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① Reference ⇒ Superconducting Ckt Companion ⇒ An Introductⁿ with worked examples
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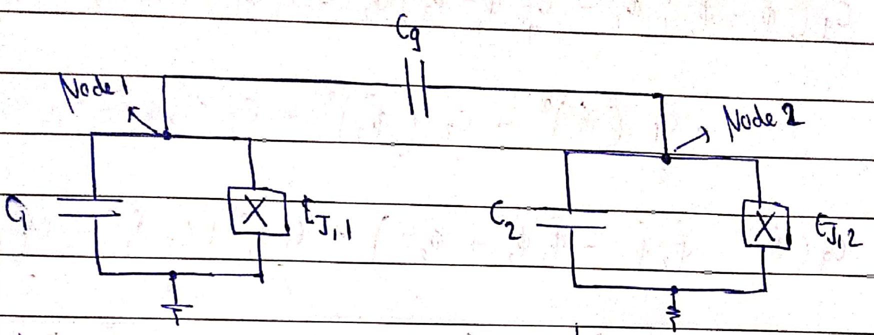
⊛ $\boxed{\text{Hamiltonians of two Qubit Coupling circuits}}$ ⇒

① $\boxed{\text{Fixed Coupling via capacitor}}$



Step ① ⇒ Applying Kirchhoff's law

@ Node 1 ⇒ flux ⇒ ϕ_1

@ Node 2 ⇒ flux ⇒ ϕ_2

Branch flux ⇒ $\phi_{cg} = \phi_1 - \phi_2$

By Kirchhoff's law ⇒ ① $\phi_1 = \phi_{C_1} = -\phi_{E_{J,1}}$

② $\phi_2 = \phi_{C_2} = -\phi_{E_{J,2}}$

③ $\phi_{Branch} = \phi_{E_{J_1}} - \phi_{E_{J_2}} + \phi_c = \tilde{\phi}$

$\boxed{\phi_{Branch} = \phi_1 - \phi_2}$

$\phi_c \approx 0$, $\tilde{\phi} = 0$ (Assumed)
↳ external flux

Step ② ⇒ Writing Lagrangian

Now, $L = T_{cap} - U_{ind} - U_{JJ}$

$$T_{cap} = \frac{C_g}{2}(\dot{\phi}_1 - \dot{\phi}_2)^2 + \frac{C_1}{2}(\dot{\phi}_1)^2 + \frac{C_2}{2}(\dot{\phi}_2)^2$$

||| K.E

$U_{inductor} = 0$, $U_{J.J} = -E_{J,1}\cos\phi_1 - E_{J_2}\cos\phi_2$

$$\therefore \boxed{L = \frac{C_g}{2}(\dot{\phi}_1 - \dot{\phi}_2)^2 + \frac{C_1}{2}(\dot{\phi}_1)^2 + \frac{C_2}{2}(\dot{\phi}_2)^2 + E_{J,1}\cos\phi_1 + E_{J_2}\cos\phi_2}$$

$\boxed{\text{Step ③}}$ ⇒ Writing Hamiltonian

$H = \sum p_i \dot{q}_i - L$, $\boxed{q_i \Rightarrow \phi_i}$ ⇒ Positⁿ / flux

$P_1 = \frac{\partial L}{\partial \dot{\phi}_1}$, $P_2 = \frac{\partial L}{\partial \dot{\phi}_2}$, $\boxed{P_i \Rightarrow \frac{\partial L}{\partial \dot{\phi}_i}}$ ⇒ conjugate Momentum

$$\boxed{H = \frac{\partial L}{\partial \dot{\phi}_1}\dot{\phi}_1 + \frac{\partial L}{\partial \dot{\phi}_2}\dot{\phi}_2 - L}$$

in vector form ⇒ $\boxed{P_i = \frac{\partial L}{\partial \dot{\phi}_i} = C\dot{\phi}_i}$

$$\boxed{\begin{aligned} P_1 &= C_g(\dot{\phi}_1 - \dot{\phi}_2) + C_1\dot{\phi}_1 \\ P_2 &= -C_g(\dot{\phi}_1 - \dot{\phi}_2) + C_2\dot{\phi}_2 \end{aligned}}$$

$$H = \frac{C_g}{2} \times 2(\dot\phi_1 - \dot\phi_2)\dot\phi_1 + C_1 \times \dot\phi_1^2 - C_2(\dot\phi_1 - \dot\phi_2)\dot\phi_2 + C_2\dot\phi_2^2$$
$$- C_g/2(\dot\phi_1 - \dot\phi_2)^2 - C_1/2(\dot\phi_1)^2 - C_2/2(\dot\phi_2)^2$$
$$- E_{J_1}\cos\phi_1 - E_{J_2}\cos\phi_2$$

$$= C_g(\dot\phi_1 - \dot\phi_2)\dot\phi_1 - C_g(\dot\phi_1 - \dot\phi_2)\dot\phi_2 + C_1\dot\phi_1^2 + C_2\dot\phi_2^2 - \frac{C_g}{2}(\dot\phi_1 - \dot\phi_2)^2$$
$$- \frac{C_1}{2}(\dot\phi_1)^2 - \frac{C_2}{2}(\dot\phi_2)^2 - E_{J_1}\cos\phi_1 - E_{J_2}\cos\phi_2$$

$$= C_g(\dot\phi_1^2 - \dot\phi_2\dot\phi_1 - \dot\phi_1\dot\phi_2 + \dot\phi_2^2) + \frac{C_1}{2}\dot\phi_1^2 + \frac{C_2}{2}\dot\phi_2^2 - \frac{C_g}{2}(\dot\phi_1^2 - 2\dot\phi_1\dot\phi_2 + \dot\phi_2^2)$$

$$\boxed{H = \frac{C_g}{2}(\dot\phi_1 - \dot\phi_2)^2 + \frac{C_1}{2}\dot\phi_1^2 + \frac{C_2}{2}\dot\phi_2^2 - E_{J_1}\cos\phi_1 - E_{J_2}\cos\phi_2}$$

Step 4 $\Rightarrow$ writing Hamiltonian in matrix form

$$\boxed{H = 1/2\, p^T C^{-1} p - E_{J_1}\cos\phi_1 - E_{J_2}\cos\phi_2}$$

where $p = \begin{pmatrix} p_1 \\ p_2 \end{pmatrix} = \begin{pmatrix} \partial L/\partial\dot\phi_1 \\ \partial L/\partial\dot\phi_2 \end{pmatrix}$ $\Rightarrow$ vector of conjugate momentum

& C = capacitance matrix = $\begin{bmatrix} C_1 + C_g & -C_g \\ -C_g & C_2 + C_g \end{bmatrix}$

$$C^{-1} = \frac{1}{C_\Sigma}\begin{bmatrix} C_2 + C_g & C_g \\ C_g & C_1 + C_g \end{bmatrix} = \begin{bmatrix} 1/C_1 & C_g/C_1C_2 \\ C_g/C_1C_2 & 1/C_2 \end{bmatrix}$$

where $C_\Sigma = \det C = C_1C_2 + C_1C_g + C_2C_g$

$\Downarrow$ assuming $C_1 \& C_2 >> C_g$

Step 5 $\Rightarrow$ Quantizing the Hamiltonian

$$\boxed{\begin{array}{l} \phi_n \rightarrow \hat\phi_n \\ p_n \rightarrow \hat p_n \\ H \rightarrow \hat H \end{array}}$$

Now consider $\Rightarrow$ $\boxed{p = C\dot\phi}$ $\Rightarrow$ generalized momentum

time derivative of generalized momentum $\Rightarrow$ $\boxed{I = C\ddot\phi}$ $\Rightarrow$ current through capacitors

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Conjugate Momentum $\hat{P}_n$ = Sum of all charges on capacitors @ around given node

∴ Define ⇒ $\hat{n}_n = -\frac{\hat{P}_n}{2e}$ ⇒ Net number of cooper pairs stored on nth node

$$\therefore \boxed{\hat{T}_{cap} = \frac{1}{2}\hat{P}^T C^{-1}\hat{P} = \frac{4e^2}{2}\hat{n}^T C^{-1}\hat{n}} \quad - (A)$$

for each diagonal element ⇒ define effective capacity energy term of the nth nodes as

$$E_{C,n} = \frac{e^2}{2}(C^{-1})_{(n,n)}, \quad E_{Cg} = \frac{e^2}{2}\left(C^{-1}\right)_{(n,m)},$$

diagonal & off diagonal elements of C^{-1}

i.e $E_{C_1} = \frac{e^2}{2} \times \left(\frac{C_2 + C_g}{C_\Sigma}\right)$, $E_{C_2} = \frac{e^2}{2}\left(\frac{C_1 + C_g}{C_\Sigma}\right)$, $E_{Cg} = \frac{e^2}{2}\left(\frac{C_g}{C_\Sigma}\right)$,

In dimensionless form ⇒ $\boxed{\hat{n}_n = \div P_n}$ $\boxed{E_{C,n} = \frac{(C^{-1})_{n,n}}{8}}$, $\boxed{E_{Cg} = \frac{(C^{-1})_{n,m}}{8}}$ - (B)

∴ Now, from ①, ⇒ converting ① in operator form,

$$\hat{H} = \frac{C_g}{2}(\hat{\dot{\phi}}_1^2) - C_g\hat{\dot{\phi}}_1\hat{\dot{\phi}}_2 + \frac{C_g\hat{\dot{\phi}}_2}{2} + \frac{C_1\hat{\dot{\phi}}_1^2}{2} + \frac{C_2\hat{\dot{\phi}}_2^2}{2} - E_{J_1}\cos\hat{\phi}_1 - E_{J_2}\cos\hat{\phi}_2$$

$$\boxed{\hat{H} = \left(\frac{C_g + C_1}{2}\right)\hat{\dot{\phi}}_1^2 + \left(\frac{C_g + C_2}{2}\right)\hat{\dot{\phi}}_2^2 - C_g\hat{\dot{\phi}}_1\hat{\dot{\phi}}_2 - E_{J_1}\cos\hat{\phi}_1 - E_{J_2}\cos\hat{\phi}_2} \quad - ③$$

→ Hamiltonian in operator form

Now from $\boxed{② \ \& (A) \ \& (B)}$

$$\boxed{\hat{H} = 4E_{C_1}\hat{n}_1^2 + 4E_{C_2}\hat{n}_2^2 - E_{J,1}\cos\hat{\phi}_1 - E_{J,2}\cos\hat{\phi}_2 + 8E_{Cg}\hat{n}_1\hat{n}_2} \quad ④$$

By taylor series expansion ⇒ $E_J\cos\phi = E_J - \frac{1}{2}E_J\phi^2 + \frac{1}{24}E_J\phi^4 + O(\phi^6)$

Neglecting higher order term & const term ⇒

$$\boxed{E_J\cos\phi \approx -\frac{1}{2}E_J\phi^2 + \frac{1}{24}E_J\phi^4}$$

$$\therefore \hat{H} = 4\left(E_{C_1}\hat{n}_1^2 + E_{C_2}\hat{n}_2^2 + 2E_{Cg}\hat{n}_1\hat{n}_2\right) + \frac{1}{2}(E_{J_1} + E_{J_2})\phi^2 - \frac{1}{24}(E_{J_1} + E_{J_2})\phi^4$$

Now, consider Hamiltonian of Simple Harmonic oscillator

i.e neglecting the coupling term & higher order (ϕ^4) term

as $\boxed{\hat{H}_{SHO} = \frac{1}{2}K_{eq}\hat{x}^2 + \frac{1}{2}m_{eq}\dot{x}^2}$ ⇒ $\boxed{\text{Keeping only these terms}}$ (4)

$$\therefore \boxed{\hat{H}_{SHO} = 4E_{C_1}\hat{n}_1^2 + 4E_{C_2}\hat{n}_2^2 + \frac{1}{2}(E_{J_1} + E_{J_2})\phi^2} = 4\sqrt{E_{C_1}\left(-\frac{E_{J_1}}{2}\right)}\left(N_1 + \frac{1}{2}\right) + 4\sqrt{E_{C_2}\left(\frac{E_{J_2}}{2}\right)}\left(N_2 + \frac{1}{2}\right)$$

$\boxed{\hat{N}_1 = b_1^\dagger b_1, \quad \hat{N}_2 = b_2^\dagger b_2}$ ← where

Define the following terms analogous to S.H.O ⇒

① Impedance $\xi_i = \sqrt{\frac{m_{eq}}{k_{eq}}} = \sqrt{\frac{4E_{C_1}}{E_{J_1}/2}}$

② $\hat{\phi}_i$ & $\hat{n}_i$ in terms of anhilatn & creatn operators ⇒

$$\boxed{\hat{\phi}_i = \sqrt{\frac{\xi_i}{2}}\,(\hat{b}_i + \hat{b}_i^{+}), \quad \hat{n}_i = \frac{i}{\sqrt{2\xi_i}}(\hat{b}_i - \hat{b}_i^{+})} \Rightarrow ©$$

Putting © in ④

$$\boxed{\begin{aligned}\hat{H} = {} & 4\left(\frac{-E_{C_1}}{2\xi_1}\right) \times (\hat{b}_1 - \hat{b}_1^{+})^2 - \frac{4E_{C_2}}{2\xi_2}(\hat{b}_2 - \hat{b}_2^{+})^2 + 8E_{Cg}\left(\frac{-1}{\sqrt{2\xi_1\xi_2}}(\hat{b}_1 - \hat{b}_1^{+})(\hat{b}_2 - \hat{b}_2^{+})\right) \\ & + \tfrac{1}{2}E_{J_1}\left(\frac{\xi_1}{2}(b_1^{+} + b_1)^2\right) + \tfrac{1}{2}E_{J_2}\left(\frac{\xi_2}{2}(b_2^{+} + b_2)\right)^2 \\ & - \frac{1}{24}E_{J_1}(b_1 + b_1^{+})^4 \times \frac{\xi_1^2}{4} - \frac{1}{24}E_{J_2}(b_2 + b_2^{+})^4 \times \frac{\xi_2^2}{4}\end{aligned}}$$

↳ ⓓ

⇒ Now using the ~~rotating wave~~ approximatn ⇒ Terms having $\boxed{\text{No. of anhilatn operators} \neq \text{No. of creatn operators}}$ can be neglected as they rotate rapidly

↳ $\boxed{\text{Non conserving terms}}$

i.e ~~Neglecting~~

$$\hat{H} = -\frac{4E_{C_1}}{2\xi_1}\left(\cancel{b_1^2} - b_1 b_1^{+} - b_1^{+}b_1 + \cancel{b_1^{+2}}\right) - \frac{4E_{C_2}}{2\xi_2}\left(\cancel{b_2^2} - \hat{b}_2\hat{b}_2^{+} - \hat{b}_2^{+}\hat{b}_2 + \cancel{b_2^{+2}}\right)$$

$$- \frac{8E_{Cg}}{\sqrt{2\xi_1\xi_2}}\left(\cancel{b_1 b_2} - b_1^{+}\hat{b}_2 + \cancel{b_1^{+}b_2^{+}} - \hat{b}_1\hat{b}_2^{+}\right) + \tfrac{1}{2}E_{J_1}\left(\frac{\xi_1}{2}\left(\cancel{b_1^{+}b_1^{+}} + b_1^{+}b_1 + b_1 b_1^{+} + \cancel{b_1 b_1}\right)\right)$$

$$+ \tfrac{1}{2}E_{J_2}\frac{\xi_2}{2}\left(\cancel{b_2^{+}b_2^{+}} + \cancel{b_2 b_2} + b_2^{+}b_2 + b_2 b_2^{+}\right) - \frac{1}{24}E_{J_1}\frac{\xi_1^2}{4}(b_1 + b_1^{+})^4 - \frac{1}{24}E_{J_2}\frac{\xi_2^2}{4}(b_2 + b_2^{+})^4$$

Putting $[b, b^{+}] = 1 \Rightarrow bb^{+} - b^{+}b = 1 \Rightarrow \boxed{b^{+}b = bb^{+} - 1} \Rightarrow \boxed{bb^{+} = b^{+}b + 1}$

& $(b + b^{+})^4 = \left(\tfrac{1}{2}\,\hat{b}\hat{b}^{+} + \hat{b}\hat{b} + \hat{b}^{+}\hat{b}\right)$ + non conserving terms

$$\therefore \hat{H} = \frac{4E_{C_1}}{2\xi_1}(2b_1^{+}b_1 + 1) + \frac{4E_{C_2}}{2\xi_2}(2b_2^{+}b_2 + 1) + \frac{8E_{Cg}}{\sqrt{2\xi_1\xi_2}}\left(\hat{b}_1\hat{b}_2^{+} + \hat{b}_1^{+}\hat{b}_2\right)$$

$$+ \tfrac{1}{4}E_{J_1}\xi_1(2b_1^{+}b_1 + 1) + \tfrac{1}{4}E_{J_2}\xi_2(2b_2^{+}b_2 + 1) - \frac{1}{24}\frac{E_{J_1}\xi_1^2}{4}\left(\tfrac{1}{2}\,\hat{b}_1^{+}\hat{b}_1^{+}\hat{b}_1\hat{b}_1 + \hat{b}_1^{+}\hat{b}_1\right)$$

$$-\frac{1}{24}\frac{E_{J_2}\,\varepsilon_2^2}{4}\left(\frac{1}{2}\hat{b}_2^+\hat{b}_2^+\hat{b}_2\hat{b}_2 + \hat{b}_1^+\hat{b}_1\right)$$

Neglecting the const. terms ⇒

$$\hat{H} = \frac{4E_{C_1}}{\varepsilon_1}b_1^+b_1 + \frac{4E_{C_2}}{\varepsilon_2}b_2^+b_2 + \frac{8E_{Cg}}{\sqrt{2\varepsilon_1\varepsilon_2}}\left(\hat{b}_1\hat{b}_2^+ + \hat{b}_1^+\hat{b}_2\right)$$

$$+\frac{1}{2}\left(E_{J_1}\varepsilon_1(b_1^+b_1)\right) + \frac{1}{2}E_{J_2}\varepsilon_2(b_2^+b_2)$$

Anharmonic term

$$-\frac{1}{24\times4}E_{J_1}\varepsilon_1^2\left(\frac{1}{2}\hat{b}_1^+\hat{b}_1^+\hat{b}_1\hat{b}_1 + \hat{b}_1^+\hat{b}_1\right) - \frac{1}{24\times4}E_{J_2}\varepsilon_2^2\left(\frac{1}{2}\hat{b}_2^+\hat{b}_2^+\hat{b}_2\hat{b}_2 + \hat{b}_2^+\hat{b}_2\right)$$

Separating $(b_1^+b_1)$, $(b_2^+b_2)$, $(\hat{b}_1\hat{b}_2^+ + \hat{b}_1^+\hat{b}_2)$, $(1/2\, b_2^+b_2^+b_2b_2)$ & $(1/2\, b_1^+b_1^+b_1b_1)$ terms

$$\therefore \hat{H} = \left(\frac{4E_{C_1}}{\varepsilon_1} + \frac{1}{2}E_{J_1}\varepsilon_1 - \frac{1}{24\times4}E_{J_1}\varepsilon_1^2\right)\hat{b}_1^+\hat{b}_1$$

$$+\left(\frac{4E_{C_2}}{\varepsilon_2} + 1/2\,E_{J_2}\varepsilon_2 - \frac{1}{24\times4}E_{J_2}\varepsilon_2^2\right)\hat{b}_2^+\hat{b}_2$$

$$+\frac{8E_{Cg}}{\sqrt{2\varepsilon_1\varepsilon_2}}\left(\hat{b}_1\hat{b}_2^+ + \hat{b}_1^+\hat{b}_2\right) - \frac{1}{24\times4\times2}E_{J_1}\varepsilon_1^2\, b_1^+b_1^+b_1b_1 - \frac{1}{24\times4\times2}E_{J_2}\varepsilon_2^2\, b_2^+b_2^+b_2b_2$$

Putting ⇒ ① ω_i ⇒ Natural freqⁿ of oscillator $= \frac{4E_{C_i}}{\varepsilon_i} + 1/2\,E_{J_i}\varepsilon_i$

② α_i ⇒ Anharmonicity of oscillator ⇒ $-\frac{\varepsilon_i^2 E_{J,i}}{8}$

③ g_{12} ⇒ Oscillator coupling strength ⇒ $\frac{8E_{Cg}}{\sqrt{2\varepsilon_1\varepsilon_2}}$

$$\therefore \hat{H} = \left(\omega_1 + \frac{\alpha_1}{12}\right)\hat{b}_1^+\hat{b}_1 + \left(\omega_2 + \frac{\alpha_2}{12}\right)\hat{b}_2^+\hat{b}_2 + g_{12}\left(\hat{b}_1\hat{b}_2^+ + \hat{b}_1^+\hat{b}_2\right)$$

$$+\frac{\alpha_1}{12}\left(\frac{\hat{b}_1^+\hat{b}_1^+\hat{b}_1\hat{b}_1}{2}\right) + \frac{\alpha_2}{12}\left(\frac{\hat{b}_2^+\hat{b}_2^+\hat{b}_2\hat{b}_2}{2}\right)$$

$$= \underbrace{\omega_1\hat{b}_1^+\hat{b}_1 + \omega_2\hat{b}_2^+\hat{b}_2}_{H_0 \Rightarrow \text{ free Hamiltonian / Non-interacting Hamiltonian}} + \underbrace{g_{12}\left(\hat{b}_1\hat{b}_2^+ + \hat{b}_1^+\hat{b}_2\right)}_{\text{Interacting Hamiltonian}} + \underbrace{\frac{\alpha_1}{12}\left(\frac{1}{2}\hat{b}_1^+\hat{b}_1^+\hat{b}_1\hat{b}_1 + \hat{b}_1^+\hat{b}_1\right) + \frac{\alpha_2}{12}\left(\frac{1}{2}\hat{b}_2^+\hat{b}_2^+\hat{b}_2\hat{b}_2 + \hat{b}_2^+\hat{b}_2\right)}_{\text{Anharmonic Hamiltonian}}$$

Total interacting Hamiltonian

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$$\boxed{\hat{H} = H_0 + H_{int} + H_{A.H.O}}$$

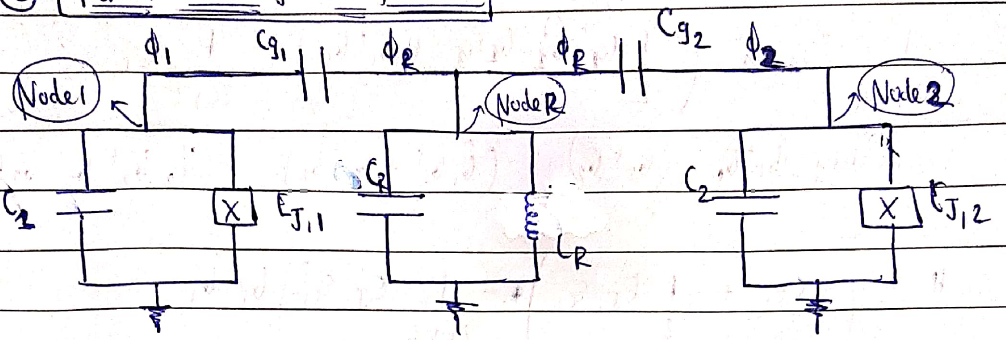
$(H_{int})_{total}$

from ① we can also express it in terms Pauli Matrices by replacing ⇒

$\boxed{(\hat{b}^{+}-\hat{b})^{2} \rightarrow -\sigma_{z}, \quad (b^{+}+b)^{2} \rightarrow -\sigma_{z}, \quad (\hat{b}^{+}+\hat{b})^{3} \rightarrow -6\sigma_{z}}$

$\boxed{(\hat{b}_i^{+}-\hat{b}_i)(\hat{b}_j^{+}-\hat{b}_j) = -\sigma_i^{y}\sigma_j^{y}}$

② Fixed coupling via resonator



Step ① ↓ Kirchhoff's Law

@ Node 1 ⇒ $\phi_{C_1} = -\phi_{E_{J,1}} = \phi_1$

@ Node 2 ⇒ $\phi_{C_R} = -\phi_{J_R} = \phi_R$

@ Node 3 ⇒ $\phi_{C_2} = -\phi_{E_{J,2}} = \phi_2$

Assuming Time independent external fluxes ⇒ $\tilde{\phi} \Rightarrow 0$

Step ② ↓ Writing Lagrangian ⇒ $L = T_{cap} - U_{ind} - U_{JJ}$

① $T_{cap} = \dfrac{C_1 \dot{\phi}_1^2}{2} + \dfrac{C_R \dot{\phi}_R^2}{2} + \dfrac{C_2 \dot{\phi}_2^2}{2} + \dfrac{C_{g_1}(\dot{\phi}_1 - \dot{\phi}_R)^2}{2} + \dfrac{C_{g_2}(\dot{\phi}_2 - \dot{\phi}_R)^2}{2}$

② $U_{ind} = \dfrac{1}{2L_R}(\phi_R)^2$

③ $U_{JJ} = -E_{J_1}\cos\phi_1 - E_{J_2}\cos\phi_2$

$$\boxed{L = \frac{C_1\dot{\phi}_1^2}{2} + \frac{C_R\dot{\phi}_R^2}{2} + \frac{C_2\dot{\phi}_2^2}{2} + \frac{C_{g_1}(\dot{\phi}_1-\dot{\phi}_R)^2}{2} + \frac{C_{g_2}(\dot{\phi}_2-\dot{\phi}_R)^2}{2} - \frac{1}{2L_R}(\phi_R)^2 + E_{J_1}\cos\phi_1 + E_{J_2}\cos\phi_2}$$

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$\boxed{\text{Step ③}} \Rightarrow$ $H = \sum p_i \dot{q}_i - L$

$$p_1 = \frac{\partial L}{\partial \dot{\phi}_1}, \quad p_R = \frac{\partial L}{\partial \dot{\phi}_R}, \quad p_2 = \frac{\partial L}{\partial \dot{\phi}_2}$$

$$p_1 = C_1\dot{\phi}_1 + C_{g_1}(\dot{\phi}_1 - \dot{\phi}_R)$$

$$p_2 = C_2\dot{\phi}_2 + C_{g_2}(\dot{\phi}_2 - \dot{\phi}_R)$$

$$p_R = -C_{g_1}(\dot{\phi}_1 - \dot{\phi}_R) - C_{g_2}(\dot{\phi}_2 - \dot{\phi}_R) + C_R\dot{\phi}_R$$

$$\begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} = \begin{bmatrix} C_1+C_{g_1} & -C_{g_1} & 0 \\ -C_{g_1} & C_{g_1}+C_{g_2}+C_R & -C_{g_2} \\ 0 & -C_{g_2} & C_2+C_{g_2} \end{bmatrix} \begin{bmatrix} \dot{\phi}_1 \\ \dot{\phi}_R \\ \dot{\phi}_2 \end{bmatrix}$$

$$[P] = [C]\,[\dot{q}]$$

$$\therefore H = \left(C_1\dot{\phi}_1 + C_{g_1}(\dot{\phi}_1 - \dot{\phi}_R)\right)\dot{\phi}_1 + \left(C_2\dot{\phi}_2 + C_{g_2}(\dot{\phi}_2 - \dot{\phi}_R)\right)\dot{\phi}_2$$
$$+\left(-C_{g_1}(\dot{\phi}_1 - \dot{\phi}_R) - C_{g_2}(\dot{\phi}_2 - \dot{\phi}_R)\right)\dot{\phi}_R + C_R\dot{\phi}_R^2$$
$$-\frac{C_1\dot{\phi}_1^2}{2} - \frac{C_R\dot{\phi}_R^2}{2} - \frac{C_2\dot{\phi}_2^2}{2} - \frac{C_{g_1}}{2}(\dot{\phi}_1 - \dot{\phi}_R)^2 - \frac{C_{g_2}}{2}(\dot{\phi}_2 - \dot{\phi}_R)^2$$
$$+\frac{1}{2L_R}(\phi_R)^2 - E_{J_1}\cos\phi_1 - E_{J_2}\cos\phi_2$$

$$= C_1\dot{\phi}_1^2 + C_{g_1}(\dot{\phi}_1)^2 - C_{g_1}\dot{\phi}_R\dot{\phi}_1 + C_2\dot{\phi}_2^2 + C_{g_2}\dot{\phi}_2^2 - C_{g_2}\dot{\phi}_R\dot{\phi}_2 + C_R\dot{\phi}_R^2$$
$$- C_{g_1}\dot{\phi}_R\dot{\phi}_1 + C_{g_1}\dot{\phi}_R^2 - C_{g_2}\dot{\phi}_2\dot{\phi}_R + C_{g_2}\dot{\phi}_R^2 - \frac{C_1\dot{\phi}_1^2}{2} - \frac{C_R\dot{\phi}_R^2}{2} - \frac{C_2\dot{\phi}_2^2}{2}$$
$$-\frac{C_{g_1}}{2}(\dot{\phi}_1^2 - 2\dot{\phi}_1\dot{\phi}_R + \dot{\phi}_R^2) - \frac{C_{g_2}}{2}(\dot{\phi}_2^2 - 2\dot{\phi}_2\dot{\phi}_R + \dot{\phi}_R^2)$$
$$+\frac{1}{2L_R}(\phi_R)^2 - E_{J_1}\cos\phi_1 - E_{J_2}\cos\phi_2$$

$$\boxed{H = \frac{C_1\dot{\phi}_1^2}{2} + \frac{C_{g_1}\dot{\phi}_1^2}{2} - C_{g_1}\dot{\phi}_1\dot{\phi}_R + \frac{C_2\dot{\phi}_2^2}{2} + \frac{C_{g_2}\dot{\phi}_2^2}{2} - C_{g_2}\dot{\phi}_R\dot{\phi}_2 + \frac{C_R\dot{\phi}_R^2}{2} + \frac{C_{g_1}\dot{\phi}_R^2}{2} + \frac{C_{g_2}\dot{\phi}_R^2}{2} + \frac{\phi_R^2}{2L_R} - E_{J_1}\cos\phi_1 - E_{J_2}\cos\phi_2} \quad -(1)$$

$\boxed{\text{Step ④}} \Rightarrow$

$$\boxed{H = \frac{1}{2}P^T C^{-1} P + \frac{\phi_R^2}{2L_R} - E_{J_1}\cos\phi_1 - E_{J_2}\cos\phi_2} \quad -(A)$$

Where P = conjugate Momentum vector $= \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} = \begin{bmatrix} C_1+C_{g_1} & -C_{g_1} & 0 \\ -C_{g_1} & C_{g_1}+C_{g_2}+C_R & -C_{g_2} \\ 0 & -C_{g_2} & C_2+C_{g_2} \end{bmatrix} \begin{bmatrix} \dot{\phi}_1 \\ \dot{\phi}_R \\ \dot{\phi}_3 \end{bmatrix}$

C^{-1} exists $\Leftarrow C =$ Capacitance Matrix

$L \Rightarrow$ Inductor Matrix $= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1/L_R & 0 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow$ inverse does not exist

__/__/____

Step 5 $\Rightarrow$ Quantizing the Hamiltonian

$$\boxed{\begin{array}{c} \phi_n \to \hat{\phi}_n \\ P_n \to \hat{P}_n \\ H \to \hat{H}_n \end{array}}$$

Now consider $\Rightarrow$ $\dot{p} = C\dot{\phi} \Rightarrow$ generalized Momentum

$$\hat{T}_{cap} = \frac{1}{2}\hat{p}^T C^{-1}\hat{p} = \frac{4e^2\hat{n}^T C^{-1}\hat{n}}{2} \Rightarrow \boxed{\frac{\hat{n}^T C^{-1}\hat{n}}{2}} \quad (2)$$

In Normalized form,

$$\boxed{E_{cn} = \frac{(C^{-1})_{n,n}}{8}, \quad E_{c,g} = \frac{(C^{-1})_{n,m}}{8}, \quad \hat{n}_n = -P_n} \quad (3)$$

$$C^{-1} = \begin{bmatrix} (C^{-1})_{11} & (C^{-1})_{12} & (C^{-1})_{13} \\ (C^{-1})_{21} & (C^{-1})_{22} & (C^{-1})_{23} \\ (C^{-1})_{31} & (C^{-1})_{32} & (C^{-1})_{33} \end{bmatrix}$$

Converting (1) in operator form

$$H = \frac{C_1\dot{\hat{\phi}}_1^2}{2} + \frac{C_{g_1}\dot{\hat{\phi}}_1^2}{2} - C_{g_1}\dot{\hat{\phi}}_1\dot{\hat{\phi}}_R + \frac{C_2\dot{\hat{\phi}}_2^2}{2} + \frac{C_{g_2}\dot{\hat{\phi}}_2^2}{2} - C_{g_2}\dot{\hat{\phi}}_R\dot{\hat{\phi}}_2 + \frac{C_R\dot{\hat{\phi}}_R^2}{2}$$

$$\frac{C_{g_2}\dot{\hat{\phi}}_R^2}{2} + \frac{C_{g_1}\dot{\hat{\phi}}_R^2}{2} + \frac{\hat{\phi}_R^2}{2L_R} - E_{J_1}\cos\phi_1 - E_{J_2}\cos\phi_2$$

$$= \left(\frac{C_1+C_{g_1}}{2}\right)\dot{\hat{\phi}}_1^2 + \left(\frac{C_2+C_{g_2}}{2}\right)\dot{\hat{\phi}}_2^2 + \frac{(C_R+C_{g_1}+C_{g_2})\dot{\hat{\phi}}_R^2}{2} + \frac{\hat{\phi}_R^2}{2L_R}$$

$$- C_{g_1}\dot{\hat{\phi}}_1\dot{\hat{\phi}}_R - C_{g_2}\dot{\hat{\phi}}_R\dot{\hat{\phi}}_2 - E_{J_1}\cos\phi_1 - E_{J_2}\cos\phi_2$$

Putting (2) & (3) in (A)

$$\hat{H} = 4E_{C_1}\hat{n}_1^2 + 4E_{C_R}\hat{n}_R^2 + 4E_{C_2}\hat{n}_2^2 + 8E_{Cg_1}\hat{n}_1\hat{n}_R + 8E_{Cg_2}\hat{n}_2\hat{n}_R$$
$$+ \frac{\hat{\phi}_R^2}{2L_R} - E_{J_1}\cos\phi_1 - E_{J_2}\cos\phi_2$$

$$\boxed{E_J\cos\phi = -\frac{1}{2}E_J\phi^2 + \frac{1}{24}E_J\phi^4}, \quad \text{Let } \frac{1}{2L_R} = E_R$$

$$\boxed{\begin{aligned}\hat{H} = {} & 4E_{C_1}\hat{n}_1^2 + 4E_{C_R}\hat{n}_R^2 + 4E_{C_2}\hat{n}_2^2 + 8E_{Cg_1}\hat{n}_1\hat{n}_R + 8E_{Cg_2}\hat{n}_2\hat{n}_R + E_R\hat{\phi}_R^2 \\ & + \frac{1}{2}E_{J_1}\phi^2 - \frac{1}{24}E_{J_1}\phi^4 + \frac{1}{2}E_{J_2}\phi^2 - \frac{1}{24}E_{J_2}\phi^4 \end{aligned}} \Rightarrow (4)$$

Now $\boxed{\hat{H}_{S.H.O} = 4E_{C_1}\hat{n}_1^2 + 4E_{C_R}\hat{n}_R^2 + 4E_{C_2}\hat{n}_2^2 + E_R\hat{\phi}_R^2 + \left(\frac{1}{2}E_{J_1}\phi_1^2 + \frac{1}{2}E_{J_2}\phi_2^2\right)}$

__/__/___

Define ① $\xi_1 = \sqrt{\dfrac{4E_{C1}}{E_{J_1}/2}}$, $\xi_2 = \sqrt{\dfrac{4E_{C2}}{E_{J_2}/2}}$ ⇒ Impedances

$\xi_R = \sqrt{\dfrac{4E_{CR}}{E_R}}$

② $\hat{\phi}_i$ & $\hat{n}_i$ in terms of anhilation & creatⁿ operators ⇒

$\boxed{i = 1, 2, R}$

$$\boxed{\hat{\phi}_i = \sqrt{\frac{\xi_i}{2}}(\hat{b}_i + \hat{b}_i^+), \quad \hat{n}_i = \frac{i}{\sqrt{2\xi_i}}(b_i - b_i^+)} \quad - (B)$$

Putting (B) in (1)

$$\boxed{\begin{aligned}\hat{H} = &-4E_{C_1} \times \frac{1}{(\sqrt{2\xi_1})^2}(b_1 - b_1^+)^2 + -4E_{CR} \times \frac{1}{(\sqrt{2\xi_R})^2}(b_R - b_R^+)^2 \\ &+ -4E_{C_2} \times \frac{1}{(\sqrt{2\xi_2})^2}(b_2 - b_2^+)^2 + 8E_{Cg_1} \times \frac{-1}{\sqrt{2\xi_1\xi_R}}(b_1 - b_1^+)(b_R - b_R^+) \\ &+ 8E_{Cg_2} \times \frac{-1}{\sqrt{2\xi_2\xi_R}}(b_2 - b_2^+)(b_R - b_R^+) + E_R\left(\frac{\xi_R}{2}\right)\left((\hat{b}_1 + \hat{b}_R^+)\right)^2 \\ &+ \frac{1}{2}E_{J_1} \times \frac{\xi_1}{2}(\hat{b}_1 + \hat{b}_1^+)^2 + \frac{1}{2}E_{J_2}\frac{\xi_2}{2}(\hat{b}_2 + \hat{b}_2^+)^2 - \frac{1}{24}E_{J_1}\left(\frac{\xi_1}{2}\right)^2 \times (b_1 + b_1^+)^4 \\ &- \frac{1}{24}E_{J_2}\left(\frac{\xi_2}{2}\right)^2(b_2 + b_2^+)^4\end{aligned}}$$

Again by R.W.A ⇒ Terms having No. of anhilatⁿ operators ≠ No. of creatⁿ operators can be neglected as they rotate rapidly & avg to zero

$$\begin{aligned}\therefore \hat{H} = &+\frac{4E_{C1}}{2\xi_1}(2b_1^+b_1 + 1) + \frac{4E_{CR}}{2\xi_R}(2b_R^+b_R + 1) + \frac{4E_{C_2}}{2\xi_2}(2b_2^+b_2 + 1) \\ &+ \frac{-8E_{Cg_1}}{\sqrt{2\xi_1\xi_R}}(-b_1^+b_R - \hat{b}_1\hat{b}_R^+) - \frac{8E_{Cg_2}}{\sqrt{2\xi_2\xi_2}}(-\hat{b}_2^+\hat{b}_R - \hat{b}_2\hat{b}_R^+) \\ &+ \frac{E_R\xi_R}{2}(b_R^+b_R + b_Rb_R^+) + \frac{1}{4}E_{J_1}\xi_1(b_1^+b_1 + b_1b_1^+) + \frac{1}{4}E_{J_2}\xi_2(b_2^+b_2 + b_2b_2^+) \\ &- \frac{1}{24}E_{J_1}\left(\frac{\xi_1^2}{4}\right)\left(\frac{1}{2}\hat{b}_1^+\hat{b}_1^+\hat{b}_1\hat{b}_1 + \hat{b}_1^+\hat{b}_1\right) - \frac{1}{24}E_{J_2}\left(\frac{\xi_2^2}{4}\right)\left(\frac{1}{2}\hat{b}_2^+\hat{b}_2^+\hat{b}_2\hat{b}_2 + \hat{b}_2^+\hat{b}_2\right)\end{aligned}$$

Neglecting const. terms ⇒

$$\hat{H} = \frac{4E_{C_1}}{\xi_1}(b_1^+b_1) + \frac{4E_{CR}}{\xi_R}(b_R^+b_R) + \frac{4E_{C_2}}{\xi_2}(b_2^+b_2) + \frac{8E_{Cg_1}}{\sqrt{2\xi_1\xi_R}}(\hat{b}_1^+\hat{b}_R + \hat{b}_1\hat{b}_R^+)$$

+ continuation on next page.

$$\hat{H} = \frac{4E_{C1}}{\xi_1}(b_1^\dagger b_1) + \frac{4E_{CR}}{\xi_R}(b_R^\dagger b_R) + \frac{4E_{C2}}{\xi_2}(b_2^\dagger b_2) + \frac{8E_{cg_1}}{\sqrt{2\xi_1\xi_R}}(\hat{b}_1^\dagger \hat{b}_R + \hat{b}_1 \hat{b}_R^\dagger)$$

$$+ \frac{8E_{cg_2}}{\sqrt{2\xi_1\xi_R}}(b_2^\dagger b_R + b_2 b_R^\dagger) + E_R\xi_R(b_R^\dagger b_R) + \frac{1}{2}E_{J_1}\xi_1(b_1^\dagger b_1) + \frac{1}{2}E_{J_2}\xi_2(b_2^\dagger b_2)$$

$$- \frac{1}{24}\frac{E_{J_1}\xi_1^2}{4}\left(\frac{1}{2}\hat{b}_1^\dagger\hat{b}_1^\dagger \hat{b}_1\hat{b}_1 + \hat{b}_1^\dagger\hat{b}_1\right) - \frac{1}{24}E_{J_2}\left(\frac{\xi_2^2}{4}\right)\left(\frac{1}{2}\hat{b}_2^\dagger\hat{b}_2^\dagger \hat{b}_2\hat{b}_2 + \hat{b}_2^\dagger\hat{b}_2\right)$$

Separating, $(\hat{b}_1^\dagger \hat{b}_1)$, $(\hat{b}_2^\dagger \hat{b}_2)$, $(\hat{b}_R^\dagger \hat{b}_R)$, $(\hat{b}_i^\dagger \hat{b}_j + \hat{b}_i \hat{b}_j^\dagger)$, Anharmonic term

$$\hat{H} = \left(\frac{4E_{C1}}{\xi_1} + \frac{1}{2}E_{J_1}\xi_1\right)b_1^\dagger b_1 + \left(\frac{4E_{CR}}{\xi_R} + E_R\xi_R\right)b_R^\dagger b_R + \left(\frac{4E_{C2}}{\xi_2} + \frac{1}{2}E_{J_2}\xi_2\right)b_2^\dagger b_2$$

$$+ \frac{8E_{cg_1}}{\sqrt{2\xi_1\xi_R}}(\hat{b}_1^\dagger\hat{b}_R + \hat{b}_1\hat{b}_R^\dagger) + \frac{8E_{cg_2}}{\sqrt{2\xi_1\xi_R}}(b_2^\dagger b_R + b_2 b_R^\dagger)$$

$$- \frac{1}{24}\frac{E_{J_1}\xi_1^2}{4}\left(\frac{1}{2}\hat{b}_1^\dagger\hat{b}_1^\dagger\hat{b}_1\hat{b}_1 + b_1^\dagger b_1\right) - \frac{1}{24}E_{J_2}\left(\frac{\xi_2^2}{4}\right)\left(\frac{1}{2}b_2^\dagger b_2^\dagger b_2 b_2 + \hat{b}_2^\dagger\hat{b}_2\right)$$

Putting, $\boxed{\omega_1 = \left(\frac{4E_{C1}}{\xi_1} + \frac{1}{2}E_{J_1}\xi_1\right)}$, $\boxed{\omega_2 = \left(\frac{4E_{C2}}{\xi_2} + \frac{1}{2}E_{J_2}\xi_2\right)}$,

$\boxed{\omega_R = \left(\frac{4E_{CR}}{\xi_R} + E_R\xi_R\right)}$, $\boxed{g_{1R} = \frac{8E_{cg_1}}{\sqrt{2\xi_1\xi_R}}, \quad g_{2R} = \frac{8E_{cg_2}}{\sqrt{2\xi_2\xi_R}}}$

$$\boxed{\alpha_i = -\frac{\xi_i^2 E_{J,i}}{8}}$$

$$\therefore \hat{H} = \omega_1 b_1^\dagger b_1 + \omega_2 b_2^\dagger b_2 + \omega_R b_R^\dagger b_R + g_{1R}(\hat{b}_1^\dagger\hat{b}_R + \hat{b}_1\hat{b}_R^\dagger)$$

$$+ g_{2R}(b_2^\dagger b_R + b_2 b_R^\dagger) + \frac{\alpha_1}{12}\left(\frac{1}{2}\hat{b}_1^\dagger\hat{b}_1^\dagger b_1 b_1 + b_1^\dagger b_1\right) + \frac{\alpha_2}{12}\left(\frac{1}{2}\hat{b}_2^\dagger\hat{b}_2^\dagger\hat{b}_2\hat{b}_2 + \hat{b}_2^\dagger\hat{b}_2\right)$$

$$\therefore \boxed{\hat{H} = \underbrace{H_1 + H_2 + H_R}_{\text{Free Hamiltonian}} + \underbrace{H_{1R} + H_{2R}}_{\text{Coupling Hamiltonian}} + \underbrace{H_{\text{Anharmonic}}}_{\text{Anharmonic Hamiltonian}} + \underbrace{H_{/2}}_{\text{Interact'n Hamiltonian}}}$$

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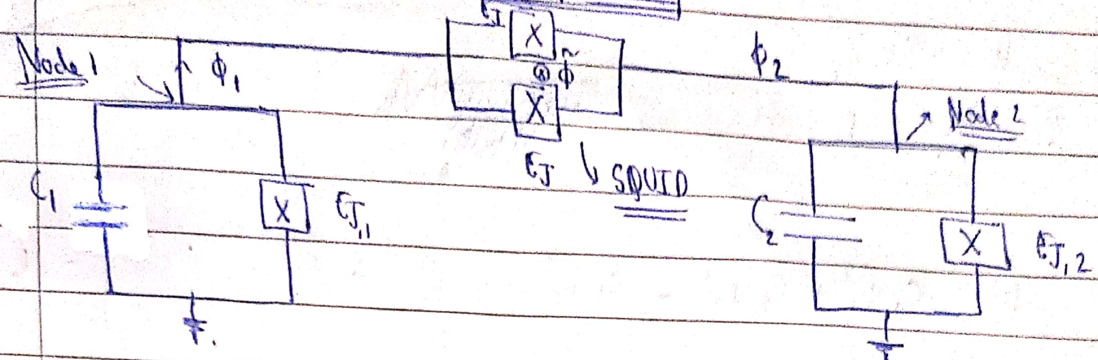
(3) Tunable coupling via a Squid | transmon



Step ① ⇒ $\phi_1 = \phi_{C_1} = -\phi_{E_{J,1}}$

Kirchhoff's law @ node ⇒ $\phi_2 = \phi_{C_2} = -\phi_{E_{J,2}}$

In squid ⇒ $\phi_{\text{above branch}} = -\phi_{\text{Below branch}} \Rightarrow \phi_1 - \phi_2 + \frac{\tilde{\phi}}{2} = -\left(\phi_2 - \phi_1 + \frac{\tilde{\phi}}{2}\right)$

Step 2 ⇒ $L = T_{cap} - V_{ind}^{\,0} - U_{squid} - U_{JJ}$

$$T_{cap} = \frac{C_1}{2}\dot{\phi}_1^2 + \frac{C_2}{2}(\dot{\phi}_2)^2,$$

$$U_{squid} = -E_J\cos\left(\phi_1 - \phi_2 + \frac{\tilde{\phi}}{2}\right) - E_J\cos\left(\phi_2 - \phi_1 + \frac{\tilde{\phi}}{2}\right)$$

$\boxed{\tilde{\phi} = \text{external magnetic flux}}$ ⇒ dividing equally b/w two arms of DC squid.

$$\boxed{U_{squid} = -2E_J\cos\left(\frac{\tilde{\phi}}{2}\right)\cos\left((\phi_1 - \phi_2)\right)}$$

$\phi_1 - \phi_2 + \tilde{\phi} = 2\pi K$ ⇒ fluxoid quantization condⁿ ⇒ reduces the DOF ~~of system~~

K ↳ integer

$$= -2E_J\cos\left(\frac{2\pi K - \phi_1 + \phi_2}{2}\right)\cos(\phi_1 - \phi_2)$$

$$= -2E_J\cos\left(\frac{\tilde{\phi}}{2}\right)\cos(\tilde{\phi}) = \boxed{-E_J\left(\cos\frac{3\tilde{\phi}}{2}\right) - E_J\cos\left(\frac{\tilde{\phi}}{2}\right)}$$

∴ Josephson energy can be dynamically tuned through ext. flux $\tilde{\phi}$

$$\boxed{U_{JJ} = -E_{J_1}\cos\phi_1 - E_{J_2}\cos(\phi_2)}$$

$$L = \frac{C_1\dot{\phi}_1^2}{2} + \frac{C_2\dot{\phi}_2^2}{2} + E_J\cos\left(\frac{3\tilde{\phi}}{2}\right) + E_J\cos\left(\frac{\tilde{\phi}}{2}\right) + E_{J_1}\cos\phi_1 + E_{J_2}\cos\phi_2$$

Step 3 $\Rightarrow$ $H = \sum p_i \dot{q}_i - L$

$p_1 = \frac{\partial L}{\partial \dot{\phi}_1}$, $p_2 = \frac{\partial L}{\partial \dot{\phi}_2}$ $\qquad q_1 = \phi_1,\ q_2 = \phi_2,$

$$H = C_1\dot{\phi}_1^2 + C_2\dot{\phi}_2^2 - \frac{C_1\dot{\phi}_1^2}{2} - \frac{C_2\dot{\phi}_2^2}{2} - E_J\cos\left(\frac{3\tilde{\phi}}{2}\right) - E_J\cos\left(\frac{\tilde{\phi}}{2}\right)$$

$$- E_{J_1}\cos\phi_1 - E_{J_2}\cos\phi_2$$

$$H = \frac{C_1\dot{\phi}_1^2}{2} + \frac{C_2\dot{\phi}_2^2}{2} - E_J\cos\left(\frac{3\tilde{\phi}}{2}\right) - E_J\cos\left(\frac{\tilde{\phi}}{2}\right) - E_{J_1}\cos\phi_1$$

$$- E_{J_2}\cos\phi_2$$

Step 4 $\Rightarrow$ $H = 1/2\, p^T C^{-1} p - E_J\cos\left(\frac{3\tilde{\phi}}{2}\right) - E_J\cos\left(\frac{\tilde{\phi}}{2}\right) - E_{J_1}\cos\phi_1 - E_{J_2}\cos\phi_2$

where $p \Rightarrow \begin{bmatrix} C_1\dot{\phi}_1 \\ C_2\dot{\phi}_2 \end{bmatrix} = \underbrace{\begin{bmatrix} C_1 & 0 \\ 0 & C_2 \end{bmatrix}}_{C} \begin{bmatrix} \dot{\phi}_1 \\ \dot{\phi}_2 \end{bmatrix}$

$$C^{-1} = \frac{1}{C_1C_2}\begin{bmatrix} C_2 & 0 \\ 0 & C_1 \end{bmatrix}$$

Step 5 $\Rightarrow$

$\phi_n \rightarrow \hat{\phi}_n$

$p_n \rightarrow \hat{p}_n$

$H \rightarrow \hat{H}$

$$H = \frac{1}{2}\hat{n}^T C^{-1}\hat{n} - E_J\cos\left(\frac{3\hat{\tilde{\phi}}}{2}\right) - E_J\cos\left(\frac{\hat{\tilde{\phi}}}{2}\right) - E_{J_1}\cos\hat{\phi}_1 - E_{J_2}\cos\hat{\phi}_2$$

$E_{C_{n,n}} = \frac{(C^{-1})_{n,n}}{8}$, $E_{C_1} = \frac{C_2}{8C_1C_2}$, $E_{C_2} = \frac{C_1}{8C_1C_2}$

$$H = 4E_{C_1}\hat{n}_1^2 + 4E_{C_2}\hat{n}_2^2 - E_J\cos\left(\frac{3\hat{\tilde{\phi}}}{2}\right) - E_J\cos\left(\frac{\hat{\tilde{\phi}}}{2}\right)$$

$$- E_{J_1}\cos\phi_1 - E_{J_2}\cos\phi_2$$

$$E_J \cos\left(\frac{3\tilde{\phi}}{2}\right) = E_J\left(1 - \frac{9\tilde{\phi}^2}{8} + \frac{81\tilde{\phi}^4}{16\times 24} + \cdots\right)$$

$$E_J \cos\left(\frac{\tilde{\phi}}{2}\right) = E_J\left(1 - \frac{\tilde{\phi}^2}{8} + \frac{\tilde{\phi}^4}{16\times 24} + \cdots\right)$$

Now, $\boxed{\hat{\tilde{\phi}} = 2\pi k - \hat{\phi}_1 + \hat{\phi}_2}$

Neglecting const & higher order terms

$$E_J \cos\left(\frac{3\hat{\tilde{\phi}}}{2}\right) = -\frac{9}{8}E_J\left(2\pi k - \hat{\phi}_1 + \hat{\phi}_2\right)^2 + \frac{E_J\, 81}{16\times 24}\left(2\pi k - \hat{\phi}_1 + \hat{\phi}_2\right)^4$$

$$E_J \cos\left(\frac{\hat{\tilde{\phi}}}{2}\right) = -\frac{E_J}{8}\left(2\pi k - \hat{\phi}_1 + \hat{\phi}_2\right)^2 + \frac{E_J\left(2\pi k - \hat{\phi}_1 + \hat{\phi}_2\right)^4}{16\times 24} + \cdots$$

$$\boxed{E_{J_i} \cos\hat{\phi}_i = -\frac{1}{2}E_J\hat{\phi}_i^2 + \frac{1}{24}E_J\hat{\phi}_i^4}$$

$$H = 4E_{C_1}\hat{n}_1^2 + 4E_{C_2}\hat{n}_2^2 + \frac{9}{8}E_J\left(2\pi k - \hat{\phi}_1 + \hat{\phi}_2\right)^2 - \frac{E_J\, 81}{24\times 16}\left(2\pi k - \hat{\phi}_1 + \hat{\phi}_2\right)^4$$
$$+ \frac{E_J}{8}\left(2\pi k - \hat{\phi}_1 + \hat{\phi}_2\right)^2 - \frac{E_J}{16\times 24}\left(2\pi k - \hat{\phi}_1 + \hat{\phi}_2\right)^4 + \frac{1}{2}E_{J_1}\hat{\phi}_1^2$$
$$+ \frac{1}{2}E_{J_2}\hat{\phi}_2^2 + \frac{1}{24}E_{J_1}\hat{\phi}_1^4 + \frac{1}{24}E_{J_2}\hat{\phi}_2^4$$

for simplicity, Neglecting the higher (fourth order) terms

$$H = 4E_{C_1}\hat{n}_1^2 + 4E_{C_2}\hat{n}_2^2 + \frac{9}{8}E_J\left(4\pi^2k^2 - 4\pi k(\hat{\phi}_1 - \hat{\phi}_2) + (\hat{\phi}_1 - \hat{\phi}_2)^2\right)$$
$$+ \frac{E_J}{8}\left(4\pi^2k^2 - 4\pi k(\hat{\phi}_1 - \hat{\phi}_2) + (\hat{\phi}_1 - \hat{\phi}_2)^2\right)$$
$$+ \frac{1}{2}E_{J_1}\hat{\phi}_1^2 + 1/2\, E_{J_2}\hat{\phi}_2^2$$

(w/o operator terms)

Neglecting the const. terms $\Rightarrow$

$$H = 4E_{C_1}\hat{n}_1^2 + 4E_{C_2}\hat{n}_2^2 + \frac{9}{8}E_J\left(-4\pi k(\hat{\phi}_1 - \hat{\phi}_2) + (\hat{\phi}_1 - \hat{\phi}_2)^2\right)$$
$$+ \frac{E_J}{8}\left(-4\pi k(\hat{\phi}_1 - \hat{\phi}_2) + (\hat{\phi}_1 - \hat{\phi}_2)^2\right) + 1/2\, E_{J_1}\hat{\phi}_1^2 + 1/2\, E_{J_2}\hat{\phi}_2^2$$

$$\boxed{H = 4E_{C_1}\hat{n}_1^2 + 4E_{C_2}\hat{n}_2^2 + \frac{\cancel{10}^{5} E_J}{\cancel{8}_4}\left(-4\pi k(\hat{\phi}_1 - \hat{\phi}_2) + (\hat{\phi}_1 - \hat{\phi}_2)^2\right) + \frac{1}{2}E_{J_1}\hat{\phi}_1^2 + \frac{1}{2}E_{J_2}\hat{\phi}_2^2}$$

Now $H_{SHO} = 4E_{C_1}\hat{n}_1^2 + 4E_{C_2}\hat{n}_2^2 + \frac{5E_J}{4}\left(\phi_1^2\right) + \frac{5E_J}{4}\left(\phi_2^2\right) + \frac{1}{2}E_{J_1}\phi_1^2 + 1/2\, E_{J_2}\phi_2^2$

Define ⇒ impedance ①

$$\boxed{\xi_1 = \sqrt{\frac{M_{eq}}{k_{eq}}} = \sqrt{\frac{4E_{C_1}}{\frac{5E_J}{4} + 1/2\, E_{J_1}}}, \quad \xi_2 = \sqrt{\frac{4E_{C_2}}{\frac{5E_J}{4} + 1/2\, E_{J_2}}}}$$

② $\hat{\phi}_i$ & $\hat{n}_i$ ⇒ in terms of anhilatⁿ & creatⁿ operators ⇒

$$\boxed{\hat{\phi}_i = \sqrt{\frac{\xi_i}{2}}\left(\hat{b}_i + \hat{b}_i^\dagger\right), \quad \hat{n}_i = \frac{i}{\sqrt{2\xi_i}}\left(\hat{b}_i - \hat{b}_i^\dagger\right)}$$

$$\hat{H} = -4E_{C_1} \times \frac{1}{\left(\sqrt{2\xi_1}\right)^2}\left(\hat{b}_1 - \hat{b}_1^\dagger\right)^2 + -4E_{C_2} \times \frac{1}{\left(\sqrt{2\xi_2}\right)^2}\left(\hat{b}_2 - \hat{b}_2^\dagger\right)^2$$

$$+ \frac{5E_J}{4}\left(-4\pi K\left(\sqrt{\frac{\xi_1}{2}}\left(b_1 + b_1^\dagger\right)\right) - 4\pi K\left(\sqrt{\frac{\xi_2}{2}}\left(b_2 + b_2^\dagger\right)\right)\right)$$

$$+ \frac{5E_J}{4}\left(\frac{\xi_1}{2}\left(\hat{b}_1 + \hat{b}_1^\dagger\right)^2 + \frac{\xi_2}{2}\left(b_2 + b_2^\dagger\right)^2 - \sqrt{\frac{\xi_1\xi_2}{2}}\left(b_1 + b_1^\dagger\right)\left(b_2 + b_2^\dagger\right)\right)$$

$$+ \frac{1}{2}E_{J_1}\left(\frac{\xi_1}{2}\left(b_1 + b_1^\dagger\right)^2\right) + 1/2\, E_{J_2}\left(\frac{\xi_2}{2}\left(b_2 + b_2^\dagger\right)^2\right)$$

By R.W.A ⇒ Terms having No. of anhilatⁿ operators ≠ No. of creatⁿ operators can be neglected as they rotate rapidly & avg out to zero (Non-conserving terms)

$$\hat{H} = -\frac{4E_{C_1}}{2\xi_1}\left(-\hat{b}_1^\dagger\hat{b}_1 - \hat{b}_1\hat{b}_1^\dagger\right) - \frac{4E_{C_2}}{2\xi_2}\left(-\hat{b}_2^\dagger\hat{b}_2 - \hat{b}_2\hat{b}_2^\dagger\right)$$

$$+ \frac{5E_J}{4}\left(\frac{\xi_1}{2}\left(\cancel{\hat{b}_1^2} + \hat{b}_1\hat{b}_1^\dagger + \hat{b}_1^\dagger\hat{b}_1 + \cancel{\hat{b}_1^{\dagger 2}}\right) + \frac{\xi_2}{2}\left(b_2 b_2^\dagger + b_2^\dagger b_2\right)\right)$$

$$- \frac{5E_J}{4}\sqrt{\frac{\xi_1\xi_2}{2}}\left(b_1^\dagger b_2 + \cancel{b_1^\dagger b_2^\dagger} + b_1 b_2 + \cancel{b_1 b_2^\dagger}\right)$$

$$+ 1/2\, E_{J_1}\frac{\xi_1}{2}\left(b_1 b_1^\dagger + b_1^\dagger b_1\right) + 1/2\, E_{J_2}\frac{\xi_2}{2}\left(b_2 b_2^\dagger + b_2^\dagger b_2\right)$$

$$+ \frac{5E_J}{4}K - 4\pi K\left(\sqrt{\frac{\xi_1}{2}}\left(b_1 + b_1^\dagger\right) + \left(b_2 + b_2^\dagger\right)\sqrt{\frac{\xi_2}{2}}\right)$$

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Neglecting const. terms & using $[b, b^\dagger] = 1$

$$\therefore \hat{H} = \frac{4E_{C_1}}{\xi_1}(\hat{b}_1^\dagger \hat{b}_1) + \frac{4E_{C_2}}{\xi_2}(b_2^\dagger b_2)$$

$$+ \frac{5E_J}{4} \times \frac{\xi_1}{2}(2\hat{b}_1^\dagger \hat{b}_1) + \frac{5E_J}{4}\frac{\xi_2}{2}(2\hat{b}_2^\dagger \hat{b}_2)$$

$$- \frac{5E_J}{4}\sqrt{\frac{\xi_1\xi_2}{2}}(b_1^\dagger b_2 + b_1^\dagger b_2) + \frac{1}{2}E_{J_1}\xi_1(b_1^\dagger b_1) + \frac{1}{2}E_{J_2}\xi_2(b_2^\dagger b_2) + \dots$$

$$= \left(\frac{4E_{C_1}}{\xi_1} + \frac{5E_J}{4}\xi_1 + \frac{1}{2}E_{J_1}\xi_1\right)(\hat{b}_1^\dagger \hat{b}_1)$$

$$+ \left(\frac{4E_{C_2}}{\xi_2} + \frac{5E_J}{4}\xi_2 + \frac{1}{2}E_{J_2}\xi_2\right)(\hat{b}_2^\dagger \hat{b}_2)$$

$$- \frac{5E_J}{4}\sqrt{\frac{\xi_1\xi_2}{2}}(b_1^\dagger b_2 + b_1^\dagger b_2)$$

$$- \frac{5E_J}{4} \times 4\pi K\left(\sqrt{\frac{\xi_1}{2}}(b_1 + b_1^\dagger) + \sqrt{\frac{\xi_2}{2}}(b_2 + b_2^\dagger)\right)$$

Putting $\Rightarrow$ $\omega_1 = \left(\frac{4E_{C_1}}{\xi_1} + \frac{5E_J}{4}\xi_1 + \frac{1}{2}E_{J_1}\xi_1\right)$, $\omega_2 = \left(\frac{4E_{C_2}}{\xi_2} + \frac{5E_J}{4}\xi_2 + \frac{1}{2}E_{J_2}\xi_2\right)$

$g_{12} = -\frac{5E_J}{4}\sqrt{\frac{\xi_1\xi_2}{2}}$, $x_1 = -\frac{5E_J}{4} \times 4\pi K \times \sqrt{\frac{\xi_1}{2}}$, $X_2 = -\frac{5E_J}{4} \times 4\pi K\sqrt{\frac{\xi_2}{2}}$

$$\boxed{\therefore \hat{H} = \underbrace{\omega_1 \hat{b}_1^\dagger \hat{b}_1 + \omega_2 \hat{b}_2^\dagger \hat{b}_2}_{H_0 \Rightarrow \text{Free Hamiltonian}} + \underbrace{g_{12}(b_1^\dagger b_2 + b_1^\dagger b_2)}_{H_{\text{interaction}}} + \underbrace{x_1(b_1 + b_1^\dagger) + X_2(b_2 + b_2^\dagger)}}$$

$+ H_{\text{anharmonic}}$