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~~② Holevo Bound ⇒~~

~~Alice has classical R.V X that she wants to send to Bob. Hence, she prepares a quantum state ρ_X~~

② Holevo Bound ⇒

① Alice has classical info source $X = 0, 1, 2, \dots n$ i.e classical R.V she wants to send to Bob. Hence she prepares quantum state ρ_X with probabilities $P_0, P_1, P_2, \dots P_n$

i.e $\boxed{\rho = \{P_X, \rho_X\}_{X=0 \text{ to } n}} = \boxed{\sum_{X=0}^{n} P_X \rho_X}$. Now Bob wants to know X, hence performs P.O.V.M measurement by elements $\{E_0, \dots E_m\} = \{E_y\}$ on ρ_X & gets classical outcome $\boxed{y = 0, 1, 2, \dots m}$

② ~~[illegible]~~ ~~Bob~~ Amount of Informat^n Bob can get about X from his measurement result y is best given by mutual info. b/w^n X & y

i.e $I(X:y)$.

③ If he wants to maximise $I(X:y)$, he should choose POVM's in such a way that maximise $I(X:y)$, & this info. can be termed as accessible Info Bob can get from y about X.

$$\therefore \boxed{I_{accessible} = \max_{\{E_y\}} I(X:y)}$$

④ In general, Bob can infer X from y completely if & only if $\boxed{I(X:y) = H(X)}$

Closeness of $I(X:y)$ with $H(X)$ is quantitative measure of how well Bob can determine $\boxed{X}$ from y.

⑤ $I(X:y) = H(X)$ is possible only when states $\boxed{\rho_X}$ are orthogonal as there are no perfect sets of POVM's possible to distinguish two Non orthogonal Quantum states.

⑥ So what is upper bound of $I(X:y)$ when $\boxed{\rho_X}$ are non-orthogonal states, i.e there are the optimum set of POVM's that gives upper bound on $I(X:y)$ when ρ_X are non-orthogonal states prepared by Alice. ⇒ This upper bound is called as Holevo Bound.

⑦ Holevo Bound or Holevo Info or Holevo Quantity = $\boxed{S(\rho) - \sum_X P_X S(\rho_X)}$ (H)

where $\boxed{\rho = \sum_X P_X \rho_X}$

(8) We know, $\boxed{I_{accessible} = \max_{\{E_y\}} I(X:Y)}$

Now, we have to prove ⇒ $\boxed{I(X:Y) \leq S(\rho) - \sum_x p_x S(\rho_x)}$

i.e $\boxed{\max_{\{E_y\}} I(X:Y) = S(\rho) - \sum_x p_x S(\rho_x) < H(X)}$ for Non orthogonal ρ_x

& $\boxed{\max_{\{E_y\}} I(X:Y) = S(\rho) - \sum_x p_x S(\rho_x) = H(X)}$ for orthogonal ρ_x

↳ Entropy of preparation of X i.e source entropy

(9) Note ⇒ for Non-orthogonal ρ_x ~~states~~,

Y can't perfectly infer X ⇐ $\boxed{I_{accessible} < H(X)}$ ⇒ consequence of no cloning theorem as,

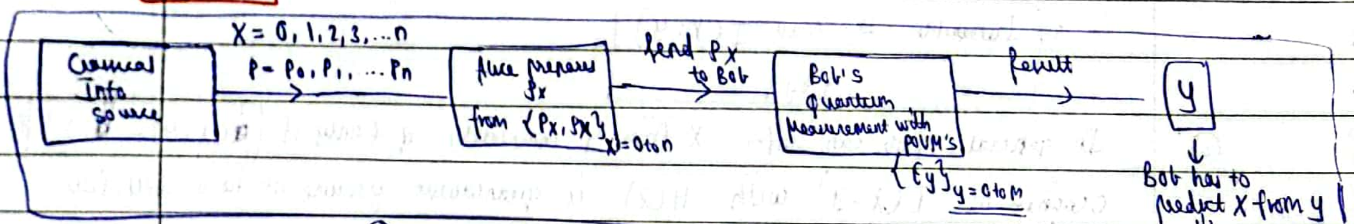
if cloning is possible i.e Bob can clone state sent by Alice n times, & for large n, ρ_x states can be nearly distinguishable with POVM with very high probability of success i.e ρ_x states nearly become orthogonal

i.e $\boxed{I_{accessible} \approx H(X)}$ if cloning is possible

(10) Hence, now we have to prove (H) $\boxed{I_{accessible} = S(\rho) - \sum_x p_x S(\rho_x)}$

i.e $\boxed{I(X:Y) \leq S(\rho) - \sum_x p_x S(\rho_x)}$

Proof ⇒



(H) Overall communication process

(1) Classical RV 'X' can be expressed as $\boxed{\rho^X = \sum_{x=0}^{n} p_x |x\rangle\langle x|}$ w.r.to some orthonormal basis $\{|x\rangle\}_{x=0}^{n}$

~~By writing~~ $\boxed{\text{Von-Neumann entropy } S(X) = \text{Shannon Entropy } H(X)}$ as ρ^X is classical state

(H) i.e $S(X) = -\mathrm{tr}(\rho^X \log \rho^X) = -\mathrm{tr}\left(\sum_{x=0}^{n} p_x \log p_x |x\rangle\langle x|\right)$

$$= -\sum_{x=0}^{n} p_x \log p_x = H(X)$$

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(2) Alice Now prepares ρ_x with probability p_x,
hence Alice's state that she sends to Bob $\Rightarrow$ $\boxed{\rho^{X\Phi} = \sum_{x=0}^{n} p_x |x\rangle\langle x| \otimes \rho_x}$

where $\Phi \Rightarrow$ represents Quantum ~~states~~ system $\boxed{\rho \Rightarrow \sum_x p_x \rho_x}$

(3) Now Alice sends $\boxed{\rho^{X\Phi}}$ to Bob but Bob only has access to $\boxed{\rho^{\Phi}}$

$\therefore \rho^{\Phi} = tr_X(\rho^{X\Phi}) = tr_X \sum_{x=0}^{n} p_x |x\rangle\langle x| \otimes \rho_x = \sum_{x=0}^{n} p_x \rho_x \underbrace{tr_X(|x\rangle\langle x|)}_{\langle x|x\rangle = 1}$

$\boxed{\text{i.e. } \rho^{\Phi} = \sum_{x=0}^{n} p_x \rho_x}$ $\Rightarrow$ Bob gets Mixed State

(4) Bob measures this mixed state with POVM elements $\boxed{\{E_y\}_{y=0}^{m}}$
& get outcomes $\boxed{y = 0, 1, 2, \dots m}$ with probability $\{q_y\}_{y=0}^{m}$ to form
classical output R.V $\underline{y}$.

(5) Consider $\underline{X, \Phi, M}$ be 3 systems $\Rightarrow$ (1) $X \Rightarrow$ Preparation System having orthonormal basis $|x\rangle$ corresponding to label $\boxed{0, \dots n}$ on possible preparation for quantum system Φ.

(2) $\Phi \Rightarrow$ Quantum system $\boxed{\rho = \sum_x p_x \rho_x}$

(3) $M \Rightarrow$ Measurement System/Apparatus of Bob having $\boxed{|y\rangle}$ basis having possible outcomes $\boxed{0, 1, \dots, n}$

(6) $\therefore$ Composite System state that represents entire communication $\Rightarrow$

$$\rho^{X\Phi M} = \sum_x p_x |x\rangle\langle x| \otimes \rho_x \otimes \underbrace{|0\rangle\langle 0|}_{\text{Before Measurement Bob's Measurement apparatus state}}$$

(7) Let $X'\Phi'M'$ be state of composite system after application of $\mathcal{E}$ (Quantum Instrument)

$$\rho^{X'\Phi'M'} = \boxed{I^{X} \otimes \mathcal{E}^{\Phi M}} \rho^{X\Phi M} = \sum_{x}\sum_{y} p_x |x\rangle\langle x| \otimes \mathcal{E}^{\Phi M}(\rho_x) \otimes |0\rangle\langle 0|$$

$$\mathcal{E}^{\Phi M}(\sigma \otimes |0\rangle\langle 0|) = \sum_{y=0}^{m} \sqrt{E_y}\, \sigma \sqrt{E_y} \otimes |y\rangle\langle y| \longrightarrow \text{O/P stored of Measurement in } M$$

where σ is one of state of $\boxed{\rho_0, \rho_1, \dots \rho_n}$ $\rightarrow$ Post Measurement State

(8)

$$\therefore \boxed{\rho^{X'\Phi'M'} = \sum_{x=0}^{n}\sum_{y=0}^{m} p_x |x\rangle\langle x| \otimes \sqrt{E_y}\, \rho_x \sqrt{E_y} \otimes |y\rangle\langle y|}$$

Now, Bob discards the $\boxed{\sqrt{E_y}\, \rho_x \sqrt{E_y}}$ state as Measurement result is already stored in $|y\rangle$

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(8) Now, $\rho^{X'M'} = Tr_{\varphi}(\rho^{X'\varphi'M'})$

$$= Tr_{\varphi} \sum_{xy} p_x |x\rangle\langle x| \otimes \sqrt{E_y}\,\rho_x\sqrt{E_y} \otimes |y\rangle\langle y|$$

$$Tr(\sqrt{E_y}\,\rho_x\sqrt{E_y}) = Tr(\rho_x E_y)$$

Now, $\boxed{P(y|x) = Tr(\rho_x E_y)}$ → probability of outcome y when given state is ρ_x

$$\therefore \rho^{X'M'} = \sum_{x,y} p_x P(y|x)\, |x\rangle\langle x| \otimes |y\rangle\langle y|$$

$$p_x P(y|x) = P(x,y) \Rightarrow \text{joint probability of } x \,\&\, y$$

$$\boxed{\rho^{X'M'} = \sum_{x,y} p(x,y)\, |x\rangle\langle x| \otimes |y\rangle\langle y|}$$

$$\therefore \rho^{X'M'} = \rho^{XY} = \sum_{x,y} P(x,y)\, |xy\rangle\langle xy|$$

$$\boxed{\begin{aligned}\rho^X &= tr_y(\rho^{XY}) = \sum_{x=0}^{n} p(x)\,|x\rangle\langle x| \\ \rho^Y &= tr_x(\rho^{XY}) = \sum_{y=0}^{m} \underbrace{p(y)}_{q_y}\,|y\rangle\langle y|\end{aligned}} \quad \Big\}\ \text{classical registers i.e. classical state}$$

Von Neumann Mutual Info betn X & Y → $\therefore S(X':M') = S(X:Y) = S(X) + S(Y) - S(X,Y)$

$$S(X) = -tr(\rho^X \log \rho^X) = -tr\left(\sum_{x=0}^{n} P(x)|x\rangle\langle x| \log\left(\sum_{x=0}^{n} P(x)|x\rangle\langle x|\right)\right)$$

$$= \boxed{-\sum_{x=0}^{n}\left(p(x)\log p(x)\right)} = H(X)$$

$$S(Y) = -tr(\rho^Y \log \rho^Y) = \boxed{\sum_{y=0}^{m} p(y)|y\rangle\langle y|} = H(Y)$$

$$S(X,Y) = -tr(\rho^{XY}\log \rho^{XY}) = \sum_{x=0,y=0}^{x=n,y=m} P(x,y)\log P(x,y) = H(X,Y)$$

as ρ^X, ρ^Y & $\rho^{X,Y}$ are classical states, corresponding Von-Neumann entropies are equal to Shannon Entropies

(11) $$\boxed{S(X':M') = S(X:Y) = H(X:Y) = I(X:Y)} \Rightarrow \text{LHS of Holevo Bound}$$

Now $S(X:\varphi) = S(X:\varphi, M) \geq S(X':\varphi':M') \geq S(X':M')$

Before Measurement

Quantum operation $\mathcal{E}$ causes info loss

or Measurement can't increase mutual info betn $X \& \varphi$

trace causes info. loss

or Removing / tracing out system can't increase mutual info

$$\therefore \boxed{S(X:\varphi) \geq \boxed{S(X':M') = S(X:Y)}} \quad - \text{(I)}$$

$S(X:\varphi) \Rightarrow$ Mutual Von Neumann Info betn X & quantum state φ

$$\rho^X = \sum_x p_x |x\rangle\langle x| \Rightarrow S(X) = H(X) \quad - (1)$$

$$\rho^\varphi = \sum_x p_x \rho_x \Rightarrow S(\varphi) = S(\rho) \quad - (2)$$

$$\rho^{X\varphi} = \sum_x p_x |x\rangle\langle x| \otimes \rho_x$$

Let spectral decompositn of ρ_x be $\rho_x = \sum_z \lambda_z^x |z^x\rangle\langle z^x|$

$z \Rightarrow$ arbitrarily chosen basis that depend on x

$$S(X,\varphi) = -\mathrm{tr}\left(\rho^{X\varphi} \log(\rho^{X\varphi})\right)$$

$$= -\mathrm{tr}\left(\sum_x p_x |x\rangle\langle x| \otimes \underbrace{\rho_x}_{\sum_z \lambda_z^x |z^x\rangle\langle z^x|} \log\left(\sum_x p_x |x\rangle\langle x| \otimes \underbrace{\rho_x}_{\sum_z \lambda_z^x |z^x\rangle\langle z^x|}\right)\right)$$

$$= -\mathrm{tr}\left(\sum_{x,z} p_x \lambda_z^x \left(|x\rangle|z^x\rangle\langle x|\langle z^x|\right) \times \log\left(\sum_{x,z} p_x \lambda_z^x \left(|x z^x\rangle\langle x z^x|\right)\right)\right)$$

$$= -\sum_{x,z} p_x \lambda_z^x \log\left(\sum_{x,z} p_x \lambda_z^x\right) = -\sum_{x,z} p_x \lambda_z^x \left(\log p_x + \log \lambda_z^x\right)$$

$$= -\sum_{x,z} p_x \lambda_z^x \left(\log p_x + \log \lambda_z^x\right)$$

$\sum_z \lambda_z^x = 1 \Rightarrow$ Normalized ρ_x

$$= -\sum_{x,z} p_x \lambda_z^x \log p_x - \sum_{x,z} p_x \underbrace{\lambda_z^x \log \lambda_z^x}_{S(\rho_x)}$$

(1)

$$= -\sum_{x,z} p_x \log p_x - \sum_{x,z} p_x S(\rho_x) = \boxed{H(X) + \sum_x p_x S(\rho_x)} \quad - (3)$$

Putting from ①, ② & ③, in Ⓐ

$$\boxed{S(X:\varphi) = S(\rho) - \sum_{k=1}^{n} p_k S(\rho_k)}$$

from Ⓘ $\Rightarrow$ $S(X:\varphi) \geq S(X:Y)$

$$\boxed{\therefore\ S(\rho) - \sum_{k=1}^{n} p_k S(\rho_k) \geq I(X:Y)}$$

$$\boxed{\therefore\ S(\rho) - \sum_{k=1}^{n} p_k S(\rho_k) = \max_{\{E_y\}} I(X:Y) = I_{\text{accessible}}}$$

③

$$\boxed{I_{\text{accessible}} = \max_{\{E_y\}} I(X:Y) = \underbrace{S(\rho) - \sum_{k=1}^{n} p_k S(\rho_k)}_{\chi}} \leq H(X)$$

To show $\boxed{\chi \leq 1}$

χ = Holevo Bound $= S(\rho) - \sum_{k=1}^{n} p_k S(\rho_k)$

as $|X_1\rangle, |X_2\rangle, |X_3\rangle, |X_4\rangle$ are pure states.

$$\boxed{\begin{array}{l} S(\rho_1) = S(|X_1\rangle\langle X_1|) = -tr(\rho_1 \log \rho_1) = 0 \\ S(\rho_2) = 0, \quad S(\rho_3) = 0, \quad S(\rho_4) = 0 \end{array}}$$

$$\chi = S(\rho) = -tr(\rho \log \rho)$$

$$\rho = \frac{1}{4}\Big(|X_1\rangle\langle X_1| + |X_2\rangle\langle X_2| + |X_3\rangle\langle X_3| + |X_4\rangle\langle X_4|\Big)$$

$$= \frac{1}{4}\Big(|0\rangle\langle 0| + 1/3\,(|0\rangle + \sqrt{2}|1\rangle)(\langle 0| + \sqrt{2}\langle 1|) + \frac{1}{3}(|0\rangle + \sqrt{2}e^{2\pi i/3}|1\rangle)(\langle 0| + \sqrt{2}e^{-2\pi i/3}\langle 1|)$$

$$+ 1/3\,[(|0\rangle + \sqrt{2}e^{4\pi i/3}|1\rangle)(\langle 0| + \sqrt{2}e^{-4\pi i/3}\langle 1|)\Big)$$

$$= \frac{1}{4}\left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + 1/3\begin{bmatrix} 1 & \sqrt{2} \\ \sqrt{2} & 2 \end{bmatrix} + 1/3\begin{bmatrix} 1 & \sqrt{2}e^{-2\pi i/3} \\ \sqrt{2}e^{2\pi i/3} & 2 \end{bmatrix} + 1/3\begin{bmatrix} 1 & \sqrt{2}e^{-4\pi i/3} \\ \sqrt{2}e^{4\pi i/3} & 2 \end{bmatrix}\right)$$

$$= \frac{1}{4}\left(\begin{bmatrix}1 & 0\\ 0 & 0\end{bmatrix} + \begin{bmatrix}1/3 & \sqrt{2}/3\\ \sqrt{2}/3 & 2/3\end{bmatrix} + \begin{bmatrix}2/3 & \frac{\sqrt{2}}{3}\left(e^{-2\pi i/3} + e^{-4\pi i/3}\right)\\ \frac{\sqrt{2}}{3}\left(e^{2\pi i/3} + e^{4\pi i/3}\right) & 4/3\end{bmatrix}\right)$$

$$= \frac{1}{4}\begin{bmatrix}6/3 & 0\\ 0 & 6/3\end{bmatrix} \Rightarrow \begin{bmatrix}1/2 & 0\\ 0 & 1/2\end{bmatrix}$$

$$\therefore \quad \chi = S(\rho) = -tr\left(\begin{pmatrix}1/2 & 0\\ 0 & 1/2\end{pmatrix}\begin{pmatrix}-1 & 0\\ 0 & -1\end{pmatrix}\right)$$

$\boxed{\chi = 1} \Rightarrow$ Holevo' Upper bound

$\therefore \boxed{I_{accesible} \leq 1}$

$$H(X) = -\left(\frac{1}{4}\log_2\left(\frac{1}{4}\right)\right) \times 4 = \boxed{2}$$

Now we have to construct E_y such that it maximizes $I(X:Y)$

$$S(X':M') = I(X:Y) = S(X') + S(Y') - S(X',Y')$$

$$\rho^{X'M'} = \rho^{XY} = \sum_{xy} P(x,y)\, |xy\rangle\langle xy|$$

Now x varies from 0 to 4,

$$\rho^{X'M'} = \sum_y P(0,y)\,|0y\rangle\langle 0y| + P(1,y)\,|1y\rangle\langle 1y| + P(2,y)\,|2y\rangle\langle 2y| + P(3,y)\,|3y\rangle\langle 3y|$$

$$= P(x)\sum_y P(y|0)\,|0y\rangle\langle 0y| + P(y|1)\,|1y\rangle\langle 1y| + P(y|2)\,|2y\rangle\langle 2y| + P(y|3)\,|3y\rangle\langle 3y|$$

$$= \frac{1}{4}\sum_{y=1}^{m} Tr(\rho_0 E_y)\,|0y\rangle\langle 0y| + Tr(\rho_1 E_y)\,|1y\rangle\langle 1y| + Tr(\rho_2 E_y)\,|2y\rangle\langle 2y| + Tr(\rho_3 E_y)\,|3y\rangle\langle 3y|$$

$$\rho^{X'M'} = \frac{1}{4}\begin{bmatrix}\begin{pmatrix}Tr(E_0\rho_0) & 0 & 0\\ 0 & Tr(E_1\rho_0) & 0\\ 0 & 0 & Tr(E_m\rho_0)\end{pmatrix} & & & 0\\ & \begin{pmatrix}Tr(E_0\rho_1) & \\ & Tr(E_m\rho_1)\end{pmatrix} & & \\ & & \ddots & \\ 0 & & & \begin{pmatrix}Tr(E_0\rho_3) & \\ & Tr(E_m\rho_3)\end{pmatrix}\end{bmatrix}_{4m\times 4m}$$

$$\rho^{X'} = Tr_{M'}\left(\rho^{X'M'}\right) = \frac{1}{4}\begin{bmatrix} \sum_{y=0}^{M} Tr(E_y\rho_0) & 0 & 0 & 0 \\ 0 & \sum_{y=0}^{M} Tr(E_y\rho_1) & 0 & 0 \\ 0 & 0 & \sum_{y=0}^{M} Tr(E_y\rho_2) & 0 \\ 0 & 0 & 0 & \sum_{y=0}^{M} Tr(E_y\rho_3) \end{bmatrix}_{4\times 4}$$

$$\rho^{Y'} = Tr_X\left(\rho^{X'M'}\right) = \frac{1}{4}\begin{bmatrix} \sum_{x=0}^{3} Tr(E_0\rho_x) & 0 & 0 \\ 0 & \sum_{x=0}^{3} Tr(E_1\rho_x) & 0 \\ 0 & 0 & \ddots \; \sum_{x=0}^{3} Tr(E_M\rho_x) \end{bmatrix}_{M\times M}$$

$$I(X:Y) = -tr(\rho^X \log \rho^X) - tr(\rho^Y \log \rho^Y) + tr(\rho^{XY} \log \rho^{XY})$$

Now, to optimise / max the $I(X:Y)$ over POVM's $\{E_y\}$,

Let us take case $M = 2 \Rightarrow E_0 + E_1 = I$

Define $E_0 = \frac{1}{2}\begin{bmatrix} 1+\cos\theta & \sin\theta\, e^{-i\phi} \\ \sin\theta\, e^{i\phi} & 1-\cos\theta \end{bmatrix}$

$E_1 = I - \frac{1}{2}\begin{bmatrix} 1+\cos\theta & \sin\theta\, e^{i\phi} \\ \sin\theta\, e^{i\phi} & 1-\cos\theta \end{bmatrix}$

The problem with above definitn is $\underline{E_0 \,\&\, E_1}$ does not have +ve real eigen values for all θ & ϕ, as E_0 & E_1, for both of them to be +ve, semidefinite it should have <u>Real positive eigen values</u>

$\therefore$ we consider only those θ & ϕ values for which eigen values of E_0 & E_1 are greater than equal to zero i.e +ve real eigen values

Now, we use python for iterating E_0 & E_1, for $0 \leq \theta \leq 180$ & $0 \leq \phi \leq 360$ over all the values of θ & ϕ such that which gives E_0 & E_1 eigen values are +ve, real & then break the loop only when,

$I(X:Y)$ is maximized

$$\boxed{I(X:Y) = -tr\left(\rho^{X'} \log \rho^{X'}\right) - tr\left(\rho^{Y'} \log \rho^{Y'}\right) + tr\left(\rho^{X'Y'} \log \rho^{X'Y'}\right)}$$

i.e $\left|I(X:Y) - \chi\right| = \underline{\text{error} < \varepsilon}$

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So for ⇒ ① Case 1 ⇒ $M = 2$, we see $I(X:Y)$ is maximized for,

$$E_0 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad E_1 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

& $I_{accessible} = \max(I(X:Y)) = \underline{0.3710}$ for $\underline{\phi = 10°, \theta = 0°}$

② Case 2 ⇒ $M = 3$, we see $I(X:Y)$ is max. for,

$$\underset{E_\theta}{E_0} = \frac{1}{2}\begin{bmatrix} 1.3090 & -0.9510 \\ -0.9510 & 0.6909 \end{bmatrix}, \quad \underset{E_{\theta+r}}{E_1} = \frac{1}{2}\begin{bmatrix} 1.3090 & -0.9510 \\ -0.9510 & 0.6909 \end{bmatrix}, \quad \underset{I - E_\theta - E_{\theta+r}}{E_2} = \begin{bmatrix} -0.3090 & 0.9510 \\ 0.9510 & -0.3090 \end{bmatrix}$$

varies from 0 to 360°

$I_{accessible} = \max I(X:Y) = \boxed{0.8110}$ for $\boxed{\theta = 72°, r = 0°, \phi = 180°}$

③ Case 3 ⇒ $M = 4$, we see $I(X:Y)$ is max for,

$$\underset{E_\theta}{E_0} = \begin{bmatrix} & \\ & \end{bmatrix}, \quad \underset{E_{\theta+r}}{E_1} = \begin{bmatrix} & \\ & \end{bmatrix}, \quad \underset{E_{\theta+r+n}}{E_2} = \begin{bmatrix} & \\ & \end{bmatrix}$$

$$\underset{I - (E_0 + E_1 + E_2)}{E_3} = \begin{bmatrix} & \\ & \end{bmatrix} \Rightarrow$$

We can increase the number of POVM elements to as much as we want, but runtime for algorithm on Python increases exponentially, ~~hence~~ as running for $M = 4$ itself will take $\boxed{11 \text{ days}}$ (if we iterate for single degree change in θ, ϕ, r, n)

$\boxed{\text{Note}}$ ⇒ Python coding of above ~~the~~ logic was implemented in collaboratⁿ with Vishal M (friend from Aerospace Engg)

$\boxed{\text{Observatⁿ}}$ ⇒ As we increase the no. of POVM's to 4 we can see increase in max $I(X:Y)$ & we have achieved the max $I(X:Y)$ nearly equal to Holevo Bound.