\pi(k+\alpha)}}{M} \sum_{k=0}^{M-1} \frac{H(k+\alpha)}{1 - e^{j2\pi(k+\alpha)/M} z^{-1}}$$

Now we view this FIR filter realization as a cascade of two filters $H(z) = H_1(z) H_2(z)$

where $H_1(z) = \frac{1}{M} \left(1 - z^{-M} e^{j2\pi\alpha} \right)$

and $H_2(z) = \sum_{k=0}^{M-1} \frac{H(k+\alpha)}{1 - e^{j2\pi(k+\alpha)/M} z^{-1}}$



UNIT 5: Digital Filter Structures

IIR filter structures
- Direct Form (I/II)
- Cascade Structure
- Parallel Form
- Ladder
- Lattice

Direct Form:

The difference equation representing an IIR filter

$$y(n) = -\sum_{k=1}^{N} a_k y(n-k) + \sum_{k=0}^{M} b_k x(n-k) \quad \text{——} (1)$$

Taking z transform on both sides

$$Y(z) = -\sum_{k=1}^{N} a_k z^{-k} Y(z) + \sum_{k=0}^{M} b_k z^{-k} X(z)$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{M} b_k z^{-k}}{1 + \sum_{k=1}^{N} a_k z^{-k}} \quad \text{——} (2)$$

Let $H(z) = H_1(z) \times H_2(z)$

$$H_1(z) = \frac{Y(z)}{W(z)} \qquad H_2(z) = \frac{W(z)}{X(z)}$$

$$H(z) = \sum_{k=0}^{N} b_k z^{-k} \cdot \frac{1}{1 + \sum_{k=1}^{N} a_k z^{-k}}$$

Therefore

$$H_1(z) = \sum_{k=0}^{N} b_k z^{-1}$$

$$H_2(z) = \frac{1}{1 + \sum_{k=1}^{N} a_k z^{-k}}$$

Let a third order system be considered

then $H_1(z) = \sum_{k=0}^{3} b_k z^{-k} = b_0 + b_1 z^{-1} + b_2 z^{-2} + b_3 z^{-3} = \frac{Y(z)}{W(z)}$ —— (3)

Taking Inverse Z transform

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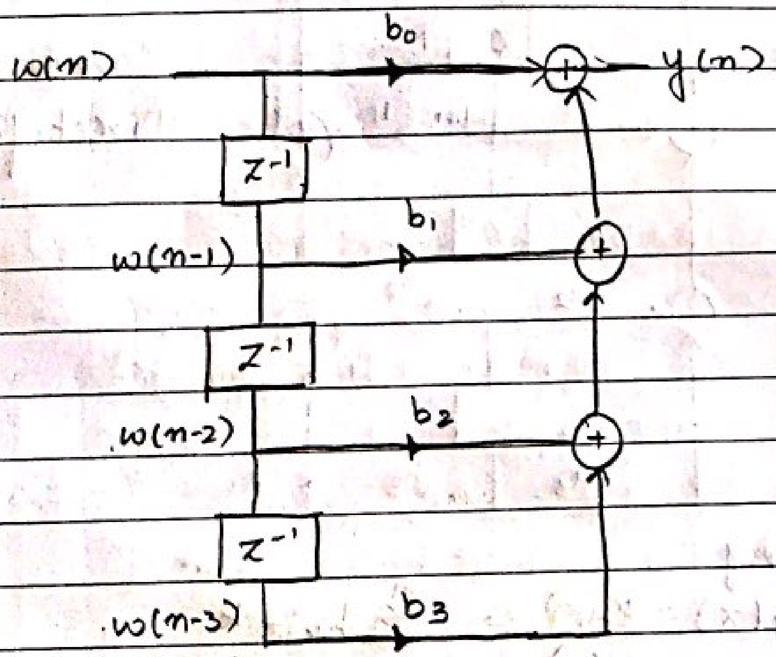
$$y(n) = b_0 w(n) + b_1 w(n-1) + b_2 w(n-2) + b_3 w(n-3) \quad \text{— (4)}$$



Filter Structure

Similarly

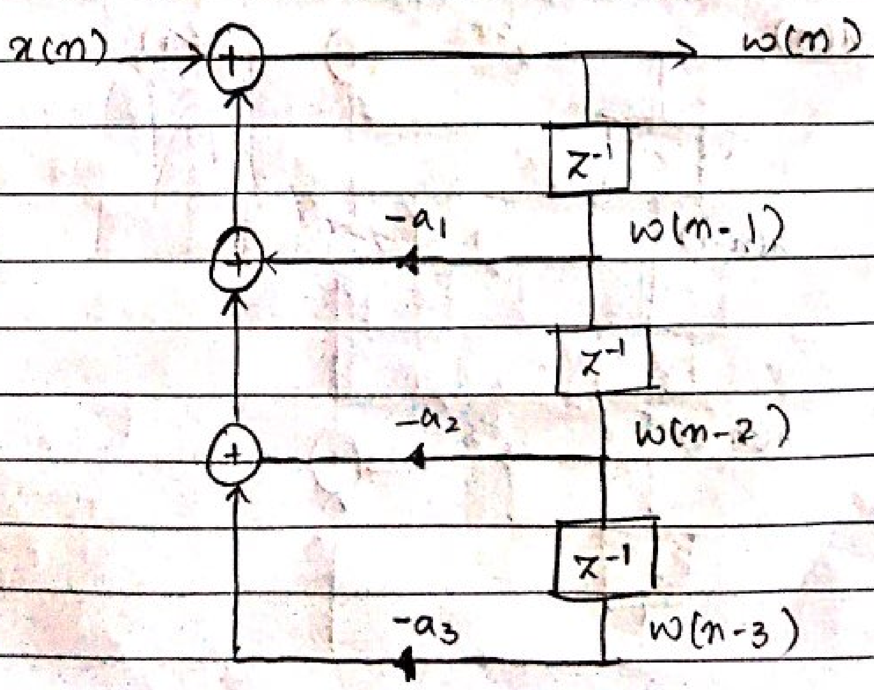
$$H_2(n) = \frac{1}{1 + \sum_{k=1}^{3} a_k z^{-k}}$$

$$\frac{W(z)}{X(z)} = \frac{1}{1 + a_1 z^{-1} + a_2 z^{-2} + a_3 z^{-3}}$$

Taking Inverse Z Transform

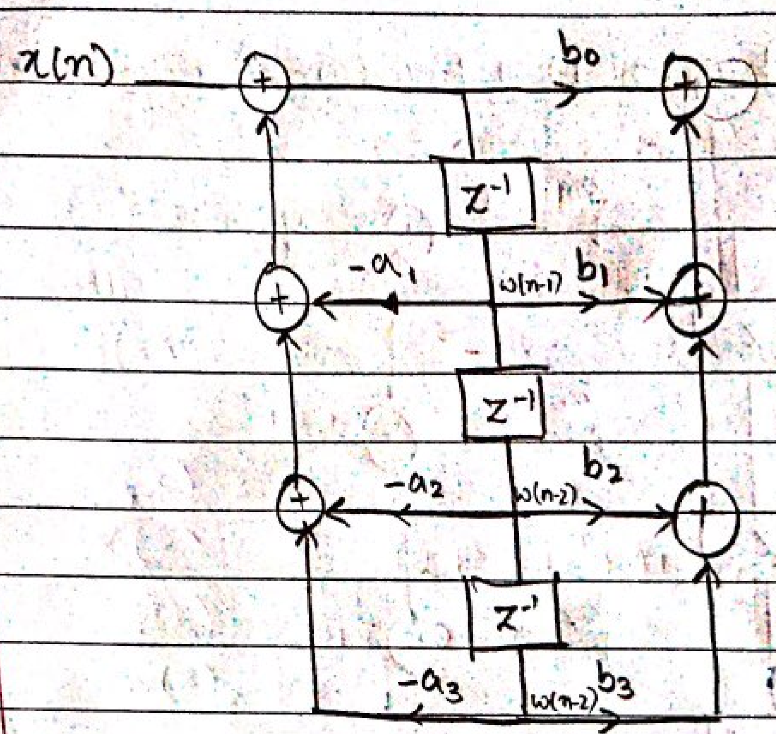
$$x(n) = w(n) + a_1 w(n-1) + a_2 w(n-2) + a_3 w(n-3)$$

$$w(n) = x(n) - a_1 w(n-1) - a_2 w(n-2) - a_3 w(n-3) \quad \text{— (5)}$$



Cannonical Form

Direct form II



considering

$$H_1(z) = \frac{W(z)}{X(z)} = \sum_{k=0}^{M} b_k z^{-k}$$

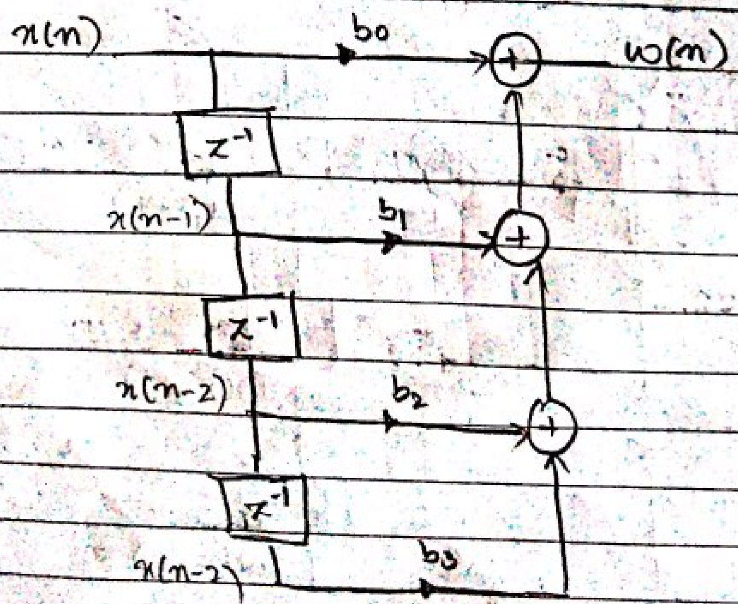
For a third order system

$$H_1(z) = \frac{W(z)}{X(z)} = \sum_{k=0}^{3} b_k z^{-k}$$

$$\frac{W(z)}{X(z)} = b_0 + b_1 z^{-1} + b_2 z^{-2} + b_3 z^{-3}$$

Taking Inverse Z transform

$$w(n) = b_0 x(n) + b_1 x(n-1) + b_2 x(n-2) + b_3 x(n-3) \quad (6)$$



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$$H_2(z) = \frac{Y(z)}{W(z)} = \frac{1}{1 + \sum_{k=1}^{N} a_k z^{-k}}$$

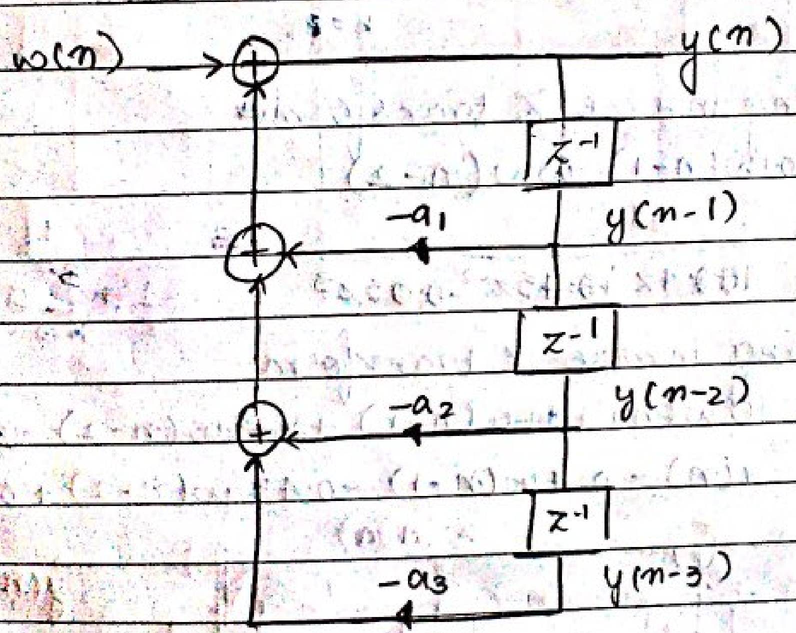
For a third order system.

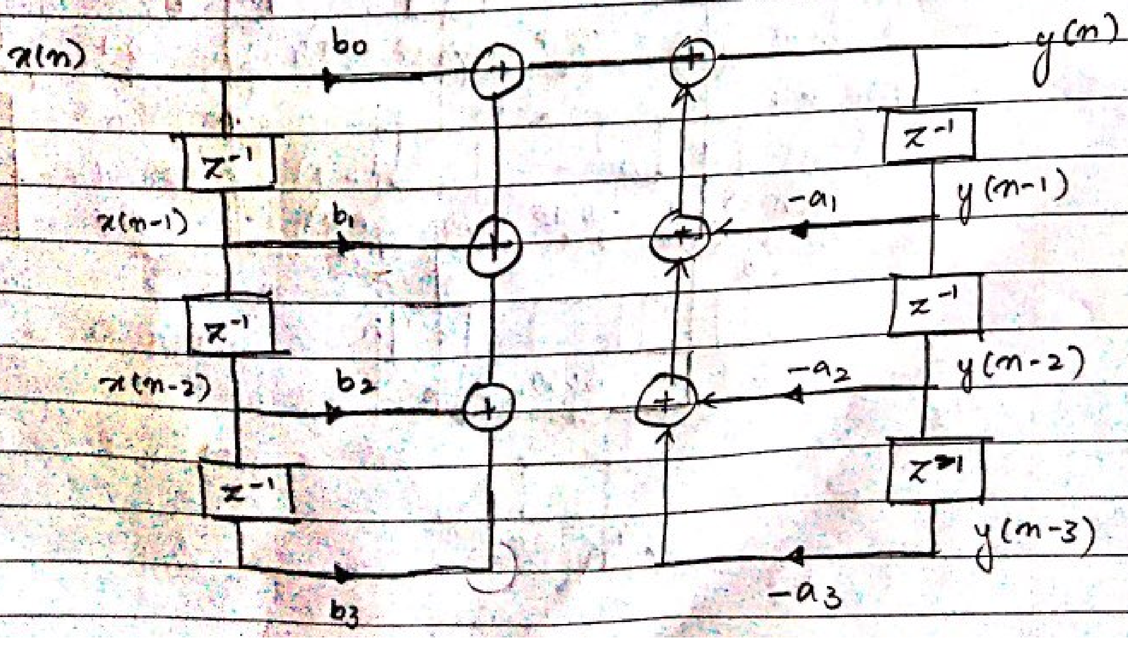
$$H_2(z) = \frac{Y(z)}{W(z)} = \frac{1}{1 + a_1 z^{-1} + a_2 z^{-2} + a_3 z^{-3}}$$

Taking inverse z transform

$$w(n) = y(n) + a_1 y(n-1) + a_2 y(n-2) + a_3 y(n-3)$$

$$y(n) = w(n) - a_1 y(n-1) - a_2 y(n-2) - a_3 y(n-3) \quad \text{— (7)}$$





Q: Simplify and draw Direct form 1 and direct form II for the given system function.

$$H(z) = \frac{z^{-1} - 3z^{-2}}{(10 - z^{-1})(1 + 0.5z^{-1} + 0.5z^{-2})}$$

Sol:

$$H(z) = \frac{z^{-1} - 3z^{-2}}{10 + 5z^{-1} + 5z^{-2} - z^{-1} - 0.5z^{-2} - 0.5z^{-3}}$$

$$\frac{Y(z)}{X(z)} = \frac{0.1z^{-1} - 0.3z^{-2}}{1 + 0.4z^{-1} + 0.45z^{-2} - 0.05z^{-3}}$$

$$\frac{Y(z)}{W(z)} = 0.1z^{-1} - 0.3z^{-2} = \sum_{k=0}^{2} b_k z^{-k}$$

Taking inverse z transform

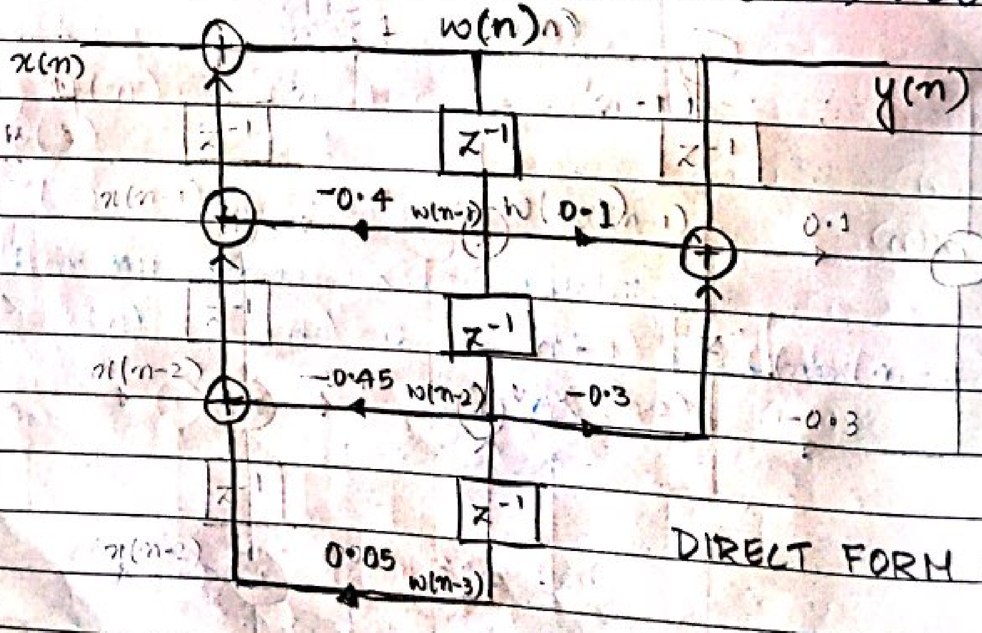
$$y(n) = 0.1w(n-1) - 0.3w(n-2)$$

$$\frac{W(z)}{X(z)} = \frac{1}{1 + 0.4z^{-1} + 0.45z^{-2} - 0.05z^{-3}} = \frac{1}{1 + \sum_{k=0}^{3} a_k z^{-k}}$$

Taking inverse z transform

$$x(n) = w(n) + 0.4w(n-1) + 0.45w(n-2) - 0.05w(n-3)$$

$$w(n) = x(n) - 0.4w(n-1) - 0.45w(n-2) + 0.05w(n-3)$$



DIRECT FORM II

considering.

$$\frac{W(z)}{X(z)} = 0.1z^{-1} - 0.3z^{-2} = \sum_{k=1}^{2} b_k z^{-k}$$

Taking inverse z transform

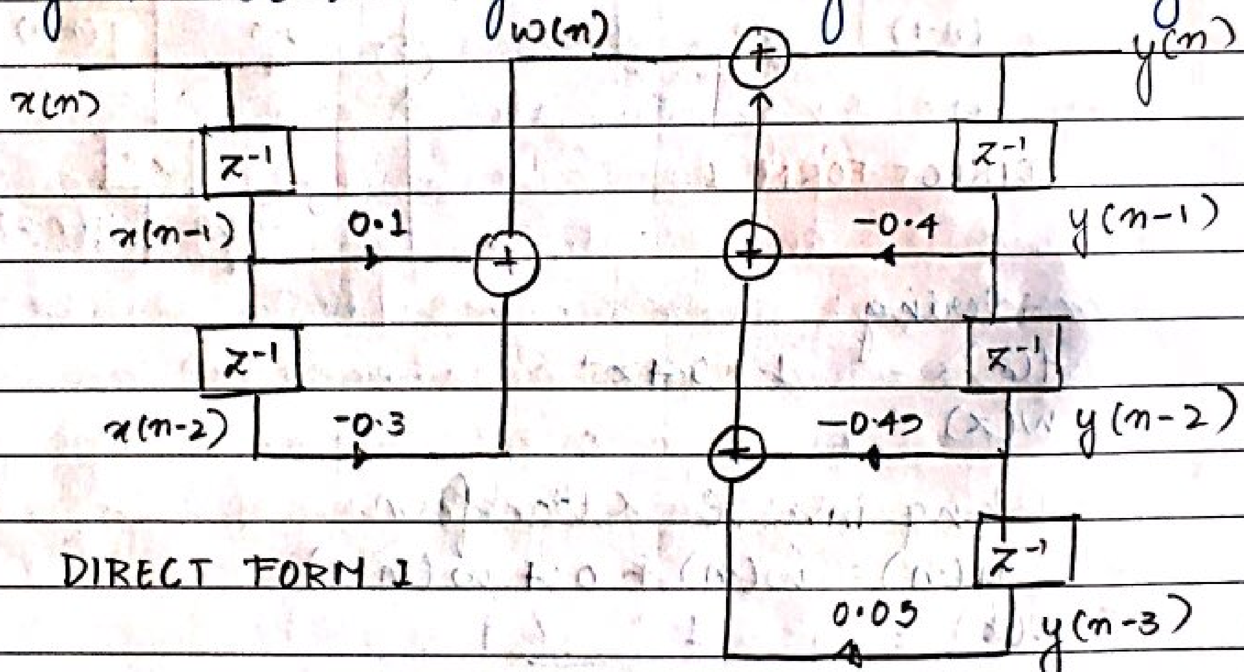
$$w(n) = 0.1x(n-1) - 0.3x(n-2)$$

$$\frac{Y(z)}{W(z)} = \frac{1}{1+0.4z^{-1}+0.45z^{-2}-0.05z^{-3}} = \frac{1}{1+\sum_{k=0}^{3} a_k z^{-k}}$$

Taking inverse z transform

$$w(n) = y(n) + 0.4y(n-1) + 0.45y(n-2) - 0.05y(n-3)$$

$$y(n) = w(n) - 0.4y(n-1) - 0.45y(n-2) + 0.05y(n-3)$$



DIRECT FORM 1

Obtain the direct form 1 and direct form 11 realisation for the system function given by.

$$H(z) = \frac{1+0.4z^{-1}}{1-0.5z^{-1}+0.06z^{-2}}$$

considering:

$$\frac{Y(z)}{X(z)} = \frac{1+0.4z^{-1}}{1-0.5z^{-1}+0.06z^{-2}}$$

$$\frac{W(z)}{X(z)} = 1+0.4z^{-1}.$$

Taking inverse z transform

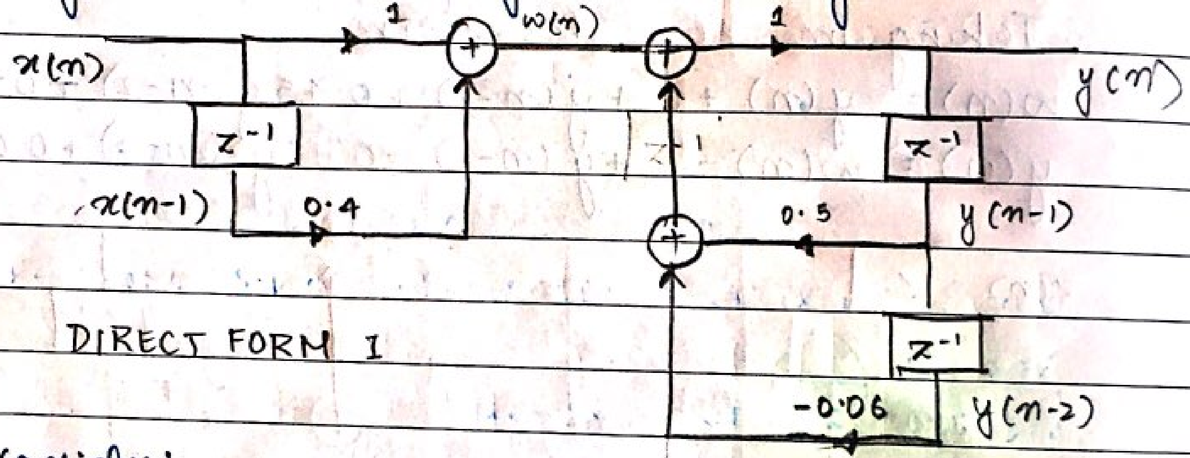
$w(n) = x(n) + 0.4\,x(n-1)$

$$\frac{Y(z)}{W(z)} = \frac{1}{1 - 0.5z^{-1} + 0.06z^{-2}}$$

Taking inverse z transform

$w(n) = y(n) - 0.5\,y(n-1) + 0.06\,y(n-2)$

$y(n) = w(n) + 0.5\,y(n-1) - 0.06\,y(n-2)$



DIRECT FORM I

considering:

$$\frac{Y(z)}{W(z)} = 1 + 0.4z^{-1}$$

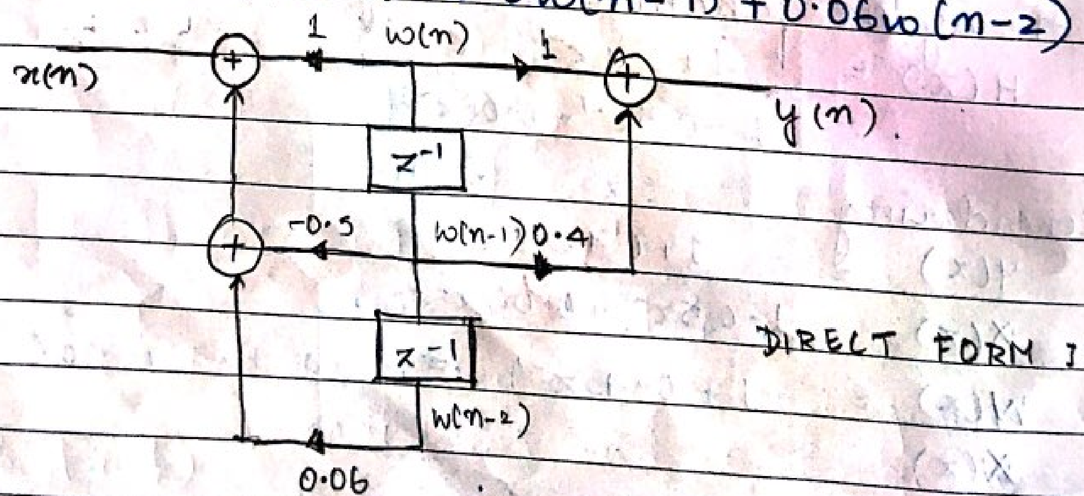
Taking inverse z transform

$y(n) = w(n) + 0.4\,w(n-1)$

$$\frac{W(z)}{X(z)} = \frac{1}{1-0.5z^{-1}+0.06z^{-2}}$$

Taking inverse z transform

$x(n) = w(n) - 0.5\,w(n-1) + 0.06\,w(n-2)$



DIRECT FORM II

Q: For the system function $H(z) = \frac{7z^2 - 5.25z + 1.375}{z^2 - 0.75z + 0.125}$ obtain direct form I and direct form II realisation.

Sol:

$$H(z) = \frac{7z^2 - 5.25z + 1.375}{z^2 - 0.75z + 0.125}$$

$$H(z) = \frac{7 - 5.25z^{-1} + 1.375z^{-2}}{1 - 0.75z^{-1} + 0.125z^{-2}}$$

considering:

$$\frac{W(z)}{X(z)} = 7 - 5.25z^{-1} + 1.375z^{-2}$$

Taking inverse z transform:

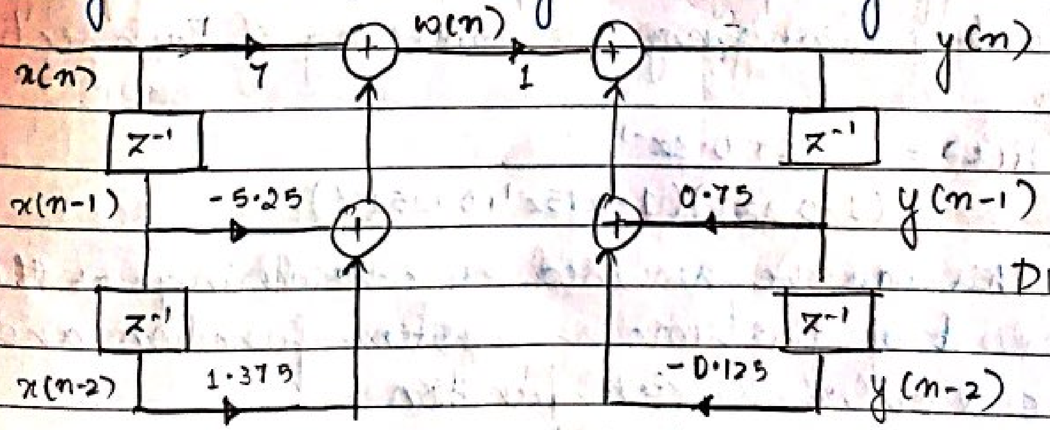
$$w(n) = 7x(n) - 5.25x(n-1) + 1.375x(n-2)$$

$$\frac{Y(z)}{W(z)} = \frac{1}{1 - 0.75z^{-1} + 0.125z^{-2}}$$

Taking inverse z transform

$$w(n) = y(n) - 0.75y(n-1) + 0.125y(n-2)$$

$$y(n) = w(n) + 0.75y(n-1) - 0.125y(n-2)$$



DIRECT FORM 1

considering

$$\frac{Y(z)}{W(z)} = 7 - 5.25z^{-1} + 1.375z^{-2}$$

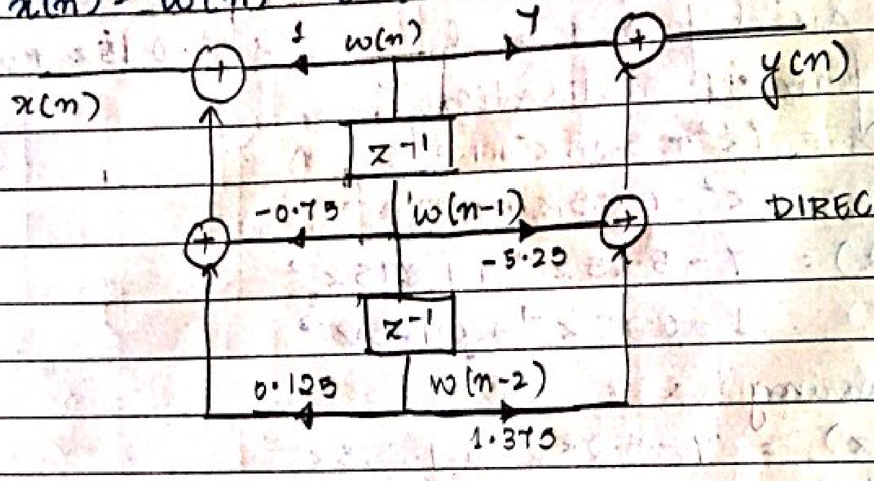
Taking inverse z transform

$$y(n) = 7w(n) - 5.25w(n-1) + 1.375w(n-2)$$

$$\frac{W(z)}{X(z)} = \frac{1}{1 - 0.75z^{-1} + 0.125z^{-2}}$$

Taking inverse z transform

$x(n) = w(n) - 0.75\,w(n-1) + 0.125\,w(n-2)$



DIRECT FORM II

★ <u>cascade structure</u>

By expressing numerator and denominator of the system function as product of polynomials of lower degree an IIR filter can be realised by cascading the lower order filter sections. $[H(z) = H_1(z)\,H_2(z)\,H_3(z)]$

Various cascaded realisation of $H(z)$ can be made by different pole zero polynomial mapping and also by changing the ordering of sub sections.

Ex:

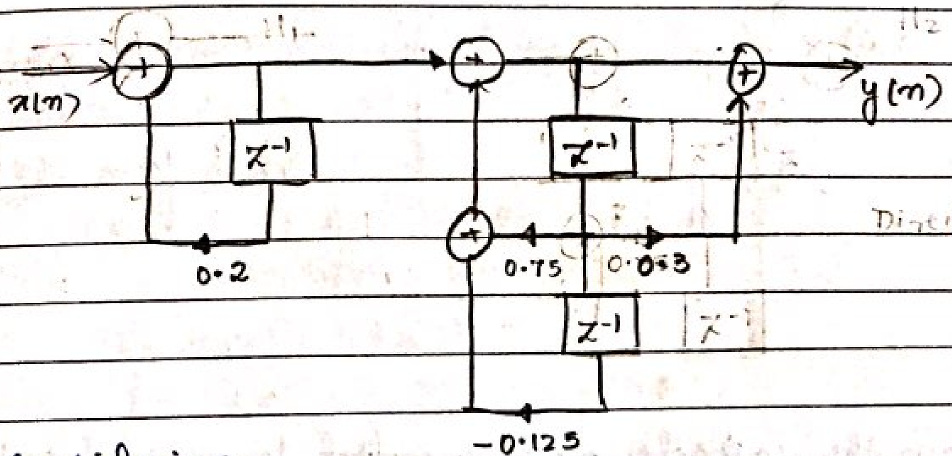
$$H(z) = \frac{1 + 0.3z^{-1}}{(1-0.2z^{-1})(1-0.75z^{-1}+0.125z^{-2})}$$

This can be realised by considering as the product of first order system function and second order system function.

$$H_1(z) = \frac{1}{1-0.2z^{-1}}$$

$$H_2(z) = \frac{1+0.3z^{-1}}{1-0.75z^{-1}+0.125z^{-2}}$$

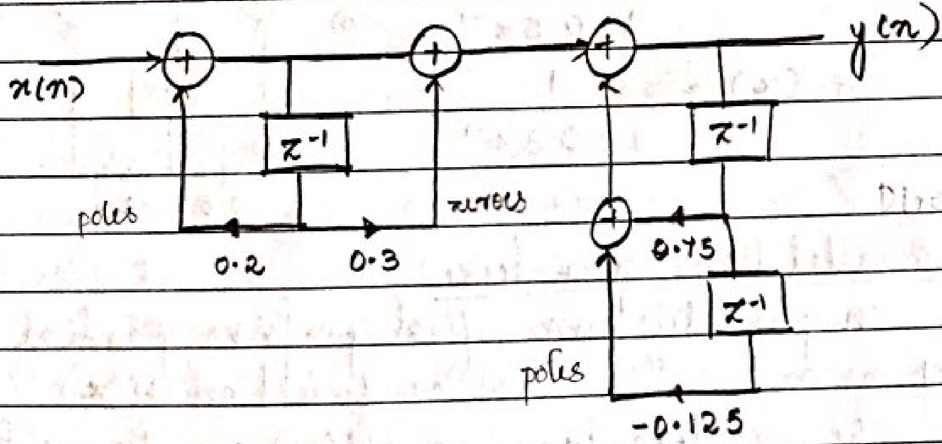
NOTE: For the poles if negative then positive in the direct form and no change in sign for the zeroes (short cut)



Considering

$$H_1(z) = \frac{1+0.3z^{-1}}{1-0.2z^{-1}}$$

$$H_2(z) = \frac{1}{1-0.75z^{-1}+0.125z^{-2}}$$

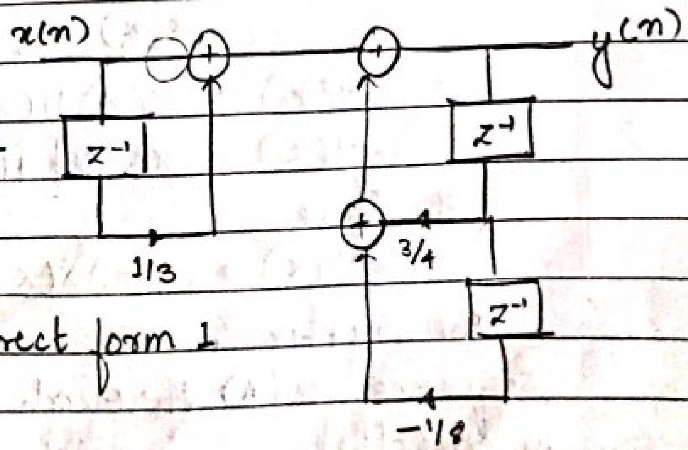


Q: $H(z) = \dfrac{1+1/3z^{-1}}{1-3/4z^{-1}+1/8z^{-2}}$ Realise $H(z)$ in direct form 1 and direct form 11.

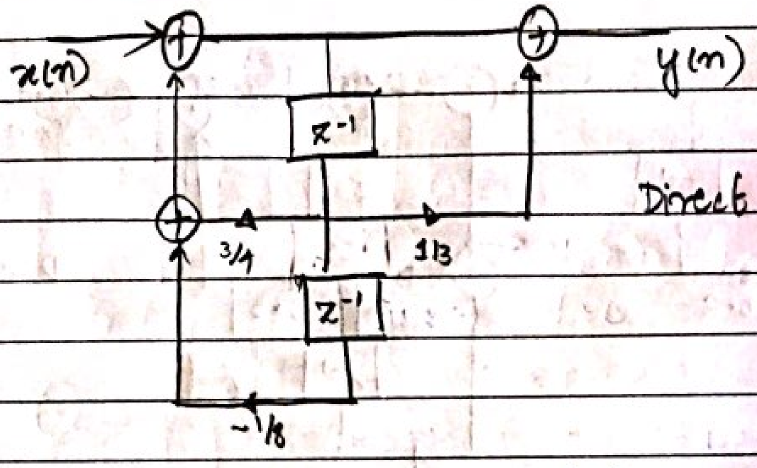
— Considering

$H_1(z) = 1 + 1/3z^{-1}$

$$H_2(z) = \frac{1}{1-3/4z^{-1}+1/8z^{-2}}$$



Direct form 1



Direct form II

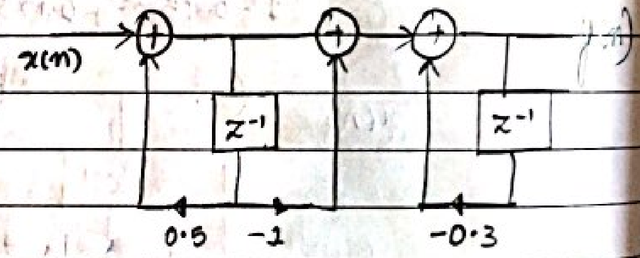
Q: Draw the cascade ~~and parallel~~ form realisation of

$$H(z) = \frac{1-z^{-1}}{(1-0.5z^{-1})(1+0.3z^{-1})}$$

→ considering

$$H_1(z) = \frac{1-z^{-1}}{1-0.5z^{-1}}$$

$$H_2(z) = \frac{1}{1+0.3z^{-1}}$$



★ **Parallel form structure:**

In parallel form, first perform partial fraction expansion of the system function $H(z)$

The system function $H(z)$ can be written as the sum of transfer functions ($H_1(z)$, $H_2(z)$ ....)

$$H(z) = H_1(z) + H_2(z) + \ldots + H_n(z)$$

wkt $H(z) = \frac{Y(z)}{X(z)}$

$$\therefore Y(z) = X(z)H(z)$$

$$Y(z) = X(z)\left[H_1(z) + H_2(z) + \cdots + H_n(z)\right]$$

Therefore

$$Y(z) = H_1(z)X(z) + H_2(z)X(z) + \cdots + H_n(z)X(z)$$

The system is realised by passing the input sequence $x(n)$ through n discreate time filters in parallel.

Adding the individual output we obtain the overall output $y(n)$.

Ex:

$$H(z) = \frac{1 + 1/3 z^{-1}}{(1 - 3/4 z^{-1} + 1/8 z^{-2})}$$

$$1 - 3/4 z^{-1} + 1/8 z^{-2}$$
$$= 1 - 1/4 z^{-1} - 1/2 z^{-1} + 1/8 z^{-2}$$
$$= 1(1 - 1/4 z^{-1}) - 1/2 z^{-1}(1 - 1/4 z^{-1})$$
$$= (1 - 1/4 z^{-1})(1 - 1/2 z^{-1})$$

Therefore

$$H(x) = \frac{1 + 1/3 z^{-1}}{(1 - 1/4 z^{-1})(1 - 1/2 z^{-1})}$$

$$= \frac{A}{1 - 1/4 z^{-1}} + \frac{B}{1 - 1/2 z^{-1}}$$

Therefore

$$1 + 1/3 z^{-1} = A(1 - 1/2 z^{-1}) + B(1 - 1/4 z^{-1})$$

at $z^{-1} = 2$

$$1 + 2/3 = A(0) + B(1 - 1/2)$$
$$B(1/2) = 5/3$$
$$B = 10/3$$

at $z^{-1} = 4$

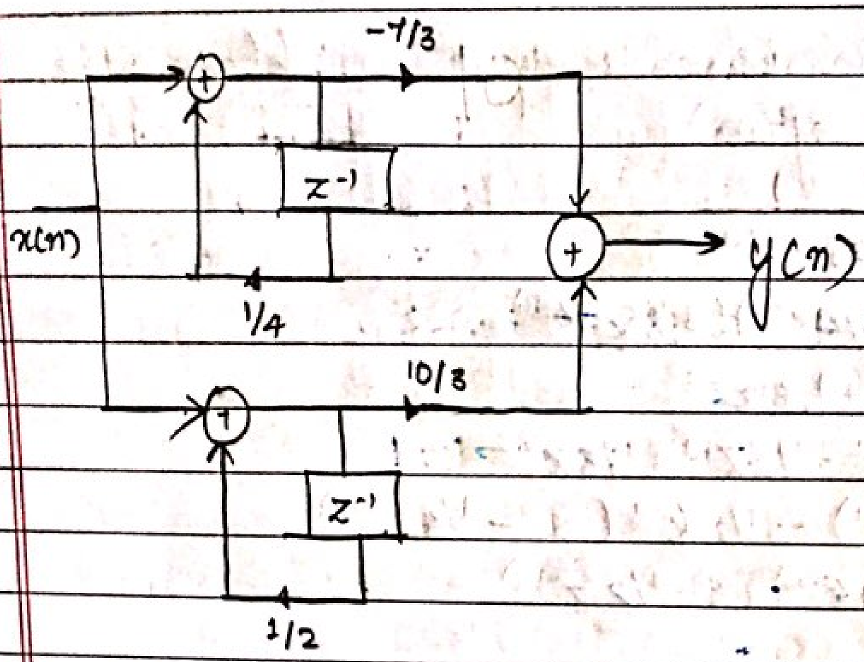
$$1 + 4/3 = A(1 - 2) + B(0)$$
$$A(-1) = 7/3$$
$$A = -7/3$$

Therefore

$$H(z) = \frac{1 + 1/3 z^{-1}}{(1 - 1/4 z^{-1})(1 - 1/2 z^{-1})} = \frac{-7/3}{1 - 1/4 z^{-1}} + \frac{10/3}{1 - 1/2 z^{-1}}$$

Here $H_1(z) = \dfrac{-7/3}{1 - 1/4 z^{-1}}$

$$H_2(z) = \frac{10/3}{1 - 1/2 z^{-1}}$$



Parallel form

Q: obtain the cascade form realisation of

$$H(z) = \frac{1+3/4\,z^{-1}+1/8\,z^{-2}}{1-5/8\,z^{-1}+1/16\,z^{-2}}$$

– considering

$$H(z) = \frac{1+3/4\,z^{-1}+1/8\,z^{-2}}{1-5/8\,z^{-1}+1/16\,z^{-2}}$$

$1+3/4z^{-1}+1/8z^{-2}$

$$= 1+1/4\,z^{-1}+1/2\,z^{-1}+1/8\,z^{-2}$$
$$= 1(1+1/4z^{-1})+1/2z^{-1}(1+1/4z^{-2})$$
$$= (1+1/4z^{-1})(1+1/2z^{-1})$$

$1-5/8z^{-1}+1/16z^{-2}$

$$= 1-1/8z^{-1}-1/2z^{-1}+1/16z^{-2}$$
$$= 1(1-1/8z^{-1})-1/2z^{-1}(1-1/8z^{-1})$$
$$= (1-1/8z^{-1})(1-1/2z^{-1})$$

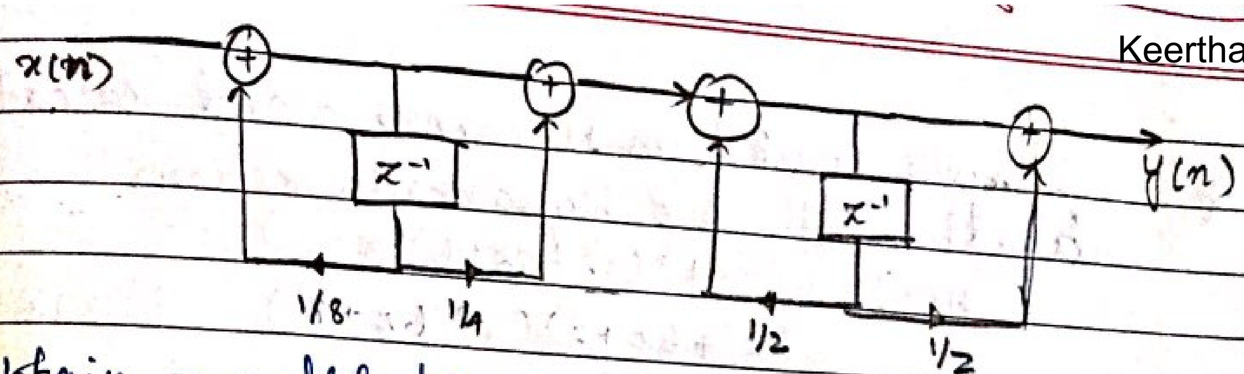
Therefore

$$H(z) = \frac{(1+1/4z^{-1})(1+1/2z^{-1})}{(1-1/8z^{-1})(1-1/2z^{-1})}$$

$$H_1(z) = \frac{(1+1/4z^{-1})}{(1-1/8z^{-1})}$$

$$H_2(z) = \frac{1+1/2z^{-1}}{1-1/2z^{-2}}$$

Keerthana Ashok



Q: Obtain parallel form realization of

$$H(z) = \frac{1-z^{-1}}{(1-0.5z^{-1})(1+0.3z^{-1})}$$

– considering

$$H(z) = \frac{1-z^{-1}}{(1-0.5z^{-1})(1+0.3z^{-1})} = \frac{A}{1-0.5z^{-1}} + \frac{B}{1+0.3z^{-1}}$$

$$\Rightarrow 1-z^{-1} = A(1+0.3z^{-1}) + B(1-0.5z^{-1})$$

at $z^{-1} = -10/3$

$$1+10/3 = A(0) + B(2.67)$$

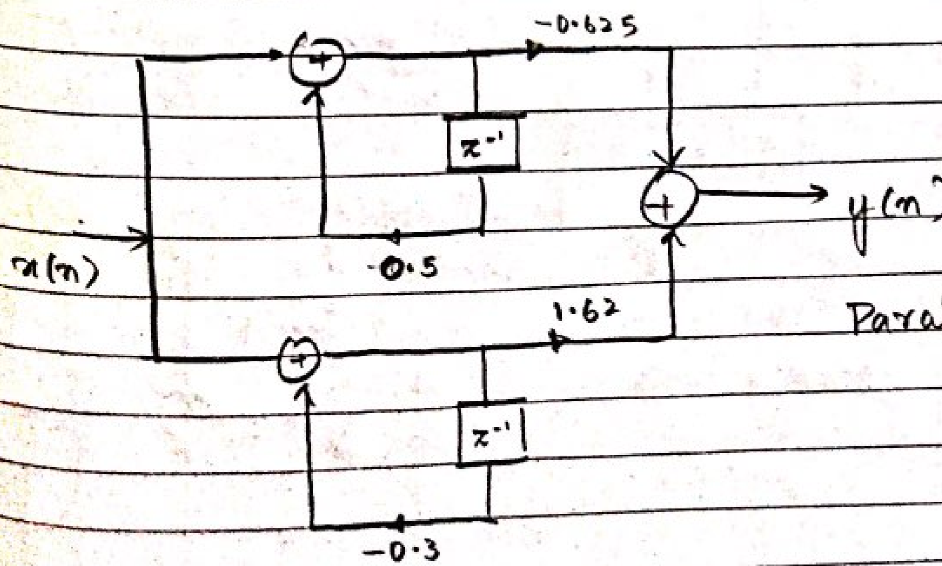
$$B = \frac{4.33}{2.67} = 1.62$$

at $z^{-1} = 2$

$$1-2 = A(1.6) + B(0)$$

$$A = \frac{-1}{1.6} = -0.625$$

$$H(z) = \frac{-0.625}{1-0.5z^{-1}} + \frac{1.62}{1+0.3z^{-1}} = H_1(z) + H_2(z)$$



Parallel form

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Q: Obtain the Direct form II cannonic and cascade realization with two biquadratic sections

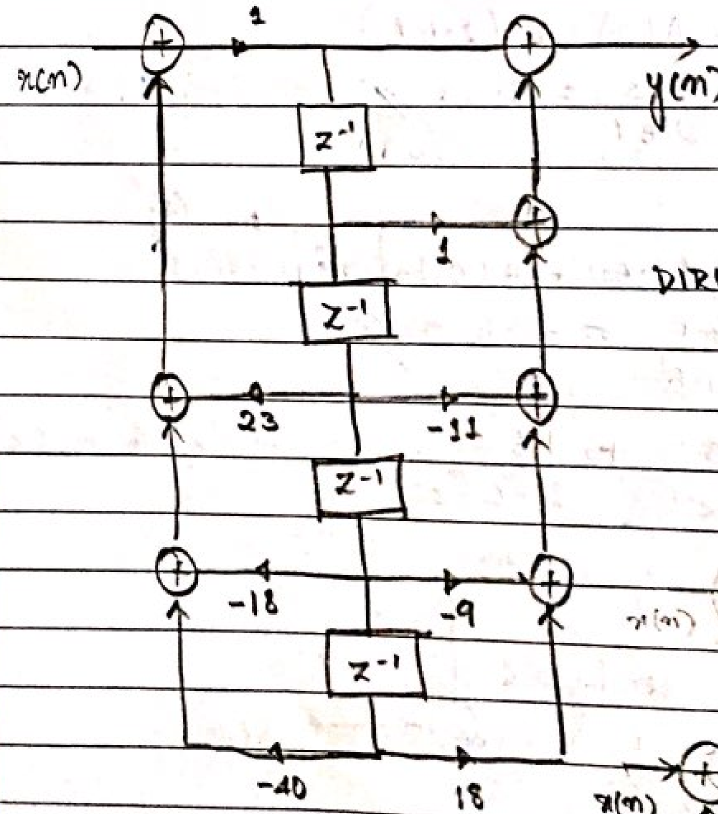
$$H(z) = \frac{(z-1)(z^2+5z+6)(z-3)}{(z^2+6z+5)(z^2-6z+8)}$$

$$H(z) = \frac{(z^2-4z+3)(z^2+5z+6)}{(z^2+6z+5)(z^2-6z+8)}$$

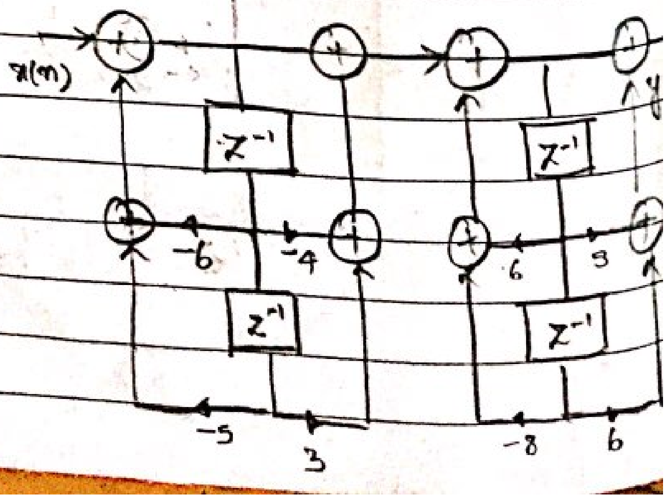
$$H(z) = \frac{(1-4z^{-1}+3z^{-2})(1+5z^{-1}+6z^{-2})}{(1+6z^{-1}+5z^{-2})(1-6z^{-1}+8z^{-2})}$$

$$H(z) = \frac{1+5z^{-1}+6z^{-2}-4z^{-1}-20z^{-2}-24z^{-3}+3z^{-2}+15z^{-3}+18z^{-4}}{1-6z^{-1}+8z^{-2}+6z^{-1}-36z^{-2}+48z^{-3}+5z^{-2}-30z^{-3}+40z^{-4}}$$

$$H(z) = \frac{1+z^{-1}-11z^{-2}-9z^{-3}+18z^{-4}}{1-23z^{-2}+18z^{-3}+40z^{-4}}$$



DIRECT FORM II

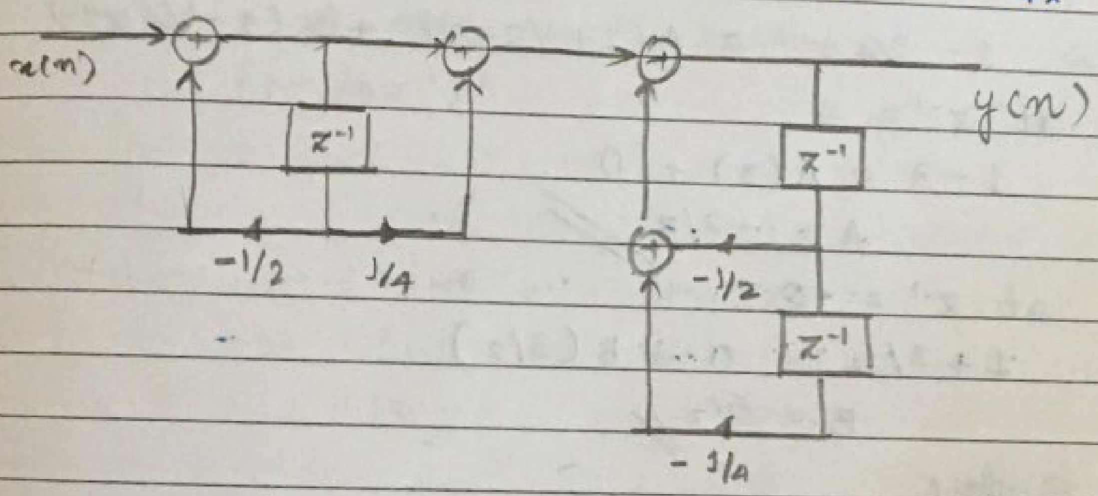


CASCADE FORM

Q: Obtain the cascade realization of

$$H(z) = \frac{1 + 1/4\, z^{-1}}{(1+1/2\, z^{-1})(1+1/2\, z^{-1}+1/4\, z^{-2})}$$

$$H_1(z) = \frac{(1+1/4\, z^{-1})}{(1+1/2\, z^{-1})} \text{ and } H_2(z) = \frac{1}{1+1/2\, z^{-1}+1/4\, z^{-2}}$$



Q: A digital filter has input $x(n) = \delta(n) + \frac{1}{4}\delta(n-1) - \frac{1}{8}\delta(n-2)$ and output $y(n) = \delta(n) - \frac{3}{4}\delta(n-1)$. Realize in parallel form.

$x(n) = \delta(n) + 1/4\, \delta(n-1) - 1/8\, \delta(n-2)$

Taking z transform

$X(z) = 1 + 1/4\, z^{-1} - 1/8\, z^{-2}$

$y(n) = \delta(n) - 3/4\, \delta(n-1)$

Taking z transform

$Y(z) = 1 - 3/4\, z^{-1}$

Therefore

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1 - 3/4\, z^{-1}}{1+1/4\, z^{-1} - 1/8\, z^{-2}}$$

$1 + 1/4\, z^{-1} - 1/8\, z^{-2}$

$= 1 + 1/2\, z^{-1} - 1/4\, z^{-1} - 1/8\, z^{-2}$

$= 1(1 + 1/2\, z^{-1}) - 1/4\, z^{-1}(1 + 1/2\, z^{-1})$

$= (1 - 1/4\, z^{-1})(1 + 1/2\, z^{-1})$

Therefore

$$H(z) = \frac{1 - 3/4\, z^{-1}}{(1 - 1/4\, z^{-1})(1 + 1/2\, z^{-1})}$$

$$= \frac{A}{1 - 1/4\, z^{-1}} + \frac{B}{1 + 1/2\, z^{-1}}$$

$$\Rightarrow \quad 1 - 3/4\, z^{-1} = A(1 + 1/2\, z^{-1}) + B(1 - 1/4\, z^{-1})$$

at $z^{-1} = 4$

$$1 - 3 = A(3) + 0$$

$$A = -2/3$$

at $z^{-1} = -2$

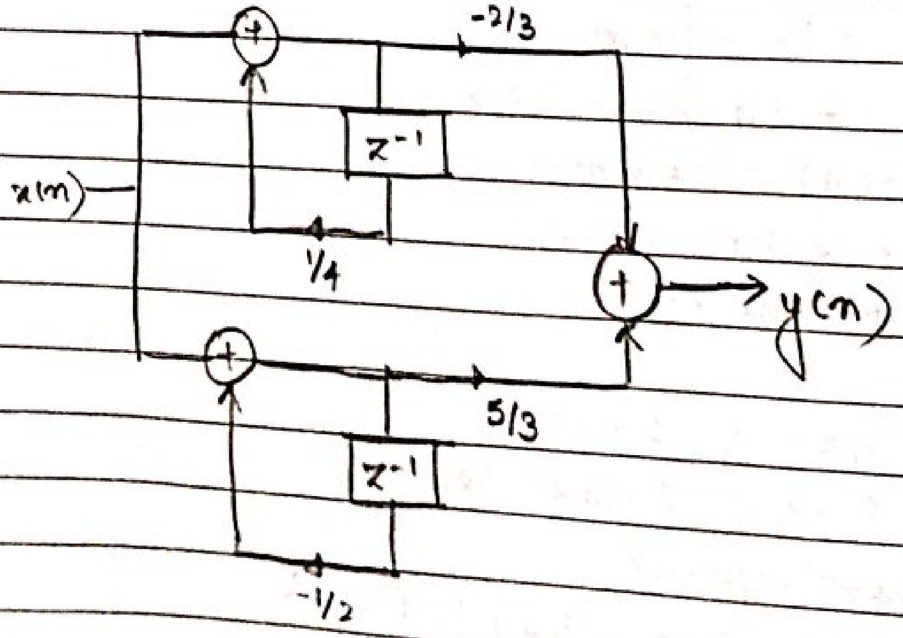
$$1 + 3/2 = 0 + B(3/2)$$

$$B = 5/3$$

Therefore

$$H(z) = \frac{-2/3}{1 - 1/4\, z^{-1}} + \frac{5/3}{1 + 1/2\, z^{-1}}$$

PARALLEL FORM



Keerthana Ashok

Q. obtain parallel form realization of the given system function : $H(z) = \dfrac{8z^3 - 4z^2 + 11z - 2}{(z - 1/4)(z^2 - z + 1/2)}$

$$H(z) = \frac{8 - 4z^{-1} + 11z^{-2} - 2z^{-3}}{(1 - 1/4 z^{-1})(1 - z^{-1} + 1/2 z^{-2})}$$

$$H(z) = \frac{8 - 4z^{-1} + 11z^{-2} - 2z^{-3}}{(1 - 1/4 z^{-1})(}$$

Scanned with CamScanner

* FIR filter structures

- Direct form
- Cascade form
- Frequency sampling form.
- Lattice
- Linear phase FIR struction

* Direct Form

Consider ~~on~~ an LTI system given a differential equation

$$y(n) = -\sum_{k=0}^{N-1} a_k\, y(n-k) + \sum_{k=0}^{M-1} b_k\, x(n-k)$$

For an FIR filter there is no feedback hence the first term is not present.

Therefore

$$y(n) = \sum_{k=0}^{M-1} b_k\, x(n-k)$$

Taking z transform on both sides.

$$Y(z) = \sum_{k=0}^{M-1} b_k\, z^{-k} X(z)$$

$$H(z) = \frac{Y(z)}{X(z)} = \sum_{k=0}^{M-1} b_k\, z^{-k}$$

Taking inverse z transform

$$h(n) = \begin{cases} b_n & ; 0 \le n \le M-1 \\ 0 & ; \text{else} \end{cases}$$

The direct form structure can be obtained by using $y(n) = \sum_{k=0}^{M-1} h(k)\, x(n-k)$

Ex: Let $M = 6$

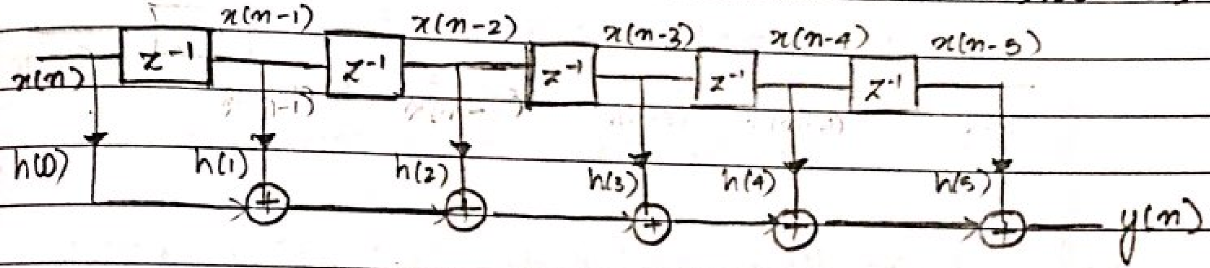
$$y(n) = \sum_{k=0}^{5} h(k)\, x(n-k)$$

Q:

*

NOTE: In direct form $M-1$ delays, memory location ~~and multipliers~~ and additions are required and M multiplication is required

$$y(n) = h(0)x(n) + h(1)x(n-1) + h(2)x(n-2) + h(3)x(n-3) + h(4)x(n-4) + h(5)x(n-5)$$

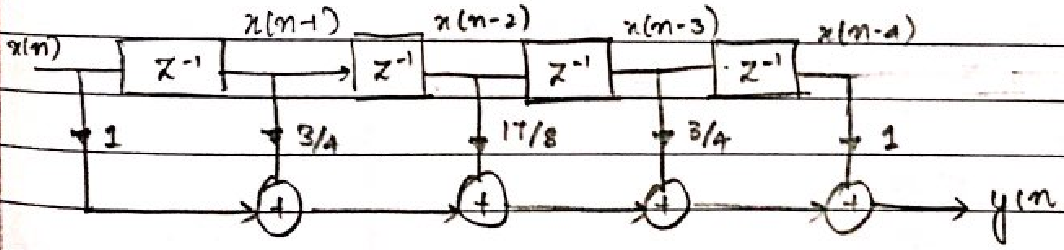


Q: Obtain the direct form realization of

$$H(z) = 1 + 3/4\, z^{-1} + 17/8\, z^{-2} + 3/4\, z^{-3} + z^{-4}$$

– Taking inverse z transform

$$y(n) = x(n) + 3/4\, x(n-1) + 17/8\, x(n-2) + 3/4\, x(n-3) + x(n-4)$$



The direct form filter realization is also called as transversal or tap delay line structure.

* **Linear Phase FIR structure:**

For FIR filter to have linear phase its unit sample response $h(n) = h(M-1-n)$, where M is the length of impulse response. For such realizations the number of multiplications can be reduced to $M/2$.

The unit impulse response is given by

$$H(z) = \sum_{n=0}^{N-1} h(n) z^{-n}$$

Ex:

Let $H(z) = 1/4 + 1/2\, z^{-1} + 3/4\, z^{-2} + 1/2\, z^{-3} + 1/4\, z^{-4}$

Here $M-1 = 4 \Rightarrow M = 5$

$h(n) = h(M-1-n)$

$h(0) = h(5-1-0) = h(4) = 1/4$

$h(1) = h(5-1-1) = h(3) = 1/2$

$h(2) = h(5-1-2) = h(2) = 3/4$

It satisfies the linear phase condition.

Q: $H(z) = 1/4 + 1/2 z^{-1} + 3/4 z^{-2} + 1/2 z^{-3} + 1/4 z^{-4}$

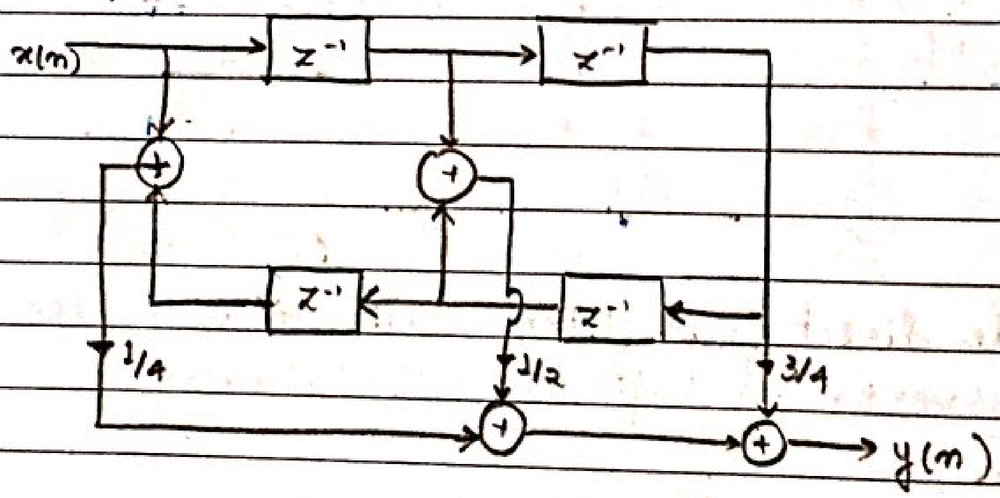
$M-1 = 4 \Rightarrow M = 5$

$h(n) = h(M-1-n) = h(4-n)$

$h(0) = h(4) = 1/4$

$h(1) = h(3) = 1/2$

$h(2) = h(2) = 3/4$



Q: $H(z) = 1/2 + 1/4 z^{-1} + 1/8 z^{-2} + 1/5 z^{-3} + 1/8 z^{-4} + 1/4 z^{-5} + 1/2 z^{-6}$

$M-1 = 6 \Rightarrow M = 7$

$h(n) = h(M-1-n) = h(7-1-n) = h(6-n)$

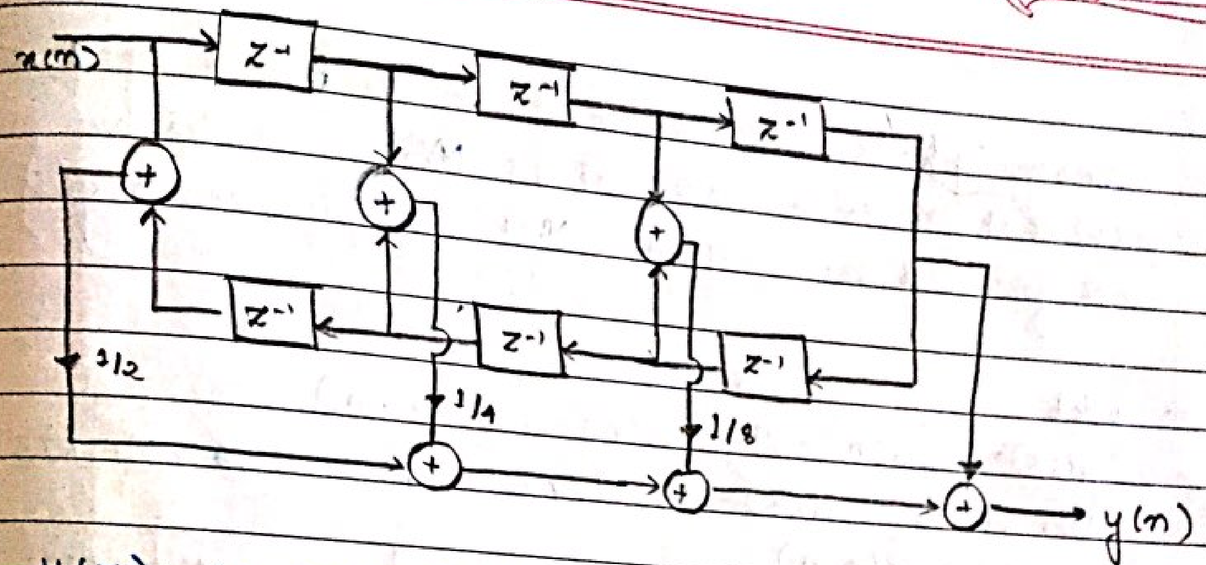
$h(0) = h(6) = 1/2$

$h(1) = h(5) = 1/4$

$h(2) = h(4) = 1/8$

$h(3) = h(3) = 1/5$

$H(z) = 1/2(1+z^{-6}) + 1/4(z^{-1} - z^{-5}) + 1/8(z^{-2} - z^{-4}) + 1/5 z^{-3}$



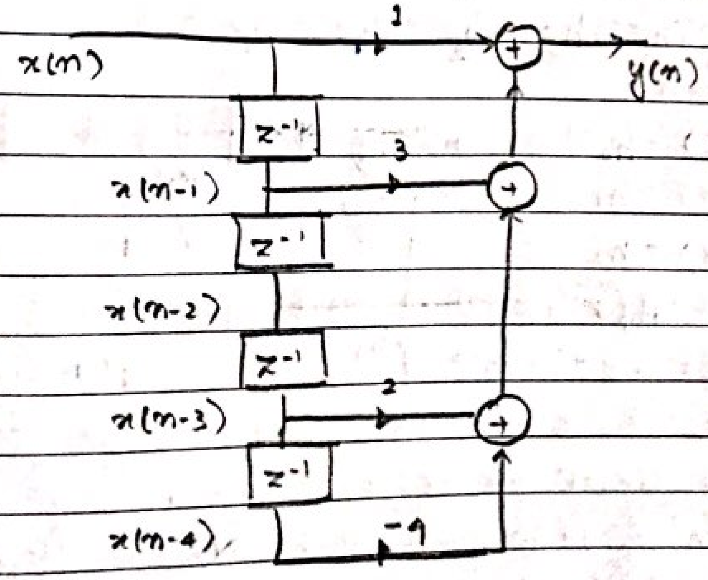
Q: $H(z) = 1 + 3z^{-1} + 2z^{-3} - 4z^{-4}$: Direct form realization of FIR filter

$$\frac{Y(z)}{X(z)} = 1 + 3z^{-1} + 2z^{-3} - 4z^{-4}$$

$$Y(z) = X(z) + 3z^{-1}X(z) + 2z^{-3}X(z) - 4z^{-4}X(z)$$

Taking inverse z transform

$$y(n) = x(n) + 3x(n-1) + 2x(n-3) - 4x(n-4)$$



Q: $H(z) = 1 + 3z^{-1} + 2z^{-3} - 4z^{-4}$: Linear phase FIR structure

$H(n) = H(M-1-n) = H(4-n)$

$H(0) \neq H(4)$ Hence it does not satisfy the

$H(1) \neq H(3)$ linear equation. Thus cannot determine the structure

Q: Linear phase FIR structure for $M=6$ and $M=7$ and prove that for even value of M, $M/2$ multiples are required and for odd values of M, $\frac{M+1}{2}$ multiples are required.

— wkt

$$H(z) = \sum_{n=0}^{M-1} h(n) z^{-n} \qquad h(n) = h(M-1-n)$$

For $M=6$: $H(z) = 1/2 + 1/7 z^{-1} + 1/6 z^{-2} + 1/6 z^{-3} + 1/7 z^{-4} + 1/2 z^{-5}$

$h(n) = h(5-n)$

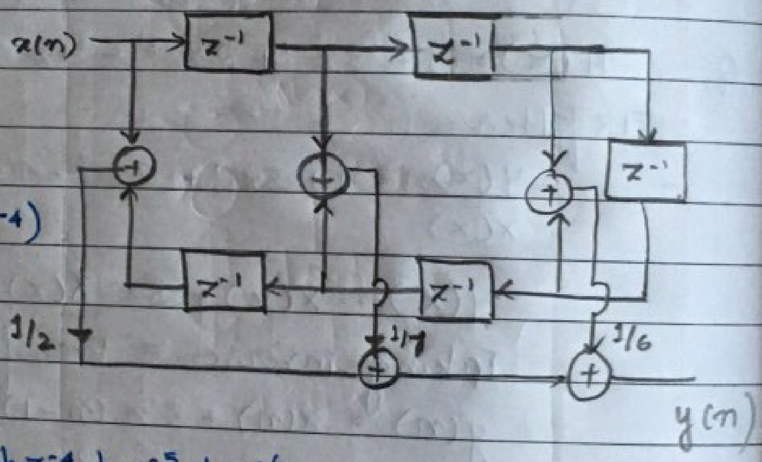
$h(0) = h(5) = 1/2$

$h(1) = h(4) = 1/7$

$h(2) = h(3) = 1/6$

$H(z) = 1/2(1+z^{-5}) + 1/7(z^{-1}+z^{-4}) + 1/6(z^{-2}+z^{-3})$

$(M/2 = 3)$



For $M=7$

$H(z) = h_0 + h_1 z^{-1} + h_2 z^{-2} + h_3 z^{-3} + h_2 z^{-4} + h_1 z^{-5} + h_0 z^{-6}$

$h(n) = h(6-n)$

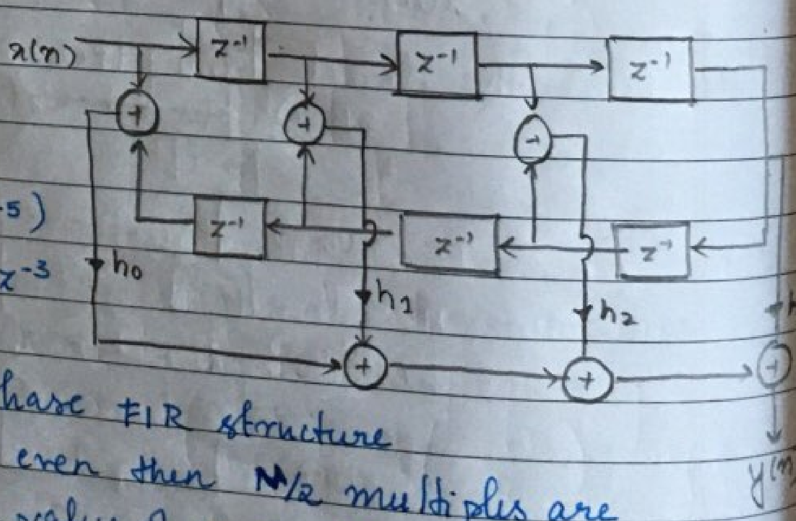
$h(0) = h(6) = h_0$

$h(1) = h(5) = h_1$

$h(2) = h(4) = h_2$

$H(z) = h_0(1+z^{-6}) + h_1(z^{-1}+z^{-5}) + h_2(z^{-2}+z^{-4}) + z^{-3}$

$((M+1)/2 = 4)$



Hence for a linear phase FIR structure if the value of M is even then $M/2$ multiples are required and if the value of M is odd then $M+1/2$ multiples are required.

* **Frequency Sampling structure :**

$$H(e^{j\omega}) = \sum_{n=0}^{M-1} h(n) e^{-j\omega n} \quad \text{Frequency Response}$$

$$\omega_k = \frac{2\pi}{M}(k+\alpha)$$

$$\text{where } \alpha = \begin{cases} 0 & ; \text{type I} \\ 1/2 & ; \text{type II} \end{cases}$$

$$\text{and} \quad k = 0 \ldots \frac{M-1}{2} \quad \text{when } M \text{ is odd}$$

$$k = 0 \ldots \frac{M}{2} - 1 \quad \text{when } M \text{ is even}$$

$$H(e^{j\omega})\Big|_{\omega=\omega_k} = \frac{2\pi}{M}(k+\alpha) = H(k+\alpha)$$

$$= \sum_{n=0}^{M-1} h(n) e^{-j\frac{2\pi}{M}(k+\alpha)n} \quad \text{--- (1)}$$

$$h(n) = \frac{1}{M}\sum_{k=0}^{M-1} H(k+\alpha) e^{j\frac{2\pi}{M}(k+\alpha)n} \quad n = 0, 1 \ldots M-1 \quad \text{--- (2)}$$

$$H(z) = \sum_{n=0}^{M-1} h(n) z^{-n}$$

$$H(z) = \sum_{n=0}^{M-1} \frac{1}{M} \sum_{k=0}^{M-1} H(k+\alpha) e^{j\frac{2\pi}{M}(k+\alpha)n} z^{-n}$$

$$H(z) = \sum_{n=0}^{M-1} \frac{1}{M} \sum_{k=0}^{M-1} H(k+\alpha) \left(e^{j\frac{2\pi}{M}(k+\alpha)} z^{-1}\right)^n$$

$$H(z) = \sum_{k=0}^{M-1} H(k+\alpha) \sum_{n=0}^{M-1} \frac{1}{M} \left(e^{j\frac{2\pi}{M}(k+\alpha)} z^{-1}\right)^n$$

$$H(z) = \sum_{k=0}^{M-1} H(k+\alpha) \frac{1}{M} \left[\frac{1 - [z^{-1} e^{j\frac{2\pi}{M}(k+\alpha)}]^M}{1 - z^{-1} e^{j\frac{2\pi}{M}(k+\alpha)}}\right]$$

$$H(z) = \sum_{k=0}^{M-1} H(k+\alpha) \frac{1}{M} \left[\frac{1 - e^{j2\pi(k+\alpha)} z^{-M}}{1 - e^{j\frac{2\pi}{M}(k+\alpha)} z^{-1}}\right]$$

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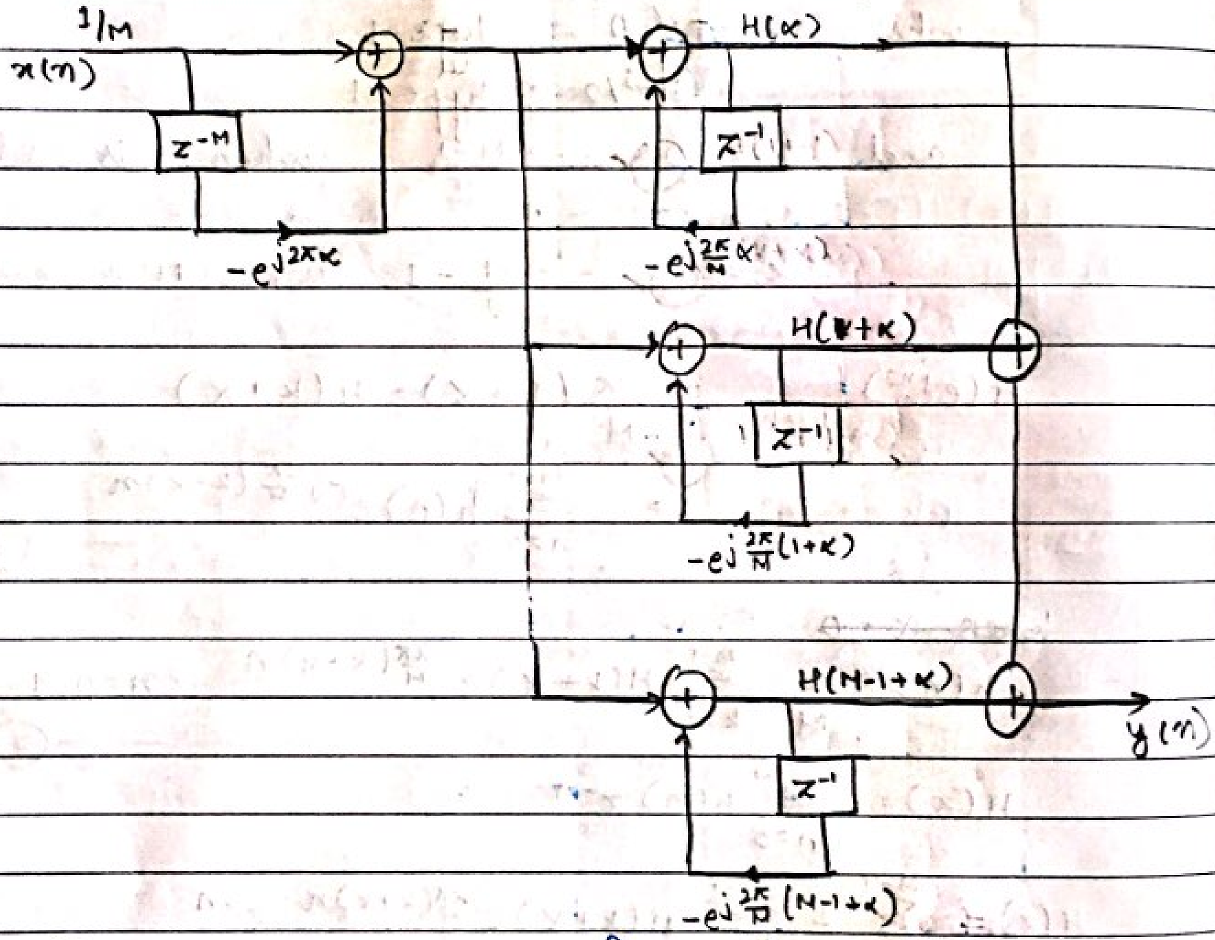
$$H(z) = \sum_{k=0}^{M-1} H(k+\alpha) \frac{1}{M} \left[\frac{1 - e^{j2\pi\alpha} z^{-M}}{1 - e^{j\frac{2\pi}{M}(k+\alpha)} z^{-1}} \right]$$

$$H(z) = \underbrace{\left[\sum_{k=0}^{M-1} \frac{H(k+\alpha)}{1 - e^{j\frac{2\pi}{M}(k+\alpha)} z^{-1}} \right]}_{H_2(z)} \underbrace{\left[\frac{1}{M} \left(1 - e^{j2\pi\alpha} z^{-M} \right) \right]}_{H_1(z)}$$



Q: Obtain the frequency sampling realization an FIR filter with $M = 16$, $\alpha = 0$ which has frequency samples $H(k+\alpha) = H(k) = H\left(\frac{2\pi k}{M}\right) = H\left(\frac{2\pi k}{16}\right) = H\left(\frac{\pi k}{18}\right)$

$$H\left(\frac{\pi k}{18}\right) = \begin{cases} 1/2 & ; \ k = 0, 1, 2 \\ 1/3 & ; \ k = 3, 4 \\ 0 & ; \ \text{otherwise} \end{cases}$$

contd

$$H_1(z) = \frac{1}{M}\left(1 - z^{-M}\underbrace{e^{-j2\pi\alpha}}_{1\ (\text{because } \alpha = 0)}\right) = \frac{1}{16}\left(1 - z^{-16}\right)$$

$$H_2(z) = \sum_{k=0}^{M-1} \frac{H(k+\alpha)}{1 - e^{j\frac{2\pi}{M}(k+\alpha)} z^{-1}} = \sum_{k=0}^{15} \frac{H(k)}{1 - e^{j\pi k/8} z^{-1}}$$

For $k = 0, 1, 2 \Rightarrow H(k) = 1/2$

For $k = 3, 4 \Rightarrow H(k) = 1/3$

For $k=0$

$$H_2(z) = \frac{1}{2}\left[\frac{1}{1 - z^{-1}}\right]$$

For $k = 1$

$$H_2(z) = \frac{1}{2}\left[\frac{1}{1 - (0.92 + 0.38j)z^{-1}}\right]$$

For $k = 2$

$$H_2(z) = \frac{1}{2}\left[\frac{1}{1 - (0.707 + 0.707j)z^{-1}}\right]$$

For $k = 3$

$$H_2(z) = \frac{1}{3}\left[\frac{1}{1 - (0.382 + 0.923j)z^{-1}}\right]$$

For $k = 4$

$$H_2(z) = \frac{1}{3}\left[\frac{1}{1 - jz^{-1}}\right]$$

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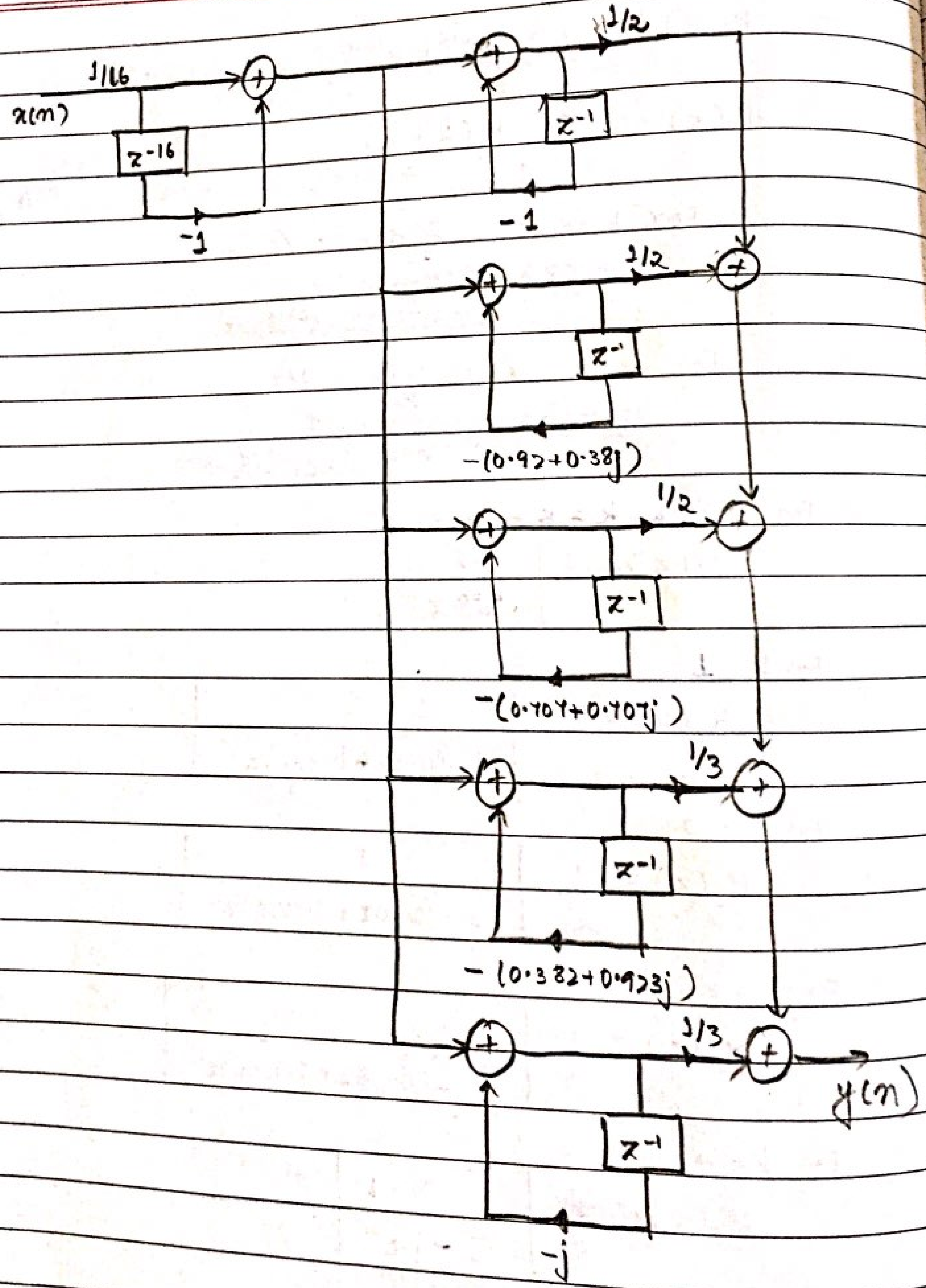

UNIT - 06

Digital Signal Processors

* **Architectural features of digital signal processors:**

- separate memory for program and data i.e., Harvard Architecture. Thus program instructions and data can be fetched at the same time but large number of pins is required which can be overcome by adopting modified Harvard Architecture which is used in DSP's these days.
- Hardware multiplier : The multiplication result is available only after one clock cycle.
- specialized instruction set.
- Multiple functional units : Usually in DSP's there are multiple ALU's which helps in parallel compitation
- on chip memory and pleripherals (including cache memory)
- Pipelining: It refers to overlapping of execution of a number of tasks.

* **Fixed Point and Floating Point Processors:**

Digital signal processing can be separated into two categories: fixed point and floating point. These designations refer to the format used to store and manipulate numeric representations of data. Fixed point DSPs are designed to represent and manipulate integers. Floating point DSPs represent and manipulate rational numbers where a number is represented with a mantissa and exponent.

Floating point processors: applications requiring large dynamic range of numbers, consumes more power and costly.

Fixed point processors: low power and low cost applications, speed is slightly higher.

* Different generations of DSP's:

Fixed point processors

1st generation: TMS320C1X
2nd generation: TMS320C5X
3rd generation: TMS320C54X
4th generation: TMS320C62X

Floating point processors

1st generation: TMS320C3X
2nd generation: TMS320C4X
3rd generation: TMS32067X

* TMS 320C67X Processors:

— TMS320C67X Architecture:

- Very long instruction word (VLIW) architecture
- CPU fetches eight instruction (each of 32 bits) at the same time and executes up to eight instructions per cycle.
- There are 8 functional units
  - two multipliers
    - .M1 and .M2: fixed or floating point multiplications
  - six ALUs
    - .D1 and .D2: only fixed point operations
    - .L1, .L2, .S1 and .S2: fixed and floating point operations
- There are two general purpose register files. Each register file contains 16 32-bit registers
  - Register file A: A0 - A15 registers
  - Register file B: B0 - B15 registers

Register pairs are used to store data that exceeds 32 bits.

- Each functional unit directly accesses the register file within its own path. The functional units are split into two halves:
  - .L1, .S1, .D1 and .M1: directly access register file A
  - .L2, .S2, .D2 and .M2: directly access register file B

- control register file can be read from or written to through the functional unit .S2 only.
- Internal memory has two level cache architecture
  - level 1 - program and data cache
  - level 11 - cache and internal memory
- On-chip peripherals include
  - two multichannel buffered serial ports (McBSPs)
  - two 32-bit general purpose timers
  - Direct memory access (DMA) controllers.
  - glueless external memory interface (EMIF)
  - host port interface (HPI)
  - interrupt selector
- Buses include:
  - a program address bus : 32 bit (PAB)
  - a program data bus : 256 bit (8 instructions) (PDB)
  - two data address buses : 32 bit each (DAB)
  - two data buses : 64 bit each (DB)
  - two 64 bit store buses.
  - two 32 bit DMA address buses
  - two 32 bit DMA data buses.

— Addressing Modes:

1. Linear and Circular Addressing:

Any register can be used for linear addressing. $A_4 - A_7$ (for functional unit .D1) and $B_4 - B_7$ (for functional unit .D2) can be used for circular addressing.

The address mode register AMR of the control register file specifies the addressing mode for each of the eight registers mentioned above.

31 (MSB) - 26 : reserved

25 - 21 : BK1 (circular buffer size)

20 - 16 : BK0 (circular buffer size)

15 - 14 : B7 mode

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13-12 : B6 mode

11-10 : B5 mode

9-8 : B4 mode

7-6 : A7 mode

5-4 : A6 mode

3-2 : A5 mode

1-0(LSB) : A4 mode

Mode bits	Mode Selected
00	linear addressing
01	circular addressing (BK0 field is used)
10	circular addressing (BK1 field is used)
11	Reserved

2. Indirect Addressing:

Indirect addressing uses load / store instruction which is used to access data in memory. Any registers in register file A or register file B can be used.

*R : Address of memory location containing data in register R.

* +R(disp) : The address is preincremented by disp. Data is accessed in memory location (R+disp). Content of R is not modified.

* ++R (disp) : The address is pre incremented by disp. Data is accessed in memory location (R+disp). Content of R is modified as (R+disp).

*R++(disp) : Address of memory location containing data is in register R. Post increment of R takes place. Content of register R is modified as (R+disp).

Instead of pre increment and postincrement we can use pre decrement and post decrement respectively. The syntax can be obtained by replacing + by −.

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– Instruction types:

1. Arithmetic Operations

ADD .S1 A2, A4, A2; Register contents A2 and A4 are added and the result is stored in A2.

SUB .L2 B2, 2, B5; Register contents B2 is decremented by 2 and the result is stored in B5.

MPY .M1 A2, B1, A1; 16 LSBs of A2 and B1 are multiplied and result is stored in A1

MPYH .M2 A5, B4, B5; 16 MSBs of A5 and B4 are multiplied are result is stored in B5

2. Move Instructions

MVKL .S1 addr, A2; loads 16 LSBs of address into 16 LSBs of register A2

MVKH .S1 addr, A2; Loads 16 MSBs of address into 16 MSBs of register A2.

3. Load and store Instructions

LDB .D1 *A3, A4 byte whose address is in register A3 is fetched from memory and loaded into register A4.

STW .D2 A2, *++B4[10] The content of register A2 is stored to the memory location whose address is specified by B4 offset by 10 words. The address register B4 is also modified.

4. Branch Instruction

ADD.L1 A0, A1, A2
[A2] B .S2 LOOP

The register contents A0 and A1 are added using functional unit .L1 and the result is stored in A2. If the result A2 is non zero branching takes place.

5. Compare Instructions:

CMPGT .L1 5, A2, A3 The content of register A2 is compared with 5. If 5 is greater than content of A2, 1 is written into register A3 else 0 is written into A3.

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There are also instructions for conversion of data from one format to another. Ex: the instructions INTSP and INTDP are used to convert an integer into the single precision and double precision values respectively.

We could load half word, full word or double word from memory using load instructions.

- **Pipelining:**

The pipline phases in a C67x processors are divided into three stages.

FETCH	DECODE	EXECUTE

A fetch packet (FP) consists of 8 32 bit instructions
There are four phases in the fetch stage:
- Program address generate : PG
- Program address send : PS
- Program access ready wait : PW
- Program fetch packet receive : PR

The decode stage consists of two phases
- Instruction dispatch : DP
- Instruction decode : DC

The number of phases in the execute stage vary depending on the instruction.

PG	PS	PW	PR	DP	DC	E1	E2	E3	E4	E5
fetch				decode		execute				

- **Interrupts:**

An interrupt is an event that stops the current process in the CPU so that the CPU is able to perform the task that needs completion due to the occurance of this event. The interrupt sources may be internal or external.

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There are 16 interrupts which can split into three groups:

1. Reset:

A low signal on the external pin RESET is used to halt the CPU and go back to known state. It has the highest priority among all the interrupts.

2. NMI:

It is normally used in case of a serious hardware problem like power failure. It has the second highest priority.

3. INT4 - INT15

These are the interrupts having priorities lower than RESET and NMI. Among them INT 4 has the highest priority and INT15 has the lowest priority. These interrupts may come from on chip peripherals or external devices. These can be also software controlled.
The twelve interrupts INT4 - INT15 are maskable ~~thus~~ thus can be disabled if it is so desired.

RESET and NMI are nonmaskable but NMI is only disabled when RESET occurs in order to prevent interruption of reset operation.

The registers controlling interrupts are

- control status register: CSR
- interrupt enable register: IER
- interrupt flag register: IFR
- interrupt set register: ISR
- interrupt clear register: ICR
- interrupt service table pointer: ISTP
- interrupt return pointer: IRP
- NMI return pointer.

Global interrupt enable GIE bit in CSR may be 1 or 0 to enable or disable maskable interrupts respectively.

Interrupts can be enabled or disabled by setting or clearing bits in IER.

The status of the interrupts can be read from IFR. If it is a 1 in the bit position for an interrupt in IFR, then the interrupt has occured.

Interrupts can be set or cleared manually by writting a 1 in the corresponding bit position in ISR or ICR respectively.

ISTP is used for locating the interrupt service routine

IRP contains the return pointer so that the CPU can return to the appropriate location to continue program execution after completion of the tasks for a maskable interrupt.

NRP is used for returning after processing of a nonmaskable interrupt.

- On Chip Peripherals:

Timers:

There are two 32 bit timers which can be used for timing and counting of events, generation of pulses and for interrupting the CPU. The clock source for the timer can be internal or external. Each timer has one input and one output pin.

Two signalling modes are possible with these timers: clock and pulse mode. In clock mode, timer output is high for one clock cycle after timer reaches the final count. In pulse mode, timer output has a 50% duty cycle.

There are three registers that configure the operation of a timer.

- timer control register: determines operating mode of timer i.e, clock or pulse mode.
- timer period register: contains a value that is equal to the number of clock cycles to be counted
- timer counter register: contains the current value of the counter.

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Multichannel Buffered serial Port (McBSP)

Two multichannel buffered serial ports allow direct interface to high speed data links. They also provide They also provide full-duplex communication and multichannel transmit as well as receive upto 128 channels. μ law and A law companding are also performed by it. The serial ports operate independently and within any serial port, framing and clocking for transmit and receive sections can be performed independently.

Serial port configuration is done through the serial port control register (SPCR) and pin control register (PCR). The transmit and receive sections are independently controlled by transmit control register (XCR) and receive control register (RCR) respectively. The data transmit pin (DX) and data receive pin (DR) are used for serial data transmission and reception.

During transmission, data is written to data transmit register (DXR) which is then transferred to transmit shift register (XSR) and finally shifted out through DX pin. During reception, data received on DR pin is shifted into receive shift register (RSR). This data is then copied to receive buffer register (RBR) and finally from RBR, it is copied to data receive register (DRR). Therefore there is double-buffering during transmission and triple buffering during reception.

Each of the serial ports can be set to transfer 8, 12, 16, 20, 24 or 32 bit words. The word length is selected independently for transmit and receive sections through the programming of XCR and RCR. Independent framing and clocking are also possible for receive and transmit sections.

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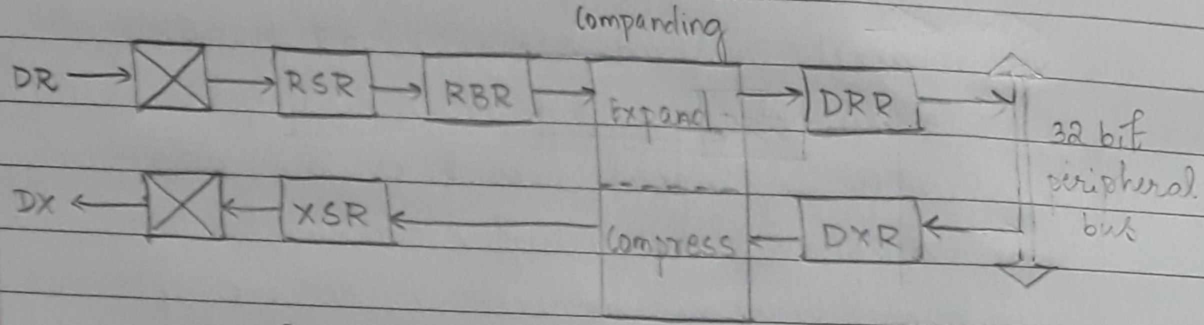

DMA controller

In order to transfer data between memory mapped regions (internal memory, internal peripherals or external devices) without intervention by the CPU, direct memory access (DMA) is used.

There are four independent programmable channels in a DMA controller. A fifth channel is available for DMA with host port interface (HPI).

There are a number of registers involved in DMA operation: DMA primary control register, DMA secondary control register, source and destination address registers and transfer counter for each channel. Also there are a number of global registers for count reload, index and address.

DMA transfer can be set up by first loading primary and secondary control register with the required values. Then source and destination and count register are set up i.e., they are loaded with the desired values. Finally writing the required bit pattern (01 or 11b) to the START field in the primary control register starts DMA. Transfers using DMA can be synchronised with events like interrupts from internal peripherals or external devices. When transfer count becomes zero, an interrupt may be sent to the CPU for desired action. Auto intialization (that resets source address, destination address and transfer counter to the initial values when block transfer is complete) is also available for repetitive transfers.