Keerthana Ashok

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UNIT - 01

Mathematical Models of Physical Systems.

control system:

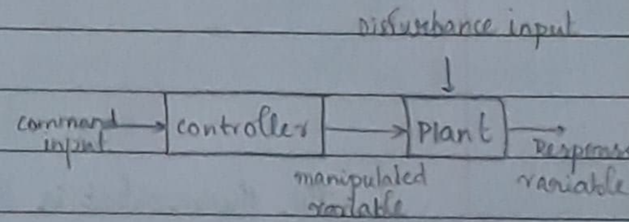
It is a means by which any quantity of interest in a machine, mechanism or other equipment is maintained or altered in a desired manner.

System:

A system is a combination or an arrangement of different physical components which act together as an entire unit to achieve certain objective.

* **open loop control system:**

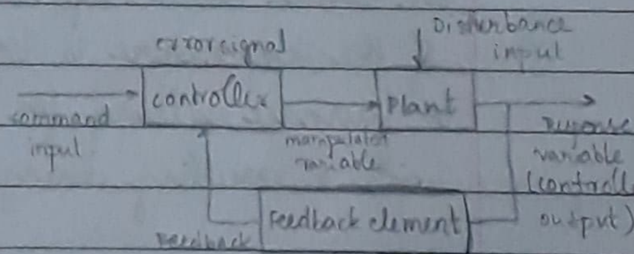
The output is unchanged if the input is unchanged.



* **closed loop control system:**

Output or part of the output is fed back to the input.

The main excitation to the system is called the command input. The signal which is output of the feedback element is called the feedback signal.



The error signal modified by controller and decides the proportional manipulated signal for the process to be controlled. This manipulation is such that error will approach zero. This signal then actuates the actual system and produces an output. As output is controlled one hence called controlled output

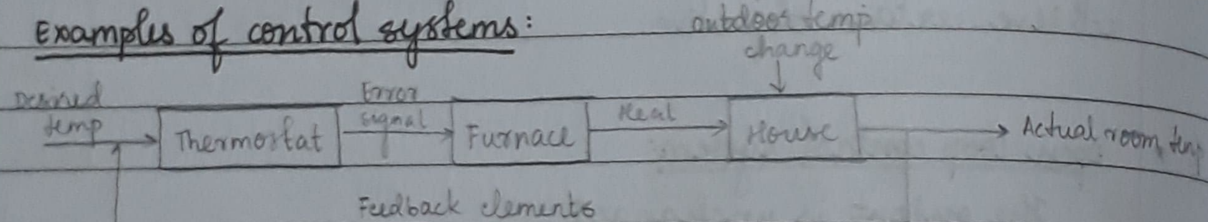
If a human is part of a controlled process and the system then such a system is called as manually controlled control system else it is automatic control system.

NOTE: Laplace Transforms

$\delta(t) = 1$	$u(t) = 1/s$	$t = 1/s^2$
$t^n = \frac{n!}{s^{n+1}}$	$e^{-at} = \frac{1}{s+a}$	$te^{-at} = \frac{1}{(s+a)^2}$
$t^n e^{-at} = \frac{n!}{(s+a)^{n+1}}$	$\sin bt = \frac{b}{s^2+b^2}$	$\cos bt = \frac{s}{s^2+b^2}$
$e^{-at}\sin bt = \frac{b}{(s+a)^2+b^2}$	$e^{-at}\cos bt = \frac{s+a}{(s+a)^2+b^2}$	$t\sin bt = \frac{2bs}{(s^2+b^2)^2}$
$t\cos bt = \frac{s^2-b^2}{(s^2+b^2)^2}$		

★ Examples of control systems:

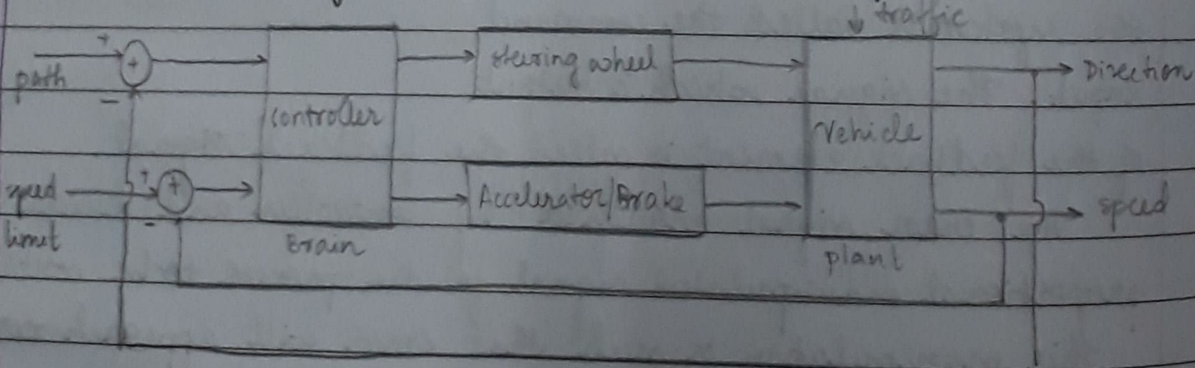
1.



ROOM HEATING SYSTEM

The thermostat is the controller. It makes comparision between the desired room temperature and the actual room temperature. If the temperature is too low the thermostat initiates a signal to furnace calling for heat. The furnace then supplies heat with a resulting change in the room temperature.

2.



The objective is to maintain the speed equal to the speed limits and follow the direction of the path.

Two command inputs and two command outputs : Multiple input Multiple Output system (MIMO system)

★ Comparison of open loop and closed loop control system:

	OPEN LOOP CONTROL SYSTEM	CLOSED LOOP CONTROL SYSTEM
1.	Any change in output has no effect on the input i.e., feedback does not exist.	Any change in output affects the input which is possible by use of feedback.
2.	Output measurement is not required for operation of system.	Output messurement is necessary.
3.	Error detector is absent.	Error detector is necessary.
4.	Inaccurate and unreliable.	Highly accurate and reliable.
5.	Highly sensitive to disturbances.	Less sensitive to disturbances.
6.	Bandwidth is small.	Bandwidth is large.
7.	Simple to construct and cheap.	Complicated to design and costly.
8.	Generally stable	Stability is a major concern.
	Ex: Time traffic control	Ex: Traffic control by sensors

Controller: The element of a system itself or external to the system which controls the plant or the process is called controller.

Plant: The portion of the system which is to be controlled or regulated is called the plant or the process.

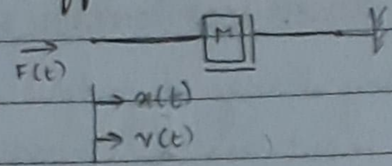
Disturbances: Disturbance is a signal which tends to adversely affect the value of the output of a system.

Mathematical Model:

The set of mathematical equations discribing the dynamic characteristics of a system is called as mathematical model. Mechanical or Electrical systems are converted to mathematical models using transfer function.

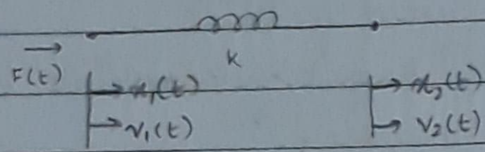
* Difference equation for physical systems:

MECHANICAL SYSTEMS



The mass element

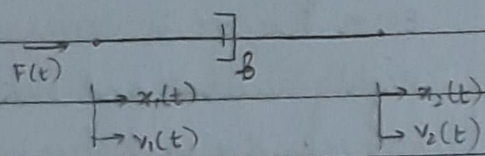
$$F = M\frac{dv}{dt} = M\frac{d^2x}{dt^2}$$



The spring element

$$F = k[x_1(t) - x_2(t)] = kx$$

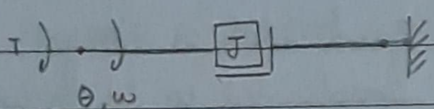
$$F = k\int[v_1(t) - v_2(t)]dt = k\int v\,dt$$



The damper element

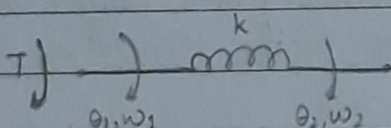
$$F = B[v_1(t) - v_2(t)] = Bv$$

$$F = B[\dot{x}_1(t) - \dot{x}_2(t)] = B\dot{x}$$



The inertial element

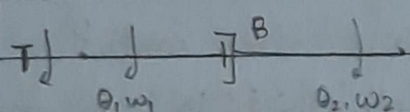
$$T = J\frac{d\omega}{dt} = J\frac{d^2\theta}{dt^2}$$



The torsional spring element

$$T = k(\theta_1 - \theta_2) = k\theta$$

$$T = k\int(\omega_1 - \omega_2)dt = k\int \omega\,dt$$

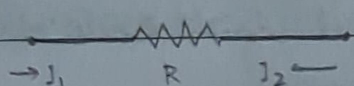


The damper element

$$T = B(\omega_2 - \omega_1) = B\omega$$

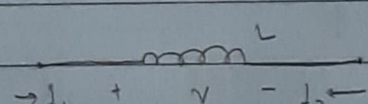
$$T = B(\dot{\theta}_1 - \dot{\theta}_2) = B\dot{\theta}$$

ELECTRICAL SYSTEMS



The resistive element

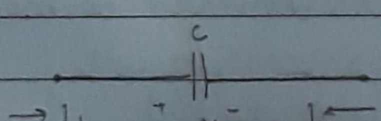
$$V = R(I_1 - I_2) = RI$$



The inductive element

$$V = L\frac{d}{dt}(I_1 - I_2) = L\frac{dI}{dt}$$

$$I = \frac{1}{L}\int V\,dt$$



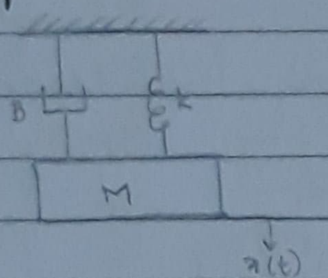
The capacitive element

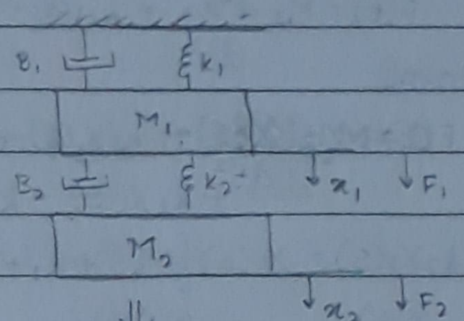
$$I = C\frac{dV}{dt}$$

$$V = \frac{1}{C}\int(I_1 - I_2)dt = \frac{1}{C}\int I\,dt$$

NOTE: The direction of displacement is same as that of application of force. The number of displacements is equivalent to the number of mass elements.

* Equivalent Mechanical system:





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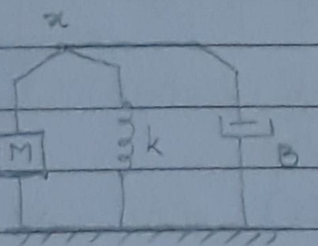
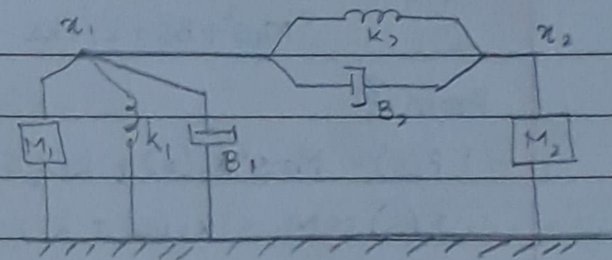




$$F = M\frac{d^2x}{dt^2} + kx + B\frac{dx}{dt}$$

Laplace transform

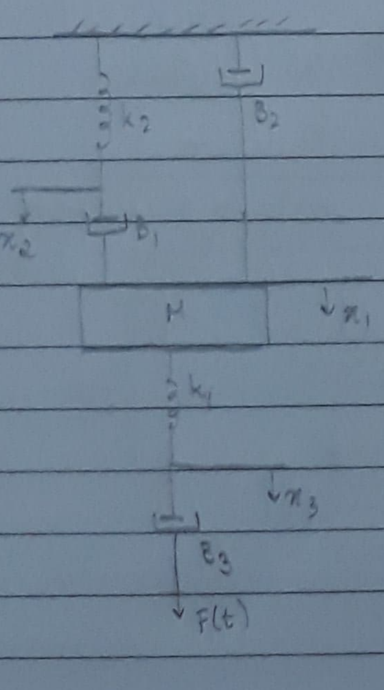
$$F(s) = Ms^2X(s) + kX(s) + BsX(s)$$

$$\frac{X(s)}{F(s)} = \frac{1}{Ms^2 + Bs + k}$$

$$F_1 = M_1\frac{d^2x_1}{dt^2} + k_1x_1 + B_1\frac{dx_1}{dt}$$

$$+ B_2\frac{d}{dt}(x_1 - x_2) + k_2(x_1 - x_2)$$

$$F_2 = M_2\frac{d^2x_2}{dt^2} + k_2(x_2 - x_1) + B_2\frac{d}{dt}(x_2 - x_1)$$



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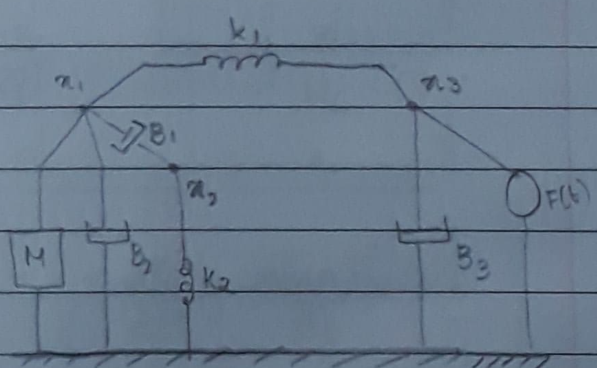


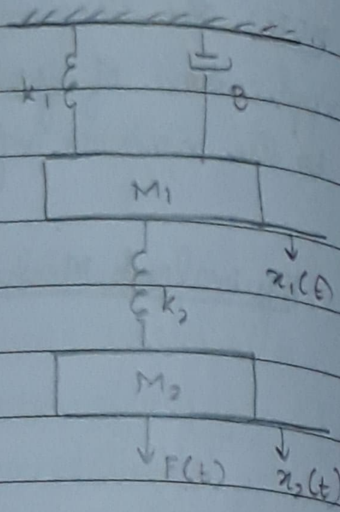
$$0 = M\frac{d^2x_1}{dt^2} + B_2\frac{dx_1}{dt} + B_1\frac{d}{dt}(x_1 - x_2) + k_1(x_1 - x_3$$

$$0 = B_1\frac{d^2(x_2 - x_1)}{dt} + k_2x_2$$

$$F(t) = k_1(x_3 - x_1) + B_3\frac{dx_3}{dt}$$

Date:

Q: Determine $X_2(s)/F(s)$



– at x_1

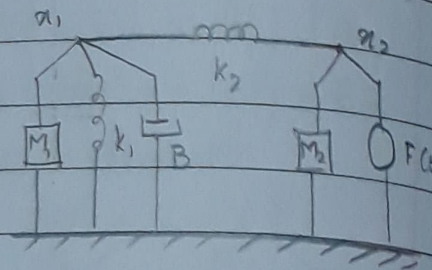
$$0 = M_1\frac{d^2x_1}{dt^2} + k_1x_1 + B\frac{dx_1}{dt} + k_2(x_1-x_2) \quad -(1)$$

at x_2

$$F(t) = M_2\frac{d^2x_2}{dt^2} + k_2(x_2-x_1) \quad -(2)$$

From (1)

$$0 = M_1 s^2 X_1(s) + k_1X_1(s) + BsX_1(s) + k_2X_1(s) - k_2X_2(s)$$



$$k_2X_2(s) = X_1(s)\left[M_1s^2+k_1+Bs+k_2\right]$$

$$X_1(s) = \frac{k_2X_2(s)}{M_1s^2+Bs+k_1+k_2}$$

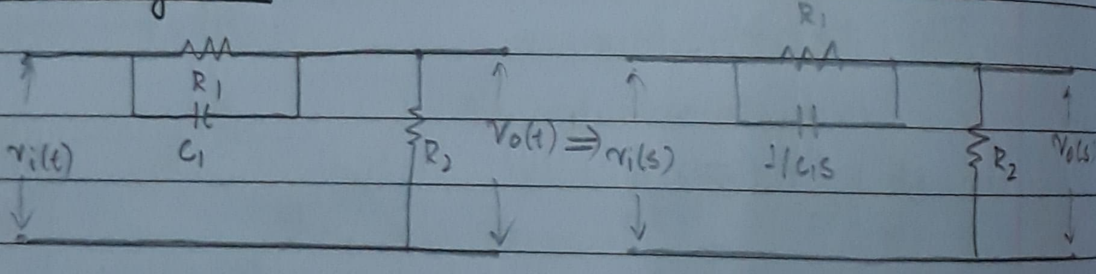
From (2)

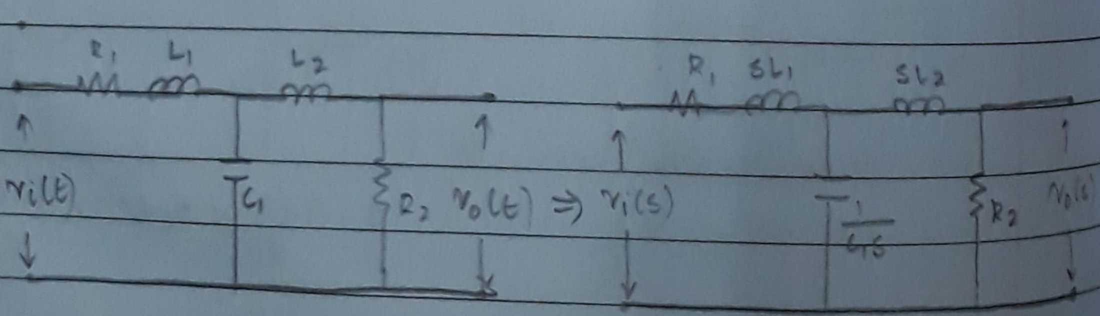
$$F(s) = M_2s^2X_2(s) + k_2X_2(s) - k_2X_1(s)$$

$$F(s) = M_2s^2X_2(s) + k_2X_2(s) - \frac{k_2^2X_2(s)}{M_1s^2+Bs+k_1+k_2}$$

$$\frac{X_2(s)}{F(s)} = \frac{1}{M_2s^2+k_2-\dfrac{k_2^2}{M_1s^2+Bs+k_1+k_2}}$$

* **Electrical systems:**

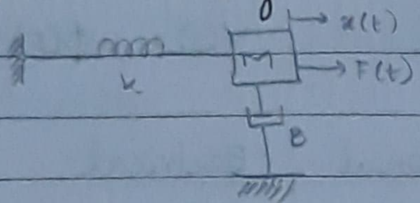




* **Analogous systems:**

There exists a fixed analogy between electrical and mechanical systems. Thus it is possible to have an electrical system which will behave exactly similar to the given mechanical system this is called electrical analogous of given mechanical system and vice versa.

Mechanical system:

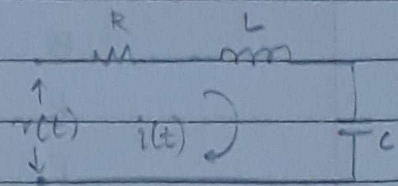


$$F(t) = M\frac{d^2x(t)}{dt^2} + B\frac{dx}{dt} + kx$$

$$\therefore F(s) = Ms^2X(s) + BsX(s) + kX(s)$$

– Force - Voltage Analogy:

By KVL



$$v(t) = i(t)R + L\frac{di(t)}{dt} + \frac{1}{C}\int i(t)\,dt$$

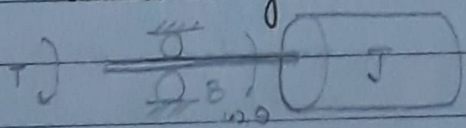
$$\therefore V(s) = I(s)R + LsI(s) + \frac{I(s)}{Cs}$$

wkt $i(t) = \frac{dq}{dt} \Rightarrow I(s) = sQ(s)$

$$V(s) = Ls^2Q(s) + RsQ(s) + \frac{Q(s)}{C}$$

Translational	Rotational	Electrical
F	T	V
M	J	L
B	B	R
k	k	1/C
x	θ	q

Rotational system

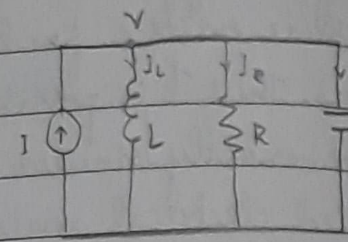


$$T = J\frac{d^2\theta}{dt^2} + B\frac{d\theta}{dt} + k\theta$$

$$T = J\frac{d\omega}{dt} + B\omega + k\int \omega\,dt$$

$$\therefore T(s) = Js^2\theta(s) + Bs\,\theta(s) + k\theta(s)$$

- **Force Current Analogy**



By KCL

$I = I_L + I_R + I_C$

$I = \frac{1}{L}\int V dt + \frac{V}{R} + C\frac{dV}{dt}$

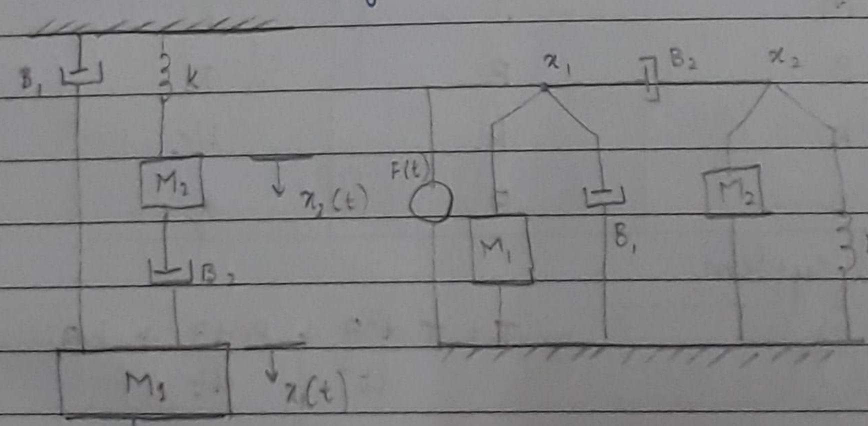
$\therefore I(s) = \frac{V(s)}{sL} + \frac{V(s)}{R} + sV(s)C$ wkt $V = \frac{d\phi}{dt}$

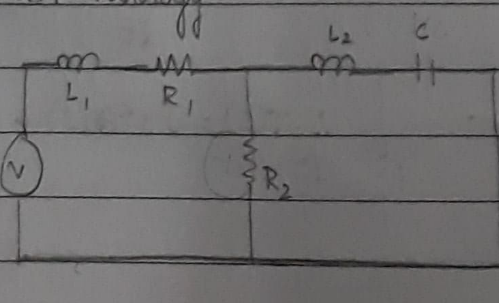
$I(s) = s^2C\phi(s) + s\frac{\phi(s)}{R} + \frac{\phi(s)}{L}$ $V(s) = s\phi(s)$

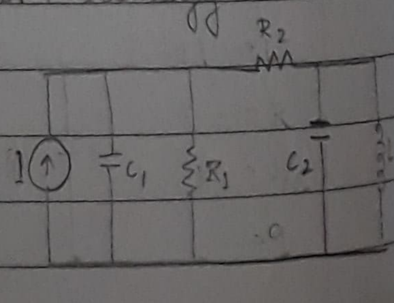
Translational	Rotational	Electrical
M	J	C
B	B	$1/R$
k	k	$1/L$
F	T	I
x	θ	ϕ

Q: Draw the equivalent mechanical system of the given system. Hence write the set of equilibrium equations for it and obtain electrical analogous circuits using
a. F-V analogy b. F-I analogy.

i.







At node x_1

$$F(t) = M_1 \frac{d^2 x_1(t)}{dt^2} + B_1 \frac{d x_1(t)}{dt} + B_2 \frac{d}{dt}(x_1(t) - x_2(t))$$

$$F(s) = M_1 s^2 X_1(s) + B_1 s X_1(s) + B_2 s(X_1(s) - X_2(s))$$

At node x_2

$$0 = M_2 \frac{d^2 x_2(t)}{dt^2} + k\, x_2(t) + B_2 \frac{d}{dt}(x_2(t) - x_1(t))$$

$$0 = M_2 s^2 X_2(s) + k X_2(s) + B_2 s(X_2(s) - X_1(s))$$

a. F-V analogy

$F \to V$; $M \to L$; $B \to R$; $k \to 1/C$; $x \to q$

$$\therefore V(s) = L_1 s^2 Q_1(s) + R_1 s Q_1(s) + R_2 s(Q_1(s) - Q_2(s))$$

$$0 = L_2 s^2 Q_2(s) + \frac{Q_2(s)}{C} + R_2 s(Q_2(s) - Q_1(s))$$

but $I(s) = sQ(s)$

$$\therefore V(s) = L_1 s I_1(s) + R_1 I_1(s) + R_2(I_1(s) - I_2(s))$$

$$0 = L_2 s I_2(s) + \frac{I(s)}{sC} + R_2(I_2(s) - I_1(s))$$

b. F-I analogy

$F \to I$; $M \to C$; $B \to 1/R$; $k \to 1/L$; $x \to \phi$

$$\therefore I(s) = C_1 s^2 \phi_1(s) + \frac{s\phi_1(s)}{R_1} + \frac{s}{R_2}(\phi_1(s) - \phi_2(s))$$

$$0 = C_2 s^2 \phi_2(s) + \frac{\phi_2(s)}{L} + \frac{s}{R_2}(\phi_2(s) - \phi_1(s))$$

but $V(s) = s\phi(s)$

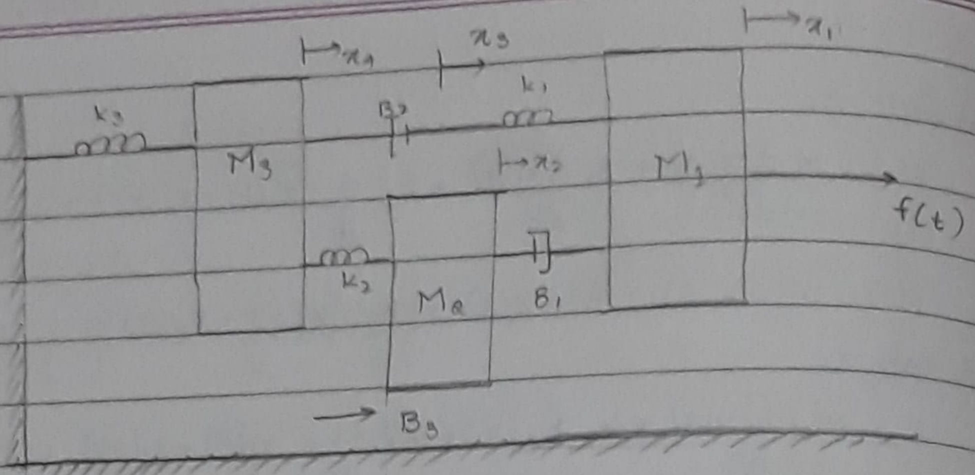
$$\therefore I(s) = C_1 s V_1(s) + \frac{V_1(s)}{R_1} + \frac{V_1(s) - V_2(s)}{R_2}$$

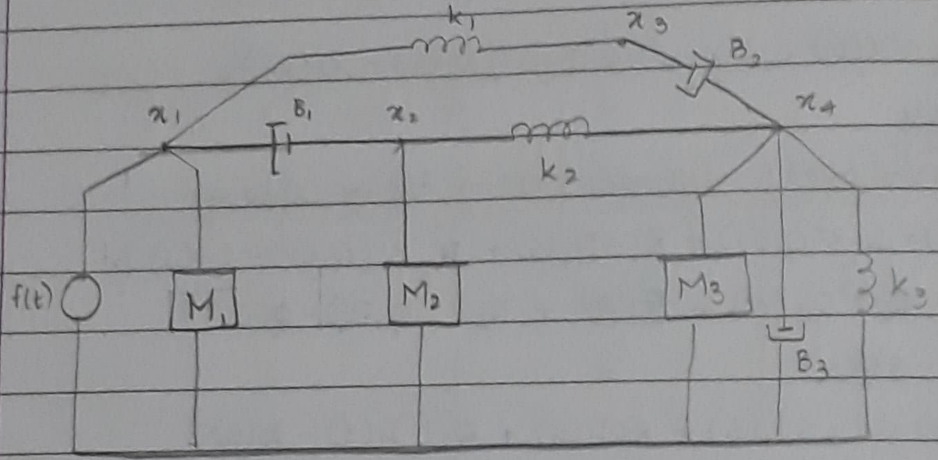
$$0 = C_2 s V_2(s) + \frac{V_2(s)}{sL} + \frac{V_2(s) - V_1(s)}{R_2}$$

NOTE: The number of loop currents in F-V analogy and number of node voltages in F-I analogy is equal to the number of displacements.

Keerthana Ashok

ii.





at node x_1

$$F(t) = M_1 \frac{d^2 x_1(t)}{dt^2} + B_1 \frac{d}{dt}\left(x_1(t) - x_2(t)\right) + k_1\left(x_1(t) - x_3(t)\right)$$

at node x_2

$$0 = M_2 \frac{d^2 x_2(t)}{dt^2} + B_1 \frac{d}{dt}\left(x_2(t) - x_1(t)\right) + k_2\left(x_2(t) - x_4(t)\right)$$

at node x_3

$$0 = B_2 \frac{d}{dt}\left(x_3(t) - x_4(t)\right) + k_1\left(x_3(t) - x_1(t)\right)$$

at node x_4

$$0 = M_3 \frac{d^2 x_4(t)}{dt^2} + B_3 \frac{dx_4(t)}{dt} + B_2 \frac{d}{dt}\left(x_4(t) - x_3(t)\right)$$

$$+ k_3 x_4(t) + k_2\left(x_4(t) - x_2(t)\right)$$

<u>F-V analogy:</u>

$F \to V$; $M \to L$; $B \to R$; $k \to 1/C$; $x \to q$

at node x_1

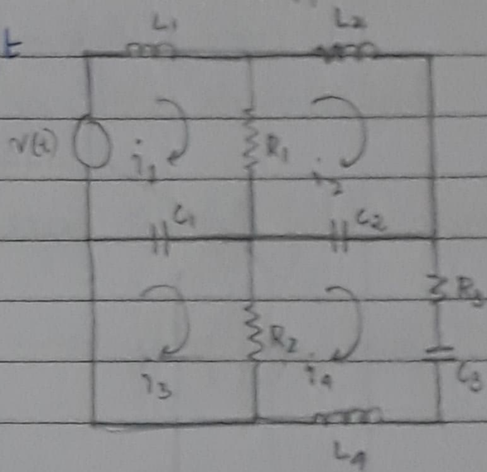
$$V(t) = L_1 \frac{d^2 q_1}{dt^2} + R_1 \frac{d}{dt}(q_1 - q_2) + \frac{1}{C_1}(q_1 - q_3)$$

$$v(t) = L_1 \frac{di_1}{dt} + R_1(i_1 - i_2) + \frac{1}{C_1}\int (i_1 - i_3)\,dt$$

at node x_2

$$0 = L_2 \frac{d^2 q_2}{dt^2} + R_1 \frac{d}{dt}(q_2 - q_1) + \frac{1}{C_2}(q_2 - q_4)$$

F-V analogy



$$0 = L_2 \frac{di_2}{dt} + R_1(i_2 - i_1) + \frac{1}{C_2}\int (i_2 - i_4)\,dt$$

at node x_3

$$0 = R_2 \frac{d}{dt}(q_3 - q_4) + \frac{1}{C_1}(q_3 - q_1)$$

$$0 = R_2(i_3 - i_4) + \frac{1}{C_1}\int (i_3 - i_1)\,dt$$

at node x_4

$$0 = L_3 \frac{d^2 q_4}{dt^2} + R_3 \frac{d}{dt} q_4 + R_2 \frac{d}{dt}(q_4 - q_3) + \frac{1}{C_3} q_4 + \frac{1}{C_2}(q_4 - q_2)$$

$$0 = L_3 \frac{di_4}{dt} + R_3 i_4 + R_2(i_4 - i_3) + \frac{1}{C_3}\int i_4\,dt + \frac{1}{C_2}\int (i_4 - i_2)\,dt$$

F-I analogy

$F \to I$; $M \to C$; $B \to 1/R$; $k \to 1/L$; $x \to \phi$

at node x_1

$$I(t) = C_1 \frac{d^2 \phi_1}{dt^2} + \frac{1}{R_1}\frac{d}{dt}(\phi_1 - \phi_2) + \frac{1}{L_1}(\phi_1 - \phi_3)$$

$$I(t) = C_1 \frac{dV_1}{dt} + \frac{1}{R_1}(V_1 - V_2) + \frac{1}{L_1}\int (V_1 - V_3)\,dt$$

at node x_2

$$0 = C_2 \frac{d^2 \phi_2}{dt^2} + \frac{1}{R_1}\frac{d}{dt}(\phi_2 - \phi_1) + \frac{1}{L_2}(\phi_2 - \phi_4)$$

$$0 = C_2 \frac{dV_2}{dt} + \frac{1}{R_1}(V_2 - V_1) + \frac{1}{L_2}\int (V_2 - V_4)\,dt$$

at node x_3

$$0 = \frac{1}{R_2}\frac{d}{dt}(\phi_3 - \phi_4) + \frac{1}{L_1}(\phi_3 - \phi_1)$$

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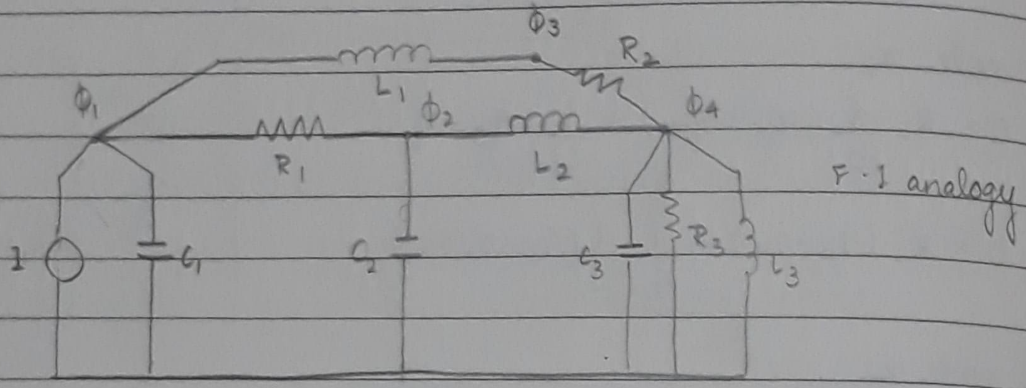
$$0 = \frac{1}{R_2}(v_3 - v_4) + \frac{1}{L_1}\int (v_3 - v_1)\,dt$$

at node x_4

$$0 = C_3 \frac{d^2\phi_4}{dt} + \frac{1}{R_3}\frac{d\phi_4}{dt} + \frac{1}{R_2}\frac{d}{dt}(\phi_4 - \phi_3) + \frac{1}{L_3}\phi_4 + \frac{1}{L_2}(\phi_4 - \phi_2)$$

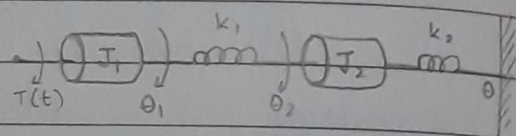
$$0 = C_3 \frac{dV_4}{dt} + \frac{1}{R_3}V_4 + \frac{1}{R_2}(V_4 - V_3) + \frac{1}{L_3}\int V_4\,dt + \frac{1}{L_2}\int (V_4 - V_2)\,dt$$



F-I analogy

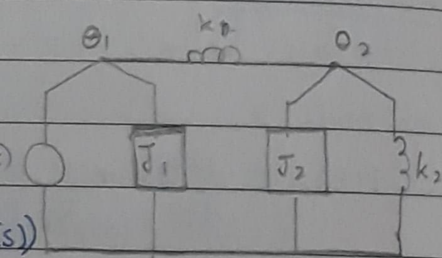
Q: Obtain the equations of moment of inertia. Find $\theta_1(s)/T(s)$.

$$T(t) = J_1 \frac{d^2\theta_1}{dt^2} + k_1(\theta_1 - \theta_2)$$



$$T(s) = J_1 s^2 \theta_1(s) + k_1(\theta_1(s) - \theta_2(s))$$

$$T(s) = (J_1 s^2 + k_2)\theta_1(s) - k_1\theta_2(s)$$



and $0 = J_2 \dfrac{d^2\theta_2}{dt^2} + k_2\theta_2 + k_1(\theta_2 - \theta_1)$

$$0 = J_2 s^2 \theta_2(s) + k_2\theta_2(s) + k_1(\theta_2(s) - \theta_1(s))$$

$$0 = (J_2 s^2 + k_2 + k_1)\theta_2(s) - k_1\theta_1(s)$$

Therefore

$$\begin{bmatrix} J_1 s^2 + k_2 & -k_1 \\ -k_1 & J_2 s^2 + k_1 + k_2 \end{bmatrix} \begin{bmatrix} \theta_1(s) \\ \theta_2(s) \end{bmatrix} = \begin{bmatrix} T(s) \\ 0 \end{bmatrix}$$

For $\theta_1(s)$

$$\Delta = (J_1 s^2 + k_2)(J_2 s^2 + k_1 + k_2) - k_1^2$$

$$\frac{\begin{vmatrix} T(s) & -k \\ 0 & J_2 s^2 + k_1 + k_2 \end{vmatrix}}{\Delta} = \theta_1(s)$$

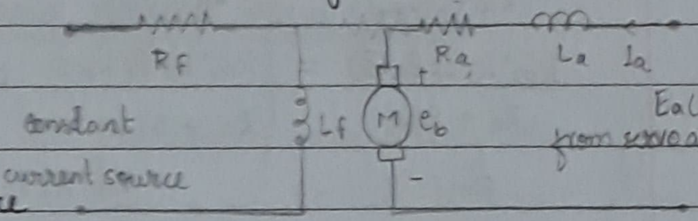
Therefore

$$\theta_1(s) = \frac{T(s)(J_2 s^2 + k_1 + k_2)}{(J_1 s^2 + k_1)(J_2 s^2 + k_1 + k_2) - k_1^2}$$

Hence $$\frac{\theta_1(s)}{T(s)} = \frac{J_2 s^2 + k_1 + k_2}{(J_1 s^2 + k_1)(J_2 s^2 + k_1 + k_2) - k^2}$$ //

* Transfer function for a dc motor using armature control:

In armature control, the input voltage E_a is applied to the armature with resistance R_a and inductance L_a. The field winding is supplied with constant current I_f. Thus armature input voltage controls the motor shaft output.



Assumptions:

a. Flux is directly proportional to current through field winding.

$$\phi_m = k_f I_f = \text{constant}.$$

b. Torque produced is proportional to product of flux and armature current.

$$T = k_m' \phi I_a$$

$$T = k_m' k_f I_f I_a$$

c. Back emf is directly proportional to shaft velocity ω_m as flux ϕ is constant as $\omega_m = d\theta(t)/dt$

$$E_b = k_b \omega_m(s) = k_b s \theta_m(s)$$

Applying Kirchhoff's laws to armature circuit.

$$E_a = E_b + I_a R_a + L_a \frac{dI_a}{dt}$$

Taking laplace transform

$$E_a(s) = E_b(s) + I_a(s)[R_a + L_a s]$$

$$I_a(s) = \frac{E_a(s) - E_b(s)}{R_a + sL_a}$$

$$I_a(s) = \frac{E_a(s) - K_b s \theta_m(s)}{R_a + sL_a}$$

$$T_m = k_m' K_f I_f I_a$$

$$T_m = k_m' k_f I_f \left[\frac{E_a - K_b s \theta_m(s)}{R_a + sL_a}\right]$$

wkt $T_m = (J_m s^2 + B_m s)\theta_m(s)$

Equating equation of T_m

$$\frac{k_m' k_f I_f E_a(s)}{R_a + sL_a} = \frac{k_m' k_f I_f k_b s\,\theta_m(s)}{R_a + sL_a} + (J_m s^2 + B_m s)\,\theta_m(s)$$

$$\frac{k_m' k_f I_f E_a(s)}{R_a + sL_a} = \left[\frac{k_m' k_f I_f k_b s}{R_a + sL_a} + J_m s^2 + B_m s\right]\theta_m(s)$$

$$\frac{\theta_m(s)}{E_a(s)} = \frac{\dfrac{k_m' k_f I_f}{R_a + sL_a}}{\dfrac{k_m' k_f I_f k_b s}{R_a + sL_a} + J_m s^2 + B_m s}$$

but $\tau_m = \dfrac{J_m}{B_m}$ and $\tau_a = \dfrac{L_a}{R_a}$

$$\frac{\theta_m(s)}{E_a(s)} = \frac{\dfrac{k_m' k_f I_f}{1 + s\tau_a}}{\dfrac{k_m' k_f I_f k_b s + (1 + \tau_m s)}{1 + s\tau_a}}$$

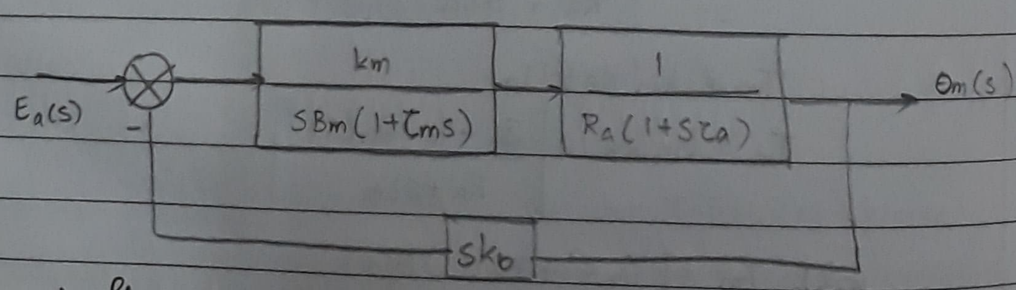
$$\frac{\theta_m(s)}{E_a(s)} = \frac{k_m' k_f I_f}{k_m' k_f I_f k_b s + (1 + \tau_m s)(1 + s\tau_a)}$$

Let $k_m' K_f = k_m$

$$\frac{\theta_m(s)}{E_a(s)} = \frac{k_m I_f}{k_m I_f k_b s + (1 + \tau_m s)(1 + \tau_a s)}$$

$$\frac{\theta_m(s)}{E_a(s)} = \frac{\dfrac{k_m}{sR_aB_m(1 + \tau_m s)(1 + \tau_a s)}}{1 + \dfrac{k_m s k_b}{sR_aB_m(1 + \tau_m s)(1 + \tau_a s)}} = \frac{G(s)}{1 + G(s)H(s)}$$

where $G(s) = \dfrac{k_m}{sR_aB_m(1 + \tau_m s)(1 + \tau_a s)}$ $\quad H(s) = sk_b$



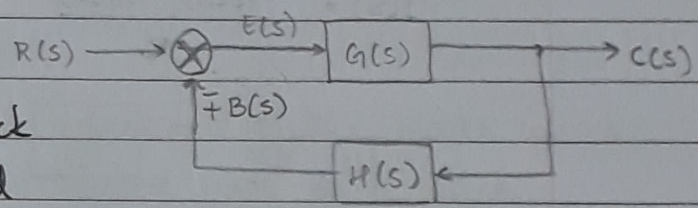
Field controlled dc motor is open loop while armature controlled is closed loop system. Hence armature controlled are preferred.

UNIT - 02

Block Diagram and Signal Flow Graphs

For a closed loop systems the function of comparing the different signals is indicated by the summing point while a point from which signal is taken for the feedback purpose is indicated by takeoff point in the block diagram.

Simple canonical form of a closed loop system contains on forward path block, one feedback block, one summing point and one takeaway point.



R(s) : Laplace of reference input

C(s) : Laplace of controlled output

E(s) : Laplace of error signal

H(s) : Equivalent feedback path transfer gain.

B(s) : Laplace of feedback signal

G(s) : Equivalent forward path transfer gain.

* **Transfer function of simple closed loop system:**

$$E(s) = R(s) \mp B(s)$$

when $B(s) = C(s)H(s)$ and $C(s) = E(s)G(s)$

$$E(s) = R(s) \mp C(s)H(s)$$

$$\frac{C(s)}{G(s)} = R(s) \mp C(s)H(s)$$

$$C(s) = G(s)R(s) \mp C(s)G(s)H(s)$$

$$C(s)\left[1 \pm G(s)H(s)\right] = G(s)R(s)$$

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1 \pm G(s)H(s)}$$

Note: + sign for negative feedback
- sign for positive feedback.

Date:

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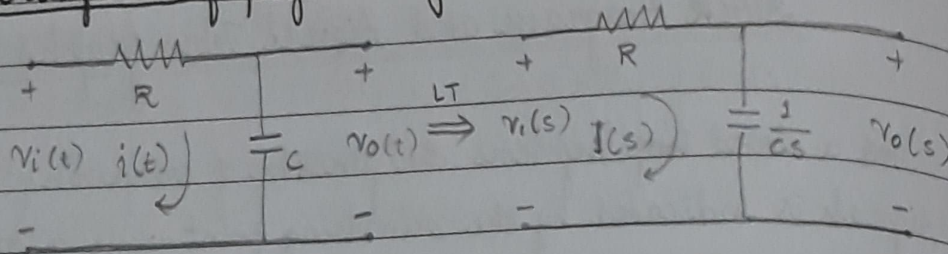
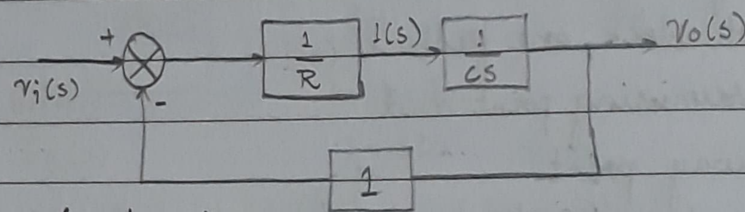
* Block diagram of physical systems:

Ex 1:



$$i(t) = \frac{v_i(t) - v_o(t)}{R}$$

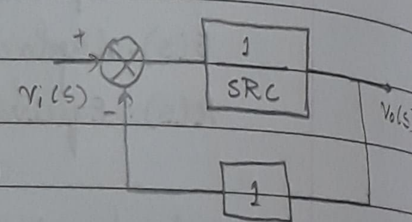
$$I(s) = \frac{v_i(s) - v_o(s)}{R} \qquad \text{Here } v_o(s) = I(s)\left(\frac{1}{Cs}\right)$$



Transfer function

$$\frac{v_o(s)}{v_i(s)} \Rightarrow \quad v_o(s) = I(s)\,(1/Cs)$$

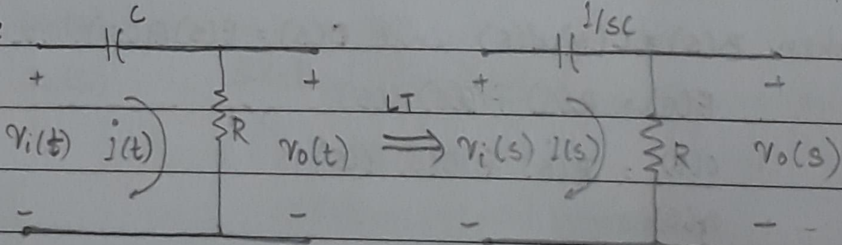
$$v_o(s) = \frac{v_i(s) - v_o(s)}{RCs}$$



$$v_o(s)\left[RCs + 1\right] = v_i(s)$$

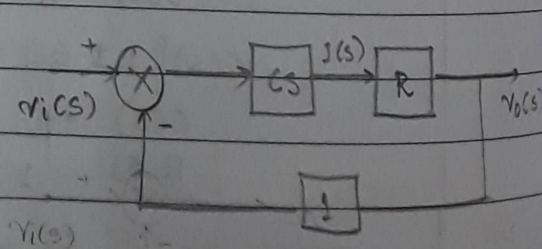
$$\frac{v_o(s)}{v_i(s)} = \frac{1}{1 + sRC}$$

Ex 2:



$$i(t) = C\frac{d}{dt}\left(v_i(t) - v_o(t)\right)$$

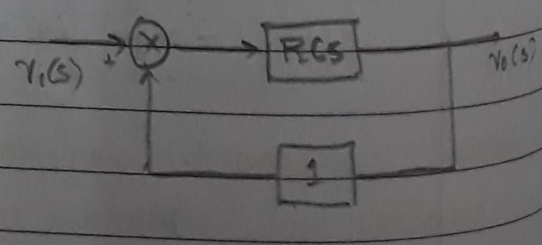
$$I(s) = Cs\left(v_i(s) - v_o(s)\right)$$



where $v_o(t) = i(t)R$

$$v_o(s) = I(s)R$$

$$v_o(s) = RCs\left(v_i(s) - v_o(s)\right)$$

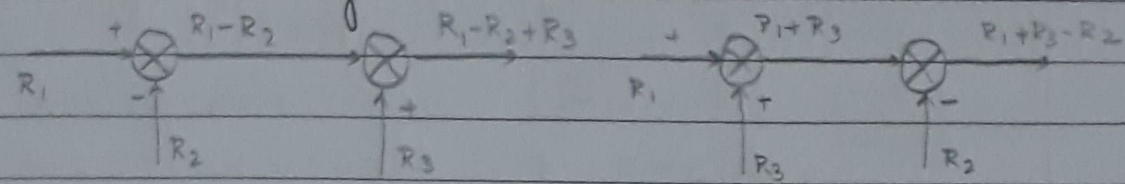


$$v_o\left[1 + RCs\right] = v_i(s)\,RCs$$

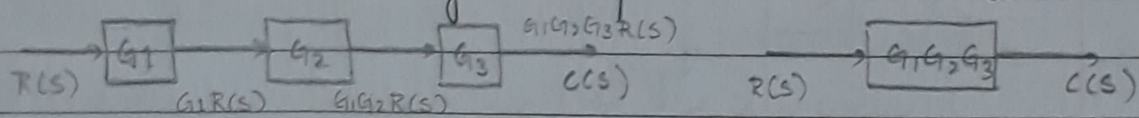
$$\frac{v_o(s)}{v_i(s)} = \frac{sRC}{1 + sRC}$$

★ Rules for block diagram reduction:

RULE 1: If there is no intermediate block between two summing points or if there is no takeoff point in between the summing points (ie summing points are connected directly to each other), then the associative law holds good.

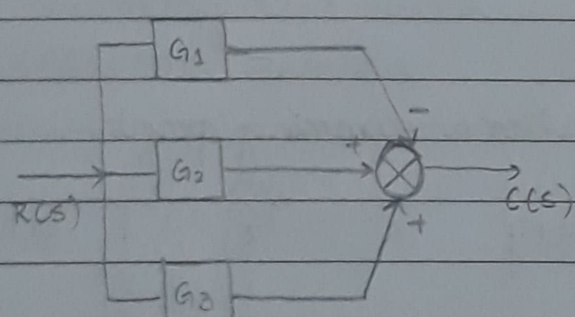


RULE 2: Blocks in series : The transfer functions of the blocks which are connected in series get multiplied with each other.

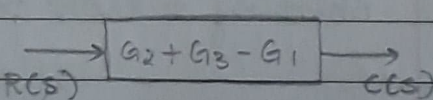


If there is a takeoff or a summing point in between the blocks, the blocks cannot be said to be in series.

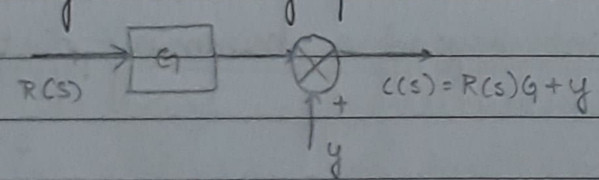
RULE 3: Blocks in parallel: The transfer functions of the blocks which are connected in parallel get added algebraically.



The direction of signal must be in the same direction.



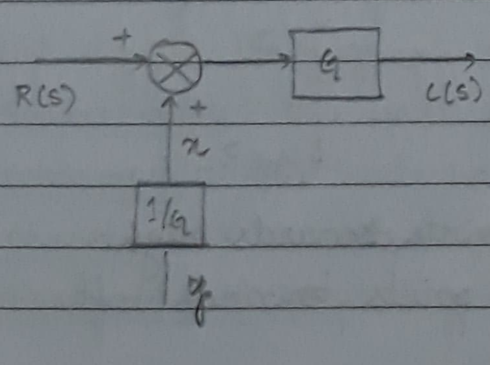
RULE 4: Shifting a summing point behind the block :



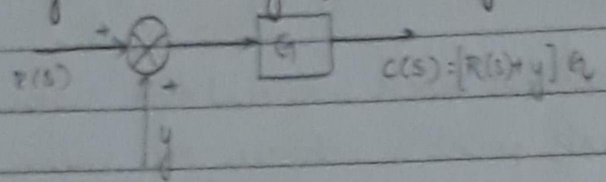
$C(s) = [R(s) + x]\,G$

$R(s)G + xG = R(s)G + y$

$\therefore\ xG = y \Rightarrow x = \dfrac{y}{G}$



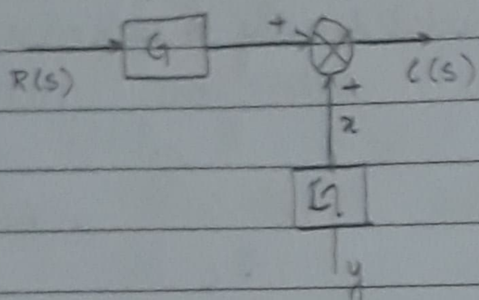
RULE 5: shifting a summing point beyond the block:



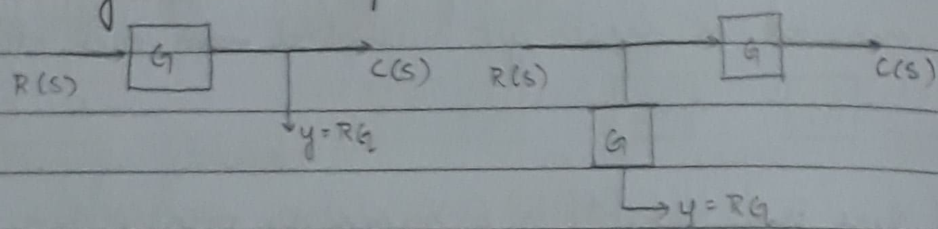
$R(s)G + x = C(s)$

$R(s)G + x = R(s)G + yG$

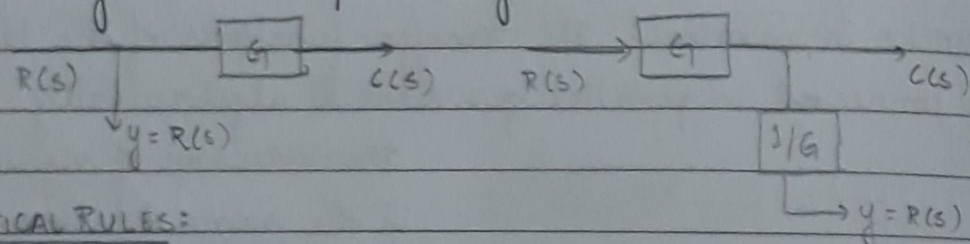
$\therefore x = yG$



RULE 6: Shifting a take off point behind the block

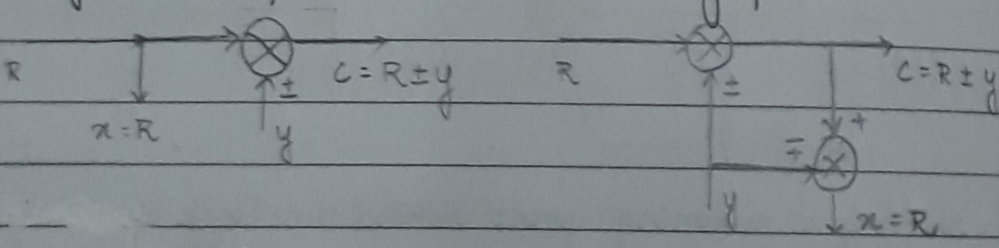


RULE 7: Shifting a takeoff point beyond the block

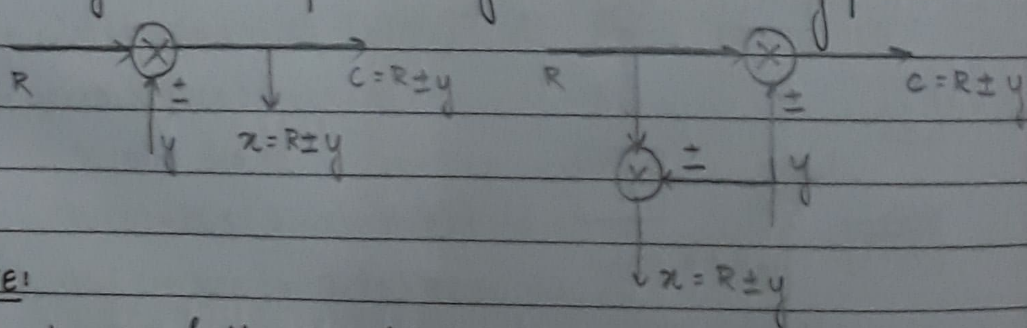


CRITICAL RULES:

RULE 8: Shifting takeoff point after a summing point.



RULE 9: Shifting take off point beyond a summing point



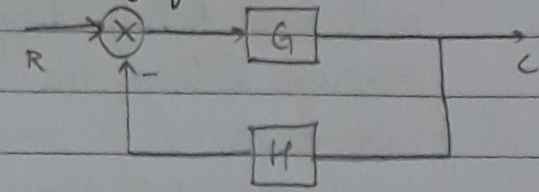
NOTE:

- try to shift takeoff points towards right
- try to shift summing points towards left.

Converting non unity feedback to unity feedback

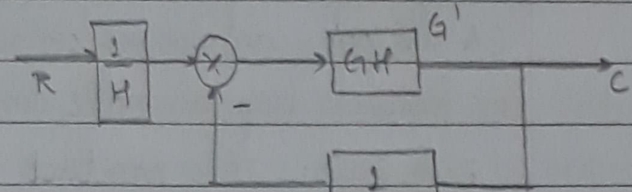
For non unity feedback

$$\frac{C}{R} = \frac{G}{1+GH}$$



For unity feed back

$$\frac{C}{R} = \frac{G'}{1+G'}$$



because $H = 1$

then $1 + GH = 1 + G'$

$$G' = GH$$

$$\frac{C}{R} = \frac{GH}{1+GH}$$

$$\left[\frac{R}{H} - C\right] GH = C$$

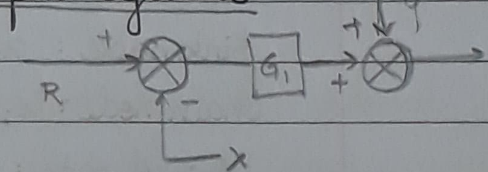
$$RG - CGH = C$$

$$C[1+GH] = RG \Rightarrow \frac{C}{R} = \frac{G}{1+GH}$$

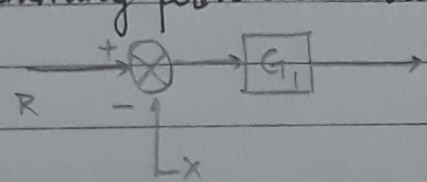
To have the same transfer function $1/H$ is introduced.

Analysis of Multiple Input Multiple Output systems:

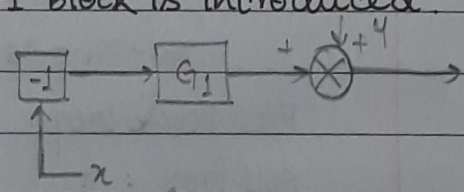
Here we consider each input separately, and assume the other inputs as zero.



Considering R(s), Y(s) is assumed zero, hence the summing point can be removed.



Considering Y(s), R(s) is assumed zero, here x is negative hence a −1 block is introduced.



* Signal flow graph representation:

The graphical representation of the variables of a set of linear algebraic equations representing the system is called signal flow graph representation.

Source node: The node having only outgoing branches

Sink node: The node having only incoming branches

Chain node: A node having incoming and outgoing branches

Forward Path: A path from input to output node.

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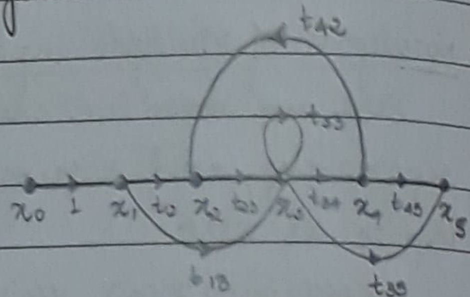
Feedback Loop: A path which originates from a particular node and terminating at the same node, travelling through atleast one other node without tracing any node twice.

Self loop: A feedback loop consisting of only one node (A self loop can not be considered while defining a forward path or feedback loop as node containing it gets traced twice.)

Path Gain: The product of branch gains while going through a forward path is known as path gain.

Non touching loops: If there is no node common in between the two or more loops are said to be non touching loops

Loop gain: The product of all the gains of the branches forming a loop is called loop gain.



Source node : x_0

Sink node : x_5

Chain node : x_1, x_2, x_3, x_4

Forward path :

$x_0 - x_1 - x_2 - x_3 - x_4 - x_5$

$x_0 - x_1 - x_3 - x_5$

$x_0 - x_1 - x_3 - x_4 - x_5$

$x_0 - x_1 - x_2 - x_3 - x_5$

Feedback loop : $x_2 - x_3 - x_4 - x_2$

Self loop : x_3

* **Mason's Gain Formula:**

$$\text{Overall transfer Function} = \frac{\sum P_k \Delta_k}{\Delta}$$

k = Number of forward paths

P_k = Gain of kth forward path

Δ = 1 − [Σ all individual feedback loop gains (including self loops)] + [Σ gain product of all possible combination of two non touching loops] − [Σ three non touching loops] + ...

Δ_k = loop gains not touching the kth forward path.

UNIT - 03

Time Domain Analysis of control systems

The output variation during the time it takes to achieve its final value is called the transient response. The time required to achieve the final value is called transient period.

The response of the system as time approaches infinity from the time at which transient response completely dies out is called steady state response.

The transient and steady state behaviour of a system is together referred to as time response.

Transient Response : $C_t(t)$

Steady state Response: C_{ss}

Total time Response

$$c(t) = C_{ss} + C_t(t)$$

The difference between the desired output and the actual output of the system is called steady state error. (e_{ss})

★ **Standard test signals:**

1. Step function : Position Function

$$u(t) = \begin{cases} 1 & t \geq 0 \\ 0 & t < 0 \end{cases}$$

2. Ramp function: Velocity Function

$$r(t) = \begin{cases} t & t \geq 0 \\ 0 & t < 0 \end{cases} \qquad r(t) = t u(t)$$

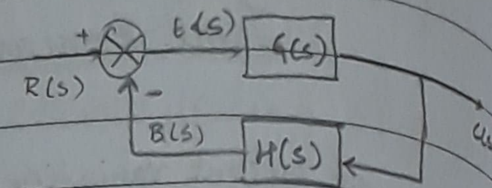
3. Parabolic function: Acceleration function

$$p(t) = \begin{cases} t^2/2 & t \geq 0 \\ 0 & t < 0 \end{cases}$$

4. Impulse function:

$$\delta(t) = \begin{cases} 1 & t = 0 \\ 0 & t \neq 0 \end{cases}$$

* <u>Steady state Error:</u>



$$E(s) = R(s) - B(s)$$

where $B(s) = H(s)C(s)$

$$E(s) = R(s) - C(s)H(s)$$

where $C(s) = E(s)G(s)$

$$E(s) = R(s) - E(s)G(s)H(s)$$

$$E(s) = \frac{R(s)}{1+G(s)H(s)}$$

E(s): error signal

B(s): feedback signal

In time domain error is $e(t)$

Therefore

steady state error

$$e_{ss} = \lim_{t \to \infty} e(t)$$

by final value theorem

wkt $\lim_{t\to\infty} F(t) = \lim_{s \to 0} sF(s)$

Therefore

$$e_{ss} = \lim_{s\to 0} sE(s)$$

$$e_{ss} = \lim_{s \to 0} \frac{sR(s)}{1+G(s)H(s)}$$

* <u>Type of a system and order of a system</u>

Type 0: $G(s)H(s) = \dfrac{k(1+T_1s)(1+T_2s)\cdots}{(1+T_as)(1+T_bs)\cdots}$

Type 1: $G(s)H(s) = \dfrac{k(1+T_1s)(1+T_2s)}{s(1+T_as)(1+T_bs)\cdots}$

Type 2: $G(s)H(s) = \dfrac{k(1+T_1s)(1+T_2s)}{s^2(1+T_as)(1+T_bs)\cdots}$

⋮

Type n: $G(s)H(s) = \dfrac{k(1+T_1s)(1+T_2s)\cdots}{s^n(1+T_as)(1+T_bs)\cdots}$

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Order of the system

The highest power of denominator or the number of poles of a system.

NOTE:

TYPE is the property of open loop transfer function $G(s)H(s)$ while ORDER is the property of closed loop transfer function $\frac{G(s)}{1 \pm G(s)H(s)}$

* **Error Constants:**

- Static error constants:

a. Reference input is step function (position function)

$$k_p = \lim_{s \to 0} G(s)H(s) \qquad e_{ss} = \frac{A}{1 + k_p}$$

positional error coefficient

b. Reference input is ramp function (velocity function)

$$k_v = \lim_{s \to 0} sG(s)H(s) \qquad e_{ss} = \frac{A}{k_v}$$

c. Reference input is parabolic function (acceleration function)

$$k_a = \lim_{s \to 0} s^2 G(s)H(s) \qquad e_{ss} = \frac{A}{k_a}$$

- Dynamic error constants: (Generalized error coefficients)

$$k_n = \lim_{s \to 0} \frac{d^n}{ds^n} \left[\frac{1}{1 + G(s)H(s)} \right]$$

$$e_{ss} = k_0 r(t) + k_1 r'(t) + \frac{k_2}{2!} r''(t) + \cdots$$

* **Analysis of First Order System:**

The transfer function of first order system is

$$G(s) = \frac{C(s)}{R(s)} = \frac{1}{1 + \tau s}$$

- Unit Step Response of First Order system

$$r(t) = u(t) \qquad R(s) = 1/s$$

$$C(s) = \frac{1}{s(1+\tau s)} = \frac{1}{s} - \frac{\tau}{1+\tau s} = \frac{1}{s} - \frac{1}{s + 1/\tau}$$

$$c(t) = 1 - e^{-t/\tau}$$

$C_{ss} = 1$; $c_t(t) = -e^{-t/\tau}$

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error response $e(t) = r(t) - c(t) = e^{-t/\tau}$

steady state error $e_{ss} = \lim_{t\to\infty} e(t) = \lim_{t\to\infty} e^{-t/\tau} = 0$

- Unit ramp response of First order system

$r(t) = t\,u(t) \quad R(s) = 1/s^2$

$$C(s) = \frac{1}{s^2(1+\tau s)} = \frac{A}{s^2} + \frac{B}{s} + \frac{C}{1+\tau s}$$

$1 = A(1+\tau s) + Bs(1+\tau s) + Cs^2$

at $s=0$: $A = 1$

at $s=-1/\tau$: $C = \tau^2$

at $s=1$: $1 = 1+\tau + B(1+\tau) + \tau^2$

$$B = \frac{-\tau(\tau+1)}{(1+\tau)} = -\tau$$

$$C(s) = \frac{1}{s^2} - \frac{\tau}{s} + \frac{\tau^2}{1+\tau s} = \frac{1}{s^2} - \frac{\tau}{s} + \tau\frac{1}{1/\tau + s}$$

$c(t) = t - \tau + \tau e^{-t/\tau}$

error response $e(t) = r(t) - c(t) = \tau - \tau e^{-t/\tau}$

steady state error $e_{ss} = \lim_{t\to\infty} e(t) = \lim_{t\to\infty} (\tau - \tau e^{-t/\tau}) = \tau$

- Unit impulse response of First Order system

$r(t) = \delta(t) \quad R(s) = 1$

$$C(s) = \frac{1}{1+\tau s} = \frac{1/\tau}{s + 1/\tau}$$

$$c(t) = \frac{1}{\tau} e^{-t/\tau}$$

error response $e(t) = r(t) - c(t) = 1 - \frac{1}{\tau} e^{-t/\tau} u(t)$

steady state error $e_{ss} = \lim_{t\to\infty} e(t) = \lim_{t\to\infty} \left[1 - \left(\frac{1}{\tau} e^{-t/\tau} \mu(t)\right)\right] = 1$

★ **Analysis of second order system**

The standard transfer function of second order system is

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2+2\xi\omega_n s+\omega_n^2}$$

The numerator need not be ω_n^2 always. It may be other constant or polynomial of s but the denominator will always have the coefficient $2\xi\omega_n$ and ω_n^2.

Ex: $\frac{C(s)}{R(s)} = \frac{10}{s^2+3s+12}$ then $2\xi\omega_n = 3$; $\omega_n^2 = 12$

– **Effect of ξ on second order system performance:**

considering input is an unit step function

$r(t) = u(t)$ $R(s) = 1/s$

wkt $\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2+2\xi\omega_n s+\omega_n^2}$

$\therefore C(s) = \frac{\omega_n^2}{s(s^2+2\xi\omega_n s+\omega_n^2)}$

For $s^2+2\xi\omega_n s+\omega_n^2$

$$s = \frac{-2\xi\omega_n \pm \sqrt{4\xi^2\omega_n^2 - 4\omega_n^2}}{2} = -\xi\omega_n \pm \omega_n\sqrt{\xi^2-1}$$

Therefore $C(s) = \frac{\omega_n^2}{s(s+\xi\omega_n - \omega_n\sqrt{\xi^2+1})(s+\xi\omega_n+\omega_n\sqrt{\xi^2-1})}$

CASE 1 $1<\xi<\infty$: Over damped.

$$C(s) = \frac{A}{s} + \frac{B}{s+\xi\omega_n-\omega_n\sqrt{\xi^2+1}} + \frac{C}{s+\xi\omega_n+\omega_n\sqrt{\xi^2-1}}$$

$$c(t) = A + Be^{-(\xi\omega_n-\omega_n\sqrt{\xi^2+1})t} + Ce^{-(\xi\omega_n+\omega_n\sqrt{\xi^2+1})t}$$

where $C_{ss} = A$

CASE 2: $\xi = 1$: critically damped.

roots : $-\omega_n, -\omega_n$

$$C(s) = \frac{\omega_n^2}{s(s+\omega_n)(s+\omega_n)} = \frac{A}{s} + \frac{B}{(s+\omega_n)^2} + \frac{C}{(s+\omega_n)}$$

$$c(t) = A + Bte^{-\omega_n t} + Ce^{-\omega_n t}$$

where $C_{ss} = A$

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CASE 3: $0 < \xi < 1$: Under damped

since $\xi < 1$ $\sqrt{\xi^2 - 1} \rightarrow \sqrt{1-\xi^2}$

roots: $-\xi\omega_n \pm j\omega_n\sqrt{1-\xi^2}$

$$C(s) = \frac{\omega_n^2}{s(s+\xi\omega_n - j\omega_n\sqrt{1-\xi^2})(s+\xi\omega_n + j\omega_n\sqrt{1-\xi^2})}$$

$$C(s) = \frac{\omega_n^2}{s(s^2+2\xi\omega_n s + \xi^2\omega_n^2 + \omega_n^2(1-\xi^2))} = \frac{\omega_n^2}{s(s^2+2\xi\omega_n s+\omega_n^2)}$$

$$C(s) = \frac{A}{s} + \frac{Bs+C}{s^2+2\xi\omega_n s+\omega_n^2}$$

$$c(t) = A + k' e^{-\xi\omega_n t} \sin(\omega_n\sqrt{1-\xi^2}\, t + \theta)$$

where $C_{ss} = A$

$\omega_n\sqrt{1-\xi^2} = \omega_d$: damped frequency of oscillations

CASE 4: $\xi = 0$: Undamped

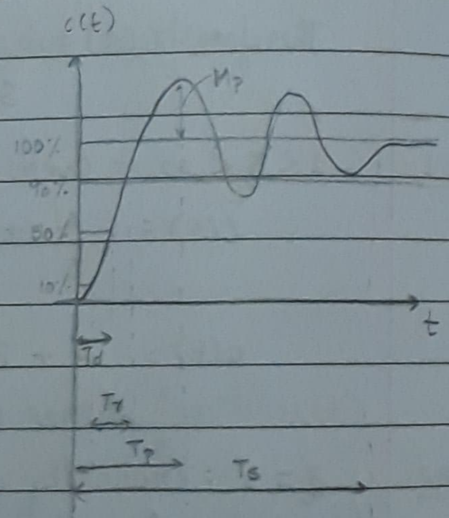
$$C(s) = \frac{\omega_n^2}{s(s^2+\omega_n^2)} = \frac{A}{s} + \frac{Bs+C}{s^2+\omega_n^2}$$

$$c(t) = A + k'' \sin(\omega_n t + \theta)$$

where $C_{ss} = A$

---

* **Time Domain Specifications:**



- **Delay time : T_d**

It is the time required for the response to reach 50% of the final value in the first attempt.

$$T_d = \frac{1+0.7\xi}{\omega_n}$$

- **Rise Time : T_r**

It is the time required for the response to rise from 10% to 90% of the final value for overdamped systems and 0 to 100% of the final value of underdamped systems.

$$T_r = \frac{\pi - \theta}{\omega_d} = \frac{\pi - \theta}{\omega_n\sqrt{1-\xi^2}}$$

- Peak Time : T_p

It is the time required for the response to reach its peak value. It is also defined as the time at which response undergoes the first over shoot which is always peak overshoot.

$$T_p = \frac{\pi}{\omega_d} = \frac{\pi}{\omega_n\sqrt{1-\xi^2}}$$

- Peak Overshoot : M_p

It is the largest error between reference input and output during the transient period.

$$M_p = c(t)\Big|_{t=T_p} - 1$$ — unit step input

$$\% M_p = e^{-\pi\xi/\sqrt{1-\xi^2}} \times 100$$

- settling Time : T_s

It is defined as the time required for the response to decrease and stay within specified percentage of its final value.

Time constant of system $= T = \frac{1}{\xi\omega_n}$

T: time required by the system output to reach 63.2% of its final value during the first attempt.

$$T_s = 4 \times \text{time constant}$$

$$T_s = \frac{4}{\xi\omega_n}$$

Peak time T_p:

Transient response of second order underdamped system

$$c(t) = 1 - \frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}} \sin(\omega_d t + \theta)$$

where $\theta = \tan^{-1}\frac{\sqrt{1-\xi^2}}{\xi}$

$c(t)$ reaches its maxima at $t = T_p$

$$\frac{d\,c(t)}{dt}\Big|_{t=T_p} = 0$$

$$\frac{d}{dt}c(t) = \frac{\xi\omega_n e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}} \sin(\omega_d t + \theta) - \frac{e^{-\xi\omega_n t}}{\sqrt{1-\xi^2}}\omega_d \cos(\omega_d t + \theta) = 0$$

$$\xi\omega_n \sin(\omega_d t + \theta) - \omega_n\sqrt{1-\xi^2}\cos(\omega_d t + \theta) = 0$$

$$\frac{\sin(\omega_d t+\theta)}{\cos(\omega_d t+\theta)} = \frac{\sqrt{1-\zeta^2}}{\zeta} \qquad \text{because } \theta = \tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta}$$

$$\tan(\omega_d t+\theta) = \tan\theta$$

$$\Rightarrow \omega_d t = n\pi$$

For first peak overshoot $n=1$

$$\omega_d T_p = \pi$$

$$T_p = \frac{\pi}{\omega_d} = \frac{\pi}{\omega_n\sqrt{1-\zeta^2}}$$

<u>Peak Overshoot</u>: M_p

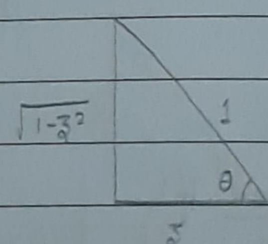
$$M_p = c(t)\Big|_{t=T_p} - 1$$

$$M_p = \left[1 - \frac{e^{-\zeta\omega_n T_p}}{\sqrt{1-\zeta^2}}\sin(\omega_d T_p+\theta)\right] - 1$$

$$M_p = -\frac{e^{-\zeta\omega_n T_p}}{\sqrt{1-\zeta^2}}\sin(\omega_d T_p+\theta)$$

but $T_p = \dfrac{\pi}{\omega_d}$

$$M_p = \frac{-e^{-\zeta\omega_n T_p}}{\sqrt{1-\zeta^2}}\sin(\pi+\theta)$$



$$M_p = \frac{e^{-\zeta\omega_n T_p}}{\sqrt{1-\zeta^2}}\sin\theta$$

$$M_p = \frac{e^{-\zeta\omega_n T_p}}{\cancel{\sqrt{1-\zeta^2}}}\cancel{\sqrt{1-\zeta^2}}$$

$$\theta = \tan^{-1}\frac{\sqrt{1-\zeta^2}}{\zeta}$$

$$M_p = e^{-\zeta\omega_n T_p}$$

$$M_p = e^{-\zeta\pi/\sqrt{1-\zeta^2}}$$

$$\% M_p = 100\, e^{-\pi\zeta/\sqrt{1-\zeta^2}}$$

<u>Rise Time</u>: T_r

$c(t)\Big|_{t=T_r} = 1$ for unit step input

$$1 = 1 - \frac{e^{-\zeta\omega_n T_r}}{\sqrt{1-\zeta^2}}\sin(\omega_d T_r+\theta)$$

$$\frac{e^{-\xi\omega_n T_r}}{\sqrt{1-\xi^2}} \sin(\omega_d T_r + \theta) = 0$$

$$\sin(\omega_d T_r + \theta) = 0$$

$$\therefore \omega_d T_r + \theta = n\pi$$

For first attempt $n = 1$

$$\omega_d T_r + \theta = \pi$$

$$T_r = \frac{\pi - \theta}{\omega_d}$$

<u>settling time</u>

The time required by the output to settle down within $\pm 2\%$ of tolerance band

$c(t)$ at $t = T_s = 0.98$

at $t = T_s$ the transient oscillatory term vanishes

$$c(t) \text{ at } t = T_s = 1 - e^{-\xi\omega_n T_s}$$

$$0.98 = 1 - e^{-\xi\omega_n T_s}$$

$$e^{-\xi\omega_n T_s} = 0.02$$

$$-\xi\omega_n T_s = -3.912$$

$$T_s = \frac{4}{\xi\omega_n} \quad \text{for } \pm 2\% \text{ tolerance.}$$

UNIT - 04

Stability Analysis of Control Systems

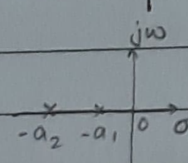
The analysis of whether the given system can reach steady state passing through the transients successfully is called stability Analysis of the system.

* **Stability of control systems:**

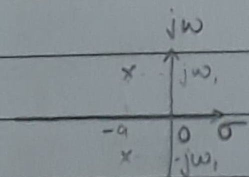
If closed loop poles are located in left half s-plane, exponential indices of the output are negative hence the exponential transient terms will vanish when $t \to \infty$. Hence this makes the system absolutely stable.

If any of the closed loop poles lie in right half of s-plane, it gives the exponential term of positive index thus the transient responce of increasing amplitude, making the system unstable.

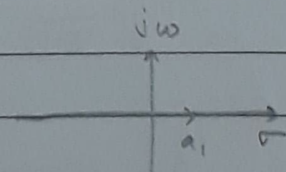
A linear time invariant system is said to be critically or marginally stable if output oscillates with constant frequency and amplitude.



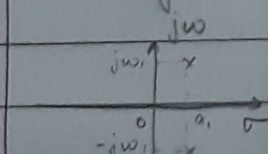
absolutely stable



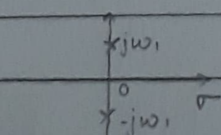
absolutely stable



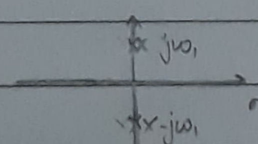
unstable



unstable



marginally or critically stable



unstable (repeated roots on imaginary axis)

* **BIBO stability**: (Bounded Input Bounded Output Stability)

→ A LTI system is said to be stable if:
  - if a bounded input to the system produces a bounded output and controllable.
  - in the absence of the input, output must tend to zero irrespective of the initial conditions.

→ A LTI system is said to be unstable if:
  - a bounded input produces an unbounded output.
  - in absence of the input, output may not return to zero. It shows certain output without input.

* **Zero input and Asymptotic Stability**:

The stability related to a system which is under zero input condition but operated under initial condition is called as zero input stability. and zero input response is the response of the system only due to the initial conditions.

If the zero input response of the system subjected to the finite initial conditions, reaches to zero as time t approaches infinity, the system is said to be zero input stable otherwise it is zero input unstable.

$c(t)$: zero input response of the system.

$$|c(t)| \leq M < \infty \quad \text{for all } t \geq t_0$$

$$\lim_{t \to \infty} |c(t)| = 0$$

As magnitude of zero input response reaches zero as $t \to \infty$, the zero input stability is also called asymptotic stability.

- Zero input or asymptotic stability depends on the poles of the system i.e., roots of the characteristic equation.
- All the roots of the characteristic equation must be located in left half of s plane for zero input or asymptotic stability.
- If a system is BIBO stable, then it must be zero input or asymptotic stable.

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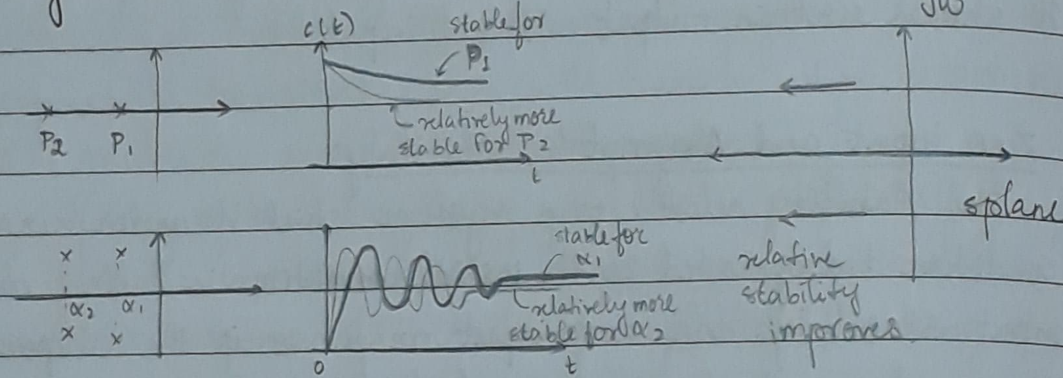
★ Relative Stability:

system is said to be relatively more stable if settling time for that system is less than that of the other system.

The roots located near the $j\omega$ axis settling time will be large. Thus the roots or pair of complex conjugate roots more away from $j\omega$ axis ie towards left half of s-plane, settling time becomes lesser and system becomes more and more stable.



★ Routh - Hurwitz Criterion:

$$\frac{C(s)}{R(s)} = \frac{b_0 s^m + b_1 s^{m-1} + \dots + b_m}{a_0 s^n + a_1 s^{n-1} + \dots + a_n}$$

characteristic equation

$$a_0 s^n + a_1 s^{n-1} + a_2 s^{n-2} + \dots + a_n = 0$$

Necessary conditions (but not sufficient)

- all the coefficients must have same sign.
- None of the cofficient vanishes (all the powers of s must be present).

- Hurwitz's Criterion:

The necessary and sufficient condition to have all roots of characteristic equation in left half splane is that the subdeterminants D_k, $k = 1, 2 \dots n$ obtained from Hurwitz's determinant H must be positive.

Hurwitz determinant :

$$H = \begin{vmatrix} a_1 & a_3 & a_5 & \cdots & a_{2n-1} \\ a_0 & a_2 & a_4 & \cdots & a_{2n-2} \\ 0 & a_1 & a_3 & \cdots & a_{2n-3} \\ 0 & a_0 & a_2 & \cdots & a_{2n-4} \\ 0 & 0 & a_1 & \cdots & a_{2n-5} \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 0 & \cdot & \cdot & \cdots & a_n \end{vmatrix}$$

where $D_1 = |a_1|$

$$D_2 = \begin{vmatrix} a_1 & a_3 \\ a_0 & a_2 \end{vmatrix}$$

$$D_3 = \begin{vmatrix} a_1 & a_3 & a_5 \\ a_0 & a_2 & a_4 \\ 0 & a_1 & a_3 \end{vmatrix}$$

For the system to be stable, all the determinants must be positive

Q: Determine the stability of the given characteristic equation by Hurwitz's method. $s^3 + s^2 + s^1 + 4 = 0$.

$$H = \begin{vmatrix} a_1 & a_3 & a_5 \\ a_0 & a_2 & a_4 \\ 0 & a_1 & a_3 \end{vmatrix} = \begin{vmatrix} 1 & 4 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 4 \end{vmatrix}$$

$D_1 = |a_1| = |1| = 1$

$$D_2 = \begin{vmatrix} a_1 & a_3 \\ a_0 & a_2 \end{vmatrix} = \begin{vmatrix} 1 & 4 \\ 1 & 1 \end{vmatrix} = 1 - 4 = -3$$

Hence the system is unstable

In this method it is difficult for higher order systems and the number of roots in the right half of splane cannot be determined.

- Routh-Hurwitz's method / Routh's array method:

s^n	a_0	a_2	a_4	a_6	....
s^{n-1}	a_1	a_3	a_5	a_7	
s^{n-2}	b_1	b_2	b_3		
s^{n-3}	c_1	c_2	c_3		
$\vdots$	$\vdots$	$\vdots$	$\vdots$		
s^0	a_n				

where

$$b_1 = \frac{a_1 a_2 - a_0 a_3}{a_1}$$

$$b_2 = \frac{a_1 a_4 - a_0 a_5}{a_1}$$

$$b_3 = \frac{a_1 a_6 - a_0 a_7}{a_1}$$

$$c_1 = \frac{b_1 a_3 - b_2 a_1}{b_1} \qquad c_2 = \frac{b_1 a_5 - b_3 a_1}{b_1}$$

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This process is continued till the coefficient for s^0 is equal to a_n.

All the terms in the first column of Routh's array must be of same sign. There should be no change in sign in the first column of Routh's array.

If there are any sign changes then

a. system is unstable

b. number of sign changes equal to number of roots lying in the right half of the s-plane.

Q: Examine the stability of given equations using Routh's method:

a. $s^3 + 6s^2 + 11s + 6 = 0$

$$\begin{array}{c|cc} s^3 & a_0 & a_2 \\ s^2 & a_1 & a_3 \\ s^1 & b_1 & b_2 \\ s^0 & c_1 & \end{array} = \begin{array}{c|cc} s^3 & 1 & 11 \\ s^2 & 6 & 6 \\ s^1 & 10 & 0 \\ s^0 & 6 & \end{array}$$

no sign change in first column, hence the system is stable

b. $s^3 + 4s^2 + s + 16 = 0$

$$\begin{array}{c|cc} s^3 & a_0 & a_2 \\ s^2 & a_1 & a_3 \\ s^1 & b_1 & b_2 \\ s^0 & c_1 & \end{array} = \begin{array}{c|cc} s^3 & 1 & 1 \\ s^2 & 4 & 16 \\ s^1 & -3 & 0 \\ s^0 & 16 & \end{array}$$

there are two sign changes, thus two roots are present in the right half of s-plane. Hence the system is unstable.

- Special Cases of Routh's Criterion:

CASE 1: First element of any of the rows of Routh's array is zero and the same remaining row contains atleast one non zero element. Then the terms in the new row becomes infinite and Routh's test fails.

Ex: $s^5 + 2s^4 + 3s^3 + 6s^2 + 2s + 1 = 0$

s^5	1	3	2
s^4	2	6	1
s^3	0	1.5	0
s^2	∞	...	....

First method: substitute a small positive number 'ε' in place of a zero occured as a first element in a row. Examine the sign change by $\lim_{\varepsilon \to 0}$

s^5	1	3	2
s^4	2	6	1
s^3	ε	1.5	0
s^2	$\frac{6\varepsilon - 3}{\varepsilon}$	1	
s^1	$\frac{\frac{9\varepsilon - 4.5}{\varepsilon} - \varepsilon}{\frac{6\varepsilon-3}{\varepsilon}}$	0	
s^0	1		

To examine sign change

$$\lim_{\varepsilon \to 0}\left(\frac{6\varepsilon - 3}{\varepsilon}\right) = 6 - \lim_{\varepsilon \to 0}\frac{3}{\varepsilon} = -\infty$$

$$\lim_{\varepsilon \to 0}\left(\frac{\frac{9\varepsilon - 4.5}{\varepsilon} - \varepsilon}{\frac{6\varepsilon - 3}{\varepsilon}}\right) = \lim_{\varepsilon \to 0}\frac{9\varepsilon - 4.5 - \varepsilon^2}{6\varepsilon - 3} = 1.5$$

s^5	1	3	2
s^4	2	6	1
s^3	0	1.5	0
s^2	$-\infty$	1	0
s^1	1.5	0	
s^0	1		

There are two sign changes hence two roots in the right half of splane. Thus the system is unstable

second method: Replace s by $1/z$ in the characteristic equation.

$$s^5 + 2s^4 + 3s^3 + 6s^2 + 2s + 1 = 0$$

$$\frac{1}{z^5} + \frac{2}{z^4} + \frac{3}{z^3} + \frac{6}{z^2} + \frac{2}{z} + 1 = 0$$

$$1 + 2z + 3z^2 + 6z^3 + 2z^4 + z^5 = 0$$

$a_5 \quad a_4 \quad a_3 \quad a_2 \quad a_1 \quad a_0$

z^5	1	6	2
z^4	2	3	1
z^3	4.5	1.5	0
z^2	2.33	1	0
z^1	-0.43	0	
z^0	1		

There are two sign changes hence two roots are in the right half of s-plane. Thus the system is unstable

Date

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CASE 2: All the elements of a row in Routh's array are zero. Then the terms of next row cannot be determined and Routh's test fails.

i. For an equation by using the coefficients of a row which is just above the row of zeros.

s^5	a	b	c
s^4	d	e	F
s^3	0	0	0

Auxiliary equation

$$A(s) = ds^4 + es^2 + F$$

(d is the coefficent of s^4 so the first term is ds^4)

(alternate powers of s so the second term is es^2 and so on)

ii. Take the derivative of the auxiliary equation wrt s

$$\frac{d\,A(s)}{ds} = 4ds^3 + 2es$$

iii. Replace row of zeros by the coefficients of $dA(s)/ds$

iv. complete the array in terms of these coefficients.

s^5	a	b	c
s^4	d	e	f
s^3	4d	2e	0

As row of zeros occur, the system may be marginally stable or unstable. To examine that find the roots of the auxiliary equation when there is no sign change.

If there are repeated roots on imaginary axis the system is unstable.

Unit 4

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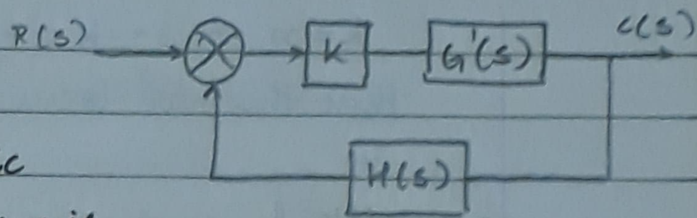
* **Root Locus:**

Root locus method is a graphical method in which movement of the poles in the s-plane is sketched when a parameter of the system is varied from zero to infinity

- **Basic concepts of Root Locus:**

Let k be assumed to be a variable parameter.



In general, the characteristic equation of a closed loop system is

$$1 + G(s)H(s) = 0$$

For the system shown

$$G(s) = kG'(s)$$

then $\quad 1 + kG'(s)H(s) = 0$

"Now the closed loop poles i.e., the roots of the above equation are now dependent on the value of k."

Now if k is varied from $-\infty$ to ∞ then for each separate value of k we will get separate set of locations of the roots of the characteristic equation. Thus the locus of the closed loop poles obtained when system gain k is varied from $-\infty$ to $+\infty$ is called Root Locus.

"When k is varied from 0 to $+\infty$ the plot is called direct root locus and when k is varied from $-\infty$ to 0, the plot obtained is called inverse root locus."

Q: consider an unity feedback system with $G(s) = K/s$. obtain its root locus.

- The characteristic equation

$$1 + G(s)H(s) = 0$$

but $H(s) = 1$ because unity feedback system.

Therefore $1 + G(s) = 0$

$$1 + k/s = 0$$

$$s + k = 0$$

The root of this equation is at $s = -k$ //

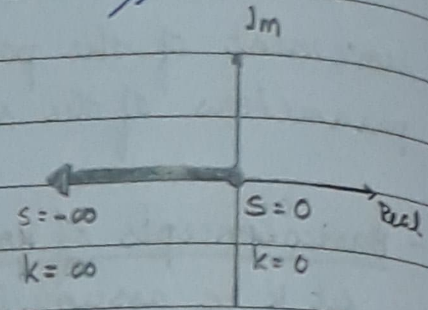
then

$k = 0 \Rightarrow s = 0$

$k = 10 \Rightarrow s = -k = -10$

⋮

$k = \infty \Rightarrow s = -k = -\infty$



Thus the root locus is the negative real axis.

- **Angle and Magnitude Condition:**

wkt the closed loop system characteristic equation is

$$1 + G(s)H(s) = 0$$

$$\Rightarrow G(s)H(s) = -1$$

As s-plane is complex

$$G(s)H(s) = -1 + 0j$$

Hence $G(s)H(s)$ is also a complex term.

Thus for any value of s if it has to be on the root locus it must satisfy the above equation.

Angle condition

$$G(s)H(s) = -1 + 0j$$

$-1 + 0j = 1 \angle \pm 180^\circ$ but the point $-1 + 0j$ is a point on negative real axis which can be traced as magnitude 1 at an angle $\pm 180^\circ, \pm 540^\circ, \pm 900^\circ$ i.e, odd multiple of 180°

$\Rightarrow \pm(2q+1) 180^\circ$

Therefore $\boxed{\angle G(s)H(s) = \pm (2q+1) 180^\circ}$ $q = 0, 1, 2 \ldots$

= Odd multiple of 180°

Thus for any point in s plane has to be on the root locus has to satisfy the above angle condition.

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Q: consider the system with $G(s)H(s) = \frac{k}{s(s+2)(s+4)}$. Find whether $s=-0.75$ is on the root locus or not using angle condition.

– For $s=-0.75$ to lie on the root locus, it has to be the root of the characteristic equation.

Using angle condition

$$\angle G(s)H(s)\Big|_{s=-0.75} = \pm(2q+1)180^\circ$$

$$\angle G(s)H(s)\Big|_{s=-0.75} = \frac{\angle k+j0}{[\angle -0.75+0j]\cdot[\angle 1.25+0j]\cdot[\angle 3.25+0j]}$$

converting to polar form and considering the angles

$$\angle G(s)H(s)\Big|_{s=-0.75} = \frac{0^\circ}{180^\circ \cdot 0^\circ \cdot 0^\circ} = -180^\circ$$

As it is an odd multiple of 180° i.e., $\pm(2q+1)180^\circ$ is satisfied it lies on the root locus.

Ex: For $s=-1+4j$

Using angle condition

$$\angle G(s)H(s)\Big|_{s=-1+4j} = \pm(2q+1)180^\circ$$

$$\angle G(s)H(s)\Big|_{s=-1+4j} = \frac{\angle k+0j}{\angle -1+4j \cdot \angle 1+4j \cdot \angle 3+4j}$$

converting to polar form and considering the angles

$$\angle G(s)H(s)\Big|_{s=-1+4j} = \frac{0^\circ}{104.03^\circ \cdot 75.96^\circ \cdot 53.13^\circ}$$

$$\angle G(s)H(s)\Big|_{s=-1+4j} = \frac{0^\circ}{233.12} = -233.13$$

Therefore the point $s=-1+4j$ cannot lie on the root locus as it did not satisfy the angle condition.

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Magnitude condition

wkt $G(s)H(s) = -1$

then $|G(s)H(s)| = |-1+0j| = 1$

K is unknown hence we cannot find $|G(s)H(s)|$ at any point in s plane. So this condition is not suitable to check the existance of a point on root locus but once we know that a point in s-plane lies on the root locus then it must ~~be~~ satisfy magnitude condition.

" so magnitude condition can be used only when a point in s-plane is confirmed for its existance on the root locus by using angle condition."

At the point which is known to be on the root locus by angle condition we can find the value of k by using magnitude condition.

$$\boxed{|G(s)H(s)|_{\substack{\text{at a point in s-plane} \\ \text{which is on root locus}}} = 1}$$

Q: From the previously consider system with $G(s)H(s) = \frac{k}{s(s+2)(s+4)}$ we know that $s = -0.75$ lies on the root locus. Use the magnitude condition to find the value of k.

— Using magnitude condition

$$|G(s)H(s)|_{s=-0.75} = 1$$

$$\frac{|k|}{|-0.75||1.25||3.25|} = 1$$

Therefore $k = 3.0468$

then $1 + G(s)H(s) = 0$

$$1 + \frac{k}{s(s+2)(s+4)} = 0$$

$$\Rightarrow s^3 + 6s^2 + 8s + k = 0$$

but wkt $s = -0.75$ is one of the three roots. Hence as we know the value of $k = 3.0468$ we can find the other two roots using

- Construction of Root Locus

(number of branches is equal to number of open loop poles)

- RULE 1:

"The root locus is always symmetrical about the real axis". This is because the roots of the characteristic equation are either real or complex conjugates or combination of both.

- RULE 2:

"As K increases from zero to infinity, each branch of the root locus originates from on open loop pole with $K=0$ and terminates either on an open loop zero or on infinity with $K=\infty$. The number of branches terminating on infinity equals the number of open loop poles minus zeros."

with M = number of open loop zeros
N = number of open loop poles

CASE 1: when $M>N$

N branches will start from each of the finite open loop pole locations and terminate at the finite location of open loop zeros. Remaining $M-N$ branches will originate from infinity and will approach to finite zeros.

CASE 2: When $M<N$

Out of N branches, M branches will start from each of the finite open loop pole locations and terminate at the finite location of open loop zeros. Remaining $N-M$ branches will originate from finite poles and will terminate at infinity.

CASE 3: When $M=N$

All the branches start from finite open loop pole locations and terminate at the finite location of open loop zeros. No branch will start or terminate at infinity.

Therefore branch direction always remains from open loop poles towards open loop zeros.

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proof

the characteristic equation in pole zero form

$$1+G(s)H(s) = 1 + k\frac{\prod_{i=1}^{m}(s+z_i)}{\prod_{j=1}^{n}(s+p_j)} = 0$$

$$k\frac{\prod_{i=1}^{m}(s+z_i)}{\prod_{j=1}^{n}(s+p_j)} = -1$$

$$k\prod_{i=1}^{m}(s+z_i) + \prod_{j=1}^{n}(s+p_j) = 0 \quad \text{— (1)}$$

$$\prod_{i=1}^{m}(s+z_i) + \frac{1}{k}\prod_{j=1}^{n}(s+p_j) = 0 \quad \text{— (2)}$$

When $k=0$, from eq 1 we get

$\prod_{j=1}^{n}(s+p_j) = 0$ For this equation the roots are the open loop poles. Hence we can conclude that the root locus branches always originate at poles.

When $k \to \infty$, from eq 2 we get

$\prod_{i=1}^{m}(s+z_i) = 0$ For this equation the roots are the open loop zeros. Hence we can conclude that the root locus branches terminate at zeros

$$\frac{\prod_{i=1}^{m}(s+z_i)}{\prod_{j=1}^{n}(s+p_j)} = \frac{-1}{k}$$

as $k \to \infty$

$$\frac{\prod_{i=1}^{m}(s+z_i)}{\prod_{j=1}^{n}(s+p_j)} = 0$$

as $s \to \infty$

$$\lim_{s\to\infty}\frac{s^m}{s^n} = 0 \Rightarrow \lim_{s\to\infty}\frac{1}{s^{n-m}} = 0$$

Thus $n-m$ branches terminate at infinity.

- **RULE 3:**

  "A point on the real axis lies on the locus if the sum of the number of open loop poles and the open loop zeros on the real axis to the right hand side of this point is odd."

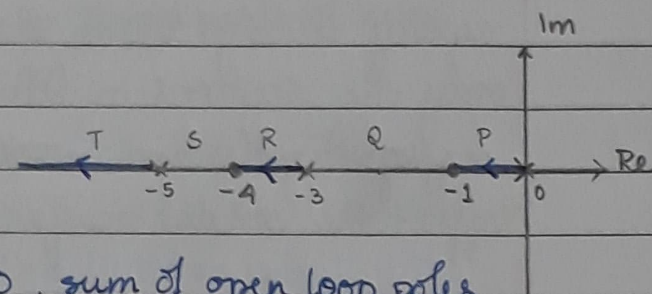
Q: For given $G(s)H(s) = \dfrac{k(s+1)(s+4)}{s(s+3)(s+5)}$ by using the rule 3 find on which sections of the real axis the root locus exists.

→ The open loop poles are

$s = 0, -3, -5$

The open loop zeros are

$s = -1, -4$



Any point P between -1 to 0, sum of open loop poles and zeroes is 1 (odd). Hence it is a root locus

Any point Q between -3 to -1, sum of open loop poles and zeroes is 2 (even). Hence it is not a root locus.

Any point R between -4 to -3, sum of open loop poles and zeros is 3 (odd). Hence it is a root locus.

Any point S between -5 to -4, sum of open loop poles and zeros is 4 (even). Hence it is not a root locus.

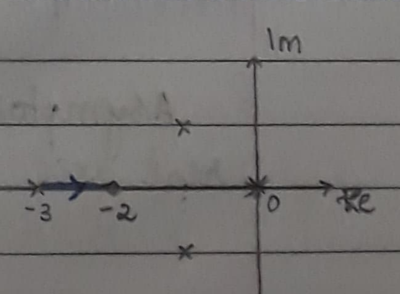
Any point T beyond -5, sum of open loop poles and zeroes is 5 (odd). Hence it is a root locus

Q: For $G(s)H(s) = \dfrac{k(s+2)}{s^2(s^2+2s+2)(s+3)}$ Find the sections of the real axis which is a root locus

→ Open loop zero : -2

Open loop pole : $0, 0, -1 \pm j, -3$

There are two poles at origin.



Any point on the positive real axis there is no poles or zeros to the right

Hence the number of poles and zeros is zero (even)

Hence it is not a root locus.

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Between -2 to 0 the right hand side sum of number of poles and zeros is 2 (even). Hence not a root locus.

Between -3 to -2 the right hand sum of number of open loop poles and zeros is 3 (odd). Hence it is a root locus.

The section to the left of -3, the right hand side sum of number of open loop poles and zeros is 4 (even). Hence not a root locus.

" This rule should not be applied to actual zeros and poles as they always lie on the root locus. It has to be applied only the sections in between them. Similarly complex conjugate poles and zeroes occur in pair hence they do not affect the condition of sum thus they are not considered".

- RULE 4: (angle of asymptotes)

The n-m branches of the root locus which tend to infinity do so along straight line asymptotes whose angles are given by

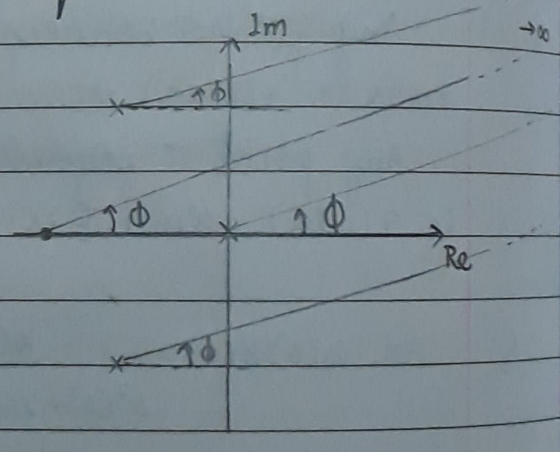
$$\boxed{\phi_A = \frac{(2q+1)180^\circ}{n-m}} \; ; \quad q = 0, 1, 2, 3 \cdots (n-m-1)$$

proof

By angle criteria

$\angle G(s)H(s) = \pm(2q+1)180^\circ$

n-m branches terminate at infinity



$$-(n-m)\phi = \pm(2q+1)180^\circ$$

$$\phi = \frac{\pm(2q+1)180^\circ}{n-m}$$

Asymptotes are always symmetrically located about the real axis.

- **RULE 5:** (location of asymptotes)

"The asymptotes cross the real axis at a point known as centroid, determined by the relationship: [sum of real parts of poles − sum of real parts of zeros] / [number of poles − number of zeros]"

" centroid is always real, it may be located on negative or positive real axis. It may or may not be the part of the root locus."

$$-\sigma_A = \frac{\Sigma \text{real parts of poles} - \Sigma \text{real parts of zeros}}{n-m}$$

Proof:

The open loop transfer function can be written as:

$$G(s)H(s) = k\frac{(s+z_1)(s+z_2)\cdots(s+z_m)}{(s+P_1)(s+P_2)\cdots(s+P_n)} \quad ; \quad m \le n$$

$$G(s)H(s) = k\frac{\left[s^m + \left(\sum_{i=1}^{m} z_i\right)s^{m-1} + \cdots + \left(\prod_{i=1}^{m} z_i\right)\right]}{\left[s^n + \left(\sum_{j=1}^{m} P_j\right)s^{n-1} + \cdots + \left(\prod_{j=1}^{m} P_j\right)\right]}$$

Dividing the denominator by the numerator, we get

$$G(s)H(s) = \frac{k}{\left[s^{n-m} + \left(\sum_{j=1}^{n} P_j - \sum_{i=1}^{m} z_i\right)s^{n-m-1} + \cdots\right]}$$

As s tends to infinity the terms with higher powers of s will dominate.

$$G(s)H(s)\Big|_{s\to\infty} \approx \frac{k}{\left[s^{n-m} + \left(\sum_{j=1}^{n} P_j - \sum_{i=1}^{m} z_i\right)s^{n-m-1}\right]}$$

considering

$$P(s) = \frac{1}{(s+\sigma_A)^{n-m}} = \frac{1}{\left[s^{n-m} + (n-m)\sigma_A s^{n-m-1} + \cdots\right]}$$

Therefore

$$(n-m)\sigma_A = \sum_{j=1}^{n} P_j - \sum_{i=1}^{m} z_i$$

The characteristic equation $1 + P(s) = 0$ has $(n-m)$ root locus branches which are straight lines passing through the points $s = -\sigma_A$ on the real axis and having angles $\frac{[(2q+1)180°]}{n-m} = \phi$

If σ_A is selected such that the function $G(s)H(s)$ behaves in the same manner as $P(s)$ for values of s approaching infinity because the first two higher order terms of their denominators are identical.

Therefore the branches of the root locus of $1+G(s)H(s)=0$ which tend to infinity, approach the straight line root locus branches of $1+P(s)=0$. Hence the straight line root loci of $1+P(s)=0$ act as asymptotes to the $(n-m)$ root locus branches of $1+G(s)H(s)=0$ which tend to infinity.

The centroid of the asymptotes is given by

$$\boxed{-\sigma_A = \frac{\sum_{j=1}^{n}(-p_j) - \sum_{i=1}^{m}(-z_i)}{n-m}}$$

Q: For $G(s)H(s) = \frac{k}{(s+1)(s+2+2j)(s+2-2j)}$, calculate angles of asymptotes and the centroid

→ The open loop poles are

$s = -1, -2-2j, -2+2j$

Number of open loop poles = 3

Number of open loop zeros = 0

$n - m = 3 - 0 = 3$ branches are approaching to infinity

Angles of asymptotes

$$\phi = \frac{(2q+1)180°}{n-m}$$

Number of asymptotes = Number of branches tending to infinity = 3

$$\phi = \frac{(2q+1)\,180^\circ}{n-m} \; ; \; q = 0, 1, 2$$

For $q = 0 \Rightarrow \phi_1 = \frac{180^\circ}{3} = 60^\circ$

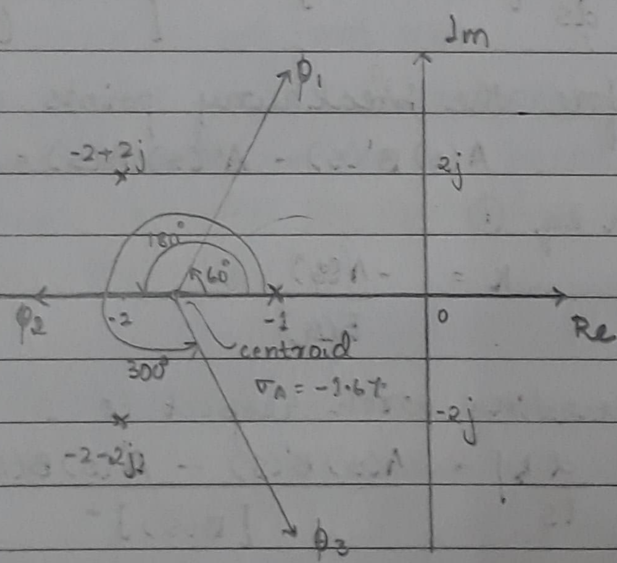
For $q = 1 \Rightarrow \phi_2 = \frac{180^\circ(3)}{3} = 180^\circ$

For $q = 2 \Rightarrow \phi_3 = \frac{5(180^\circ)}{3} = 300^\circ$

All these asymptotes intersect at a common point on the real axis called centroid.

$$\sigma_A = \frac{\Sigma \text{ real parts of poles} - \Sigma \text{ real parts of zeros}}{n-m}$$

$$\sigma_A = \frac{(-1-2-2)-(0)}{3} = \frac{-5}{3} = -1.67$$



- **RULE 6 :** (Break away point)

"Break away point is a point on the root locus where multiple roots of the characteristic equation occurs for a particular value of k."

Proof

Assuming that the characteristic equation $1 + G(s)H(s) = 0$ has a multiple root at $s = -b$ of multiplicity r.

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$$1 + G(s)H(s) = (s+b)^r A_1(s)$$

where $A_1(s)$ does not contain the factor $(s+b)$

differentiating wrt s

$$\frac{d}{ds}\left[G(s)H(s)\right] = r(s+b)^{r-1}A_1(s) + (s+b)^r A_1'(s)$$

$$\frac{d}{ds}\left[G(s)H(s)\right] = (s+b)^{r-1}\left[r A_1(s) + (s+b)A_1'(s)\right]$$

at $s = -b$

$$\frac{d}{ds}\left[G(s)H(s)\right] = 0$$

It implies that this equation has a root at least order one at the same location as the multiple root of the original characteristic equation.

Thus the breakaway points are the roots of this equation.

$$1 + G(s)H(s) = 1 + k\frac{B(s)}{A(s)} = 0 \quad \text{—— (1)}$$

differenting with respect to s

$$\frac{d}{ds}\left[G(s)H(s)\right] = k\left[\frac{A(s)B'(s) - A'(s)B(s)}{(A(s))^2}\right] = 0$$

Therefore the breakaway points are given by the roots of

$$A(s)B'(s) - A'(s)B(s) = 0 \quad \text{—— (2)}$$

From eq (1)

$$k = \frac{-A(s)}{B(s)}$$

differenting with respect to s

$$\frac{dk}{ds} = \frac{A(s)B'(s) - A'(s)B(s)}{[B(s)]^2}$$

From eq (2) the numerator of the above equation is zero

Therefore $\frac{dk}{ds} = 0$ (used to determine the breakaway points)

A breakaway point may involve two or more branches. The actual breakaway points are those roots of the equation at which the root locus angle criterion is met.

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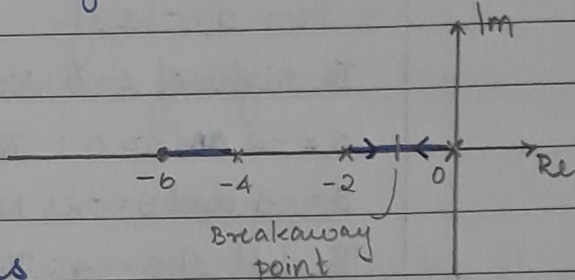
General predictions about existance of breakaway points

- If there are adjacenty placed poles on the real axis and the real axis between them is a part of the root locus then there exists mininum one breakaway point in between adjacently placed poles.
- If there are adjacenty placed zeros on the real axis and section of the real axis inbetween them is a part of the root locus then there exists minimum one breakaway point inbetween adjacently placed zeros.

Q: For $G(s)H(s) = \dfrac{k(s+6)}{s(s+2)(s+4)}$ how many minimum breakaway points exist?



→ Open loop poles : $s = 0, -2, -4$

Open loop zeros : $s = -6$

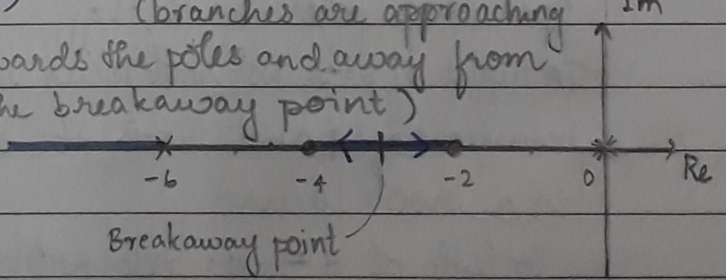
The adjacantly placed pairs of poles are $s=0$ and $s=-2$: root locus

and $s=-2$ and $s=-4$: not root locus

Hence the section of the real axis between -2 to 0 is a part of root locus thus minimum one breakaway point exists between them. (branches are approaching towards breakaway point from the poles)

Q: For $G(s)H(s) = \dfrac{k(s+2)(s+4)}{s^2(s+6)}$, how many minimum break-away points exist?

(branches are approaching towards the poles and away from the breakaway point)



→ Open loop zeros

$s = -2, -4$

Open loop poles

$s = 0, 0, -6$

to right of $s=0$: Not root locus

$s=0$ to $s=2$: not root locus

$s=-4$ to $s=-2$: root locus

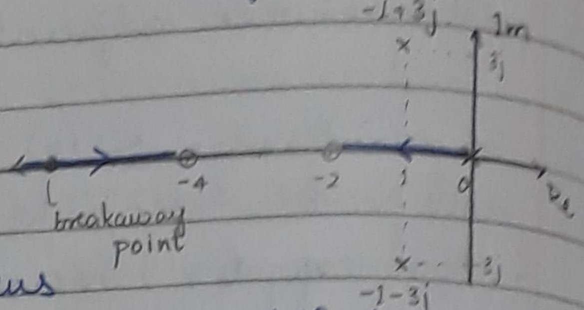
$s=-6$ to $s=-4$: not root locus

$s=-6$ to left of it : root locus

The adjacently placed poles which are part of root locus is $s=-4$ and $s=-2$. Hence there is a minimum of one breakaway point between them.

– If there is a zero on the real axis and to the left of that zero ~~there~~ are no pole or zero existing on the real axis and complete real axis to the left of this zero is a part of the root locus then there exists minimum one breakaway point to the left of that zero.

Q: For $G(s)H(s) = \dfrac{K(s+2)(s+4)}{s(s^2+2s+20)}$, how many minimum breakaway points exist?



→ Open loop zeros

$s = -2, -4$

Open loop poles

$s = 0, -1 \pm 3j$

To right of $s = 0$: Not root locus

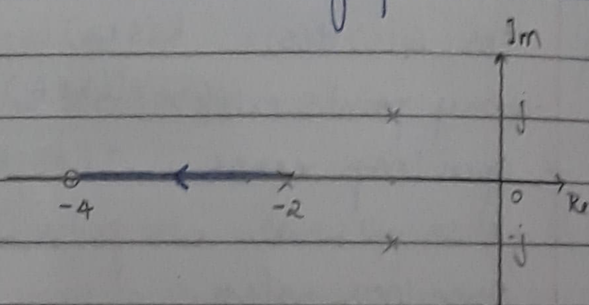
$s = -2$ and $s = 0$: Root locus

$s = -4$ and $s = -2$: Not root locus

To left of $s = -4$: Root locus

To the left of open loop zero $s = -4$ there is not open loop pole or zero and is also a part of the root locus. Hence there is minimum of one breakaway point.

Q: For $G(s)H(s) = \dfrac{k(s+4)}{(s+2)(s^2+2s+2)}$, how many minimum breakaway points exist?



→ Open loop zeros: $s = -4$

Open loop poles: $s = -2, -1 \pm j$

To right of $s = -2$ Not root locus

between $s = -4$ and $s = -2$ it is a root locus.

To left of $s = -4$: not root locus.

There is no open loop pole or zero to the left of open loop zero $s = -4$ but it is not a root locus hence the is not breakaway point present to the left of that zero. Therefore no breakaway point is possible.

- **Determination of breakaway point**

STEP 1: construct the characteristic equation $1+G(s)H(s)=0$ of the system.

STEP 2: From this equation separate the terms involving k and terms involving 's'. Write the value of k in terms of s.

$$k = f(s)$$

STEP 3: Differentiate above equation wrt 's' equate it to zero.

$$\frac{dk}{ds} = 0$$

STEP 4: Roots of the equation $\frac{dk}{ds}=0$ gives us the breakaway points.

If value of k is positive it is valid for the root locus and if value of k is negative it is invalid for direct locus but are valid for inverse root locus.

Q: For $G(s)H(s) = \frac{k}{s(s+1)(s+4)}$, determine the coordinates of valid breakaway points.

- characteristic equation

$$1 + G(s)H(s) = 0$$

$$1 + \frac{k}{s(s+1)(s+4)} = 0$$

$$k = -s(s+1)(s+4) = -s^3 - 5s^2 - 4s$$

$$\frac{dk}{ds} = -3s^2 - 10s - 4 = 0$$

$$\therefore 3s^2 + 10s + 4 = 0$$

$$\therefore \text{Breakaway points} = \frac{-10 \pm \sqrt{100-48}}{6} = -0.46, -2.86$$

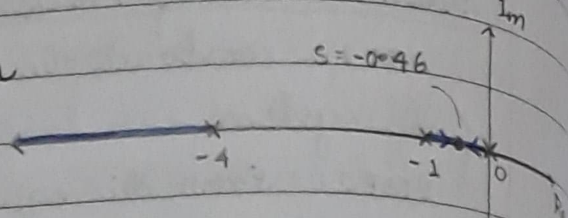
For $s = -0.46$; $k = 0.8793$

$s = -2.86$; $k = -6.064$

Therefore for $s = -0.46$ k is positive. Hence $s = -0.46$ is the valid breakaway point for the root locus.

Date

Root locus approaches and leaves breakaway point at an angle $\pm \frac{180^\circ}{n}$.



Here the number of branches approaching is 2.

$\therefore$ Angle of approaching $= \pm 90^\circ$

- **RULE 7:**

" The intersection of root locus branches with the imaginary axis can be determined by using Routh Criterion.

Procedure

STEP1: consider the characteristic equation $1 + G(s)H(s) = 0$

STEP2: construct Routh's array in terms of k.

STEP3: Determine $k_{marginal}$ i.e., value of k which creates one of the rows of Routh's array as row of zeros, except the row of s^0.

STEP4: construct auxillary equation $A(s) = 0$ by using coefficients of a row which is just above the row of zeros.

STEP5: Roots of auxillary equation $A(s) = 0$ for $k = k_{mar}$ are the intersection points of the root locus with imaginary axis.

Ex: $G(s)H(s) = \frac{k}{s(s+1)(s+4)}$

characteristic equation is given by

$$1 + G(s)H(s) = 1 + \frac{k}{s(s+1)(s+4)} = 0$$

i.e., $s^3 + 5s^2 + 4s + k = 0$

Routh's array

s^3	1	4
s^2	5	k
s^1	$\frac{20-k}{5}$	0
s^0	k	

$k_{max} = 20$

$\therefore A(s) = 5s^2 + k = 0$

$k = k_{max} = 20$

$5s^2 + 20 = 0$

$s^2 = -4 \Rightarrow s = \pm 2j$

Therefore $s = \pm 2j$ are the points of intersection of root locus with imaginary axis.

If kmax is positive, root locus intersects with imaginary axis. But if kmax is negative, root locus does not intersect with imaginary axis and lies totally in left half of s-plane.

- RULE 8:

"Angle of departure at complex conjugate poles and angle of arrival at complex conjugate zeros".

- A branch always leaves from a open loop pole, this angle at which it departs is called angle of departure and is denoted as ϕ_d.

$$\phi_d = \pm(2q+1)180° - \phi$$

where $\phi = \Sigma\phi_p - \Sigma\phi_z$

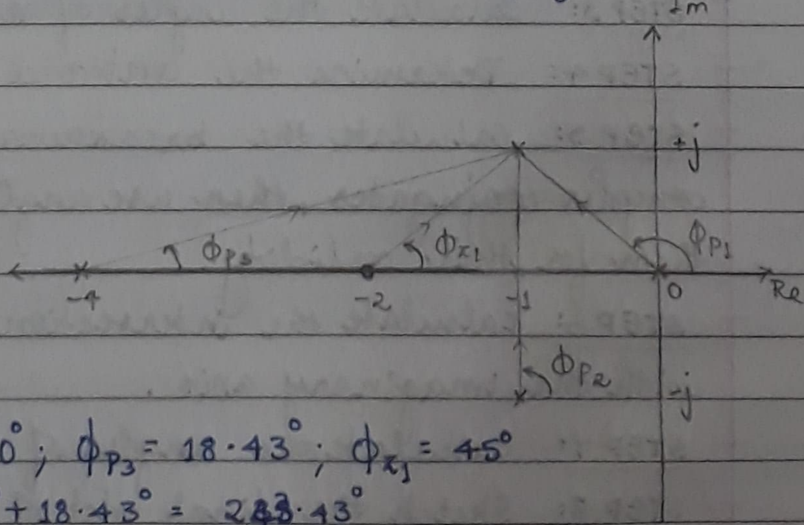
Q: For $G(s)H(s) = \dfrac{k(s+2)}{s(s+4)(s^2+2s+2)}$, calculate angles of departures at complex conjugate poles.

- Open loop poles

$s = 0, -4, -1 \pm j$

Open loop zeros

$s = -2$



To find ϕ_d at $-1+j$

$\Sigma\phi_p = \phi_{P_1} + \phi_{P_2} + \phi_{P_3}$

$\Sigma\phi_z = \phi_{z_1}$

$\phi_{P_1} = 135°;\ \phi_{P_2} = 90°;\ \phi_{P_3} = 18.43°;\ \phi_{z_1} = 45°$

$\Sigma\phi_p = 135° + 90° + 18.43° = 243.43°$

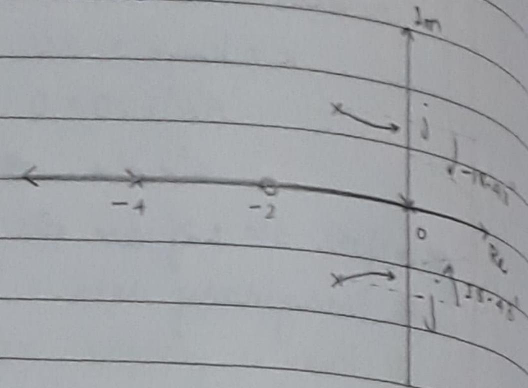
$\Sigma\phi_z = 45°$

$$\phi = \Sigma\phi_p - \Sigma\phi_z = 243.43 - 45^\circ = 198.43^\circ$$

$$\therefore \phi_d = 180^\circ - \phi = 180^\circ - 198.43^\circ = -18.43^\circ$$

Therefore root locus branch leaving this pole will depart tangentially to the line whose angle is given by $\phi_d = -18.43^\circ$.

For $-1-j$, the sign of ϕ_d will be just opposite as root locus is symmetrical about real axis.



- A branch always terminates at a open loop zero, this angle at which it arrives is called angle of arrival and is denoted as ϕ_a.

$$\phi_a = 180^\circ + \phi$$

where $\phi = \Sigma\phi_p - \Sigma\phi_z$

GENERAL STEPS TO SOLVE THE PROBLEMS ON ROOT LOCUS:

- **STEP 1:** From $G(s)H(s)$ find the number of open loop poles and zeros. Also number of branches.
- **STEP 2:** Draw the pole-zero plot. Identify sections of real axis for existance of root locus. Predict the number of breakaway points by general predictions.
- **STEP 3:** Calculate the angles of asymptotes.
- **STEP 4:** Determine the centroid.
- **STEP 5:** Calculate the breakaway points. If they are complex conjugates, then use angle criterion to check them for their validity.
- **STEP 6:** Calculate the intersection points of root locus with the imaginary axis.
- **STEP 7:** Calculate the angles of departures or arrivals.
- **STEP 8:** Sketch the final plot of the root locus.

$\phi = \Sigma\phi_p - \Sigma\phi_z = 243.43 - 45^\circ = 198.43^\circ$

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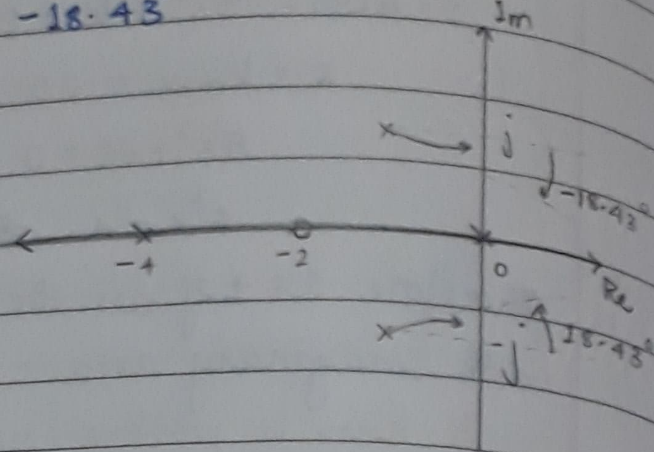

- A branch always terminates at a open loop zero, this angle at which it arrives is called angle of arrival and is denoted as ϕ_a.

$$\phi_a = 180^\circ + \phi$$
$$\text{where } \phi = \Sigma\phi_p - \Sigma\phi_z$$

GENERAL STEPS TO SOLVE THE PROBLEMS ON ROOT LOCUS:

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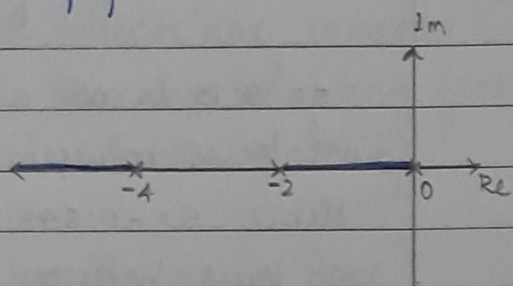
- **STEP 8:** Sketch the final plot of the root locus.

Q: For a unity feedback system, $G(s) = \dfrac{k}{s(s+4)(s+2)}$ sketch the rough nature of the root locus showing all details on it. comment on the stability of the system.

— Given: $G(s)H(s) = \dfrac{k}{s(s+4)(s+2)}$

open loop poles: $s = 0, -2, -4 \Rightarrow n = 3$
Open loop zeros: none. $\Rightarrow m = 0$
No finite zero hence all $n - m = 3$ braches terminate at infinity whereas start at open loop poles.

Between $s=0$ and $s=-2$ there is a possibility of one breakaway point. sections of real axis are considered as part of root locus as to right side sum of poles and zeros is odd.



Angle of asymptotes

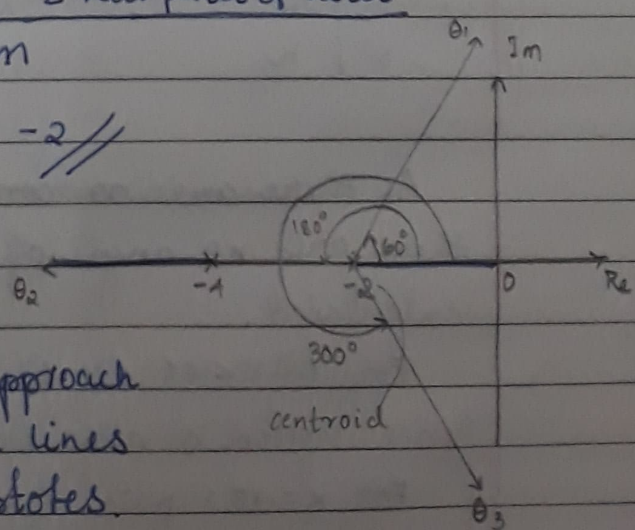
$$\theta_A = \frac{(2q+1)180^\circ}{n-m} \quad ; \quad q = 0, 1, 2$$

$$\therefore \theta_1 = \frac{180^\circ}{3} = 60^\circ \quad ; \quad \theta_2 = \frac{3(180^\circ)}{3} = 180^\circ \quad ; \quad \theta_3 = \frac{5(180^\circ)}{3} = 300^\circ$$

Centroid

$$\sigma_A = \frac{\Sigma \text{ real parts of poles} - \Sigma \text{ real parts of zeros}}{n-m}$$

$$\sigma_A = \frac{-2-4-0}{3} = \frac{-6}{3} = -2$$



Branches will approach to ∞ along these lines which are asymptotes.

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Breakaway points

$$1+G(s)H(s)=0$$

$$1+\frac{k}{s(s+4)(s+2)}=0$$

$$k=-s(s+4)(s+2) = -s^3-6s^2-8s$$

$$\frac{dk}{ds}=-3s^2-12s-8=0$$

$$\therefore 3s^2+12s+8=0$$

Therefore breakaway points are

$$\frac{-12\pm\sqrt{144-96}}{6} = -0.845, -3.15$$

$s=-3.15$ is not a breakaway point as there is no root locus between $s=-2$ and $s=-4$.

Hence $s=-0.845$ is a breakaway point as it lies on the root locus between $s=0$ and $s=-2$.

For $s=-0.845$; $k=3.079$

Intersection point with imaginary axis.

Characteristic equation

$$s^3+6s^2+8s+k=0$$

Routh's array

s^3	1	8
s^2	6	K
s^1	$\frac{k-48}{6}$	0
s^0	k	

$k_{max}=48$

$A(s)=6s^2+k=0$

$k=k_{max}=48$

$6s^2+48=0$

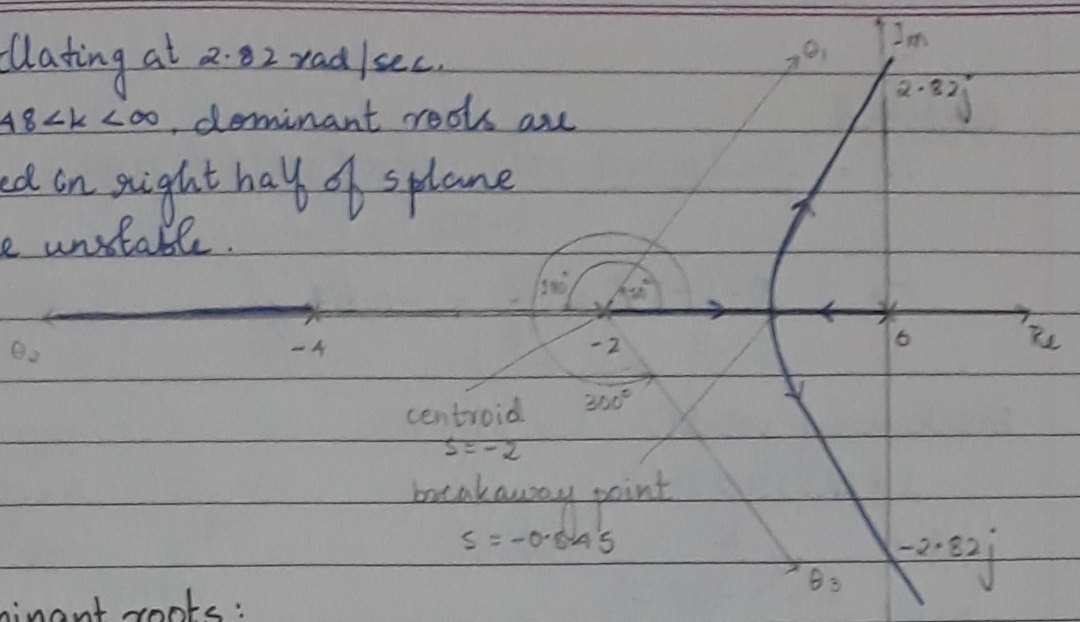
$s^2=-8 \Rightarrow s=\pm 2.828j$ //

As there are no complex conjugate poles or zeros there are no angle of departure or arrival to be calculated.

For $0<K<48$ all the roots are in left half of s-plane hence system is absolutely stable.

For $k=48$: pair of dominant roots on imaginary axis with remaining root in left half. Hence marginally stable

oscillating at 2.82 rad/sec.
For $48 < k < \infty$, dominant roots are located on right half of s plane hence unstable.



Dominant roots:

stability is predicted by the locations of the dominant roots. The dominant roots are those which are located closest to the imaginary axis. The branches starting from such roots which are dominant decide the stability.

Q: Sketch the root locus for the system having

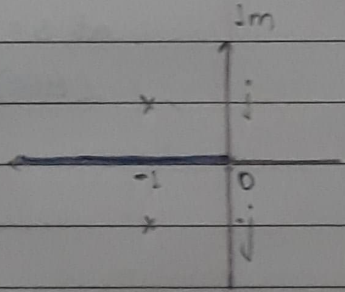
$$G(s)H(s) = \frac{k}{s(s^2+2s+2)}$$

– Open loop poles : $s = 0, -1 \pm j$ ⇒ $n = 3$

Open loop zeros : none ⇒ $m = 0$

Therefore n branches start from poles and $n-m = 3$ branches terminate at infinity.

From $s=0$ to $s \to \infty$ there is one open loop pole on the right hand side i.e., sum of open loop poles and zeros is odd. Hence it is a root locus.



Angles of asymptotes.

$$\theta_A = \frac{(2q+1)180^\circ}{n-m} \quad ; q = 0,1,2$$

$$\theta_1 = \frac{180^\circ}{3} = 60^\circ \quad ; \quad \theta_2 = \frac{3(180^\circ)}{3} = 180^\circ \quad ; \quad \theta_3 = \frac{5(180^\circ)}{3} = 300^\circ$$

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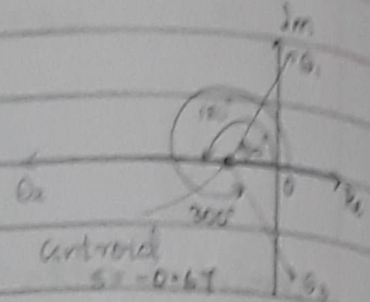
Centroid

$$\sigma_A = \frac{\Sigma \text{ real parts of poles} - \Sigma \text{ real parts of zeros}}{n-m}$$

$$\sigma_A = \frac{0-1-1-0}{3} = \frac{-2}{3} = -0.67$$



Breakaway points.

$$1 + G(s)H(s) = 0$$

$$1 + \frac{k}{s(s^2+2s+2)} = 0$$

$$k = -s(s^2+2s+2) = -s^3 - 2s^2 - 2s$$

$$\therefore \frac{dk}{ds} = -3s^2 - 4s - 2 = 0$$

$$\therefore 3s^2 + 4s + 2 = 0$$

$$\text{Breakaway points} = \frac{-4 \pm \sqrt{16-24}}{6} = -0.67 \pm 0.47j$$

For $s = -0.67 + 0.47j$

by angle criteria

$$\angle G(s)H(s) = \pm(2q+1)180 \; ; \; q = 0, 1, 2 \ldots$$

$$G(s)H(s) = \frac{k}{s(s+1-j)(s+1+j)}$$

at $s = -0.67 + 0.47j$

$$\angle G(s)H(s) = \frac{\angle k + 0j}{\angle -0.67+0.47j \; \angle 0.33-0.53j \; \angle 0.33+1.47j}$$

$$= \frac{0^\circ}{\angle 144.95^\circ \; \angle -58.09^\circ \; \angle 77.34^\circ} = -164.2^\circ$$

This is not an odd multiple of 180°. Hence point is not on the root locus and hence there is no breakaway point existing for this system.

(This is applicable to $s = -0.67 - 0.47j$ as well as they are symmetric about real axis)

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Intersection with imaginary axis.

characteristic equation

$$s^3 + 2s^2 + 2s + k = 0$$

Routh's array

s^3	1	2
s^2	2	k
s^1	$\frac{4-k}{2}$	0
s^0	k	

$k_{max} = 4$

$A(s) = 2s^2 + k = 0$

$k_{max} = k = 4$

$2s^2 + 4 = 0$

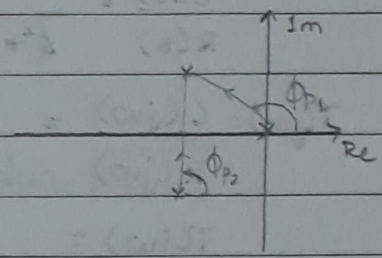
$s^2 = -2 \Rightarrow s = \pm 1.414j$

Angle of departure

For $-1+j$

$\Sigma \phi_p = \phi_{p_1} + \phi_{p_2}$

$= 135^\circ + 90^\circ = 225^\circ$

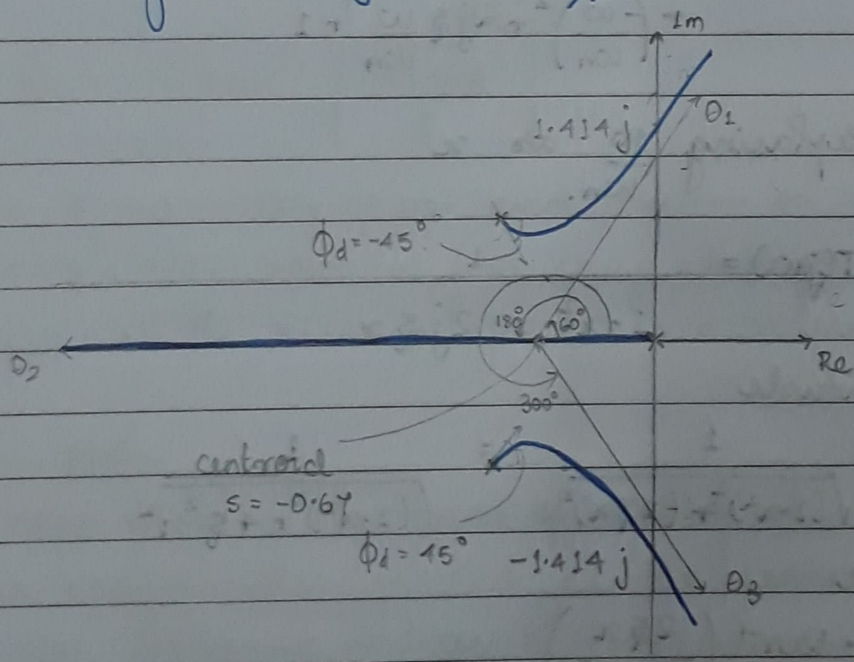


$\Sigma \phi_z = 0$

$\therefore \phi = \Sigma\phi_p - \Sigma\phi_z = 225^\circ$

$\phi_d = 180^\circ - \phi = 180^\circ - 225 = -45^\circ$

Similarly at $-1-j$, $\phi_d = 45^\circ$



For $0 < k < 4$: all roots in left half splane : stable

at $k = 4$: dominant roots on imaginary axis; system is marginally stable, oscillating with 1.414 rad/sec.

at $k > 4$: dominant roots in right half splane : unstable.

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UNIT - 05

Frequency Domain Analysis of Control Systems

Estimation of specifications for a second order system or corelation between time domain and frequency domain for second order system: Pg 24.

Frequency domain specifications: Pg 34

Bode Plots: Pg 30

Gain and phase margins
stability analysis using Bode plot } Pg 34 to 36

Q: A unity feedback control system has $G(s) = \frac{80}{s(s+2)(s+20)}$. Draw the Bode plot. Determine GM and PM, ω_{gc} and ω_{pc}. Comment on the stability.

→ given:

$$G(s)H(s) = \frac{80}{s(s+2)(s+20)} \quad \text{unity feedback} \therefore H(s) = 1$$

$$= \frac{80}{s(2)\left[\frac{s}{2}+1\right]20\left[\frac{s}{20}+1\right]} = \frac{2}{s\left(\frac{s}{2}+1\right)\left(\frac{s}{20}+1\right)}$$

Therefore

$k = 2$; 1 pole at origin

$\frac{1}{(1+\frac{s}{2})}$ here $T_1 = \frac{1}{2}$ $\Rightarrow \omega_{c_1} = \frac{1}{T_1} = 2$

$\frac{1}{(1+\frac{s}{20})}$ here $T_2 = \frac{1}{20}$ $\Rightarrow \omega_{c_2} = \frac{1}{T_2} = 20$

<u>Magnitude plot analysis:</u>

For $k = 2$

$$20 \log k = 20 \log 2 = 6\,dB$$

1 pole at origin

straight line of slope -20dB/decade passing through intersection point $\omega = 1$ at 0dB.

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Shift intersection point of $\omega = 1$ at 0dB on 20 log k line and draw parallel to -20 dB/decade line drawn. This will continue as a result of k and 1/s till first corner frequency occurs i.e, $\omega_{c_1} = 2$.

At $\omega_{c_1} = 2$, as there is simple pole it will contribute the rate of -20dB/decade hence resultant slope after $\omega_{c_1} = 2$ becomes -20 dB/decade - 20dB/decade = -40dB/decade. This is addition of k, 1/s, 1/(1+s/2). This will continue till it intersects next corner frequency i.e $\omega_{c_2} = 20$

At $\omega_{c_2} = 20$, there is a simple pole contributing -20dB/dec and hence resultant slope after $\omega_{c_2} = 20$ becomes -40-20 = -60 dB/decade.

Phase Angle Plot:

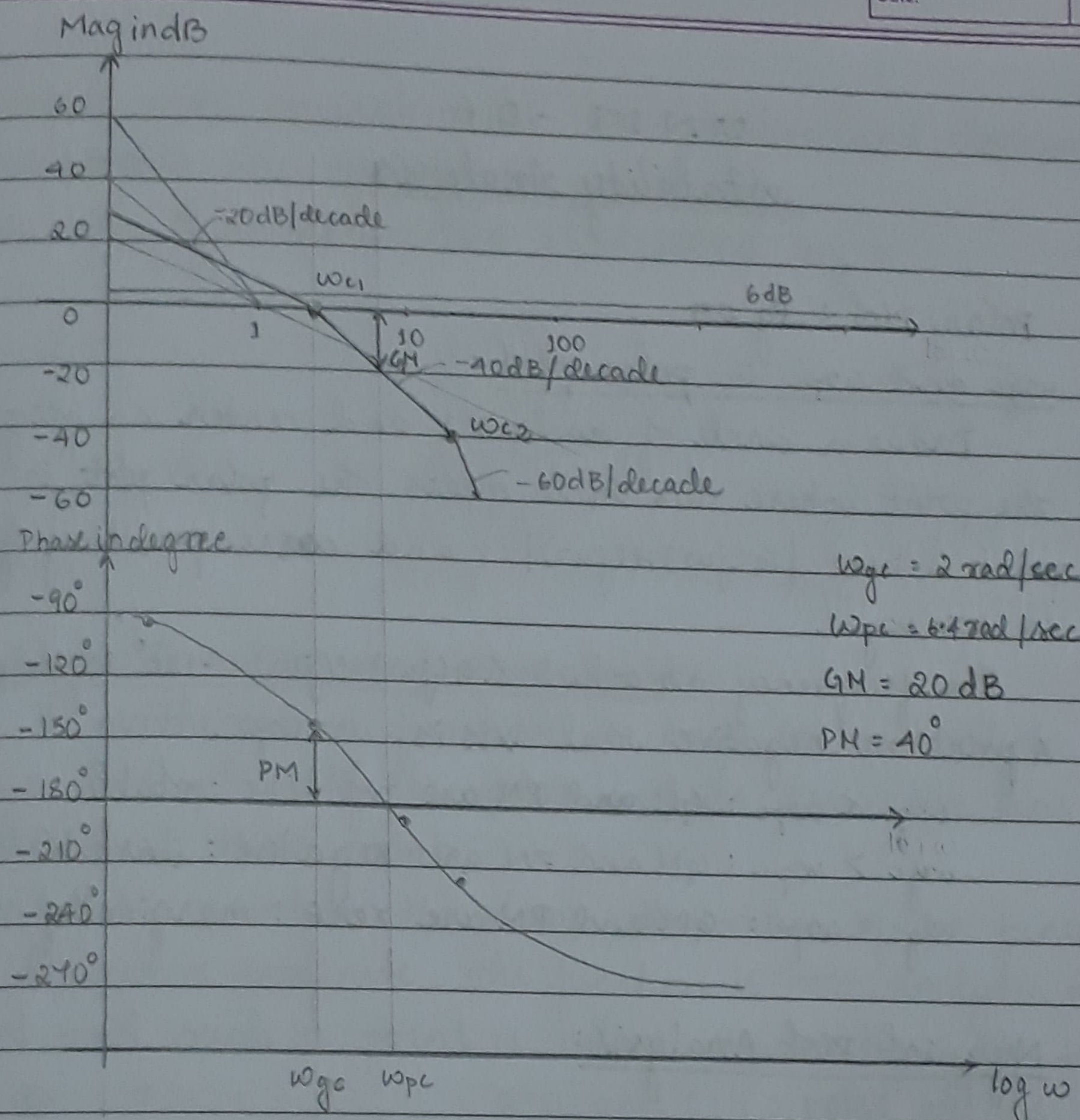
$$G(j\omega)H(j\omega) = \frac{2}{j\omega\left(1+\frac{j\omega}{2}\right)\left(1+\frac{j\omega}{20}\right)}$$

$$\angle G(j\omega)H(j\omega) = 0^\circ - 90^\circ - \tan^{-1}\left(\frac{\omega}{2}\right) - \tan^{-1}\left(\frac{\omega}{20}\right)$$

$$\phi_R = -90^\circ - \tan^{-1}\left(\frac{\omega}{2}\right) - \tan^{-1}\left(\frac{\omega}{20}\right)$$

ω	ϕ_R
0.2	-96.27°
2	-140.7°
8	-187.76°
20	-219.28°
∞	-270°

$\omega_{gc} < \omega_{pc}$: GM and PM are positive. Hence the given system is stable.



$\omega_{gc} = 2$ rad/sec

$\omega_{pc} = 6.4$ rad/sec

GM = 20 dB

PM = 40°

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UNIT 5

★ Frequency Response analysis:

- correlation between time domain and frequency domain for second order system:

$$G(s) = \frac{\omega_n^2}{s(s+2\xi\omega_n)} \quad ; \quad H(s) = 1$$

$$G(j\omega) = \frac{\omega_n^2}{j\omega(j\omega+2\xi\omega_n)} \quad ; \quad H(j\omega) = 1$$

The closed loop transfer function in time domain is

$$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2+2\xi\omega_n s+\omega_n^2}$$

$$T(j\omega) = \frac{C(j\omega)}{R(j\omega)} = \frac{\omega_n^2}{(-j\omega)^2+2\xi\omega_n(j\omega)+\omega_n^2}$$

$$T(j\omega) = \frac{\omega_n^2}{-\omega^2+2j\xi\omega\omega_n+\omega_n^2}$$

dividing by ω_n^2

$$T(j\omega) = \frac{1}{-\left(\frac{\omega}{\omega_n}\right)^2+2j\xi\frac{\omega}{\omega_n}+1}$$

Replacing $\frac{\omega}{\omega_n}$ as x

$$T(j\omega) = \frac{1}{1-x^2+2j\xi x}$$

Magnitude

$$M = \frac{1}{\sqrt{(1-x^2)^2+(2\xi x)^2}} = \frac{1}{\sqrt{(1-x^2)^2+4\xi^2x^2}}$$

Phase

$$\phi = -\tan^{-1}\left(\frac{2\xi x}{1-x^2}\right)$$

Maximum magnitude

$$\frac{dM}{dx} = 0.$$

$$\frac{dM}{dx} = \frac{-1}{2\left[(1-x^2)^2+4\xi^2x^2\right]^{3/2}}\left[2(1-x^2)(-2x)+8\xi^2x\right] = 0$$

$$(1-x^2)2x - 4\xi^2 x = 0$$

$$(1-x^2) = 2\xi^2$$

$$x_r = \sqrt{1-2\xi^2}$$ at which magnitude is max

substituting

$$M_r = \frac{1}{\sqrt{[1-(1-2\xi^2)]^2+4\xi^2(1-2\xi^2)}}$$

$$= \frac{1}{\sqrt{4\xi^4+4\xi^2(1-2\xi^2)}}$$

$$= \frac{1}{2\xi^2\sqrt{\xi^2+1-2\xi^2}}$$

$$M_r = \frac{1}{2\xi^2\sqrt{1-\xi^2}} \qquad \omega_r = \omega_n\sqrt{1-2\xi^2}$$

$$\phi_r = -\tan^{-1}\left[\frac{2\xi\sqrt{1-2\xi^2}}{1-(1-2\xi^2)}\right]$$

$$= -\tan^{-1}\left[\frac{2\xi\sqrt{1-2\xi^2}}{2\xi^2}\right]$$

$$\phi_r = -\tan^{-1}\left[\frac{\sqrt{1-2\xi^2}}{\xi}\right]$$

as $\xi \to 0$, $\omega_r \to \omega_n$ and $M_r \to \infty$

For $0 < \xi < 1/\sqrt{2}$ resonant frequency always has a value less than ω_n and the resonant peak has a value greater than 1.

Unit 5

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- **Bode Plot:**

It consists of two plots

- magnitude plot
- phase angle plot.

The transfer function $G(j\omega)$ is represented by

$$G(j\omega) = |G(j\omega)| e^{j\phi(\omega)}$$

$$\log_e G(j\omega) = \log_e |G(j\omega)| + j\phi(\omega)$$

Magnitude plot

$20\log_{10} |G(j\omega)|$ versus $\log_{10}\omega$

Phase plot

$\phi(\omega)$ versus $\log_{10}\omega$

$$G(s) = k_s \frac{(s+T_a)(s+T_b)\ldots\ldots}{(s+T_1)(s+T_2)\ldots(s^2+2\xi s\omega_n+\omega_n^2)}$$

$$G(s) = k_s \frac{(1+sT_a)(1+sT_b)\ldots\ldots}{(1+sT_1)(1+sT_2)\ldots}$$

$$G(j\omega) = k_s \frac{(1+j\omega T_a)(1+j\omega T_b)\ldots\ldots}{(1+j\omega T_1)(1+j\omega T_2)}$$

$$|G(j\omega)| = k_s \frac{|(1+j\omega T_a)|\,|(1+j\omega T_b)|\ldots}{|(1+j\omega T_1)|\,|(1+j\omega T_2)|\ldots}$$

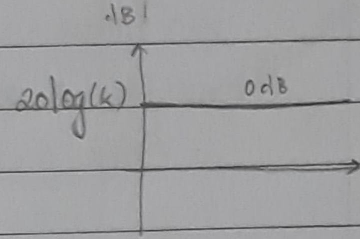
$$20\log_{10}|G(j\omega)| = 20\log_{10}k + 20\log_{10}|(j\omega T_a+1)| + 20\log_{10}|(1+j\omega T_b)| \cdots - 20\log_{10}|(1+j\omega T_1)| - 20\log_{10}|(1+j\omega T_2)| \cdots$$

and

$$\phi(\omega) = \tan^{-1}(\omega T_a) + \tan^{-1}(\omega T_b) + \cdots - \tan^{-1}(\omega T_1) - \tan^{-1}(\omega T_2) \cdots$$

Basic Bode Plots:

1. constant k : $|G(j\omega)|_{dB} = 20\log|k|$
   straight line : slope = 0 dB



2. $\frac{k}{s}$ or $\frac{k}{s^n}$

$$G(s) = \frac{k}{s} \Rightarrow G(j\omega) = \frac{k}{j\omega} \Rightarrow |G(j\omega)| = \frac{k}{\omega}$$

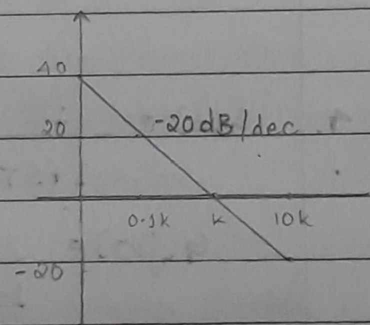
$$20\log_{10}|G(j\omega)| = 20\log_{10}\left(\frac{k}{\omega}\right)$$

$$\phi = \angle G(j\omega) = \tan^{-1}\left(\frac{-k/\omega}{0}\right) = -90^\circ$$

$\omega = k/10$ $\quad 20\log_{10}|G(j\omega)| = 20$

$\omega = k$ $\quad 20\log_{10}|G(j\omega)| = 0$

$\omega = 10k$ $\quad 20\log_{10}|G(j\omega)| = -20$



$$G(s) = \frac{k}{s^n} \Rightarrow G(j\omega) = \frac{k}{(j\omega)^n} \Rightarrow |G(j\omega)| = \frac{k}{\omega^n}$$

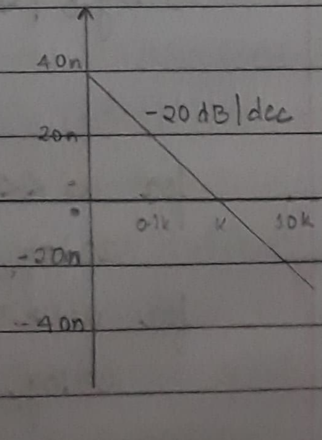
$$20\log_{10}|G(j\omega)| = 20\log_{10}\frac{k}{\omega^n} = 20n\log_{10}\frac{k^{1/n}}{\omega}$$

$$\phi = \angle G(j\omega) = -90^\circ$$

$\omega = k^{1/n}/10$ $\quad 20\log_{10}|G(j\omega)| = 20n$

$\omega = k^{1/n}$ $\quad 20\log_{10}|G(j\omega)| = 0$

$\omega = 10k^{1/n}$ $\quad 20\log_{10}|G(j\omega)| = -20n$



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3. ks or ks^n

$$G(s) = ks \Rightarrow G(j\omega) = kj\omega \Rightarrow |G(j\omega)| = k\omega$$

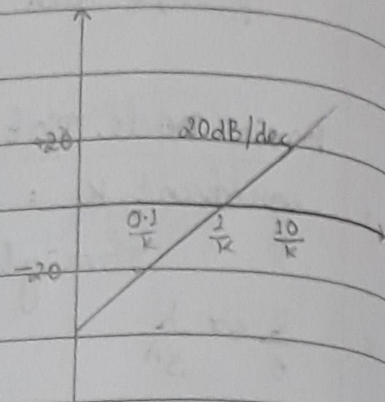
$$20\log_{10}|G(j\omega)| = 20\log_{10} k\omega$$

$$\phi = \angle G(j\omega) = \tan^{-1}\left(\frac{k\omega}{0}\right) = 90^\circ$$

$\omega = 1/10k$: $20\log_{10}|G(j\omega)| = -20dB$

$\omega = 1/k$: $20\log_{10}|G(j\omega)| = 0$

$\omega = 10/k$: $20\log_{10}|G(j\omega)| = 20dB$



$$G(s) = ks^n \Rightarrow G(j\omega) = k(j\omega)^n$$

$$\Rightarrow |G(j\omega)| = k\omega^n$$

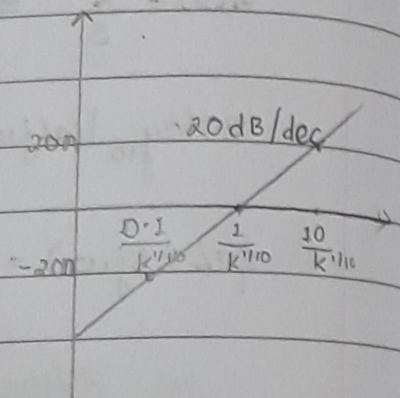
$$20\log_{10}|G(j\omega)| = 20n\log_{10} k^{1/10}\omega$$

$$\phi = \angle G(j\omega) = \tan^{-1}\left(\frac{k\omega^n}{0}\right) = 90^\circ$$

$\omega = 1/10k^{1/10}$: $20\log_{10}|G(j\omega)| = -20n\,dB$

$\omega = 1/k^{1/10}$: $20\log_{10}|G(j\omega)| = 0$

$\omega = 10/k^{1/10}$: $20n\log_{10}|G(j\omega)| = 20n\,dB$



4. $G(s) = \dfrac{1}{1+sT}$

$$G(j\omega) = \frac{1}{1+j\omega T} \Rightarrow G(j\omega) = \frac{1}{\sqrt{1+\omega^2T^2}}$$

$$\phi = \angle G(j\omega) = -\tan^{-1}\left(\frac{\omega T}{1}\right) = -\tan^{-1}(\omega t)$$

$$20\log_{10}|G(j\omega)| = 20\log_{10}\frac{1}{\sqrt{1+\omega^2T^2}}$$

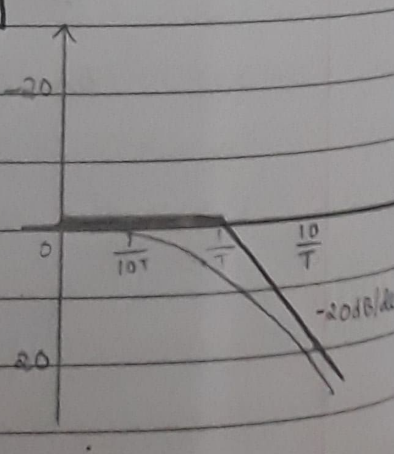
$$= -10\log_{10}[1+\omega^2T^2]$$

for $\omega << 1/T$ ω^2T^2 is neglected

$$20\log_{10}|G(j\omega)| = 0$$

for $\omega >> 1/T$ 1 is neglected

$$20\log_{10}|G(j\omega)| = -20\log(\omega T)$$



5. $G(s) = 1 + sT$

$G(s) = 1 + sT \Rightarrow G(j\omega) = 1 + j\omega T$

$\Rightarrow |G(j\omega)| = \sqrt{1 + \omega^2 T^2}$

$20\log_{10}|G(j\omega)| = 20\log_{10}\sqrt{1+\omega^2T^2}$

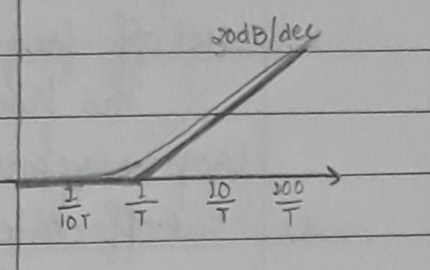
$= 10\log_{10}[1+\omega^2T^2]$

for $\omega << 1/T$ ω^2T^2 is neglected

$20\log_{10}|G(j\omega)| = 0$

for $\omega >> 1/T$ 1 is neglected

$20\log_{10}|G(j\omega)| = 20\log(\omega T)$



★ STEPS TO SKETCH THE BODE PLOT:

1. Express given G(S)H(S) into time constant form
2. Draw a line of 20 log k dB.
3. Draw a line of appropriate slope representing poles or zeros at origin passing through the intersection point of ω=1 and 0 dB.
4. Shift this intersection point on 20 log k line and draw parallel line to the line drawn in step 3. This is addition of constant k and number of poles or zero at origin.
5. Change the slope of this line at various corner frequencies by appropriate value i.e., depending upon which factor is occuring at corner frequency. For a simple pole slope must be changed by -20 dB/decade for a simple zero by +20dB/decade. Do not draw these individual lines. Change the slope of line obtained in step 5 by respective value and draw line with resultant slope. Continue this line till it intersects next corner frequency line change the slope and continue.
6. Prepare the phase angle table and obtain the table of ω and resultant phase angle ϕ_R by actual calculation. Plot these points and draw the smooth curve obtaining the phase plot.

NOTE: At every corner frequency slope of resultant line must change.

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* **Frequency response specifications:**

1. Bandwidth

The range of frequencies over which the system will respond satisfactorily.

2. cut off frequency.

The frequency at which the magnitude of the closed loop response is 3dB down from its zero frequency value.

3. cut off rate

The slope of the resultant magnitude curve near the cutoff frequency.

4. Resonant peak: M_r

It is the maximum value of magnitude of the closed loop frequency response.

5. Resonant frequency: ω_r

The frequency at which resonant peak M_r occurs in closed loop frequency response.

6. Gain cross over frequency: ω_{gc}

The frequency at which magnitude of $G(j\omega)H(j\omega)$ is unity.

7. Phase cross over frequency: ω_{pc}

The frequency at which phase angle of $G(j\omega)H(j\omega)$ is $-180°$.

8. Gain Margin: GM

In root locus as gain k is increased the system stability reduces and for a certain value of k it becomes mariginally stable. Thus gain margin is defined as the margin in gain allowable by which gain can be increased till system reaches the verge of instability.

$$GM = 20\log \frac{1}{|G(j\omega)H(j\omega)|_{\omega=\omega_{pc}}}$$

$$GM = -20\log |G(j\omega)H(j\omega)|_{\omega=\omega_{pc}}$$

More positive the GM, more stable the system.

9. Phase Margin: PM

It is possible to introduce phase lag in the system i.e., negative angles without affecting magnitude plot of $G(j\omega)H(j\omega)$. The amount of additional phase lag which can be introduced in the system till system reaches on the verge of instability is called phase margin.

The positive phase margin means negative angle introduction in system is possible before system becomes unstable. Such system is stable system.

The negative phase margin means present negative phase lag should be changed by adding positive angle hence phase margin is said to be negative and system is unstable.

$$PM = \left[\angle G(j\omega)H(j\omega)\big|_{at\ \omega=\omega_{gc}}\right] - (-180^\circ)$$

$$PM = 180^\circ + \angle G(j\omega)H(j\omega)\big|_{at\ \omega=\omega_{gc}}$$

★ Calculation of GM and PM from Bode Plot:

→ Extend $\omega = \omega_{pc}$ line upwards till its intersects resultant magnitude plot. The magnitude corresponding to this point is $|G(j\omega)H(j\omega)|_{\omega=\omega_{pc}}$

The difference between 0dB and magnitude at that point is gain margin. If the point is below 0, GM is positive and if the point is above 0, GM is negative.

→ Extend $\omega = \omega_{gc}$ line towards the phase angle plot. Note the point of intersection.

The distance between this point and -180° line is the phase margin. If the point is above -180° line, PM is positive and if the point is below -180° line, PM is negative.

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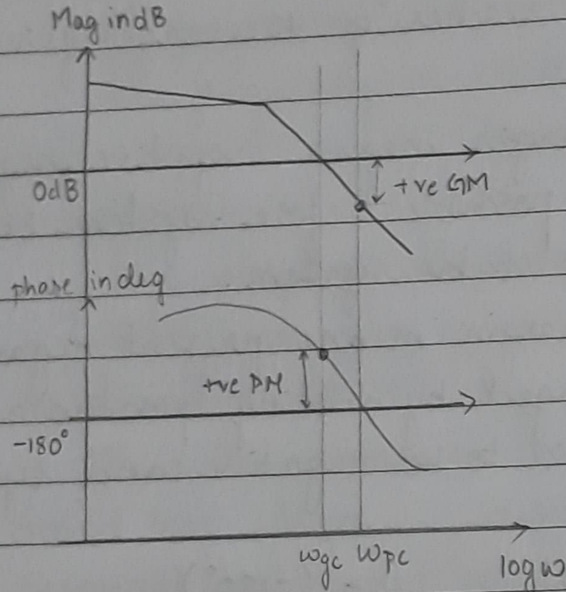
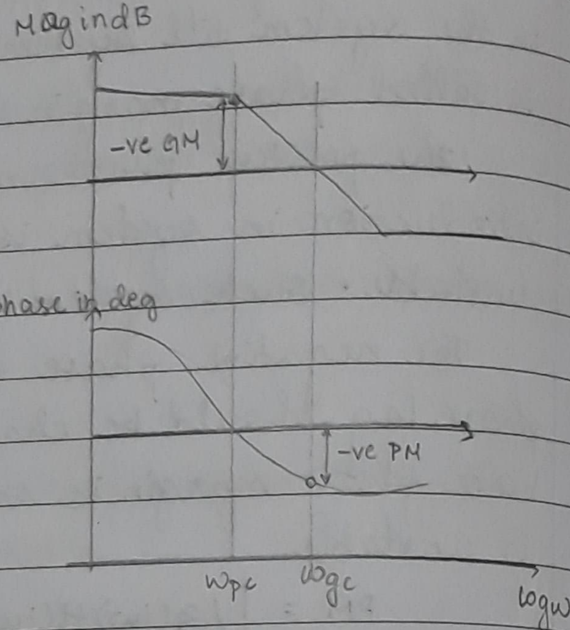
PM and GM positive : stable $\omega_{gc} < \omega_{pc}$

PM and GM negative : unstable $\omega_{gc} > \omega_{pc}$

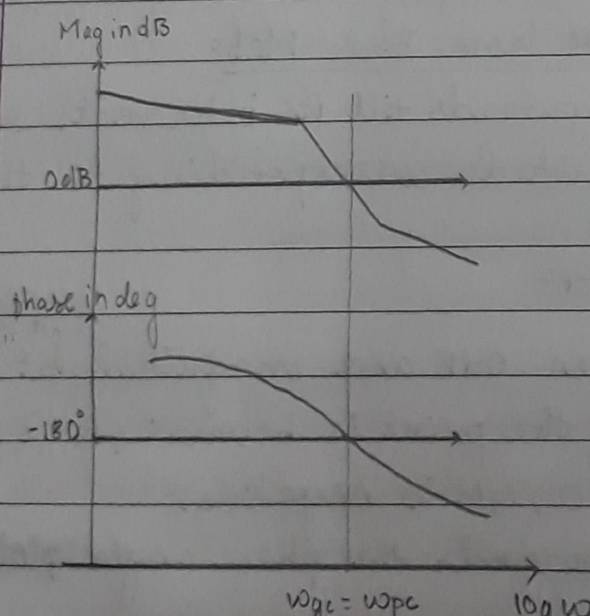
PM and GM are zero : marginally stable ($\omega_{gc} = \omega_{pc}$)





$\omega_{gc} < \omega_{pc}$: GM and PM are positive → stable system

$\omega_{gc} > \omega_{pc}$: GM and PM are negative → unstable system



$\omega_{gc} = \omega_{pc}$: GM and PM are zero → marginally stable

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UNIT - 06

Stability Analysis

UNIT 6

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- Polar Plots:

Polar plot is the locus of tips of the phasors of various magnitudes plotted at the corresponding phase angles for different values of frequencies from 0 to ∞.

So a polar plot starts at point representing magnitude and phase angle for $\omega = 0$ and terminates at a point representing magnitude and phase angle for $\omega = \infty$

Q: consider a system with open loop transfer function as $G(s)H(s) = 10/s$. Obtain its polar plot.

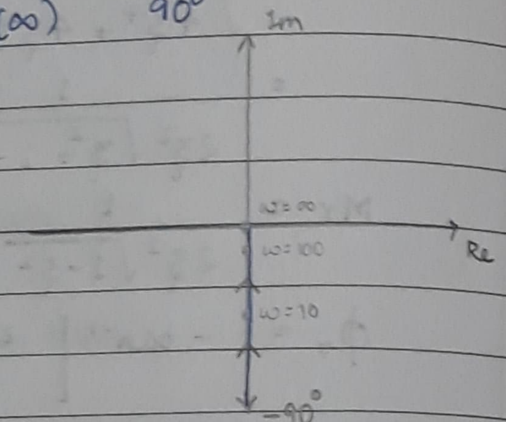
- Frequency domain transfer function is obtained by replacing s by jω

$$G(j\omega)H(j\omega) = \frac{10}{j\omega} = \frac{10+0j}{0+\omega j}$$

$$|G(j\omega)H(j\omega)| = M = \frac{10}{\omega}$$

$$\angle G(j\omega)H(j\omega) = \phi = \frac{\tan^{-1}(0)}{\tan^{-1}(\infty)} = \frac{0^\circ}{90^\circ} = -90^\circ$$

ω	M	ϕ
0	∞	-90°
10	1	-90°
100	0.1	-90°
∞	0	-90°



so the plot starts at ∞ at angle -90° and then terminates at origin at angle -90° (negative imaginary axis).

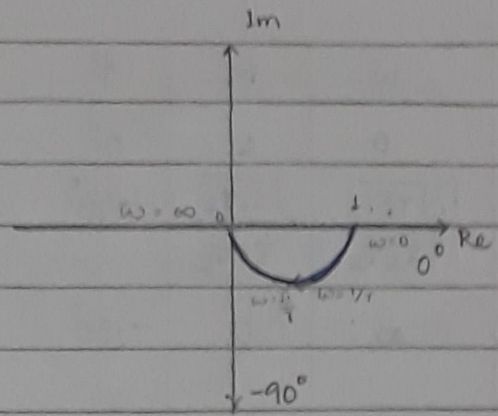
Q: consider a system with open loop transfer function $G(s)H(s) = \frac{1}{1+Ts}$ where T is constant. Obtain its polar plot

- $$G(j\omega)H(j\omega) = \frac{1}{1+j\omega T} = \frac{1+0j}{1+\omega Tj}$$

$$|G(j\omega)H(j\omega)| = \frac{1}{\sqrt{1+\omega^2T^2}}$$

$$\angle G(j\omega)H(j\omega) = \frac{\tan^{-1}(0)}{\tan^{-1}(\omega T)} = \frac{0^\circ}{\tan^{-1}(\omega T)} = -\tan^{-1}(\omega T)$$

ω	M	ϕ
0	1	0°
$1/T$	$1/\sqrt{2}$	-45°
$\sqrt{3}/T$	$1/2$	-60°
∞	0	-90°



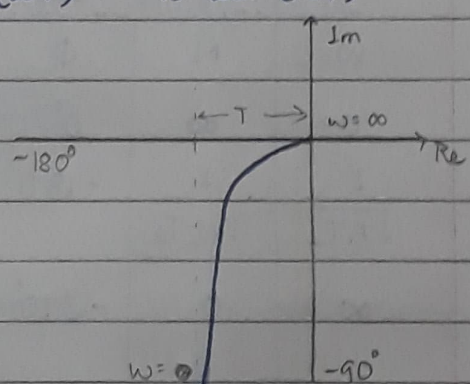
Q: consider a system with open loop transfer function $G(s)H(s) = \frac{1}{s(1+sT)}$. obtain its polar plot.

$$G(j\omega)H(j\omega) = \frac{1}{j\omega(1+j\omega T)} = \frac{1+0j}{0+\omega j} \cdot \frac{1+0j}{1+\omega T j}$$

$$|G(j\omega)H(j\omega)| = \frac{1}{\omega\sqrt{1+\omega^2T^2}}$$

$$\angle G(j\omega)H(j\omega) = \frac{\tan^{-1}(0)}{\tan^{-1}(\infty)} \frac{\tan^{-1}(0)}{\tan^{-1}(\omega T)} = \frac{0^\circ}{90^\circ \tan^{-1}(\omega T)} = -90^\circ - \tan^{-1}(\omega T)$$

ω	M	ϕ
0	∞	-90°
$1/T$	T	-135°
$\sqrt{3}/T$	$T/2\sqrt{3}$	-150°
∞	0	-180°



Q: consider a system with a transfer function of $G(s)H(s) = \frac{1}{(1+sT_1)(1+sT_2)}$. obtain its polar plot

$$G(j\omega)H(j\omega) = \frac{1}{(1+j\omega T_1)(1+j\omega T_2)} = \frac{1+0j}{1+\omega T_1 j} \cdot \frac{1+0j}{1+\omega T_2 j}$$

$$|G(j\omega)H(j\omega)| = \frac{1}{\sqrt{1+\omega^2T_1^2}} \cdot \frac{1}{\sqrt{1+\omega^2T_2^2}} = \frac{1}{\sqrt{(1+\omega^2T_1^2)(1+\omega^2T_2^2)}}$$

$$\angle G(j\omega)H(j\omega) = \frac{\tan^{-1}(0)}{\tan^{-1}(\omega T_1)} \frac{\tan^{-1}(0)}{\tan^{-1}(\omega T_2)} = \frac{0^\circ}{\tan^{-1}(\omega T_1) \cdot \tan^{-1}(\omega T_2)} = -\tan^{-1}(\omega T_1) - \tan^{-1}(\omega T_2)$$

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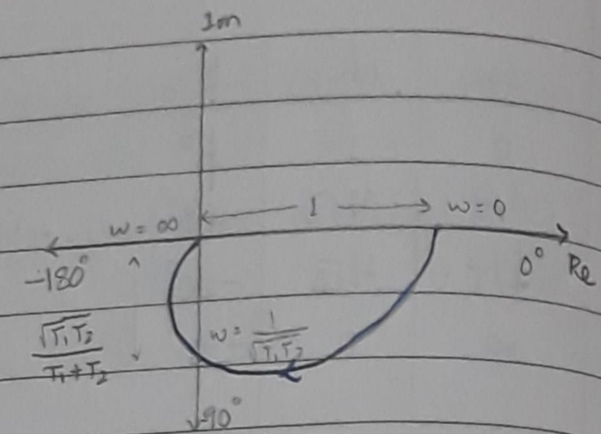
ω	M	ϕ
0	1	0°
$\frac{1}{\sqrt{T_1T_2}}$	$\frac{\sqrt{T_1T_2}}{T_1+T_2}$	-90°
∞	0	-180°



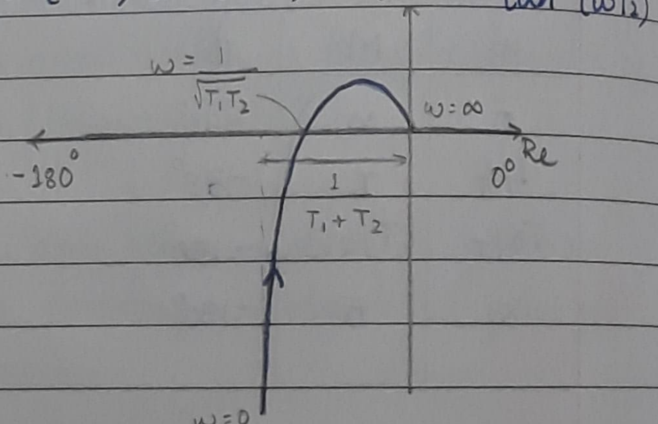
Q: consider a system with open loop transfer function $G(s)H(s) = \frac{1}{s(1+sT_1)(1+sT_2)}$. Obtain its polar plot.

$$G(j\omega)H(j\omega) = \frac{1}{j\omega(1+j\omega T_1)(1+j\omega T_2)}$$

$$|G(j\omega)H(j\omega)| = \frac{1}{\omega\sqrt{(1+\omega^2T_1^2)(1+\omega^2T_2^2)}}$$

$$\angle G(j\omega)H(j\omega) = \frac{\tan^{-1}(0)}{\tan^{-1}(\omega)\tan^{-1}(\omega T_1)\tan^{-1}(\omega T_2)} = -\tan^{-1}(\omega)-\tan^{-1}(\omega T_1)-\tan^{-1}(\omega T_2)$$

ω	M	ϕ
0	∞	0°
$\frac{1}{\sqrt{T_1T_2}}$	$\frac{1}{T_1+T_2}$	-180°
∞	0	-270°



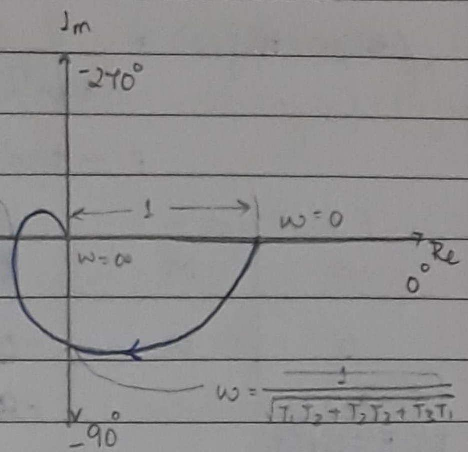
Q: consider a system with open loop transfer function $G(s)H(s) = \frac{1}{(1+sT_1)(1+sT_2)(1+sT_3)}$. Obtain its polar plot.

$$G(j\omega)H(j\omega) = \frac{1}{(1+j\omega T_1)(1+j\omega T_2)(1+j\omega T_3)}$$

$$|G(j\omega)H(j\omega)| = \frac{1}{\sqrt{(1+\omega^2T_1^2)(1+\omega^2T_2^2)(1+\omega^2T_3^2)}}$$

$$\angle G(j\omega)H(j\omega) = \frac{\tan^{-1}(0)}{\tan^{-1}(\omega T_1)\tan^{-1}(\omega T_2)\tan^{-1}(\omega T_3)} = -\tan^{-1}(\omega T_1)-\tan^{-1}(\omega T_2)-\tan^{-1}(\omega T_3)$$

ω	M	ϕ
0	1	0°
$\frac{1}{\sqrt{T_1T_2+T_2T_3+T_3T_1}}$		-90°
$\sqrt{\frac{T_1+T_2+T_3}{T_1T_2T_3}}$		-180°
∞		-270°



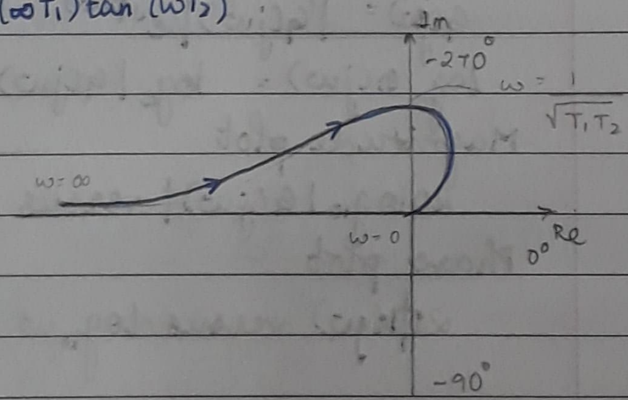
Q: consider a system with open loop transfer function $G(s)H(s) = \frac{1}{s^2(1+sT_1)(1+sT_2)}$ obtain its polar plot.

$$G(j\omega)H(j\omega) = \frac{1}{(j\omega)^2(1+j\omega T_1)(1+j\omega T_2)} = \frac{1}{-\omega^2(1+j\omega T_1)(1+j\omega T_2)}$$

$$|G(j\omega)H(j\omega)| = \frac{1}{\omega^2\sqrt{(1+\omega^2T_1)(1+\omega^2T_2)}}$$

$$\angle G(j\omega)H(j\omega) = \frac{\tan^{-1}(0)}{\tan^{-1}(0)\tan^{-1}(\omega T_1)\tan^{-1}(\omega T_2)} = -\tan^{-1}(\omega T_1) - \tan^{-1}(\omega T_2)$$

ω	M	ϕ
0	∞	0°
$\frac{1}{\sqrt{T_1T_2}}$		-270°
∞	0	0



Q: consider a system with open loop transfer function $G(s)H(s) = \frac{1}{s^2(1+sT_1)(1+sT_2)(1+sT_3)}$. obtain its polar plot.

$$G(j\omega)H(j\omega) = \frac{1}{-\omega^2(1+j\omega T_1)(1+j\omega T_2)(1+j\omega T_3)}$$

$$|G(j\omega)H(j\omega)| = \frac{1}{\omega^2\sqrt{(1+\omega^2T_1^2)(1+\omega^2T_2^2)(1+\omega^2T_3^2)}}$$

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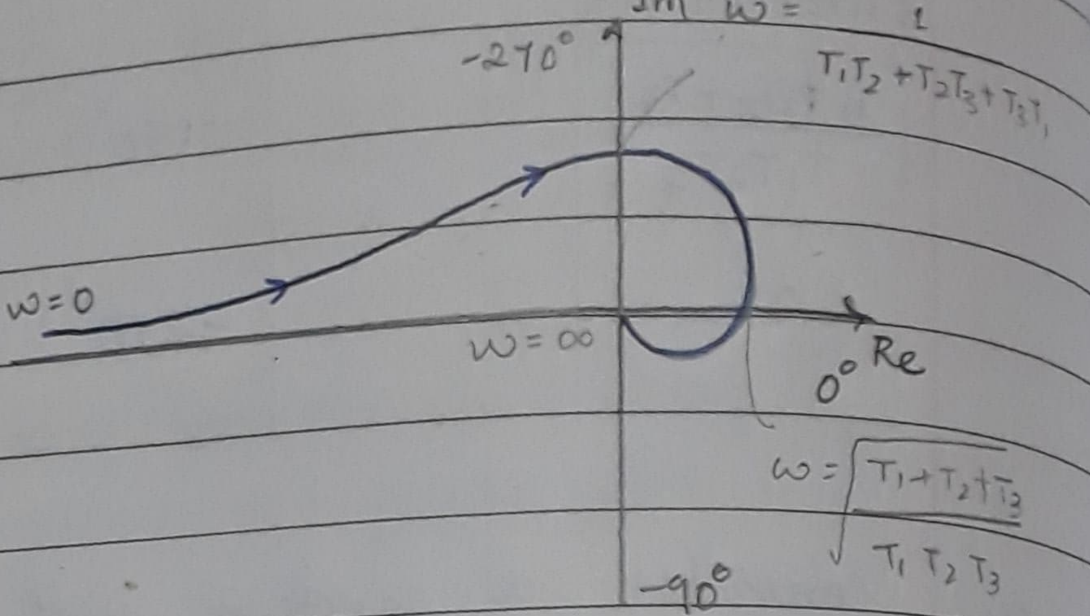
$$\angle G(j\omega)H(j\omega) = \frac{\tan^{-1}(0)}{\tan^{-1}(\infty)\tan^{-1}(\omega T_1)\tan^{-1}(\omega T_2)\tan^{-1}(\omega T_2)}$$

$$= -\tan^{-1}(\omega T_1) - \tan^{-1}(\omega T_2) - \tan^{-1}(\omega T_2)$$

ω	M	ϕ
0	∞	-0°
$\frac{1}{T_1T_2+T_2T_3+T_3T_1}$		-270°
$\sqrt{\frac{T_1+T_2+T_3}{T_1T_2T_3}}$		0°
∞	0	-270°



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* **ω_{gc} and ω_{pc} in polar plot:**

Draw a circle of radius 1 and center as origin. The point where this circle meets the polar plot is the point where $|G(j\omega)H(j\omega)| = 1$ and corresponding frequency is $\omega = \omega_{gc}$.

The frequency at which $\angle G(j\omega)H(j\omega) = -180^\circ$ is ω_{pc}. A point on negative real axis is $\omega = \omega_{pc}$.

$\omega_{gc} < \omega_{pc}$: GM and PM are positive : stable

$\omega_{gc} > \omega_{pc}$: GM and PM are negative : unstable

$\omega_{gc} = \omega_{pc}$: GM and PM are zero : marginally stable

* **Nyquist Plot Analysis:**

- **Pole - Zero Configuration:**

Poles of $1 + G(s)H(s)$ = open loop poles of a system

Zeros of $1 + G(s)H(s)$ = closed loop poles of a system.

So from Nyquist analysis, the system is absolutely stable if all the zeros of the $1 + G(s)H(s)$ i.e, closed loop poles of the system are located in the left half of s plane.

- **Encirclement:**

A point is said to be encircled by a closed path if it is found to lie inside that closed path.

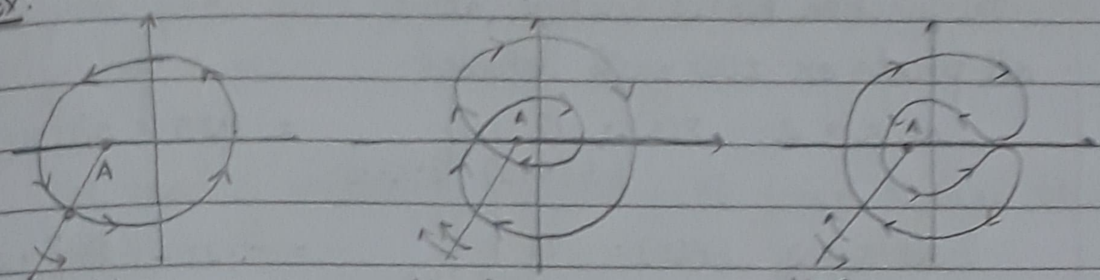
counting number of encirclements

Draw a vector from a point whose encirclements are to be determined. Identify the number of intersections of this vector with a closed path. The oppositely directed encirclements cancel each other. The remaining encirclements gives us the number of encirclements of that point.

Anticlockwise encirclements are positive and clockwise encirclements are negative.

Number of encirclements is denoted by N

Ex:



N = +1   N = -2   N = 0

- Analytic Function and singularities:

A mathematical function is said to be analytic at a point in a plane if its value and its derivative has finite existance at that point.

If at a point in the plane, the value of the function or its derivative is infinite, the function is non analytic at that point and such a point is called sigularity of the function.

Ex: $F(s) = \frac{25}{s(s+1)}$ $s=0$ and $s=-1$: poles are its singularities as $F(s) = \infty$ at $s = 0, -1$

Ex: $F(s) = \sqrt{s}$ $s = \pm 5$ for 25. Hence the function is not single valued.

- Mapping theorem or principle of argument:

The mapped locus $z'(s)$ encircles the new origin of F-plane as many times as the difference between the number of zeros and poles of $F(s)$ which are encircled by $z(s)$ path in s plane.

i.e., $N = Z - P$

N: number of encirclements of origin of F plane by $z'(s)$ path

P: number of poles of $F(s)$ encircled by $z(s)$ path in s plane

Z: number of zeros of $F(s)$ encircled by $z(s)$ path in s plane

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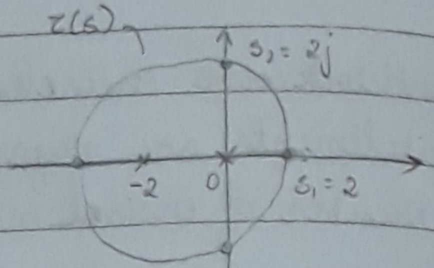
Ex : $F(s) = \frac{10}{s(s+2)}$

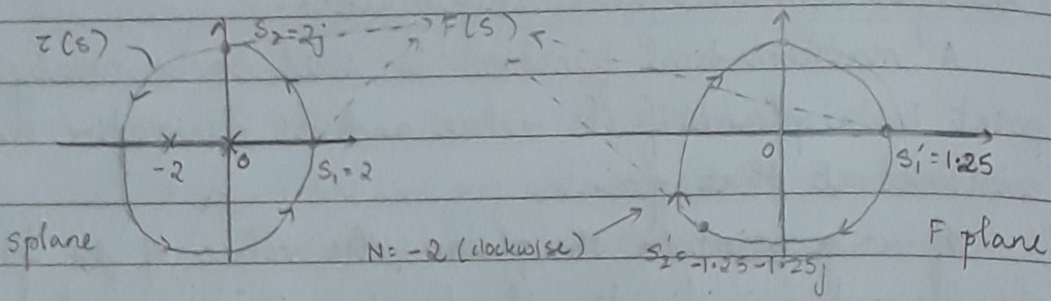


$Z = 0 \quad P = 2$

the function $F(s)$ is analytic at all points on $z(s)$ path selected.

at $s_1 = 2$ $\quad F(s) = \frac{10}{s(s+2)}\Big|_{s=s_1=2} = 1.25 = s_1'$

at $s_2 = 2j$ $\quad F(s) = \frac{10}{s(s+2)}\Big|_{s=s_2=2j} = -1.25 - 1.25j = s_2'$



Nyquist Stability Criterion:

By mapping theorem

$$N = Z - P$$

Z: number of zeros of $1 + G(s)H(s)$ encircled by Nyquist path in splane.

Nyquist path is entire right half of splane From $s = +j\infty$ to $s = -j\infty$ of radius ∞.

Z : number of zeros of $1 + G(s)H(s)$ is the closed loop poles of system. Hence $Z = 0$ for stability has no zero of $1 + G(s)H(s)$ should be in right of s plane.

Therefore Nyquist stability criterion is

$$\boxed{N = -P}$$

Nyquist criterion states that " For absolute stability of the system the number of encirclements of the new origin of F-plane by Nyquist plot must be equal to number of poles of $1 + G(s)H(s)$ i.e., poles of $G(s)H(s)$ which are in the right half of splane and in clockwise direction ".

Ex: $G(s)H(s) = \frac{10}{s(s+1)}$ P = Number of poles of $G(s)H(s)$ which are on right half of s plane.

$N = -P = 0$ the nyquist plot obtained by mapping Nyquist path from s plane to F plane should not encircle origin of F plane.

NOTE:

For ease of mapping Nyquist path from s plane to F plane instead of $1 + G(s)H(s)$, $G(s)H(s)$ is only considered

then $N = -P$

where P = number of poles of $G(s)H(s)$ in right half of s-plane.

N = number of encirclements of a critical point $-1+0j$ of F plane by Nyquist plot instead of number of encirclements of an origin.

Nyquist contour

Section I: $s = +j\infty$ to $s = +j0$ Poles at the origin

section II: $s = +j0$ to $s = -j0$ Section I and Section II

section III: $s = -j0$ to $s = -j\infty$ are mirror images

section IV: $s = -j\infty$ to $s = +j\infty$ (semicircle) $+90$ to -90°

Q: A unity feedback system has a loop transfer function $G(s) = \frac{50}{(s+1)(s+2)}$. Use Nyquist criterion to determine the system stablity.

$$G(s)H(s) = \frac{50}{(s+1)(s+2)}$$

Put $s = j\omega$

$$G(j\omega)H(j\omega) = \frac{50}{(j\omega+1)(j\omega+2)}$$

$$M = |G(j\omega)H(j\omega)| = \frac{50}{\sqrt{1+\omega^2}\sqrt{4+\omega^2}}$$

$$\phi = \angle G(j\omega)H(j\omega) = -\tan^{-1}\omega - \tan^{-1}\frac{\omega}{2}$$