

UNIT - 1

Fourier Series

★ Periodic Function:

Any function that repeats for certain time interval is called periodic function.

Ex: sine and cosine are periodic functions of period 2π .
 $f(x+n\pi) = f(x)$ with period T .

NOTE:

$$1. \int e^{ax} \cos bx dx = \frac{e^{ax}}{a^2 + b^2} [a \cos bx + b \sin bx]$$

$$2. \int e^{ax} \sin bx dx = \frac{e^{ax}}{a^2 + b^2} [a \sin bx - b \cos bx]$$

$$3. \cos n\pi = \cos(-n\pi) = (-1)^n$$

$$4. \sin n\pi = \sin(-n\pi) = 0$$

★ Fourier Series:

Fourier Series Expansion for the period λ in the interval $[-\lambda, \lambda]$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{\lambda}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{\lambda}\right)$$

$$\text{where } a_0 = \frac{1}{\lambda} \int_{-\lambda}^{\lambda} f(x) dx$$

$$a_n = \frac{1}{\lambda} \int_{-\lambda}^{\lambda} f(x) \cos\left(\frac{n\pi x}{\lambda}\right) dx$$

$$b_n = \frac{1}{\lambda} \int_{-\lambda}^{\lambda} f(x) \sin\left(\frac{n\pi x}{\lambda}\right) dx$$

* Bernoulli's Rule:

$$\int u v dx = u \int v dx - u' \int \int v dx + u'' \int \int \int v dx - \dots$$

Q1: $\int x \cos nx dx$

Sol:

$$\begin{aligned} &= x \int \cos nx dx - \int \sin nx dx \\ &= x \left(\frac{\sin nx}{n} \right) - \left(-\frac{\cos nx}{n^2} \right) \\ &= \frac{x \sin nx}{n} + \frac{\cos nx}{n^2} // \end{aligned}$$

Q2: $\int x^2 \sin nx dx$

Sol:

$$\begin{aligned} &= x^2 \int \sin nx dx - 2x \int \sin nx dx + 2 \int \int \sin nx dx \\ &= n^2 \left(-\frac{\cos nx}{n} \right) - 2x \left(-\frac{\sin nx}{n^2} \right) + 2 \left(\frac{\cos nx}{n^3} \right) \\ &= -\frac{n^2 \cos nx}{n} + \frac{2x \sin nx}{n^2} + \frac{2 \cos nx}{n^3} // \end{aligned}$$

Q3: $\int e^{2x} \cos nx dx$

Sol:

$$= \frac{e^{2x}}{4+n^2} [2 \cos nx + n \sin nx] //$$

Q4: $\int (x-x^2) \sin nx dx$

Sol:

$$\begin{aligned} &= (x-x^2) \int \sin nx dx - (1-2x) \int \int \sin nx dx + (-2) \int \int \int \sin nx dx \\ &= (x-x^2) \left(-\frac{\cos nx}{n} \right) - (1-2x) \left(-\frac{\sin nx}{n^2} \right) - 2 \left(\frac{\cos nx}{n^3} \right) \\ &= -\frac{(x-x^2) \cos nx}{n} + \frac{(1-2x) \sin nx}{n^2} - \frac{2 \cos nx}{n^3} // \end{aligned}$$

Q1:

Obtain the Fourier series for the following functions:
 $f(x) = x - x^2$ in $(-1, 1)$

Sol:

The Fourier series is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right) \quad (1)$$

$$\text{where } a_0 = \frac{1}{l} \int_{-l}^l f(x) dx$$

$$a_0 = \frac{1}{1} \int_{-1}^1 (x - x^2) dx$$

$$a_0 = \left[\frac{x^2}{2} - \frac{x^3}{3} \right]_{-1}^1$$

$$a_0 = \left[\frac{1}{2} - \frac{1}{3} - \frac{1}{2} - \frac{1}{3} \right]$$

$$a_0 = -\frac{2}{3} \Rightarrow \frac{a_0}{2} = -\frac{1}{3} //$$

$$a_n = \frac{1}{l} \int_{-l}^l f(x) \cos\left(\frac{n\pi x}{l}\right) dx$$

$$= \frac{1}{1} \int_{-1}^1 (x - x^2) \cos n\pi x dx$$

$$= \left[(x - x^2) \left(\frac{\sin n\pi x}{n\pi} \right) - (1 - 2x) \left(-\frac{\cos n\pi x}{n^2\pi^2} \right) + (-2) \left(\frac{-\sin n\pi x}{n^3\pi^3} \right) \right]_{-1}^1$$

$$= \frac{1}{n\pi} \left[(x - x^2) \cancel{\sin n\pi x} + (1 - 2x) \cancel{\cos n\pi x} + 2 \cancel{\sin n\pi x} \right]_{-1}^1$$

$$= \frac{1}{n^2\pi^2} \left[(1 - 2x) \cos n\pi x \right]_{-1}^1$$

$$= \frac{1}{n^2\pi^2} \left[(-1)(-1)^n - (3)(-1)^n \right]$$

$$a_n = \frac{-4(-1)^n}{n^2\pi^2} = \frac{4(-1)^{n+1}}{n^2\pi^2} //$$

$$b_n = \frac{1}{l} \int_{-l}^l f(x) \sin\left(\frac{n\pi x}{l}\right) dx$$

$$= \frac{1}{l} \int_{-l}^l (x-x^2) \sin n\pi x dx$$

$$= \left[(x-x^2) \left(-\frac{\cos n\pi x}{n\pi} \right) - (1-2x) \left(-\frac{\sin n\pi x}{n^2\pi^2} \right) + (-2) \left(\frac{\cos n\pi x}{n^3\pi^3} \right) \right]_{-l}^l$$

$$= \frac{1}{n\pi} \left[-(x-x^2) \cos n\pi x + (1-2x) \frac{\sin n\pi x}{n\pi} - \frac{2 \cos n\pi x}{n^2\pi^2} \right]_{-l}^l$$

$$= \frac{1}{n\pi} \left[0 - \frac{2(-1)^n}{n^2\pi^2} - 2(-1)^n + \frac{2(-1)^n}{n^2\pi^2} \right]$$

$$b_n = \frac{2(-1)^{n+1}}{n\pi} //$$

Substituting a_0, a_n and b_n in eq ①

$$f(x) = \frac{-1}{3} + \sum_{n=1}^{\infty} \frac{4(-1)^{n+1}}{n^2\pi^2} \cos n\pi x + \sum_{n=1}^{\infty} \frac{2(-1)^{n+1}}{n\pi} \sin n\pi x //$$

Q2: $f(x) = e^{-x}$ in $-l \leq x \leq l$

Sol:

The Fourier series is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right) --- ①$$

$$a_0 = \frac{1}{l} \int_{-l}^l f(x) dx$$

$$= \frac{1}{l} \int_{-l}^l e^{-x} dx$$

$$= \frac{1}{l} \left[-e^{-x} \right]_{-l}^l$$

$$= \frac{1}{l} [-e^{-l} + e^l]$$

$$\frac{a_0}{2} = \frac{\sinhl}{l} //$$

$$\begin{aligned}
 a_n &= \frac{1}{l} \int_{-l}^l f(x) \cos\left(\frac{n\pi x}{l}\right) dx \\
 &= \frac{1}{l} \int_{-l}^l e^{-x} \cos\left(\frac{n\pi x}{l}\right) dx \\
 &= \frac{1}{l} \left[\frac{e^{-x}}{1 + (n\pi/l)^2} \left(\frac{n\pi}{l} \cos\left(\frac{n\pi x}{l}\right) + \frac{n\pi}{l} \sin\left(\frac{n\pi x}{l}\right) \right) \right]_{-l}^l \\
 &= \frac{1}{l} \left[\frac{x^2}{l^2 + n^2 \pi^2} \left(-e^{-x} \cos\left(\frac{n\pi x}{l}\right) \right) \Big|_{-l}^l \right] \\
 &= \frac{l}{l^2 + n^2 \pi^2} \left[-e^{-l} (-1)^n + e^l (-1)^n \right] \\
 &= \frac{2l(-1)^n}{l^2 + n^2 \pi^2} \left[\frac{e^l - e^{-l}}{2} \right]
 \end{aligned}$$

$$a_n = 2 \sinhl \cdot \frac{l(-1)^n}{l^2 + n^2 \pi^2} //$$

$$\begin{aligned}
 b_n &= \frac{1}{l} \int_{-l}^l f(x) \sin\left(\frac{n\pi x}{l}\right) dx \\
 &= \frac{1}{l} \int_{-l}^l e^{-x} \sin\left(\frac{n\pi x}{l}\right) dx \\
 &= \frac{1}{l} \left[\frac{e^{-x}}{1 + (n\pi/l)^2} \left(-\sin\left(\frac{n\pi x}{l}\right) - \frac{n\pi}{l} \cos\left(\frac{n\pi x}{l}\right) \right) \right]_{-l}^l \\
 &= \frac{n\pi}{l^2} \left[\frac{x^2}{l^2 + n^2 \pi^2} \left(-e^x \cos\left(\frac{n\pi x}{l}\right) \right) \Big|_{-l}^l \right] \\
 &= \frac{n\pi}{l^2 + n^2 \pi^2} \left[-e^l (-1)^n + e^{-l} (-1)^n \right] \\
 &= -\frac{2n\pi(-1)^n}{l^2 + n^2 \pi^2} \left[\frac{e^l - e^{-l}}{2} \right]
 \end{aligned}$$

$$b_n = -\frac{2n\pi(-1)^n}{l^2 + n^2 \pi^2} \sinhl //$$

Substituting a_0, a_n and b_n in eq ①

$$f(x) = \frac{\sinhl}{l} + \sum_{n=1}^{\infty} \frac{2l(-1)^n}{l^2 + n^2 \pi^2} \cos\frac{n\pi x}{l} - \sum_{n=1}^{\infty} \frac{2n\pi(-1)^n}{l^2 + n^2 \pi^2} \sinhl \sin\frac{n\pi x}{l} //$$

Q3: $f(x) = x^2 - 2$ in $(-2 \leq x \leq 2)$

Sol: The Fourier series is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right)$$

$$a_0 = \frac{1}{l} \int_{-l}^l f(x) dx$$

$$= \frac{1}{2} \int_{-2}^2 (x^2 - 2) dx$$

$$= \frac{1}{2} \left[\frac{x^3}{3} - 2x \right]_{-2}^2$$

$$= \frac{1}{2} \left[\frac{8}{3} - 4 + \frac{8}{3} - 4 \right]$$

$$= \frac{1}{2} \left[-\frac{8}{3} \right]$$

$$\frac{a_0}{2} = \frac{-2}{3}$$

$$a_n = \frac{1}{l} \int_{-l}^l f(x) \cos\left(\frac{n\pi x}{l}\right) dx$$

$$= \frac{1}{2} \int_{-2}^2 (x^2 - 2) \cos\left(\frac{n\pi x}{2}\right) dx$$

$$= \frac{1}{2} \left[(x^2 - 2) \left(\sin\left(\frac{n\pi x}{2}\right) \right) \frac{2}{n\pi} - 2x \left(-\cos\left(\frac{n\pi x}{2}\right) \right) \frac{4}{n^2\pi^2} + 2 \left(-\sin\left(\frac{n\pi x}{2}\right) \right) \right]_{-2}^2$$

$$= \frac{1}{n\pi} \left[(x^2 - 2) \sin\left(\frac{n\pi x}{2}\right) + \frac{4x}{n\pi} \cos\left(\frac{n\pi x}{2}\right) - \frac{8}{n^2\pi^2} \sin\left(\frac{n\pi x}{2}\right) \right]_{-2}^2$$

$$= \frac{4}{n^2\pi^2} \left[x \cos\left(\frac{n\pi x}{2}\right) \right]_{-2}^2$$

$$= \frac{4}{n^2\pi^2} \left[2(-1)^n + 2(-1)^n \right]$$

$$a_n = \frac{16(-1)^n}{n^2\pi^2}$$

$$\begin{aligned}
 b_n &= \frac{1}{\pi} \int_{-2}^2 f(x) \sin\left(\frac{n\pi x}{2}\right) dx \\
 &= \frac{1}{2} \int_{-2}^2 (x^2 - 2) \sin\left(\frac{n\pi x}{2}\right) dx \\
 &= \frac{1}{2} \left[\left(x^2 - 2 \right) \left(-\cos\left(\frac{n\pi x}{2}\right) \right) \cdot \frac{2}{n\pi} - 2x \left(-\sin\left(\frac{n\pi x}{2}\right) \right) \cdot \frac{4}{n^2\pi^2} + 2 \left(\cos\left(\frac{n\pi x}{2}\right) \right) \cdot \frac{8}{n^3\pi^3} \right]_{-2}^2 \\
 &= \frac{1}{n\pi} \left[-(x^2 - 2) \cos\left(\frac{n\pi x}{2}\right) + \frac{4x}{n^2\pi^2} \sin\left(\frac{n\pi x}{2}\right) + \frac{8}{n^3\pi^3} \cos\left(\frac{n\pi x}{2}\right) \right]_{-2}^2 \\
 &= \frac{1}{n\pi} \left[-(2)(-1)^n + \frac{8}{n^2\pi^2} (-1)^n + (2)(-1)^n - \frac{8}{n^2\pi^2} (-1)^n \right]
 \end{aligned}$$

$$\underline{b_n = 0}$$

Substituting a_0, a_n and b_n

$$f(x) = \frac{-2}{3} + \sum_{n=1}^{\infty} \frac{16(-1)^n}{n^2\pi^2} \cos\left(\frac{n\pi x}{2}\right) //$$

NOTE:

- It is also possible to deduce a particular series from the Fourier series of the given function $f(x)$ in a given interval. We have to substitute a suitable value for x in a given interval. Normally it takes the value to be either of the end points or the middle point. The resulting series will be equal to the value discussed in convergence.
- If $f(x)$ is discontinuous at x , then the series converges to $\frac{1}{2} [f(x^+) + f(x^-)]$, where $f(x^+)$ and $f(x^-)$ are respectively right hand limit and left hand limit of $f(x)$.

Q1: Obtain the Fourier series for the function

$$f(x) = \begin{cases} -1 & \text{in } -1 < x < 0 \\ x & \text{in } 0 < x < 1 \end{cases}$$

Hence deduce the sum of the reciprocal squares of the odd integers is equal to $\frac{\pi^2}{8}$.

Sol: The Fourier series is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right)$$

$$a_0 = \frac{1}{l} \int_{-l}^l f(x) dx$$

$$= \frac{1}{l} \int_{-l}^0 f(x) dx + \frac{1}{l} \int_0^l f(x) dx$$

$$= \frac{1}{l} \int_{-l}^0 -l dx + \frac{1}{l} \int_0^l x dx$$

$$= -l \left[x \right]_{-l}^0 + \frac{1}{l} \left[\frac{x^2}{2} \right]_0^l$$

$$= -l(0+l) + \frac{1}{l} \left(\frac{l^2}{2} \right)$$

$$= -l + \frac{l}{2}$$

$$\frac{a_0}{2} = \frac{-l}{4} //$$

$$a_n = \frac{1}{l} \int_{-l}^l f(x) \cos\left(\frac{n\pi x}{l}\right) dx$$

$$= \frac{1}{l} \left[\int_{-l}^0 f(x) \cos\left(\frac{n\pi x}{l}\right) dx + \int_0^l f(x) \cos\left(\frac{n\pi x}{l}\right) dx \right]$$

$$= \frac{1}{l} \left[\int_{-l}^0 -l \cos\left(\frac{n\pi x}{l}\right) dx + \int_0^l x \cos\left(\frac{n\pi x}{l}\right) dx \right]$$

$$= \frac{1}{l} \left[-l \left(\sin\left(\frac{n\pi x}{l}\right) \right) \Big|_{-l}^0 + x \left(\sin\left(\frac{n\pi x}{l}\right) \right) \Big|_{-l}^0 + \left(\cos\left(\frac{n\pi x}{l}\right) \right) \Big|_{-l}^0 \right]$$

$$= \frac{l}{n^2 \pi^2} \left[\cos\left(\frac{n\pi x}{l}\right) \right]_0^l$$

$$= \frac{l}{n^2 \pi^2} [(-1)^n - 1]$$

$$a_n = \frac{[(-1)^n - 1] l}{n^2 \pi^2} //$$

$$\begin{aligned}
 b_n &= \frac{1}{l} \int_{-l}^l f(x) \sin\left(\frac{n\pi x}{l}\right) dx \\
 &= \frac{1}{l} \left[\int_{-l}^0 f(x) \sin\left(\frac{n\pi x}{l}\right) dx + \int_0^l f(x) \sin\left(\frac{n\pi x}{l}\right) dx \right] \\
 &= \frac{1}{l} \left[\int_{-l}^0 -l \sin\left(\frac{n\pi x}{l}\right) dx + \int_0^l x \sin\left(\frac{n\pi x}{l}\right) dx \right] \\
 &= \frac{1}{l} \left[-l \left(-\cos\left(\frac{n\pi x}{l}\right) \right) \Big|_0^l + \left[x \left(-\cos\left(\frac{n\pi x}{l}\right) \right) \Big|_0^l - \left(-\sin\left(\frac{n\pi x}{l}\right) \right) \Big|_0^l \right] \right] \\
 &= \frac{1}{n\pi} \left[l \cos\left(\frac{n\pi x}{l}\right) \Big|_{-l}^0 - x \cos\left(\frac{n\pi x}{l}\right) \Big|_0^l \right] \\
 &= \frac{1}{n\pi} \left[l(1) - l(-1)^n - l(-1)^n + 0 \right]
 \end{aligned}$$

$$b_n = \frac{l}{n\pi} [1 - 2(-1)^n] //$$

Substituting a_0 , a_n and b_n

$$f(x) = \frac{-l}{4} + \sum_{n=1}^{\infty} \frac{[(-1)^n - 1]l}{n^2\pi^2} \cos\left(\frac{n\pi x}{l}\right) + \sum_{n=1}^{\infty} \frac{[1 - 2(-1)^n]l}{n\pi} \sin\left(\frac{n\pi x}{l}\right) //$$

To deduce the given series, in the Fourier series choose x value equal to 0, since the given function is discontinuous at $x=0$ the series converges at $x=0$

$$f(0) = \frac{f(x^+) + f(x^-)}{2} = \frac{x-l}{2} = \frac{0-l}{2} = \frac{-l}{2} //$$

$$f(0) = \frac{-l}{4} + \frac{l}{\pi^2} \sum_{n=1}^{\infty} \frac{[(-1)^n - 1]}{n^2} + \frac{l}{\pi} \sum_{n=1}^{\infty} \frac{[1 - 2(-1)^n]}{n} (0)$$

$$\frac{-l}{2} = \frac{-l}{4} + \frac{l}{\pi^2} \sum_{n=1}^{\infty} \frac{[(-1)^n - 1]}{n^2}$$

$$\text{But } 1 - (-1)^n = \begin{cases} 0 & \text{for } n \text{ is even} \\ 2 & \text{for } n \text{ is odd} \end{cases}$$

$$\frac{-l}{2} = \frac{-l}{4} + \frac{l}{\pi^2} \sum_{n=1,3,5}^{\infty} \frac{2}{n^2}$$

$$\frac{l}{4} = \frac{2l}{\pi^2} \left[\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots \right]$$

$$\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots // \text{ Hence Proved}$$

* Even and Odd Function : (in interval $[-1, 1]$)

If $f(-x) = f(x)$ is an even function such as x^2 ,
and if $f(-x) = -f(x)$ it is an odd function such as x .

Ex: $f(x) = x^2$ in $-1 < x < 1$

replace x by $(-x)$

$$f(-x) = (-x)^2$$

$$f(-x) = x^2$$

$$f(-x) = f(x)$$

Hence x^2 is an even function

NOTE:

- Even function \times Even function } Even function
- Odd function \times Odd function } Even function
- Even function \times Odd function } Odd function
- Odd function \times Even function } Odd function
- $\int_{-a}^a f(x) dx = \begin{cases} 2 \int_0^a f(x) dx & \text{when } f(x) \text{ is even function} \\ 0 & \text{when } f(x) \text{ is odd function} \end{cases}$

We know that

$$a_0 = \frac{1}{\pi} \int_{-1}^1 f(x) dx$$

$$a_n = \frac{1}{\pi} \int_{-1}^1 f(x) \cos\left(\frac{n\pi x}{\pi}\right) dx$$

$$b_n = \frac{1}{\pi} \int_{-1}^1 f(x) \sin\left(\frac{n\pi x}{\pi}\right) dx$$

CASE 1: When $f(x)$ is an even function

$$a_0 = \frac{2}{l} \int_0^l f(x) dx$$

$$a_n = \frac{2}{l} \int_0^l f(x) \cos\left(\frac{n\pi x}{l}\right) dx$$

$$b_n = 0$$

CASE 2: When $f(x)$ is an odd function

$$a_0 = 0$$

$$a_n = 0$$

$$b_n = \frac{2}{l} \int_0^l f(x) \sin\left(\frac{n\pi x}{l}\right) dx$$

NOTE:

- If $f(x)$ is a discontinuous function such that

$$f(x) = \begin{cases} \phi(x) & \text{in } -l < x < 0 \\ \psi(x) & \text{in } 0 < x < l \end{cases}$$

then if $\phi(-x) = \psi(x)$ then it is even function
and if $\phi(-x) = -\psi(x)$ then it is odd function.

Q1: Obtain the Fourier series for the function $f(x) = x^2$ in $-l < x < l$

Hence deduce $\frac{\pi^2}{12} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \dots$

Sol: $f(x) = x^2$

Replacing x by $-x$

$$f(-x) = (-x)^2$$

$$f(-x) = x^2$$

$f(-x) = f(x) \therefore$ the given function is even function

Hence:

$$a_0 = \frac{2}{l} \int_0^l f(x) dx$$

$$= \frac{2}{l} \int_0^l x^2 dx$$

$$a_0 = \frac{2}{l} \left[\frac{x^3}{3} \right]_0^l$$

$$= \frac{2}{l} \left[\frac{l^3}{3} \right]$$

$$a_0 = \frac{2l^2}{3} \Rightarrow a_0 = \frac{l^2}{3} //$$

$$a_n = \frac{2}{l} \int_0^l f(x) \cos\left(\frac{n\pi x}{l}\right) dx$$

$$= \frac{2}{l} \int_0^l x^2 \cos\left(\frac{n\pi x}{l}\right) dx$$

$$= \frac{2}{l} \left[x^2 \left(\sin \frac{n\pi x}{l} \right) \frac{l}{n\pi} - 2x \left(-\cos \frac{n\pi x}{l} \right) \frac{l^2}{n^2\pi^2} + \text{...} \right]_0^l$$

$$= \frac{2}{l} \left[\frac{2\pi l^2}{n^2\pi^2} \cos \frac{n\pi x}{l} \right]_0^l$$

$$= \frac{4l}{n^2\pi^2} [l \cos n\pi - 0]$$

$$a_n = \frac{4l^2(-1)^n}{n^2\pi^2} //$$

$$b_n = 0 //$$

The Fourier series is given by

$$f(x) = \frac{l^2}{3} + \sum_{n=1}^{\infty} \frac{4l^2(-1)^n}{n^2\pi^2} \cos\left(\frac{n\pi x}{l}\right)$$

$$\text{let } x=0$$

$$f(0) = 0$$

$$f(0) = \frac{l^3}{3} + \frac{4l^2}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos 0$$

$$\text{But } (-1)^n = \begin{cases} 1 & \text{for } n \text{ is even} \\ -1 & \text{for } n \text{ is odd} \end{cases}$$

$$\therefore 0 = \frac{l^3}{3} + \frac{4l^2}{\pi^2} \left[\sum_{n=1,3,5}^{\infty} \frac{-1}{n^2} + \sum_{n=2,4,6}^{\infty} \frac{1}{n^2} \right]$$

$$-\frac{l^3}{3} = \frac{4l^2}{\pi^2} \left[-\frac{1}{1^2} - \frac{1}{3^2} + \frac{1}{2^2} + \frac{1}{4^2} \dots \right]$$

$$\frac{-\pi}{12} = -\frac{1}{1^2} + \frac{1}{2^2} - \frac{1}{3^2} + \frac{1}{4^2} - \dots$$

$$\therefore \frac{\pi}{12} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \dots //$$

Q2: $f(x) = |x|$ in $-l < x < l$, deduce $\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots$

Sol: $f(x) = |x|$

replacing x by $-x$

$$f(-x) = |-x|$$

$$f(-x) = |x|$$

$$f(-x) = f(x)$$

Hence \therefore the given function is even function.

$$a_0 = \frac{2}{l} \int_0^l f(x) dx$$

$$= \frac{2}{l} \int_0^l x dx$$

$$= \frac{2}{l} \left[\frac{x^2}{2} \right]_0^l$$

$$= \frac{l^2}{l}$$

$$a_0 = l \Rightarrow \frac{a_0}{2} = \frac{l}{2} //$$

$$a_n = \frac{2}{l} \int_0^l f(x) \cos \left(\frac{n\pi x}{l} \right) dx$$

$$= \frac{2}{l} \int_0^l x \cos \left(\frac{n\pi x}{l} \right) dx$$

$$= \frac{2}{l} \left[x \left(\sin \frac{n\pi x}{l} \right) \Big|_0^l - \left(-\cos \frac{n\pi x}{l} \right) \Big|_0^l \right]$$

$$= \frac{2l}{n^2 \pi^2} \left[\cos \frac{n\pi l}{l} \right]_0^l$$

considering
 $f(x) = \begin{cases} -x & \text{in } -l < x < 0 \\ x & \text{in } 0 < x < l \end{cases}$

$$= \frac{2l}{n^2\pi^2} [\cos n\pi - \cos 0]$$

$$a_n = \frac{2l [(-1)^n - 1]}{n^2\pi^2} //$$

$$b_n = 0 //$$

Therefore the Fourier series is given by

$$f(x) = \frac{d}{2} + \sum_{n=1}^{\infty} \frac{2l [(-1)^n - 1]}{n^2\pi^2} \cos\left(\frac{n\pi x}{l}\right)$$

$$\text{let } n = 0$$

$$f(0) = 0$$

$$f(0) = \frac{d}{2} + \frac{2l}{\pi^2} \sum_{n=1}^{\infty} \frac{[(-1)^n - 1]}{n^2}$$

$$\text{But } [(-1)^n - 1] = \begin{cases} -2 & \text{for } n \text{ is odd} \\ 0 & \text{for } n \text{ is even} \end{cases}$$

$$\therefore 0 = \frac{d}{2} + \frac{2l}{\pi^2} \left[\sum_{n=2,4,6,\dots}^{\infty} \frac{-2}{n^2} \right]$$

$$\frac{d}{2} = \frac{4l}{\pi^2} \left[\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots \right]$$

$$\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots //$$

$$\text{Q3: } f(x) = \begin{cases} -k & \text{in } -2 < x < 0 \\ k & \text{in } 0 < x < 2 \end{cases} \quad \text{Deduce: } \frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} + \dots$$

Sol: consider

$$\phi(x) = -k$$

replacing x by $-x$

$$\phi(-x) = -k$$

$\phi(-x) = -\psi(x)$: The given function is odd function.

Hence

$$a_0 = 0 //$$

$$a_n = 0 //$$

$$\begin{aligned}
 b_n &= \frac{2}{\pi} \int_0^{\pi} f(x) \sin\left(\frac{n\pi x}{\pi}\right) dx \\
 &= \frac{2}{\pi} \int_0^{\pi} k \sin\left(\frac{n\pi x}{\pi}\right) dx \\
 &= k \left[\left(-\cos\left(\frac{n\pi x}{\pi}\right) \cdot \frac{\pi}{n\pi} \right) \right]_0^{\pi} \\
 &= -\frac{2k}{n\pi} [\cos n\pi - \cos 0]
 \end{aligned}$$

$$b_n = -\frac{2k}{n\pi} [(-1)^n - 1] //$$

Therefore the Fourier series is

$$f(x) = \sum_{n=1}^{\infty} -\frac{2k}{n\pi} [(-1)^n - 1] \sin\left(\frac{n\pi x}{\pi}\right)$$

$$\text{let } x=1$$

$$f(1) = k$$

$$f(1) = -\frac{2k}{\pi} \sum_{n=1,3,5,\dots}^{\infty} -\frac{2}{n} \sin \frac{n\pi}{2}$$

$$\text{But } [(-1)^n - 1] = \begin{cases} -2 & \text{for } n \text{ is odd} \\ 0 & \text{for } n \text{ is even} \end{cases}$$

$$\therefore k = +\frac{4k}{\pi} \left[\frac{1}{1} \sin \frac{\pi}{2} + \frac{1}{3} \sin \frac{3\pi}{2} + \frac{1}{5} \sin \frac{5\pi}{2} + \dots \right]$$

$$\frac{\pi}{4} = 1 + \frac{1}{3} (-1) + \frac{1}{5} (1) + \dots$$

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} \dots //$$

$$\text{Q4: } f(x) = \begin{cases} 1 + 4x/3 & \text{in } -3/2 < x < 0 \\ 1 - 4x/3 & \text{in } 0 < x < 3/2 \end{cases} \quad \text{Deduce: } \frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots$$

Sol: consider

$$\phi(x) = 1 + 4x/3$$

replacing x by $-x$

$$\phi(-x) = 1 - 4x/3$$

$\phi(-x) = \psi(x)$: the given function is even func
Hence

$$\begin{aligned} a_0 &= \frac{2}{\pi} \int_0^{\pi} f(x) dx \\ &= \frac{2}{\pi} \int_0^{\pi/2} \left(1 - \frac{4x}{3}\right) dx \\ &= \frac{4}{3} \left[x - \frac{2x^2}{3} \right] \Big|_0^{\pi/2} \\ &= \frac{4}{3} \left[\frac{\pi}{2} - \frac{3}{2} \right] \end{aligned}$$

$$a_0 = 0 //$$

$$\begin{aligned} a_n &= \frac{2}{\pi} \int_0^{\pi} f(x) \cos \left(\frac{n\pi x}{\pi} \right) dx \\ &= \frac{2}{\pi} \int_0^{\pi/2} \left(1 - \frac{4x}{3}\right) \cos \left(\frac{3n\pi x}{2} \right) dx \\ &= \frac{4}{3} \left[\left(1 - \frac{4x}{3}\right) \frac{\sin \frac{3n\pi x}{2}}{\frac{3n\pi}{2}} \right] \Big|_0^{\pi/2} + \frac{4}{3} \left(-\cos \frac{3n\pi x}{2} \right) \frac{3}{4n^2\pi^2} \Big|_0^{\pi/2} \\ &= \frac{-4}{n^2\pi^2} \left[\cos \frac{2n\pi x}{3} \right] \Big|_0^{\pi/2} \\ &= \frac{-4}{n^2\pi^2} [\cos n\pi - \cos 0] \end{aligned}$$

$$a_n = \frac{4}{n^2\pi^2} [1 - (-1)^n] //$$

$$b_n = 0 //$$

Therefore the Fourier series is

$$f(x) = \sum_{n=1}^{\infty} \frac{4}{n^2\pi^2} [1 - (-1)^n] \cos \frac{2n\pi x}{3}$$

$$\text{But } [1 - (-1)^n] = \begin{cases} 0 & \text{for } n \text{ is even} \\ 2 & \text{for } n \text{ is odd} \end{cases}$$

$$f(x) = \frac{4}{12\pi^2} \sum_{n=1,3,5}^{\infty} \frac{2}{n^2} \cos \frac{2n\pi x}{3}$$

$$f(0) = 1$$

$$f(0) = \frac{4}{\pi^2} \sum_{n=1,3,5}^{\infty} \frac{2}{n^2}$$

$$1 = \frac{8}{\pi^2} \left[\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots \right]$$

$$\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots //$$

* Fourier Series in $[0, 2l]$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \left(\frac{n\pi x}{l} \right) + \sum_{n=1}^{\infty} b_n \sin \left(\frac{n\pi x}{l} \right)$$

$$\text{where } a_0 = \frac{1}{l} \int_0^{2l} f(x) dx$$

$$a_n = \frac{1}{l} \int_0^{2l} f(x) \cos \left(\frac{n\pi x}{l} \right) dx$$

$$b_n = \frac{1}{l} \int_0^{2l} f(x) \sin \left(\frac{n\pi x}{l} \right) dx$$

Q1: $f(x) = 2x - x^2$ in $0 < x < 3$

Sol: The Fourier series is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \left(\frac{n\pi x}{l} \right) + \sum_{n=1}^{\infty} b_n \sin \left(\frac{n\pi x}{l} \right) \quad (1)$$

$$\text{where } a_0 = \frac{1}{l} \int_0^{2l} f(x) dx$$

$$a_0 = \frac{1}{3/2} \int_0^3 (2x - x^2) dx$$

$$= \frac{2}{3} \left[x^2 - \frac{x^3}{3} \right]_0^3$$

$$a_0 = 0 //$$

$$\begin{aligned}
 a_n &= \frac{1}{l} \int_0^{2l} f(x) \cos\left(\frac{n\pi x}{l}\right) dx \\
 &= \frac{1}{3/2} \int_0^3 (2x-x^2) \cos\left(\frac{2n\pi x}{3}\right) dx \\
 &= \frac{2}{3} \left[(2x-x^2) \left(\sin\frac{2n\pi x}{3} \right) \frac{3}{2n\pi} - (2-2x)(-\cos\frac{2n\pi x}{3}) \right]_0^3 \\
 &\quad + (-2) \left(-\sin\frac{2n\pi x}{3} \right) \frac{27}{8n^2\pi^2} \Big|_0^3 \\
 &= \frac{3}{8n^2\pi^2} \left[(1-x) \cos\frac{2n\pi x}{3} \right]_0^3 \\
 &= \frac{3}{n^2\pi^2} \left[(1-3) \cos 2n\pi - (1) \cos 0 \right] \\
 &= \frac{3}{n^2\pi^2} [-2(1)-1]
 \end{aligned}$$

$$a_n = \frac{-9}{n^2\pi^2} //$$

$$\begin{aligned}
 b_n &= \frac{1}{l} \int_0^{2l} f(x) \sin\left(\frac{n\pi x}{l}\right) dx \\
 &= \frac{1}{3/2} \int_0^3 (2x-x^2) \sin\left(\frac{2n\pi x}{3}\right) dx \\
 &= \frac{2}{3} \left[(2x-x^2) \left(-\cos\frac{2n\pi x}{3} \right) \frac{3}{2n\pi} - (2-2x) \left(-\sin\frac{2n\pi x}{3} \right) \frac{9}{4n^2\pi^2} \right. \\
 &\quad \left. + (-2) \left(\cos\frac{2n\pi x}{3} \right) \frac{27}{8n^3\pi^3} \right]_0^3 \\
 &= \frac{2}{3} \left[\frac{8}{2n\pi} \left[(x^2-2x) \cos\frac{2n\pi x}{3} - \frac{9}{4n^2\pi^2} \cos\frac{2n\pi x}{3} \right] \right]_0^3
 \end{aligned}$$

$$= \frac{1}{n\pi} \left[(9-6) \cos 2nx - \frac{9}{4n^2\pi^2} \cos 2nx \right] = 0 + \frac{9}{4n^2\pi^2} \cos 0 \\ = \frac{1}{n\pi} [3(1)]$$

$$b_n = \frac{3}{n\pi} //$$

Substituting a_0, a_n and b_n in eq ①

$$f(x) = \sum_{n=1}^{\infty} \frac{-9}{n^2\pi^2} \cos \left(\frac{2n\pi x}{3} \right) + \sum_{n=1}^{\infty} \frac{3}{n\pi} \sin \left(\frac{2n\pi x}{3} \right) //$$

* Even and Odd Functions in $(0, 2l)$:

A function is said to be an even function in $(0, 2l)$ if
 $f(2l-x) = f(x)$.

A function is said to be an odd function in $(0, 2l)$ if
 $f(2l-x) = -f(x)$.

Q1: $f(x) = 2x - x^2$ in $0 \leq x \leq 2$

Sol: Replacing x by $2l-x$ where $l=1$.

$$f(2-x) = 2(2-x) - (2-x)^2$$

$$f(2-x) = 4 - 2x - x^2 - 4 + 4x$$

$$f(2-x) = -x^2 + 2x$$

$f(2-x) = f(x)$ ∴ the given function is an even function.

Hence

$$a_0 = \frac{2}{1} \int_0^1 f(x) dx$$

$$= \frac{2}{1} \int_0^1 (2x - x^2) dx$$

$$= 2 \left[x^2 - \frac{x^3}{3} \right]_0^1$$

$$= 2 \left[1 - \frac{1}{3} \right]$$

$$\frac{a_0}{2} = \frac{2}{3} //$$

$$a_n = \frac{2}{l} \int_0^l f(x) \cos\left(\frac{n\pi x}{l}\right) dx$$

$$= \frac{2}{l} \int_0^l (2x-x^2) \cos n\pi x dx$$

$$= 2 \left[(2x-x^2) \left(\frac{\sin n\pi x}{n\pi} \right) - (2-2x) \left(\frac{-\cos n\pi x}{n^2\pi^2} \right) + (-2) \left(\frac{\sin n\pi x}{n^2\pi^2} \right) \right]_0^l$$

$$= \frac{4}{n^2\pi^2} \left[(1-x) \cos n\pi x \right]_0^l$$

$$= \frac{4}{n^2\pi^2} [0 - (1) \cos 0]$$

$$a_n = \frac{-4}{n^2\pi^2} //$$

$$b_n = 0 //$$

Hence the Fourier series is

$$f(x) = \frac{2}{3} + \sum_{n=1}^{\infty} \frac{4}{n^2\pi^2} \cos n\pi x //$$

$$Q2: f(x) = \frac{l-x}{2} \text{ in } 0 < x < 2l$$

$$\text{Hence deduce } \frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots$$

Sol: The Fourier series is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right) - \textcircled{1}$$

Replace x by $2l-x$

$$f(2l-x) = \frac{l-(2l-x)}{2}$$

$$f(2l-x) = \frac{l-2l+x}{2}$$

$$f(2l-x) = -\frac{l+x}{2}$$

Hence $f(2l-x) = -f(x)$: the given function is odd function

$$a_0 = 0 //$$

$$a_n = 0 //$$

$$b_n = \frac{2}{l} \int_0^l f(x) \sin\left(\frac{n\pi x}{l}\right) dx$$

$$= \frac{2}{l} \int_0^l \frac{l-x}{2} \sin\left(\frac{n\pi x}{l}\right) dx$$

$$= \frac{2}{l} \left[\frac{l-x}{2} \left(-\cos \frac{n\pi x}{l} \right) \Big|_{n\pi} + \frac{1}{2} \left(-\sin \frac{n\pi x}{l} \right) \frac{l^2}{n^2 \pi^2} \Big|_0^l \right]$$

$$= \frac{2}{l} \left[\frac{-l(l-n)}{n\pi} \Big|_0^l \right]$$

$$= \frac{-2}{n\pi} [0 \cos n\pi - l \cos 0]$$

$$b_n = \frac{-1}{n\pi} [0 - l] \quad , \quad b_n = \frac{l}{n\pi} //$$

Substituting a_0, a_n and b_n in eq ①

$$f(x) = \sum_{n=1}^{\infty} \frac{1}{n\pi} \sin \frac{n\pi x}{l}$$

$$f(x) = \frac{l}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \sin \frac{n\pi x}{l}$$

$$\text{Let } x = l/2$$

$$f(l/2) = \frac{l - l/2}{2} = \frac{l}{4}$$

$$f(l/2) = \frac{l}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \sin \frac{n\pi}{2}$$

$$\frac{l}{4} = \frac{l}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \sin \frac{n\pi}{2}$$

$$\frac{l}{4} = \left[\frac{1}{1} \sin \frac{\pi}{2} + \frac{1}{2} \sin \frac{\pi}{2} + \frac{1}{3} \sin \frac{3\pi}{2} + \frac{1}{4} \sin \frac{2\pi}{2} + \dots \right]$$

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \dots //$$

Q3: $f(x) = \left(\frac{1-x}{2}\right)^2$ in $[0, 2]$

Deduce: $\frac{\pi^2}{6} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \dots$

Sol: The Fourier series is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right)$$

Given $f(x) = \left(\frac{1-x}{2}\right)^2$

Replacing x by $(2x - x)$ where $l = 1$

$$f(2-x) = \left(\frac{1-(2-x)}{2}\right)^2$$

$$f(2-x) = \left(\frac{1-2+x}{2}\right)^2$$

$$f(2-x) = \left(\frac{-1+x}{2}\right)^2$$

$f(2-x) = f(x) \therefore$ the given function is even function

Hence

$$a_0 = \frac{2}{l} \int_0^l f(x) dx$$

$$= 2 \int_0^1 \left(\frac{1-x}{2}\right)^2 dx$$

$$= 2 \int_0^1 \left(\frac{1}{4} + \frac{x^2}{4} - \frac{x}{2}\right) dx$$

$$= 2 \left[\frac{x}{4} + \frac{x^3}{12} - \frac{x^2}{4} \right]_0^1$$

$$= \frac{2}{4} \left[1 + \frac{1}{3} - 1 \right]$$

$$\frac{a_0}{2} = \frac{1}{12} //$$

$$\begin{aligned}
 a_n &= \frac{2}{4} \int_0^4 f(x) \cos\left(\frac{n\pi x}{4}\right) dx \\
 &= 2 \int_0^4 \left(\frac{1-x}{2}\right)^2 \cos n\pi x dx \\
 &= \frac{1}{2} \left[\left(1-x\right)^2 \left(\frac{\sin n\pi x}{n\pi}\right) - 2(1-x)(-1) \left(-\frac{\cos n\pi x}{n^2\pi^2}\right) \right. \\
 &\quad \left. + 2 \left(-\frac{\sin n\pi x}{n^3\pi^3}\right) \right] \Big|_0^4 \\
 &= \frac{1}{2} \left[\frac{-2}{n^2\pi^2} \left[(1-x)\cos n\pi x\right] \Big|_0^4 \right] \\
 &= \frac{-1}{n^2\pi^2} [0 - (1)\cos 0]
 \end{aligned}$$

$$a_n = \frac{1}{n^2\pi^2} //$$

$$b_n = 0 //$$

Substituting a_0, a_n and b_n in eq ①

$$f(x) = \frac{1}{12} + \sum_{n=1}^{\infty} \frac{1}{n^2\pi^2} \cos n\pi x$$

$$f(x) = \frac{1}{12} + \frac{1}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2} \cos n\pi x$$

$$\text{Let } x = 0$$

$$f(0) = \frac{1}{4}$$

$$f(0) = \frac{1}{12} + \frac{1}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2} \quad (1)$$

$$\therefore \frac{1}{4} = \frac{1}{12} + \frac{1}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2}$$

$$\frac{1}{4} - \frac{1}{12} = \frac{1}{\pi^2} \left[\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \dots \right]$$

$$\frac{\pi^2}{6} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \dots //$$

$$Q4: f(x) = \begin{cases} \pi x & \text{in } 0 \leq x \leq 1 \\ \pi(2-x) & \text{in } 1 \leq x \leq 2 \end{cases}$$

Hence deduce $\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots$

$$\text{or } \frac{\pi^2}{8} = \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2}$$

Sol: The Fourier series is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right)$$

Let $\phi(x) = \pi x$ and $\psi(x) = \pi(2-x)$

Replacing x by $2l-x$ in $\phi(x)$ where $l=1$

$$\phi(2-x) = \pi(2-x)$$

$\phi(2-x) = \psi(x) \therefore$ the given function is even function

Hence

$$a_0 = \frac{2}{l} \int_0^l f(x) dx$$

$$= \frac{2}{1} \int_0^1 \pi x dx$$

$$= 2\pi \left[\frac{x^2}{2} \right]_0^1$$

$$a_0 = \pi \Rightarrow \frac{a_0}{2} = \frac{\pi}{2} //$$

$$a_n = \frac{2}{l} \int_0^l f(x) \cos\left(\frac{n\pi x}{l}\right) dx$$

$$= \frac{2}{1} \int_0^1 \pi x \cos n\pi x dx$$

$$= 2\pi \left[x \left(\frac{\sin n\pi x}{n\pi} \right) - \left(-\frac{\cos n\pi x}{n^2\pi^2} \right) \right]_0^1$$

$$= \frac{2\pi}{n^2\pi^2} \left[\cos n\pi x \right]_0^1$$

$$= \frac{2}{n^2\pi} [\cos n\pi - \cos 0]$$

$$a_n = \frac{2[(-1)^n - 1]}{n^2\pi} //$$

$$b_n = 0 //$$

Substituting a_0, a_n and b_n in eq ①

$$f(x) = \frac{\pi}{2} + \sum_{n=1}^{\infty} \frac{2[(-1)^n - 1]}{n^2\pi} \cos nx$$

$$\text{but } [(-1)^n - 1] = \begin{cases} -2 & \text{for } n \text{ is odd} \\ 0 & \text{for } n \text{ is even} \end{cases}$$

$$f(x) = \frac{\pi}{2} + \frac{2}{\pi} \sum_{n=1,3}^{\infty} \frac{-2}{n^2} \cos nx$$

$$\text{let } n=0$$

$$f(0) = 0$$

$$f(0) = \frac{\pi}{2} - \frac{4}{\pi} \sum_{n=1,3}^{\infty} \frac{1}{n^2}$$

$$\therefore 0 = \frac{\pi}{2} - \frac{4}{\pi} \sum_{n=1,3}^{\infty} \frac{1}{n^2}$$

$$\frac{\pi}{2} = \frac{4}{\pi} \left[\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots \right]$$

$$\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots //$$

$$\text{or } \frac{\pi^2}{8} = \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} //$$

$$\text{Q5: } f(x) = \begin{cases} 2-x & \text{in } 0 \leq x \leq 4 \\ 6-x & \text{in } 4 \leq x \leq 8 \end{cases}$$

$$\text{Hence deduce } \frac{\pi^2}{8} = \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2}$$

Sol: The Fourier series is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{4}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{4}\right) \quad \text{--- ①}$$

Replacing x by $2l-x$ in $\phi(x)$ where Keerthana Ashok

$$\phi(2-x) = 2-(2-x)$$

$$\phi(2-x) = 2-2+x$$

$$\phi(2-x) = -x$$

$\phi(2-x) = -\psi(x) \therefore$ It is an odd function.

Hence

$$a_0 = 0 //$$

$$a_n = 0 //$$

$$b_n = \frac{2}{\pi} \int_0^{\pi} f(x) \sin\left(\frac{n\pi x}{\pi}\right) dx$$

$$= \frac{2}{4} \int_0^{\pi} (2-x) \sin\left(\frac{n\pi x}{4}\right) dx$$

$$= \frac{1}{2} \left[(2-x) \left(-\cos \frac{n\pi x}{4} \right) \frac{4}{n\pi} - (-1) \left(-\sin \frac{n\pi x}{4} \right) \frac{16}{n^2\pi^2} \right]_0^{\pi}$$

$$= \frac{1}{2} \left[\frac{-4^2}{n\pi} \left[(2-\pi) \cos \frac{n\pi x}{4} \right] \right]_0^{\pi}$$

$$= \frac{-2}{n\pi} \left[(-2) \cos n\pi - 2 \cos 0 \right]$$

$$= \frac{-2}{n\pi} \left[-2(-1)^n - 2(1) \right]$$

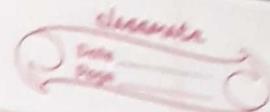
$$b_n = \frac{4}{n\pi} [(-1)^n + 1] //$$

Substituting a_0, a_n and b_n in eq, ①

$$f(x) = \sum_{n=1}^{\infty} \frac{4}{n\pi} [(-1)^n + 1] \sin\left(\frac{n\pi x}{4}\right)$$

but $[-1]^n + 1 = \begin{cases} 0 & \text{for } n \text{ is odd} \\ 2 & \text{for } n \text{ is even} \end{cases}$

$$f(x) = \sum_{n=2,4}^{\infty} \frac{4}{n\pi} (2) \sin\left(\frac{n\pi x}{4}\right)$$



Q6: $f(x) = \begin{cases} 2-x & \text{in } 0 \leq x \leq 4 \\ x-6 & \text{in } 4 \leq x \leq 8 \end{cases}$

Deduce: $\frac{\pi^2}{8} = \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2}$

Sol:

The Fourier series is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right) \quad \text{--- (1)}$$

Here $l = 4$

Let $\phi(x) = 2-x$ and $\psi(x) = x-6$

Replacing x by $2l-x$ in $\phi(x)$

$$\phi(8-x) = 2-(8-x)$$

$$\phi(8-x) = 2-8+x$$

$$\phi(8-x) = x-6$$

$\phi(8-x) = \psi(x) \therefore$ The given function is an even function

Hence

$$a_0 = \frac{2}{l} \int_0^l f(x) dx$$

$$\begin{aligned} a_0 &= \frac{2}{4} \int_0^4 (2-x) dx \\ &= \frac{1}{2} \left[2x - \frac{x^2}{2} \right]_0^4 \\ &= \frac{1}{2} [8 - 8] \end{aligned}$$

$$a_0 = 0 //$$

$$\begin{aligned} a_n &= \frac{2}{l} \int_0^l f(x) \cos\left(\frac{n\pi x}{l}\right) dx \\ &= \frac{2}{4} \int_0^4 (2-x) \cos \frac{n\pi x}{4} dx \\ &= \frac{1}{2} \left[(2-x) \left(\sin \frac{n\pi x}{4} \right) \frac{4}{n\pi} + \left(-\cos \frac{n\pi x}{4} \right) \frac{16}{n^2\pi^2} \right]_0^4 \\ &= \frac{1}{2} \left[\frac{-16}{n^2\pi^2} \left(\cos \frac{n\pi x}{4} \right) \Big|_0^4 \right] \\ &= \frac{-8}{n^2\pi^2} [\cos n\pi - \cos 0] \\ &= \frac{-8}{n^2\pi^2} [(-1)^n - 1] \end{aligned}$$

$$a_n = \frac{8[1 - (-1)^n]}{n^2\pi^2} //$$

$$b_n = 0 //$$

Substituting a_0, a_n and b_n in eq ①

$$f(x) = \sum_{n=1}^{\infty} \frac{8}{n^2\pi^2} [1 - (-1)^n] \cos \frac{n\pi x}{4}$$

but $[1 - (-1)^n] = \begin{cases} 2 & \text{if } n \text{ is odd} \\ 0 & \text{if } n \text{ is even} \end{cases}$

$$f(x) = \frac{8}{\pi^2} \sum_{n=1,3}^{\infty} \frac{2}{n^2} \cos \frac{n\pi x}{4}$$

let $x = 0$

$$f(0) = 2$$

$$f(0) = \frac{16}{\pi^2} \sum_{n=1,3}^{\infty} \frac{1}{n^2}$$

$$\therefore 2 = \frac{16}{\pi^2} \sum_{n=1,3}^{\infty} \frac{1}{n^2}$$

$$\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots$$

Since $1, 3, 5, \dots$ can be denoted as $(2n-1)$

$$\frac{\pi^2}{8} = \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2}$$

Q7: $f(x) = \begin{cases} 1-x & \text{in } 0 < x < 1 \\ 0 & \text{in } 1 < x < 2 \end{cases}$

Hence deduce i) $\frac{\pi^2}{8} = \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2}$; ii) $\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots$

Sol: The Fourier series is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right) \quad \text{--- (1)}$$

Here $f(x)$ is neither an even function nor an odd function.

$$a_0 = \frac{1}{l} \int_0^{2l} f(x) dx$$

$$= \frac{1}{1} \int_0^2 f(x) dx$$

$$= \int_0^1 f(x) dx + \int_1^2 f(x) dx$$

$$= \int_0^1 (1-x) dx + 0$$

$$= x - \frac{x^2}{2} \Big|_0^1$$

$$\frac{a_0}{2} = \frac{1}{4} //$$

$$\begin{aligned}
 a_n &= \frac{1}{l} \int_0^{2l} f(x) \cos\left(\frac{n\pi x}{l}\right) dx \\
 &= \frac{1}{l} \int_0^2 f(x) \cos n\pi x dx \\
 &= \int_0^1 f(x) \cos n\pi x dx + \int_1^2 f(x) \cos n\pi x dx \\
 &= \int_0^1 (1-x) \cos n\pi x dx + 0 \\
 &= \left[(1-x) \left(\frac{\sin n\pi x}{n\pi} \right) - (-1) \left(\frac{-\cos n\pi x}{n^2\pi^2} \right) \right]_0^1 \\
 &= \frac{-1}{n^2\pi^2} \left[\cos n\pi x \right]_0^1 \\
 &= \frac{-1}{n^2\pi^2} [\cos n\pi - \cos 0] \\
 &= \frac{-1}{n^2\pi^2} [(-1)^n - 1]
 \end{aligned}$$

$$a_n = \frac{[1 - (-1)^n]}{n^2\pi^2} //$$

$$\begin{aligned}
 b_n &= \frac{1}{l} \int_0^{2l} f(x) \sin\left(\frac{n\pi x}{l}\right) dx \\
 &= \frac{1}{l} \int_0^2 f(x) \sin n\pi x dx \\
 &= \int_0^1 f(x) \sin n\pi x dx + \int_1^2 f(x) \sin n\pi x dx \\
 &= \int_0^1 (1-x) \sin n\pi x dx + 0
 \end{aligned}$$

$$\begin{aligned}
 &= \left[(1-x) \left(-\frac{\cos nx}{n\pi} \right) - (-1) \left(-\frac{\sin nx}{n^2\pi^2} \right) \right]_0^1 \\
 &= \left[\frac{x-1}{n\pi} \cos nx \right]_0^1 \\
 &= \left[0 + \frac{\cos 0}{n\pi} \right]
 \end{aligned}$$

$$b_n = \frac{1}{n\pi} //$$

Substituting a_0 , a_n and b_n in eq ①.

$$f(x) = \frac{1}{4} + \sum_{n=1}^{\infty} \frac{[1-(-1)^n]}{n^2\pi^2} \cos nx + \sum_{n=1}^{\infty} \frac{1}{n\pi} \sin nx$$

$$\text{but } [1-(-1)^n] = \begin{cases} 2 & \text{for } n \text{ is odd} \\ 0 & \text{for } n \text{ is even} \end{cases}$$

$$f(x) = \frac{1}{4} + \sum_{n=1,3}^{\infty} \frac{2}{n^2\pi^2} \cos nx + \sum_{n=1}^{\infty} \frac{1}{n\pi} \sin nx \quad \text{--- ②}$$

i] In eq ②, let $x = 1$

$$f(1) = 0$$

$$f(1) = \frac{1}{4} + \frac{2}{\pi^2} \sum_{n=1,3}^{\infty} \frac{\cos n}{n^2}$$

$$0 = \frac{1}{4} + \frac{2}{\pi^2} \sum_{n=1,3}^{\infty} \frac{1}{n^2} \cos n$$

$$-\frac{\pi^2}{8} = \sum_{n=1,3}^{\infty} \frac{(-1)^n}{n^2}$$

$$-\frac{\pi^2}{8} = \frac{-1}{1^2} - \frac{1}{3^2} - \frac{1}{5^2} - \dots$$

$$\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots$$

$$\therefore \frac{\pi^2}{8} = \sum_{n=1}^{\infty} \frac{1}{(2n-1)^2} //$$

ii] let $x = 1/2$

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$$f(1/2) = 1 - 1/2 = 1/2$$

$$\text{from eq } ② \quad f(1/2) = \frac{1}{4} + \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \sin \frac{n\pi}{2}$$

$$\therefore \frac{1}{2} = \frac{1}{4} + \frac{1}{\pi} \sum_{n=1}^{\infty} \frac{1}{n} \sin \frac{n\pi}{2}$$

$$\frac{1}{4} = \frac{1}{\pi} \left[\frac{1}{1} \sin \frac{\pi}{2} + \frac{1}{2} \sin \pi + \frac{1}{3} \sin \frac{3\pi}{2} + \frac{1}{4} \sin 2\pi + \dots \right]$$

$$\frac{\pi}{4} = 1 + \frac{1}{3}(-1) + \frac{1}{5}(1) + \frac{1}{7}(-1) + \dots$$

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \dots$$

* Half Range Fourier Series in (0, l):

In an interval of length $2l$ in general a periodic function of x will have a Fourier expansion containing cosine and sine terms. Many times it becomes necessary to have an expression containing only cosine or sine terms.

In the half range Fourier series, the function must be defined in the interval of the form $(0, l)$ which is to be regarded as half the interval.

Half range cosine series:

$$\text{It is defined as } f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \left(\frac{n\pi x}{l} \right)$$

$$\text{where } a_0 = \frac{2}{l} \int_0^l f(x) dx$$

$$a_n = \frac{2}{l} \int_0^l f(x) \cos \left(\frac{n\pi x}{l} \right) dx$$

Half range sine series:

$$\text{It is defined as } f(x) = \sum_{n=1}^{\infty} b_n \sin \left(\frac{n\pi x}{l} \right)$$

where $b_n = \frac{2}{l} \int_0^l f(x) \sin\left(\frac{n\pi x}{l}\right) dx$.

Q1: Obtain the sine half range Fourier series of $f(x) = x^2$ for $0 < x < l$.

Sol: Half range sine series is given by

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right)$$

where $b_n = \frac{2}{l} \int_0^l f(x) \sin\left(\frac{n\pi x}{l}\right) dx$

Here $l = \pi$

$$b_n = \frac{2}{\pi} \int_0^\pi x^2 \sin nx dx$$

$$= \frac{2}{\pi} \left[x^2 \left(-\frac{\cos nx}{n} \right) - 2x \left(-\frac{\sin nx}{n^2} \right) + 2 \left(\frac{\cos nx}{n^3} \right) \right]_0^\pi$$

$$= \frac{2}{n\pi} \left[-\pi^2 \cos n\pi + 2 \frac{\cos n\pi}{n^2} \right]_0^\pi$$

$$= \frac{2}{n\pi} \left[-\pi^2 (-1)^n + \frac{2}{n^2} \cos n\pi + 0 - \frac{2}{n^2} \cos 0 \right]$$

$$= \frac{2}{n\pi} \left[-\pi^2 (-1)^n + \frac{2}{n^2} (-1)^n - \frac{2}{n^2} \right]$$

$$b_n = \frac{2}{n\pi} \left[\frac{-\pi^2 (-1)^n}{n} + \frac{2}{n^3} [(-1)^n - 1] \right] //$$

$$\therefore f(x) = \sum_{n=1}^{\infty} \frac{2}{n\pi} \left[\frac{-\pi^2 (-1)^n}{n} + \frac{2}{n^3} [(-1)^n - 1] \right] \sin nx //$$

Q2: Expand $f(x) = 2x-1$ as a cosine half range Fourier series in $0 < x < 1$.

sol: Half range cosine series is given by
 $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right)$

Here $l = 1$

$$\text{where } a_0 = \frac{2}{l} \int_0^l f(x) dx$$

$$a_0 = \frac{2}{1} \int_0^1 (2x-1) dx$$

$$= 2 [x^2 - x]_0^1$$

$$= 2 [1 - 1]$$

$$\underline{a_0 = 0}$$

$$a_n = \frac{2}{l} \int_0^l f(x) \cos\left(\frac{n\pi x}{l}\right) dx$$

$$= \frac{2}{1} \int_0^1 (2x-1) \cos n\pi x dx$$

$$= 2 \left[(2x-1) \left(\frac{\sin n\pi x}{n\pi} \right) - 2 \left(-\frac{\cos n\pi x}{n^2\pi^2} \right) \right]_0^1$$

$$= \frac{4}{n^2\pi^2} [\cos n\pi x]_0^1$$

$$= \frac{4}{n^2\pi^2} [\cos n\pi - \cos 0]$$

$$a_n = \frac{4}{n^2\pi^2} [(-1)^n - 1] //$$

$$\therefore f(x) = \sum_{n=1}^{\infty} \frac{4}{n^2\pi^2} [(-1)^n - 1] \cos n\pi x //$$

Q3: Find the cosine half range Fourier series of
 $f(x) = (x-1)^2$ in $0 \leq x \leq 1$.

Hence deduce: $\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots$

Sol:

Half range cosine series is given by
 $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right)$

Here $l = 1$

$$\text{where } a_0 = \frac{2}{l} \int_0^l f(x) dx$$

$$a_0 = \frac{2}{1} \int_0^1 (x-1)^2 dx$$

$$= 2 \int_0^1 (x^2 - 2x + 1) dx$$

$$= 2 \left[\frac{x^3}{3} - x^2 + x \right]_0^1$$

$$= 2 \left[\frac{1}{3} - 1 + 1 \right]$$

$$\frac{a_0}{2} = \frac{1}{3}$$

$$a_n = \frac{2}{l} \int_0^l f(x) \cos\left(\frac{n\pi x}{l}\right) dx$$

$$= \frac{2}{1} \int_0^1 (x-1)^2 \cos n\pi x dx$$

$$= 2 \left[(x-1)^2 \left(\frac{\sin n\pi x}{n\pi} \right) - 2(x-1) \left(-\frac{\cos n\pi x}{n^2\pi^2} \right) + 2 \left(\frac{\sin n\pi x}{n^3\pi^3} \right) \right]_0^1$$

$$= \frac{4}{n^2\pi^2} \left[(x-1) \cos n\pi x \right]_0^1$$

$$= \frac{4}{n^2\pi^2} [0 + \cos 0]$$

$$a_n = \frac{4}{n^2\pi^2}$$

$$\therefore f(x) = \frac{1}{3} + \sum_{n=1}^{\infty} \frac{4}{n^2\pi^2} \cos n\pi x$$

$$\text{let } x = 0$$

$$f(0) = 1$$

$$f(0) = \frac{1}{3} + \frac{4}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2}$$

$$\therefore 1 = \frac{1}{3} + \frac{4}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2}$$

$$\frac{2}{3} = \frac{4}{\pi^2} \left[\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \dots \right] \quad \text{--- } ①$$

$$\frac{\pi^2}{6} = \frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \dots$$

Let $x=1$

$$f(1)=0$$

$$f(1) = \frac{1}{3} + \frac{4}{\pi^2} \sum_{n=1}^{\infty} \frac{1}{n^2} \cos nx$$

$$\therefore 0 = \frac{1}{3} + \frac{4}{\pi^2} \sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}$$

$$\frac{-1}{3} = \frac{4}{\pi^2} \left[\frac{-1}{1^2} + \frac{1}{2^2} - \frac{1}{3^2} + \frac{1}{4^2} - \dots \right]$$

$$\frac{\pi^2}{12} = \frac{1}{1^2} - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} \dots \quad \text{--- } ②$$

Adding eq. ① and eq. ②

$$\frac{\pi^2}{6} + \frac{\pi^2}{12} = 2 \left[\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots \right]$$

$$\frac{\pi^2}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots$$



Q4: Find half range cosine series for the function

$$f(x) = \begin{cases} kx & \text{in } 0 \leq x \leq \pi/2 \\ k(\pi-x) & \text{in } \pi/2 \leq x \leq \pi \end{cases}$$

$$\text{Deduce: } f(x) = \frac{\pi k}{4} - \frac{2k}{\pi} \left(\frac{\cos 2x}{1^2} + \frac{\cos 6x}{3^2} + \frac{\cos 10x}{5^2} \right)$$

sol:

Half range cosine series is given by
 $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right)$

where $a_0 = \frac{2}{l} \int_0^l f(x) dx$ here $l = \pi$

$$a_0 = \frac{2}{\pi} \int_0^\pi f(x) dx$$

$$= \frac{2}{\pi} \left[\int_0^{\pi/2} f(x) dx + \int_{\pi/2}^\pi f(x) dx \right]$$

$$= \frac{2}{\pi} \left[\int_0^{\pi/2} kx dx + \int_{\pi/2}^\pi k(\pi-x) dx \right]$$

$$= \frac{2}{\pi} \left[\frac{kx^2}{2} \Big|_0^{\pi/2} + \left(k\pi x - \frac{kx^2}{2} \right) \Big|_{\pi/2}^\pi \right]$$

$$= \frac{2k}{\pi} \left[\frac{\pi^2}{8} + \pi^2 - \frac{\pi^2}{2} - \frac{\pi^2}{2} + \frac{\pi^2}{8} \right]$$

$$= \frac{2k}{\pi} \left[\frac{\pi^2}{4} \right]$$

$$\frac{a_0}{2} = \frac{k\pi}{4} //$$

$$a_n = \frac{2}{l} \int_0^l f(x) \cos\left(\frac{n\pi x}{l}\right) dx$$

$$= \frac{2}{\pi} \int_0^\pi f(x) \cos nx dx$$

$$= \frac{2}{\pi} \left[\int_0^{\pi/2} f(x) \cos nx dx + \int_{\pi/2}^\pi f(x) \cos nx dx \right]$$

$$= \frac{2}{\pi} \left[\int_0^{\pi/2} kx \cos nx dx + \int_{\pi/2}^\pi k(\pi-x) \cos nx dx \right]$$

$$= \frac{2k}{\pi} \left[\left[x \left(\frac{\sin nx}{n} \right) - \left(-\frac{\cos nx}{n^2} \right) \right] \Big|_0^{\pi/2} + \left[(\pi-x) \left(\frac{\sin nx}{n} \right) + \left(-\frac{\cos nx}{n^2} \right) \right] \Big|_{\pi/2}^\pi \right]$$

$$= \frac{2k}{\pi} \left[\frac{\pi}{2n} \frac{\sin n\pi}{2} + \frac{1}{n^2} \cos \frac{n\pi}{2} - 0 - \frac{\cos 0}{n^2} + 0 - \frac{\cos n\pi}{n^2} \right]$$

$$- \frac{\pi}{2n} \frac{\sin n\pi}{2} + \frac{1}{n^2} \cos \frac{n\pi}{2} \right]$$

$$= \frac{2k}{\pi} \left[\frac{2}{n^2} \cos \frac{n\pi}{2} - \frac{1}{n^2} - \frac{(-1)^n}{n^2} \right]$$

$$a_n = \frac{-2k}{\pi} \left[\frac{1}{n^2} + \frac{(-1)^n}{n^2} - \frac{2}{n^2} \cos \frac{n\pi}{2} \right] //$$

$$\therefore f(x) = \frac{k\pi}{4} - \frac{2k}{\pi} \sum_{n=1}^{\infty} \left[\frac{1+(-1)^n - 2\cos n\pi/2}{n^2} \right] \cos nx$$

but $1+(-1)^n = \begin{cases} 0 & \text{for } n \text{ is odd} \\ 2 & \text{for } n \text{ is even} \end{cases}$

$$f(x) = \frac{k\pi}{4} - \frac{2k}{\pi} \sum_{n=2,4,\dots}^{\infty} \left[\frac{2 - 2\cos n\pi/2}{n^2} \right] \cos nx$$

$$f(x) = \frac{k\pi}{4} - \frac{4k}{\pi} \sum_{n=2,4,\dots}^{\infty} \left[\frac{1 - \cos n\pi/2}{n^2} \right] \cos nx$$

but $1 - \cos \frac{n\pi}{2} = \begin{cases} 2 & \text{for } n = 2, 6, 10, \dots \\ 0 & \text{for } n = 4, 8, 12, \dots \end{cases}$

$$\therefore f(x) = \frac{k\pi}{4} - \frac{4k}{\pi} \sum_{n=2,6,\dots}^{\infty} \left[\frac{2}{n^2} \cos nx \right]$$

$$f(x) = \frac{k\pi}{4} - \frac{8k}{\pi} \sum_{n=2,6,\dots}^{\infty} \left[\frac{1}{n^2} \cos nx \right]$$

$$f(x) = \frac{k\pi}{4} - \frac{8k}{\pi} \left[\frac{1}{2^2} \cos 2x + \frac{1}{6^2} \cos 6x + \frac{1}{10^2} \cos 10x + \dots \right]$$

$$f(x) = \frac{k\pi}{4} - \frac{2k}{\pi} \left[\frac{\cos 2x}{12} + \frac{\cos 6x}{32} + \frac{\cos 10x}{52} + \dots \right]$$

Q5: Find the Half range cosine series for

$$f(x) = \begin{cases} 1 & \text{in } 0 < x < 1 \\ x & \text{in } 1 < x < 2 \end{cases}$$

Sol: Half range cosine series is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \left(\frac{n\pi x}{l} \right) \quad \text{Here } l=2$$

$$\begin{aligned}
 a_0 &= \frac{2}{l} \int_0^l f(x) dx \\
 &= \frac{2}{2} \int_0^2 f(x) dx \\
 &= \int_0^1 f(x) dx + \int_1^2 f(x) dx \\
 &= \int_0^1 1 dx + \int_1^2 x dx \\
 &= x \Big|_0^1 + \frac{x^2}{2} \Big|_1^2 \\
 &= 1 + 2 - \frac{1}{2}
 \end{aligned}$$

$$a_0 = \frac{5}{2} \Rightarrow a_0 = \frac{5}{4} //$$

$$\begin{aligned}
 a_n &= \frac{2}{l} \int_0^l f(x) \cos\left(\frac{n\pi x}{l}\right) dx \\
 &= \frac{2}{2} \int_0^2 f(x) \cos\left(\frac{n\pi x}{2}\right) dx \\
 &= \int_0^1 f(x) \cos\left(\frac{n\pi x}{2}\right) dx + \int_1^2 f(x) \cos\left(\frac{n\pi x}{2}\right) dx \\
 &= \int_0^1 \cos\left(\frac{n\pi x}{2}\right) dx + \int_1^2 x \cos\left(\frac{n\pi x}{2}\right) dx \\
 &= \left(\sin\frac{n\pi x}{2} \right) \frac{2}{n\pi} \Big|_0^1 + \left[x \left(\sin\frac{n\pi x}{2} \right) \frac{2}{n\pi} + \left(\cos\frac{n\pi x}{2} \right) \frac{4}{n^2\pi^2} \right]_1^2
 \end{aligned}$$

$$\begin{aligned}
 &= \frac{2}{n\pi} \left[\sin\frac{n\pi}{2} \Big|_0^1 + \left(x \sin\frac{n\pi x}{2} + \frac{2}{n\pi} \cos\frac{n\pi x}{2} \right) \Big|_1^2 \right] \\
 &= \frac{2}{n\pi} \left[\cancel{\sin\frac{n\pi}{2}} - 0 + 2 \cancel{\sin\frac{n\pi}{2}} + \frac{2}{n\pi} \cos\frac{n\pi}{2} - \cancel{\sin\frac{n\pi}{2}} - \frac{2}{n\pi} \cos\frac{n\pi}{2} \right]
 \end{aligned}$$

$$= \frac{2}{n\pi} \left[\frac{2}{n\pi} \left(\cos n\pi - \cos \frac{n\pi}{2} \right) \right]$$

$$a_n = \frac{4}{n^2\pi^2} \left[(-1)^n - \cos \frac{n\pi}{2} \right] //$$

$$\therefore f(x) = \frac{5}{4} + \frac{4}{\pi^2} \sum_{n=1}^{\infty} \frac{[(-1)^n - \cos nx/2]}{n^2} \cos \frac{nx}{2}$$

Q6: Find the half range sine series for the function
 $f(x) = \begin{cases} 1 & \text{in } 0 < x < \pi/2 \\ -1 & \text{in } \pi/2 < x < \pi \end{cases}$

Sol: The half range sine series is given by
 $f(x) = \sum_{n=1}^{\infty} b_n \sin \left(\frac{n\pi x}{l} \right)$ here $l = \pi$

$$\text{where } b_n = \frac{2}{\pi} \int_0^\pi f(x) \sin \left(\frac{n\pi x}{\pi} \right) dx$$

$$b_n = \frac{2}{\pi} \int_0^\pi f(x) \sin nx dx$$

$$= \frac{2}{\pi} \left[\int_0^{\pi/2} f(x) \sin nx dx + \int_{\pi/2}^{\pi} f(x) \sin nx dx \right]$$

$$= \frac{2}{\pi} \left[\int_0^{\pi/2} \sin nx dx - \int_{\pi/2}^{\pi} \sin nx dx \right]$$

$$= \frac{2}{\pi} \left[\frac{-\cos nx}{n} \Big|_0^{\pi/2} + \frac{\cos nx}{n} \Big|_{\pi/2}^{\pi} \right]$$

$$= \frac{2}{n\pi} \left[-\cos \frac{n\pi}{2} + \cos 0 + \cos n\pi - \cos \frac{n\pi}{2} \right]$$

$$b_n = \frac{2}{n\pi} \left[1 + (-1)^n - 2 \cos \frac{n\pi}{2} \right] //$$

$$\therefore f(x) = \sum_{n=1}^{\infty} \frac{2}{n\pi} \left[1 + (-1)^n - 2 \cos \frac{n\pi}{2} \right] \sin nx //$$

Q7: Find the half range sine series

$$f(x) = \begin{cases} \frac{1}{4} - x & \text{in } 0 < x < \frac{1}{2} \\ \frac{x-3}{4} & \text{in } \frac{1}{2} < x < 1 \end{cases}$$

Sol:

The half range sine series is given by
 $f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right)$ here $l=1$

$$\text{where } b_n = \frac{2}{l} \int_0^l f(x) \sin\left(\frac{n\pi x}{l}\right) dx$$

$$\begin{aligned} b_n &= \frac{2}{1} \left[\int_0^1 f(x) \sin n\pi x dx \right] \\ &= 2 \left[\int_0^{1/2} f(x) \sin n\pi x dx + \int_{1/2}^1 f(x) \sin n\pi x dx \right] \\ &= 2 \left[\int_0^{1/2} \left(\frac{1-x}{4}\right) \sin n\pi x dx + \int_{1/2}^1 \left(\frac{x-3}{4}\right) \sin n\pi x dx \right] \\ &= 2 \left[\left(\frac{1}{4} - x \right) \left(-\frac{\cos n\pi x}{n\pi} \right) + \left(-\frac{\sin n\pi x}{n^2\pi^2} \right) \Big|_0^{1/2} \right. \\ &\quad \left. + \left(\frac{x-3}{4} \right) \left(-\frac{\cos n\pi x}{n\pi} \right) - \left(-\frac{\sin n\pi x}{n^2\pi^2} \right) \Big|_{1/2}^1 \right] \\ &= 2 \left[\frac{1}{4n\pi} \cos \frac{n\pi}{2} - \frac{1}{n^2\pi^2} \sin \frac{n\pi}{2} + \frac{1}{4n\pi} \cos 0 + 0 \right. \\ &\quad \left. - \frac{1}{4n\pi} \cos n\pi + \frac{1}{n^2\pi^2} \sin n\pi - \frac{1}{4n\pi} \frac{\cos n\pi}{2} - \frac{1}{n^2\pi^2} \frac{\sin n\pi}{2} \right] \end{aligned}$$

$$= \frac{2}{n\pi} \left[\frac{1}{4} - \frac{(-1)^n}{4} - \frac{2 \sin n\pi/2}{n\pi} \right]$$

$$b_n = \frac{1}{2n\pi} \left[1 - (-1)^n - \frac{8 \sin n\pi}{n\pi} \right] //$$

$$\therefore f(x) = \sum_{n=1}^{\infty} \left[\frac{1}{2\pi n} (1 - (-1)^n) - \frac{4}{n^2\pi^2} \sin \frac{n\pi}{2} \right] \sin n\pi x //$$

Q8:

Find the half range sine series

$$f(x) = \begin{cases} kx & \text{in } 0 \leq x \leq l/2 \\ k(l-x) & \text{in } l/2 \leq x \leq l \end{cases}$$

Sol:

Half range sine series is given by

$$\text{Q8: } f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right)$$

Keerthana

$$\text{where } b_n = \frac{2}{l} \int_0^l f(x) \sin\left(\frac{n\pi x}{l}\right) dx$$

$$b_n = \frac{2}{l} \left[\int_0^{l/2} f(x) \sin\left(\frac{n\pi x}{l}\right) dx + \int_{l/2}^l f(x) \sin\left(\frac{n\pi x}{l}\right) dx \right]$$

$$= \frac{2}{l} \left[\int_0^{l/2} kx \sin\left(\frac{n\pi x}{l}\right) dx + \int_{l/2}^l k(l-x) \sin\left(\frac{n\pi x}{l}\right) dx \right]$$

$$= \frac{2k}{l} \left[\left[x \left(-\cos \frac{n\pi x}{l} \right) \frac{l}{n\pi} + \left(\sin \frac{n\pi x}{l} \right) \frac{l^2}{n^2\pi^2} \right]_0^{l/2} \right.$$

$$\quad \quad \quad \left. + \left[(l-x) \left(-\cos \frac{n\pi x}{l} \right) \frac{l}{n\pi} + \left(-\sin \frac{n\pi x}{l} \right) \frac{l^2}{n^2\pi^2} \right] \right]_{l/2}^l$$

$$= \frac{2k}{l} \left[\frac{l}{n\pi} \left[-\frac{l}{2} \cos \frac{n\pi}{2} + \frac{l}{n\pi} \sin \frac{n\pi}{2} \right] + 0 - 0 - 0 - 0 + \frac{l}{2} \cos \frac{n\pi}{2} + \frac{l}{2} \sin \frac{n\pi}{2} \right]$$

$$= \frac{2k}{n\pi} \left[\frac{2l}{n\pi} \sin \frac{n\pi}{2} \right]$$

$$b_n = \frac{4kl}{n^2\pi^2} \sin \frac{n\pi}{2} //$$

$$\therefore f(x) = \sum_{n=1}^{\infty} \frac{4kl}{n^2\pi^2} \sin \frac{n\pi}{2} \sin \frac{n\pi x}{l} //$$

Q9: Find the half range cosine and sine series for the following functions:

i. $f(x) = \begin{cases} x & \text{in } 0 < x < \pi/2 \\ \pi - x & \text{in } \pi/2 < x < \pi \end{cases}$

Sol: a. Half range cosine series is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos \frac{n\pi x}{l} \quad \text{Here } l = \pi$$

$$\text{where } a_0 = \frac{2}{l} \int_0^l f(x) dx$$

$$\begin{aligned}
 a_0 &= \frac{2}{\pi} \int_0^\pi f(x) dx \\
 &= \frac{2}{\pi} \left[\int_0^{\pi/2} f(x) dx + \int_{\pi/2}^\pi f(x) dx \right] \\
 &= \frac{2}{\pi} \left[\int_0^{\pi/2} x dx + \int_{\pi/2}^\pi (\pi - x) dx \right] \\
 &= \frac{2}{\pi} \left[\frac{x^2}{2} \Big|_0^{\pi/2} + \left[\pi x - \frac{x^2}{2} \right] \Big|_{\pi/2}^\pi \right] \\
 &= \frac{2}{\pi} \left[\frac{\pi^2}{8} + \pi^2 - \frac{\pi^2}{2} - \frac{\pi^2}{2} + \frac{\pi^2}{8} \right] \\
 &= \frac{2}{\pi} \left[\frac{\pi^2}{4} \right]
 \end{aligned}$$

$$\frac{a_0}{2} = \frac{\pi}{4} //$$

$$\begin{aligned}
 a_n &= \frac{2}{\pi} \int_0^\pi f(x) \cos nx dx \\
 &= \frac{2}{\pi} \int_0^\pi f(x) \cos nx dx \\
 &= \frac{2}{\pi} \left[\int_0^{\pi/2} f(x) \cos nx dx + \int_{\pi/2}^\pi f(x) \cos nx dx \right] \\
 &= \frac{2}{\pi} \left[\int_0^{\pi/2} x \cos nx dx + \int_{\pi/2}^\pi (\pi - x) \cos nx dx \right] \\
 &= \frac{2}{\pi} \left[\left[x \left(\frac{\sin nx}{n} \right) - \left(-\frac{\cos nx}{n^2} \right) \right] \Big|_0^{\pi/2} + \left[(\pi - x) \left(\frac{\sin nx}{n} \right) + \left(-\frac{\cos nx}{n^2} \right) \right] \Big|_{\pi/2}^\pi \right] \\
 &= \frac{2}{\pi} \left[\frac{\pi}{2n} \sin \frac{n\pi}{2} + \frac{1}{n^2} \cos \frac{n\pi}{2} - 0 - \frac{1}{n^2} \cos 0 \right. \\
 &\quad \left. + 0 - \frac{1}{n^2} \cos n\pi - \frac{\pi}{2n} \sin \frac{n\pi}{2} + \frac{1}{n^2} \cos \frac{n\pi}{2} \right]
 \end{aligned}$$

$$a_n = \frac{2}{n^2 \pi} \left[2 \cos \frac{n\pi}{2} - 1 - (-1)^n \right] //$$

$$\therefore f(x) = \frac{\pi}{4} + \sum_{n=1}^{\infty} \frac{2}{n^2 \pi} \left[2 \cos \frac{n\pi}{2} - 1 - (-1)^n \right] //$$

b. Half range sine series is given by
 $f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right)$ here $l=\pi$

$$\text{where } b_n = \frac{2}{l} \int_0^l f(x) \sin\left(\frac{n\pi x}{l}\right) dx$$

$$b_n = \frac{2}{\pi} \int_0^\pi f(x) \sin nx dx$$

$$= \frac{2}{\pi} \left[\int_0^{\pi/2} f(x) \sin nx dx + \int_{\pi/2}^\pi f(x) \sin nx dx \right]$$

$$= \frac{2}{\pi} \left[\int_0^{\pi/2} x \sin nx dx + \int_{\pi/2}^\pi (\pi-x) \sin nx dx \right]$$

$$= \frac{2}{\pi} \left[\left[x \left(-\frac{\cos nx}{n} \right) - \left(-\frac{\sin nx}{n^2} \right) \right] \Big|_0^{\pi/2} \right.$$

$$\quad \quad \quad \left. + \left[(\pi-x) \left(-\frac{\cos nx}{n} \right) + \left(-\frac{\sin nx}{n^2} \right) \right] \Big|_{\pi/2}^\pi \right]$$

$$= \frac{2}{\pi} \left[\frac{-\pi}{2n} \frac{\cos n\pi}{2} + \frac{1}{n^2} \frac{\sin n\pi}{2} + 0 - 0 \right.$$

$$\quad \quad \quad \left. + 0 - 0 + \frac{\pi}{2n} \frac{\cos n\pi}{2} + \frac{1}{n^2} \frac{\sin n\pi}{2} \right]$$

$$= \frac{2}{\pi} \left[\frac{2}{n^2} \frac{\sin \frac{n\pi}{2}}{2} \right]$$

$$b_n = \frac{4}{n^2\pi} \frac{\sin \frac{n\pi}{2}}{2} //$$

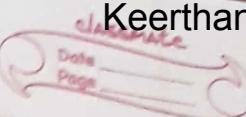
$$\therefore f(x) = \sum_{n=1}^{\infty} \frac{4}{n^2\pi} \frac{\sin \frac{n\pi}{2}}{2} \sin nx //$$

ii. $f(x) = \ln x - x^2$ in $0 \leq x \leq 1$

Sol: a. Half range cosine series is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right)$$

Here $l=1$



$$\begin{aligned}
 a_0 &= \frac{2}{l} \int_0^l f(x) dx \\
 &= \frac{2}{l} \int_0^l (\ln x - x^2) dx \\
 &= 2 \left[\frac{x \ln x}{2} - \frac{x^3}{3} \right]_0^l \\
 &= 2 \left[\frac{l}{2} - \frac{1}{3} \right]
 \end{aligned}$$

$$\frac{a_0}{2} = \frac{l}{2} - \frac{1}{3} //$$

$$\begin{aligned}
 a_n &= \frac{2}{l} \int_0^l f(x) \cos\left(\frac{n\pi x}{l}\right) dx \\
 &= \frac{2}{l} \int_0^l (\ln x - x^2) \cos n\pi x dx \\
 &= 2 \left[(\ln x - x^2) \frac{\sin n\pi x}{n\pi} - (l-2x) \left(\frac{-\cos n\pi x}{n^2\pi^2} \right) + (-2) \left(\frac{-\sin n\pi x}{n^3\pi^3} \right) \right]_0^l \\
 &= 2 \left[0 + \frac{(l-2)}{n^2\pi^2} \cos n\pi x + 0 - 0 - \frac{l}{n^2\pi^2} \cos 0 - 0 \right]
 \end{aligned}$$

$$a_n = \frac{2}{n^2\pi^2} \left[(l-2)(-1)^n - l \right] //$$

$$\therefore f(x) = \frac{l}{2} - \frac{1}{3} + \sum_{n=1}^{\infty} \frac{2}{n^2\pi^2} \left[(l-2)(-1)^n - l \right] //$$

b. Half range sine series is given by

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right) \text{ here } l=1$$

$$\text{where } b_n = \frac{2}{l} \int_0^l f(x) \sin\left(\frac{n\pi x}{l}\right) dx$$

$$b_n = \frac{2}{l} \int_0^l (\ln x - x^2) \sin n\pi x dx$$

$$= 2 \left[(\ln x - x^2) \left(\frac{-\cos n\pi x}{n\pi} \right) - (l-2x) \left(\frac{-\sin n\pi x}{n^2\pi^2} \right) + (-2) \left(\frac{\cos n\pi x}{n^3\pi^3} \right) \right]_0^l$$

$$\begin{aligned}
 &= 2 \left[-\frac{(l-1)}{n\pi} \cos nx + 0 - \frac{2}{n\pi^2} \cos nx + 0 - 0 + \frac{2}{n^3\pi^3} \cos nx \right] \\
 &= \frac{2}{n\pi} \left[-(l-1)(-1)^n + \frac{2(-1)^n}{n^2\pi^2} + \frac{2}{n^3\pi^3} \right] \\
 b_n &= \frac{2}{n\pi} \left[\frac{2}{n^2\pi^2} [(-1)^n + 1] - (l-1) \right] // \\
 \therefore f(x) &= \sum_{n=1}^{\infty} \frac{2}{n\pi} \left[\frac{2}{n^2\pi^2} [(-1)^n + 1] - (l-1) \right] \sin nx //
 \end{aligned}$$

* complex form of Fourier series: ($c_0 c_{+2l}$)

The complex form of Fourier series is defined as

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{inx/l}$$

$$\text{where } c_n = \frac{1}{2l} \int_c^{c+2l} f(x) e^{-inx/l} dx$$

Q1: Find the complex form of Fourier series for the function
 $f(x) = e^x$ in $-\pi < x < \pi$

$$\begin{aligned}
 \text{sol: } c_n &= \frac{1}{2\pi} \int_c^{c+2l} f(x) e^{-inx/l} dx \quad \text{here } l=\pi \\
 &= \frac{1}{2\pi} \int_{-\pi}^{\pi} e^x e^{-inx/\pi} dx \\
 &= \frac{1}{2\pi} \int_{-\pi}^{\pi} e^{x(1-in)} dx \\
 &= \frac{1}{2\pi} \left[\frac{e^{x(1-in)}}{1-in} \right] \Big|_{-\pi}^{\pi} \\
 &= \frac{1}{2\pi} \left[\frac{1}{1-in} \times \frac{1+in}{1+in} \right] (e^{\pi(1-in)} - e^{-\pi(1-in)})
 \end{aligned}$$

$$= \frac{1}{2\pi} \left[\frac{1+in}{1+n^2} \right] (e^x \cdot e^{-inx} - e^{-x} e^{inx})$$

wkt $e^{i\theta} = \cos\theta + i\sin\theta$

$$e^{-i\theta} = \cos\theta - i\sin\theta$$

$$= \frac{1+in}{2\pi(1+n^2)} \left[e^x (\cos n\pi - i\sin n\pi) - e^{-x} (\cos n\pi + i\sin n\pi) \right]$$

$$= \frac{1+in}{2\pi(1+n^2)} [e^x \cos n\pi - e^{-x} \cos n\pi]$$

$$= \frac{1+in}{\pi(1+n^2)} \cos n\pi \left[\frac{e^x - e^{-x}}{2} \right]$$

$$C_n = \frac{1+in}{\pi(1+n^2)} (-1)^n \sin n\pi$$

$$\therefore f(x) = \sum_{n=-\infty}^{\infty} \frac{1+in}{\pi(1+n^2)} (-1)^n \sin n\pi e^{inx}$$

Q2: Find the complex form of Fourier series for the function.
 $f(x) = e^{-x}$ in $(-1, 1)$.

sol: $f(x) = \sum_{n=-\infty}^{\infty} C_n e^{inx}$

$$\text{where } C_n = \frac{1}{2l} \int_c^{c+2l} f(x) e^{-inx} dx \quad \text{Here } l=1$$

$$C_n = \frac{1}{2} \int_{-1}^1 e^{-x} e^{-inx} dx$$

$$= \frac{1}{2} \int_{-1}^1 e^{-x(1+inx)} dx$$

$$= \frac{1}{2} \left[\frac{e^{-x(1+inx)}}{-1-inx} \right]_{-1}^1$$

$$= \frac{-1}{2(1+in\pi)} [e^{-(1+in\pi)} - e^{(1+in\pi)}]$$

$$\begin{aligned}
 &= \frac{-1}{2(1+i\pi)} [e^{-1}e^{inx} - e^1 e^{inx}] \\
 &= \frac{-1}{2(1+i\pi)} e^{inx} [e^{-1} - e^1] \\
 &= \frac{-1}{(1+i\pi)} (\cos n\pi + i \sin n\pi) \left[\frac{e^1 - e^{-1}}{2} \right] \\
 &= \left[\frac{1}{1+i\pi} \times \frac{1-i\pi}{1-i\pi} \right] \cos n\pi \sinh 1.
 \end{aligned}$$

$$c_n = \left[\frac{1-i\pi}{1+n^2\pi^2} \right] (-1)^n \sinh 1 //$$

$$\therefore f(x) = \sum_{n=-\infty}^{\infty} \frac{\sinh 1 (-1)^n (1-i\pi)}{1+n^2\pi^2} e^{inx} //$$

* Practical Harmonic Analysis:

TYPE I: π is in degree

Period 2π

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx + \sum_{n=1}^{\infty} b_n \sin nx$$

First Harmonic

$$f(x) = \frac{a_0}{2} + a_1 \cos x + b_1 \sin x$$

Second Harmonic

$$f(x) = \frac{a_0}{2} + a_1 \cos x + a_2 \cos 2x + b_1 \sin x + b_2 \cos 2x$$

Third Harmonic

$$\begin{aligned}
 f(x) = & \frac{a_0}{2} + a_1 \cos x + a_2 \cos 2x + a_3 \cos 3x + b_1 \sin x \\
 & + b_2 \sin 2x + b_3 \sin 3x
 \end{aligned}$$

$$\text{where } a_0 = \frac{2}{N} \sum y$$

Q1:

sd:

$$a_n = \frac{2}{N} \sum y \cos nx$$

$$b_n = \frac{2}{N} \sum y \sin nx$$

Q1: Determine the constant term and first cosine and sine terms of the Fourier series.

| | | | | | | | | |
|-------------|---|---------------|----|---------------|-----|---------------|-----|---------------|
| x° | 0 | 45 | 90 | 135 | 180 | 225 | 240 | 315 |
| y | 2 | $\frac{3}{2}$ | 1 | $\frac{1}{2}$ | 0 | $\frac{1}{2}$ | 1 | $\frac{3}{2}$ |

| | | | | | | | | | |
|------------|-------|---|--------|---|---------|---|---------|----|---------|
| <u>sd:</u> | 4cosx | 2 | 1.0606 | 0 | -0.3535 | 0 | -0.3535 | 0 | 1.0606 |
| | 4sinx | 0 | 1.0606 | 1 | 0.3535 | 0 | -0.3535 | -1 | -1.0606 |

First harmonic of Fourier series is given by
 $f(x) = \frac{a_0}{2} + a_1 \cos x + b_1 \sin x$

$$\text{where } a_0 = \frac{2}{N} \sum y$$

$$a_0 = \frac{2}{8} (8) \Rightarrow a_0 = 2 \Rightarrow \frac{a_0}{2} = 1 //$$

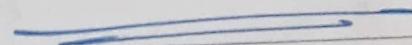
$$a_1 = \frac{2}{N} \sum y \cos x$$

$$a_1 = \frac{2}{8} (3.4142) \Rightarrow a_1 = 0.8535 //$$

$$b_1 = \frac{2}{N} \sum y \sin x$$

$$b_1 = \frac{2}{8} (0) \Rightarrow b_1 = 0 //$$

$$\therefore f(x) = 1 + 0.8535 \cos x$$



Q2: Express y as Fourier series upto the third harmonic.

| x | 0 | $\pi/3$ | $2\pi/3$ | π | $4\pi/3$ | $5\pi/3$ |
|-----|------|---------|----------|-------|----------|----------|
| y | 1.98 | 1.30 | 1.05 | 1.30 | -0.88 | -0.25 |

Sol: Here the interval of x is $0 \leq x \leq 2\pi$ and the value at $x=0$ and $x=2\pi$ must be same by the periodic properties i.e., $f(x+2\pi) = f(x)$.

In the given problem value of y at $x=0$ and $x=2\pi$ both are given thus we must take any one of the two.

| x | 0 | $\pi/3$ | $2\pi/3$ | π | $4\pi/3$ | $5\pi/3$ |
|-------------|------|---------|----------|-------|----------|----------|
| x° | 0 | 60 | 120 | 180 | 240 | 300 |
| y | 1.98 | 1.30 | 1.05 | 1.30 | -0.88 | -0.25 |
| $y \cos x$ | 1.98 | 0.65 | -0.525 | -1.30 | 0.44 | -0.125 |
| $y \cos 2x$ | 1.98 | -0.65 | -0.525 | -1.30 | 0.44 | 0.125 |
| $y \cos 3x$ | 1.98 | -1.3 | 1.05 | -1.30 | -0.88 | 0.25 |
| $y \sin x$ | 0 | 1.1258 | 0.9093 | 0 | 0.7621 | 0.2165 |
| $y \sin 2x$ | 0 | 1.1258 | -0.9093 | 0 | -0.7621 | 0.2165 |
| $y \sin 3x$ | 0 | 0 | 0 | 0 | 0 | 0 |

Third harmonic of fourier series is given by

$$f(x) = \frac{a_0}{2} + a_1 \cos x + a_2 \cos 2x + a_3 \cos 3x + b_1 \sin x + b_2 \sin 2x + b_3 \sin 3x$$

$$a_0 = \frac{2}{N} \sum y = \frac{2}{6} (4.5) = 1.5 //$$

$$a_1 = \frac{2}{N} \sum y \cos x = \frac{2}{6} (1.12) = 0.3733 //$$

$$a_2 = \frac{2}{N} \sum y \cos 2x = \frac{2}{6} (2.67) = 0.89 //$$

$$a_3 = \frac{2}{N} \sum y \cos 3x = \frac{2}{6} (-0.2) = -0.0667 //$$

$$b_1 = \frac{2}{N} \sum y \sin x = \frac{2}{6} (3.0137) = 1.0045 //$$

$$b_2 = \frac{2}{N} \sum y \sin 2x = \frac{2}{6} (-0.3291) = -0.1097 //$$

$$b_3 = \frac{2}{N} \sum y \sin 3x = \frac{2}{6} (0) = 0 //$$

$$\therefore f(x) = 0.75 + 0.3733 \cos x + 0.89 \cos 2x - 0.667 \cos 3x \\ + 1.0045 \sin x - 0.1097 \sin 2x //$$

TYPE 2: Period of 2L.

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{L}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{L}\right)$$

$$\text{put } \theta = \frac{n\pi x}{L}$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\theta + \sum_{n=1}^{\infty} b_n \sin n\theta$$

First Harmonic

$$f(x) = \frac{a_0}{2} + a_1 \cos \theta + b_1 \sin \theta$$

Second Harmonic

$$f(x) = \frac{a_0}{2} + a_1 \cos \theta + a_2 \cos 2\theta + a_3 \cos 3\theta + b_1 \sin \theta + b_2 \sin 2\theta + b_3 \sin 3\theta$$

$$\text{where } a_0 = \frac{2}{N} \sum y$$

$$a_n = \frac{2}{N} \sum y \cos n\theta$$

$$b_n = \frac{2}{N} \sum y \sin n\theta$$

- Q1: Obtain the constant term and the coefficients of the first cosine and sine terms in the Fourier expansion of y from the following data:

| | | | | | | |
|----------|---|----|----|----|----|----|
| <u>x</u> | 0 | 1 | 2 | 3 | 4 | 5 |
| y | 9 | 18 | 24 | 28 | 26 | 20 |

S1: By the data a value lies in the interval $0 \leq x \leq 6$.
 Therefore length of the interval is 6 i.e., $N=6$, comparing
 the period $2l$

$$\text{we have } 2l=6 \Rightarrow l=3$$

The Fourier series of period $2l$ is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right)$$

$$\text{put } \theta = \frac{\pi x}{l}$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\theta + \sum_{n=1}^{\infty} b_n \sin n\theta$$

First harmonic of Fourier series is

$$f(x) = \frac{a_0}{2} + a_1 \cos \theta + b_1 \sin \theta \quad \text{where } \theta = \frac{\pi x}{3}$$

| | | | | | | | |
|----------------------------|-----------|------------|-------------|-------------|-------------|-------------|----------|
| <u>x</u> | 0 | 1 | 2 | 3 | 4 | 5 | |
| $\theta = \frac{\pi x}{3}$ | 0° | 60° | 120° | 180° | 240° | 300° | |
| y | 9 | 18 | 24 | 28 | 26 | 20 | 125 |
| $y \cos \theta$ | 9 | 9 | -12 | -28 | -13 | 10 | -25 |
| $y \sin \theta$ | 0 | 15.5884 | 20.7846 | 0 | -22.5166 | -17.3205 | -13.4641 |

$$a_0 = \frac{2}{N} \sum y = \frac{2}{6} (125) = \underline{\underline{41.6667}}$$

$$a_1 = \frac{2}{N} \sum y \cos \theta = \frac{2}{6} (-25) = \underline{\underline{-8.3333}}$$

$$b_1 = \frac{2}{N} \sum y \sin \theta = \frac{2}{6} (-13.4641) = \underline{\underline{-4.488}}$$

$$\therefore f(x) = 20.8333 - 2.3333 \cos \theta - 4.488 \sin \theta //$$

Find the Fourier series of y upto second harmonic from the following data.

| | | | | | | | |
|-----|-----|------|------|------|------|------|-----|
| x | 0 | 2 | 4 | 6 | 8 | 10 | 12 |
| y | 9.0 | 18.2 | 24.4 | 27.8 | 27.5 | 22.0 | 9.0 |

The values of y at $x=0$ and $x=12$ are same hence the interval of x is $(0, 12)$ i.e., $0 \leq x \leq 12$ and we shall neglect the value of y for $x = 12$ in the process of calculation, comparing this with the period $2l$. the length of the interval is 12. Comparing with the period of $2l$

$$\text{we have } 2l = 12 \Rightarrow l = 6 //$$

The Fourier series of period $2l$ is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right)$$

$$\text{put } \theta = \frac{\pi x}{l}$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\theta + \sum_{n=1}^{\infty} b_n \sin n\theta$$

Second harmonic of Fourier series is

$$f(x) = \frac{a_0}{2} + a_1 \cos \theta + a_2 \cos 2\theta + b_1 \sin \theta + b_2 \sin 2\theta$$

| x | 0 | 2 | 4 | 6 | 8 | 10 | |
|---------------------|-----------|------------|-------------|-------------|-------------|-------------|---------|
| $\theta = \pi x/12$ | 0° | 60° | 120° | 180° | 240° | 300° | |
| y | 9.0 | 18.2 | 24.4 | 27.8 | 27.5 | 22.0 | 128.9 |
| $y \cos \theta$ | 9.0 | 9.1 | -12.2 | -24.8 | -13.75 | 11.0 | -24.65 |
| $y \cos 2\theta$ | 9.0 | -9.1 | -12.2 | 27.8 | -13.75 | -11.0 | -9.25 |
| $y \sin \theta$ | 0 | 15.7616 | 21.1310 | 0 | -23.8156 | -19.0525 | -5.9755 |
| $y \sin 2\theta$ | 0 | 15.7616 | -21.1310 | 0 | 23.8156 | -19.0525 | -0.6063 |

$$a_1 = \frac{2}{N} \sum y \cos \theta = \frac{2}{6} (-24.65) = \underline{\underline{-8.2164}}$$

$$a_2 = \frac{2}{N} \sum y \cos 2\theta = \frac{2}{6} (-9.25) = \underline{\underline{-3.0833}}$$

$$b_1 = \frac{2}{N} \sum y \sin \theta = \frac{2}{6} (-5.9155) = \underline{\underline{-1.9918}}$$

$$b_2 = \frac{2}{N} \sum y \sin 2\theta = \frac{2}{6} (-0.6063) = \underline{\underline{-0.2021}}$$

$$\therefore f(x) = 21.4833 - 8.2164 \cos \theta - 3.0833 \cos 2\theta$$

$$-1.9918 \sin \theta - 0.2021 \sin 2\theta //$$

Q3: Express y as a Fourier series upto third harmonic from the given data.

| | | | | | | |
|-----|---|---|----|---|---|---|
| x | 0 | 1 | 2 | 3 | 4 | 5 |
| y | 4 | 8 | 15 | 7 | 6 | 2 |

Sol: By the data x value lies on the interval $0 \leq x < 6$. $[0, 6)$
Therefore by comparing with period $2l$

$$2l = 6 \Rightarrow l = \underline{\underline{3}}$$

Fourier series of period $2l$ is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right)$$

$$\text{put } \theta = \frac{\pi x}{l} = \frac{\pi x}{3}$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\theta + \sum_{n=1}^{\infty} b_n \sin n\theta$$

Third harmonic of Fourier series is given by

$$f(x) = \frac{a_0}{2} + a_1 \cos \theta + a_2 \cos 2\theta + a_3 \cos 3\theta + b_1 \sin \theta$$

| $\theta = \pi x/3$ | 0 | 1 | 2 | 3 | 4 | 5 | |
|--------------------|---|------------|-------------|-------------|-------------|-------------|---------|
| x | 0 | 60° | 120° | 180° | 240° | 300° | |
| y | 4 | 8 | 15 | 4 | 6 | 2 | 42 |
| $y \cos \theta$ | 4 | 4 | -7.5 | -7 | -3 | 1 | -8.5 |
| $y \cos 2\theta$ | 4 | -4 | -7.5 | +7 | -3 | -1 | -4.5 |
| $y \cos 3\theta$ | 4 | -8 | 15 | -7 | 6 | -2 | 8 |
| $y \sin \theta$ | 0 | 6.9282 | 12.99 | 0 | -5.1961 | -1.732 | 12.9901 |
| $y \sin 2\theta$ | 0 | 6.9282 | -12.99 | 0 | 5.1961 | -1.732 | -2.5941 |
| $y \sin 3\theta$ | 0 | 0 | 0 | 0 | 0 | 0 | 0 |

$$a_0 = \frac{2}{N} \sum y = 14 //$$

$$a_1 = \frac{2}{N} \sum y \cos \theta = \frac{2}{6} (-8.5) = -2.833 //$$

$$a_2 = \frac{2}{N} \sum y \cos 2\theta = \frac{2}{6} (-4.5) = -1.5 //$$

$$a_3 = \frac{2}{N} \sum y \cos 3\theta = \frac{2}{6} (8) = 2.667 //$$

$$b_1 = \frac{2}{N} \sum y \sin \theta = \frac{2}{6} (12.9901) = 4.33 //$$

$$b_2 = \frac{2}{N} \sum y \sin 2\theta = \frac{2}{6} (-2.5941) = -0.8659 //$$

$$b_3 = \frac{2}{N} \sum y \sin 3\theta = \frac{2}{6} (0) = 0 //$$

$$\therefore f(x) = 14 - 2.833 \cos \theta - 1.5 \cos 2\theta + 2.667 \cos 3\theta + 4.33 \sin \theta - 0.8659 \sin 2\theta$$

Q4: Obtain the constant term and coefficient of $\sin \theta$ and $\sin 2\theta$ in the Fourier expansion of y from the given data

| θ° | 0 | 60 | 120 | 180 | 240 | 300 | 360 |
|----------------|---|-----|------|------|------|------|-----|
| y | 0 | 9.2 | 14.4 | 17.8 | 17.3 | 11.7 | 0 |

| x° | 0 | 60 | 120 | 180 | 240 | 300 | |
|------------------|---|---------|----------|------|----------|----------|---------|
| y | 0 | 7.2 | 14.4 | 17.8 | 17.3 | 11.7 | 70.4 |
| $y \sin \theta$ | 0 | -7.9674 | 12.4401 | 0 | -14.9822 | -10.1324 | -9.6465 |
| $y \sin 2\theta$ | 0 | 7.9674 | -12.4401 | 0 | 14.9822 | -10.1324 | 20.3465 |

Q6:

$$a_0 = \frac{2}{N} \sum y = \frac{2}{6} (70.4) = 23.467 //$$

$$a_1 = \frac{2}{N} \sum y \sin \theta = \frac{2}{6} (-9.6465) = -1.558 //$$

$$a_2 = \frac{2}{N} \sum y \sin 2\theta = \frac{2}{6} (20.3465) = 0.1151 //$$

Sol:

Q5: Given a following table

| θ° | 0 | 60 | 120 | 180 | 240 | 300 |
|----------------|-----|-----|-----|-----|-----|-----|
| y | 7.9 | 7.2 | 3.6 | 0.5 | 0.9 | 6.8 |

Obtain the Fourier series neglecting the terms higher than first harmonic

| θ° | 0 | 60 | 120 | 180 | 240 | 300 |
|-----------------|-----|--------|--------|------|---------|---------|
| y | 7.9 | 7.2 | 3.6 | 0.5 | 0.9 | 6.8 |
| $y \cos \theta$ | 7.9 | 3.6 | -1.8 | -0.5 | -0.45 | 3.4 |
| $y \sin \theta$ | 0 | 6.2354 | 3.1144 | 0 | -0.7794 | -5.8889 |

$$a_0 = \frac{2}{N} \sum y = \frac{2}{6} (26.9) = 8.967 //$$

$$a_1 = \frac{2}{N} \sum y \cos \theta = \frac{2}{6} (12.25) = 4.083 //$$

$$a_2 = \frac{2}{N} \sum y \sin \theta = \frac{2}{6} (2.6848) = 0.895 //$$

$$\therefore f(x) = 4.4835 + 4.083 \cos \theta + 0.895 \sin \theta //$$

Q6: Obtain the constant term and first three coefficients in the Fourier cosine series for y using the following data:

| | | | | | | |
|-----|---|---|----|---|---|---|
| x | 0 | 1 | 2 | 3 | 4 | 5 |
| y | 4 | 8 | 15 | 7 | 6 | 2 |

Sol: Fourier cosine series: By the data x values lies in the interval $0 \leq x \leq 6$ i.e., $(0, 6)$ Half range series.

$$\text{Comparing } (0, l) \Rightarrow l = 6 //$$

Fourier series of period l is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right) + \sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right)$$

$$\text{put } \theta = \frac{\pi x}{l} = \frac{\pi x}{6}$$

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos n\theta + \sum_{n=1}^{\infty} b_n \sin n\theta$$

The third harmonic is given by

$$f(x) = \frac{a_0}{2} + a_1 \cos \theta + a_2 \cos 2\theta + a_3 \cos 3\theta + b_1 \sin \theta \\ + b_2 \sin 2\theta + b_3 \sin 3\theta$$

| | | | | | | |
|----------------------------|---|-------|------|----|-----|--------|
| x | 0 | 1 | 2 | 3 | 4 | 5 |
| $\theta = \frac{\pi x}{6}$ | 0 | 30 | 60 | 90 | 120 | 150 |
| y | 4 | 8 | 15 | 7 | 6 | 2 |
| $y \cos \theta$ | 4 | 6.928 | 4.5 | 0 | -3 | -1.732 |
| $y \cos 2\theta$ | 4 | 4 | -4.5 | -7 | -3 | 1 |
| $y \cos 3\theta$ | 4 | 0 | -15 | 0 | 6 | 0 |

~~$y \sin \theta$~~

~~$y \sin 2\theta$~~ $a_0 = \frac{2}{N} \sum y = \frac{2}{6} (42) = 14 //$

~~$y \sin 3\theta$~~ $a_1 = \frac{2}{N} \sum y \cos \theta = \frac{2}{6} (13.696) = 4.565 //$

$a_2 = \frac{2}{N} \sum y \cos 2\theta = \frac{2}{6} (-8.5) = -7.833 //$

$$a_3 = \frac{2}{N} \sum y \cos 3\theta = \frac{2}{6} (-5) = -1.667 //$$

UNIT - 2

Partial Differential Equations

* Order and Degree:

i. $\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0$ order = 2 degree = 1

ii. $\frac{\partial^2 z}{\partial x \partial y} = xy$ order = 2 degree = 1

iii. $\frac{\partial^2 z}{\partial x^2} + \frac{\partial z}{\partial y} = xy$ order = 2 degree = 1

* Formation of a PDE by eliminating arbitrary constant and arbitrary functions:

Many problems in vibration of strings, heat conduction, electrostatics involve two or more variables, analysis of these problems leads to partial derivatives and equations involving them.

An equation involving one or more partial derivatives for a function of two or more variables is called a partial differential equations.

The order of the PDE is the order of the highest derivative and the degree of the PDE is the degree of the highest order derivative after clearing the equation of fractional powers.

Given a relation of the form $f(x, y, z, a, b) = 0$, where z is a function of (x, y) and a, b are arbitrary constants, we differentiate the given relation with respect to x and y partially and eliminate the arbitrary constants a, b to form the PDE. In case the number of arbitrary constants are more than the number of independent

variables we need appropriate number of partial derivatives of second order or higher also.

Suppose z is a function of two arbitrary functions we have to find partial derivatives up to the second order and use the necessary partial derivatives of second order to form the PDE by eliminating the arbitrary functions.

The following are standard notations when z is a function of two independent variables

$$P = \frac{\partial z}{\partial x}, \quad q = \frac{\partial z}{\partial y}, \quad r = \frac{\partial^2 z}{\partial x^2}$$

$$s = \frac{\partial^2 z}{\partial x \partial y}, \quad t = \frac{\partial^2 z}{\partial y^2}$$

Q1: Find the PDE by eliminating arbitrary constants

$$z = (x+a)(y+b) \quad \text{--- (1)}$$

Sol: diff partially w.r.t x, y

$$P = \frac{\partial z}{\partial x} = (y+b) \quad (\cancel{(x+a)}) \quad \text{--- (2)}$$

$$q = \frac{\partial z}{\partial y} = (x+a) \quad \text{--- (3)}$$

Substitute eq (2) and (3) in (1)

$z = pq$ is the required PDE.

$$Q2: 2z = \frac{x^2}{a^2} + \frac{y^2}{b^2} \quad \text{--- (1)}$$

Sol: diff eq (1) partially w.r.t x and y

$$\frac{\partial z}{\partial x} = \frac{x}{a^2} \Rightarrow P = \frac{x}{a^2} \Rightarrow a^2 = \frac{x}{P}$$

$$\frac{\partial z}{\partial y} = \frac{y}{b^2} \Rightarrow q = \frac{y}{b^2} \Rightarrow b^2 = \frac{y}{q}$$

substituting a^2 and b^2 in eq ①

$$2x = \frac{x^2}{x/p} + \frac{y^2}{y/q}$$

$2x = xp + yq$ // is the required PDE

Q3:

$$ax^2 + by^2 + z^2 = 1$$

Sol:

$$z^2 = 1 - ax^2 - by^2 \quad \text{--- ①}$$

diff eq ① wrt x and y

$$2z \frac{\partial z}{\partial x} = -2ax$$

$$xp = -2ax$$

$$a = -\frac{zp}{x}$$

$$\bullet 2z \frac{\partial z}{\partial y} = -2by$$

$$zp = -by$$

$$b = -\frac{zp}{y}$$

substituting a and b in eq ①

$$z^2 = 1 - \left(-\frac{zp}{x}\right)x^2 - \left(-\frac{zp}{y}\right)y^2$$

$z^2 = 1 + zpx + zqy$ // is the required PDE.

Q4: $z = a \log(x^2 + y^2) + b^2 \quad \text{--- ①}$

Sol:

diff eq ① wrt x

$$\frac{\partial z}{\partial x} = \frac{a}{x^2 + y^2} (2x)$$

$$p = \frac{2ax}{x^2 + y^2} \rightarrow a = \frac{p(x^2 + y^2)}{2x}$$

diff eq ① w.r.t y

$$\frac{\partial z}{\partial y} = \frac{a}{(x^2 + y^2)} \quad (2y)$$

$$q_y = \frac{2ay}{(x^2 + y^2)} \quad \text{--- } ③$$

Dividing eq ② and eq ③

$$\frac{P}{q} = \frac{-2ax}{(x^2 + y^2)} \cdot \frac{(x^2 + y^2)}{-2ay}$$

$$\frac{P}{q} = \frac{x}{y} \Rightarrow \underline{py = xq} \text{ is the required PDE}$$

Q5: $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$

sol: $\frac{z^2}{c^2} = 1 - \frac{x^2}{a^2} - \frac{y^2}{b^2} \quad \text{--- } ①$

diff eq ① w.r.t x

$$\frac{\partial z}{c^2} \left(\frac{\partial z}{\partial x} \right) = -\frac{2x}{a^2}$$

$$\frac{zp}{c^2} = -\frac{x}{a^2} \Rightarrow \frac{1}{a^2} = \frac{-zp}{xc^2} \quad \text{--- } ②$$

diff eq ① w.r.t y

$$\frac{\partial z}{c^2} \left(\frac{\partial z}{\partial y} \right) = -\frac{2y}{b^2}$$

$$\frac{xq}{c^2} = -\frac{y}{b^2} \Rightarrow \frac{1}{b^2} = -\frac{xq}{yc^2} \quad \text{--- } ③$$

Substituting eq ② and ③ in eq ①

$$\frac{z^2}{c^2} = 1 - \frac{x^2}{a^2} \left(\frac{zp}{xc^2} \right) - \frac{y^2}{b^2} \left(\frac{xq}{yc^2} \right)$$

$$z^2 = c^2 + zpx + xqy$$

Q6:

sol:

Q7:

sol:

diff eq @ w.r.t x

$$\frac{-1}{a^2} = \frac{1}{c^2} [x\gamma + p^2]$$

$$\frac{-1}{a^2} = \frac{1}{c^2} [x\gamma + p^2]$$

$$\frac{x\gamma}{a^2} = \frac{1}{c^2} [x\gamma + p^2] \quad \text{from eq ②}$$

$x\gamma = x\gamma + np^2$ is the required PDE

Q6: $Z = xy + y\sqrt{x^2 - a^2} + b \quad \text{--- ①}$

SL: diff eq ① w.r.t x

$$P = \frac{\partial Z}{\partial x} = y + y \frac{1}{2\sqrt{x^2 - a^2}} (2x)$$

$$P = y + \frac{yx}{\sqrt{x^2 - a^2}} \quad \text{--- ②}$$

diff eq ① w.r.t y

$$Q = \frac{\partial Z}{\partial y} = x + \sqrt{x^2 - a^2}$$

$$Q - x = \sqrt{x^2 - a^2} \quad \text{--- ③}$$

Substituting eq ③ in eq ②

$$P = y + \frac{xy}{\sqrt{x^2 - a^2}} //$$

Q7 Find the PDE by eliminating arbitrary function.

$$Z = f(x^2 + y^2) \quad \text{--- ①}$$

SL: diff eq ① w.r.t x

$$P = \frac{\partial Z}{\partial x} = f'(x^2 + y^2) 2x \quad \text{--- ②}$$

diff eq ① wrt y

$$q = \frac{\partial z}{\partial y} = f'(x^2 + y^2) 2y \quad \text{--- } ③$$

Dividing eq ② by eq ③

$$\frac{P}{q} = \frac{f'(\frac{1}{x^2} + \log y) 2x}{f'(\frac{1}{x^2} + \log y) 2y}$$

$$\underline{P} \underline{Y} = q \underline{x}$$

Q8: $z = y^2 + 2f\left(\frac{1}{x} + \log y\right) \quad \text{--- } ①$

Sol: diff wrt x

$$P = \frac{\partial z}{\partial x} = 2f'\left(\frac{1}{x} + \log y\right) \cdot -\frac{1}{x^2}$$

$$\underline{P} \underline{x}^2 = -2f'\left(\frac{1}{x} + \log y\right) \quad \text{--- } ②$$

diff wrt y

$$q = \frac{\partial z}{\partial y} = 2y + 2f'\left(\frac{1}{x} + \log y\right) \frac{1}{y}$$

$$\underline{q} \underline{y} = 2f'\left(\frac{1}{x} + \log y\right) \quad \text{--- } ③$$

Substituting eq ③ in eq ②

$$\underline{P} \underline{x}^2 = -\underline{(q - 2y)} \underline{y}$$

Q9: $z = e^{ax+by} f(ax-by) \quad \text{--- } ①$

Sol: diff eq ① wrt x

$$\frac{\partial z}{\partial x} = e^{ax+by} f'(ax-by) \cdot (+a) + f(ax-by) e^{ax+by} (a)$$

$$P = ae^{ax+by} [f(ax-by) + f'(ax-by)] \quad \text{--- } ②$$

diff eq ① wrt y

$$\frac{\partial z}{\partial y} = e^{ax+by} f'(ax-by)(-b) + f(ax-by) e^{ax+by} (b)$$

$$q = be^{ax+by} [f(ax-by) - f'(ax-by)] \quad \text{--- } ③$$

Multiplying a to eq ③ and b to eq ②

$$\text{eq } ② : bp = abe^{ax+by} [f(ax-by) + f'(ax-by)] \quad \text{--- } ④$$

$$\text{eq } ③ : qg = abe^{ax+by} [f(ax-by) - f'(ax-by)] \quad \text{--- } ⑤$$

Adding eq ④ and eq ⑤

$$pb = abe^{ax+by}$$

$$pb + aq = 2abe^{ax+by} f(ax-by)$$

$$pb + aq = 2ab \frac{e^{ax+by}}{e^{ax+by}} \quad \text{from eq ①}$$

$$\underline{pb + aq = 2abx}$$

$$\text{Q10: } lx + my + nz = \phi(x^2 + y^2 + z^2) \quad \text{--- } ①$$

sd: diff eq ① wrt x

$$l + n \frac{\partial x}{\partial x} = \phi'(x^2 + y^2 + z^2) \left(2x + 2z \frac{\partial z}{\partial x} \right)$$

$$l + np = 2\phi'(x^2 + y^2 + z^2)(x + xp) \quad \text{--- } ②$$

diff eq ② wrt y

$$m + n \frac{\partial z}{\partial y} = \phi'(x^2 + y^2 + z^2) \left(2y + 2z \frac{\partial z}{\partial y} \right)$$

$$m + nq = 2\phi'(x^2 + y^2 + z^2)(y + xq) \quad \text{--- } ③$$

Dividing eq ② by eq ③

$$\frac{l+np}{m+nq} = \frac{x+xp}{y+yq}$$

$$(l+np)(y+q) = (a+xp)(m+nq)$$

$$ly + lzq + npy + npq = am + anq + mxp + nxp q$$

$$p(ny - mx) + q(lz - na) = ma - ly$$

NOTE:

To form the PDE from $\phi(u, v) = 0$ where u and v are functions of (x, y) are as follows.

$$\frac{\partial \phi}{\partial u} \frac{\partial u}{\partial x} + \frac{\partial \phi}{\partial v} \frac{\partial v}{\partial x} = 0$$

$$\frac{\partial \phi}{\partial u} \frac{\partial u}{\partial y} + \frac{\partial \phi}{\partial v} \frac{\partial v}{\partial y} = 0$$

Q1: $\phi(x+y+z, x^2+y^2-z^2) = 0$

Sol: It is of the form $\phi(u, v) = 0$

$$u = x+y+z$$

$$v = x^2+y^2-z^2$$

$$\frac{\partial \phi}{\partial u} \frac{\partial u}{\partial x} + \frac{\partial \phi}{\partial v} \frac{\partial v}{\partial x} = 0 \quad \text{--- (1)}$$

$$\frac{\partial \phi}{\partial u} \frac{\partial u}{\partial y} + \frac{\partial \phi}{\partial v} \frac{\partial v}{\partial y} = 0 \quad \text{--- (2)}$$

From eq (1) $\frac{\partial \phi}{\partial u} \frac{\partial u}{\partial x} = -\frac{\partial \phi}{\partial v} \frac{\partial v}{\partial x} \quad \text{--- (3)}$

From eq (2) $\frac{\partial \phi}{\partial u} \frac{\partial u}{\partial y} = -\frac{\partial \phi}{\partial v} \frac{\partial v}{\partial y} \quad \text{--- (4)}$

Dividing eq (3) by eq (4)

$$\frac{\frac{\partial u}{\partial x}}{\frac{\partial u}{\partial y}} = \frac{\frac{\partial v}{\partial x}}{\frac{\partial v}{\partial y}} \quad \text{--- (5)}$$

diff u w.r.t x

$$\frac{\partial u}{\partial x} = 1 + \frac{\partial z}{\partial x} = 1+p$$

diff u w.r.t y

$$\frac{\partial u}{\partial y} = 1 + \frac{\partial z}{\partial y} = 1+q$$

diff v w.r.t x

$$\frac{\partial v}{\partial x} = 2x - z \frac{\partial z}{\partial x} = 2x - 2px$$

diff v w.r.t y

$$\frac{\partial v}{\partial y} = 2y - z \frac{\partial z}{\partial y} = 2y - 2qz$$

substituting in eq ⑤

$$\begin{matrix} 1+p \\ 1+q \end{matrix} = \begin{matrix} 2x-2px \\ 2y-2qz \end{matrix}$$

$$(1+p)(y-qz) = (1+q)(x-pz)$$

$$y + py - qz - pqz = x - px + pqz + qx$$

$$\underline{p(y+z) - q(x+z) = x - y}$$

$$Q2: \phi(xy + z^2, x+y+z) = 0$$

Sol: It is of the form $\phi(u, v) = 0$

$$\text{Here } u = xy + z^2$$

$$v = x + y + z$$

$$\frac{\partial \phi}{\partial u} \frac{\partial u}{\partial x} + \frac{\partial \phi}{\partial v} \frac{\partial v}{\partial x} = 0 \quad \text{--- } ①$$

$$\frac{\partial \phi}{\partial u} \frac{\partial u}{\partial y} + \frac{\partial \phi}{\partial v} \frac{\partial v}{\partial y} = 0 \quad \text{--- } ②$$

$$\text{From eq } ① \quad \frac{\partial \phi}{\partial u} \frac{\partial u}{\partial x} = - \frac{\partial \phi}{\partial v} \frac{\partial v}{\partial x} \quad \text{--- } ③$$

From eq ② $\frac{\partial \phi}{\partial u} \frac{\partial u}{\partial y} = - \frac{\partial \phi}{\partial v} \frac{\partial v}{\partial y}$ — ④

Dividing eq ③ by eq ④

$$\frac{\frac{\partial u}{\partial x}}{\frac{\partial v}{\partial x}} = \frac{\frac{\partial v}{\partial x}}{\frac{\partial u}{\partial x}} \quad \text{here} \quad \text{eq ⑤}$$

diff u wrt x

$$\frac{\partial u}{\partial x} = y + 2x \frac{\partial z}{\partial x} = y + 2xp$$

diff u wrt y

$$\frac{\partial u}{\partial y} = x + 2z \frac{\partial x}{\partial y} = x + 2zq$$

diff v wrt x

$$\frac{\partial v}{\partial x} = 1 + \frac{\partial z}{\partial x} = 1 + p$$

diff v wrt y

$$\frac{\partial v}{\partial y} = 1 + \frac{\partial z}{\partial y} = 1 + q$$

Substituting in eq ⑤

$$\frac{y + 2zp}{x + 2zq} = \frac{y + p}{1 + q}$$

$$(y + 2zp)(1 + q) = (x + 2zq)(1 + p)$$

$$y + qy + 2pqz + 2pz = x + xp + 2qz + 2pqz$$

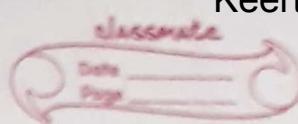
$$\cancel{x} \rightarrow p(x - 2z) + q(y - 2z) = y - x$$

Q3: $f(x^2 + 2yz, y^2 + 2zx) = 0$

Sol: It is of the form $\phi(u, v) = 0$

where $u = x^2 + 2yz$

$v = y^2 + 2zx$



$$\frac{\partial f}{\partial u} \frac{\partial u}{\partial x} + \frac{\partial f}{\partial v} \frac{\partial v}{\partial x} = 0 \Rightarrow \frac{\partial f}{\partial u} \frac{\partial u}{\partial x} = - \frac{\partial f}{\partial v} \frac{\partial v}{\partial x} \quad \text{--- (1)}$$

$$\frac{\partial f}{\partial u} \frac{\partial u}{\partial y} + \frac{\partial f}{\partial v} \frac{\partial v}{\partial y} = 0 \Rightarrow \frac{\partial f}{\partial u} \frac{\partial u}{\partial y} = - \frac{\partial f}{\partial v} \frac{\partial v}{\partial y} \quad \text{--- (2)}$$

Dividing eq (1) by eq (2)

$$\frac{\frac{\partial f}{\partial u} \frac{\partial u}{\partial x}}{\frac{\partial f}{\partial u} \frac{\partial u}{\partial y}} = \frac{\frac{\partial f}{\partial v} \frac{\partial v}{\partial x}}{\frac{\partial f}{\partial v} \frac{\partial v}{\partial y}} \quad \text{--- (3)}$$

diff u wrt x

$$\frac{\partial u}{\partial x} = 2x + 2y \frac{\partial z}{\partial x} = 2(x+yz)$$

diff u wrt y

$$\frac{\partial u}{\partial y} = 2z + 2y \frac{\partial z}{\partial y} = 2(z+yz)$$

diff v wrt x

$$\frac{\partial v}{\partial x} = 2z + 2x \frac{\partial z}{\partial x} = 2(z+xz)$$

diff v wrt y

$$\frac{\partial v}{\partial y} = 2y + 2x \frac{\partial z}{\partial y} = 2(y+xz)$$

substituting in eq (3)

$$\frac{2(x+yz)}{2(z+yz)} = \frac{2(z+xz)}{2(y+xz)}$$

$$(x+yz)(y+xz) = (z+xz)(z+yz)$$

$$xy + x^2z + y^2z + xyz^2 = z^2 + xzy^2 + xzx^2 + xyz^2$$

$$p(y^2 - zx) + q(z^2 - xp) = x^2 - xy$$

Q: $x = yf(x) + x\phi(y) \quad \dots \textcircled{1}$

Sol: diff x wrt x

$$p = \frac{\partial x}{\partial x} = yf'(x) + \phi(y) \quad \dots \textcircled{2}$$

diff x wrt y

$$q = \frac{\partial x}{\partial y} = f(x) + x\phi'(y) \quad \dots \textcircled{3}$$

diff x wrt x^2

$$\frac{\partial p}{\partial x} = g_1 = yf''(x) \quad \dots \textcircled{4}$$

diff x wrt y^2

$$\frac{\partial q}{\partial y} = t = x\phi''(y) \quad \dots \textcircled{5}$$

diff p wrt y

$$s = f'(x) + \phi'(y) \quad \dots \textcircled{6}$$

From eq. $\textcircled{2}$

$$f'(x) = \frac{p - \phi(y)}{y}$$

From eq. $\textcircled{3}$

$$\phi'(x) = \frac{q - f(x)}{x}$$

Substituting in eq. $\textcircled{6}$

$$s = \frac{p - \phi(y)}{y} + \frac{q - f(x)}{x}$$

$$xys = xp - x\phi(y) + yq - yf(x)$$

$$xys = xp + yq - z. \quad (\text{From eq. } \textcircled{1})$$

$$\underline{x = xp + yq - xys}$$

$$z = x f_1(x+y) + f_2(x+y) \quad \textcircled{1}$$

diff z wrt x

$$P = \frac{\partial z}{\partial x} = f_1(x+y) + x f_1'(x+y) + f_2'(x+y) \quad \textcircled{2}$$

diff z wrt y

$$Q = \frac{\partial z}{\partial y} = x f_1'(x+y) + f_2'(x+y) \quad \textcircled{3}$$

diff z wrt x^2

$$R = \frac{\partial P}{\partial x} = f_1'(x+y) + x f_1''(x+y) + f_1''(x+y) + f_2''(x+y)$$

$$R = 2f_1'(x+y) + x f_1''(x+y) + f_2''(x+y) \quad \textcircled{4}$$

diff z wrt y^2

$$T = \frac{\partial Q}{\partial y} = x f_1''(x+y) + f_2''(x+y) \quad \textcircled{5}$$

diff z wrt x and y

$$S = \frac{\partial P}{\partial y} = f_1'(x+y) + x f_1''(x+y) + f_2''(x+y) \quad \textcircled{6}$$

Substituting eq $\textcircled{5}$ in eq $\textcircled{4}$

$$R = 2f_1'(x+y) + T$$

Substituting eq $\textcircled{6}$ in eq $\textcircled{6}$

$$S = f_1'(x+y) + T$$

$$R = 2f_1'(x+y) + T$$

$$\underline{2S = 2f_1'(x+y) + 2T}$$

$$\underline{\underline{R - 2S = -T}}$$

$$\frac{\partial^2 z}{\partial x^2} - 2 \frac{\partial^2 z}{\partial x \partial y} + \frac{\partial^2 z}{\partial y^2} = 0 \quad /$$

Q6: $z = f(y+x) + g(y+2x) \quad \text{--- } ①$

sol: diff z wrt x

$$P = \frac{\partial z}{\partial x} = f'(y+x) + 2g'(y+2x) \quad \text{--- } ②$$

diff z wrt y

$$q = \frac{\partial z}{\partial y} = f'(y+x) + g'(y+2x) \quad \text{--- } ③$$

diff z wrt x^2

$$r = \frac{\partial P}{\partial x} = f''(y+x) + 4g''(y+2x) \quad \text{--- } ④$$

diff z wrt y^2

$$s = \frac{\partial q}{\partial y} = f''(y+x) + g''(y+2x) \quad \text{--- } ⑤$$

diff z wrt xy

$$\Delta = \frac{\partial P}{\partial y} = f''(y+x) + 2g''(y+2x) \quad \text{--- } ⑥$$

Subtracting eq ④ and eq ⑥

$$r-t = 2g''(y+2x)$$

Subtracting eq ⑥ and eq ⑤

$$s-t = g''(y+2x)$$

$$\Rightarrow r-t = 2(s-t)$$

$$r-t = 2s-2t$$

$$r=2s-t$$

$$\underline{r-2s+t=0}$$

$$\underline{\underline{\frac{\partial^2 z}{\partial x^2} - 2 \frac{\partial^2 z}{\partial x \partial y} + \frac{\partial^2 z}{\partial y^2} = 0}}$$

$$z = f(y-2x) + g(2y-x)$$

diff z wrt x

$$P = \frac{\partial z}{\partial x} = -2f'(y-2x) - g'(2y-x) \quad \textcircled{1}$$

diff z wrt y

$$Q = \frac{\partial z}{\partial y} = f'(y-2x) + 2g'(2y-x) \quad \textcircled{2}$$

diff z wrt x^2

$$R = \frac{\partial P}{\partial x} = 4f''(y-2x) + g''(2y-x) \quad \textcircled{3}$$

diff z wrt y^2

$$S = \frac{\partial Q}{\partial y} = f''(y-2x) + 4g''(2y-x) \quad \textcircled{4}$$

diff z wrt x and y

$$T = \frac{\partial P}{\partial y} = -2f''(y-2x) - 2g''(2y-x) \quad \textcircled{5}$$

$$R + 2S = -3g''(2y-x)$$

$$\text{similarly } 2T + S = -6g''(2y-x)$$

$$\therefore \frac{R+2S}{2T+S} = \frac{-1}{2}$$

$$2R+4S = -2T-S$$

$$2R+5S+2T = 0$$

$$2 \frac{\partial^2 z}{\partial x^2} + 5 \frac{\partial^2 z}{\partial x \partial y} + 2 \frac{\partial^2 z}{\partial y^2} = 0$$

* Solution of Non-Homogeneous PDE by direct Integration Method

In this method we find the dependent variable which be the solution, by removing the differential operator through the process of integration.

Ex: If u is function of x, y, z , then the solution

$$u(x, y, z)$$

$$\frac{\partial u}{\partial x} = x^2 \rightarrow u = \frac{x^3}{3} + f(y, z)$$

Q1: Solve

$$\frac{\partial^2 u}{\partial x^2} = x + y$$

Sol: Given $\frac{\partial^2 u}{\partial x^2} + x + y$

$$\frac{\partial}{\partial x} \left(\frac{\partial u}{\partial x} \right) = x + y$$

Integrating wrt x by keeping y as constant.

$$\frac{\partial u}{\partial x} = \frac{x^2}{2} + xy + f(y)$$

Integrating wrt x by keeping y as constant

$$u = \frac{x^3}{6} + \frac{x^2y}{2} + xf(y) + g(y)$$

Q2 solve: $\frac{\partial^2 z}{\partial x \partial y} = \frac{x}{4} + a$

Sol: Given $\frac{\partial}{\partial x} \left(\frac{\partial z}{\partial y} \right) = \frac{x}{4} + a$

Integrating wrt x by keeping y as constant

$$\int \frac{\partial}{\partial x} \left(\frac{\partial z}{\partial y} \right) dx = \int \frac{x}{4} dx + f(a)$$

$$\frac{\partial z}{\partial y} = \frac{x^2}{2y} + ax + f(y)$$

Integrating w.r.t y by keeping x as constant.

$$\int \frac{\partial z}{\partial y} dy = \int \frac{x^2}{2y} dy + \int ax dy + \int f(y) dy$$

$$z = \frac{x^2}{2} \log y + axy + y F(y) + g(x) \quad \text{where } F(y) = \int f(y) dy$$

Solve: $\frac{\partial^2 u}{\partial x \partial t} = e^{-t} \cos x$ given that $u=0$ when $t=0$

and $\frac{\partial u}{\partial t} = 0$ at $x=0$. Also show that $u \rightarrow \sin x$ as $t \rightarrow \infty$.

Given: $\frac{\partial}{\partial x} \left(\frac{\partial u}{\partial t} \right) = e^{-t} \cos x$

Integrating w.r.t x by keeping t as constant

$$\int \frac{\partial}{\partial x} \left(\frac{\partial u}{\partial t} \right) dx = \int e^{-t} \cos x dx$$

$$\frac{\partial u}{\partial t} = e^{-t} \sin x + f(t) \quad \text{--- (1)}$$

Integrating w.r.t t keeping x as constant

$$\int \frac{\partial u}{\partial t} dt = \int e^{-t} \sin x dt + \int f(t) dt$$

where $\int f(t) dt = F(t)$

$$u = -e^{-t} \sin x + tF(t) + g(x) \quad \text{--- (2)}$$

given that $\frac{\partial u}{\partial t} = 0$ at $x=0$

From eq (1) $0 = e^{-t} \sin 0 + f(t)$
 $\Rightarrow f(t) = 0$

given that $u=0$ at $t=0$

From eq (2) $0 = -e^0 \sin 0 + 0(F(t)) + g(x)$
 $g(x) = \sin x$

Substituting $f(t)$ in eq ①

$$\frac{du}{dt} = e^{-t} \sin x$$

Substituting $g(x)$ in eq ②

$$u = -e^t \sin x + \sin x$$

$$u = \sin x (1 - e^{-t})$$

when

$t \rightarrow \infty$ it implies $e^{-t} = 0$

$$\therefore u = \sin x \text{ hence } u \rightarrow \underline{\sin x}$$

Q4: solve: $\frac{\partial^2 z}{\partial x \partial y} = \frac{x}{y}$ subjected to the conditions

$$\text{i. } \frac{\partial z}{\partial x} = \log y \text{ when } y=1 \text{ and } z=0$$

$$\text{ii. } z=0$$

$$\text{iii. } \frac{\partial z}{\partial x} = \log x \text{ when } x=1.$$

Sol: Given: $\frac{\partial}{\partial y} \left(\frac{\partial z}{\partial x} \right) = \frac{x}{y}$

Integrating w.r.t y and keeping x as constant

$$\int \frac{\partial}{\partial y} \left(\frac{\partial z}{\partial x} \right) dy = \int \frac{x}{y} dy$$

$$\frac{\partial z}{\partial x} = x \log y + f(x) \quad \text{--- ①}$$

Integrating w.r.t x and keeping y as constant

$$\int \frac{\partial z}{\partial x} dx = \int x \log y dx + \int f(x) dx \quad \text{where } \int f(x) dx = F(x)$$

$$z = \frac{x^2}{2} \log y + F(x) + g(y) \quad \text{--- ②}$$

i. Given that $\frac{\partial z}{\partial x} = \log x$ when $y=1$

From eq ① $\log x = x \log 1 + f(x)$

$$f(x) = \log x$$

$$F(x) = \int f(x) dx \quad \int \log x dx = x \log x - x$$

$\therefore x=0$ when $x=1$

$$\text{From eq. } ② \quad 0 = \frac{1}{2} \log y + x \log x - 1 + g(y)$$

$$g(y) = 1 - \frac{1}{2} \log y$$

Substituting in eq. ②

$$x = \frac{x^2}{2} \log y + x \log x - x + 1 - \frac{1}{2} \log y \quad /$$

$$\text{Solve: } \frac{\partial^2 z}{\partial x \partial y} = \sin x \sin y \text{ for which } \frac{\partial z}{\partial y} = -2 \sin y \text{ when}$$

$$x=0 \text{ and } z=0 \text{ if } y=(2n+1) \frac{\pi}{2}.$$

$$\text{Given: } \frac{\partial}{\partial x} \left(\frac{\partial z}{\partial y} \right) = \sin x \sin y$$

Integrating wrt x by keeping y as constant

$$\int \frac{\partial}{\partial x} \left(\frac{\partial z}{\partial y} \right) dx = \int \sin x \sin y dx$$

$$\frac{\partial z}{\partial y} = -\sin y \cos x + f(y) \quad \text{--- } ①$$

Integrating wrt xy by keeping x as constant

$$\int \frac{\partial z}{\partial y} dy = \int \sin y \cos x dy + \int f(y) dy \quad \text{where} \\ \int f(y) dy = F(y)$$

$$z = \cos x \cos y + F(y) + g(x) \quad \text{--- } ②$$

~~Given that~~

$$\frac{\partial z}{\partial y} = -2 \sin y \quad \text{when } x=0$$

$$\text{From eq. } ① \quad -2 \sin y = -\sin y \cos 0 + f(y)$$

$$f(y) = -\sin y \Rightarrow F(y) = -\cos y$$

Given that $x=0$ if $y=(2n+1)\frac{\pi}{2}$
From eq. ②

$$0 = \cos x \cos(2n+1)\frac{\pi}{2} + \cos(2n+1)\frac{\pi}{2} g(x)$$

$$g(x) = -\cos x$$

Substituting in eq. ②

$$x = \cancel{\cos x \cos y} + \cos y$$

Q6: $\frac{\partial^2 z}{\partial x^2} = xy$

i. $\frac{\partial z}{\partial x} = \log(1+y)$ when $x=1$

ii. $z=0$ when $x=0$

Sol: Given: $\frac{\partial^2 z}{\partial x^2} = xy \Rightarrow \frac{\partial}{\partial x} \left(\frac{\partial z}{\partial x} \right) = xy$

Integrating w.r.t x keeping y as constant.

$$\int \frac{\partial}{\partial x} \left(\frac{\partial z}{\partial x} \right) dx = \int xy dx$$

$$\frac{\partial z}{\partial x} = \frac{x^2 y}{2} + f(y) \quad \text{--- ①}$$

Integrating w.r.t x keeping y as constant.

$$\int \frac{\partial z}{\partial x} dx = \int \left(\frac{x^2 y}{2} + f(y) \right) dx$$

$$z = \frac{x^3 y}{6} + x f(y) \quad \text{--- ②}$$

Given that $\frac{\partial z}{\partial x} = \log(1+y)$ when $x=0$

From eq. ① $\frac{\partial z}{\partial x} = \log(1+y) = \frac{y}{2} + f(y) \Rightarrow f(y) = \log(1+y) - \frac{y}{2}$

given that $z = 0$ when $x = 0$

From eq @ $0 = 0$

Substituting in eq ⑤

$$z = \frac{x^3 y}{6} + x \left(\log(1+y) - \frac{y}{2} \right)$$

$$z = \underline{\underline{\frac{x^3 y}{6} + x \log(1+y) - \frac{xy}{2}}}$$

solution of Homogeneous PDE: involving derivatives with respect to one independent variable:

suppose that the dependent variable has been differentiated partially with respect to the independent variable, then the PDE can be treated as ordinary differential equations. The arbitrary constant in the solution are replaced by arbitrary function of the other variable. This gives a solution for the PDE.

Given $\frac{\partial^2 z}{\partial x^2} + z = 0$, Given that $z = 0$, $z = e^y$ and $\frac{\partial z}{\partial x} = 1$.

Given $\frac{\partial^2 z}{\partial x^2} + z = 0$

$$D^2 z + z = 0$$

$$(D^2 + 1)z = 0$$

The auxiliary equation is given by

$$m^2 + 1 = 0$$

$$m^2 = -1$$

$m = \pm i$ are the roots.

The solution is given by

$$z = C_1 \cos x + C_2 \sin x$$

$$z = f(y) \cos x + g(y) \sin x \quad \text{--- } ①$$

diff eq ① w.r.t x

$$\frac{\partial z}{\partial x} = -f(y) \sin x + g(y) \cos x \quad ②$$

Given conditions. $z = e^y$ at $x=0$ in eq ①

$$e^y = f(y) \cos 0 + g(y) \sin 0$$

$$e^y = f(y)$$

Substituting $f(y)$ in eq ②

$$\frac{\partial z}{\partial x} = -e^y \sin x + g(y) \cos x$$

Given $\frac{\partial z}{\partial x} = 1$ at $x=0$ in eq ②

$$\Rightarrow 1 = +e^y \sin 0 + g(y) \cos 0$$

$$\therefore g(y) = 1$$

Substituting $f(y)$ and $g(y)$ in eq ①

$$z = e^y \cos x + \sin x$$

Q2: $\frac{\partial^2 z}{\partial y^2} = z$ given that $y=0$, $z=e^x$ and $\frac{\partial z}{\partial y} = 0$.

Sol: Given $\frac{\partial^2 z}{\partial y^2} - z = 0$

$$D^2 z - z = 0$$

$$(D^2 - 1) z = 0$$

The auxiliary equation is

$$m^2 - 1 = 0$$

$$m^2 = 1$$

$m = \pm 1$ are the roots.

The solution is given by.

$$z = C_1 e^y + C_2 e^{-y}$$

$$z = f(x)e^y + g(x)e^{-y}$$

 ①

Q3:



diff eq ① wrt y

$$\frac{\partial z}{\partial y} = f(x)e^y - g(x)e^{-y} \quad \text{--- ②}$$

Given condition $y=0$ and $z=e^x$

$$\text{From eq ① } e^x = f(x)e^0 + g(x)e^{-0}$$

$$e^x = f(x) + g(x) \quad \text{--- ③}$$

Given condition $\frac{\partial z}{\partial y} = 0$ at $y=0$

From eq ②

$$e^{-x} = f(x)e^0 - g(x)e^0$$

$$e^{-x} = f(x) - g(x) \quad \text{--- ④}$$

Adding eq ③ and eq ④

$$2f(x) = e^x + e^{-x}$$

$$f(x) = \frac{e^x + e^{-x}}{2} = \cosh x$$

Substituting in eq ④

$$g(x) = \frac{e^x - e^{-x}}{2} = \sinh x$$

Substituting $f(x)$ and $g(x)$ in eq ①

$$z = \frac{e^x e^y}{2} + \frac{e^x e^{-y}}{2} \quad z = \cosh x e^y + \sinh x e^{-y}$$

$$z = \frac{e^{x+y}}{2} + \frac{e^{x-y}}{2}$$

Q3: $\frac{\partial^2 u}{\partial x^2} + u = 0$ given conditions

i. $u(0,y) = \sqrt{e}$

ii. $\frac{\partial u}{\partial x}(0,y) = 1$

Sol: Given: $\frac{\partial^2 u}{\partial x^2} + u = 0$

$$\frac{\partial^2 u}{\partial x^2} + u = 0$$

$$(\frac{\partial^2}{\partial x^2} + 1)u = 0$$

The auxiliary equation is

$$m^2 + 1 = 0$$

$$m^2 = -1$$

$m = \pm i$ are the roots

∴ the solution is given by

$$u = c_1 \cos x + c_2 \sin x$$

$$u = f(y) \cos x + g(y) \sin x$$

— ①

diff eq ① w.r.t x

$$\frac{\partial u}{\partial x} = -f(y) \sin x + g(y) \cos x$$

— ②

Given condition $u = \sqrt{e}$ at $x = 0$

From eq ①

$$\sqrt{e} = f(y) \cos 0 + g(y) \sin 0$$

$$f(y) = \sqrt{e}$$

Given condition $\frac{\partial u}{\partial x} = 1$ at $x = 0$

From eq ②

$$1 = -f(y) \sin 0 + g(y) \cos 0$$

$$g(y) = 1$$

Substituting in eq ①

$$u = \sqrt{e} \cos x + \sin x$$

—————

Q4: Solve: $\frac{\partial^3 x}{\partial x^3} + \frac{\partial x}{\partial x} = 0$, given that $x=0, \frac{\partial x}{\partial x}=0, \frac{\partial^2 x}{\partial x^2}$

when $x=0$.

Sol! Given: $\frac{\partial^3 z}{\partial x^3} + 4 \frac{\partial z}{\partial x} = 0$

$$D^3 z + 4Dz = 0$$

$$D(D^2 + 4)z = 0$$

The auxiliary equation is

$$m(m^2 + 4) = 0$$

$$\underline{m=0} \quad m^2 + 4 = 0$$

$$m^2 = -4$$

$$m = \pm 2i$$

: The solution is

$$z = C_1 e^0 + C_2 \cos 2x + C_3 \sin 2x$$

$$z = f(y) + g(y) \cos 2x + h(y) \sin 2x \quad \text{--- } ①$$

diff eq ① wrt x

$$\frac{\partial z}{\partial x} = -2g(y) \sin 2x + 2h(y) \cos 2x \quad \text{--- } ②$$

~~given conditions are zero at x=0~~

~~From eq~~

diff wrt x (eq ②)

$$\frac{\partial^2 z}{\partial x^2} = -4g(y) \cos 2x - 4h(y) \sin 2x \quad \text{--- } ③$$

diff eq ③ wrt x

$$\frac{\partial^3 z}{\partial x^3} = 8g(y) \sin 2x - 8h(y) \cos 2x \quad \text{--- } ④$$

Given condition $z=0$ at $x=0$

From eq ①

$$0 = f(y) + g(y) \cos 0 + h(y) \sin 0$$

$$0 = f(y) + g(y) \quad \text{--- } ④$$

Given condition $\frac{\partial z}{\partial x} = 0$ at $x=0$

From eq ②

$$0 = -2g(y) \sin 0 + 2h(y) \cos 0$$

$$\Rightarrow h(y) = 0 \quad -\textcircled{5}$$

$$\text{Given condition } \frac{\partial^2 x}{\partial n^2} = 4 \text{ at } n=0$$

Because as (3)

From eq. ③

$$A = -4g(y)\cos\theta - 4h(y)\sin\theta$$

$$\Rightarrow g(y) = -1 \quad \textcircled{G}$$

Substituting eq. ⑥ in eq. ④

$$f(y) \stackrel{?}{=} 1$$

substituting $f(y)$, $g(y)$ and $h(y)$ in eq(1)

$$x = 1 - \cos 2n$$

Q5: solve: $\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial u}{\partial x}$ using the substitution $\frac{\partial u}{\partial x} = v$.

Sol: Given: $\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial u}{\partial x}$

$$\frac{\partial u}{\partial y} \left(\frac{\partial u}{\partial x} \right) = \frac{\partial u}{\partial x}$$

Given $\frac{\partial u}{\partial n} = \gamma$

$$\therefore \frac{\partial \mathbf{r}^u}{\partial \mathbf{q}} = \mathbf{v}$$

$$\Rightarrow \frac{\partial v}{\partial y} - v = 0$$

$$Dy - v = 0$$

$$(D-1)v = 0$$

The auxiliary equation is

13

m = 1 is the root

\therefore The solution is given by

$$v = ce^y$$

$$v = f(x) e^y$$

$$\text{but } v = \frac{\partial u}{\partial x}$$

$$\therefore \frac{\partial u}{\partial x} = f(x) e^y$$

Integrating wrt x

$$\int \frac{\partial u}{\partial x} dx = \int f(x) e^y dx$$

$$u = c^y F(x) + g(y) \quad \text{where } \int f(x) dx = F(x)$$

Q6: Solve: $\frac{\partial^3 u}{\partial x^2 \partial y} - \frac{\partial u}{\partial y} = 0$ using the substitution $\frac{\partial u}{\partial y} = v$.

Sol: Given: $\frac{\partial^2}{\partial x^2} \left(\frac{\partial u}{\partial y} \right) - \frac{\partial u}{\partial y} = 0$

$$\Rightarrow \frac{\partial^2 v}{\partial x^2} - v = 0$$

$$D^2 v - v = 0$$

$$(D^2 - 1)v = 0$$

The auxiliary equation is

$$m^2 - 1 = 0$$

$$m^2 = 1$$

$m = \pm 1$ are the roots

∴ The solution is given by

$$v = C_1 e^x + C_2 e^{-x}$$

$$v = f(y)e^x + g(y)e^{-x}$$

$$\text{but } v = \frac{\partial u}{\partial y}$$

$$\therefore \frac{\partial u}{\partial y} = f(y)e^x + g(y)e^{-x}$$

Integrating wrt y

$$\int \frac{\partial u}{\partial y} dy = \int f(y)e^x dy + \int g(y)e^{-x} dy$$

$$\underline{u = F(y) e^{ax} + G(y) e^{-ax} + h(y)} \text{ where } F(y) = \int f(y) dy \\ G(y) = \int g(y) dy$$

Q1: Solve: $\frac{\partial^2 z}{\partial x^2} = a^2 z$; given that when $x=0$, $\frac{\partial z}{\partial x} = a \sin y$

and $\frac{\partial z}{\partial y} = 0$, $\frac{\partial^2 z}{\partial x^2} \neq \frac{\partial^2 z}{\partial y^2}$

Sol: Given: $\frac{\partial^2 z}{\partial x^2} - a^2 z = 0$

$$D^2 z - a^2 z = 0$$

$$(D^2 - a^2) z = 0$$

The auxiliary equation is

$$m^2 - a^2 = 0$$

$$m^2 = a^2$$

$m = \pm a$ are the roots

∴ The solution is given by

$$z = C_1 e^{ax} + C_2 e^{-ax}$$

$$z = f(y) e^{ay} + g(y) e^{-ay} \quad \text{--- (1)}$$

diff eq (1) wrt x

$$\frac{\partial z}{\partial x} = af(y)e^{ay} - ag(y)e^{-ay} \quad \text{--- (2)}$$

diff eq (1) wrt y

$$\frac{\partial z}{\partial y} = e^{ay} f'(y) + e^{-ay} g'(y) \quad \text{--- (3)}$$

$$\text{at } x=0 \quad \frac{\partial z}{\partial x} = a \sin y \quad (\text{given})$$

From eq (2)

$$a \sin y = a f(y) e^0 - a g(y) e^0$$

$$f(y) - g(y) = \sin y \quad \text{--- (4)}$$

$$\text{at } x=0 \quad \frac{\partial z}{\partial y} = 0 \quad (\text{given})$$

From eq (3)

$$0 = e^0 f'(y) + e^0 g'(y)$$

$$f'(y) + g'(y) = 0$$

Integrating w.r.t y

$$f(y) + g(y) = k \quad \text{--- (5)}$$

Adding eq (4) and eq (5)

$$2f(y) = k + \sin y$$

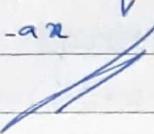
$$f(y) = \frac{k + \sin y}{2}$$

Substituting in eq (3)

$$g(y) = k - \frac{k + \sin y}{2} = \frac{k - \sin y}{2}$$

Substituting $f(y)$ and $g(y)$ in eq (1)

$$z = \frac{k + \sin y}{2} e^{ax} + \frac{k - \sin y}{2} e^{-ax}$$



* Method of Separation of Variables:

This method is applicable for solving a linear homogeneous PDE involving derivatives with respect to independent variables. Solution of the PDE is determined by ordinary differential equation.

Q1: Solve: $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 2(x+y)u$ by the method of separation of variables.

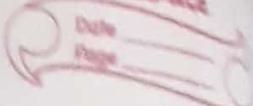
Given $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 2(x+y)u$

Let us assume $u = xy \quad \text{--- (1)}$

$$\frac{\partial(xy)}{\partial x} + \frac{\partial(xy)}{\partial y} = 2(x+y)xy$$

$$y \frac{\partial x}{\partial x} + x \frac{\partial y}{\partial y} = 2(x+y)xy$$

Dividing both sides by xy



$$\frac{1}{x} \frac{dx}{dx} + \frac{1}{y} \frac{dy}{dy} = 2(x+y)$$

$$\frac{1}{x} \frac{dx}{dx} - 2x = 2y - \frac{1}{y} \frac{dy}{dy}$$

Equating both sides to constant k

$$\frac{1}{x} \frac{dx}{dx} - 2x = k$$

$$2y - \frac{1}{y} \frac{dy}{dy} = k$$

$$\frac{1}{x} \frac{dx}{dx} = k + 2x$$

$$\frac{1}{y} \frac{dy}{dy} = 2y - k$$

$$\frac{1}{x} \cancel{\frac{dx}{dx}} = (k+2x) dx$$

$$\frac{1}{y} dy = (2y - k) dy$$

Integrating

$$\int \frac{1}{x} dx = \int (k+2x) dx$$

Integrating

$$\int \frac{1}{y} dy = \int (2y - k) dy$$

$$\log x = x^2 + kx + c_1 \\ \Rightarrow x = e^{(x^2+kx+c_1)}$$

$$\log y = y^2 - ky + c_2 \\ \Rightarrow y = e^{(y^2-ky+c_2)}$$

Substituting x and y in eq ①

$$u = e^{(x^2+kx+c_1)} e^{(y^2-ky+c_2)}$$

$$u = e^{(c_1+c_2)} e^{(x^2+y^2-kx-ky)}$$

$$\therefore u = c e^{(x^2+y^2-kx-ky)} \text{ where } e^{(c_1+c_2)} = c$$

Q2: Solve by the method of separation of variables

$$\frac{du}{dx} = 2 \frac{du}{dt} + u \text{ where } u(x, 0) = 6e^{-3x}$$

sol:

$$\underline{\underline{\frac{du}{dx} = 2 \frac{du}{dt} + u}}$$

Let us assume $u = xt$ ————— ①

$$\therefore \frac{\partial (xt)}{\partial x} = 2 \frac{\partial (xt)}{\partial t} + xt$$

$$T \frac{dx}{dx} = 2x \frac{dt}{dt} + xt$$

Dividing both sides by x^2

$$\frac{1}{x} \frac{dx}{dt} = \frac{2}{T} \frac{dT}{dt} + 1$$

equating both sides to constant k

$$\frac{1}{x} \frac{dx}{dt} = k$$

$$\frac{1}{x} dx = k dt$$

Integrating

$$\int \frac{1}{x} dx = \int k dt$$

$$\log x = kx + C_1$$

$$\Rightarrow x = e^{kx + C_1}$$

$$\frac{2}{T} \frac{dT}{dt} + 1 = k$$

$$\frac{1}{T} \frac{dT}{dt} = \frac{k-1}{2}$$

$$\frac{1}{T} dt = \frac{k-1}{2} dt$$

Integrating

$$\int \frac{1}{T} dt = \int \left(\frac{k-1}{2} \right) dt$$

$$\log T = \left(\frac{k-1}{2} \right) t + C_2$$

$$\Rightarrow T = e^{\left(\frac{k-1}{2} \right) t + C_2}$$

Substituting x and T in eq ①

$$u = e^{kx + C_1} \cdot e^{\left(\frac{k-1}{2} \right) t + C_2}$$

$$= e^{kx + \left(\frac{k-1}{2} \right) t} \cdot e^{C_1 + C_2}$$

$$u = C e^{kx + \left(\frac{k-1}{2} \right) t} \quad \text{where } C = e^{C_1 + C_2}$$

given $u(x=0) = 6e^{-3x}$

i.e., $u = 6e^{-3x}$ at $t_f = 0$

$$\therefore 6e^{-3x} = Ce^{kx}$$

$$\Rightarrow C = 6 \text{ and } k = -3$$

$$\therefore u = 6 e^{-3x - 2t}$$

Q3: solve $py^3 + qx^2 = 0$ by method of separation of variables.

sol: given: $y^3 \frac{dx}{dx} + x^2 \frac{dx}{dy} = 0$

Assume $x = yz$ — ①

$$y^3 \frac{d(xz)}{dx} + x^2 \frac{d(xz)}{dy} = 0$$

$$y^3 z \frac{dx}{dx} + x^2 z + x^2 \times \frac{dy}{dy} = 0$$

dividing both sides by $x^2 y^3 z$

$$\frac{1}{x^2 z} \frac{dx}{dx} = - \frac{1}{y^3 z} \frac{dy}{dy}$$

equating both sides to constant k.

$$\frac{1}{x^2 z} \frac{dx}{dx} = k$$

$$-\frac{1}{y^3 z} \frac{dy}{dy} = k$$

$$\frac{1}{x} \frac{dx}{dx} = x^2 k$$

$$\frac{1}{y} dy = -y^3 k dy$$

Integrating

$$\int \frac{1}{x} \frac{dx}{dx} = \int x^2 k dx$$

$$\log x = \frac{kx^3}{3} + C_1$$

$$x = e^{\frac{kx^3}{3} + C_1}$$

Integrating

$$\int \frac{1}{y} dy = - \int y^3 k dy$$

$$\log y = -\frac{ky^4}{4} + C_2$$

$$y = e^{-\frac{ky^4}{4} + C_2}$$

Substituting x and y in eq ①

$$z = e^{kx^3/3 + C_1} e^{-ky^4/4 + C_2}$$

$$= e^{k(x^3/3 - y^4/4)} e^{C_1 + C_2}$$

$$z = e e^{k(x^3/3 - y^4/4)}$$



$\frac{\partial^2 z}{\partial x^2} - 2 \frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = 0$ by method of variable separation.

$$\text{given: } \frac{\partial^2 z}{\partial x^2} - 2 \frac{\partial z}{\partial x} + \frac{\partial z}{\partial y} = 0$$

Assume $z = xy \quad \dots \textcircled{1}$

$$\frac{\partial^2 (xy)}{\partial x^2} - 2 \frac{\partial (xy)}{\partial x} + \frac{\partial (xy)}{\partial y} = 0$$

$$y \frac{d^2 x}{dx^2} - 2y \frac{dx}{dx} + x \frac{dy}{dy} = 0$$

dividing both sides by xy

$$\frac{1}{x} \frac{d^2 x}{dx^2} - \frac{2}{x} \frac{dx}{dx} + \frac{1}{y} \frac{dy}{dy} = 0$$

$$\frac{1}{x} \frac{d^2 x}{dx^2} - \frac{2}{x} \frac{dx}{dx} = -\frac{1}{y} \frac{dy}{dy} .$$

Equating both sides to constant k

$$\frac{1}{x} \frac{d^2 x}{dx^2} - \frac{2}{x} \frac{dx}{dx} = k \quad -\frac{1}{y} \frac{dy}{dy} = k$$

$$\frac{d^2 x}{dx^2} - 2 \frac{dx}{dx} - kx = 0 \quad \frac{-1}{y} dy = -k dy$$

$$D^2 x - 2Dx - kx = 0$$

$$x(D^2 - 2D - k) = 0 \quad \because \frac{d^2}{dx^2} = D^2$$

Auxiliary equation

$$m^2 - 2m - k = 0$$

$$m = \frac{2 \pm \sqrt{4 + 4k}}{2}$$

Integrating

$$\int \frac{1}{y} dy = - \int k dy$$

$$\log y = -ky + C_1$$

$$y = e^{-ky + C_1}$$

$$m = 1 \pm \sqrt{1+k}$$

\therefore the solution is given by

$$x = C_2 e^{(1+\sqrt{1+k})x} + C_3 e^{(1-\sqrt{1+k})x}$$

Substituting x and y in eq $\textcircled{1}$

$$z = (C_2 e^{(1+\sqrt{1+k})x} + C_3 e^{(1-\sqrt{1+k})x})(e^{-ky + C_1})$$

$$z = \underbrace{\left[A e^{\frac{(1+\sqrt{1+k})x}{2}} + B e^{\frac{(1-\sqrt{1+k})x}{2}} \right]}_{\text{---}} e^{-ky} \quad \text{where } A = C_2 e^y \\ B = C_3 e^y$$

Q5: solve by the method of separation of variables

$$\frac{\partial^2 z}{\partial x^2} = \frac{\partial z}{\partial y} + 2z \quad \text{subjected to the conditions}$$

$$\text{given: } z(0,y) = 0, z_x(0,y) = e^{2y}$$

Sol: given $\frac{\partial^2 z}{\partial x^2} = \frac{\partial z}{\partial y} + 2z$

Assume $z = xy \quad \dots \textcircled{1}$

$$\frac{\partial^2 (xy)}{\partial x^2} = \frac{\partial (xy)}{\partial y} + 2xy$$

$$y \frac{d^2 x}{dx^2} = x \frac{dy}{dy} + 2xy$$

dividing both sides by xy

$$\frac{1}{x} \frac{d^2 x}{dx^2} = \frac{1}{y} \frac{dy}{dy} + 2$$

Equating both sides to constant k

$$\frac{1}{x} \frac{d^2 x}{dx^2} = k$$

$$\frac{1}{y} \frac{dy}{dy} + 2 = k$$

$$\cancel{\frac{1}{x} D^2 x} - kx = 0$$

$$\frac{1}{y} \frac{dy}{dy} = k - 2$$

$$(D^2 - k)x = 0$$

$$\frac{1}{y} dy = (k-2)dy$$

Auxiliary equation

$$m^2 - k = 0$$

Integrating

$$m^2 = k \Rightarrow m = \pm \sqrt{k}$$

$$\int \frac{1}{y} dy = \int (k-2) dy$$

\therefore the solution is

$$x = C_1 e^{-\sqrt{k}x} + C_2 e^{\sqrt{k}x} \text{ ~~and zero~~}$$

$$\log y = (k-2)y + C_3$$

$$y = e^{(k-2)y + C_3}$$

Substituting x and y in eq ①

$$z = (e^{\sqrt{k}x} + c_2 e^{\sqrt{k}x}) (e^{(k-2)y} + c_3)$$

$$z = [A e^{-\sqrt{k}x} + B e^{\sqrt{k}x}] e^{(k-2)y} \quad \text{--- ② where } A = c_1 e^{c_3}$$

$$B = c_2 e^{c_3}$$

diff wrt x

$$z_x = [-A \sqrt{k} e^{-\sqrt{k}x} + \sqrt{k} B e^{\sqrt{k}x}] e^{(k-2)y} \quad \text{--- ③}$$

given that $x=0$ when $y=0$

From eq ②

$$0 = (A e^0 + B e^0) e^{(k-2)y}$$

$$0 = e^{(k-2)y} (A + B)$$

\Rightarrow since $e^{(k-2)y} \neq 0$

$$\therefore A + B = 0 \Rightarrow B = -A$$

Substituting in eq ②

$$z = [A e^{-\sqrt{k}x} - A e^{\sqrt{k}x}] e^{(k-2)y}$$

$$z = A [e^{-\sqrt{k}x} - e^{\sqrt{k}x}] e^{(k-2)y} \quad \text{--- ④}$$

given that $z_x = e^{2y}$ at $x=0$

From eq ③

$$e^{2y} = [-A \sqrt{k} e^0 + A \sqrt{k} e^0] e^{(k-2)y}$$

$$e^{2y} = -2A \sqrt{k} e^{(k-2)y}$$

$$\Rightarrow 2 = k-2 \Rightarrow -2A \sqrt{k} = 1$$

$$\therefore k = 4$$

$$A = \frac{-1}{2\sqrt{k}} = \frac{-1}{2\sqrt{4}} = \frac{1}{4} /$$

Substituting k and A in eq ④

$$z = \frac{1}{4} [e^{-2x} - e^{2x}] e^{+2y}$$

Q6: Solve by method of separation of variables.

$$4 \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 3u \quad \text{given that } u(0, y) = 2e^{5y}$$

Sol: given $4 \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} = 3u$

Assume $u = xy \quad \dots \textcircled{1}$

$$\therefore 4 \frac{\partial(xy)}{\partial x} + \frac{\partial(xy)}{\partial y} = 3xy$$

$$4y \frac{dx}{dx} + x \frac{dy}{dy} = 3xy$$

dividing both sides by xy .

$$\frac{4}{x} \frac{dx}{dx} + \frac{1}{y} \frac{dy}{dy} = 3$$

$$\frac{4}{x} \frac{dx}{dx} = 3 - \frac{1}{y} \frac{dy}{dy}$$

Equating both sides to constant k

$$\frac{4}{x} \frac{dx}{dx} = k$$

$$3 - \frac{1}{y} \frac{dy}{dy} = k$$

$$\frac{4}{x} dx = k dx$$

$$\frac{1}{4} \frac{dy}{dy} = 3 - k$$

Integrating

$$4 \log x = kx + C_1$$

$$\log x = \underline{kx + C_1}$$

$$\cancel{x} = e^{\frac{kx}{4} + C_2}$$

Substituting x and y in eq \textcircled{1}

$$u = \left(e^{\frac{kx}{4} + C_2}\right) \left(e^{(3-k)y + C_3}\right)$$

$$u = e^{C_2 + C_3} e^{\left(\frac{kx}{4} + (3-k)y\right)}$$

$$u = c e^{\frac{kx}{4} + (3-k)y}$$

where $c = e^{C_2 + C_3}$

Integrating

$$\int \frac{1}{4} \frac{dy}{dy} = \int (3-k) dy$$

$$\log y = (3-k)y + C_3$$

$$y = e^{(3-k)y + C_3}$$

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given $u = 2e^{5y}$ at $x = 0$
 $2e^{5y} = C e^{(0+(3-k)y)}$
 $\Rightarrow C = 2 \Rightarrow (3-k)y = 5y$
 $3 - k = 5$
 $k = -2$

$$\therefore u = 2e^{-x/2 + 5y}$$

Various possible solution of one dimensional wave equation:

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

Assume $w = xt$

$$\frac{\partial^2 (xt)}{\partial t^2} = c^2 \frac{\partial^2 (xt)}{\partial x^2}$$

$$x \frac{d^2 t}{dt^2} = c^2 t \frac{d^2 x}{dx^2}$$

Dividing both sides by $xc^2 t$

$$\frac{1}{c^2 t} \frac{d^2 t}{dt^2} = \cancel{x} \frac{1}{x} \frac{d^2 x}{dx^2}$$

Equating both sides to constant k

$$\frac{1}{c^2 t} \frac{d^2 t}{dt^2} = k \quad \frac{1}{x} \frac{d^2 x}{dx^2} = k$$

~~Integrating~~

$$\frac{d^2 t}{dt^2} = k c^2 t$$

$$\frac{d^2 x}{dx^2} = k x$$

$$D^2 t - k c^2 t = 0$$

$$(D^2 - k) t = 0$$

————— ②

$$D^2 t - k c^2 t = 0$$

$$(D^2 - k c^2) t = 0$$

————— ①

CASE 1: Suppose $k = 0$ in eq ① and eq ②

$$D^2 t = 0$$

$$D^2 x = 0$$

Auxillary equation

$$m^2 = 0$$

$$\underline{\underline{m = 0, 0}}$$

Auxillary equation

$$m^2 = 0$$

$$\underline{\underline{m = 0, 0}}$$

$$\therefore T = (C_1 + C_2 t) e^{0t} \quad X = (C_3 + C_4 x) e^{0x}$$

wkt $u = XT$

$\therefore u = (C_3 + C_4 x)(C_1 + C_2 t)$ is the solution.

- CASE 2: suppose k is positive, that is $k = p^2$ in eq(1) and eq(2)

$$(D^2 - p^2 c^2)T = 0$$

$$(CD^2 - p^2)T = 0$$

Auxiliary equation

$$m^2 - p^2 c^2 = 0$$

$$m^2 = p^2 c^2$$

$$\underline{m = \pm pc}$$

Auxiliary equation

$$m^2 - p^2 = 0$$

$$m^2 = p^2$$

$$\underline{m = \pm p}$$

$$\therefore T = C_1 e^{pct} + C_2 e^{-pct}$$

wkt $u = XT$

$\therefore u = (C_3 e^{px} + C_4 e^{-px})(C_1 e^{pct} + C_2 e^{-pct})$ is the solution.

- CASE 3: suppose k is negative, that is $k = -p^2$ in eq(1) and eq(2)

$$(D^2 + p^2 c^2)T = 0$$

$$(D^2 + p^2)T = 0$$

Auxiliary equation

$$m^2 + p^2 c^2 = 0$$

$$m^2 = -p^2 c^2$$

$$\underline{m = \pm pc i}$$

Auxiliary equation

$$m^2 + p^2 = 0$$

$$m^2 = -p^2$$

$$\underline{m = \pm pi}$$

$$\therefore T = C_1 \cos pct + C_2 \sin pct$$

wkt $u = XT$

$$\therefore X = C_3 \cos px + C_4 \sin px$$

$\therefore u = (C_1 \cos pct + C_2 \sin pct)(C_3 \cos px + C_4 \sin px)$ is the solution.

LAGRANGE'S LINEAR PDE:

The equation of the form $P_p + Q_q = R$ is called Lagrange's linear equation which has a solution of the form $\phi(u, v) = 0$.

Procedure:

Given the PDE of the form $\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$ known as

the auxiliary equation. This system of equation can be solved as follows.

- We can consider suitable pairs which can be put in the forms like

$$f(x)dx = g(y)dy$$

$$g(y)dy = h(z)dz$$

$$h(z)dz = f(x)dx$$

so that by integration we can get the relations in the form (x, y) (y, z) (z, x) .

- We have a property in the ratio and proportion that a ratio $\frac{a_1}{b_1} = \frac{a_2}{b_2} = \frac{a_3}{b_3} = \frac{k_1 a_1 + k_2 a_2 + k_3 a_3}{k_1 b_1 + k_2 b_2 + k_3 b_3}$.

With reference to the auxiliary equation we try to find multipliers k_1, k_2, k_3 such that $\frac{k_1 dx + k_2 dy + k_3 dz}{k_1 P + k_2 Q + k_3 R}$

- Integrating the 2 new expressions we obtain two relations connecting x, y, z .

Suppose $u(x, y, z) = c_1$ and $v(x, y, z) = c_2$ are the two relations obtained then $\phi(u, v) = 0$ is the general solution of the PDE.

Q1. solve $xP + yQ = z$.

Sol:

The given equation is of the form $Pp + Qq = R$
 Auxiliary equation is given by

$$\frac{dx}{P} + \frac{dy}{Q} = \frac{dz}{R}$$

$$\therefore \frac{dx}{n} = \frac{dy}{y} = \frac{dz}{z}$$

Considering

$$\frac{dx}{n} = \frac{dy}{y}$$

$$\int \frac{dx}{n} = \int \frac{dy}{y}$$

$$\log n = \log y + \log C_1$$

$$\log n - \log y = \log C_1$$

$$\log \frac{n}{y} = \log C_1$$

$$\therefore u = \frac{n}{y} = C_1$$

considering

$$\frac{dy}{y} = \frac{dz}{z}$$

$$\int \frac{dy}{y} = \int \frac{dz}{z}$$

$$\log y = \log z + \log C_2$$

$$\log y - \log z = \log C_2$$

$$\log \frac{y}{z} = \log C_2$$

$$\therefore v = \frac{y}{z} = C_2$$

$$\phi(u, v) = 0$$

$\therefore \phi\left(\frac{x}{y}, \frac{y}{z}\right) = 0$ is the solution for the given PDE.

Q2: $p \cot x + q \cot y = \cot z$

Sol:

The given equation is of the form $Pp + Qq = R$

Auxiliary equation is given by

$$\frac{dx}{n} = \frac{dy}{y} = \frac{dz}{R}$$

$$\frac{dx}{\cot x} = \frac{dy}{\cot y} = \frac{dz}{\cot z}$$

considering

$$\frac{dx}{\cot x} = \frac{dy}{\cot y}$$

$$\tan x dx = \tan y dy$$

Integrating

$$\int \tan x dx = \int \tan y dy$$

$$-\log |\cos x| = -\log |\cos y| + \log c_1$$

$$\log |\cos y| - \log |\cos x| = \log c_1$$

$$\log \frac{\cos y}{\cos x} = \log c_1$$

$$\therefore u = \frac{\cos y}{\cos x} = c_1$$

$$\phi(u, v) = 0$$

$$\therefore \phi\left(\frac{\cos y}{\cos x}, \frac{\cos z}{\cos y}\right) = 0 \text{ is the solution for the given PDE.}$$

considering

$$\frac{dy}{\cot y} = \frac{dz}{\cot z}$$

$$\tan y dy = \tan z dz$$

Integrating

$$\int \tan y dy = \int \tan z dz$$

$$-\log |\cos y| = -\log |\cos z| + \log c_2$$

$$\log |\cos z| - \log |\cos y| = \log c_2$$

$$\log \frac{\cos z}{\cos y} = \log c_2$$

$$\therefore v = \frac{\cos z}{\cos y} = c_2$$

Solve:

Q3:

$$(y-z)p + (z-x)q = (x-y)$$

Sol:

The given equation is of the form $Pp + Qq = R$
 Auxiliary equation is given by

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

$$\frac{dx}{y-x} = \frac{dy}{z-x} = \frac{dz}{x-y}$$

Considering multiplier $k_1 = k_2 = k_3 = 1$ (To get denominator = 0)

$$\begin{aligned} \frac{dx}{y-x} &= \frac{dy}{z-x} = \frac{dz}{x-y} = \frac{dx + dy + dz}{(y-x) + (z-x) + (x-y)} \\ &= \frac{dx + dy + dz}{0} \end{aligned}$$

$$\int dx + \int dy + \int dz = 0$$

$$u = x + y + z = C_1$$

Considering multipliers $k_1 = x$ and $k_2 = y$ and $k_3 = z$

$$\begin{aligned} \frac{dx}{y-x} &= \frac{dy}{z-x} = \frac{dz}{x-y} = \frac{x dx + y dy + z dz}{xy - xz + yz - xy + xz - yz} \\ &= \frac{x dx + y dy + z dz}{0} \end{aligned}$$

$$\int x dx + \int y dy + \int z dz = 0$$

$$v = \frac{x^2}{2} + \frac{y^2}{2} + \frac{z^2}{2} = C_2$$

Thus $\phi(u, v) = 0$

$$\therefore \phi(x+y+z, \frac{x^2+y^2+z^2}{2}) = 0$$

$$Q4: x(y^2 - z^2)p + y(z^2 - x^2)q = z(x^2 - y^2)$$

Sol: The given equation is of the form $Pp + Qq = R$
The auxillary equation is given by

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}$$

$$\frac{dx}{x(y^2 - z^2)} = \frac{dy}{y(z^2 - x^2)} = \frac{dz}{z(x^2 - y^2)}$$

considering multipliers $k_1 = x, k_2 = y$ and $k_3 = z$

$$\begin{aligned} \frac{dx}{x(y^2 - z^2)} &= \frac{dy}{y(z^2 - x^2)} = \frac{dz}{z(x^2 - y^2)} = \frac{x dx + y dy + z dz}{x^2 y^2 - x^2 z^2 + y^2 z^2 - x^2 y^2 + x^2 z^2 - y^2 z^2} \\ &= \frac{x dx + y dy + z dz}{0} \end{aligned}$$

$$\therefore \int x dx + \int y dy + \int z dz = 0$$

$$u = \frac{x^2}{2} + \frac{y^2}{2} + \frac{z^2}{2} = C_1$$

considering multipliers $k_1 = 1/x, k_2 = 1/y, k_3 = 1/z$

$$\begin{aligned} \frac{dx}{x(y^2 - z^2)} &= \frac{dy}{y(z^2 - x^2)} = \frac{dz}{z(x^2 - y^2)} = \frac{dx/x + dy/y + dz/z}{y^2 - x^2 + z^2 - x^2 + x^2 - y^2} \\ &= \frac{dx/x + dy/y + dz/z}{0} \end{aligned}$$

$$\therefore \int \frac{dx}{x} + \int \frac{dy}{y} + \int \frac{dz}{z} = 0$$

$$v = \log|x| + \log|y| + \log|z| = C_2$$

$$v = \log|xyz| = C_2$$

wkt $\phi(u, v) = 0$

$$\therefore \phi\left(\frac{x^2 + y^2 + z^2}{2}, \log|xyz|\right) = 0$$

$$(mx-ny) \frac{\partial z}{\partial x} + (nx-lx) \frac{\partial z}{\partial y} = (ly-mx)$$

The given equation is of the form $Pp+Qq=R$
 The auxiliary equation is given by

$$\frac{dz}{P} = \frac{dy}{Q} = \frac{dx}{R}$$

$$\frac{dx}{mx-ny} = \frac{dy}{nx-lx} = \frac{dz}{ly-mx} = 0$$

Considering multipliers $k_1 = x, k_2 = y, k_3 = z$

$$\frac{dx}{mx-ny} = \frac{dy}{nx-lx} = \frac{dz}{ly-mx} = \frac{x dx + y dy + z dz}{mx^2 - nxy + nxz - lyx + lyz - mz^2}$$

$$= \frac{x dx + y dy + z dz}{0}$$

$$\therefore \int x dx + \int y dy + \int z dz = 0$$

$$u = \frac{x^2}{2} + \frac{y^2}{2} + \frac{z^2}{2} = C_1$$

Considering multipliers $k_1 = l, k_2 = m, k_3 = n$

$$\frac{dx}{mx-ny} = \frac{dy}{nx-lx} = \frac{dz}{ly-mx} = \frac{l dx + m dy + n dz}{mlx - n ly + mnz - mlz + ln y - mn x}$$

$$= \frac{l dx + m dy + n dz}{0}$$

$$\therefore \int l dx + \int m dy + \int n dz = 0$$

$$v = lx + my + nz = C_2$$

Solution is given by $\phi(u, v) = 0$

$$\therefore \phi \left(\frac{x^2 + y^2 + z^2}{2}, lx + my + nz \right) = 0 //$$

* Applications of PDE: (continued)

By fitting solution of one dimensional wave equation
 $u(x, t) = (A \cos px + B \sin px)(C \cos pt + D \sin pt)$
 and

Two dimensional Laplace equation

$$u(x, t) = (A \cos px + B \sin px)(C e^{pt} + D e^{-pt})$$

Q1: Solve the wave equation $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$ given that

$$u(0, t) = 0, u(l, t) = 0,$$

$$\frac{du}{dt} = 0 \text{ when } t=0 \text{ and } u(x, 0) = U_0 \sin \frac{\pi x}{l}$$

~~sol:~~ The solution of wave equation is given by

$$u(x, t) = (A \cos px + B \sin px)(C \cos pt + D \sin pt) \quad \textcircled{1}$$

Applying the given condition $u(0, t) = 0$ in eq. \textcircled{1}

$$u(0, t) = (A \cos 0 + B \sin 0)(C \cos pt + D \sin pt) = 0$$

$$\Rightarrow A(C \cos pt + D \sin pt) = 0$$

$$\therefore \underline{A = 0}$$

Applying the given condition $u(l, t) = 0$ in eq. \textcircled{1}

$$u(l, t) = (A \cos lp + B \sin lp)(C \cos pt + D \sin pt) = 0$$

$$\Rightarrow B \sin lp (C \cos pt + D \sin pt) = 0$$

$$\therefore B \sin lp = 0$$

Since $A = 0 \Rightarrow B \neq 0$

$$\therefore \sin lp = 0$$

$$\sin lp = \sin n\pi t$$

$$lp = n\pi$$

$$p = \frac{n\pi}{l}$$

Substituting p and A in eq. \textcircled{1}

$$u(x, t) = \left(B \sin \frac{n\pi x}{l} \right) \left(C \cos \frac{n\pi ct}{l} + D \sin \frac{n\pi ct}{l} \right) \quad \textcircled{2}$$

partially diff eq ⑤ wrt t

$$\frac{\partial u}{\partial t} = B \sin \frac{n\pi x}{l} \left(-\frac{C n \pi c}{l} \sin \frac{n\pi ct}{l} + \frac{D n \pi c}{l} \cos \frac{n\pi ct}{l} \right)$$

$$\frac{\partial u}{\partial t} = \frac{B n \pi c}{l} \sin \frac{n\pi x}{l} \left(\frac{D \cos \frac{n\pi ct}{l}}{l} - \frac{C \sin \frac{n\pi ct}{l}}{l} \right)$$

Applying the condition $\frac{\partial u}{\partial t} = 0$ when $t=0$

$$0 = \frac{B n \pi c}{l} \sin \frac{n\pi x}{l} \left(D \cos 0 - C \sin 0 \right)$$

$$\frac{DB n \pi c}{l} \sin \frac{n\pi x}{l} = 0$$

Since $B \neq 0 \Rightarrow \underline{D=0}$

Substituting D in eq ②

$$u(x, t) = \frac{B}{l} \sin \frac{n\pi x}{l} \cdot C \cos \frac{n\pi ct}{l}$$

Let BC = b_n

$$u(x, t) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} \cos \frac{n\pi ct}{l} \quad \text{--- ③}$$

Applying the given condition $u(x, 0) = u_0 \sin \frac{\pi x}{l}$ to eq ③

$$u(x, 0) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l} \cos 0$$

$$u_0 \sin \frac{\pi x}{l} = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l}$$

$$u_0 \sin \frac{\pi x}{l} = b_1 \sin \frac{\pi x}{l} + b_2 \sin \frac{2\pi x}{l} + b_3 \sin \frac{3\pi x}{l} + \dots$$

$$\therefore u_0 \sin \frac{\pi x}{l} = b_1 \sin \frac{\pi x}{l}$$

$$\therefore \underline{b_1 = u_0}$$

Substituting b_1 in eq ③

$$u(x, t) = u_0 \sin \frac{\pi x}{l} \cos \frac{\pi ct}{l}$$

Q2: solve the wave equation $\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$

given that

$$u(0, t) = 0, u(l, t) = 0, \frac{\partial u}{\partial t} = 0 \text{ when } t=0 \text{ and } u(x, 0) = u_0 \sin^3 \frac{\pi x}{l}$$

Sol: * Applying the given condition $u(x, 0) = u_0 \sin^3 \frac{\pi x}{l}$ in eq ③

$$u_0 \sin^3 \frac{\pi x}{l} = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{l}$$

$$u_0 \left(\frac{3 \sin \frac{\pi x}{l}}{4} - \frac{1}{4} \sin^3 \frac{\pi x}{l} \right) = b_1 \frac{\sin \frac{\pi x}{l}}{l} + b_2 \frac{\sin \frac{2\pi x}{l}}{l} + b_3 \frac{\sin \frac{3\pi x}{l}}{l} + \dots$$

Equating the coefficients

$$\frac{3}{4} u_0 \frac{\sin \frac{\pi x}{l}}{l} = b_1 \frac{\sin \frac{\pi x}{l}}{l}$$

$$\Rightarrow b_1 = \frac{3}{4} u_0 //$$

$$\frac{-1}{4} u_0 \frac{\sin \frac{3\pi x}{l}}{l} = b_3 \frac{\sin \frac{3\pi x}{l}}{l}$$

$$\Rightarrow b_3 = -\frac{u_0}{4} //$$

Substituting in eq ③

$$u(x, t) = \frac{3u_0}{4} \frac{\sin \frac{\pi x}{l}}{l} - \frac{u_0}{4} \frac{\sin \frac{3\pi x}{l}}{l} \cos \frac{\pi c t}{l} \cos \frac{3\pi c t}{l}$$

UNIT 3

Fourier Transforms

The infinite Fourier transform or complex Fourier transform of a real value function $f(x)$ is defined as

$$F[f(x)] = \int_{-\infty}^{\infty} f(x) e^{iux} dx = F(u)$$

On integration we obtain a function of 'u' which is denoted by $F(u)$. The inverse Fourier transform of $F(u)$ is defined by the integral,

$$F^{-1}[F(u)] = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(u) e^{-iux} du = f(x)$$

On integration we obtain a function of 'x'.

Q1: Find the complex Fourier transform of the function

$$f(x) = \begin{cases} 1 & \text{for } |x| \leq a \\ 0 & \text{for } |x| > a \end{cases}$$

Deduce $\int_0^{\infty} \frac{\sin x}{x} dx$.

Q2: By definition

$$F(u) = \int_{-\infty}^{\infty} f(x) e^{iux} dx$$

$$= \int_{-a}^a 1 \cdot e^{iux} dx$$

$$= \left[\frac{e^{iux}}{iu} \right]_{-a}^a$$

$$= \frac{1}{iu} [e^{iau} - e^{-iau}]$$

$$= -\frac{i}{u} [e^{iau} - e^{-iau}]$$

NOTE:
 $e^{iu} + e^{-iu} = (\cos u + i \sin u)$
 $+ (\cos u - i \sin u)$
 $e^{iu} + e^{-iu} = 2 \cos u$

2) $e^{iu} - e^{-iu} = (\cos u + i \sin u)$
 $- (\cos u - i \sin u)$
 $e^{iu} - e^{-iu} = 2i \sin u$

$$F(u) = \frac{-i}{u} (2i \sin \omega u)$$

$$F(u) = \frac{2 \sin \omega u}{u}$$

By Inverse Fourier transform,

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(u) e^{-iux} du$$

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{2 \sin \omega u}{u} e^{-iux} du$$

$$\text{Let } x = 0$$

$$f(0) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\sin \omega u}{u} du$$

since $x=0$ is a point of continuity of $f(x)$, the value of $f(x)$ at $x=0$ implies $f(0)=1$ because $f(x)$ value is 1 for $|x| \leq a$

$$1 = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\sin \omega u}{u} du$$

$$\int_{-\infty}^{\infty} \frac{\sin \omega u}{u} du = \pi$$

$$2 \int_0^{\infty} \frac{\sin \omega u}{u} du = \pi \quad (\because f(x) \text{ is even})$$

put $a=1$, change u to x

$$\int_0^{\infty} \frac{\sin x}{x} dx = \frac{\pi}{2}$$

Q2: $f(x) = \begin{cases} x & \text{for } |x| \leq a \\ 0 & \text{for } |x| > a \end{cases}$

Find the complex Fourier transform for the function given where a is a positive constant.

Sol: By definition

$$F(u) = \int_{-\infty}^{\infty} f(x) e^{iux} dx$$

$$f(x) = \int_{-\infty}^x x \cdot e^{iux} dx$$

$$F(u) = \left[x \frac{e^{iux}}{iu} - \frac{e^{iux}}{i^2 u^2} \right]_{-\infty}^x$$

$$F(u) = \frac{1}{iu} \left[x e^{iux} \right]_{-\infty}^x - \frac{1}{i^2 u^2} \left[e^{iux} \right]_{-\infty}^x$$

$$F(u) = \frac{-i}{u} \left[x e^{iux} + x e^{-iux} \right] + \frac{1}{u^2} \left[e^{iux} - e^{-iux} \right]$$

$$F(u) = \frac{-i}{u} \left[x (e^{iux} + e^{-iux}) \right] + \frac{1}{u^2} (2i \sin ux)$$

$$F(u) = \frac{-ix}{u} (2 \cos ux) + \frac{1}{u^2} (2i \sin ux)$$

—————

Q3: $f(x) = \begin{cases} 1-x^2, & \text{for } |x| \leq 1 \\ 0, & \text{for } |x| > 1 \end{cases}$

Find the Fourier transform of $f(x)$ and hence find the value of

i. $\int_0^\infty \frac{x \cos x - \sin x}{x^3} dx$

ii. $\int_0^\infty \frac{x \cos x - \sin x}{x^3} \cos\left(\frac{x}{2}\right) dx$

Sol: By definition

$$F(u) = \int_{-\infty}^{\infty} f(x) e^{iux} dx$$

$$F(u) = \int_{-1}^1 (1-x^2) e^{iux} dx$$

$$= \left[\frac{e^{iux}}{iu} - \left(\frac{x^2 e^{iux}}{iu} - \int x^2 e^{iux} dx + 2 \int \frac{e^{iux}}{iu^2} dx \right) \right]_{-1}^1$$

$$\begin{aligned}
 &= \left[\frac{e^{iux}}{iu} - \frac{\pi^2 e^{iux}}{iu} + 2\pi \frac{e^{iux}}{i^2 u^2} - \frac{2e^{iux}}{i^3 u^3} - \frac{2e^{iux}}{i^3 u^3} \right] \Big|_1 \\
 &= \left[\frac{e^{iux}}{iu} (1-\pi^2) - \frac{2\pi e^{iux}}{u^2} - \frac{2e^{iux}}{i^3 u^3} \right] \Big|_1 \\
 &= \left[\frac{e^{iu}}{iu} (0) - \frac{2e^{iu}}{u^2} - \frac{2e^{iu}}{i^3 u^3} - \frac{e^{-iu}}{iu} (0) - \frac{2e^{-iu}}{u^2} + \frac{2e^{-iu}}{i^3 u^3} \right] \\
 &= -\frac{2}{u^2} (e^{iu} + e^{-iu}) - \frac{2}{i^3 u^3} (e^{iu} - e^{-iu}) \\
 &= -\frac{2}{u^2} (2\cos u) - \frac{2}{i^3 u^3} (2i \sin u) \\
 &= -\frac{4\cos u}{u^2} + \frac{4\sin u}{u^3}
 \end{aligned}$$

$$F(u) = \frac{-4u\cos u + 4\sin u}{u^3} = \frac{4(\sin u - u\cos u)}{u^3}$$

i] ~~Def~~ By inverse Fourier transform

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(u) e^{-iux} du$$

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} 4 \frac{(-u\cos u + \sin u)}{u^3} e^{-iux} du \quad \text{--- (1)}$$

Let $x=0$

$$f(0) = \frac{2}{\pi} \int_{-\infty}^{\infty} \frac{-u\cos u + \sin u}{u^3} du$$

Since $x=0$ is a point of continuity of $f(x)$, the value of $f(x)$ at $x=0$ implies $f(0)=1$ because $f(x)$ value is $1-x^2$ for $|x| \leq 1$

$$\therefore 1 = \frac{2}{\pi} \int_{-\infty}^{\infty} \frac{-u\cos u + \sin u}{u^3} du$$

$$1 = \frac{4}{\pi} \int_0^{\infty} \frac{-u\cos u + \sin u}{u^3} du \quad (\because f(x) \text{ is an even function})$$

change u to x

$$\int_0^\infty \frac{-x \cos x + \sin x}{x^3} dx = \frac{\pi}{4}$$

$$\therefore \int_0^\infty \frac{x \cos x - \sin x}{x^3} dx = -\frac{\pi}{4} //$$

iii) Let us substitute $x = \frac{1}{2}u$ in eq. ①

$$f\left(\frac{1}{2}\right) = \frac{2}{\pi} \int_{-\infty}^0 \frac{-u \cos u + \sin u}{u^3} e^{-\frac{\pi u}{2}} du$$

$$\frac{1}{4} = \frac{2}{\pi} \int_{-\infty}^0 \frac{-u \cos u + \sin u}{u^3} \left[\cos \frac{u}{2} - i \sin \frac{u}{2} \right] du$$

$$\frac{3\pi}{8} = \int_{-\infty}^0 \frac{-u \cos u + \sin u}{u^3} \cos \frac{u}{2} du \quad (\text{considering only real part})$$

$$\frac{3\pi}{8} = 2 \int_0^\infty -u \cos u + \sin u \cos \frac{u}{2} du \quad (\because f(x) \text{ is an even function})$$

$$\frac{3\pi}{16} = \int_0^\infty -u \cos u + \sin u \cos \frac{u}{2} du$$

changing u by x

$$\int_0^\infty \frac{x \cos x - \sin x}{x^3} \cos \frac{x}{2} dx = -\frac{3\pi}{16} //$$

Q4: Find Fourier transform of

$$f(x) = \begin{cases} 1 - |x| & \text{for } |x| < 1 \\ 0 & \text{for } |x| > 1 \end{cases}$$

Hence deduce $\int_0^\infty \frac{\sin t}{t^2} dt = \frac{\pi}{2}$

Sol:

By definition

$$F(u) = \int_{-\infty}^{\infty} f(x) e^{iux} dx$$

$$= \int_{-1}^1 (1 - |x|) e^{iux} dx$$

$$= \int_{-1}^0 (1+x) e^{iux} dx + \int_0^1 (1-x) e^{iux} dx$$

$$\begin{aligned}
 &= \left[\frac{e^{iux}}{iu} + \pi e^{iux} - \frac{e^{iux}}{i^2 u^2} \right]_0^\infty + \left[\frac{e^{iux}}{iu} - \pi e^{iux} + \frac{e^{iux}}{i^2 u^2} \right]_0^\infty \\
 &= \left[\frac{1}{iu} + 0 + \frac{1}{u^2} - \frac{e^{-iu}}{iu} + \frac{e^{-iu}}{iu} - \frac{e^{iu}}{u^2} \right] \\
 &\quad + \left[\frac{e^{iu}}{iu} - \frac{e^{iu}}{iu} + \frac{e^{iu}}{u^2} - \frac{1}{iu} + 0 + \frac{1}{u^2} \right] \\
 &= \frac{2}{u^2} - \frac{1}{u^2} (e^{iu} + e^{-iu}) \\
 &= \frac{2 - 2 \cos u}{u^2}
 \end{aligned}$$

$$F(u) = \frac{2(1-\cos u)}{u^2}$$

$$F(u) = \frac{2(2\sin^2 u/2)}{u^2} = \frac{4}{u^2} \frac{\sin^2 u}{2}$$

By inverse fourier transform

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(u) e^{iux} dx$$

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{4}{u^2} \frac{\sin^2 u}{2} e^{iux} dx$$

$$f(x) = \frac{1}{2\pi} 2 \int_0^{\infty} \frac{4}{u^2} \frac{\sin^2 u}{2} e^{iux} dx \quad (\because F(u) \text{ is an even function})$$

$$f(x) = \frac{4}{\pi} \int_0^{\infty} \frac{1}{u^2} \frac{\sin^2 u}{2} e^{iux} dx$$

$$\text{Let } x = 0$$

$$f(0) = \frac{4}{\pi} \int_0^{\infty} \frac{1}{u^2} \frac{\sin^2 u}{2} du$$

$$1 = \frac{4}{\pi} \int_0^{\infty} \frac{1}{u^2} \frac{\sin^2 u}{2} du$$

replace u by x

$$\frac{\pi}{4} = \int_0^{\infty} \frac{1}{x^2} \frac{\sin^2 x}{2} dx$$

Replacing u by at
 $du = a dt$

as $x \rightarrow \infty$ $t \rightarrow \infty$

$u \rightarrow 0$ $t \rightarrow 0$

$$\frac{\pi}{4} = 2 \int_0^{\infty} \frac{4}{t^2} \sin^2 t dt$$

$$\frac{\pi}{2} = \int_0^{\infty} \frac{\sin^2 t}{t^2} dt //$$

* NOTE:

$$1. F_s(u) = \int_0^{\infty} f(x) \sin ux dx$$

If $f(x)$ is defined
for all positive values
of x then.

$$2. f(x) = \frac{2}{\pi} \int_0^{\infty} F_s(u) \sin ux du$$

1. Fourier sine transform
2. Inverse Fourier sine transform
3. Fourier cosine transform
4. Inverse Fourier cosine transform
is given as.

$$3. F_c(u) = \int_0^{\infty} f(x) \cos ux dx$$

$$4. f(x) = \frac{2}{\pi} \int_0^{\infty} F_c(u) \cos ux du$$

Find the fourier sine and cosine transform of

$$f(x) = \begin{cases} x & 0 < x < 2 \\ 0 & \text{elsewhere} \end{cases}$$

Fourier sine transform

$$F_s(u) = \int_0^{\infty} f(x) \sin ux dx$$

$$F_s(u) = \int_0^{\infty} x \sin ux dx$$

$$= \left[x \left(\frac{-\cos ux}{u} \right) - \left(\frac{-\sin ux}{u^2} \right) \right]_0^2$$

$$F_s(u) = -\frac{2\cos 2u}{u} + \frac{\sin 2u}{u^2} = \frac{\sin 2u - 2u \cos 2u}{u^2} //$$

Fourier cosine transform

$$F_c(u) = \int_0^\infty f(x) \cos ux dx$$

$$F_c(u) = \int_0^x x \cos ux dx$$

$$= \left[x \left(\frac{\sin ux}{u} \right) - \left(\frac{-\cos ux}{u^2} \right) \right]_0^x$$

$$= \frac{2 \sin 2u}{u} + \frac{\cos 2u}{u^2} - \frac{1}{u^2}$$

$$F_c(u) = \frac{\cos 2u + 2u \sin 2u - 1}{u^2}$$

Q2: Find the Fourier sine and cosine transform of $e^{-\alpha x}$, $x > 0$.

Sol: Fourier sine transform

$$F_s(u) = \int_0^\infty f(x) \sin ux dx$$

$$F_s(u) = \int_0^x e^{-\alpha x} \sin ux dx$$

$$= \frac{e^{-\alpha x}}{\alpha^2 + u^2} [x \cos ux - u \sin ux] \Big|_0^x$$

$$= \frac{e^{-\alpha x}}{\alpha^2 + u^2} [-\alpha \cos ux - u \sin ux + x] //$$

Fourier cosine transform

$$F_c(u) = \int_0^\infty f(x) \cos ux dx$$

$$F_c(u) = \int_0^x e^{-\alpha x} \cos ux dx$$

$$= \frac{e^{-\alpha x}}{\alpha^2 + u^2} [-\alpha \cos ux + u \sin ux] \Big|_0^x$$

$$= \frac{e^{-\alpha x}}{\alpha^2 + u^2} [-\alpha \cos ux + u \sin ux - u] //$$

Find the equivalent Fourier transform of e^{-x^2}

Q3:
Fourier cosine transform

$$F_c(u) = \int_0^\infty f(x) \cos ux dx$$

$$F_c(u) = \int_0^\infty e^{-x^2} \cos ux dx$$

The integral is evaluated using Leibniz rule (diff wrt u)

$$F'_c(u) = \int_0^\infty e^{-x^2} (-x \sin ux) dx$$

$$F'_c(u) = \int_0^\infty -x e^{-x^2} \sin ux dx$$

$$= \frac{1}{2} \int_0^\infty -2x e^{-x^2} \sin ux dx$$

$$= \frac{1}{2} \left[(\sin ux (e^{-x^2})) \Big|_0^\infty - \int_0^\infty e^{-x^2} (u \cos ux) dx \right]$$

$$F'_c(u) = \frac{1}{2} [-u F_c(u)]$$

$$F'_c(u) = \frac{-u F_c(u)}{2}$$

$$\frac{F'_c(u)}{F_c(u)} = \frac{-u}{2}$$

Integrating wrt u

$$\int \frac{F'_c(u)}{F_c(u)} du = \int \frac{-u}{2} du$$

$$\log F_c(u) = \frac{-u^2}{4} + C$$

If $u=0$

$$F_c(u) = e^{-u^2/4} e^C$$

$$F_c(u) = k e^{-u^2/4}$$

$$F_c(0) = k e^0$$

$$\therefore F_c(0) = \int_0^\infty e^{-x^2} \cos(0x) dx = k$$

$$\frac{\sqrt{\kappa}}{2} = k$$

$$\therefore F_c(u) = \frac{\sqrt{\kappa}}{2} e^{-u^2/4}$$

Q4: Find the fourier cosine transform of $\frac{1}{1+x^2}$

Sol: Fourier cosine transform

$$F_c(u) = \int_0^\infty f(x) \cos ux dx$$

$$F_c(u) = \int_0^\infty \frac{1}{1+x^2} \cos ux dx \quad \text{--- (1)}$$

diff eq (1) wrt u (Leibniz rule)

$$F_c'(u) = \int_0^\infty \frac{1}{1+x^2} (-x \sin ux) dx$$

$$F_c'(u) = - \int_0^\infty \frac{x}{1+x^2} \sin ux dx$$

$$F_c'(u) = - \int_0^\infty \frac{1+x^2-1}{x(1+x^2)} \sin ux dx$$

$$F_c'(u) = - \int_0^\infty \frac{1+x^2}{x(1+x^2)} \sin ux dx + \int_0^\infty \frac{\sin ux}{x(1+x^2)} dx$$

$$F_c'(u) = - \int_0^\infty \frac{\sin ux}{x} dx + \int_0^\infty \frac{\sin ux}{x(1+x^2)} dx$$

$$F_c'(u) = -\frac{\pi}{2} + \int_0^\infty \frac{\sin ux}{x(1+x^2)} dx \quad \text{--- (2)}$$

diff eq (2) wrt u

$$F_c''(u) = \int_0^\infty \frac{-2x \cos ux}{x(1+x^2)} dx$$

From eq (1)

$$F_c''(u) = F_c(u)$$

$$F_c''(u) - F_c(u) = 0$$

$$(D^2 - 1) F_c(u) = 0$$

$$\text{AE: } m^2 - 1 = 0$$

$$m^2 = 1$$

$$m = \pm 1$$

$$F_c(u) = C_1 e^u + C_2 e^{-u} \quad \text{--- (3)}$$

diff eq (3) wrt u

$$F'_c(u) = C_1 e^u - C_2 e^{-u} \quad \text{--- (4)}$$

Put $u=0$ in eq (3) and eq (4)

$$F_c(0) = C_1 + C_2$$

$$\int_0^\infty \frac{\cos ux}{1+x^2} dx = \int_0^\infty \frac{\cos 0}{1+x^2} dx = C_1 + C_2$$

$$\int_0^\infty \frac{1}{1+x^2} dx = [\tan^{-1} x]_0^\infty = C_1 + C_2$$

$$\tan^{-1}(\infty) - \tan^{-1}0 = C_1 + C_2$$

$$\frac{\pi}{2} - 0 = C_1 + C_2$$

$$\therefore C_1 + C_2 = \frac{\pi}{2} \quad \text{--- (5)}$$

From eq (2) and eq (4)

$$F'_c(0) = C_1 - C_2$$

$$\frac{-\pi}{2} + \int_0^\infty \frac{\sin 0}{x(1+x^2)} dx = C_1 - C_2$$

$$C_1 - C_2 = -\frac{\pi}{2} \quad \text{--- (6)}$$

Adding eq (5) and eq (6)

$$2C_1 = 0 \quad \text{From eq (5)} \quad C_1 = \frac{\pi}{2}$$

$$\underline{\underline{C_1 = 0}}$$

Substituting C_1 and C_2 in eq (3)

$$F_c(u) = \frac{\pi}{2} e^{-u}$$

Q5: $\frac{1}{\pi(1+x^2)}$. Find the Fourier sine transform.

Sol: Fourier sine transform

$$F_s(u) = \int_0^\infty f(x) \sin ux dx$$

$$F_s(u) = \int_0^\infty \frac{1}{\pi(1+x^2)} \sin ux dx \quad \textcircled{1}$$

diff eq, ① wrt u

$$F'_s(u) = \int_0^\infty \frac{1}{\pi(1+x^2)} x \cos ux dx$$

$$F'_s(u) = \int_0^\infty \frac{\cos ux}{1+x^2} dx \quad \textcircled{2}$$

diff eq, ② wrt u

$$F''_s(u) = \int_0^\infty \frac{-x \sin ux}{1+x^2} dx$$

$$F''_s(u) = - \int_0^\infty \frac{1+x^2-1}{\pi(1+x^2)} \sin ux dx$$

$$F''_s(u) = - \int_0^\infty \frac{1+x^2}{\pi(1+x^2)} \sin ux dx + \int_0^\infty \frac{\sin ux}{\pi(1+x^2)} dx$$

$$F''_s(u) = - \int_0^\infty \frac{\sin ux}{x} dx + \int_0^\infty \frac{\sin ux}{x(1+x^2)} dx$$

$$F''_s(u) = -\frac{\pi}{2} + \int_0^\infty \frac{\sin ux}{x(1+x^2)} dx \quad \textcircled{3}$$

Substituting eq, ① in eq, ③

$$F''_s(u) = -\frac{\pi}{2} + F_s(u)$$

$$F''_s(u) - F_s(u) = -\frac{\pi}{2}$$

$$(D^2 - 1) F_s(u) = -\pi/2$$

$$AE : m^2 - 1 = 0$$

$$m^2 = 1 \Rightarrow m = \underline{\underline{\pm 1}}$$

$$CF = C_1 e^u + C_2 e^{-u}$$

$$P.I. = \frac{-\pi/2}{D^2 - 1} \quad \text{Replacing } D \text{ by } 0$$

$$P.I. = \frac{-\pi/2}{-1} = \frac{\pi}{2}$$

$$F_s(u) = CF + P.I.$$

$$F_s(u) = C_1 e^u + C_2 e^{-u} + \frac{\pi}{2} \quad \text{--- (3)}$$

diff eq (3) wrt u

$$F'_s(u) = C_1 e^u - C_2 e^{-u} \quad \text{--- (4)}$$

$$\text{Put } u=0$$

$$\text{From eq (3)} \quad F_s(0) = C_1 + C_2 + \frac{\pi}{2} \quad \text{--- (5)} \Rightarrow C_1 + C_2 = -\pi/2$$

$$\text{From eq (4)} \quad F'_s(0) = C_1 - C_2 \quad \text{--- (6)} \Rightarrow C_1 - C_2 = \pi/2$$

Adding eq (5) and eq (6)

$$2C_1 = 0 \quad C_2 = -\pi/2$$

$$\underline{\underline{C_1 = 0}}$$

Substituting ~~eq~~ C_1 and C_2 in eq (3)

$$F_s(u) = -\frac{\pi}{2} e^{-u} + \frac{\pi}{2}$$

$$F_s(u) = \frac{\pi}{2} (1 - e^{-u}) //$$

Q6: Find the fourier sine transform of $f(x) = e^{-|x|}$ and also evaluate $\int_0^\infty \frac{x \sin mx}{1+x^2} dx$, $m > 0$.

Q7: Fourier sine transform

$$F_s(u) = \int_0^\infty f(x) \sin ux dx$$

$$F_s(u) = \int_0^\infty e^{-|x|} \sin ux dx$$

$$F_s(u) = \left[\frac{e^{-x}}{1+u^2} (-\sin ux - u \cos ux) \right]_0^\infty$$

$$F_s(u) = \frac{1}{1+u^2} [0 - e^0(-u)]$$

$$F_s(u) = \frac{u}{1+u^2}$$

By inverse fourier sine transform

$$f(x) = \frac{2}{\pi} \int_0^\infty F_s(u) \sin ux du$$

$$f(x) = \frac{2}{\pi} \int_0^\infty \frac{u \sin ux}{1+u^2} du$$

$$e^{-x} = \frac{2}{\pi} \int_0^\infty \frac{u \sin ux}{1+u^2} du$$

$$\int_0^\infty \frac{u \sin ux}{1+u^2} du = \frac{e^{-x}\pi}{2}$$

Replacing u by πx and x by m

$$\int_0^\infty \frac{\pi x \sin mx}{1+\pi^2 x^2} dx = \frac{e^{-m\pi}}{2}$$

UNIT - 4

Numerical Methods 1

* Gauss - Seidel Method:

$$\left. \begin{array}{l} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3 \end{array} \right\} \begin{array}{l} \text{System of linear} \\ \text{equations.} \end{array}$$

$$\Rightarrow x_1 = \frac{1}{a_{11}} [b_1 - a_{12}x_2 - a_{13}x_3]$$

$$x_2 = \frac{1}{a_{22}} [b_2 - a_{21}x_1 - a_{23}x_3]$$

$$x_3 = \frac{1}{a_{33}} [b_3 - a_{31}x_1 - a_{32}x_2]$$

System of equations must be diagonally dominant.

$$|a_{11}| > |a_{12}| + |a_{13}|$$

$$|a_{22}| > |a_{21}| + |a_{23}|$$

$$|a_{33}| > |a_{31}| + |a_{32}|$$

Initial Approximation

$$x_1^{(0)} = 0 \quad x_2^{(0)} = 0 \quad x_3^{(0)} = 0$$

First Iteration

$$x_1^{(1)} = \frac{1}{a_{11}} [b_1 - a_{12}x_2^{(0)} - a_{13}x_3^{(0)}]$$

$$x_2^{(1)} = \frac{1}{a_{22}} [b_2 - a_{21}x_1^{(1)} - a_{23}x_3^{(0)}]$$

$$x_3^{(1)} = \frac{1}{a_{33}} [b_3 - a_{31}x_1^{(1)} - a_{32}x_2^{(1)}]$$

Q1: solve the system of equations using Gauss-Seidel Method.

$$x + y + 54z = 110$$

$$27x + 6y - z = 85$$

$$6x + 15y + 2z = 72$$

Sol: Rearranging

$$27x + 6y - z = 85$$

$$6x + 15y + 2z = 72$$

$$x + y + 54z = 110$$

$$\text{let } x^{(0)} = 0; y^{(0)} = 0; z^{(0)} = 0$$

$$x = \frac{1}{27} [85 - 6y + z]$$

$$y = \frac{1}{15} [72 - 6x - 2z]$$

$$z = \frac{1}{54} [110 - x - y]$$

First Iteration

$$x^{(1)} = \frac{1}{27} [85 - 6y^{(0)} + z^{(0)}] = \frac{1}{27} [85 - 0 + 0] = \underline{\underline{3.1481}}$$

$$y^{(1)} = \frac{1}{15} [72 - 6x^{(1)} - 2z^{(0)}] = \frac{1}{15} [72 - 6(3.1481) - 0] = \underline{\underline{3.5407}}$$

$$z^{(1)} = \frac{1}{54} [110 - x^{(1)} - y^{(1)}] = \frac{1}{54} [110 - 3.1481 - 3.5407] = \underline{\underline{1.9131}}$$

Second Iteration

$$x^{(2)} = \frac{1}{27} [85 - 6y^{(1)} + z^{(1)}] = \underline{\underline{2.4324}}$$

$$y^{(2)} = \frac{1}{15} [72 - 6x^{(2)} - 2z^{(1)}] = \underline{\underline{3.5720}}$$

$$z^{(2)} = \frac{1}{54} [110 - x^{(2)} - y^{(2)}] = \underline{\underline{1.9258}}$$

Third Iteration

$$x^{(3)} = \frac{1}{27} [85 - 6y^{(2)} + z^{(2)}] = \underline{\underline{2.4256}}$$

$$y^{(3)} = \frac{1}{15} [72 - 6x^{(3)} - 2z^{(2)}] = \underline{\underline{3.5729}}$$

$$z^{(3)} = \frac{1}{54} [110 - x^{(3)} - y^{(3)}] = \underline{\underline{1.9259}}$$

Fourth Iteration

$$x^{(4)} = \frac{1}{27} [85 - 6y^{(3)} + z^{(3)}] = \underline{\underline{2.4255}}$$

$$y^{(4)} = \frac{1}{15} [72 - 6x^{(4)} - 2z^{(3)}] = \underline{\underline{3.5730}}$$

$$z^{(4)} = \frac{1}{54} [110 - x^{(4)} - y^{(4)}] = \underline{\underline{1.9259}}$$

Hence $x = \underline{\underline{2.425}} \quad y = \underline{\underline{3.573}} \quad z = \underline{\underline{1.925}}$

Q2: $5x + 2y + z = 12$ consider the initial approximation as
 $x + 4y + 2z = 15$ $(1, 0, 3)$.
 $x + 2y + 5z = 20$

Sol:Rearrange

$$x = \frac{1}{5} [12 - 2y - z]$$

$$y = \frac{1}{4} [15 - x - 2z]$$

$$z = \frac{1}{5} [20 - x - 2y]$$

given: $x^{(0)} = 1 \quad y^{(0)} = 0 \quad z^{(0)} = 3$

First Iteration

$$x^{(1)} = \frac{1}{5} [12 - 2y^{(0)} - z^{(0)}] = \frac{1}{5} [12 - 2(0) - 3] = \underline{\underline{1.8000}}$$

$$y^{(1)} = \frac{1}{4} [15 - x^{(0)} - 2z^{(0)}] = \frac{1}{4} [15 - 1.8 - 2(3)] = \underline{\underline{1.8000}}$$

$$z^{(1)} = \frac{1}{5} [20 - x^{(1)} - 2y^{(1)}] = \frac{1}{5} [20 - 1.8 - 2(1.8)] = \underline{\underline{2.9200}}$$

Second Iteration

$$x^{(2)} = \frac{1}{5} [12 - 2y^{(1)} - z^{(1)}] = \underline{\underline{1.0960}}$$

$$y^{(2)} = \frac{1}{4} [15 - x^{(2)} - 2z^{(1)}] = \underline{\underline{2.0160}}$$

$$z^{(2)} = \frac{1}{5} [20 - x^{(2)} - 2y^{(2)}] = \underline{\underline{2.9744}}$$

Third Iteration

$$x^{(3)} = \frac{1}{5} [12 - 2y^{(2)} - z^{(2)}] = \underline{\underline{0.9984}}$$

$$y^{(3)} = \frac{1}{4} [15 - x^{(3)} - 2z^{(2)}] = \underline{\underline{2.0131}}$$

$$z^{(3)} = \frac{1}{5} [20 - x^{(3)} - 2y^{(3)}] = \underline{\underline{2.9950}}$$

Fourth Iteration

$$x^{(4)} = \frac{1}{5} [12 - 2y^{(3)} - z^{(3)}] = \underline{\underline{0.9957}}$$

$$y^{(4)} = \frac{1}{4} [15 - x^{(4)} - 2z^{(3)}] = \underline{\underline{2.0035}}$$

$$z^{(4)} = \frac{1}{5} [20 - x^{(4)} - 2y^{(4)}] = \underline{\underline{2.9994}}$$

Hence $x = \underline{\underline{0.995}}$ $y = \underline{\underline{2.003}}$ $z = \underline{\underline{2.999}}$

* Relaxation Method:

Consider a system of equations

$$a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = b_1$$

$$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = b_2$$

$$a_{31}x_1 + a_{32}x_2 + a_{33}x_3 = b_3$$

This system of linear equations must be diagonally dominant

Define residual as

$$R_1 = a_{11}x_1 + a_{12}x_2 + a_{13}x_3 - b_1$$

$$R_2 = a_{21}x_1 + a_{22}x_2 + a_{23}x_3 - b_2$$

$$R_3 = a_{31}x_1 + a_{32}x_2 + a_{33}x_3 - b_3$$

Relaxation method aims in reducing the values of the residuals as close to zero as possible by modifying the value of the variables at every stage.

i: solve the system of equation by relaxation method.

$$12x_1 + x_2 + x_3 = 31$$

$$2x_1 + 8x_2 - x_3 = 24$$

$$3x_1 + 4x_2 + 10x_3 = 58$$

: considering residuals

$$R_1 = 12x_1 + x_2 + x_3 - 31$$

$$R_2 = 2x_1 + 8x_2 - x_3 - 24$$

$$R_3 = 3x_1 + 4x_2 + 10x_3 - 58$$

$$x_1 = \Delta x_1 = 2 //$$

$$x_2 = \Delta x_2 = 3 //$$

$$x_3 = \Delta x_3 = 4 //$$

| Δx_1 | Δx_2 | Δx_3 | R_1 | R_2 | R_3 |
|--------------|--------------|--------------|-------|-------|-------|
| 0 | 0 | 0 | -31 | -24 | -58 |
| 0 | 0 | 6 | -25 | -30 | 2 |
| 0 | 4 | 0 | -21 | 2 | 18 |
| 2 | 0 | 0 | 3 | 6 | 24 |
| 0 | 0 | -2 | 1 | 8 | 4 |
| 0 | -1 | 0 | 0 | 0 | 0 |

Q2:

$$10x - 2y - 3z = 205$$

$$-2x + 10y - 2z = 154$$

$$-2x - y + 10z = 120$$

Sol: Residuals are

$$R_1 = 10x - 2y - 3z - 205$$

$$R_2 = -2x + 10y - 2z - 154$$

$$R_3 = -2x - y + 10z - 120$$

$$x = \sum \Delta x = 32$$

$$y = \sum \Delta y = 26$$

$$z = \sum \Delta z = 21$$

| Δx | Δy | Δz | R_1 | R_2 | R_3 |
|------------|------------|------------|-------|-------|-------|
| 0 | 0 | 0 | -205 | -154 | -120 |
| 20 | 0 | 0 | -5 | -194 | -160 |
| 0 | 20 | 0 | -45 | 6 | -180 |
| 0 | 0 | 18 | -99 | -30 | 0 |
| 10 | 0 | 0 | 1 | -50 | -20 |
| 0 | 5 | 0 | -9 | 0 | -25 |
| 0 | 0 | 2 | -15 | -4 | -5 |
| 81 | 0 | 0 | -5 | -6 | -7 |
| 0 | 0 | 81 | -8 | -8 | 3 |
| 1 | 0 | 0 | 2 | -10 | 1 |
| 0 | 1 | 0 | 0 | 0 | 0 |

Q1:

Sol:

Q3: $9x - 2y + z = 50$

$$x + 5y - 3z = 18$$

$$-2x + 2y + 7z = 19$$

| Δx | Δy | Δz | R_1 | R_2 | R_3 |
|------------|------------|------------|-------|-------|-------|
| 0 | 0 | 0 | -50 | -18 | -19 |
| 6 | 0 | 0 | 4 | -12 | -31 |
| 0 | 0 | 4 | 8 | -24 | -3 |
| 0 | 5 | 0 | -2 | 1 | 7 |
| 0 | 0 | -1 | -3 | 4 | 0 |
| 0 | -1 | 0 | -1 | -1 | -2 |
| 0 | 0 | 0 | 0 | 0 | 0 |
| 0 | 0 | 0 | 0 | 0 | 0 |

Sol: Residuals are

$$R_1 = 9x - 2y + z - 50$$

$$R_2 = x + 5y - 3z - 18$$

$$R_3 = -2x + 2y + 7z - 19$$

* Rayleigh's Power Method:

It is used to find the largest Eigen value and its corresponding Eigen vector.

Initial Eigen vector can be considered as

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \text{ or } \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \text{ or } \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \text{ or } \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

Q1: Find the largest Eigen value and corresponding Eigen vector for the following matrices by Rayleigh's Power method.

$$A = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix}$$

λ - Eigen Value

x - Eigen Vector

Sol: Let $x^{(0)} = [1 \ 0 \ 0]'$

$$AX^{(0)} = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 0 \\ 0.5 \end{bmatrix} = \lambda^{(0)} x^{(0)}$$

$$AX^{(1)} = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0.5 \end{bmatrix} = \begin{bmatrix} 2.5 \\ 0 \\ 2 \end{bmatrix} = 2.5 \begin{bmatrix} 1 \\ 0 \\ 0.8 \end{bmatrix} = \lambda^{(1)} x^{(1)}$$

$$AX^{(2)} = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0.8 \end{bmatrix} = \begin{bmatrix} 2.8 \\ 0 \\ 2.6 \end{bmatrix} = 2.8 \begin{bmatrix} 1 \\ 0 \\ 0.928 \end{bmatrix} = \lambda^{(2)} x^{(2)}$$

$$AX^{(3)} = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0.928 \end{bmatrix} = \begin{bmatrix} 2.928 \\ 0 \\ 2.856 \end{bmatrix} = 2.928 \begin{bmatrix} 1 \\ 0 \\ 0.975 \end{bmatrix} = \lambda^{(3)} x^{(3)}$$

$$AX^{(4)} = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0.975 \end{bmatrix} = \begin{bmatrix} 2.975 \\ 0 \\ 2.95 \end{bmatrix} = 2.975 \begin{bmatrix} 1 \\ 0 \\ 0.991 \end{bmatrix} = \lambda^{(4)} x^{(4)}$$

$$AX^{(5)} = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0.991 \end{bmatrix} = \begin{bmatrix} 2.991 \\ 0 \\ 2.98 \end{bmatrix} = 2.991 \begin{bmatrix} 1 \\ 0 \\ 0.996 \end{bmatrix} = \lambda^{(6)} x^{(6)}$$

$$AX^{(6)} = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0.996 \end{bmatrix} = \begin{bmatrix} 2.996 \\ 0 \\ 2.992 \end{bmatrix} = 2.996 \begin{bmatrix} 1 \\ 0 \\ 0.998 \end{bmatrix} = \lambda^{(7)} x^{(7)}$$

\therefore the largest eigen value = 2.996

and eigen vector = $[1 \ 0 \ 0.998]'$

Q2: $A = \begin{bmatrix} 25 & 1 & 2 \\ 1 & 3 & 0 \\ 2 & 0 & -4 \end{bmatrix}$

sol: let $x^{(0)} = [1 \ 0 \ 0]'$

$$AX^{(0)} = \begin{bmatrix} 25 & 1 & 2 \\ 1 & 3 & 0 \\ 2 & 0 & -4 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 25 \\ 1 \\ 2 \end{bmatrix} = 25 \begin{bmatrix} 1 \\ 0.04 \\ 0.08 \end{bmatrix} = \lambda^{(1)} x^{(1)}$$

$$AX^{(1)} = \begin{bmatrix} 25 & 1 & 2 \\ 1 & 3 & 0 \\ 2 & 0 & -4 \end{bmatrix} \begin{bmatrix} 1 \\ 0.04 \\ 0.08 \end{bmatrix} = \begin{bmatrix} 25.2 \\ 1.12 \\ 1.68 \end{bmatrix} = 25.2 \begin{bmatrix} 1 \\ 0.044 \\ 0.067 \end{bmatrix} = \lambda^{(2)} x^{(2)}$$

$$AX^{(2)} = \begin{bmatrix} 25 & 1 & 2 \\ 1 & 3 & 0 \\ 2 & 0 & -4 \end{bmatrix} \begin{bmatrix} 1 \\ 0.044 \\ 0.067 \end{bmatrix} = \begin{bmatrix} 25.176 \\ 1.132 \\ 1.732 \end{bmatrix} = 25.176 \begin{bmatrix} 1 \\ 0.044 \\ 0.068 \end{bmatrix} = \lambda^{(3)} x^{(3)}$$

$$AX^{(3)} = \begin{bmatrix} 25 & 1 & 2 \\ 1 & 3 & 0 \\ 2 & 0 & -4 \end{bmatrix} \begin{bmatrix} 1 \\ 0.044 \\ 0.068 \end{bmatrix} = \begin{bmatrix} 25.18 \\ 1.132 \\ 1.728 \end{bmatrix} = 25.18 \begin{bmatrix} 1 \\ 0.044 \\ 0.068 \end{bmatrix} = \lambda^{(4)} x^{(4)}$$

$$AX^{(4)} = \begin{bmatrix} 25 & 1 & 2 \\ 1 & 3 & 0 \\ 2 & 0 & -4 \end{bmatrix} \begin{bmatrix} 1 \\ 0.044 \\ 0.068 \end{bmatrix} = \begin{bmatrix} 25.18 \\ 1.132 \\ 1.728 \end{bmatrix} = 25.18 \begin{bmatrix} 1 \\ 0.044 \\ 0.068 \end{bmatrix} = \lambda^{(5)} x^{(5)}$$

\therefore The largest Eigen value = 25.18

and eigen vector = $[1 \ 0.044 \ 0.068]'$

Q3:

$$A = \begin{bmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

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sol: let $x^{(0)} = [1 \ 0 \ 0]'$

$$AX^{(0)} = \begin{bmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = 1 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \lambda^{(1)} x^{(1)}$$

$$AX^{(1)} = \begin{bmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \\ 0 \end{bmatrix} = -1 \begin{bmatrix} 1 \\ 0.428 \\ 0 \end{bmatrix} = \lambda^{(2)} x^{(2)}$$

$$AX^{(2)} = \begin{bmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0.428 \\ 0 \end{bmatrix} = \begin{bmatrix} 3.568 \\ 1.856 \\ 0 \end{bmatrix} = 3.568 \begin{bmatrix} 1 \\ 0.52 \\ 0 \end{bmatrix} = \lambda^{(3)} x^{(3)}$$

$$AX^{(3)} = \begin{bmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0.52 \\ 0 \end{bmatrix} = \begin{bmatrix} 4.12 \\ 2.04 \\ 0 \end{bmatrix} = 4.12 \begin{bmatrix} 1 \\ 0.495 \\ 0 \end{bmatrix} = \lambda^{(4)} x^{(4)}$$

$$AX^{(4)} = \begin{bmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0.495 \\ 0 \end{bmatrix} = \begin{bmatrix} 3.91 \\ 1.99 \\ 0 \end{bmatrix} = 3.91 \begin{bmatrix} 1 \\ 0.501 \\ 0 \end{bmatrix} = \lambda^{(5)} x^{(5)}$$

$$AX^{(5)} = \begin{bmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0.501 \\ 0 \end{bmatrix} = \begin{bmatrix} 4.006 \\ 2.002 \\ 0 \end{bmatrix} = 4.006 \begin{bmatrix} 1 \\ 0.499 \\ 0 \end{bmatrix} = \lambda^{(6)} x^{(6)}$$

$$AX^{(6)} = \begin{bmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0.499 \\ 0 \end{bmatrix} = \begin{bmatrix} 3.994 \\ 1.998 \\ 0 \end{bmatrix} = 3.994 \begin{bmatrix} 1 \\ 0.500 \\ 0 \end{bmatrix} = \lambda^{(7)} x^{(7)}$$

$$AX^{(7)} = \begin{bmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0.5 \\ 0 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \\ 0 \end{bmatrix} = 4 \begin{bmatrix} 1 \\ 0.5 \\ 0 \end{bmatrix} = \lambda^{(8)} x^{(8)}$$

$$AX^{(8)} = \begin{bmatrix} 1 & 6 & 1 \\ 1 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0.5 \\ 0 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \\ 0 \end{bmatrix} = 4 \begin{bmatrix} 1 \\ 0.5 \\ 0 \end{bmatrix} = \lambda^{(9)} x^{(9)}$$

The largest Eigen value = 4
 and eigen vector = $[1 \ 0.5 \ 0]'$

Q4: $A = \begin{bmatrix} 5 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & 5 \end{bmatrix}$ Take $[1 \ 0 \ 0]'$

Sol: Here $X^{(0)} = [0 \ 1 \ 0]'$

$$AX^{(0)} = \begin{bmatrix} 5 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 5 \\ 0 \\ 1 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 \\ 0 \\ 0.2 \end{bmatrix} = \lambda^{(1)} X^{(1)}$$

$$AX^{(1)} = \begin{bmatrix} 5 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0.2 \end{bmatrix} = \begin{bmatrix} 5.2 \\ 0 \\ 2 \end{bmatrix} = 5.2 \begin{bmatrix} 1 \\ 0 \\ 0.384 \end{bmatrix} = \lambda^{(2)} X^{(2)}$$

$$AX^{(2)} = \begin{bmatrix} 5 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0.384 \end{bmatrix} = \begin{bmatrix} 5.384 \\ 0 \\ 2.92 \end{bmatrix} = 5.384 \begin{bmatrix} 1 \\ 0 \\ 0.542 \end{bmatrix} = \lambda^{(3)} X^{(3)}$$

$$AX^{(3)} = \begin{bmatrix} 5 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0.542 \end{bmatrix} = \begin{bmatrix} 5.542 \\ 0 \\ 3.71 \end{bmatrix} = 5.542 \begin{bmatrix} 1 \\ 0 \\ 0.669 \end{bmatrix} = \lambda^{(4)} X^{(4)}$$

$$AX^{(4)} = \begin{bmatrix} 5 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0.669 \end{bmatrix} = \begin{bmatrix} 5.669 \\ 0 \\ 4.345 \end{bmatrix} = 5.669 \begin{bmatrix} 1 \\ 0 \\ 0.766 \end{bmatrix} = \lambda^{(5)} X^{(5)}$$

$$AX^{(5)} = \begin{bmatrix} 5 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0.766 \end{bmatrix} = \begin{bmatrix} 5.766 \\ 0 \\ 4.83 \end{bmatrix} = 5.766 \begin{bmatrix} 1 \\ 0 \\ 0.837 \end{bmatrix} = \lambda^{(6)} X^{(6)}$$

$$AX^{(6)} = \begin{bmatrix} 5 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0.837 \end{bmatrix} = \begin{bmatrix} 5.837 \\ 0 \\ 5.185 \end{bmatrix} = 5.837 \begin{bmatrix} 1 \\ 0 \\ 0.999 \end{bmatrix} = \lambda^{(7)} X^{(7)}$$

* Regula-Falsi Method / Method of False Position:

Let $f(x)=0$ possess a real root in (a, b) .

Let $f(a) < 0$ and $f(b) > 0$.

Consider two points A $[a, f(a)]$ and B $[b, f(b)]$ on the curve.

Line joining two points is given by :

$$\frac{y - y_1}{x - x_1} = \frac{y_2 - y_1}{x_2 - x_1}$$

$$\Rightarrow \frac{y - f(a)}{x - a} = \frac{f(b) - f(a)}{b - a}$$

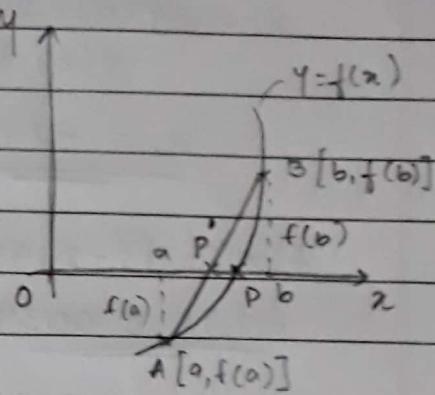
Since P' crosses x axis, $y = 0$

$$\therefore \frac{-f(a)}{x - a} = \frac{f(b) - f(a)}{b - a}$$

$$(x - a)[f(b) - f(a)] = -f(a)[b - a]$$

$$x[f(b) - f(a)] - af(b) + af(a) = -bf(a) + af(a)$$

$$\therefore x = \frac{af(b) - bf(a)}{f(b) - f(a)}$$



Q1: Find the approximate root of $x^3 - 3x + 4 = 0$ using the method of False Position.

$$f(x) = x^3 - 3x + 4 = 0$$

$$f(0) = 4 \quad f(-1) = 6$$

Roots lies in between (-2, 3)

$$f(1) = 2 \quad f(-2) = 2 > 0$$

$$f(-2.1) = 1.039$$

$$f(2) = 6 \quad f(-3) = -14 < 0$$

$$f(-2.2) = -0.048$$

Roots lies in between (-2.1, -2.2)

$$\therefore a = -2.1; f(a) = 1.039$$

$$b = -2.2; f(b) = -0.048$$

$$x_1 = \frac{af(b) - bf(a)}{f(b) - f(a)}$$

1st Approximation

$$x_1 = \frac{af(b) - bf(a)}{f(b) - f(a)} = \frac{-2.1(-0.048) - (-2.2)(1.039)}{-0.048 - 1.039}$$

$$\underline{\underline{x_1 = -2.1955}}$$

$$f(x_1) = f(-2.1955)$$

$$\underline{\underline{f(x_1) = 0.0034}}$$

$$f(-2.2) = -0.048 < 0$$

\therefore The root lies in $(-2.1955, -2.2)$

2nd Approximation

$$x_2 = \frac{af(b) - bf(a)}{f(b) - f(a)} = \frac{-2.1955(-0.048) - (-2.2)(0.0034)}{-0.048 - 0.0034}$$

$$\underline{\underline{x_2 = -2.1958}}$$

Q2: $\tanh x + \tan x = 0$ in $(2, 3)$. Consider 3 iterations.

Sol:

$$f(x) = \tanh x + \tan x$$

$$a = 2 \quad f(a) = -1.221$$

$$b = 3 \quad f(b) = 0.852$$

1st Approximation

$$x_1 = \frac{af(b) - bf(a)}{f(b) - f(a)} = \frac{2(0.852) - 3(-1.221)}{0.852 + 1.221}$$

$$\underline{\underline{x_1 = 2.5890}}$$

$$f(x_1) = \underline{\underline{f(2.5890) = 0.3724}}$$

$$f(2) = -1.221 < 0$$

\therefore The root lies between $[2.589, 2]$

$$a = 2.589$$

$$f(a) = 0.3724$$

$$b = 2$$

$$f(b) = -1.221$$

2nd Approximation

$$x_2 = \frac{af(b) - bf(a)}{f(b) - f(a)} = \frac{2.589(-1.221) - 2(0.372)}{-1.221 - 0.372} = 2.4514$$

$$\underline{\underline{x_2 = 2.4514}}$$

$$f(x_2) = f(2.4514) = 0.1595 > 0$$

$$f(2) = -1.221 < 0$$

\therefore The root lies between $(2.4514, 2)$

$$a = 2.4514 \quad f(a) = 0.1595$$

$$b = 2 \quad f(b) = -1.221$$

3rd Approximation

$$x_3 = \frac{af(b) - bf(a)}{f(b) - f(a)} = \frac{2.4514(-1.221) - 2(0.1595)}{-1.221 - 0.1595}$$

$$\underline{\underline{x_3 = 2.3992}}$$

$$Q3: \lambda \log_{10} x - 1.2 = 0$$

$$\underline{\underline{f(x) = \lambda \log_{10} x - 1.2}}$$

$$f(1) = -1.2$$

$$f(2) = -0.597 < 0$$

$$f(3) = 0.281 > 0$$

Root lies between $(2, 3)$

$$f(2.7) = -0.0353 \quad a = 2.7$$

$$f(2.8) = 0.0520 \quad b = 2.8$$

The root lies in between $(2.7, 2.8)$

1st Approximation

$$x_1 = \frac{af(b) - bf(a)}{f(b) - f(a)} = \frac{2.7(0.0520) - 2.8(-0.0353)}{0.052 + 0.0353}$$

$$\underline{\underline{x_1 = 2.7404}}$$

$$f(x_1) = f(2.7404) = -0.0003 < 0$$

\therefore The root lies between $(2.4404, 2.8)$

2nd Approximation

$$x_2 = \frac{af(b) - bf(a)}{f(b) - f(a)} = \frac{2.4404(+0.0520) - 2.8(-0.0003)}{0.0520 + 0.0003}$$

$$\underline{\underline{x_2 = 2.4405}}$$

$$f(x_2) = f(2.4405) = -0.0001 < 0$$

\therefore The root lies between $(2.4405, 2.8)$

3rd Approximation

$$x_3 = \frac{af(b) - bf(a)}{f(b) - f(a)} = \frac{2.4405(+0.052) - 2.8(-0.0001)}{0.052 + 0.0001}$$

$$\underline{\underline{x_3 = 2.4406}}$$

$$Q4: x^4 + 2x^2 - 16x + 5 = 0 \text{ in } (0, 1).$$

Sol: $f(x) = x^4 + 2x^2 - 16x + 5$

$$a = 0 \quad f(0) = 5$$

$$b = 1 \quad f(1) = -8$$

$$f(0.3) = 0.3881 > 0$$

$$f(0.4) = -1.0544 < 0$$

\therefore The root lies between $(0.3, 0.4)$

1st Approximation

$$x_1 = \frac{af(b) - bf(a)}{f(b) - f(a)} = \frac{0.3(-1.0544) - 0.4(0.3881)}{-1.0544 - 0.3881}$$

$$\underline{\underline{x_1 = 0.3269}}$$

$$f(x_1) = f(0.3269) = -0.0052 < 0$$

\therefore The root lies in between $(0.3, 0.3269)$

2nd Approximation

$$x_2 = \frac{af(b) - bf(a)}{f(b) - f(a)} = \frac{0.3(-0.0052) - 0.3269(0.3881)}{-0.0052 - 0.3881}$$

$$\underline{x_2 = 0.3265}$$

$$f(x_2) = f(0.3265) = \underline{0.0005} > 0$$

The root lies between $(0.3265, 0.3269)$

3rd Approximation

$$x_3 = af(b) - bf(a) = \frac{0.3265(-0.0052)}{f(b) - f(a)} - 0.3269(0.0005) \\ -0.0052 - 0.0052$$

$$\underline{x_3 = 0.3265}$$

* Newton-Raphson Method:

Let $f(x) = 0$

$x_0 \rightarrow$ approximate value

$h \rightarrow$ step length

$x_1 = x_0 + h \rightarrow$ exact value

$$f(x_0 + h) = 0$$

$$f(x_0) + h f'(x_0) + \frac{h^2}{2!} f''(x_0) + \dots = 0 \quad (\text{Taylor's Series})$$

~~error~~

Since h is very small, h^2, h^3, \dots can be neglected

$$f(x_0) + h f'(x_0) = 0$$

$$h = -\frac{f(x_0)}{f'(x_0)}$$

$$x_1 = x_0 + h$$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Q1:

Use Newton-Raphson method to find a real root.
 $x^4 + x^3 - 7x^2 - x + 5 = 0$

Sol:

$$f(1) = -1$$

$$f(2) = -1 < 0$$

$$f(3) = 47 > 0$$

The root lies in between (2, 3)

Let $x_0 = 2$

$$f(x_0) = f(2) = -1$$

$$f'(x_0) = f'(2) = 15$$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$x_1 = 2 - \frac{-1}{15} = \frac{31}{15} = 2.0666 //$$

$$f(x_1) = f(2.0666) = 0.1036$$

$$f'(x_1) = f'(2.0666) = 18.1845$$

$$x_2 = x_1 - \frac{f(x_1)}{f'(x_1)}$$

$$x_2 = 2.0666 - \frac{0.1036}{18.1845} = 2.0609 //$$

$$f(x_2) = f(2.0609) = 0.0008$$

$$f'(x_2) = f'(2.0609) = 17.9024$$

$$x_3 = x_2 - \frac{f(x_2)}{f'(x_2)}$$

$$x_3 = 2.0609 - \frac{0.0008}{17.9024} = 2.0608 //$$

Q2: Desire an iterative formula for finding k^{th} root of a positive number and hence find 4^{th} root of 22.

$$x = \sqrt[k]{N}$$

$$\Rightarrow x = N^{1/k}$$

$$x^k = N \Rightarrow x^k - N = 0$$

$$\therefore f(x) = x^k - N ; f'(x) = kx^{k-1}$$

Newton-Raphson formula

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$= x_n - \frac{x_n^k - N}{kx_n^{k-1}}$$

$$= \frac{kx_n^{k-1}x_n - x_n^k + N}{kx_n^{k-1}}$$

$$= \frac{kx_n^k - x_n^k + N}{kx_n^{k-1}}$$

| | |
|---|--------------------|
| $x_{n+1} = \frac{(k-1)x_n^k + N}{kx_n^{k-1}}$ | Iterative formula. |
|---|--------------------|

4^{th} root of 22. : $\sqrt[4]{22}$

$$\sqrt[4]{16} < \sqrt[4]{22} < \sqrt[4]{81}$$

$\sqrt[4]{22}$ is nearer to $\sqrt[4]{16} = 2$

$$\sqrt[4]{22} : x_0 = 2 ; N = 22 ; k = 4$$

$$x_{n+1} = \frac{(4-1)x_n^4 + 22}{4x_n^{4-1}} \quad \therefore x_{n+1} = \frac{(k-1)x_n^k + N}{kx_n^{k-1}}$$

$$x_{n+1} = \frac{3x_n^4 + 22}{4x_n^3}$$

$$x_{n+1} = \frac{1}{4} \left[3x_n + \frac{22}{x_n^3} \right]$$

1st iteration

$$x_1 = \frac{1}{4} \left[3x_0 + \frac{22}{x_0^3} \right]$$

$$x_1 = \frac{1}{4} \left[3(2) + \frac{22}{8} \right]$$

$$\underline{\underline{x_1 = 2.1875}}$$

2nd iteration

$$x_2 = \frac{1}{4} \left[3x_1 + \frac{22}{x_1^3} \right]$$

$$x_2 = \frac{1}{4} \left[3(2.1875) + \frac{22}{(2.1875)^3} \right]$$

$$\underline{\underline{x_2 = 2.1660}}$$

3rd iteration

$$x_3 = \frac{1}{4} \left[3x_2 + \frac{22}{x_2^3} \right]$$

$$x_3 = \frac{1}{4} \left[3(2.166) + \frac{22}{(2.166)^3} \right]$$

$$\underline{\underline{x_3 = 2.1657}}$$

4th iteration

$$x_4 = \frac{1}{4} \left[3x_3 + \frac{22}{x_3^3} \right]$$

$$x_4 = \frac{1}{4} \left[3(2.1657) + \frac{22}{(2.1657)^3} \right]$$

$$\underline{\underline{x_4 = 2.1657}}$$

$$\therefore \underline{\underline{\sqrt{22} = 2.1657}}$$

Q3: Derive an iterative formula to find reciprocal of square root of a positive number. (22).

$$x = \frac{1}{\sqrt{N}}$$

$$x = \frac{1}{N^{1/2}}$$

$$\Rightarrow x^2 = \frac{1}{N} \Rightarrow \frac{x^2 - 1}{N} = 0$$

$$\therefore f(x) = x^2 - \frac{1}{N} ; f'(x) = 2x$$

Newton Raphson Formula

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$x_{n+1} = x_n - \frac{x_n^2 - 1/N}{2x_n}$$

$$x_{n+1} = \frac{2x_n^2 - x_n^2 + 1/N}{2x_n}$$

$$x_{n+1} = \frac{x_n^2 + 1/N}{2x_n}$$

$$x_{n+1} = \frac{1}{2} \left[x_n + \frac{1}{Nx_n} \right]$$

Iterative formula

$$\frac{1}{\sqrt{16}} < \frac{1}{\sqrt{22}} < \frac{1}{\sqrt{25}}$$

$$\frac{1}{\sqrt{22}} \text{ is nearer to } \frac{1}{\sqrt{25}} = \frac{1}{5} = 0.2$$

$$\therefore x_0 = 0.2, N = 22$$

1st iteration

$$x_1 = \frac{1}{2} \left[x_0 + \frac{1}{N x_0} \right]$$

$$x_1 = \frac{1}{2} \left[0.2 + \frac{1}{22(0.2)} \right]$$

$$\underline{\underline{x_1 = 0.2136}}$$

2nd iteration

$$x_2 = \frac{1}{2} \left[x_1 + \frac{1}{N x_1} \right]$$

$$x_2 = \frac{1}{2} \left[0.2136 + \frac{1}{22(0.2136)} \right]$$

$$\underline{\underline{x_2 = 0.2132}}$$

3rd iteration

$$x_3 = \frac{1}{2} \left[x_2 + \frac{1}{N x_2} \right]$$

$$x_3 = \frac{1}{2} \left[0.2132 + \frac{1}{22(0.2132)} \right]$$

$$\underline{\underline{x_3 = 0.2132}}$$

$$\therefore \frac{1}{\sqrt[3]{22}} = 0.2132 \quad \cancel{\cancel{}}$$

Q4: Find an iterative formula for cube root of a positive number and hence find cube root of 12.

$$x = \sqrt[3]{N}$$

$$x = N^{1/3}$$

$$\Rightarrow x^3 = N \Rightarrow x^3 - N = 0$$

$$\therefore f(x) = x^3 - N \Rightarrow f'(x) = 3x^2$$

Newton Raphson Formula

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

$$x_{n+1} = x_n - \frac{x_n^3 - N}{3x_n^2}$$

$$x_{n+1} = \frac{3x_n^3 - x_n^3 + N}{3x_n^2}$$

$$x_{n+1} = \frac{2x_n^3 + N}{3x_n^2}$$

$$x_{n+1} = \frac{1}{3} \left[2x_n + \frac{N}{x_n^2} \right]$$

Iterative formula

$$\sqrt[3]{8} < \sqrt[3]{12} < \sqrt[3]{24}$$

$\sqrt[3]{12}$ is nearer to $\sqrt[3]{8} = 2$

$$\therefore x_0 = 2 ; N = 12$$

1st iteration

$$x_1 = \frac{1}{3} \left[2x_0 + \frac{N}{x_0^2} \right]$$

$$x_1 = \frac{1}{3} \left[2(2) + \frac{12}{4} \right]$$

$$x_1 = \underline{\underline{2.3333}}$$

2nd Iteration

$$x_2 = \frac{1}{3} \left[2x_1 + \frac{N}{x_1^2} \right]$$

$$x_2 = \frac{1}{3} \left[2(2.3333) + \frac{12}{(2.3333)^2} \right]$$

$$x_2 = \underline{\underline{2.2902}}$$

3rd iteration.

$$x_3 = \frac{1}{3} \left[2x_2 + \frac{N}{x_2^2} \right]$$

$$x_3 = \frac{1}{3} \left[2(2.2902) + \frac{12}{(2.2902)^2} \right]$$

$$\underline{\underline{x_3 = 2.2894}}$$

4th iteration

$$x_4 = \frac{1}{3} \left[2x_3 + \frac{N}{x_3^2} \right]$$

$$x_4 = \frac{1}{3} \left[2(2.2894) + \frac{12}{(2.2894)^2} \right]$$

$$\underline{\underline{x_4 = 2.2894}}$$

Hence $\underline{\underline{\sqrt[3]{12} = 2.2894}}$

UNIT - 5

Numerical Methods - 2

★ FORWARD AND BACKWARD DIFFERENCE:

Consider $y = f(x)$, for different values of x i.e., $x_0, x_1, x_2, \dots, x_n$ corresponding y values will be $y_0, y_1, y_2, \dots, y_n$.

Interpolation is a process where we find a particular value in the set of values given. We consider two differences to find the particular value.

- i. Forward Difference: denoted as Δ

Forward difference table

| x | y | Δy | $\Delta^2 y$ | \dots | $\Delta^n y$ |
|-----------|-----------|--------------------------------------|--------------------|---------|----------------|
| x_0 | y_0 | | | | |
| x_1 | y_1 | $y_1 - y_0 = \Delta y_0$ | $\Delta^2 y_0$ | | |
| x_2 | y_2 | $y_2 - y_1 = \Delta y_1$ | $\Delta^2 y_1$ | | |
| x_3 | y_3 | $y_3 - y_2 = \Delta y_2$ | | | $\Delta^n y_0$ |
| : | : | | | | |
| x_{n-1} | y_{n-1} | $y_{n-1} - y_{n-2} = \Delta y_{n-1}$ | $\Delta^2 y_{n-2}$ | | |
| x_n | y_n | | | | |

Newton's Forward Interpolation Formula

$$y_r = y(x_0 + rh) = y_0 + r\Delta y_0 + \frac{r(r-1)}{2!} \Delta^2 y_0 + \frac{r(r-1)(r-2)}{3!} \Delta^3 y_0 + \dots$$

where $r = \frac{x - x_0}{h}$ $x \rightarrow$ value to be found
 $x_0 \rightarrow$ initial value

$h \rightarrow$ step length

- ii. Backward Difference: Denoted as ∇

Backward Difference Table

| x | y | ∇y | $\nabla^2 y$ | \dots | $\nabla^n y$ |
|-----------|-----------|------------|------------------------------|----------------|----------------|
| x_0 | y_0 | | $y_1 - y_0 = \nabla y_1$ | $\nabla^2 y_2$ | |
| x_1 | y_1 | | $y_2 - y_1 = \nabla y_2$ | $\nabla^2 y_3$ | |
| x_2 | y_2 | | $y_3 - y_2 = \nabla y_3$ | | $\nabla^n y_n$ |
| x_3 | y_3 | | | | |
| \vdots | \vdots | | | | |
| x_{n-1} | y_{n-1} | | $y_n - y_{n-1} = \nabla y_n$ | $\nabla^2 y_n$ | |
| x_n | y_n | | | | |

Newton's Backward Interpolation Formula

$$y_r = y(x_0 + rh) = y_n + r\nabla y_n + \frac{r(r-1)}{2!} \nabla^2 y_n + \frac{r(r-1)(r-2)}{3!} \nabla^3 y_n + \dots$$

where $r = \frac{x - x_n}{h}$. $x \rightarrow$ value to be found
 $x_n \rightarrow$ final value
 $h \rightarrow$ step length

Newton's Forward Interpolation Formula

$$y_r = y(x_0 + rh) = y_0 + r\Delta y_0 + \frac{r(r-1)}{2!} \Delta^2 y_0 + \frac{r(r-1)(r-2)}{3!} \Delta^3 y_0 + \dots$$

diff wrt r

$$y'_r = y'(x_0 + rh)h = 0 + \Delta y_0 + \frac{2r-1}{2!} \Delta^2 y_0 + \frac{3r^2-6r+2}{3!} \Delta^3 y_0 + \dots$$

$$y'(x_0 + rh) = \frac{1}{h} \left[\Delta y_0 + \frac{2r-1}{2} \Delta^2 y_0 + \frac{3r^2-6r+2}{6} \Delta^3 y_0 + \dots \right]$$

diff wrt n

$$y''_r = y''(x_0 + rh).h = \frac{1}{h} \left[0 + \frac{2}{2} \Delta^2 y_0 + \frac{6r-6}{6} \Delta^3 y_0 + \dots \right]$$

$$y''(x_0 + rh) = \frac{1}{h^2} \left[\Delta^2 y_0 + (r-1) \Delta^3 y_0 + \dots \right]$$

Q1:

Use appropriate interpolation formula to compute as indicated for the following data.

| | | | | | | |
|-----|------|------|------|------|------|------|
| x | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 | 0.6 |
| y | 2.68 | 3.04 | 3.38 | 3.68 | 3.96 | 4.21 |

(compute $y(0.25)$,

(value lies in the first half hence forward polation is considered)

Sol:

| x | y | Δy | $\Delta^2 y$ | $\Delta^3 y$ | $\Delta^4 y$ | $\Delta^5 y$ |
|-----|------|------------|--------------|--------------|--------------|--------------|
| 0.1 | 2.68 | | | | | |
| 0.2 | 3.04 | 0.36 | | | | |
| 0.3 | 3.38 | 0.34 | -0.02 | | | |
| 0.4 | 3.68 | 0.30 | -0.04 | -0.02 | 0.04 | |
| 0.5 | 3.96 | 0.28 | -0.02 | 0.02 | -0.03 | -0.04 |
| 0.6 | 4.21 | 0.25 | -0.03 | -0.01 | | |

Newton's Forward Interpolation Formula

$$y_r = y(x_0 + rh) = y_0 + r \Delta y_0 + \frac{r(r-1)}{2!} \Delta^2 y_0 + \frac{r(r-1)(r-2)}{3!} \Delta^3 y_0 \\ + \frac{r(r-1)(r-2)(r-3)}{4!} \Delta^4 y_0 + \frac{r(r-1)(r-2)(r-3)(r-4)}{5!} \Delta^5 y_0$$

$$r = \frac{x - x_0}{h} = \frac{0.25 - 0.1}{0.1} = 1.5$$

$$y_r = 2.68 + \frac{(1.5)(0.36)}{2} + \frac{(1.5)(1.5-1)(-0.02)}{6} + \frac{(1.5)(1.5-1)(1.5-2)(-0.02)}{6} \\ + \frac{(1.5)(1.5-1)(1.5-2)(1.5-3)(0.04)}{24} + \frac{(1.5)(1.5-1)(1.5-2)(1.5-3)(1.5-4)(-0.01)}{120}$$

$$= 2.68 + 0.54 - 0.0075 + 0.0012 + 0.0009 + 0.0008$$

$$\underline{y_0 = 3.215}$$

Q2: Compute $f(1.1)$ and $f'(1.1)$

| x | 1 | 1.2 | 1.4 | 1.6 | 1.8 | 2.0 |
|-----|-----|-------|-------|-------|-------|------|
| y | 0.0 | 0.128 | 0.544 | 1.296 | 2.432 | 4.00 |

| x | Δy | $\Delta^2 y$ | $\Delta^3 y$ | $\Delta^4 y$ | $\Delta^5 y$ |
|-----|------------|--------------|--------------|--------------|--------------|
| 1 | 0.0 | 0.128 | | | |
| 1.2 | 0.128 | 0.416 | 0.288 | 0.048 | 0 |
| 1.4 | 0.544 | 0.752 | 0.336 | -0.048 | 0 |
| 1.6 | 1.296 | 1.136 | 0.384 | 0.048 | 0 |
| 1.8 | 2.432 | 1.568 | 0.432 | | |
| 2.0 | 4.00 | | | | |

Newton's Forward Interpolation formula

$$r = \frac{x - x_0}{h} = \frac{1.1 - 1.0}{0.2} = 0.5$$

$$y_r = y(x_0 + rh) = y_0 + r\Delta y_0 + \frac{r(r-1)}{2!} \Delta^2 y_0 + \frac{r(r-1)(r-2)}{3!} \Delta^3 y_0$$

$$y = 0 + 0.5(0.128) + \frac{0.5(0.5-1)(0.288)}{2} + \frac{0.5(0.5-1)(0.5-2)}{6}(0.048)$$

$$y = 0.064 - 0.0285 + 0.003$$

$$f(1.1) = y = \underbrace{0.0325}$$

$$y' = f'(y) = \frac{1}{h} \left[\Delta y_0 + \frac{2r-1}{2} \Delta^2 y_0 + \frac{3r^2-6r+2}{3} \Delta^3 y_0 \right]$$

$$y' = f'(1.1) = \frac{1}{0.2} \left[0.128 + \frac{2(0.5)-1}{2} (0.288) + \frac{(3(0.5)^2-6(0.5)+2)}{3} (0.048) \right]$$

$$f'(1.1) = \frac{1}{0.2} [0.128 + 0 - 0.004]$$

$$\underline{\underline{f'(1.1) = 0.6200}}$$

Q3: $f(0) = 0, f(2) = 4, f(4) = 56, f(6) = 204, f(8) = 496, f(10) = 980$
 Find the interpolating polynomial and hence evaluate $f(3)$ and $f(7)$.

| x | y | I diff | II diff | III diff | IV diff | V diff |
|----|-----|--------|---------|----------|---------|--------|
| 0 | 0 | | | | | |
| 2 | 4 | 4 | | | | |
| 4 | 56 | 52 | 18 | | | |
| 6 | 204 | 148 | 96 | 48 | 0 | |
| 8 | 496 | 292 | 144 | 48 | 0 | 0 |
| 10 | 980 | 484 | 192 | | | |

Nekaten's Forward Interpolation formula

$$y_0 = y(x_0 + rh) = y_0 + r_1 \Delta y_0 + \frac{r(r-1)}{2!} \Delta^2 y_0 + \frac{r(r-1)(r-2)}{3!} \Delta^3 y_0.$$

$$r = \frac{x - x_0}{h} = \frac{x - 0}{2} = \frac{x}{2}$$

$$y = 0 + \frac{x}{2} (4) + \frac{x}{2} \left(\frac{x}{2}-1\right) \frac{1}{2!} (18) + \frac{x}{2} \left(\frac{x}{2}-1\right) \left(\frac{x}{2}-2\right) \frac{1}{3!} (48)$$

$$y = 2x + 6x(x-2) + x(x-2)(x-4)$$

$$y = 2x + 6x^2 - 12x + (x^2 - 2x)(x-4)$$

$$y = 6x^2 + x^3 - 4x^2 - 2x^2 + 8x - 10x$$

$$y = x^3 - 5x^2 + 8x$$

$$y = x^3 - 2x$$

$$f(x) = y = x^3 - 2x$$

$$f(3) = 3^3 - 2(3) = 21$$

~~$$f(7) = 7^3 - 2(7) = 329$$~~

For $f(7)$ consider backward difference

Q4: Find the number of students securing

- less than 45 marks
- between 40 and 45 marks

| Marks | 30-40 | 40-50 | 50-60 | 60-70 | 70-80 |
|----------------|-------|-------|-------|-------|-------|
| No of students | 31 | 42 | 51 | 85 | 31 |

Sol: Number of students securing less than 40 is 31 students
 $\therefore < 40 = 31$

$$\text{similarly } < 50 = 31 + 42 = 73$$

$$< 60 = 73 + 51 = 124$$

$$< 70 = 124 + 85 = 159$$

$$< 80 = 159 + 31 = 190$$

Newton's forward difference table

| x | y | Δy_0 | $\Delta^2 y_0$ | $\Delta^3 y_0$ | $\Delta^4 y_0$ |
|-----|-----|--------------|----------------|----------------|----------------|
| 40 | 31 | 42 | | | |
| 50 | 73 | | 9 | | |
| 60 | 124 | 51 | -16 | -25 | |
| 70 | 159 | 35 | -4 | 12 | |
| 80 | 190 | 31 | | | |

Newton's forward interpolation formula.

$$y_r = y(x_0 + rh) = y_0 + r\Delta y_0 + \frac{r(r-1)}{2!} \Delta^2 y_0 + \frac{r(r-1)(r-2)}{3!} \Delta^3 y_0 + \frac{r(r-1)(r-2)(r-3)}{4!} \Delta^4 y_0$$

$$r = \frac{x - x_0}{h} = \frac{45 - 40}{10} = 0.5$$

$$y_r = 31 + 0.5(42) + \frac{0.5(0.5-1)(9)}{2} + \frac{0.5(0.5-1)(0.5-2)(-25)}{6} + \frac{0.5(0.5-1)(0.5-2)(0.5-3)(34)}{24}$$

$$Y_1 = 31 + 21 - 1.125 - 1.5625 - 1.4953$$

$$Y_2 = 44.8672$$

$$\underline{Y_3 \approx 48}$$

$$b. f(45) - f(40) = 48 - 31 = 17$$

17 students have secured between 40 and 45.

Q5: A rod is rotating in a plane. The following table is the angle θ radians through which the rod has turned for various values of the time 't' sec.

| t | 0 | 0.2 | 0.4 | 0.6 | 0.8 | 1.0 | 1.2 |
|----------|---|------|------|------|------|------|------|
| θ | 0 | 0.12 | 0.49 | 1.12 | 2.02 | 3.20 | 4.67 |

Calculate the angular velocity and angular acceleration of the rod when $t = 0.68$ sec

| <u>sol:</u> | t | θ | $\Delta\theta_0$ | $\Delta^2\theta_0$ | $\Delta^3\theta_0$ | $\Delta^4\theta_0$ | $\Delta^5\theta_0$ | $\Delta^6\theta_0$ |
|-------------|-----|----------|------------------|--------------------|--------------------|--------------------|--------------------|--------------------|
| | 0 | 0 | | | | | | |
| | 0.2 | 0.12 | 0.12 | | | | | |
| | 0.4 | 0.49 | 0.37 | 0.25 | 0.01 | 0 | | |
| | 0.6 | 1.12 | 0.63 | 0.26 | 0.01 | 0 | 0 | |
| | 0.8 | 2.02 | 0.90 | 0.24 | 0.01 | 0 | 0 | |
| | 1.0 | 3.20 | 1.18 | 0.28 | 0.01 | 0 | | |
| | 1.2 | 4.67 | 1.47 | 0.29 | | | | |

$$\text{Ans} \quad g_c = \frac{x - x_0}{h} = \frac{0.6 - 0}{0.2} = 3$$

$$\omega = \frac{d\theta}{dt} = \dot{\theta}_0 = \frac{1}{h} \left[\Delta\theta_0 + \frac{(2r-1)}{2!} \Delta^2\theta_0 + \frac{(3r^2-6r+2)}{3!} \Delta^3\theta_0 \right]$$

$$\omega = \frac{1}{0.2} \left[0.12 + \frac{(2 \times 3 - 1)}{2} (0.25)^2 + \frac{(3(3)^2 - 6(3) + 2)}{6} (0.01)^2 \right]$$

$$\omega = \frac{1}{0.2} \left[0.12 + 0.6250 + 0.0083 \right]$$

$$v = 3.816 \text{ rad/sec}$$

$$a = \frac{d^2\theta}{dt^2} = \ddot{\theta} = \frac{1}{h^2} \left[\Delta^2 \theta_0 + \frac{(6n-6)}{3!} \Delta^3 \theta_0 \right]$$

$$a = \frac{1}{0.2^2} \left[0.25 + \frac{(6(3)-6)}{6} (0.01) \right]$$

$$a = \frac{1}{0.2^2} [0.25 + 0.02]$$

$$a = 6.35 \text{ rad/sec}^2$$

* Lagrange's Interpolation Formula:

$$y = f(x) = \frac{(x-x_0)(x-x_1)\dots(x-x_n)}{(x_0-x_1)(x_0-x_2)\dots(x_0-x_n)} y_0 + \frac{(x-x_0)(x-x_1)(x-x_3)\dots(x-x_n)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)\dots(x_1-x_n)} y_1 \\ + \dots + \frac{(x-x_0)(x-x_1)\dots(x-x_{n-1})}{(x_n-x_0)(x_n-x_1)\dots(x_n-x_{n-1})} y_n$$

Q1: Use Lagrange's interpolation formula to find an interpolating polynomial. Hence find $f(5)$

| x | $1 x_0$ | $3 x_1$ | $4 x_2$ | $6 x_3$ |
|--------|---------|---------|----------|-----------|
| $f(x)$ | $3 y_0$ | $9 y_1$ | $30 y_2$ | $132 y_3$ |

$$\text{Sol: } y = f(x) = \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} y_0 + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} y_1 \\ + \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} y_2 + \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} y_3$$

$$f(x) = \frac{(x-3)(x-4)(x-6)}{(1-3)(1-4)(1-6)} (3) + \frac{(x-1)(x-4)(x-6)}{(3-1)(3-4)(3-6)} (9)$$

$$+ \frac{(x-1)(x-3)(x-6)}{(4-1)(4-3)(4-6)} (30) + \frac{(x-1)(x-3)(x-4)}{(6-1)(6-3)(6-4)} (132)$$

$$f(x) = -0.1(x-3)(x-4)(x-6) + 1.5(x-1)(x-4)(x-6) \\ - 5(x-1)(x-3)(x-6) + 4 \cdot 4(x-1)(x-3)(x-4)$$

wkt

$$(x-a)(x-b)(x-c) = x^3 - (a+b+c)x^2 + (ab+bc+ca)x - abc$$

$$\therefore f(x) = -0.1[x^3 - (13x^2 + 54x - 12)] \\ + 1.5[x^3 - (11x^2 + 34x - 24)] \\ - 5[x^3 - (10x^2 + 24x - 18)] \\ + 4 \cdot 4[x^3 - (8x^2 + 19x - 12)]$$

$$f(x) = -0.1x^3 + 1.3x^2 - 5.4x + 1.2 \\ + 1.5x^3 - 16.5x^2 + 51x - 36 \\ - 5x^3 + 50x^2 - 135x + 90 \\ + 4 \cdot 4x^3 - 35.2x^2 + 83.6x - 52.8$$

$$f(x) = \underline{\underline{0.8x^3 - 0.4x^2 - 5.8x + 8.4}}$$

$$f(5) = (0.8)5^3 - 0.4(5)^2 - 5.8(5) + 8.4 = 69.4 //$$

| x | 0^{x_0} | 1^{x_1} | 2^{x_2} | 5^{x_3} |
|--------|-----------|-----------|-----------|-----------|
| $f(x)$ | $2y_0$ | $3y_1$ | $12y_2$ | $147y_3$ |

Find $f(3)$

sol:

$$y = f(x) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} y_0 + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} y_1 \\ + \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} y_2 + \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} y_3$$

$$f(x) = \frac{(x-1)(x-2)(x-5)}{(0-1)(0-2)(0-5)} (2) + \frac{(x-0)(x-2)(x-5)}{(1-0)(1-2)(1-5)} (3) \\ + \frac{(x-0)(x-1)(x-5)}{(2-0)(2-1)(2-5)} (12) + \frac{(x-0)(x-1)(x-2)}{(5-0)(5-1)(5-2)} (147)$$

$$f(x) = -0.2(x-1)(x-2)(x-5) + 0.45(x-0)(x-2)(x-5) \\ -2(x-0)(x-1)(x-5) + 2.45(x-0)(x-1)(x-2)$$

since

$$(x-a)(x-b)(x-c) = x^3 - (a+b+c)x^2 + (ab+bc+ca)x - abc$$

$$f(x) = -0.2[x^3 - 8x^2 + 17x - 10] \\ + 0.45[x^3 - 4x^2 + 10x - 0] \\ - 2[x^3 - 6x^2 + 5x - 0] \\ + 2.45[x^3 - 3x^2 + 2x - 0]$$

$$f(x) = -0.2x^3 + 1.6x^2 - 3.4x + 2 \\ + 0.45x^3 - 5.25x^2 + 4.5x + 0 \\ - 2x^3 + 12x^2 - 10x + 0 \\ + 2.45x^3 - 4.35x^2 + 4.9x - 0$$

$$f(x) = x^3 + x^2 - x + 2 // \\ f(3) = 3^3 + 3^2 - 3 + 2 = 35 //$$

| | x_0 | x_1 | x_2 | x_3 | |
|-----|--------|--------|--------|--------|------------------|
| Q3: | x | 654 | 658 | 659 | 661 |
| | $f(x)$ | 2.8156 | 2.8182 | 2.8189 | 2.8202 |
| | y_0 | y_1 | y_2 | y_3 | compute $y(656)$ |

so1:

$$y = f(x) = \frac{(x-x_1)(x-x_2)(x-x_3)}{(x_0-x_1)(x_0-x_2)(x_0-x_3)} y_0 + \frac{(x-x_0)(x-x_2)(x-x_3)}{(x_1-x_0)(x_1-x_2)(x_1-x_3)} y_1 \\ + \frac{(x-x_0)(x-x_1)(x-x_3)}{(x_2-x_0)(x_2-x_1)(x_2-x_3)} y_2 + \frac{(x-x_0)(x-x_1)(x-x_2)}{(x_3-x_0)(x_3-x_1)(x_3-x_2)} y_3$$

$$f(656) = \frac{(656-658)(656-659)(656-661)}{(654-658)(654-659)(654-661)} (2.8156) \\ + \frac{(656-654)(656-659)(656-661)}{(658-654)(658-659)(658-661)} (2.8182) \\ + \frac{(656-654)(656-658)(656-661)}{(659-654)(659-658)(659-661)} (2.8189)$$

$$+ \frac{(656 - 654)(656 - 658)(656 - 659)}{(661 - 654)(661 - 658)(661 - 659)} (2.8202)$$

$$\begin{aligned} f(656) &= \frac{(-2)(-3)(-5)(2.8156)}{(-4)(-5)(-7)} + \frac{(2)(-3)(-5)(2.8182)}{(4)(-1)(-3)} \\ &+ \frac{(2)(-2)(-5)(2.8189)}{(5)(1)(-2)} + \frac{(2)(-2)(-3)(2.8202)}{(4)(3)(2)} \end{aligned}$$

$$f(656) = 0.6033 + 4.0455 - 5.6388 + 0.8057$$

$$\underline{\underline{f(656) = 2.8164}}$$

* Newton's Divided difference Formula:

Divided difference table.

| x | $y = f(x)$ | 1 DD | 1L DD | |
|-----------|------------------------|---|--|---------------------------|
| x_0 | $y_0 = f(x_0)$ | | | |
| x_1 | $y_1 = f(x_1)$ | $f(x_0, x_1) = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$ | $f(x_0, x_1, x_2) = \frac{f(x_1, x_2) - f(x_0, x_1)}{x_2 - x_0}$ | |
| x_2 | $y_2 = f(x_2)$ | $f(x_1, x_2) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$ | $f(x_1, x_2, x_3) = \frac{f(x_2, x_3) - f(x_1, x_2)}{x_3 - x_1}$ | $f(x_0, x_1, \dots, x_n)$ |
| x_3 | $y_3 = f(x_3)$ | $f(x_2, x_3) = \frac{f(x_3) - f(x_2)}{x_3 - x_2}$ | | $\dots, x_{n-1}, x_n)$ |
| | | : | : | |
| | | : | : | |
| | | : | : | |
| | | | | |
| x_{n-1} | $y_{n-1} = f(x_{n-1})$ | | $f(x_{n-2}, x_{n-1}, x_n) = \frac{f(x_{n-1}, x_n) - f(x_{n-2}, x_{n-1})}{x_n - x_{n-2}}$ | |
| | | $f(x_{n-1}, x_n) = \frac{f(x_n) - f(x_{n-1})}{x_n - x_{n-1}}$ | | |
| | $y_n = f(x_n)$ | | | |

Q11: Newton's Divided Difference Formula

$$y = f(x_0) + (x - x_0) f(x_0, x_1) + (x - x_0)(x - x_1) f(x_0, x_1, x_2) \\ + (x - x_0)(x - x_1)(x - x_2) f(x_0, x_1, x_2, x_3) + \dots$$

Q1: Use Newton's general interpolation formula to

compute $f(310)$

| | | | | |
|--------|--------|--------|--------|--------|
| x | 300 | 304 | 305 | 307 |
| $f(x)$ | 2.4771 | 2.4829 | 2.4843 | 2.4871 |

| <u>sol:</u> | x | $f(x)$ | 1 DD | II DD | III DD |
|-------------|-----|--------|---------|--------------------|----------|
| | 300 | 2.4771 | | | |
| | 304 | 2.4829 | 0.00145 | 0.00001 | |
| | 305 | 2.4843 | 0.0014 | -0.00001 | 0.000001 |
| | 307 | 2.4871 | 0.0014 | 0 | |

$$y = f(x_0) + (x - x_0) f(x_0, x_1) + (x - x_0)(x - x_1) f(x_0, x_1, x_2) \\ + (x - x_0)(x - x_1)(x - x_2) f(x_0, x_1, x_2, x_3) + \dots$$

$$y = 2.4771 + (310 - 300) [0.00145] + (310 - 300)(310 - 304) [0.00001] \\ + (310 - 300)(310 - 304)(310 - 305) (0.000001)$$

$$y = 2.4771 + 0.0145 - 0.0006 + 0.0003$$

$$\underline{y = 2.4913}$$

| | | | | | | | |
|-----|--------|----|-----|-----|-----|------|------|
| Q2: | x | 4 | 5 | 7 | 10 | 11 | 13 |
| | $f(x)$ | 48 | 100 | 294 | 900 | 1210 | 2028 |

compute $f(8)$, and $f'(8)$ and $f''(8)$.

| <u>sol:</u> | x | $f(x)$ | I DD | II DD | III DD | IV DD | V DD |
|-------------|-----|--------|------|-------|--------|-------|------|
| | 4 | 48 | 52 | | | | |
| | 5 | 100 | | 15 | 1 | | |
| | 7 | 294 | 197 | 21 | 1 | 0 | 0 |
| | 10 | 900 | 202 | 27 | 1 | 0 | |
| | 11 | 1210 | 310 | | 1 | | |
| | 13 | 2028 | 409 | 33 | | | |

$$y = f(x_0) + (x-x_0)f(x_0, x_1) + (x-x_0)(x-x_1)f(x_0, x_1, x_2) \\ + (x-x_0)(x-x_1)(x-x_2)f(x_0, x_1, x_2, x_3)$$

$$y = 48 + (8-4)52 + (8-4)(8-5)15 + (8-4)(8-5)(8-7)1$$

$$y = 48 + 208 + 180 + 12$$

$$\underline{\underline{y = 448}}$$

$$f(x) = f(x_0) + (x-x_0)f(x_0, x_1) + (x-x_0)(x-x_1)f(x_0, x_1, x_2) \\ + (x-x_0)(x-x_1)(x-x_2)f(x_0, x_1, x_2, x_3).$$

diff w.r.t x

$$f'(x) = 0 + f(x_0, x_1) + [(x-x_0) + (x-x_1)]f(x_0, x_1, x_2) \\ + [(x-x_0)(x-x_1) + (x-x_2)(x-x_0) + (x-x_1)(x-x_2)] \\ f(x_0, x_1, x_2, x_3).$$

$$f'(8) = 52 + [(8-4) + (8-5)]15 + [(8-4)(8-5) + (8-7)(8-4) + (8-5)(8-7)]$$

$$f'(8) = 52 + 105 + 19$$

$$\underline{\underline{f'(8) = 176}}$$

$$\text{d}^2 f \text{ wrt } x \\ f''(x) = 2f(x_0, x_1, x_2) + [(x-x_1) + (x-x_0) + (x-x_1) + (x-x_2)] f(x_0, x_1, x_2, x_3)$$

$$+ (x-x_2) + (x-x_0) f(x_0, x_1, x_2, x_3)$$

$$f''(18) = 2(15) + [18-5) + (18-4) + (18-5) + (18-7) + (18-7) + (18-4)]$$

$$f''(18) = 30 + 76$$

$$\underline{\underline{f''(18) = 106}}$$

Q3:

| | | | | | |
|-----|------|----|---|---|------|
| x | -4 | -1 | 0 | 2 | 5 |
| y | 1245 | 33 | 5 | 9 | 1335 |

Determine $f(x)$ as a polynomial.

~~sol:~~

| x | y | I DD | II DD | III DD | IV DD |
|-----|------|------|-------|--------|-------|
| -4 | 1245 | -404 | | | |
| -1 | 33 | -28 | 948 | -14 | 3 |
| 0 | 5 | 2 | 10 | 13 | |
| 2 | 9 | | 88 | | |
| 5 | 1335 | 442 | | | |

$$y = f(x) = f(x_0) + (x-x_0) f(x_0, x_1) + (x-x_0)(x-x_1) f(x_0, x_1, x_2)$$

$$+ (x-x_0)(x-x_1)(x-x_2) f(x_0, x_1, x_2, x_3)$$

$$+ (x-x_0)(x-x_1)(x-x_2)(x-x_3) f(x_0, x_1, x_2, x_3, x_4)$$

$$f(x) = 1245 + (x+4)(-404) + (x+4)(x+1)(948)$$

$$+ (x+4)(x+1)x(-14) + (x+4)(x+1)x(x-2)(3)$$

$$f(x) = 1245 - 404x - 1616 + 94x^2 + 440x + 346$$

$$- 14x^3 - 70x^2 - 56x + 3x^4 + 15x^3 + 12x^2 - 6x^3 - 30x^2 - 21$$

$$\underline{\underline{f(x) = 3x^4 + 5x^3 + 6x^2 - 14x + 5}}$$

* Numerical Integration:

1. - Simpson's $\frac{1}{3}$ rd Rule: $I = \int_a^b f(x) dx$

$$I = \frac{h}{3} \left[(y_0 + y_n) + 4(y_1 + y_3 + y_5 + \dots) + 2(y_2 + y_4 + y_6 + \dots) \right]$$

$$\text{where } h = \frac{b-a}{n}$$

Q1: Use Simpson's $\frac{1}{3}$ rule to evaluate

$$\int_0^5 \frac{dx}{4x+5} \text{ taking 10 equal parts.}$$

Hence find $\log_e 5$.

Sol: $h = \frac{b-a}{n} = \frac{5-0}{10} = 0.5$

$$y = \frac{1}{4x+5}$$

$$h = \frac{5-0}{10} = \underline{\underline{0.5}}$$

| x | 0 | 0.5 | 1 | 1.5 | 2 | 2.5 | 3 | 3.5 | 4 | 4.5 | 5 |
|----------------------|-----|--------|--------|--------|--------|--------|--------|--------|--------|--------|------|
| $y = \frac{1}{4x+5}$ | 0.2 | 0.1428 | 0.1111 | 0.0909 | 0.0769 | 0.0666 | 0.0588 | 0.0526 | 0.0476 | 0.0434 | 0.04 |

$$I = \frac{0.5}{3} \left[\frac{h}{3} \left(y_0 + y_{10} \right) + 4(y_1 + y_3 + y_5 + y_7 + y_9) + 2(y_2 + y_4 + y_6 + y_8) \right]$$

$$I = \frac{0.5}{3} \left[(0.2 + 0.04) + 4(0.1428 + 0.0909 + 0.0666 + 0.0526 + 0.0434) + 2(0.1111 + 0.0769 + 0.0588 + 0.0476) \right]$$

$$I = 0.1666 \left[0.24 + 1.5852 + 0.5888 \right]$$

$$\underline{\underline{I = 0.4023}}$$

$$\begin{aligned}
 \int_0^5 \frac{dx}{4x+5} &= \frac{1}{4} \int_0^5 \frac{1}{(x+\frac{5}{4})} dx = \frac{1}{4} \left[\log\left(x + \frac{5}{4}\right) \right]_0^5 \\
 &= \frac{1}{4} \left[\log \frac{25}{4} - \log \frac{5}{4} \right] \\
 &= \frac{1}{4} \left[\frac{\log(25/4)}{\log(5/4)} \right] \\
 &= \frac{1}{4} \log 5
 \end{aligned}$$

Comparing with numerical solution

$$\frac{1}{4} \log_e 5 = 0.4023$$

$$\underline{\log_e 5 = 1.6092}$$

Q2: Evaluate $\int_0^{0.6} e^{-x^2} dx$ taking $n = 6$

Sol: $h = \frac{b-a}{n}$ Here $n = 6$

$$h = \frac{0.6-0}{6} = 0.1$$

| x | 0 | 0.1 | 0.2 | 0.3 | 0.4 | 0.5 | 0.6 |
|----------------|-------|--------|--------|--------|--------|--------|--------|
| $y = e^{-x^2}$ | 1 | 0.9900 | 0.9607 | 0.9139 | 0.8521 | 0.7788 | 0.6976 |
| | y_0 | y_1 | y_2 | y_3 | y_4 | y_5 | y_6 |

$$\begin{aligned}
 I &= \frac{h}{3} \left[(y_0 + y_6) + 4(y_1 + y_3 + y_5) + 2(y_2 + y_4) \right] \\
 &= \frac{0.1}{3} \left[(1 + 0.6976) + 4(0.99 + 0.9139 + 0.7788) + 2(0.9607 + 0.8521) \right] \\
 &= \frac{0.1}{3} [1.6976 + 10.7308 + 3.6256] \\
 I &= \underline{0.5351}
 \end{aligned}$$

Q3: $I = \int_0^{\pi/2} \sqrt{\cos \theta} d\theta \quad n=6$

Sol: $h = \frac{b-a}{n}$ here $n=6$

$$h = \frac{\pi/2 - 0}{6} = \frac{\pi}{12} \approx 15^\circ$$

| x | 0° | 15° | 30° | 45° | 60° | 75° | 90° |
|--------------------------|-----------|------------|------------|------------|------------|------------|------------|
| $y = \sqrt{\cos \theta}$ | 1 | 0.9828 | 0.9306 | 0.8408 | 0.7071 | 0.5087 | 0 |
| | y_0 | y_1 | y_2 | y_3 | y_4 | y_5 | y_6 |

$$I = \frac{h}{3} \left[(y_0 + y_6) + 4(y_1 + y_5 + y_3) + 2(y_2 + y_4) \right]$$

$$= \frac{3.14}{12 \times 3} \left[(1+0) + 4(0.9828 + 0.8408 + 0.5087) + 2(0.9306 + 0.7071) \right]$$

$$= \frac{3.14}{36} \left[1 + 9.3292 + 3.2454 \right]$$

$$\underline{I = 1.1866}$$

2 - Simpson's 3/8 th Rule:

$$I = \frac{3h}{8} \left[(y_0 + y_n) + 3(y_1 + y_2 + y_4 + y_5 + y_7 + \dots) + 2(y_3 + y_6 + \dots) \right]$$

Q1: Use Simpson's 3/8 th rule to evaluate $\int_0^{0.3} \sqrt{1-8x^3} dx$ by taking 7 ordinates = 6 equal parts.

Sol: $h = \frac{b-a}{n} = \frac{0.3-0}{6} = \frac{0.05}{\underline{\underline{}}}$

| x | 0 | 0.05 | 0.1 | 0.15 | 0.2 | 0.25 | 0.3 |
|---------------------|-------|--------|--------|--------|--------|--------|--------|
| $y = \sqrt{1-8x^3}$ | 1 | 0.9994 | 0.9959 | 0.9864 | 0.9674 | 0.9354 | 0.8854 |
| | y_0 | y_1 | y_2 | y_3 | y_4 | y_5 | y_6 |

$$I = \frac{3h}{8} \left[(y_0 + y_6) + 3(y_1 + y_2 + y_4 + y_5) + 2(y_3) \right]$$

$$I = \frac{3(0.05)}{8} \left[(1+0.8854) + 3(0.9994 + 0.9959 + 0.9674 + 0.9359) + 2(0.9864) \right]$$

$$I = \frac{0.15}{8} [1.8854 + 11.6943 + 1.9728]$$

$$\underline{\underline{I = 0.2916}}$$

Q2: $\int_2^8 \frac{dx}{\log_{10} x}$ taking $h=1$

Sol: $h = \frac{b-a}{n} \Rightarrow n = \frac{b-a}{h} = \frac{8-2}{1} = \underline{\underline{6}}$

| x | $\underline{\underline{0}}$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---------------------------------|-----------------------------|-----|--------|--------|--------|--------|--------|--------|--------|
| $\underline{\underline{h = 1}}$ | $\underline{\underline{0}}$ | | 3.3219 | 2.0959 | 1.6609 | 1.4306 | 1.2850 | 1.1832 | 1.1073 |
| $\log_{10} x$ | | | y_0 | y_1 | y_2 | y_3 | y_4 | y_5 | y_6 |

$$I = \frac{3h}{8} \left[(y_0 + y_6) + 3(y_1 + y_2 + y_4 + y_5) + 2(y_3) \right]$$

$$= \frac{3(1)}{8} \left[(0.4292 + 1.2850) + 3(1.4292 + 1.1073 + 1.1832 + 1.4306) \right]$$

$$= \frac{3}{8} [4.4292 + 18.675 + 2.8612]$$

$$\underline{\underline{I = 9.4340}}$$

Q3: $\int_0^3 \frac{dx}{(1+x)^2}$ taking $n=3$

Sol: $h = \frac{b-a}{n} = \frac{3-0}{3} = 1$

| x | 0 | 1 | 2 | 3 |
|-------------------------|-------|-------|-------|-------|
| $y = \frac{1}{(1+x^2)}$ | y_0 | y_1 | y_2 | y_3 |

$$I = \frac{3h}{8} [(y_0 + y_3) + 3(y_1 + y_2)]$$

$$I = \frac{3(1)}{8} [(1+0.0625) + 3(0.25+0.1111)]$$

$$\underline{\underline{I = 0.8046}}$$

* Weddle's Rule

$$I = \frac{3h}{10} [y_0 + 5y_1 + y_2 + 6y_3 + y_4 + 5y_5 + y_6]$$

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$$I = \frac{3h}{10} \sum_k y_i \quad \text{where } k = 1, 5, 1, 6, 1, 5, 2, 5, 1, 6, 1, 5$$

Q1: Use Weddle's rule to evaluate the following.

$$\int_{\frac{5}{4}}^{\frac{5}{2}} \log_e x dx \text{ taking 6 equal strips}$$

$$\underline{\text{sol:}} \quad h = \frac{b-a}{n} = \frac{5.2-4}{6} = 0.2$$

| x | 4 | 4.2 | 4.4 | 4.6 | 4.8 | 5.0 | 5.2 |
|----------------|--------|--------|--------|--------|--------|--------|--------|
| $y = \log_e x$ | 1.3862 | 1.4350 | 1.4816 | 1.5260 | 1.5686 | 1.6094 | 1.6486 |

$$I = \frac{3h}{10} [y_0 + 5y_1 + y_2 + 6y_3 + y_4 + 5y_5 + y_6]$$

$$= \frac{3(0.2)}{10} [1.3862 + 5(1.4350) + 1.4816 + 6(1.5260) + 1.5686 \\ + 5(1.6094) + 1.6486]$$

$$\underline{\underline{I = 1.8247}}$$

Q2: $\int_0^1 \frac{x dx}{1+x^2}$ taking 7 ordinates and hence find $\log_e 2$

$$\text{Sol: } h = \frac{b-a}{n} = \frac{1-0}{6} = 0.1666 = \frac{1}{6}$$

| $x =$ | 0 | $1/6$ | $2/6$ | $3/6$ | $4/6$ | $5/6$ | 1 |
|-----------------------|---|--------|--------|--------|--------|--------|--------|
| $y = \frac{x}{1+x^2}$ | 0 | 0.1621 | 0.3000 | 0.4000 | 0.4615 | 0.4918 | 0.5000 |

$$I = \frac{3h}{10} [y_0 + 5y_1 + y_2 + 6y_3 + y_4 + 5y_5 + y_6]$$

$$I = \frac{3}{6 \times 10} [0 + 5(0.1621) + 0.3 + 6(0.4) + 0.4615 + 5(0.4918) + 0.5]$$

$$\underline{\underline{I = 0.3465}}$$

$$\int_0^1 \frac{x dx}{1+x^2} = \frac{1}{2} \int_0^1 \frac{2x}{1+x^2} dx$$

$$0.3465 = \frac{1}{2} \log_e |1+x^2| \Big|_0^1$$

$$0.6930 = [\log_e 2 - \log_e 1]$$

$$\cancel{\cancel{0.6930}} \Rightarrow \underline{\underline{\log_e 2 = 0.6930}}$$

Q3: $\int_0^{\pi/2} \frac{\sin x}{\sqrt{\sin x + \sqrt{\sin x}}} dx$ taking $n=6$

$$\text{Sol: } h = \frac{b-a}{n} = \frac{\pi/2 - 0}{6} = \frac{\pi}{12}$$

| x | 0 | $\pi/12$ | $\pi/6$ | $\pi/4$ | $\pi/3$ | $5\pi/12$ | $\pi/2$ |
|--------------------------|-------|----------|---------|---------|---------|-----------|---------|
| $y = \sqrt{\sin x}$ | 0 | 0.3410 | 0.4314 | 0.5 | 0.5682 | 0.6589 | 1 |
| $\sqrt{\sin x + \cos x}$ | y_0 | y_1 | y_2 | y_3 | y_4 | y_5 | y_6 |

$$I = \frac{3h}{10} [y_0 + 5y_1 + y_2 + 6y_3 + y_4 + 5y_5 + y_6]$$

$$= \frac{3\pi}{12 \times 10} [0 + 5(0.3410) + 0.4317 + 6(0.5) + 0.5682 + 5(0.6589) + 1]$$

$$\underline{I = 0.7853}$$

Q4: A curve is drawn to pass through the points given by the following table.

| x | 1 | 1.5 | 2 | 2.5 | 3 | 3.5 | 4 |
|-----|---|-----|-----|-----|---|-----|-----|
| y | 2 | 2.4 | 2.7 | 2.8 | 3 | 2.6 | 2.1 |

Estimate the area bounded by the curve between the x axis and the lines $x=1$, $x=4$ by applying Weddell's rule

$$I = \frac{3h}{10} [y_0 + 5y_1 + y_2 + 6y_3 + y_4 + 5y_5 + y_6]$$

$$I = \frac{3(0.5)}{10} [2 + 5(2.4) + 2.7 + 6(2.8) + 3 + 5(3.5) + 2.1]$$

$$\underline{I = 7.44 \text{ sq units}}$$

5: A solid of revolution is formed by rotating about the x axis, the area between the x axis, the lines $x=0$ and $x=1$ and a curve through the points with the coordinates

| x | 0.00 | 0.25 | 0.50 | 0.75 | 1.0 |
|-----|------|--------|--------|--------|--------|
| y | 1 | 0.9896 | 0.9589 | 0.9089 | 0.8415 |

Estimate the volume of the solid formed using

Sol:

$$h = 0.25$$

Simpson's $\frac{1}{3}$ rd rule

$$I = \frac{h}{3} [(y_0 + y_4) + 4(y_1 + y_3) + 2(y_2)]$$

$$I = \frac{0.25}{3} [(1 + 0.8415) + 4(0.9896 + 0.9089) + 2(0.9589)]$$

$$\underline{I = 0.9461} \quad (\text{is the solution}) \quad \text{where } n = 4 \text{ multiple of 2.}$$

Simpson's $\frac{3}{8}$ th rule

$$I = \frac{3h}{8} [(y_0 + y_4) + 3(y_1 + y_2) + 2(y_3)]$$

$$\underline{I = \frac{3(0.25)}{8} [(1 + 0.8415) + 3(0.9896 + 0.9589) + 2(0.9089)]}$$

$$\underline{\underline{I = 0.8910}} \quad X$$

For $\frac{1}{3}$ rd : n is multiples of 2

For $\frac{3}{8}$ th : n is multiples of 3

For Weddell's : n is multiples of 5.

UNIT - 6

Z Transforms

If $u_n = f(n)$ defined for all $n = 0, 1, 2, \dots$ and $u_n = 0$ if $n < 0$, then Z transforms of u_n denoted by $Z_T[u_n]$ and is defined as

$$Z_T[u_n] = \sum_{n=0}^{\infty} u_n z^{-n} = \bar{u}(z)$$

$$Z_T[u_n] = \bar{u}(z)$$

$$\Rightarrow u_n = Z_T^{-1}[\bar{u}(z)] \quad \text{Inverse Z-transforms}$$

* Z-transforms of some standard functions:

$$\begin{aligned} 1. \quad Z_T[k^n] &= \sum_{n=0}^{\infty} k^n z^{-n} \\ &= \sum_{n=0}^{\infty} \left(\frac{k}{z}\right)^n \\ &= 1 + \frac{k}{z} + \left(\frac{k}{z}\right)^2 + \left(\frac{k}{z}\right)^3 + \dots \end{aligned}$$

$$\text{wkt } 1 + r + r^2 + \dots = \frac{1}{1-r} \quad (\text{GP})$$

$$Z_T[k^n] = \frac{1}{1 - k/z}$$

| |
|----------------------------|
| $Z_T[k^n] = \frac{z}{z-k}$ |
|----------------------------|

$$\begin{aligned} 2. \quad Z_T[1] &= \sum_{n=0}^{\infty} 1 \cdot z^{-n} \\ &= \sum_{n=0}^{\infty} \frac{1}{z^n} \\ &= 1 + \frac{1}{z} + \frac{1}{z^2} + \dots \end{aligned}$$

$$\therefore Z_T[1] = \frac{1}{1 - 1/z}$$

$$Z_T[1] = \frac{z}{z-1}$$

3. show that $Z_T[n^k] = -z \frac{d}{dz} Z_T(n^{k-1})$

considering RHS

$$\text{RHS: } -z \frac{d}{dz} Z_T(n^{k-1}) = -z \frac{d}{dz} \sum_{n=0}^{\infty} n^{k-1} z^{-n}$$

$$= -z \sum_{n=0}^{\infty} n^{k-1} \frac{d}{dz} z^{-n}$$

$$= -z \sum_{n=0}^{\infty} n^{k-1} (-n) z^{-n-1}$$

$$= z \sum_{n=0}^{\infty} n^k z^{-n-1}$$

$$= z \sum_{n=0}^{\infty} \frac{n^k z^{-n}}{z}$$

$$= \sum_{n=0}^{\infty} n^k z^{-n}$$

$$= \underline{\underline{Z_T[n^k]}} = \text{LHS} \quad \text{Hence proved}$$

4. $Z_T[n] = \sum_{n=0}^{\infty} n z^{-n}$

but $\sum_{n=0}^{\infty} n^k z^{-n} = -z \frac{d}{dz} Z_T(n^{k-1})$

Taking $k=1$

$$Z_T[n] = -z \frac{d}{dz} Z_T[1]$$

$$z_T[n] = -z \frac{d}{dz} \left[\frac{z}{z-1} \right]$$

$$= -z \left[\frac{(z-1) - z}{(z-1)^2} \right]$$

$$\boxed{z_T[n] = \frac{z}{(z-1)^2}}$$

5. $z_T[n^2]$

$$\text{wkt } z_T[n^k] = -z \frac{d}{dz} z_T[n^{k-1}]$$

Here $n = 2$

$$z_T[n^2] = -z \frac{d}{dz} z_T[n]$$

$$\text{wkt } z_T[n] = \frac{z}{(z-1)^2}$$

$$\therefore z_T[n^2] = -z \frac{d}{dz} \left[\frac{z}{(z-1)^2} \right]$$

$$= -z \left[\frac{(z-1)^2 - 2(z-1)z}{(z-1)^4} \right]$$

$$= -z \left[\frac{z-1 - 2z}{(z-1)^3} \right]$$

$$\boxed{z_T[n^2] = \frac{z^2 + z}{(z-1)^3}}$$

6. $z_T[n^3]$

$$\text{wkt } z_T[n^k] = -z \frac{d}{dz} z_T[n^{k-1}]$$

Here $n = 3$

$$z_T[n^3] = -z \frac{d}{dz} z_T[n^2]$$

$$\text{wkt } z_T[n^2] = \frac{z^2 + z}{(z-1)^3}$$

$$\begin{aligned}
 \therefore Z_T[n^3] &= -z \frac{d}{dz} \left[\frac{z^2+z}{(z-1)^3} \right] \\
 &= -z \left[\frac{(z^2+z)^3}{(z-1)^6} - 3(z-1)^2(z^2+z) \right] \\
 &= -z \left[\frac{z^6+3z^5-3z^2-3z}{(z-1)^4} \right] \\
 Z_T[n^3] &= \boxed{\frac{z^3+4z^2-3z}{(z-1)^4}}
 \end{aligned}$$

* Linearity Property:

If U_n and V_n be any two discrete valued functions
then $Z_T[C_1 U_n + C_2 V_n] = C_1 Z_T[U_n] + C_2 Z_T[V_n]$

Considering

$$LHS = Z_T[C_1 U_n + C_2 V_n]$$

$$= \sum_{n=0}^{\infty} [C_1 U_n + C_2 V_n] z^{-n}$$

$$= \sum_{n=0}^{\infty} C_1 U_n z^{-n} + \sum_{n=0}^{\infty} C_2 V_n z^{-n}$$

$$= C_1 \sum_{n=0}^{\infty} U_n z^{-n} + C_2 \sum_{n=0}^{\infty} V_n z^{-n}$$

$$= C_1 Z_T[U_n] + C_2 Z_T[V_n] = RHS$$

* Damping Rule:

If $Z_T[U_n] = \bar{u}(z)$, then:

$$1. Z_T[k^n U_n] = \bar{u}\left[\frac{z}{k}\right]$$

$$2. Z_T[k^{-n} U_n] = \bar{u}[kz]$$

Proof 1:

$$\begin{aligned} Z_T [k^n u_n] &= \sum_{n=0}^{\infty} [k^n u_n] z^{-n} \\ &= \sum_{n=0}^{\infty} u_n \left[\frac{z}{k} \right]^{-n} \end{aligned}$$

$$Z_T [k^n u_n] = \bar{u} \left[\frac{z}{k} \right]$$

Proof 2:

$$\begin{aligned} Z_T [k^{-n} u_n] &= \sum_{n=0}^{\infty} [k^{-n} u_n] z^{-n} \\ &= \sum_{n=0}^{\infty} u_n [z k]^{-n} \end{aligned}$$

$$Z_T [k^{-n} u_n] = \bar{u} [k z]$$

* Shifting Property:

Right shifting Property:

If $Z_T [u_n] = \bar{u}(z)$, then $Z_T [u_{n-k}] = z^{-k} \bar{u}(z)$

considering

$$LHS = Z_T [u_{n-k}] = \sum_{n=k}^{\infty} u_{n-k} z^{-n} \quad \text{for all } n < k \quad u_{n-k} = 0$$

$$= \sum_{n=k}^{\infty} u_0 z^{-k} + u_1 z^{-k+1} + u_2 z^{-k+2} + \dots$$

$$= z^{-k} [u_0 + u_1 z^{-1} + u_2 z^{-2} + \dots]$$

$$= z^{-k} \left[\sum_{n=0}^{\infty} u_n z^{-n} \right]$$

$$Z_T [u_{n-k}] = z^{-k} \bar{u}(z) = RHS.$$

Left Shifting Property:

If $Z_T[u_n] = \bar{u}(z)$ then $Z_T[u_{n+k}] = z^k \bar{u}(z) - u_0 - u_1 z^{-1} - u_2 z^{-2} - \dots - u_{k-1} z^{-(k-1)}$

considering

$$\text{LHS: } Z_T[u_{n+k}] = \sum_{n=0}^{\infty} u_{n+k} z^{-n}$$

$$= z^k \sum_{n=0}^{\infty} u_{n+k} z^{-(n+k)}$$

$$= z^k [u_k z^{-k} + u_{k+1} z^{-(k+1)} + u_{k+2} z^{-(k+2)} + \dots]$$

Adding and subtracting $u_0 + u_1 z^{-1} + u_2 z^{-2} + \dots$

$$= z^k [u_0 + u_1 z^{-1} + u_2 z^{-2} + \dots + u_k z^{-k} + u_{k+1} z^{-(k+1)} + \dots - u_0 - u_1 z^{-1} - u_2 z^{-2} - \dots - u_{k-1} z^{-(k-1)}]$$

$$= z^k [\sum u_n z^{-n} - (u_0 + u_1 z^{-1} + u_2 z^{-2}, \dots)]$$

$$Z_T[u_{n+k}] = z^k [\bar{u}(z) - u_0 - u_1 z^{-1} - u_2 z^{-2} - \dots - u_{k-1} z^{-(k-1)}]$$

CASE 1: $Z_T[u_{n+1}] = z [\bar{u}(z) - u_0]$

CASE 2: $Z_T[u_{n+2}] = z^2 [\bar{u}(z) - u_0 - u_1 z^{-1}]$

* $Z_T[(n+1)^2]$

$$\text{wkt: } Z_T[n^2] = \frac{z^2 + z}{(z-1)^2} = \bar{u}(z)$$

$$u_n = n^2 \Rightarrow u_{n+1} = (n+1)^2$$

$$Z_T[u_n] = \bar{u}(z)$$

$$Z_T[u_{n+1}] = z [\bar{u}(z) - u_0]$$

$$Z_T [(n+1)^2] = z \left[\frac{z^2 + z}{(z-1)^3} \right] - 0$$

$$Z_T [(n+1)^2] = \frac{z^3 + z^2}{(z-1)^3}$$

* show that $Z_T \left[\frac{1}{n!} \right] = e^{1/z}$

LHS

$$\begin{aligned} Z_T \left[\frac{1}{n!} \right] &= \sum_{n=0}^{\infty} \frac{1}{n!} z^{-n} \\ &= 1 + \frac{z^{-1}}{1!} + \frac{z^{-2}}{2!} + \frac{z^{-3}}{3!} + \dots \\ &= 1 + \frac{1}{z} + \frac{1}{2z^2} + \frac{1}{6z^3} + \dots \end{aligned}$$

$$\text{RHS } e^{\frac{t}{z}} = 1 + \frac{t}{z} + \frac{t^2}{2!} + \frac{t^3}{3!}$$

$$\therefore Z_T \left[\frac{1}{n!} \right] = e^{1/z} //$$

* Obtain Z_T of $\cos n\theta$ and $\sin n\theta$. Hence deduce Z_T of $a^n \cos n\theta$, $a^n \sin n\theta$.

$$\text{wkt } e^{i\theta} = \cos n\theta + i \sin n\theta$$

 $Z_T[\cos n\theta]$

$$e^{i\theta} = [e^{i\theta}]^n = a^n \text{ where } a = e^{i\theta}$$

$$Z_T[a^n] = \frac{z}{z-a}$$

$$\begin{aligned} Z_T[(e^{i\theta})^n] &= \frac{z}{z-e^{i\theta}} \times \frac{z-e^{-i\theta}}{z-e^{-i\theta}} \\ &= \frac{z^2 - ze^{-i\theta}}{z^2 - ze^{-i\theta} - ze^{i\theta} + 1} \\ &= \frac{z[z - (\cos\theta - i\sin\theta)]}{z^2 - z(e^{-i\theta} + e^{i\theta}) + 1} \end{aligned}$$

$$= \frac{z[(x-\cos\theta) + i(\sin\theta)]}{x^2 - 2x\cos\theta + 1}$$

$$Z_T[\cos n\theta + i\sin n\theta] = \frac{z[(x-\cos\theta)]}{x^2 - 2x\cos\theta + 1} + i \frac{z\sin\theta}{x^2 - 2x\cos\theta + 1}$$

$$Z_T[\cos n\theta] + i Z_T[\sin n\theta] = \frac{z[(x-\cos\theta)]}{x^2 - 2x\cos\theta + 1} + i \frac{z\sin\theta}{x^2 - 2x\cos\theta + 1}$$

$$\therefore Z_T[\cos n\theta] = \frac{z[(x-\cos\theta)]}{x^2 - 2x\cos\theta + 1} //$$

$$Z_T[\sin n\theta] = \frac{z\sin\theta}{x^2 - 2x\cos\theta + 1} //$$

$$\text{let } Z_T[\cos n\theta] = \bar{u}(x)$$

$$Z_T[\sin n\theta] = \bar{v}(x)$$

$$Z_T[a^n \cos n\theta] = \bar{u}\left[\frac{z}{a}\right]$$

By damping rule

$$Z_T[a^n \sin n\theta] = \bar{v}\left[\frac{z}{a}\right]$$

$$\therefore Z_T[a^n \cos n\theta] = \frac{\bar{I}/a[z/a - \cos\theta]}{(z/a)^2 - 2z\cos\theta/a + 1}$$

$$= \frac{az[x-\cos\theta]}{x^2 - 2xz\cos\theta + a^2} //$$

$$\therefore Z_T[a^n \sin n\theta] = \frac{z\sin\theta/a}{(z/a)^2 - 2z\cos\theta/a + 1}$$

$$= \frac{az\sin\theta}{x^2 - 2xz\cos\theta + a^2} //$$

* Obtain Z_T of $\cosh n\theta$ and $\sinh n\theta$

$$\cosh n\theta = \frac{e^{n\theta} + e^{-n\theta}}{2} = \frac{1}{2} [(e^\theta)^n + (e^{-\theta})^n]$$

$$\frac{1}{2} Z_T [(e^\theta)^n + (e^{-\theta})^n] = \frac{1}{2} [Z_T [(e^\theta)^n] + Z_T [(e^{-\theta})^n]]$$

$$= \frac{1}{2} \left[\frac{z}{z-e^\theta} + \frac{z}{z-e^{-\theta}} \right]$$

$$= \frac{z}{2} \left[\frac{z-e^{-\theta} + z-e^\theta}{z^2 - ze^{-\theta} - ze^\theta + 1} \right]$$

$$= \frac{z}{2} \left[\frac{2z - (e^\theta + e^{-\theta})}{z^2 - z(e^\theta + e^{-\theta}) + 1} \right]$$

$$= \frac{z}{2} \left[\frac{2z - 2\cosh\theta}{z^2 - 2z\cosh\theta + 1} \right] = z \left[\frac{z - \cosh\theta}{z^2 - 2z\cosh\theta + 1} \right] //$$

similarly

$$\sinh n\theta = \frac{e^{n\theta} - e^{-n\theta}}{2} = \frac{1}{2} [(e^\theta)^n - (e^{-\theta})^n]$$

$$\frac{1}{2} Z_T [(e^\theta)^n - (e^{-\theta})^n] = \frac{1}{2} [Z_T [(e^\theta)^n] - Z_T [(e^{-\theta})^n]]$$

$$= \frac{1}{2} \left[\frac{z}{z-e^\theta} - \frac{z}{z-e^{-\theta}} \right]$$

$$= \frac{z}{2} \left[\frac{z-e^{-\theta} - z+e^\theta}{z^2 - ze^{-\theta} - ze^\theta + 1} \right]$$

$$= \frac{z}{2} \left[\frac{e^\theta - e^{-\theta}}{z^2 - z(e^\theta + e^{-\theta}) + 1} \right]$$

$$= \frac{z}{2} \left[\frac{2 \sinh\theta}{z^2 - 2z\cosh\theta + 1} \right] = z \left[\frac{\sinh\theta}{z^2 - 2z\cosh\theta + 1} \right] //$$

* Find Z_T of $\cos\left(\frac{n\pi}{2} + \frac{\pi}{4}\right)$

$$\cos\left(\frac{n\pi}{2} + \frac{\pi}{4}\right) = \cos\frac{n\pi}{2} \cos\frac{\pi}{4} - \sin\frac{n\pi}{2} \sin\frac{\pi}{4}$$

$$\begin{aligned} \cos\left(\frac{n\pi}{2} + \frac{\pi}{4}\right) &= \cos\frac{n\pi}{2} \cdot \frac{1}{\sqrt{2}} - \sin\frac{n\pi}{2} \cdot \frac{1}{\sqrt{2}} \\ &= \frac{1}{\sqrt{2}} \left(\cos\frac{n\pi}{2} - \sin\frac{n\pi}{2} \right) \end{aligned}$$

$$\begin{aligned} Z_T \left[\cos\left(\frac{n\pi}{2} + \frac{\pi}{4}\right) \right] &= \frac{1}{\sqrt{2}} \left[Z_T \left[\cos\frac{n\pi}{2} \right] - Z_T \left[\sin\frac{n\pi}{2} \right] \right] \\ &= \frac{1}{\sqrt{2}} \left[Z_T \left[e^{in\pi/2} \right] \right] \end{aligned}$$

$$e^{in\pi/2} = \cos\frac{n\pi}{2} + i \sin\frac{n\pi}{2}$$

$$\begin{aligned} Z_T \left[(e^{in\pi/2})^n \right] &= \frac{z}{z - e^{in\pi/2}} \times \frac{z - e^{-in\pi/2}}{z - e^{-in\pi/2}} \\ &= \frac{z [z - e^{-in\pi/2}]}{z^2 - ze^{-in\pi/2} - ze^{in\pi/2} + 1} \end{aligned}$$

$$= \frac{z [z - e^{-in\pi/2}]}{z^2 - z(e^{in\pi/2} + e^{-in\pi/2}) + 1} = \frac{z [z - e^{in\pi/2}]}{z^2 - 2z \cos\pi/2 + 1}$$

$$= \frac{z [z - e^{in\pi/2}]}{z^2 + 1}$$

$$= \frac{z^2}{z^2 + 1} - \frac{ze^{in\pi/2}}{z^2 + 1}$$

$$= \frac{z^2}{z^2 + 1} - \frac{z(\cos\pi/2 - i\sin\pi/2)}{z^2 + 1}$$

$$= \frac{z^2}{z^2 + 1} + i \frac{z}{z^2 + 1}$$

$$\therefore Z_T \left[\cos\frac{n\pi}{2} \right] = \frac{z^2}{z^2 + 1} \quad \therefore Z_T \left[\sin\frac{n\pi}{2} \right] = \frac{z}{z^2 + 1}$$

$$\begin{aligned}\therefore Z_T \left[\cos\left(\frac{n\pi}{2} + \frac{\pi}{4}\right) \right] &= \frac{1}{\sqrt{2}} \left[\frac{z^2}{z^2+1} - \frac{z}{z^2+1} \right] \\ &= \frac{1}{\sqrt{2}} \left[\frac{z^2-z}{z^2+1} \right] //\end{aligned}$$

* Find Z_T of $\sin(3n+5)$

$$- Z_T [\sin(3n+5)]$$

$$\sin(3n+5) = \sin 3n \cos 5 + \cos 3n \sin 5$$

$$\begin{aligned}Z_T [\sin(3n+5)] &= Z_T [\sin 3n \cos 5] + Z_T [\cos 3n \sin 5] \\ &= \cos 5 Z_T [\sin 3n] + \sin 5 Z_T [\cos 3n]\end{aligned}$$

——— (1)

$$e^{im} = \cos 3n + i \sin 3n$$

$$\begin{aligned}Z_T [(e^{3i})^n] &= \frac{z}{z-e^{3i}} \times \frac{z-e^{-3i}}{z-e^{-3i}} \\ &= \frac{z [z-e^{-3i}]}{z^2 - z e^{-3i} - z e^{3i} + 1} \\ &= \frac{z [z-e^{-3i}]}{z^2 - z(e^{3i} + e^{-3i}) + 1} \\ &= \frac{z [z-e^{-3i}]}{z^2 - 2z \cos 3 + 1} \\ &= \frac{z [z - (\cos 3 - i \sin 3)]}{z^2 - 2z \cos 3 + 1} \\ &= \frac{z^2 - z \cos 3}{z^2 - 2z \cos 3 + 1} + i \frac{z \sin 3}{z^2 - 2z \cos 3 + 1}\end{aligned}$$

$$\therefore Z_T [\cos 3n] = \frac{z(z-\cos 3)}{z^2 - 2z \cos 3 + 1} //$$

$$Z_T [\sin 3n] = \frac{z \sin 3}{z^2 - 2z \cos 3 + 1} //$$

Substituting in eq ①

$$\begin{aligned}
 \therefore Z_T[\sin(3t+5)] &= \cos 5 \frac{z \sin 3}{z^2 - 2z \cos 3 + 1} + \sin 5 \frac{(z(z - \cos 3))}{z^2 - 2z \cos 3 + 1} \\
 &= \frac{z \sin 3 \cos 5 + z^2 \sin 5 - z \cos 3 \sin 5}{z^2 - 2z \cos 3 + 1} \\
 &= \frac{z(\sin 3 \cos 5 - \cos 3 \sin 5) + z^2 \sin 5}{z^2 - 2z \cos 3 + 1} \\
 &= \frac{z \sin(3-5) + z^2 \sin 5}{z^2 - 2z \cos 3 + 1} \\
 &= \frac{z(\pi \sin 5 - \sin 2)}{z^2 - 2z \cos 3 + 1}
 \end{aligned}$$

* ~~error made in expression~~

* Initial Value Theorem:

State and Prove Initial Value Theorem also find expressions for u_1, u_2, u_3 .

Statement:

If $Z_T[u_n] = \bar{u}(z)$ then $\lim_{z \rightarrow \infty} \bar{u}(z) = u_0$

Proof:

$$\text{wkt } Z_T[u_n] = \sum_{n=0}^{\infty} u_n z^{-n}$$

$$\begin{aligned}
 Z_T[u_n] &= \bar{u}(z) = u_0 + u_1 z^{-1} + u_2 z^{-2} + u_3 z^{-3} + \dots \\
 \bar{u}(z) &= u_0 + \frac{u_1}{z} + \frac{u_2}{z^2} + \frac{u_3}{z^3} + \dots \quad \text{--- ①}
 \end{aligned}$$

Applying limit

$$\lim_{z \rightarrow \infty} \bar{u}(z) = \lim_{z \rightarrow \infty} \left[u_0 + \frac{u_1}{z} + \frac{u_2}{z^2} + \frac{u_3}{z^3} + \dots \right]$$

$$\therefore \lim_{z \rightarrow \infty} \bar{u}(z) = u_0$$

considering eq ①

$$\bar{u}(z) - u_0 = \frac{u_1}{z} + \frac{u_2}{z^2} + \frac{u_3}{z^3} + \dots$$

$$z [\bar{u}(z) - u_0] = u_1 + \frac{u_2}{z} + \frac{u_3}{z^2} + \dots$$

Applying limit

$$\lim_{z \rightarrow \infty} [z [\bar{u}(z) - u_0]] = \lim_{z \rightarrow \infty} \left[u_1 + \frac{u_2}{z} + \frac{u_3}{z^2} + \dots \right]$$

$$\lim_{z \rightarrow \infty} z [\bar{u}(z) - u_0] = u_1$$

From eq ①

$$\bar{u}(z) - u_0 - \frac{u_1}{z} = \frac{u_2}{z^2} + \frac{u_3}{z^3} + \frac{u_4}{z^4} + \dots$$

$$z^2 \left[\bar{u}(z) - u_0 - \frac{u_1}{z} \right] = u_2 + \frac{u_3}{z} + \frac{u_4}{z^2} + \dots$$

Applying limit

$$\lim_{z \rightarrow \infty} z^2 \left[\bar{u}(z) - u_0 - \frac{u_1}{z} \right] = \lim_{z \rightarrow \infty} \left[u_2 + \frac{u_3}{z} + \frac{u_4}{z^2} + \dots \right]$$

$$\lim_{z \rightarrow \infty} z^2 \left[\bar{u}(z) - u_0 - \frac{u_1}{z} \right] = u_2$$

From eq ①

$$\bar{u}(z) - u_0 - \frac{u_1}{z} - \frac{u_2}{z^2} = \frac{u_3}{z^3} + \frac{u_4}{z^4} + \dots$$

$$z^3 \left[\bar{u}(z) - u_0 - \frac{u_1}{z} - \frac{u_2}{z^2} \right] = u_3 + \frac{u_4}{z} + \dots$$

Applying limit

$$\lim_{z \rightarrow \infty} z^3 \left[\bar{u}(z) - u_0 - \frac{u_1}{z} - \frac{u_2}{z^2} \right] = \lim_{z \rightarrow \infty} \left[u_3 + \frac{u_4}{z} + \dots \right]$$

$$\lim_{z \rightarrow \infty} z^3 \left[\bar{u}(z) - u_0 - \frac{u_1}{z} - \frac{u_2}{z^2} \right] = u_3$$

* Final Value Theorem:

State and prove final value theorem

Statement:

$$\text{If } z_T[u_n] = \bar{u}(z) \text{ then } \lim_{z \rightarrow 1} [(z-1)\bar{u}(z)] = \lim_{n \rightarrow \infty} u_n$$

Proof:

$$\text{wkt } z_T[u_{n+1}] = z[\bar{u}(z) - u_0] \quad \dots \quad (1)$$

$$z_T[u_n] = \bar{u}(z) \quad \dots \quad (2)$$

Subtracting eq (2) from eq (1)

$$z_T[u_{n+1}] - z_T[u_n] = z[\bar{u}(z) - u_0] - \bar{u}(z)$$

$$z_T[u_{n+1} - u_n] = \bar{u}(z)(z-1) - zu_0$$

$$\sum_{n=0}^{\infty} [u_{n+1} - u_n] z^{-n} = (z-1)\bar{u}(z) - zu_0$$

Applying limit $z \rightarrow 1$ on both sides

$$\lim_{z \rightarrow 1} \sum_{n=0}^{\infty} [u_{n+1} - u_n] z^{-n} = \lim_{z \rightarrow 1} (z-1)\bar{u}(z) - zu_0$$

$$\sum_{n=0}^{\infty} [u_{n+1} - u_n] = \lim_{z \rightarrow 1} [(z-1)\bar{u}(z)] - u_0$$

$$\lim_{z \rightarrow 1} [(z-1)\bar{u}(z)] = \sum_{n=0}^{\infty} [u_{n+1} - u_n] + u_0$$

Expressing $\bar{u}(z)$

$$= u_0 + \lim_{m \rightarrow \infty} \sum_{n=0}^m [u_{n+1} - u_n]$$

$$= u_0 + \lim_{m \rightarrow \infty} [u_1 + u_2 + \dots + u_{m+1} - u_0 - u_1 - u_2 - \dots - u_m]$$

$$= u_0 + \lim_{m \rightarrow \infty} [-u_0 + u_{m+1}]$$

$$\lim_{z \rightarrow 1} [(z-1)\bar{u}(z)] = \lim_{m \rightarrow \infty} u_{m+1}$$

as $m \rightarrow \infty$

$m+1 \rightarrow \infty$

: R.O.P.S.

$$\therefore \underset{m \rightarrow \infty}{\lim} u_{m+1} \approx \underset{m \rightarrow \infty}{\lim} u_{m+1}$$

$$\therefore \boxed{\underset{x \rightarrow 1}{\lim} [(x-1) \bar{u}(x)] = \underset{n \rightarrow \infty}{\lim} u_n}$$

Q: If $\bar{u}(x) = \frac{2x^2 + 3x + 4}{(x-3)^3}$, find u_1, u_2, u_3

S1: Initial value theorem

$$u_0 = \underset{x \rightarrow \infty}{\lim} \bar{u}(x) = \underset{x \rightarrow \infty}{\lim} \left[\frac{2x^2 + 3x + 4}{(x-3)^3} \right]$$

$$u_0 = \underset{x \rightarrow \infty}{\lim} \frac{x^2 [2 + 3/x + 4/x^2]}{x^3 (1 - 3/x)^3}$$

$$u_0 = \underset{x \rightarrow \infty}{\lim} \frac{2 + 3/x + 4/x^2}{x (1 - 3/x)^3} = 0 //$$

$$u_1 = \underset{x \rightarrow \infty}{\lim} x [\bar{u}(x) - u_0]$$

$$u_1 = \underset{x \rightarrow \infty}{\lim} x \left[\frac{2x^2 + 3x + 4}{(x-3)^3} - 0 \right]$$

$$u_1 = \underset{x \rightarrow \infty}{\lim} \frac{x [x^2 (2 + 3/x + 4/x^2)]}{x^3 (1 - 3/x)^3} = 2 //$$

$$u_2 = \underset{x \rightarrow \infty}{\lim} x^2 \left[\bar{u}(x) - u_0 - \frac{u_1}{x} \right]$$

$$u_2 = \underset{x \rightarrow \infty}{\lim} x^2 \left[\frac{2x^2 + 3x + 4}{(x-3)^3} - 0 - \frac{2}{x} \right]$$

$$u_2 \rightarrow \underset{x \rightarrow \infty}{\lim} \frac{x^3}{x^3} \left[\frac{2 + 3/x + 4/x^2}{(1 - 3/x)^3} - 2 \right]$$

$$u_2 = \underset{x \rightarrow \infty}{\lim} \frac{2x^4 + 3x^3 + 4x^2 - 2x(x-3)^3}{(x-3)^3}$$

$$u_2 = \underset{x \rightarrow \infty}{\lim} \frac{2x^4 + 3x^3 + 4x^2 - 2x(x^3 - 9x^2 + 27x - 27)}{(x-3)^3}$$

$$U_1 = \lim_{z \rightarrow \infty} \frac{2z^4 + 3z^3 + 4z^2 - 2z^4 + 18z^3 - 54z^2 + 54z}{(z-3)^3}$$

$$U_2 = \lim_{z \rightarrow \infty} \frac{21z^3 - 50z^2 + 54z}{(z-3)^3}$$

$$U_2 = \lim_{z \rightarrow \infty} z^3 \frac{(21 - 50/z + 54/z^2)}{z^3 (1 - 3/z)^3} = 21 //$$

$$U_3 = \lim_{z \rightarrow \infty} z^3 \left[\bar{u}(z) - U_0 - \frac{U_1}{z} - \frac{U_2}{z^2} \right]$$

$$U_3 = \lim_{z \rightarrow \infty} z^3 \left[\frac{2z^2 + 3z + 4}{(z-3)^3} - 0 - \frac{2}{z} - \frac{21}{z^2} \right]$$

$$U_3 = \lim_{z \rightarrow \infty} \left[\frac{2z^5 + 3z^4 + 4z^3 - 2z^5 + 18z^4 - 42z^3 + 54z^2 - 21z^4 + 189z^3 - 231z^2}{(z-3)^3} \right]$$

$$U_3 = \lim_{z \rightarrow \infty} \left[\frac{2z^5 + 3z^4 + 4z^3 - (2z^2 + 21z)(z^3 - 9z^2 + 21z - 21)}{(z-3)^3} \right]$$

$$U_3 = \lim_{z \rightarrow \infty} \left[\frac{2z^5 + 3z^4 + 4z^3 - 2z^5 + 18z^4 - 42z^3 + 54z^2 - 21z^4 + 189z^3 - 231z^2}{(z-3)^3} \right] \quad 567z$$

$$U_3 = \lim_{z \rightarrow \infty} \left[\frac{139z^3 - 177z^2 - 567z}{(z-3)^3} \right]$$

$$U_3 = \lim_{z \rightarrow \infty} z^3 \left[\frac{139 - 177/z - 567/z^2}{z^3 (1 - 3/z)^3} \right] = 139 //$$

Q: If $\bar{u}(z) = \frac{2z^2 + 5z + 14}{(z-1)^4}$, show that $U_0 + 6U_2 + 1 = U_3$

$$\text{Sol: } U_0 = \lim_{z \rightarrow \infty} \bar{u}(z)$$

$$U_0 = \lim_{z \rightarrow \infty} \left[\frac{2z^2 + 5z + 14}{(z-1)^4} \right]$$

$$U_0 = \lim_{z \rightarrow \infty} z^4 \left[\frac{2 + 5/z + 14/z^2}{z^4 (1 - 1/z)^4} \right] = 0 //$$

$$u_1 = \lim_{z \rightarrow \infty} z [\bar{u}(z) - u_0]$$

$$u_1 = \lim_{z \rightarrow \infty} z \left[\frac{2z^2 + 5z + 14}{(z-1)^4} - 0 \right]$$

$$u_1 = \lim_{z \rightarrow \infty} z \left[\frac{z^2 (2 + 5/z + 14/z^2)}{z^4 (1 - 1/z)^4} \right] = 0 //$$

$$u_2 = \lim_{z \rightarrow \infty} z^2 \left[\bar{u}(z) - u_0 - \frac{u_1}{z} \right]$$

$$u_2 = \lim_{z \rightarrow \infty} z^2 \left[\frac{2z^2 + 5z + 14}{(z-1)^4} - 0 - 0 \right]$$

$$u_2 = \lim_{z \rightarrow \infty} z^4 \left[\frac{2 + 5/z + 14/z^2}{z^4 (1 - 1/z)^4} \right] = 2 //$$

$$u_3 = \lim_{z \rightarrow \infty} z^3 \left[\bar{u}(z) - u_0 - \frac{u_1}{z} - \frac{u_2}{z^2} \right]$$

$$u_3 = \lim_{z \rightarrow \infty} z^3 \left[\frac{2z^2 + 5z + 14}{(z-1)^4} - 0 - 0 - \frac{2}{z^2} \right]$$

$$u_3 = \lim_{z \rightarrow \infty} \left[\frac{2z^3 + 5z^4 + 14z^3 - 2z(z^4 - 4z^3 + 6z^2 - 4z + 1)}{(z-1)^4} \right]$$

$$u_3 = \lim_{z \rightarrow \infty} \left[\frac{2z^5 + 5z^4 + 14z^3 - 2z(z^4 - 4z^3 + 6z^2 - 4z + 1)}{(z-1)^4} \right]$$

$$u_3 = \lim_{z \rightarrow \infty} \left[\frac{2z^5 + 5z^4 + 14z^3 - 2z^5 + 8z^4 - 12z^3 + 8z^2 - 2z}{(z-1)^4} \right]$$

$$u_3 = \lim_{z \rightarrow \infty} \left[\frac{13z^4 + 2z^3 + 8z^2 - 2z}{(z-1)^4} \right]$$

$$u_3 = \lim_{z \rightarrow \infty} z^4 \left[\frac{13 + 2/z + 8/z^2 - 2/z^3}{z^4 (1 - 1/z)^4} \right] = 13 //$$

$$u_0 + 6u_2 + 1 = u_3$$

Substituting

$$0 + 6(2) + 1 = 13 = u_3 \quad \text{Hence proved.}$$

* Prove that $Z_T[nU_n] = -z \frac{d}{dz} [Z_T(u_n)]$
 Hence show that
 $Z_T[n \cos n\theta] = \frac{z \cos(z^2 + 1) - 2z^2}{(z^2 - 2z \cos \theta + 1)^2}$

- Considering RHS

$$\begin{aligned}
 -z \frac{d}{dz} [Z_T(u_n)] &= -z \frac{d}{dz} \left[\sum_{n=0}^{\infty} u_n z^{-n} \right] \\
 &= -z \frac{d}{dz} \left[u_0 + \frac{u_1}{z} + \frac{u_2}{z^2} + \dots \right] \\
 &= -z \left[\frac{-u_1}{z^2} - \frac{2u_2}{z^3} - \frac{3u_3}{z^4} + \dots \right] \\
 &= \frac{u_1}{z} + \frac{2u_2}{z^2} + \frac{3u_3}{z^3} + \dots \\
 &= \sum_{n=0}^{\infty} n u_n z^{-n} = Z_T[n U_n] = \text{LHS}
 \end{aligned}$$

$$\therefore Z_T[n \cos n\theta] = -z \frac{d}{dz} [Z_T(\cos n\theta)]$$

$$\text{wkt } Z_T[\cos n\theta] = \frac{z[z - \cos \theta]}{z^2 - 2z \cos \theta + 1}$$

$$= -z \frac{d}{dz} \left[\frac{z^2 - z \cos \theta}{z^2 - 2z \cos \theta + 1} \right]$$

$$= -z \left[\frac{(z^2 - 2z \cos \theta + 1)(2z - \cos \theta) - (z^2 - z \cos \theta)(2z - 2 \cancel{\cos \theta})}{(z^2 - 2z \cos \theta + 1)^2} \right]$$

$$= -z \left[\frac{2z^3 - z^2 \cos \theta - 4z^2 \cos \theta + 2z \cos^2 \theta + 2z - \cos \theta}{(z^2 - 2z \cos \theta + 1)^2} \right.$$

$$\quad \left. - 2z^3 + 2z^2 \cos \theta + \cancel{2z^2 \cos^2 \theta} + 2z^2 \cos \theta - 2z \cos^2 \theta + \cancel{2z \cos \theta} \right]$$

$$= \frac{z^3 \cos \theta - 2z^2 + z \cos \theta}{(z^2 - 2z \cos \theta + 1)^2}$$

$$= \frac{z \cos \theta (z^2 + 1) - 2z^2}{(z^2 - 2z \cos \theta + 1)^2} // \text{Hence proved}$$



Inverse Z-Transforms:

Z-transform is given by

$$Z_T[u_n] = \bar{u}(z)$$

Taking inverse

$$u_n = Z_T^{-1}[\bar{u}(z)]$$

Formulas

$$1. Z_T^{-1}\left[\frac{z}{z-1}\right] = 1$$

$$2. Z_T^{-1}\left[\frac{z}{z-k}\right] = k^n$$

$$3. Z_T^{-1}\left[\frac{kz}{(z-k)^2}\right] = nk^n$$

$$4. Z_T^{-1}\left[\frac{kz^2 + zk^2}{(z-k)^3}\right] = n^2 k^n$$

Q: Find inverse Z-transform of $\frac{8x - x^3}{(4-x)^3}$

$$\text{Sol: } Z_T^{-1}\left[\frac{8x - x^3}{(4-x)^3}\right]$$

$$\Rightarrow \frac{8x - x^3}{(4-x)^3} = \frac{x^3 - 8x}{(x-4)^3} \quad z/2kx$$

$$Z_T^{-1}\left[\frac{z}{z-4}\right] = 4^n$$

$$Z_T^{-1}\left[\frac{4x}{(z-4)^2}\right] = 4n$$

$$Z_T^{-1}\left[\frac{4x^2 + 16x}{(z-4)^3}\right] = n^2 4^n$$

$$\frac{x^3 - 8x}{(x-4)^3} = A \frac{x}{z-4} + B \frac{4x}{(z-4)^2} + C \frac{(4x^2 + 16x)}{(z-4)^3} \quad \text{--- (1)}$$

$$\frac{x^3 - 8x}{(x-4)^3} = A x (x-4)^2 + B 4x (x-4) + C (4x^2 + 16x) \quad (z-4)^3$$

$$z^3 - 8z = Az(z-4)^2 + 4Bz(z-4) + C(4z^2 + 16z)$$

$$z^3 - 8 = A(z-4)^2 + 4B(z-4) + C(4z + 16)$$

Comparing the coefficients

$$\underline{\underline{A = 1}}$$

$$\underline{\underline{16A - 16B - 16C = 8}}$$

$$\underline{\underline{16(B+C) = 8}}$$

$$\underline{\underline{B+C = \frac{1}{2}}}$$

$$\text{put } z = 4$$

$$32C = 8$$

$$\underline{\underline{C = \frac{1}{4}}}$$

$$\underline{\underline{B = \frac{-7}{4}}}$$

$$0 = -8A + 4B + 4C$$

Substituting in eq ①

$$\underline{\underline{\frac{z^3 - 8z}{(z-4)^3} = \frac{z}{z-4} + \frac{+1}{4} \frac{4z}{(z-4)^2} + \frac{1}{4} \frac{4z+16z}{(z-4)^3}}}$$

$$\begin{aligned} Z_T^{-1} \left[\frac{z^3 - 8z}{(z-4)^3} \right] &= Z_T^{-1} \left[\frac{z}{z-4} \right] + \frac{+1}{4} Z_T^{-1} \left[\frac{4z}{(z-4)^2} \right] + \frac{1}{4} Z_T^{-1} \left[\frac{4z+16z}{(z-4)^3} \right] \\ &= 4^n + \frac{1}{4} 4^n n + \frac{1}{4} 4^n n^2 \\ &= \frac{4^n}{4} [4 + 7n + n^2] // \end{aligned}$$

* Difference equation:

An equation of the form

$$a_n y_{mn} + a_{n-1} y_{n+n-1} + a_{n-2} y_{n+n-2} + \dots + a_2 y_{n+2} + a_1 y_{n+1} + a_0 y_n = 0$$

where $n = 0, 1, 2, 3, \dots$ $a_0, a_1, a_2, \dots, a_r$ are constants,

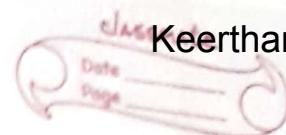
It represents linear difference equation.

Q: $U_{n+2} - 5U_{n+1} + 6U_n = 2^n$

Sol: Applying \mathcal{Z} -transform

$$Z_T[U_{n+2}] - 5Z_T[U_{n+1}] + 6Z_T[U_n] = Z_T[2^n]$$

$$Z_T^2 \left[\frac{U_2 - U_0}{U_0} - U_1 z^{-1} \right] - 5Z_T \left[\frac{U_1 - U_0}{U_0} - U_0 z^{-1} \right] + 6U_0 = \frac{z}{z-2}$$



$$[z^2 + 5z] u_0 + [6 + z^2 - 5z] \bar{u}(z) - u_1 z = \frac{z}{z-2}$$

$$\bar{u}(z) = \frac{z}{z-2} + u_1 z + (z^2 - 5z) u_0$$

$$= \frac{z^2 - 5z + 6}{z-3}$$

$$= \frac{z}{(z-2)(z^2 - 5z + 6)} + u_1 \frac{z}{z^2 - 5z + 6} + u_0 \frac{z(z-5)}{z^2 - 5z + 6} \quad \textcircled{1}$$

$$\bar{u}(z) = p(z) + u_1 q_1(z) + u_0 r(z)$$

$$p(z) = \frac{z}{(z-2)(z-3)} = \frac{z}{(z-2)^2(z-3)}$$

$$p(z) = \frac{\star}{z(z-2)^2(z-3)} = \frac{A}{z-3} + \frac{B}{(z-2)} + \frac{C}{(z-2)^2}$$

$$\star = A(z-2)^2 + B(z-3)(z-2) + C(z-3)$$

$$\text{Put } z = 2$$

Equating coefficients of z^2

$$1 = -C \Rightarrow C = -1$$

$$A + B = 0$$

$$\text{Put } z = 3$$

$$B = -A = -1$$

$$1 = A \Rightarrow A = 1$$

Substituting

$$\frac{1}{(z-2)^2(z-3)} = \frac{1}{z-3} - \frac{1}{z-2} - \frac{1}{(z-2)^2}$$

$$p(z) = \frac{z}{(z-2)^2(z-3)} = \frac{z}{z-3} - \frac{z}{z-2} - \frac{z}{(z-2)^2}$$

$$Z_T^{-1} \left[\frac{z}{(z-2)^2(z-3)} \right] = Z_T^{-1} \left[\frac{z}{z-3} \right] - Z_T^{-1} \left[\frac{z}{z-2} \right] - Z_T^{-1} \left[\frac{z}{(z-2)^2} \right]$$

$$= 3^n - 2^n - \frac{z^n n}{2} \quad \text{not equal}$$

$$g(z) = \frac{z}{z^2 - 5z + 6}$$

$$\frac{g(z)}{z} = \frac{1}{(z-2)(z-3)} = \frac{A}{(z-2)} + \frac{B}{(z-3)}$$

$$1 = A(z-3) + B(z-2)$$

$$\text{Put } z=3 \quad \text{Put } z=2$$

$$\underline{\underline{B=1}} \quad \underline{\underline{A=-1}}$$

Substituting

$$\frac{z}{(z-2)(z-3)} = \frac{-z}{z-2} + \frac{z}{z-3}$$

$$z^{-1} \left[\frac{z}{(z-2)(z-3)} \right] = -z^{-1} \left[\frac{z}{z-2} \right] + z^{-1} \left[\frac{z}{z-3} \right]$$

$$= -2^n + 3^n //$$

$$g(z) = \frac{z(z-2)}{z^2 - 5z + 6} = \frac{z(z-2)}{(z-2)(z-3)}$$

$$g(z) = \frac{z \cancel{(z-2)}}{z-3}$$

$$z^{-1} \left[\frac{z}{z-3} \right] = 3^n //$$

Substituting in eq ①

$$u_n = \left(3^n - 2^n - \frac{2^n n}{2} \right) u_0 + (-2^n + 3^n) u_1 + 3^n u_0 //$$