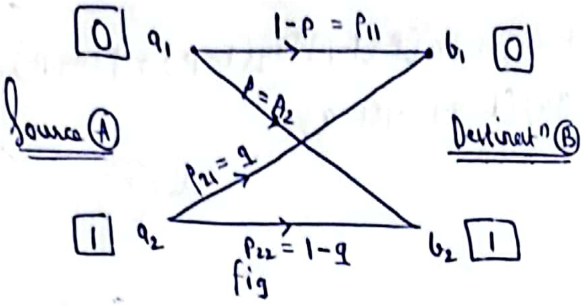
Q1 (a) ⇒ Binary Asymmetric channel ⇒

Channel Matrix ⇒ $P = \begin{bmatrix} 1-p & p \\ q & 1-q \end{bmatrix} = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix}$



fig

Let probability of source A & Destinatn B, be

$$\boxed{P(a_1) = \alpha,\ P(a_2) = 1-\alpha,\ P(b_1) = \beta,\ P(b_2) = 1-\beta}$$

Define entropies of channel ⇒

$H_1 = -P_{11}\log P_{11} - P_{12}\log P_{12}$ ... (channels related to a_1 as i/p) - (1)

$H_2 = -P_{21}\log P_{21} - P_{22}\log P_{22}$ ... (" " to a_2 as input) - (2)

Entropy of Destinatn ⇒ $H(B) = -\beta\log\beta - (1-\beta)\log(1-\beta)$

Condit^nal entropy ⇒ $H(B|A) = \alpha H_1 + (1-\alpha) H_2$

$= H_2 + \alpha(H_1 - H_2)$ - (3)

Mutual Info ⇒ $\boxed{I(A:B) = H(B) - H(B|A) = H(B) - H_2 - \alpha(H_1 - H_2)}$ - (4)

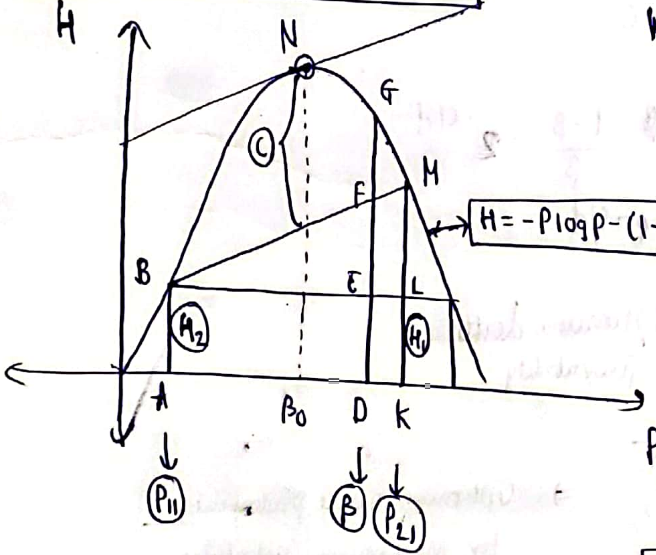
from Fig (1) ⇒ $\beta = \alpha P_{11} + (1-\alpha)P_{21} = \alpha(1-p) + (1-\alpha)q$

$$\Rightarrow \boxed{\alpha = \frac{\beta - q}{1-p-q}} \quad - (5)$$

Now, Channel capacity by def^n ⇒ $\boxed{C = \max_{P(a_i)} I(A:B)}$

i.e We have to find optimum probability of source $P(a_1)$ i.e α for which $I(A:B)$ is max.

Method I ⇒ Graphical Method ⇒



We have to find point N i.e point @ which $I(A:B)$ becomes max i.e optimum α & β

$LM = KM - KL$

$LM = H_1 - H_2$

$\frac{BE}{BL} = \frac{EF}{LM} = \frac{\beta - P_{21}}{P_{11} - P_{21}} = \alpha$

$EF = \alpha(H_1 - H_2)$

$FG = DG - DE - EF$

$= H(B) - H_2 - \alpha(H_1 - H_2)$

$\boxed{FG = I(A:B)}$

$\boxed{\text{C} \Rightarrow \text{Draw } \parallel^{el} \text{ to BM \& tangent to curve.}}$

Optimal probability distribut" of source ⇒ Point $\parallel^{el}$ to BM touches curve @ N.

$$\boxed{P(a_1) = \alpha_0 = \frac{\beta_0 - q}{1-p-q}}$$

(#)

$$\therefore \boxed{\begin{aligned} C = \max_{\alpha} I(A:B) &= -\beta_0 \log \beta_0 - (1-\beta_0)\log(1-\beta_0) + \alpha_0\big((1-p)\log(1-p) + p\log p\big) \\ &+ (1-\alpha_0)\, q\log q + (1-\alpha_0)(1-q)\log(1-q) \end{aligned}}$$

↳ expression for C ↳ (I)

Method 2 ⇒ Analytical Method ⇒

To max. C ⇒ $\boxed{\frac{dI}{d\alpha} = 0}$,

$$\boxed{\begin{aligned} \alpha = \frac{\beta - q}{1-p-q} \;\Rightarrow\; &\beta = \alpha(1-p-q) + q \\ &\frac{d\beta}{d\alpha} = (1-p-q) \end{aligned}}$$

$$I(A:B) = -\beta\log\beta - \log(1-\beta) + \beta\log(1-\beta) + \alpha\big(\log(1-p) - p\log(1-p) + p\log p\big) + (q\log q) - \alpha q\log q + \big(\log(1-q) - \alpha\log(1-q)\big)(1-q)$$

$$\frac{dI}{d\alpha} = 0 \Rightarrow -\beta \times \frac{1}{\beta} \times \frac{d\beta}{d\alpha} - \log\beta\frac{d\beta}{d\alpha} - (1-\beta)\cdot\frac{1}{(1-\beta)}(-1)\frac{d\beta}{d\alpha} + \frac{d\beta}{d\alpha}\log(1-\beta)$$
$$+ \big(\log(1-p) - p\log(1-p) + p\log p - q\log q - \log(1-q) + q\log(1-q)\big)$$

Let this const term = $\underline{\underline{C}}$ for simplicity ↳ (A)

$$0 = -\cancel{\frac{d\beta}{d\alpha}} - \log\beta\frac{d\beta}{d\alpha} + \cancel{\frac{d\beta}{d\alpha}} + \frac{d\beta}{d\alpha}\log(1-\beta) + C$$

$\boxed{\frac{d\beta}{d\alpha} = 1-p-q = d \Rightarrow \text{const (again for simplicity)}}$ — (B)

$$0 = \big(-\log\beta + \log(1-\beta)\big)d + C$$

$$-\frac{C}{d} = \log\left(\frac{1-\beta}{\beta}\right) \Rightarrow \frac{1-\beta}{\beta} = 2^{-C/d}$$

$$\frac{1}{\beta} - 1 = 2^{-C/d}$$

$$\boxed{\beta_0 = \frac{1}{1+2^{-C/d}}}$$ ⇒ Optimum destinat" probability

$$\therefore \boxed{\alpha_0 = \frac{\beta - q}{d} = \frac{\left(\frac{1}{1+2^{-C/d}} - q\right)}{d}}$$ ⇒ Optimum source probability for maximizing probability

Putting values of ⓒ & ⓓ & optimum probabilities ~~is~~ of source in terms of p, q are ⇒ H_1 & H_2

$$\boxed{P(a_1)_{opt} = \alpha_0 = \frac{1}{1+b}} \;\; ; \text{ where } \boxed{b = \frac{ep-(1-p)}{q-e(1-q)}} \;, \text{ where } \boxed{e = 2^{\left(\frac{H_2-H_1}{1-p-q}\right)}}$$

↳ ⓒ ↳ ⓓ ↳ ⓔ

$$\boxed{\beta_0 = \alpha_0(1-p-q)+q} \quad - Ⓕ$$

Putting ⓒ, ⓓ, ⓔ, & Ⓕ in Ⓘ we get a closed form expression for $\boxed{C}$

↓ Req. Ans

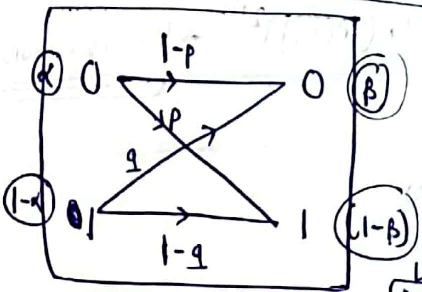
Ⓗ i.e

$$\boxed{C = -\beta_0\log\beta_0 - (1-\beta_0)\log(1-\beta_0) + \alpha_0\left[(1-p)(\log(1-p)) + p\log p\right] + (1-\alpha_0)\left[q\log q + (1-q)\log(1-q)\right]}$$

↳ for above mentioned, $\boxed{\beta_0, \alpha_0}$

Q1 ⓑ $0 \to \underline{000}$, $1 \to \underline{111}$

	Channel O/P	Probability
We are sending 0	000	$(1-p)^3$
	001, 010, 100	$(1-p)^2 p$
	110, 101, 011	$p^2(1-p)$
	111	$(p)^3$
We are sending 1	111	$(1-q)^3$
	110, 101, 011	$(1-q)^2 q$
	001, 010, 100	$q^2(1-q)$
	000	q^3



$\boxed{P(0)+P(1)=1}$

Let $\boxed{P(0)=\alpha, \; P(1) = 1-\alpha}$

$$P(\text{error}) = P(e) = P(e|0)\,P(0) + P(e|1)\times P(1)$$

$$= \left(\left(p^2(1-p)\times 3 + p^3\right)\times P(0)\right) + \left(\left(3q^2(1-q)+q^3\right)P(1)\right)$$

$$= \left(3p^2(1-p)+p^3\right)\underbrace{P(0)}_{(\alpha)} + \left(3q^2(1-q)+q^3\right)\underbrace{P(1)}_{(1-\alpha)}$$

$$= \left(3p^2 - 3p^3 + p^3\right)\alpha + \left(3q^2 - 3q^3 + q^3\right)(1-\alpha)$$

Ⓗ $$= \left(3p^2 - 2p^3\right)\alpha + \left(3q^2 - 2q^3\right) - 3q^2\alpha + 2\alpha q^3$$

$$P(e) = \left(3p^2 - 2p^3 - 3q^2 + 2q^3\right)\alpha + \left(3q^2 - 2q^3\right)$$

for $\alpha = 1/2$ ⇒ equi-probable 0 & 1 ⇒ $\boxed{P(e) = 3/2\,(p^2+q^2) - (p^3+q^3)}$

Now, if we directly send 0 & 1 over channels, w/o encoding to 000 & 111

then $P(e) = P(e|0)\,P(0) + P(e|1)\times P(1)$

$$= p\times 1/2 + q\times 1/2 = \boxed{\frac{p+q}{2}}$$

i.e for 0th layer ⓐ no concatenation, $\boxed{P(e_0) = p+q}$

Now, $P(e_1) = 3/2\,(p^2+q^2) - (p^3+q^3)$

~~Now~~. $P(e_0) = \frac{p+q}{2}$, $P(e_0)^2 = \left(\frac{p+q}{2}\right)^2$, $(P(e_0))^3 = \left(\frac{p+q}{2}\right)^3$

$$P(e_1) = 3/2\left(\left(\frac{p+q}{2}\right)^2 - pq\right) - \left[8\left(\frac{p+q}{2}\right)^3 - 24p^2q - 24pq^2\right]$$

$$= 3/2\,(P(e_0)^2 - pq) - 8(P(e_0))^3 - 24p^2q - 24pq^2$$

$$\boxed{P(e_1) = 3/2\,(P(e_0)^2) - 8(P(e_0))^3 - \underbrace{3/2\,pq - 24p^2q - 24pq^2}_{\text{terms of } pq}}$$

$\Rightarrow$ 1st layer of concatenation i.e $0 \to 000$, $1 \to 111$

$$\therefore \boxed{P(e_2) = 3/2\,(P(e_1))^2 - 8(P(e_1))^3 + \text{terms of } p\,\&\,q}$$

$\Rightarrow$ 2nd layer of concatenation i.e $0 \to 000\,000$, $1 \to 111\,111$

$= O(P(e_0))^4 \to$ (Most significant term)

$\therefore$ Error exponents

$$\boxed{P(e_n) = 3/2\,(P(e_n))^2 - 8(P(e_n))^3 + \dots}$$

$\Rightarrow$ nth layer of concatenation

$\Rightarrow O(P(e_0))^{2^n} \Rightarrow$ $\boxed{\text{error exponent} \Rightarrow 2^n}$

$$\boxed{0 \to \underset{1^{st}}{000}\ \underset{2^{nd}}{000} \dots \underset{nth}{000}}$$

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(2) ~~[illegible]~~)

Q2 ⇒ ① 3.5.1

$$H_1 H_2 \, CNOT \, H_1 H_2 = |+\rangle\langle+| \otimes I + |-\rangle\langle-| \otimes Z$$

⇒ Consider LHS ⇒ operating on $|00\rangle$

$$H_1 H_2 \, CNOT \, H_1 H_2 |0_1 0_2\rangle = H_1 H_2 \, CNOT \, |+_1 +_2\rangle$$

$$= H_1 H_2 \, CNOT \left(\frac{1}{2}\left(|00\rangle + |01\rangle + |10\rangle + |11\rangle\right)\right)$$

$$= H_1 H_2 \left(\frac{1}{2}\left(|00\rangle + |01\rangle + |11\rangle + |10\rangle\right)\right)$$

$$= H_1 H_2 |+_1 +_2\rangle = |00\rangle \quad - ①$$

$$\left(|+\rangle\langle+| \otimes I + |-\rangle\langle-| \otimes Z\right)|0_1 0_2\rangle$$

In Matrix form

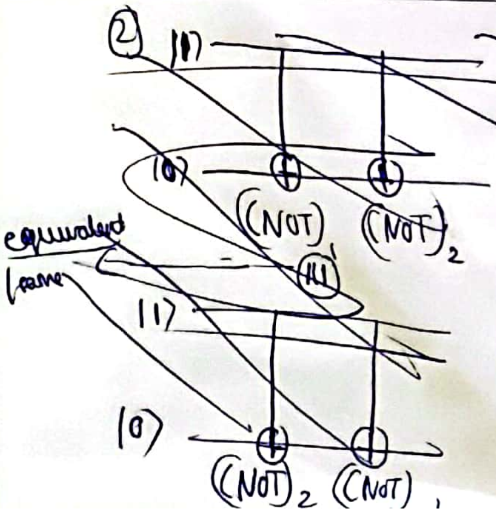
$$\left(\frac{1\times1}{\sqrt2\sqrt2}\begin{pmatrix}1\\1\end{pmatrix}\begin{pmatrix}1 & 1\end{pmatrix} \otimes \begin{pmatrix}1&0\\0&1\end{pmatrix} + \frac{1}{2}\begin{pmatrix}1\\-1\end{pmatrix}\begin{pmatrix}1 & -1\end{pmatrix} \otimes \begin{pmatrix}1&0\\0&-1\end{pmatrix}\right)\begin{bmatrix}1\\0\\0\\0\end{bmatrix}$$

$$\Rightarrow \left[\frac{1}{2}\begin{pmatrix}1&1\\1&1\end{pmatrix} \otimes \begin{pmatrix}1&0\\0&1\end{pmatrix} + \frac{1}{2}\begin{pmatrix}1&-1\\-1&1\end{pmatrix} \otimes \begin{pmatrix}1&0\\0&-1\end{pmatrix}\right]\begin{bmatrix}1\\0\\0\\0\end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix}1&0&1&0\\0&1&0&1\\1&0&1&0\\0&1&0&1\end{bmatrix} + \frac{1}{2}\begin{pmatrix}1&0&-1&0\\0&-1&0&1\\-1&0&+1&0\\0&1&0&-1\end{pmatrix} \begin{bmatrix}1\\0\\0\\0\end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix}2&0&0&0\\0&0&0&2\\0&0&2&0\\0&2&0&0\end{bmatrix}\begin{bmatrix}1\\0\\0\\0\end{bmatrix} = \begin{bmatrix}1\\0\\0\\0\end{bmatrix} = |00\rangle \quad - ②$$

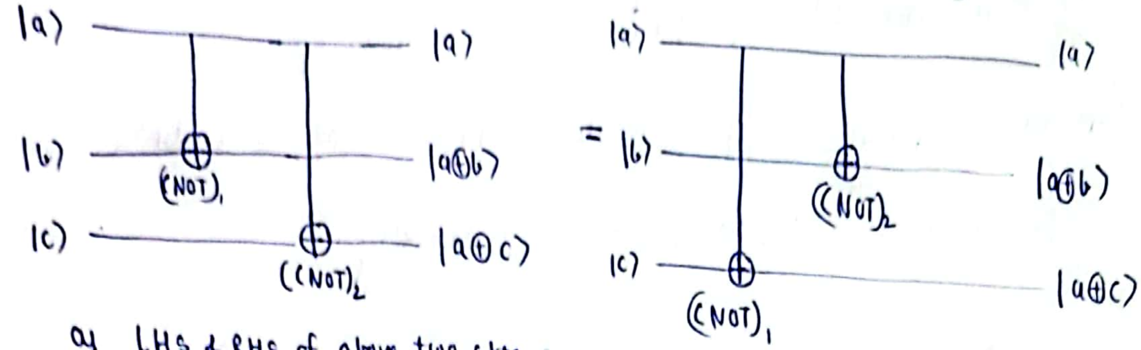
Hence proved from ① + ②

② 

~~$[(CNOT)_1, (CNOT)_2] = 0$~~

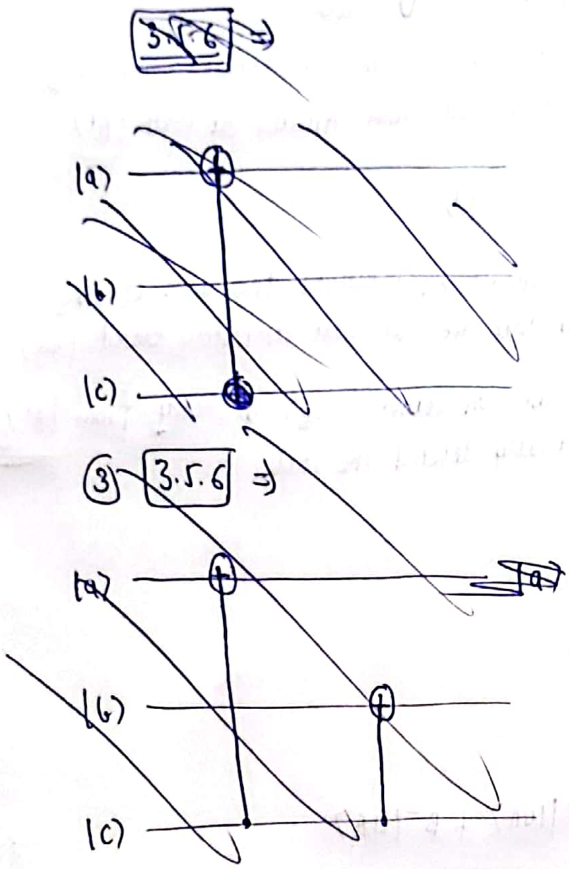
~~$CNOT_1 \, CNOT_2 - CNOT_2 \, CNOT_1 = 0$~~

② 3.5.5 ⇒ ⓐ Same control bits but different target bits

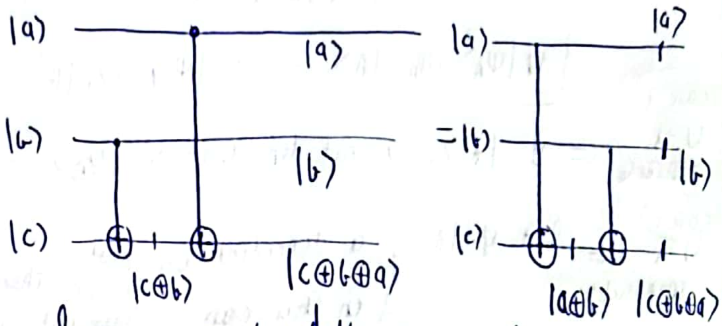


as LHS & RHS of above two ckts are equal hence $[(CNOT)_1, (CNOT)_2] = 0$

i.e two CNOT gates with same control bit commute



③ 3.5.6 ⇒



Same target bit but different control bits

as LHS & RHS of above ckts are same

$[(CNOT)_1, (CNOT)_2] = 0$

i.e they commute.

④ 3.5.13 ⇒

Proof by example ⇒

① Consider Bell state ⇒ $|\psi\rangle_{AB} = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$

Check for Normality ⇒ $\langle\psi_{AB}|\psi_{AB}\rangle = \frac{1}{\sqrt{2}}(\langle 00| + \langle 11|) \times \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$

$= \frac{1}{2}(1+0+0+1) = \underline{1}$ ⇒ Hence Bell state is normalized

Check for orthogonality ⇒ Consider another Bell state $|\phi_{AB}\rangle = \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle)$

Overlap of $|\psi_{AB}\rangle$ with $|\phi_{AB}\rangle$ ⇒ $\langle\psi_{AB}|\phi_{AB}\rangle = \frac{1}{\sqrt{2}}(\langle 00| + \langle 11|) \times \frac{1}{\sqrt{2}}(|00\rangle - |11\rangle)$

$= 1/2\,(1 + 0 + 0 - 1) = \boxed{0}$ ⇒ Hence $|\psi\rangle_{AB}$ & $|\phi\rangle_{AB}$ are orthogonal as. $\langle \psi_{AB} | \psi_{AB} \rangle = 1$, $\langle \psi_{AB} | \phi_{AB} \rangle = 0$

$\boxed{\text{Bell states } |\psi^{+}_{AB}\rangle \text{ \& } |\phi^{+}_{AB}\rangle \text{ are orthonormal}}$

Similarly $\Rightarrow$ $|\psi^{-}_{AB}\rangle$ & $|\phi^{-}_{AB}\rangle$ also satisfy orthonormality condt^{n}s

$\therefore$ In general $\Rightarrow$ $\boxed{\langle \phi_{zx} | \phi_{z'x'} \rangle = \delta(z, z')\,\delta(x, x')}$

Q ③. ① Completeness of coherent states ⇒

$$\boxed{\frac{1}{\pi}\int d^2\alpha\, |\alpha\rangle\langle\alpha| = \mathbb{I}} \Rightarrow \text{Overcompleteness}$$

$D(\alpha)|0\rangle = |\alpha\rangle$

$D(\beta)|0\rangle = |\beta\rangle$

There will be some displacement operator that will take state from $|\alpha\rangle$ to $|\beta\rangle$

$\boxed{D(\beta_\alpha)|\alpha\rangle = |\beta\rangle}$ ⇒ Hence just by displacing $|\alpha\rangle$, it is possible to get $|\beta\rangle$

Hence $|\alpha\rangle$ & $|\beta\rangle$ states are **not linearly independent.**

As coherent states are not orthogonal (proved later in ②),

Any coherent state can be expanded in terms of all other linearly dependent non-orthogonal coherent state basis.

④ Proof ⇒ Since $|\alpha\rangle$ is continuous state ⇒ $\boxed{|\alpha\rangle = e^{-1/2|\alpha|^2}\sum_{n=0}^{\infty}\frac{\alpha^n}{\sqrt{n!}}|n\rangle}$

$$\frac{1}{\pi}\int d^2\alpha|\alpha\rangle\langle\alpha| = \frac{1}{\pi}\sum_{n,m}\frac{1}{\sqrt{n!m!}}|n\rangle\langle m|\int d^2\alpha\, e^{-|\alpha|^2}\alpha^n(\alpha^*)^m$$

The integral on the R.H.S of above eqn can be solved using polar coordinates.

$$\alpha = |\alpha|e^{i\phi} = re^{i\phi}$$

$$d^2\alpha = d\,\mathrm{Re}(\alpha)\, d\,\mathrm{Im}(\alpha)$$

$$\therefore \int d^2\alpha\, e^{-|\alpha|^2}\alpha^n(\alpha^*)^m = \int_0^\infty r\,dr\, e^{-r^2} r^{n+m}\underbrace{\int_0^{2\pi}d\phi\, e^{i(n-m)\phi}}_{2\pi\delta_{nm}} = 2\pi\int_0^\infty r\,dr\,e^{-r^2}r^{2n}$$

Putting $r^2 = t$, $2r\,dr = dt$ ⇒

$$2\pi\int_0^\infty r\,dr\, e^{-r^2}r^{2n} = \pi\int_0^\infty e^{-t}t^n\,dt = \pi\,\Gamma(n+1) = \pi n!$$

$$\therefore \frac{1}{\pi}\int d^2\alpha|\alpha\rangle\langle\alpha| = \frac{1}{\cancel{\pi}}\sum_n\frac{1}{n!}|n\rangle\langle n|\cancel{\pi}\, n!$$

$$= \sum_n |n\rangle\langle n|$$

We know that fock states form complete basis

$$\Rightarrow \boxed{\sum_n |n\rangle\langle n| = \mathbb{I}}$$

$$\therefore \boxed{\frac{1}{\pi}\int d^2\alpha|\alpha\rangle\langle\alpha| = \mathbb{I}} \Rightarrow |\alpha\rangle \Rightarrow \text{coherent state forms overcomplete basis}$$

② Coherent state $|\alpha\rangle$ is generated from vacuum state $|0\rangle$ by using a displacement operator $\underline{D(\alpha)}$

$$|\alpha\rangle = D(\alpha)|0\rangle = e^{-1/2|\alpha|^2} e^{\alpha a^\dagger} e^{\alpha^* a} |0\rangle$$

Consider another coherent state $|\beta\rangle$,

$$\langle\beta|\alpha\rangle = \langle 0|D^\dagger(\beta)\, D(\alpha)|0\rangle$$

$$= \langle 0| e^{-1/2|\alpha|^2} e^{\alpha a^\dagger} e^{-\alpha^* a} e^{-\beta a^\dagger} e^{\beta^* a}\, e^{-1/2|\beta|^2}|0\rangle$$

$$= \langle 0| e^{-\beta a^\dagger} e^{\beta^* a} e^{\alpha a^\dagger} e^{-\alpha^* a}|0\rangle\, e^{-1/2(|\alpha|^2+|\beta|^2)}$$

Now, $e^{-\beta a^\dagger} = \left(1-\beta a^\dagger + \frac{1}{2!}(\beta^* a)^2 - \dots\right)$

$e^{-\alpha^* a} = \left(1-\alpha^* a + \frac{1}{2!}(\alpha^* a)^2 \dots\right)$ & similarly the middle two operators $e^{\beta^* a}$ & $e^{\alpha a^\dagger}$

$$\therefore \langle 0|\left(1-\beta a^\dagger + 1/2(\beta a^\dagger)^2 - \dots\right)\left(1+\beta^* a + 1/2(\beta^* a)^2 \dots\right)$$
$$\left(1+\alpha a^\dagger + \frac{1}{2!}(\alpha a^\dagger)^2 \dots\right)\left(1-\alpha^* a + 1/2!(\alpha^* a)^2 \dots\right)|0\rangle\; e^{-1/2(|\alpha|^2+|\beta|^2)}$$

as $\langle 0|(a^\dagger) = 0$, $a|0\rangle = 0$

eqn reduces to ⇒

$$\langle 0|\left(1+\beta^* a + 1/2(\beta^* a)^2 + \dots\right)\left(1+\alpha a^\dagger + 1/2!(\alpha a^\dagger)^2 \dots\right)|0\rangle\; e^{-1/2(|\alpha|^2+|\beta|^2)}$$

$\langle 0|a = (a^\dagger|0\rangle)^\dagger = (|1\rangle)^\dagger = \underline{\langle 1|}$, $a|n\rangle = \sqrt{n}\,|n-1\rangle$

$a^\dagger|0\rangle = \underline{|1\rangle}$ $\qquad a^\dagger|n\rangle = \sqrt{n+1}\,|n+1\rangle$

$$= \left(\langle 0| + \beta^*\langle 1| + 1/2(\beta^*)^2\sqrt{2}\langle 2| + \dots\right)\left(|0\rangle + \alpha|1\rangle + 1/2\,\alpha^2 \times \sqrt{2}\,|2\rangle \dots\right) e^{-1/2(|\alpha|^2+|\beta|^2)}$$

$$\therefore \langle\beta|\alpha\rangle = \left(1+\beta^*\alpha + \frac{1}{2!}(\beta^*\alpha)^2 + \dots\right) e^{-1/2(|\alpha|^2+|\beta|^2)}$$

overlap $\boxed{\langle\beta|\alpha\rangle = e^{-1/2(|\alpha|^2+|\beta|^2)}\, e^{\beta^*\alpha}}$

Now, $\boxed{|\alpha-\beta|^2 = |\alpha|^2 + |\beta|^2 - \alpha\beta^* - \alpha^*\beta}$

$|\langle\beta|\alpha\rangle|^2$ = transitn probability
$= \underline{e^{-|\alpha-\beta|^2}}$

Hence as $\langle\beta|\alpha\rangle$ = overlap of two coherent state $\neq 0$ ∴ those two coherent states are not orthogonal

CamScanner

③ Average of the photon number ⇒ $\langle n \rangle$

$$\langle \alpha | N | \alpha \rangle = \langle \alpha | \hat{a}^{\dagger} \hat{a} | \alpha \rangle$$

By defⁿ ⇒ coherent state is eigen vector of annihilation operator $\hat{a}$, with eigen value α
|state (some complex no.)

$$\therefore \hat{a}|\alpha\rangle = \alpha|\alpha\rangle \quad , \quad \langle\alpha|\hat{a}^{\dagger} = \langle\alpha|\alpha^{*}$$

$$\therefore \langle\alpha|\hat{a}^{\dagger}\hat{a}|\alpha\rangle = \langle\alpha|\alpha^{*}\alpha|\alpha\rangle$$

$$= |\alpha|^2 \langle\alpha|\alpha\rangle$$

as $|\alpha\rangle$ is normalized state, $\langle\alpha|\alpha\rangle = 1$

$$\boxed{\langle n \rangle = |\alpha|^2}$$

Variance of the photon number ⇒ $\underset{\text{Std. deviatⁿ}}{\Delta n} = \sqrt{\langle n^2\rangle - (\langle n\rangle)^2}$

$$\langle n^2 \rangle = \langle\alpha|(\hat{a}^{\dagger}\hat{a})^2|\alpha\rangle$$

$$= \langle\alpha|\hat{a}^{\dagger}\hat{a}\,\hat{a}^{\dagger}\hat{a}|\alpha\rangle$$

$$= \langle\alpha|\alpha^{*}\hat{a}\,\hat{a}^{\dagger}\alpha|\alpha\rangle$$

$$= |\alpha|^2\langle\alpha|\hat{a}\,\hat{a}^{\dagger}|\alpha\rangle = |\alpha|^2\langle\alpha|\hat{a}^{\dagger}a + 1|\alpha\rangle$$

$$= |\alpha|^2\langle\alpha|\hat{a}^{\dagger}\hat{a}|\alpha\rangle + |\alpha|^2$$

$$= |\alpha|^4 + |\alpha|^2$$

$$\Delta n = \sqrt{|\alpha|^4 + |\alpha|^2 - |\alpha|^4} = |\alpha|$$

$$\therefore \boxed{\text{Variance} = (\Delta n)^2 = |\alpha|^2}$$

---

④ By defⁿ ⇒ ~~Hamiltonian~~ positⁿ & Momentum operators can be written in terms of annihilatⁿ & creatⁿ operators ⇒

$$\hat{x} = \sqrt{\frac{\hbar}{2}}\left(\hat{a} + \hat{a}^{\dagger}\right) \quad , \quad \hat{p} = \sqrt{\frac{\hbar}{2}}\, i\left(\hat{a} - \hat{a}^{\dagger}\right)$$

$$\Delta x = \sqrt{\langle x^2\rangle - (\langle x\rangle)^2} = \sqrt{\langle\alpha|x^2|\alpha\rangle - (\langle\alpha|x|\alpha\rangle)^2}$$

$$= \sqrt{\langle\alpha|\left((\hat{a}+\hat{a}^{\dagger})\sqrt{\frac{\hbar}{2}}\right)^2|\alpha\rangle - \left(\langle\alpha|(\hat{a}+\hat{a}^{\dagger})\sqrt{\frac{\hbar}{2}}|\alpha\rangle\right)^2}$$

$$= \sqrt{\frac{\hbar}{2}}\sqrt{\langle\alpha|\hat{a}^2 + \hat{a}\hat{a}^{\dagger} + \hat{a}^{\dagger}\hat{a} + (\hat{a}^{\dagger})^2|\alpha\rangle - (\langle\alpha|\hat{a}|\alpha\rangle + \langle\alpha|\hat{a}^{\dagger}|\alpha\rangle)^2}$$

$$= \sqrt{\frac{\hbar}{2}}\sqrt{(\alpha)^2 + (\alpha^{*})^2 + 2\alpha^{*}\alpha + 1 - (\alpha+\alpha^{*})^2}$$

$$= \sqrt{\frac{\hbar}{2}}\sqrt{(\alpha)^2 + (\alpha^{*})^2 + 2\alpha^{*}\alpha - \alpha^2 - \alpha^{*2} - 2\alpha^{*}\alpha + 1} = \sqrt{\frac{\hbar}{2}}$$

$$\Delta p = \sqrt{\langle p^2\rangle - (\langle p\rangle)^2} = \sqrt{\langle\alpha|\left(i(\hat{a}-\hat{a}^{+})\sqrt{\frac{\hbar}{2}}\right)^2|\alpha\rangle - \left(\langle\alpha|i(\hat{a}-\hat{a}^{+})\times\sqrt{\frac{\hbar}{2}}|\alpha\rangle\right)^2}$$

$$= \sqrt{\frac{\hbar}{2}}\sqrt{-\langle\alpha|\,\hat{a}^2-\hat{a}\hat{a}^{+}-\hat{a}^{+}\hat{a}+(\hat{a}^{+})^2|\alpha\rangle + \left(\langle\alpha|\,\hat{a}-\hat{a}^{+}|\alpha\rangle\right)^2}$$

$$= \sqrt{\frac{\hbar}{2}}\sqrt{-\left(|\alpha|^2+(\alpha^{+})^2-2\alpha^{+}\alpha\right)+\left(\alpha-\alpha^{+}\right)^2}$$

$$\boxed{\Delta p = \sqrt{\frac{\hbar}{2}}}$$

$$\therefore \quad \boxed{\Delta x\,\Delta p = \sqrt{\frac{\hbar}{2}}\times\sqrt{\frac{\hbar}{2}} = \frac{\hbar}{2}}$$

$\Rightarrow$ Hence coherent states satisfy the upper bound of Heisenberg Uncertainty principle

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~~Q1~~ Q4 ① $\boxed{\text{No deleting theorem}}$ ⇒

~~It is impossible that there exist a universal "deleter" U having~~

It is not possible to delete a Quantum state completely. i.e we can apply a unitary operator U on it to ~~delete~~ erase it but its info can be recovered from Anicilla bit state $|A'\rangle$

$$\boxed{U|\psi_A\rangle|\psi_B\rangle|A\rangle_C \longrightarrow |\psi\rangle_A|0\rangle_B|A'\rangle_C} \quad -①$$

case 1 U is possible ⇐ if $|A'\rangle_C$ is not dependent of $|\psi_B\rangle$ ⇒ then we can't recover the info. of $|\psi_B\rangle$ in this case, we have completely erased $|\psi_B\rangle$

case 2 U is impossible ⇐ But if $|A'\rangle_C$ is dependent on $|\psi_B\rangle$ then we can recover $|\psi_B\rangle$ in theory from $|A'\rangle_C$ & in this case we have not actually deleted the state

U ⇒ Universal deletor

Proof ⇒

$$\left.\begin{aligned} U|00A\rangle &= |00A_0\rangle \\ U|11A\rangle &= |10A_1\rangle \end{aligned}\right\} \text{By def}^n$$

Let $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle$

Considering LHS of ①

$$|\psi\psi A\rangle = \alpha^2|00A\rangle + \alpha\beta|01A\rangle + \beta\alpha|10A\rangle + \beta^2|11A\rangle$$

$$U|\psi\psi A\rangle = \alpha^2|00A_0\rangle + \beta^2|10A_1\rangle + \left(\alpha\beta U|01A\rangle + \beta\alpha U|10A\rangle\right) - (A)$$

Considering RHS of ① ⇒ $|\psi\rangle|0A'\rangle = \alpha|00A'\rangle + \beta|10A'\rangle \quad - (B)$

∴ (A) = (B) ⇒ Let $|\phi\rangle = \frac{1}{\sqrt{2}}\left(|01A\rangle + |10A\rangle\right)$

$$\alpha^2|00A_0\rangle + \beta^2|10A_1\rangle + \sqrt{2}\alpha\beta U|\phi\rangle = \alpha|00A'\rangle + \beta|10A'\rangle$$

Assume $|A'\rangle = c|A_0\rangle + d|A_1\rangle$

$$\alpha^2|00A_0\rangle + \beta^2|10A_1\rangle + \sqrt{2}\alpha\beta U|\phi\rangle = c\alpha|00A_0\rangle + \alpha d|00A_1\rangle + c\beta|10A_0\rangle + \beta d|10A_1\rangle$$

⇒ comparing the two ⇒ $\boxed{c = \alpha,\ d = \beta}$

for this case ⇒ $\boxed{\sqrt{2}\alpha\beta U|\phi\rangle = \alpha\beta|00A_1\rangle + \beta\alpha|10A_0\rangle}$ ⇒ $\boxed{U|\phi\rangle = \frac{|00A_1\rangle + |10A_0\rangle}{\sqrt{2}}}$

& for this case as $c = \alpha$, $d = \beta$,

$$\boxed{|A'\rangle = \alpha|A_0\rangle + \beta|A_1\rangle}$$

If we look @ $\boxed{|\psi\rangle = \alpha|0\rangle + \beta|1\rangle}$, we can clearly see that info of $|\psi\rangle$ is stored in $|A'\rangle$ i.e $|A'\rangle$ is essentially $|\psi\rangle$ in different basis

Hence we cannot delete $|\psi\rangle$ completely as it can be recovered from $|A'\rangle$

9④ ② $\boxed{\text{Proof by thought experiment}}$ ⇒

Consider a scenario ⇒

A + B are two friends who are curious about Mars. One day B gets an opportunity to go to Mars, but A can't go. But A wants to know info that B gets on Mars instantaneously i.e w/o any wait.

So B comes up with a solutⁿ + gives a Quantum cloner to A. A + B share entangled Bell pair state with each other.

A makes a list of questⁿs that he wants to know instantaneously from B, as soon as B knows it on Mars.

eg ⇒ ① Does Mars contain Water?

② Does Mars contain Habitat in which Humans can survive?

+ so on.

$\boxed{\text{No. of questⁿs A is curious about = No. of entangled pairs shared b/wⁿ A + B}}$

$\boxed{\text{Now A + B decide apriori that B has to measure his Bell state in } |+\rangle, |-\rangle \text{ basis if Answer to questⁿ is yes, + in } |0\rangle, |1\rangle \text{ basis if answer is No}}$

$$|\phi^+\rangle_{AB} = \frac{1}{\sqrt{2}}\left(|00\rangle_{AB} + |11\rangle_{AB}\right) = \frac{1}{\sqrt{2}}\left(|++\rangle_{AB} + |--\rangle_{AB}\right) = \frac{1}{\sqrt{2}}\left(\underbrace{|aa\rangle_{AB} + |bb\rangle_{AB}}\right)$$

Orthonormal basis a + b

Now, B is on Mars + finds out the Answer to first questⁿ is Yes, + hence Measures his Bell state in $|+\rangle, |-\rangle$ basis + get o/p as $|+\rangle$ state. (i.e A should get $|+\rangle$ if he/she measures measurement in $|+\rangle, |-\rangle$ basis)

Note ⇒ Assume A + B share a clock showing exact same timing + they have apriori decided on what timing should B measure his state.

So Now, at same timing (a little 1 to 2 second latter) A measures his state. but Before Measuring his/her State he has created 5 to 6 copies of his shared entangled State.

Now, A doesn't know which basis to Measure in so he/she measures in $|0\rangle + |1\rangle$ basis + suppose gets o/p as $|0\rangle$.

Now A takes his/her cloned entangled pair state + again measures in $|0\rangle + |1\rangle$ basis, he gets o/p as $|1\rangle$, so he knows that B has measured his/her state in $|+\rangle + |-\rangle$ basis + Answer to 1ˢᵗ questⁿ is $\boxed{\text{Yes}}$

But if A gets $|0\rangle$ again, then he/she has to repeat measurement on cloned entangled state until he/she reach high probability of getting $|0\rangle$ then answer to first questⁿ is $\boxed{\text{No}}$, as A becomes sure that B has measured in $|0\rangle + |1\rangle$ basis.

~~Now Assuming that A takes~~ Now, consider distance b/wⁿ A + B i.e earth + Mars = 132.57 million km

$\Rightarrow 10^8$

i.e light can travel it in $\boxed{458\text{ sec}}$

Assuming A takes less time for repeated measurement than 458 sec, hence we get faster than light communicatn in this thought expt if A has Quantum clones that clones his entangled pair that he/she shares with B exactly

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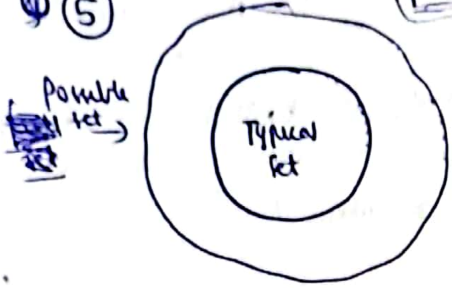
Q 5

Method 1

Let us consider inductn Method $\Rightarrow$



① $\boxed{n=2} \Rightarrow$

Full set $\Rightarrow \{00, 01, 10, 11\}$

Possible set $\Rightarrow \{00, 01, 10\}$

(No two consecutive 1's) for given source

Let 0 occurs with probability p in possible set
& 1 " " " q " "
$\hookrightarrow 1-p$

$\therefore$ Probability that source generates an element from possible set = 1

$$\therefore \underset{(00)}{p\times p} + \underset{(01)}{pq} + \underset{(10)}{qp} = 1 \Rightarrow p^2 + p - p^2 + p - p^2 = 1$$

$$2p - p^2 = 1$$

$$p^2 - 2p + 1 = 0$$

$$\boxed{p = 1, p = 1} \Rightarrow q = 0$$

$\therefore \boxed{\text{Typical set} \Rightarrow \{00\}}$

Size of typical set $\Rightarrow$ $2^{nH(X)}$, $H(X) \Rightarrow$ entropy of source $= \sum_x -P(x)\log_2 P(x)$

$\Rightarrow 2^{2\times 0}$

$\Rightarrow 2^0 = 1$

sample entropy

00: $\Rightarrow -\sum p\ln p = -1\log_2 1 - 1\log_2 1 = 0$

$01, 10$: $-\sum p\ln p = -1\log_2 1 - 0\log_2 0 = 0$

$\boxed{\therefore \text{Size of typical set} = \boxed{1}}$

$\therefore \boxed{H(X) = 0}$

② $\boxed{n=3} \Rightarrow$ Full set $\Rightarrow \{000, 001, 010, 011, 100, 101, 110, 111\}$

Possible set $\Rightarrow \{000, 001, 010, 100, 101\}$

Now, $\sum$ Probability in possible set = 1

$$p^3 + 3p^2q + q^2p = 1$$

$$p^3 + 3p^2(1-p) + (1-p)^2p = 1$$

$$p^3 + 3p^2 - 3p^3 + p - 2p^2 + p^3 = 1$$

$$p^2 - p^3 + p = 1 \Rightarrow \boxed{p \simeq 1} \Rightarrow \boxed{\text{Typical set} \Rightarrow \{000\}}$$

Size of Typical set ⇒ $2^{nH(X)} = 2^{n\times 0} = \underline{\underline{1}}$

③ $\boxed{n=4}$ ⇒

$\boxed{\text{Full set}}$ ⇒ {0000, 0001, 0010, 0011, 0100, 0101, 0110, 0111, 1000, 1001, 1010, 1011, 1100, 1101, 1110, 1111}

$\boxed{\text{Possible set}}$ ⇒ {0000, 0001, 0010, 0100, 0101, 1000, 1001, 1010}.

$$p^4 + 4p^3(1-p) + 3p^2(1-p)^2 = 1$$

$$\Rightarrow p^4 + 4p^3 - 4p^4 + 3p^2(1-2p+p^2) = 1$$

$$p^4 + 4p^3 - 4p^4 + 3p^2 - 6p^3 + 3p^4 = 1$$

$$\boxed{-2p^3 + 3p^2 = 1} \Rightarrow \boxed{p \simeq 1}$$

∴ $\boxed{\text{Typical set}}$ ⇒ {0000}, ⇒ $H(x) = 0$, Sample entropy also = $\underline{\underline{0}}$, $H_{0000} = (-p\ln p)\times 4 = \underline{\underline{0}}$

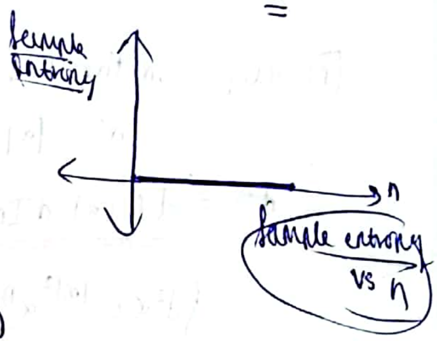
∴ Size of typical set ⇒ $\boxed{1} = \underline{\underline{2^{n\times 0}}}$

④ for n realisation of such random sequence,

Size of typical set = $2^{nH(X)}$

$\boxed{H(X) = \text{Entropy of source} = 0}$

$\boxed{\therefore \text{Size of typical set} = \underline{\underline{1}} = \{\underbrace{00\ldots0}_{n\text{ times}}\}}$



Sample entropy as function of n = $\underline{\underline{0}}$

for large n ⇒ $n \to \infty$

Sample entropy ⇒ $-p\log p - p\log p \ldots - p\log p \ldots = -np\log p = \underline{\underline{0}}$
(000...0 n times)

Q5) ⇒ **Method 2**

Note ⇒ Method 1 is accurate Method as we are calculating the probabilities of '0' occurring from source $= p$ & probabilities of '1' occurring in the source q from possible set itself.

In python, the max value I checked for is $\boxed{n = 20}$; it was throwing $p = 1$ & $q = 0$ ⇒ So again typical set is $\{\underbrace{00\ldots0}_{20\text{ times}}\}$

Sample entropy ⇒ $(-1\ln 1) \times 20 = 0$

Now **Method 2** ⇒ i.e taking only one possible seq. from a possible set,

ex ⇒ $n = 4$ ⇒ possible set ⇒ { 0000, 0001, 0010, 0100, 0101, 1000, 1001, 1010 }

Now, let us take the case that 0 & 1 for 1st bit is equiprobable i.e $p = 1/2$

ex ⇒ ~~0000~~ 0001 ⇒ seq

$p = 1/2$ $p = 1/2$ $p = 1$ $p = 1/2$

as only one possibility as 11 can't exist

⇒ Sample entropy for this seq ⇒ $\dfrac{-\frac{1}{2}\log_2\frac{1}{2} - \frac{1}{2}\log_2\frac{1}{2} - 1\log_2 1 - \frac{1}{2}\log_2\frac{1}{2}}{n}$

$$= \frac{-\frac{3}{2}\log_2\frac{1}{2}}{4} = \frac{3/2}{4} \Rightarrow 3/8 \Rightarrow \boxed{0.375}$$

, size of typical set $= 2^{4 \times 0.375} = 2.828$

∴ for large n ⇒ from plot it is clear that, Sample entropy converges to $\boxed{0.340}$ ⇒ true entropy $H(x)$

Observatn for large n

ex ⇒ $n = 5000$

Size of typical set $= 2^{n H(x)}$

$= 2^{0.340 \times 5000}$

$= 2^{1700}$