$\boxed{\text{QT 207 Assignment}}$ $\Rightarrow$

① Generic $U \Rightarrow \quad U = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$$UU^{+} = I \Rightarrow \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} a^{+} & c^{+} \\ b^{+} & d^{+} \end{pmatrix} = I$$

$|a|^2 + |b|^2 = 1, \quad ac^{+} + bd^{+} = 0, \quad a^{+}c + db^{+} = 0, \quad |c|^2 + |d|^2 = 1$

$\boxed{a, b, c, d \in \mathbb{C}}$

A pair of complex no. $a, b \in \mathbb{C}$ satisfying $|a|^2 + |b|^2 = 1$ can always be parametrized as, as $a = |a|e^{i\alpha}$

$a = e^{i\alpha_{11}}\cos\theta, \quad b = e^{i\alpha_{12}}\sin\theta$ for some real coefficients $\alpha_{ij}, \theta \in \mathbb{R}$

It follows that using Normalization constraint $\Rightarrow U = \begin{pmatrix} e^{i\alpha_{11}}\cos\theta & e^{i\alpha_{12}}\sin\theta \\ e^{i\alpha_{21}}\sin\theta & e^{i\alpha_{22}}\cos\theta \end{pmatrix}$

Using the orthogonality condition, $\boxed{X_1 X_2^T = 0}$, X_1 & X_2 are eigen vectors of U

$$e^{i(\alpha_{11} - \alpha_{12})} + e^{i(\alpha_{21} - \alpha_{22})} = 0$$

i.e $\boxed{\alpha_{11} = \alpha_{12} + \alpha_{21} - \alpha_{22} + \pi}$

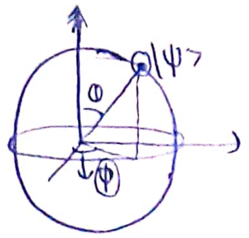
$\therefore$ We conclude that U is parameterized by 4 real parameters $\Rightarrow \boxed{\theta, \alpha_{12}, \alpha_{21}, \alpha_{22}}$

i.e for 2×2 Matrix $\Rightarrow 4 \Rightarrow$ $\boxed{\text{for } N\times N \text{ Matrix} \Rightarrow \boxed{N^2} \text{ real parameters}}$

Now consider a general state $|\psi\rangle$ on Bloch sphere $\Rightarrow$ $\boxed{|\psi\rangle = \cos\frac{\theta}{2}|0\rangle + e^{i\phi}\sin\theta/2\,|1\rangle}$

$\hookrightarrow$ pure state

$$\rho = |\psi\rangle\langle\psi| = \left(\cos\theta/2\,|0\rangle + e^{i\phi}\sin\theta/2\,|1\rangle\right)\left(\cos\theta/2\,\langle 0| + e^{i\phi}\sin\theta/2\,\langle 1|\right)$$



$$= \cos^2\theta/2\,|0\rangle\langle 0| + \cos\theta/2\, e^{i\phi}\sin\theta/2\,|1\rangle\langle 1|$$
$$+ e^{i\phi}\sin\theta/2 \times \cos\theta/2\,|1\rangle\langle 0| + (e^{i\phi}\sin\theta/2)^2\,|1\rangle\langle 1|$$

$$= \cos^2\theta/2 \begin{pmatrix}1\\0\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} + \cos\theta/2\, e^{i\phi}\sin\theta/2 \begin{pmatrix}0\\1\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix}$$
$$+ e^{i\phi}\sin\theta/2\cos\theta/2 \begin{pmatrix}0\\1\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix} + (e^{i\phi}\sin\theta/2)^2 \begin{pmatrix}0\\1\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix}$$

$$\boxed{\rho = \begin{bmatrix} \cos^2\theta/2 & \cos\theta/2\sin\theta/2\, e^{i\phi} \\ \cos\theta/2\sin\theta/2\, e^{+i\phi} & \sin^2\theta/2 \end{bmatrix}_{2\times 2}} \Rightarrow \text{Real parameters } \boxed{\theta, \phi}$$

$\therefore$ for 2×2 Normalized pure density Matrix $\Rightarrow$ No. of Real parameters $\boxed{2}$

i.e $\quad 2N - 2 \Rightarrow 2\times2 - 2 = \boxed{2}$

In general, if $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$,

$$\rho = |\psi\rangle\langle\psi| = \begin{pmatrix} |\alpha|^2 & \alpha\beta^{*} \\ \alpha^{*}\beta & |\beta|^2 \end{pmatrix} \Rightarrow$$

$\boxed{\begin{array}{l} 1^{st} \text{ condition} \Rightarrow ① \text{ Normalization} \Rightarrow |\alpha|^2 + |\beta|^2 = 1 \\ 2^{nd} \text{ condition } ② \ \rho \text{ is Hermitian} \\ 3^{rd} \text{ condition } ③ \ \rho^2 = \rho \end{array}}$

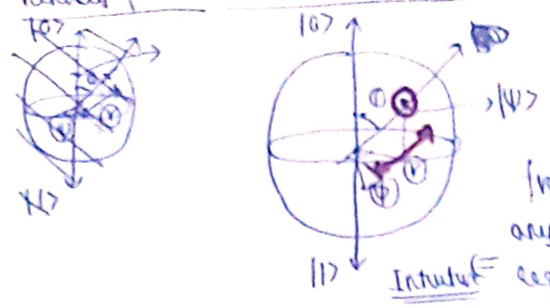
where $\boxed{\alpha, \beta \in \mathbb{C}}$, if one complex parameter is required to parametrize ρ_{pure}

$(\alpha \text{ or } \beta) \Rightarrow$ i.e 2 real parameters

$\downarrow$ $a + ib$ $\quad\quad \boxed{(a, b)}$

for 2x2 Normalized mixed state,

(ex) ⇒ Density Matrix inside a Bloch sphere



$$\rho = \sum_n p_n |\psi\rangle\langle\psi|$$

$$\boxed{\rho = \begin{bmatrix} |\alpha|^2 & 0 \\ 0 & |\beta|^2 \end{bmatrix}} \Rightarrow \text{Normalized mixed density Matrix}$$

from fig it is clear that for representing any state $|\psi\rangle$ inside Bloch sphere we (1+)

Intuitively require $\boxed{\theta, \phi \ \& \ r}$ ⇒ $\boxed{3 \text{ real parameters}}$

i.e for 2x2 Mixed Normalized density matrix ⇒ $N^2 - 1$

⇒ $4 - 1$ ⇒ 3 real parameters

In general 1st constraint ⇒ Normalization ⇒ $\boxed{|\alpha|^2 + |\beta|^2 = 1}$ ⇒ α (or) β ⇒ $\mathbb{C}$ ⇒ 2 real parameters

2nd constraint ⇒ $\boxed{r < 1}$ ⇒ 1 real parameter

radius while representing on bloch sphere

⇒ $\boxed{3 \text{ parameters}}$

∴ In general for NxN Normalized mixed state ⇒ $\boxed{N^2 - 1}$ real parameters

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2 (a) Let $U = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}$ a simple $\boxed{2\times2 \text{ unitary Matrix}}$ whose determinant is $\boxed{1}$

(1) Closure under multiplication

$$\begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}\begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix} = \underbrace{\begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}}_{[A]} \rightarrow \boxed{|A| = 1} \ \& \ \boxed{[A] \in U} \text{ as } \boxed{AA^\dagger = I}$$

⇒ Closed under multiplication

(2) Identity ⇒ (2x2) ⇒ $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \in \boxed{U}$ ⇒ Identity exists

(3) Inverse ⇒ $A^{-1} = \frac{1}{|A|}\begin{bmatrix} 0 & -i \\ -i & 0 \end{bmatrix} = \frac{1}{1}\begin{bmatrix} 0 & -i \\ -i & 0 \end{bmatrix}$ → $|A^{-1}| = 0 - (i^2) = \boxed{1}$

$\boxed{A^{-1} \in U}$ ⇒ Inverse exists

∴ $\boxed{([U], \underset{\text{multiplication}}{\times}) \text{ is a group}}$

(b) $$e^{\left(i\frac{\theta}{2}\hat{n}\cdot\vec{\sigma}\right)} = \sum_{n=0}^{n=\infty} \frac{\left(i\frac{\theta}{2}\hat{n}\cdot\vec{\sigma}\right)^n}{n!}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}\left(\frac{\theta}{2}\hat{n}\cdot\vec{\sigma}\right)^{2n} + \sum_{n=0}^{\infty} \frac{i(-1)^n}{(2n+1)!}\left(\frac{\theta}{2}\hat{n}\cdot\vec{\sigma}\right)^{2n+1}$$

$$\sigma_n^2 \ \boxed{= (\hat{n}\cdot\vec{\sigma})^2 = I} \quad \text{(from property of pauli matrices)}$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}\left(\frac{\theta}{2}\right)^{2n} + \sum_{n=0}^{\infty} \frac{i(-1)^n}{(2n+1)!}\left(\left(\frac{\theta}{2}\right)^{2n+1} (\hat{n}\cdot\vec{\sigma})^{2n} \times (\hat{n}\cdot\vec{\sigma})\right)$$

$$= \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}\left(\frac{\theta}{2}\right)^{2n} + \sum_{n=0}^{\infty} \frac{i(-1)^n}{(2n+1)!}\left(\left(\frac{\theta}{2}\right)^{2n+1} (\hat{n}\cdot\vec{\sigma})\right)$$

$\therefore$ $e^{i\alpha}$ $\left(\cos\left(\frac{\theta}{2}\right)[I] + i\sin\left(\frac{\theta}{2}\right)(\hat{n}.\vec{\sigma}) \right) = \boxed{e^{\left(i\left(\frac{\theta}{2}\right)\hat{n}.\vec{\sigma}\right)}}$

can be ignored (global phase)

Now any matrix can be expanded in terms of $\boxed{I, \sigma_x, \sigma_y, \sigma_z}$

Considering $\boxed{[U] \text{ given in Sir's Notes}}$

$$\cos\frac{\theta}{2}[I] + i\sin\frac{\theta}{2}(\hat{n}.\vec{\sigma}) = \begin{bmatrix} \cos\theta/2 & 0 \\ 0 & \cos\theta/2 \end{bmatrix} - i\sin\theta/2 \left[-\sin\phi\, \hat{i} + \cos\phi\, \hat{j} + 0\hat{k} \right].$$

$$\left[\sigma_1 \hat{i} + \sigma_2 \hat{j} + \sigma_3 \hat{k} \right]$$

$$= \begin{bmatrix} \cos\theta/2 & 0 \\ 0 & \cos\theta/2 \end{bmatrix} - i\sin\theta/2 \left[-\sin\phi\, \underset{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}}{\sigma_1} + \cos\phi\, \underset{\begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}}{\sigma_2} + 0 \right]$$

$$= \begin{bmatrix} \cos\theta/2 & 0 \\ 0 & \cos\theta/2 \end{bmatrix} - \begin{bmatrix} 0 & i\sin\theta/2 \sin\phi \\ -i\sin\theta/2 \sin\phi & 0 \end{bmatrix} + \begin{bmatrix} 0 & \cos\phi \sin\theta/2 \\ -\cos\phi\sin\theta/2 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} \cos\theta/2 & \sin\theta/2\, e^{i\phi} \\ -\sin\theta/2\, e^{i\phi} & \cos\theta/2 \end{bmatrix} \Rightarrow \underline{\text{Unitary Matrix}}$$

$$\boxed{U_{2\times2} = e^{\left(i\frac{\theta}{2}\hat{n}.\vec{\sigma}\right)} = \cos(\theta/2)\, I + i\sin(\theta/2)\, \hat{n}.\vec{\sigma}}$$

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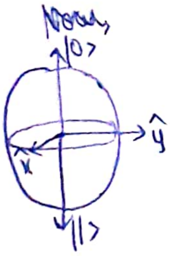
③ Let $|\psi\rangle = \cos\theta/2\, |0\rangle + e^{i\phi}\sin\theta/2\, |1\rangle$ (Pure state

① Let the guess be $|\phi\rangle$ → (which can take any value on Bloch sphere)

for $\boxed{\text{simplicity}}$ let $\boxed{|\phi\rangle = |0\rangle}$ $\boxed{(+z \text{ axis})}$

Probability of $|\psi\rangle$ to be $|0\rangle$ state

$$P_\phi \Rightarrow |\langle\phi|\psi\rangle|^2 = \langle 0|\psi\rangle^2 = \cos^2\theta/2 = \boxed{\frac{1+\cos\theta}{2}}$$



Now, Any random state $|\phi\rangle$ in $\hat{x}-\hat{y}$ plane has fidelity $= \boxed{1/2}$

As it has equally likely chance to be in the upper hemisphere as well as lower hemisphere

Average value / Expectation value of fidelity

$$\langle F\rangle \Rightarrow \frac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi} (P_\phi) \times \underset{①}{R^2}\sin\theta\, d\theta\, d\phi = \frac{1}{4\pi}[2\pi - 0]\int_0^{\pi}\left(\frac{1+\cos\theta}{2}\right)\sin\theta\, d\theta$$

unit radius of Bloch sphere

$$= \frac{1}{2}\int_0^{\pi} \cos^2\theta/2 \times 2\sin\theta/2\cos\theta/2\, d\theta$$

$$= \frac{1}{2}\int_0^{\pi} \cos^3\theta/2 \sin\theta/2 \times (2)\, d\theta$$

$$= 1/2 \int_0^{\pi/2} \cos^3\theta/2 \sin\theta/2 = \boxed{1/2}$$

⑥ Now the measurement along z direction / axis (corresponding to projection operators is producing a mixed state, $\rho_m = P_+ \langle\psi|P_+|\psi\rangle + P_-\langle\psi|P_-|\psi\rangle$

$$P_+ = |0\rangle\langle 0|, \quad P_- = |1\rangle\langle 1|$$

$$\therefore \rho_m = |0\rangle\langle 0|\left(\cos\theta/2\langle 0| + e^{-i\phi}\sin\theta/2\langle 1|\right)\left(|0\rangle\langle 0|\right)\left(\cos\theta/2\, |0\rangle + e^{i\phi}\sin\theta/2|1\rangle\right)$$

$$+ |1\rangle\langle 1|\left(\cos\theta/2\, \langle 0| + e^{-i\phi}\sin\theta/2\, \langle 1|\right)\left(|1\rangle\langle 1|\right)\left(\cos\theta/2|0\rangle + e^{i\phi}\sin\theta/2|1\rangle\right)$$

$$\rho_m = |0\rangle\langle 0| \cos^2\theta/2 + |1\rangle\langle 1| \sin^2\theta/2$$ $\Rightarrow$ mixed state

$\therefore$ Now ~~Average fidelity~~ $\rightarrow \boxed{\langle\psi|\rho_m|\psi\rangle}$ $= (\cos\theta/2 \langle 0| + e^{-i\phi}\sin\theta/2\langle 1|)$

~~Probability~~ / Expectation value of Mixed state

$$(|0\rangle\langle 0|\cos^2\theta/2 + |1\rangle\langle 1|\sin^2\theta/2)$$
$$(\cos\theta/2|0\rangle + e^{i\phi}\sin\theta/2|1\rangle)$$

$$= \boxed{\cos^4\theta/2 + \sin^4\theta/2}$$

Now (Average fidelity over the sphere) $\Rightarrow$

$$\langle f\rangle = \frac{1}{4\pi}\int_0^{2\pi} d\phi \int_0^{\pi} \sin\theta \left(\cos^4\theta/2 + \sin^4\theta/2\right) d\theta$$

$$= \boxed{2/3}$$

(c) As per the given questⁿ $\Rightarrow$ ① $f_a = |\langle\phi|\psi\rangle|^2 = \cos^2\theta/2 = \boxed{\frac{1+\cos\theta/2}{2}}$

$\boxed{\langle f_a\rangle = 1/2}$

② $f_b = \langle\psi|\rho_m|\psi\rangle = \cos^4\theta/2 + \sin^4\theta/2$, $\boxed{\langle f_b\rangle = 2/3}$

$\therefore \boxed{f_b > f_a}$

$\because$ In case ① we are randomly guessing the state $|\psi\rangle$ to $|\phi\rangle$ hence depending on our guess the closeness to the actual value has range ~~to~~ $\boxed{0 \text{ to } 1}$ i.e Average chance of correct guessing $= \boxed{1/2}$. i.e Average fidelity $= 1/2$

But In case ②, by measurement of the state in $\pm z$ direction we are getting a mixed state which is always a better approximatⁿ of given state than randomly guessing the states & hence when we calculate its closeness (fidelity) to actual ~~guess~~ state it is always greater than case 1.

(5) $|\phi^+\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$ $\quad |\psi^+\rangle = \frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$

$|\phi^-\rangle = \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle)$ $\quad |\psi^-\rangle = \frac{1}{\sqrt{2}}(|01\rangle - |10\rangle)$

(1) $\rho_{\phi^+} = |\phi^+\rangle\langle\phi^+| = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)\frac{1}{\sqrt{2}}(\langle 00| + \langle 11|)$

$= \frac{1}{2}(|00\rangle\langle 00| + |00\rangle\langle 11| + |11\rangle\langle 00| + |11\rangle\langle 11|)$

$= \frac{1}{2}\Big(\begin{pmatrix}1\\0\end{pmatrix}\otimes\begin{pmatrix}1\\0\end{pmatrix}(10)\otimes(10) + \begin{pmatrix}1\\0\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}(01)\times(01)$

$+ \begin{pmatrix}0\\1\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix}(10)(10) + \begin{pmatrix}0\\1\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix}(1\ 0)(1\ 0)\Big)$

$= \frac{1}{2}\Big[\begin{pmatrix}1\\0\\0\\0\end{pmatrix}(1000) + \begin{pmatrix}1\\0\\0\\0\end{pmatrix}(0001) + \begin{pmatrix}0\\0\\0\\1\end{pmatrix}(1000)$

$+ \begin{pmatrix}0\\0\\0\\1\end{pmatrix}(1000)\Big]$

$= 1/2\left[\begin{bmatrix}1&0&0&0\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{bmatrix} + \begin{bmatrix}0&0&0&1\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{bmatrix} + \begin{bmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\1&0&0&0\end{bmatrix}\right.$

$\left. + \begin{bmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\1&0&0&0\end{bmatrix}\right]$

$$\boxed{\rho_{\phi^+} = 1/2\begin{bmatrix}1&0&0&1\\0&0&0&0\\0&0&0&0\\1&0&0&1\end{bmatrix}}$$

Similarly,

(2) $\rho_{\psi^+} = |\psi^+\rangle\langle\psi^+| = 1/\sqrt{2}(|01\rangle + |10\rangle)\left(\frac{1}{\sqrt{2}}(\langle 01| + \langle 10|)\right)$

$= 1/2\begin{bmatrix}0&0&0&0\\0&1&1&0\\0&1&1&0\\0&0&0&0\end{bmatrix}$

(3) $\rho_{\phi^-} = |\phi^-\rangle\langle\phi^-| = 1/\sqrt{2}(|00\rangle - |11\rangle)\left(\frac{1}{\sqrt{2}}(\langle 00| - \langle 11|\right)$

$= 1/2\begin{bmatrix}1&0&0&-1\\0&0&0&0\\0&0&0&0\\-1&0&0&1\end{bmatrix}$

(4) $\rho_{\psi^-} = |\psi^-\rangle\langle\psi^-| = 1/\sqrt{2}(|01\rangle - |10\rangle) \times 1/\sqrt{2}(|01\rangle - |10\rangle)$

$= 1/2\begin{bmatrix}0&0&0&0\\0&1&-1&0\\0&-1&1&0\\0&0&0&0\end{bmatrix}$

$$\rho_{\psi^+} = \frac{1}{2}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \frac{1}{2}\left[\frac{I \otimes I - \sigma_{z_1} \otimes \sigma_{z_2}}{2} + \left(\frac{\sigma_{x_1} \otimes \sigma_{x_2} + \sigma_{y_1} \otimes \sigma_{y_2}}{2} \right) \right]$$

$$\rho_{\psi^-} = \frac{1}{2}\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \frac{1}{2}\left[\frac{I \otimes I - \sigma_{z_1} \otimes \sigma_{z_2}}{2} - \left(\frac{x_1 \otimes x_2 + \sigma_{y_1} \otimes \sigma_{y_2}}{2} \right) \right]$$

$$\rho_{\phi^+} = \frac{1}{2}\begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix} = \frac{1}{2}\left[\left(\frac{I \otimes I + \sigma_{z_1} \otimes \sigma_{z_2}}{2} \right) + \left(\frac{\sigma_{x_1} \otimes x_2 - \sigma_{y_1} \otimes \sigma_{y_2}}{2} \right) \right]$$

$$\rho_{\phi^-} = \frac{1}{2}\begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 1 \end{bmatrix} = \frac{1}{2}\left[\left(\frac{I \otimes I + \sigma_{z_1} \otimes \sigma_{z_2}}{2} \right) - \left(\frac{\sigma_{x_1} \otimes \sigma_{x_2} - \sigma_{y_1} \otimes \sigma_{y_2}}{2} \right) \right]$$

Required answer

where $(\sigma_{z_1} \otimes \sigma_{z_2}) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

$$(\sigma_{x_1} \otimes \sigma_{x_2}) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$$

$$(\sigma_{y_1} \otimes \sigma_{y_2}) = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & +1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix}$$

$$(I \otimes I) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(5) Given state $\Rightarrow |X\rangle_{ABE} = |00\rangle_{AB}|e_{00}\rangle_E + |01\rangle_{AB}|e_{01}\rangle_E + |10\rangle_{AB}|e_{10}\rangle_E + |11\rangle_{AB}|e_{11}\rangle_E$

Applying measurement constraint $M(\sigma_1^A \sigma_1^B \otimes I_E) = -1$ ~~is~~

$\langle X|_{ABE}\, \sigma_1^A \sigma_1^B \otimes I_E\, |X\rangle_{ABE} \Rightarrow (\langle 00|_{AB}\langle e_{00}|_E + \langle 01|_{AB}\langle e_{01}|_E + \langle 10|_{AB}\langle e_{10}|_E + \langle 11|_{AB}\langle e_{11}|_E)$

$(|11\rangle_{AB}|e_{00}\rangle_E + |10\rangle_{AB}|e_{01}\rangle_E + |01\rangle_{AB}|e_{10}\rangle_E + |00\rangle_{AB}|e_{11}\rangle_E)$

$$\boxed{\Rightarrow \langle e_{00}|e_{11}\rangle + \langle e_{01}|e_{10}\rangle + \langle e_{10}|e_{01}\rangle + \langle e_{11}|e_{11}\rangle = (-1) \Rightarrow \text{(given)}} \quad -(1)$$

Now measurement constraint $M(\sigma_3^A \sigma_3^B \otimes I_E) = -1$ is imposed on state

$\langle X|_{ABE}\, \sigma_3^A \sigma_3^B \otimes I_E\, |X\rangle_{ABE} \Rightarrow (\langle 00|_{AB}\langle e_{00}|_E + \langle 01|_{AB}\langle e_{01}|_E + \langle 10|_{AB}\langle e_{10}|_E + \langle 11|_{AB}\langle e_{11}|_E)$

$(|00\rangle_{AB}|e_{00}\rangle_E - |01\rangle_{AB}|e_{01}\rangle_E - (|10\rangle_{AB}|e_{10}\rangle_E + |11\rangle_{AB}|e_{11}\rangle_E)$

$$= \boxed{\langle e_{00}|e_{00}\rangle - \langle e_{01}|e_{01}\rangle - \langle e_{10}|e_{10}\rangle + \langle e_{11}|e_{11}\rangle = (-1)} \quad -(2)$$

$=$ for (1) $-$ (2) $\Rightarrow \boxed{\begin{matrix}|e_{00}\rangle = |e_{11}\rangle \\ |e_{01}\rangle = -|e_{10}\rangle\end{matrix}} \Rightarrow$ (A)

for (1) + (2) & substituting (A)

$$\boxed{\langle e_{01}|e_{01}\rangle - \langle e_{00}|e_{00}\rangle = 1/2} \Rightarrow -(3)$$

Assuming $\langle X|X\rangle = 1$ i.e $\langle X|$ state is normalized

$$\boxed{\langle e_{00}|e_{00}\rangle + \langle e_{01}|e_{01}\rangle + \langle e_{10}|e_{10}\rangle + \langle e_{11}|e_{11}\rangle = 1} \quad -(4)$$

Putting (A) in (4)

$$\boxed{\langle e_{00}|e_{00}\rangle + \langle e_{01}|e_{01}\rangle = 1/2} \quad -(5)$$

from (3) & (5) $\Rightarrow \boxed{\langle e_{01}|e_{01}\rangle = 1/2}$ & $\boxed{|e_{00}\rangle = |e_{11}\rangle = 0}$

~~$\langle e_{01}|e_{01}\rangle = 1/2$~~

$$\therefore \boxed{|X\rangle_{ABE} = |01\rangle_{AB}|e_{01}\rangle_E + |10\rangle_{AB}|e_{10}\rangle_E}$$

$$\boxed{|X\rangle_{ABE} = (|01\rangle_{AB} - |10\rangle_{AB})|e_{01}\rangle_E} \quad \text{(from (A))}$$

Now By normalizing the state $\Rightarrow$

$$\boxed{|X\rangle_{ABE} = \frac{1}{\sqrt{2}}(|01\rangle_{AB} - |10\rangle_{AB})|e_{01}\rangle_E} \Rightarrow \boxed{\text{Bell singlet state}}$$

⑥ $|\phi\rangle_{AB} = \frac{1}{\sqrt{6}}\left[|0\rangle \otimes |0\rangle + \sqrt{2}|1\rangle + |1\rangle \otimes (\sqrt{2}|0\rangle + |1\rangle)\right]$

$$|\phi\rangle_{AB} = \frac{1}{\sqrt{6}}\left[|00\rangle + \sqrt{2}|01\rangle + \sqrt{2}|10\rangle + |11\rangle\right]$$

$$\therefore \rho_{AB} = |\phi\rangle\langle\phi| = \frac{1}{6}\Big[|00\rangle\langle 00| + \sqrt{2}|01\rangle\langle 00| + \sqrt{2}|10\rangle\langle 00| + |11\rangle\langle 00|$$
$$+ \sqrt{2}|00\rangle\langle 01| + 2|01\rangle\langle 01| + 2|10\rangle\langle 01| + \sqrt{2}|11\rangle\langle 01|$$
$$+ \sqrt{2}|00\rangle\langle 10| + 2|01\rangle\langle 10| + 2|10\rangle\langle 10| + \sqrt{2}|11\rangle\langle 10|$$
$$+ |00\rangle\langle 11| + \sqrt{2}|01\rangle\langle 11| + \sqrt{2}|10\rangle\langle 11| + |11\rangle\langle 11|\Big]$$

$$\therefore \rho_A = Tr_B(\rho_{AB}) = \frac{1}{6}\Big[|0\rangle\langle 0| + \sqrt{2}|1\rangle\langle 0| + 2|0\rangle\langle 0|$$
$$+ \sqrt{2}|1\rangle\langle 0| + \sqrt{2}|0\rangle\langle 1| + 2|1\rangle\langle 1|$$
$$+ \sqrt{2}|0\rangle\langle 1| + |1\rangle\langle 1|\Big]$$
$$= 1/6\left[3|0\rangle\langle 0| + 2\sqrt{2}|1\rangle\langle 0| + 2\sqrt{2}|0\rangle\langle 1| + 3|1\rangle\langle 1|\right]$$
$$= 1/6\begin{bmatrix} 3 & 2\sqrt{2} \\ 2\sqrt{2} & 3 \end{bmatrix}$$

$$\therefore \lambda^2 - \lambda + 0.027 = 0 \Rightarrow \lambda = \boxed{0.98, 0.027}$$

Schmidt No $= 2$

$\boxed{\text{Schmidt Basis}} \Rightarrow$ Eigen vectors $\Rightarrow$

$$|X_1\rangle = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \quad |X_2\rangle = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

Normalizing $|X_1\rangle$ & $|X_2\rangle$

$$|X_1\rangle = \frac{|0\rangle + |1\rangle}{2} = |+\rangle, \quad |X_2\rangle = |-\rangle$$

$\Rightarrow$ Set of orthonormal vectors for $\boxed{|\phi_B\rangle}$

$$|\phi_{B_1}\rangle = \frac{(\langle X_1| \otimes I_B)}{\sqrt{\lambda_1}} \times (|\phi\rangle_{AB})$$

$$= \frac{1+\sqrt{2}}{\sqrt{2}\times\sqrt{3+2\sqrt{2}}}|+\rangle = \frac{1}{\sqrt{2}}|+\rangle$$

$$|\phi_{B_2}\rangle = \frac{\langle X_2| \otimes I_B)}{\sqrt{\lambda_2}} \times (|\phi\rangle_{AB}) = \frac{1-\sqrt{2}}{\sqrt{2}\times\sqrt{3-2\sqrt{2}}}|-\rangle$$

$$\boxed{|\phi\rangle_{AB} = \sum_i \sqrt{\lambda_i}\,|X_i\rangle \otimes |\phi_{B_i}\rangle = \sqrt{\lambda_1}\,|X_1\rangle \otimes |\phi_{B_1}\rangle + \sqrt{\lambda_2}\,|X_2\rangle \otimes |\phi_{B_2}\rangle}$$