~~② Holevo Bound ⇒~~

~~Alice has classical R.V X that she wants to send to Bob. Hence, she prepares a quantum state ρ_X~~

② **Holevo Bound ⇒**

① Alice has classical info source $X = 0, 1, 2, \ldots n$ i.e classical R.V she wants to send to Bob. Hence she prepares quantum state ρ_X with probabilities $p_0, p_1, p_2, \cdots p_n$

i.e $\boxed{\rho = \{ p_X, \rho_X \}_{X=0 \text{ to } n}} = \boxed{\sum_{X=0}^{n} p_X \rho_X}$. Now Bob wants to know X, hence performs P.O.V.M measurement by elements $\{ E_0, \cdots E_m \} = \{ E_y \}$ on ρ_X & gets classical outcome $\boxed{y = 0, 1, 2, \cdots m}$

② ~~Mutual Info ⇒~~ ~~Bob~~ Amount of Informat^n Bob can get about X from his measurement result y is best given by mutual info. b/w^n X & y

i.e $I(X:y)$.

③ If he wants to maximise $I(X:y)$, he should choose POVM's in such a way that maximise $I(X:y)$, & this info. can be termed as accessible Info Bob can get from y about X.

$\therefore \boxed{I_{accessible} = \max_{\{E_y\}} I(X:y)}$

④ In general, Bob can infer X from y completely if & only if $\boxed{I(X:y) = H(X)}$

Closeness of $I(X:y)$ with $H(X)$ is quantitative measure of how well Bob can determine X from y.

⑤ $I(X:y) = H(X)$ is possible only when states $\boxed{\rho_X}$ are orthogonal as there are no perfect sets of POVM's possible to distinguish two Non orthogonal Quantum States.

⑥ So what is upper bound of $I(X:y)$ when $\boxed{\rho_X}$ are non-orthogonal states, i.e there are the optimum set of POVM's that gives upper bound on $I(X:y)$ when ρ_X are non-orthogonal states prepared by Alice. ⇒ This upper bound is called as Holevo Bound.

⑦ Holevo Bound or Holevo Info or Holevo Quantity = $\boxed{S(\rho) - \sum_{x} p_x S(\rho_x)}$ (H)

where $\boxed{\rho = \sum_{x} p_x \rho_X}$

(8) We know, $\boxed{I_{accessible} = \max_{\{E_y\}} I(X:Y)}$

Now, we have to prove ⇒ $\boxed{I(X:Y) \leq S(\rho) - \sum_x p_x S(\rho_x)}$

i.e $\boxed{\max_{\{E_y\}} I(X:Y) = S(\rho) - \sum_x p_x S(\rho_x) < H(X)}$ for Non orthogonal ρ_x

& $\boxed{\max_{\{E_y\}} I(X:Y) = S(\rho) - \sum_x p_x S(\rho_x) = H(X)}$ for orthogonal ρ_x

Entropy of preparation of X i.e source entropy

(9) Note ⇒ for $\boxed{\text{Non-orthogonal } \rho_x \text{ s}}$,

∴ Y can't perfectly infer X ⇐ $\boxed{I_{accessible} < H(X)}$ ⇒ consequence of no cloning theorem as,

if cloning is possible i.e Bob can clone state sent by Alice n times,

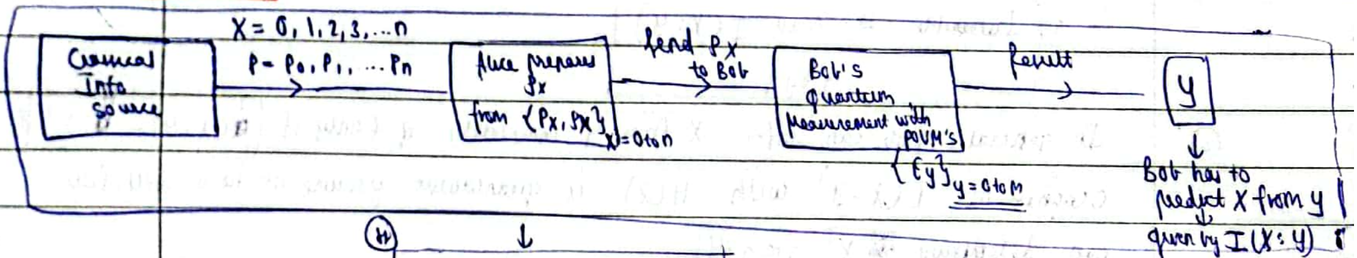
& for large n, ρ_x states can be nearly distinguishable with POVM with very high probability of success i.e ρ_x states nearly become orthogonal

i.e $\boxed{I_{accessible} \approx H(X)}$ if cloning is possible

(10) Hence, now we have to prove (H) $\boxed{I_{accessible} = S(\rho) - \sum_x p_x S(\rho_x)}$

i.e $\boxed{I(X:Y) \leq S(\rho) - \sum_x p_x S(\rho_x)}$

Proof ⇒



(H) Overall communication process

(1) Classical RV 'X' can be expressed as $\boxed{\rho^X = \sum_{x=0}^{n} p_x |x\rangle\langle x|}$ w.r.to some orthonormal basis $\{|x\rangle\}_{x=0}^{n}$

~~By writing~~ $\boxed{\text{Von-Neumann entropy } S(X) = \text{Shannon Entropy } H(X)}$ as ρ^X is classical state

(H) i.e

$$\boxed{S(X) = -\mathrm{tr}(\rho^X \log \rho^X) = -\mathrm{tr}\left(\sum_{x=0}^{n} p_x \log p_x |x\rangle\langle x|\right) = -\sum_{x=0}^{n} p_x \log p_x = H(X)}$$

(2) Alice Now prepares ρ_x with probability p_x,
hence Alice's state that she sends to Bob $\Rightarrow$ $\boxed{\rho^{X\phi} = \sum_{x=0}^{n} p_x |x\rangle\langle x| \otimes \rho_x}$

where $\phi \Rightarrow$ represents Quantum ~~states~~ system $\boxed{\rho \Rightarrow \sum_x p_x \rho_x}$

(3) Now Alice sends $\boxed{\rho^{X\phi}}$ to Bob but Bob only has access to $\boxed{\rho^{\phi}}$

$\therefore \rho^{\phi} = tr_X(\rho^{X\phi}) = tr_X \sum_{x=0}^{n} p_x |x\rangle\langle x| \otimes \rho_x = \sum_{x=0}^{n} p_x \rho_x \underbrace{tr_X(|x\rangle\langle x|)}_{\langle x|x\rangle = 1}$

$\boxed{\text{i.e. } \rho^{\phi} = \sum_{x=0}^{n} p_x \rho_x}$ $\Rightarrow$ Bob gets mixed state

(4) Bob measures this mixed state with POVM elements $\boxed{\{E_y\}_{y=0}^{m}}$
& get outcomes $\boxed{y = 0, 1, 2, \ldots m}$ with probability $\{q_y\}_{y=0}^{m}$ to form classical output R.V $\underline{y}$.

(5) Consider $\underline{X, \phi, M}$ be 3 systems $\Rightarrow$ (1) $X \Rightarrow$ Preparation System having orthonormal basis $|x\rangle$ corresponding to label $\boxed{0, \ldots n}$ on possible preparation for quantum system ϕ.

(2) $\phi \Rightarrow$ Quantum system $\boxed{\rho = \sum_x p_x \rho_x}$

(3) $M \Rightarrow$ Measurement System/Apparatus of Bob having $\boxed{|y\rangle \text{ basis}}$ having possible outcomes $\boxed{0, 1, \ldots, n}$

(6) $\therefore$ Composite system state that represents entire communication $\Rightarrow$

$$\rho^{X\phi M} = \sum_x p_x |x\rangle\langle x| \otimes \rho_x \otimes |0\rangle\langle 0|$$

$\hookrightarrow$ Before Measurement Bob's Measurement apparatus state

(7) Let $X'\phi'M'$ be state of composite system after application of $\mathcal{E}$ (Quantum Instrument)

$$\rho^{X'\phi'M'} = \boxed{I^X \otimes \mathcal{E}^{\phi M}} \rho^{X\phi M} = \sum_{x}\sum_{y} p_x |x\rangle\langle x| \otimes \mathcal{E}^{\phi M}(\rho_x) \otimes |0\rangle\langle 0|$$

$$\boxed{\mathcal{E}^{\phi M}(\sigma \otimes |0\rangle\langle 0|) = \sum_{y=0}^{m} \sqrt{E_y}\, \sigma \sqrt{E_y} \otimes |y\rangle\langle y|}$$

$\rightarrow$ o/p stored of measurement in M

where σ is one of state of $\boxed{\rho_0, \rho_1, \ldots \rho_n}$ $\rightarrow$ Post Measurement State

(H)

$$\therefore \boxed{\rho^{X'\phi'M'} = \sum_{x=0}^{n}\sum_{y=0}^{m} p_x |x\rangle\langle x| \otimes \sqrt{E_y}\, \rho_x \sqrt{E_y} \otimes |y\rangle\langle y|}$$

Now, Bob discards the $\boxed{\sqrt{E_y}\, \rho_x \sqrt{E_y}}$ state as Measurement result is already stored in $|y\rangle$

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(8) Now, $\rho^{X'M'} = Tr_{\varphi}(\rho^{X'\varphi'M'})$

$$= Tr_{\varphi} \sum_{xy} p_x |x\rangle\langle x| \otimes \sqrt{E_y}\, \rho_x \sqrt{E_y} \otimes |y\rangle\langle y|$$

$$Tr\left(\sqrt{E_y}\, \rho_x \sqrt{E_y}\right) = Tr(\rho_x E_y)$$

Now, $\boxed{P(y|x) = Tr(\rho_x E_y)}$

$\rightarrow$ probability of outcome (y) when given state is (ρ_x)

$$\therefore \rho^{X'M'} = \sum_{x,y} p_x P(y|x)\, |x\rangle\langle x| \otimes |y\rangle\langle y|$$

$$p_x P(y|x) = P(x,y) \Rightarrow \text{joint probability of } x \,\&\, y$$

$$\boxed{\rho^{X'M'} = \sum_{x,y} p(x,y)\, |x\rangle\langle x| \otimes |y\rangle\langle y|}$$

$$\therefore \rho^{X'M'} = \rho^{XY} = \sum_{x,y} P(x,y)\, |xy\rangle\langle xy|$$

$$\boxed{\begin{aligned} \rho^X &= tr_y(\rho^{XY}) = \sum_{x=0}^{n} p(x)\,|x\rangle\langle x| \\ \rho^Y &= tr_X(\rho^{XY}) = \sum_{y=0}^{m} p(y)\,|y\rangle\langle y| \end{aligned}} \Bigg\} \text{classical registers i.e classical state}$$

(q_y)

Von Neumman Mutual Info bet^n X & Y $\rightarrow$

$$S(X':M') = S(X:Y) = S(X) + S(Y) - S(X,Y)$$

$$S(X) = -tr(\rho^X \log \rho^X) = -tr\left(\sum_{x=0}^{n} P(x)\,|x\rangle\langle x| \log\left(\sum_{x=0}^{n} P(x)|x\rangle\langle x|\right)\right)$$

$$= -\boxed{\sum_{x=0}^{n}\left(p(x)\log p(x)\right)} = H(X)$$

$$S(Y) = -tr(\rho^Y \log \rho^Y) = \boxed{\sum_{y=0}^{m} p(y)|y\rangle\langle y|} = H(Y)$$

$$S(X,Y) = -tr(\rho^{XY}\log\rho^{XY}) = \sum_{x=0,y=0}^{x=n,y=m} P(x,y)\log P(x,y) = H(X,Y)$$

as ρ^X, ρ^Y & $\rho^{X,Y}$ are classical states, corresponding Von-neumann entropies are equal to Shannon Entropies

(11) $\boxed{S(X':M') = S(X:Y) = H(X:Y) = I(X:Y)}$ $\Rightarrow$ LHS of Holevo Bound

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Now $S(X:\varphi) = S(X:\varphi, M) \geq S(X':\varphi':M') \geq S(X':M')$

Before Measurement

Quantum operation $\mathcal{E}$ causes info loss

or Measurement can't increase mutual info betⁿ X & φ

Trace causes info. loss

or Removing / tracing out systems can't increase mutual info

$$\therefore \boxed{S(X:\varphi) \geq \boxed{S(X':M') = S(X:Y)}} \quad - (\mathrm{I})$$

$S(X:\varphi) \Rightarrow$ Mutual Von Neumann Info betⁿ X & quantum state φ

$$\rho^X = \sum_x p_x |x\rangle\langle x| \Rightarrow S(X) = H(X) \quad - (1)$$

$$\rho^\varphi = \sum_x p_x \rho_x \Rightarrow S(\varphi) = S(\rho) \quad - (2)$$

$$\rho^{X\varphi} = \sum_x p_x |x\rangle\langle x| \otimes \rho_x$$

Let spectral decompositⁿ of ρ_x be $\rho_x = \sum_z \lambda_z^x |z^x\rangle\langle z^x|$

$z \Rightarrow$ arbitrarily chosen bases that depend on x

$$S(X,\varphi) = -tr\left(\rho^{X\varphi} \log(\rho^{X\varphi})\right)$$

$$= -tr\left(\sum_x p_x |x\rangle\langle x| \otimes \underbrace{\rho_x}_{\sum_z \lambda_z^x |z^x\rangle\langle z^x|} \log\left(\sum_x p_x |x\rangle\langle x| \otimes \underbrace{\rho_x}_{\sum_z \lambda_z^x |z^x\rangle\langle z^x|}\right)\right)$$

$$= -tr\left(\sum_{x,z} p_x \lambda_z^x \left(|x\rangle|z^x\rangle\langle x|\langle z^x|\right) \times \log\left(\sum_{x,z} p_x \lambda_z^x \left(|x z^x\rangle\langle x z^x|\right)\right)\right)$$

$$= -\sum_{x,z} p_x \lambda_z^x \log\left(\sum_{x,z} p_x \lambda_z^x\right) = -\sum_{x,z} p_x \lambda_z^x \left(\log p_x + \log \lambda_z^x\right)$$

$$= -\sum_{x,z} p_x \lambda_z^x \left(\log p_x + \log \lambda_z^x\right)$$

$$\sum_z \lambda_z^x = 1 \Rightarrow \text{Normalized } \rho_x$$

$$= -\sum_{x,z} p_x \lambda_z^x \log p_x - \sum_{x,z} p_x \underbrace{\lambda_z^x \log \lambda_z^x}_{S(\rho_x)}$$

$$= -\sum_{x,z} p_x \log p_x - \sum_{x,z} p_x S(\rho_x) = \boxed{H(X) + \sum_x p_x S(\rho_x)} \quad - (3)$$

_/_/_

putting from ①, ② & ③, in Ⓐ

$$\boxed{S(X:\varphi) = S(\rho) - \sum_{k=1}^{n} p_k S(\rho_k)}$$

from Ⓘ $\Rightarrow$ $S(X:\varphi) \geq S(X:Y)$

$$\boxed{\therefore S(\rho) - \sum_{k=1}^{n} p_k S(\rho_k) \geq I(X:Y)}$$

$$\boxed{\therefore S(\rho) - \sum_{k=1}^{n} p_k S(\rho_k) = \max_{\{E_y\}} I(X:Y) = I_{\text{accessible}}}$$

③ $\boxed{I_{\text{accessible}} = \max_{\{E_y\}} I(X:Y) = \underbrace{S(\rho) - \sum_{k=1}^{n} p_k S(\rho_k)}_{\chi}} \leq H(X)$

To show $\boxed{\chi \leq 1}$

χ = Holevo Bound $= S(\rho) - \sum_{k=1}^{n} p_k S(\rho_k)$

as $|X_1\rangle, |X_2\rangle, |X_3\rangle, |X_4\rangle$ are pure states

$$\boxed{\begin{aligned} &S(\rho_1) = S(|X_1\rangle\langle X_1|) = -tr(\rho_1 \log \rho_1) = 0 \\ &S(\rho_2) = 0, \quad S(\rho_3) = 0, \quad S(\rho_4) = 0 \end{aligned}}$$

$$\chi = S(\rho) = -tr(\rho \log \rho)$$

$$\rho = \frac{1}{4}\Big(|X_1\rangle\langle X_1| + |X_2\rangle\langle X_2| + |X_3\rangle\langle X_3| + |X_4\rangle\langle X_4|\Big)$$

$$= \frac{1}{4}\Big(|0\rangle\langle 0| + 1/3\,(|0\rangle + \sqrt{2}|1\rangle)(\langle 0| + \sqrt{2}\langle 1|) + \frac{1}{3}(|0\rangle + \sqrt{2}e^{2\pi i/3}|1\rangle)(\langle 0| + \sqrt{2}e^{-2\pi i/3}\langle 1|)$$

$$+ 1/3\,[(|0\rangle + \sqrt{2}e^{4\pi i/3}|1\rangle)(\langle 0| + \sqrt{2}e^{-4\pi i/3}\langle 1|)\Big)$$

$$= \frac{1}{4}\left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + 1/3\begin{bmatrix} 1 & \sqrt{2} \\ \sqrt{2} & 2 \end{bmatrix} + 1/3\begin{bmatrix} 1 & \sqrt{2}e^{-2\pi i/3} \\ \sqrt{2}e^{2\pi i/3} & 2 \end{bmatrix} + 1/3\begin{bmatrix} 1 & \sqrt{2}e^{-4\pi i/3} \\ \sqrt{2}e^{4\pi i/3} & 2 \end{bmatrix}\right)$$

$$= \frac{1}{4}\left(\begin{bmatrix}1&0\\0&0\end{bmatrix} + \begin{bmatrix}1/3 & \sqrt{2}/3\\ \sqrt{2}/3 & 2/3\end{bmatrix} + \begin{bmatrix}2/3 & \sqrt{2}/3\left(e^{-2\pi i/3}+e^{-4\pi i/3}\right)\\ \sqrt{2}/3\left(e^{2\pi i/3}+e^{4\pi i/3}\right) & 4/3\end{bmatrix}\right)$$

$$= \frac{1}{4}\begin{bmatrix}6/3 & 0\\0 & 6/3\end{bmatrix} \Rightarrow \begin{bmatrix}1/2 & 0\\ 0 & 1/2\end{bmatrix}$$

$$\therefore \quad \chi = S(\rho) = -\mathrm{tr}\left(\begin{pmatrix}1/2 & 0\\0 & 1/2\end{pmatrix}\begin{pmatrix}-1 & 0\\0 & -1\end{pmatrix}\right)$$

$$\boxed{\chi = 1} \Rightarrow \underline{\text{Holevo' Upper bound}}$$

$$\therefore \boxed{I_{accessible} \leq 1}$$

$$H(X) = -\left(1/4 \log_2 (1/4)\right) \times 4 = \boxed{2}$$

Now we have to construct E_y such that it maximizes $I(X:Y)$

$$S(X':M') = I(X:Y) = S(X') + S(Y') - S(X'Y')$$

$$\rho^{X'M'} = \rho^{XY} = \sum_{xy} P(x,y)\,|xy\rangle\langle xy|$$

Now x varies from $\underline{0 \text{ to } 4}$,

$$\rho^{X'M'} = \sum_y P(0,y)\,|0y\rangle\langle 0y| + P(1,y)\,|1y\rangle\langle 1y| + P(2,y)\,|2y\rangle\langle 2y| + P(3,y)\,|3y\rangle\langle 3y|$$

$$= P(x)\sum_y P(y|0)\,|0y\rangle\langle 0y| + P(y|1)\,|1y\rangle\langle 1y| + P(y|2)\,|2y\rangle\langle 2y| + P(y|3)\,|3y\rangle\langle 3y|$$

$$= \frac{1}{4}\sum_{y=1}^{m} \mathrm{Tr}(\rho_0 E_y)\,|0y\rangle\langle 0y| + \mathrm{Tr}(\rho_1 E_y)\,|1y\rangle\langle 1y| + \mathrm{Tr}(\rho_2 E_y)\,|2y\rangle\langle 2y| + \mathrm{Tr}(\rho_3 E_y)\,|3y\rangle\langle 3y|$$

$$\rho^{X'M'} = \frac{1}{4}\begin{bmatrix}\begin{pmatrix}\mathrm{Tr}(E_0\rho_0) & 0 & 0\\ 0 & \mathrm{Tr}(E_1\rho_0) & 0\\ 0 & 0 & \mathrm{Tr}(E_m\rho_0)\end{pmatrix} & & 0 \\ & \begin{pmatrix}\mathrm{Tr}(E_0\rho_1) & \\ & \ddots \ \mathrm{Tr}(E_m\rho_1)\end{pmatrix} & \\ 0 & & \begin{pmatrix}\mathrm{Tr}(E_0\rho_3) & \\ & \ddots \ \mathrm{Tr}(E_m\rho_3)\end{pmatrix}\end{bmatrix}_{4m\times 4m}$$

$$\rho^{X'} = Tr_{M'}(\rho^{X'M'}) = \frac{1}{4}\begin{bmatrix} \sum_{y=0}^{M} Tr(E_y\rho_0) & 0 & 0 & 0 \\ 0 & \sum_{y=0}^{M} Tr(E_y\rho_1) & 0 & 0 \\ 0 & 0 & \sum_{y=0}^{M} Tr(E_y\rho_2) & 0 \\ 0 & 0 & 0 & \sum_{y=0}^{M} Tr(E_y\rho_3) \end{bmatrix}_{4\times 4}$$

$$\rho^{Y'} = Tr_X(\rho^{X'M'}) = \frac{1}{4}\begin{bmatrix} \sum_{x=0}^{3} Tr(E_0\rho_x) & 0 & 0 \\ 0 & \sum_{x=0}^{3} Tr(E_1\rho_x) & 0 \\ 0 & 0 & \sum_{x=0}^{3} Tr(E_M\rho_x) \end{bmatrix}_{M\times M}$$

$$I(X:Y) = -tr(\rho^X \log \rho^X) - tr(\rho^Y \log \rho^Y) + tr(\rho^{XY} \log \rho^{XY})$$

Now, to optimize / max the $I(X:Y)$ over POVM's $\{E_y\}$,

Let us take case $M=2 \Rightarrow E_0 + E_1 = I$

Define $E_0 = \frac{1}{2}\begin{bmatrix} 1+\cos\theta & \sin\theta e^{-i\phi} \\ \sin\theta e^{i\phi} & 1-\cos\theta \end{bmatrix}$

$E_1 = I - \frac{1}{2}\begin{bmatrix} 1+\cos\theta & \sin\theta e^{i\phi} \\ \sin\theta e^{i\phi} & 1-\cos\theta \end{bmatrix}$

The problem with above definitn is $\underline{E_0 \& E_1}$ does not have +ve real eigen values for all θ & ϕ, as E_0 & E_1, for both of them to be +ve, semidefinite it should have $\underline{\text{Real positive eigen values}}$

$\therefore$ we consider only those θ & ϕ values for which eigen values of E_0 & E_1 are greater than equal to zero i.e +ve real eigen values

Now, we use python for iterating E_0 & E_1, for $0 \leq \theta \leq 180$ & $0 \leq \phi \leq 360$ over all the values of θ & ϕ such that which gives E_0 & E_1 eigen values are +ve, real & then break the loop only when,

$I(X:Y)$ is maximized

$$\boxed{I(X:Y) = -tr(\rho^{X'} \log \rho^{X'}) - tr(\rho^{Y'} \log \rho^{Y'}) + tr(\rho^{X'Y'} \log \rho^{X'Y'})}$$

i.e $|I(X:Y) - \chi| = \underline{\text{error} < \varepsilon}$

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So for ⇒ ① **Case 1** ⇒ $M=2$, we see $I(X:Y)$ is maximized for,

$$E_0 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad E_1 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

& $I_{accessible} = \max(I(X:Y)) = \underline{0.3710}$ for $\underline{\phi = 10^\circ, \theta = 0^\circ}$

② **Case 2** ⇒ $M=3$, we see $I(X:Y)$ is max. for,

$$\underset{\overset{\parallel}{E_\theta}}{E_0} = \frac{1}{2}\begin{bmatrix} 1.3090 & -0.9510 \\ -0.9510 & 0.6909 \end{bmatrix}, \quad \underset{\overset{\parallel}{E_{\theta+r}}}{E_1} = \frac{1}{2}\begin{bmatrix} 1.3090 & -0.9510 \\ -0.9510 & 0.6909 \end{bmatrix}, \quad \underset{\overset{\downarrow}{I - E_\theta - E_{\theta+r}}}{E_2} = \begin{bmatrix} -0.3090 & 0.9510 \\ 0.9510 & -0.3090 \end{bmatrix}$$

varies from 0 to 360°

$I_{accessible} = \max I(X:Y) = \boxed{0.8110}$ for $\boxed{\theta = 72^\circ,\ r = 0^\circ,\ \phi = 180^\circ}$

③ **Case 3** ⇒ $M=4$, we see $I(X:Y)$ is max for,

$$\underset{\overset{\downarrow}{E_\theta}}{E_0} = \begin{bmatrix} & \\ & \end{bmatrix}, \quad \underset{\overset{\downarrow}{E_{\theta+r}}}{E_1} = \begin{bmatrix} & \\ & \end{bmatrix}, \quad \underset{\overset{\parallel}{E_{\theta+r+n}}}{E_2} = \begin{bmatrix} & \\ & \end{bmatrix}$$

$$\underset{\overset{\parallel}{I-(E_0+E_1+E_2)}}{E_3} = \begin{bmatrix} & \\ & \end{bmatrix} \Rightarrow$$

We can increase the number of POVM elements to as much as we want, but runtime for algorithm on python increases exponentially, ~~hence~~ as running for $M=4$ itself will take $\boxed{11 \text{ days}}$ (if we iterate for single degree change in θ, ϕ, r, n)

$\boxed{\text{Note}}$ ⇒ Python coding of above ~~the~~ logic was implemented in collaboratⁿ with Vishal M (friend from Aerospace Engg)

$\boxed{\text{Observatⁿ}}$ ⇒ As we increase the no. of POVM's to 4 we can see increase in max $I(X:Y)$ & we have achieved the max $I(X:Y)$ nearly equal to Holevo Bound.

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Q1

9.1.1 ⇒ To prove $\|N(\rho) - N(\sigma)\|_1 \leq \|\rho - \sigma\|_1$

⇒ we know, we can define any Noisy quantum channel with,

Kraus operators ⇒ $N(\rho) = \sum_{i=0}^{m} A_i \rho A_i^{+}$

Similarly $N(\sigma) = \sum_{i=0}^{m} A_i \sigma A_i^{+}$

$\boxed{\sum_{i=0}^{m} A_i A_i^{+} = I}$

$\|\rho - \sigma\|_1 = \underbrace{Tr\sqrt{(\rho-\sigma)^{+}(\rho-\sigma)}}_{①} = \sum_{n}$ eigenvalues of $(\rho - \sigma)$ for n dimensional Hilbert space

$\|N(\rho) - N(\sigma)\|_1 = \| \sum_{i=0}^{m} A_i \rho A_i^{+} - \sum_{i=0}^{m} A_i \sigma A_i^{+} \|$

Method 1 ⇒

Now we don't have Info about the Quantum channel, so,

we can consider it as Depolarizing channel ⇒ (Pessimistic Model)

$\rho \to (1-p)\rho + p\Pi$, $\sigma \to (1-p)\sigma + p\underbrace{\Pi}_{I/2}$

$\therefore \|N(\rho) - N(\sigma)\|_1 = \|(1-p)\rho + p\Pi - (1-p)\sigma + p\Pi)\|_1$

$= \|(1-p)(\rho - \sigma)\|_1 = |1-p| \, \|\rho - \sigma\|_1$

$\boxed{\leq \|(\rho - \sigma)\|_1}$ (Hence proved)

Method 2 ⇒ $N(\rho) - N(\sigma) = N(\rho - \sigma) - Tr_E\{U^{A\to AE}(\rho - \sigma)U^{+A\to AE}\}$ (U ⇒ Isometry Map)

~~$\|N(\rho) - N(\sigma)\|_1$~~

Let us consider ⇒ $\|\rho - \sigma\|_1$ ⇒ we know,

$$\|\rho - \sigma\|_1 = 2 \max_{0 \leq \Lambda \leq I} Tr(\Lambda(\rho - \sigma))$$

Now, we have Isometry Map $U^{A\to AE}$ that maps $(\rho - \sigma)$ in A system to joint state in system AE.

Hence $Tr(\Lambda(\rho - \sigma)) \xrightarrow{\text{becomes}} Tr_A(\Lambda_A Tr_E(U(\rho - \sigma)U^{+}))$ for quantum channel Maps

$\therefore \|N(\rho) - N(\sigma)\|_1 = 2 \max_{0 \leq \Lambda_A \leq I} Tr_A\{(\Lambda_A \cdot Tr_E(U(\rho - \sigma)U^{+}))\}$

$= 2 Tr_A\{(\Lambda_A \otimes I_E(U(\rho - \sigma)U^{+}))$

$\leq \|U(\rho - \sigma)U^{+}\| = \|\rho - \sigma\|_1$ (as $\|UMU^{+}\| = \|M\|$ ⇒ Isometry property)

Method 2 $\Rightarrow$ $\boxed{N(\rho)-N(\sigma) = N(\rho-\sigma) - Tr_E\left(U(\rho-\sigma)U^\dagger\right)}$

$$\|N(\rho^A)-N(\sigma^A)\|_1 = Tr_A\left(\left(\sqrt{\left(N(\rho^A-\sigma^A)-Tr_E\left(U(\rho-\sigma)U^\dagger\right)\right)^\dagger}\left(N(\rho^A-\sigma^A)-Tr_E(U\rho\sigma U^\dagger)\right)\right)\right)$$

$\boxed{\text{Using (1)}}$

$$\leq Tr_A\left[\left(N(\rho^A-\sigma^A)-Tr_E\left(U(\rho-\sigma)U^\dagger\right)\right)\right]$$

(equal only if $\left(N(\rho^A-\sigma^A)-Tr_E\left(U(\rho-\sigma)U^\dagger\right)\right)$ is Hermitian Matrix)

$$\leq Tr_A\left(\left(N(\rho^A-\sigma^A)\right)\right) - Tr_A\left(Tr_E\left(U(\rho-\sigma)U^\dagger\right)\right)$$

$N(\rho^A-\sigma^A) \Rightarrow$ Trace preserving Map $\Rightarrow Tr_A\left(N(\rho^A-\sigma^A)\right) = Tr_A(\rho^A-\sigma^A)$

$$\leq Tr_A(\rho^A-\sigma^A) - Tr_A\left(Tr_E U(\rho-\sigma)U^\dagger\right)$$

$$\leq \|\rho^A-\sigma^A\|$$

$$\boxed{\|N(\rho^A)-N(\sigma^A)\|_1 \leq \|\rho^A-\sigma^A\| - (+\text{ve quantity})}$$

$$\Downarrow$$

$$\boxed{\|N(\rho^A)-N(\sigma^A)\| \leq \|\rho^A-\sigma^A\|}$$

9.2.4 To prove $\Rightarrow F(\rho,\sigma) = Tr\left(\sqrt{\sqrt{\rho}\,\sigma\sqrt{\rho}}\right)^2$

By Ulhman's theorem

$$F(\rho,\sigma) = \|\sqrt{\rho}\,\sqrt{\sigma}\|_1^2$$

$$= \left(Tr\sqrt{(\sqrt{\rho}\sqrt{\sigma})^\dagger(\sqrt{\rho}\sqrt{\sigma})}\right)^2$$

$$= \left(Tr\sqrt{\sqrt{\sigma}^\dagger\sqrt{\rho}^\dagger\sqrt{\rho}\sqrt{\sigma}}\right)^2$$

$\boxed{\rho^\dagger = \rho,\quad \sigma^\dagger = \sigma}$ (Hermitian)

$$\boxed{F(\rho,\sigma) = \left(Tr\sqrt{\sqrt{\sigma}\,\rho\sqrt{\sigma}}\right)^2}$$

We know,

$$\boxed{F(\rho,\sigma) = F(\sigma,\rho) \Rightarrow \left(Tr\sqrt{\sqrt{\rho}\,\sigma\sqrt{\sigma}}\right)^2}$$

9.2.5 $F(\rho,\sigma) = F(U\rho U^\dagger, U\sigma U^\dagger)$

$\Rightarrow$

By Ulhmann's-theorem $\Rightarrow$

$$F(\rho,\sigma) = \| \sqrt{\rho}\sqrt{\sigma} \|_1^2 = \max_U |\langle \phi_\rho | U^R \otimes I^A | \phi_\sigma \rangle^{RA}|^2$$

Let $|\phi\rangle^{RA}$ be purificat^n of ρ_A $\Rightarrow$ $Tr_R(|\phi\rangle^{RA}\langle\phi|) = \rho_A$

$\therefore$ $U_A|\phi\rangle^{RA}$ purificat^n of $U_A\rho_A U_A^\dagger$ $\Rightarrow$ $Tr_R(U_A|\phi\rangle^{RA}\langle\phi|U_A^\dagger) = U_A\rho_A U_A^\dagger$

$\therefore$ from Ulhmann's theorem $\Rightarrow$

$$F(U\rho U^\dagger, U\sigma U^\dagger) = \max_{U_R} |\langle\phi|^{RA} U_A (U_R \otimes I_A) U_A^\dagger |\phi\rangle^{RA}|^2$$

$$\boxed{= \max_{U_R} |\langle \phi_\rho | (U_R \otimes I_A) | \phi_\sigma \rangle|^2 = F(\rho,\sigma)}$$

OR $\Rightarrow$ $F(\rho,\sigma) = \left(Tr\left(\sqrt{\sqrt{\rho}\sigma\sqrt{\rho}}\right)\right)^2$

$$F(U\rho U^\dagger, U\sigma U^\dagger) = \left(Tr\left(\sqrt{\sqrt{U\rho U^\dagger}\, U\sigma U^\dagger \sqrt{U\rho U^\dagger}}\right)\right)^2$$

We know, $\boxed{\|\rho\| = \|U\rho U^\dagger\|}$ (if U is isometry)

$\therefore$ $\|\rho\sigma\| = \|U\rho U^\dagger U\sigma U^\dagger\|$

$\therefore$ $F(\rho,\sigma) = \|\sqrt{\rho}\sqrt{\sigma}\|_1^2 = \|\sqrt{U\rho U^\dagger}\sqrt{U\sigma U^\dagger}\|^2 = F(U\rho U^\dagger, U\sigma U^\dagger)$

$\boxed{9.2.6}$ $\Rightarrow$

$$F(\rho,\sigma) \leq F(N(\rho), N(\sigma))$$

$\Rightarrow$ The fidelity is non decreasing w.r.to partial trace $\Rightarrow$

$F(\rho_{AB}, \sigma_{AB}) \leq F(\rho_A, \sigma_A)$

$\therefore$ $F(\rho_{AE}, \sigma_{AE}) \leq F(N(\rho), N(\sigma))$

$$N(\rho) = Tr_E\left(U_{AE}\left(|0\rangle_E\langle 0| \otimes \rho_A\right)U_{AE}^\dagger\right) = Tr_E(\rho_{AE})$$

$$N(\sigma) = Tr_E\left(U_{AE}\left(|0\rangle_E\langle 0| \otimes \sigma_A\right)U_{AE}^\dagger\right) = Tr_E(\sigma_{AE})$$

Now, $F(\rho_{AE}, \sigma_{AE}) = F(|0\rangle\langle 0| \otimes \rho_A, |0\rangle\langle 0| \otimes \sigma_A)$

Now, $F(\rho_1 \otimes \rho_2, \sigma_1 \otimes \sigma_2) = F(\rho_1,\sigma_1)F(\rho_2,\sigma_2)$ (from Multiplicativity property)

$\therefore$ $F(|0\rangle\langle 0| \otimes \rho_A, |0\rangle\langle 0| \otimes \sigma_A) = F(|0\rangle_E\langle 0|, |0\rangle_E\langle 0|)F(\rho_A,\sigma_A)$

$= \boxed{F(\rho_A,\sigma_A)}$ $\boxed{\text{as } (F(\rho,\rho) = 1)}$

$\boxed{9.4.2} \Rightarrow \quad Tr\{\Lambda_k^S \rho_k^S\} \geq 1-\epsilon$

from ex $\Rightarrow$ 5.4.1 $\Rightarrow$ We know, there exists $U^{S\to SS'}$ $\Rightarrow$ coherent measurement such that,

$$\langle \phi_k|^{RS} \langle k|^{S'} U^{S\to SS'} |\phi_k\rangle^{RS} \geq 1-\epsilon \quad - (*)$$

$\boxed{Tr_R(|\phi_k\rangle\langle\phi_k|) = \rho_k} \Rightarrow |\phi_k\rangle^{RA}$ is purification of ρ_k

Now to prove $\Rightarrow D^{A\to AK}$ exists such that,

$$\|D^{A\to AK}(\phi_k^{RA}) - \phi_k^{RA} \otimes |k\rangle\langle k|^K\| \leq 2\sqrt{\epsilon}$$

from (*) in given Notation

$$|\langle \phi_{RA}^k| \otimes \langle k|U_{A\to AK}|\phi_{RA}^k\rangle| \geq 1-\epsilon$$

where,

$$D_{A\to AK}(\phi_{RA}^k) = U_{A\to AK}|\phi_{RA}^k\rangle\langle\phi_{RA}^k| U^\dagger_{A\to AK} \quad - (A)$$

~~Consider~~, $\|D_{A\to AK}(\phi_{RA}^k) - \phi_{RA}^k \otimes |k\rangle\langle k|\|_1 \Rightarrow$ LHS of Proof

~~[illegible]~~

$$\|D_{A\to AK}(\phi_{RA}^k) - (\phi_{RA}^k)\otimes|k\rangle\langle k|\|_1 \leq 2\{1 - F(D_{A\to AK}(\phi_{RA}^k), |\phi_{RA}^k\rangle\otimes|k\rangle \cdot \langle\phi_{RA}^k|\otimes\langle k|)\}^{1/2}$$

$$= 2\left(1 - |\langle\phi_{RA}^k|\otimes\langle k|U_{A\to AK}|\phi_{RA}^k\rangle|^2\right)^{1/2} \quad \text{(from (A))}$$

$$\leq 2(1-(1-\varepsilon)^2)^{1/2} = 2\sqrt{\varepsilon(2-\epsilon)}$$

$$\boxed{\|D_{A\to AK}(\phi_{RA}^k) - \phi_{RA}^k \otimes |k\rangle\langle k|\| \leq 2\sqrt{\epsilon}}$$

$$11.6.3 \Rightarrow \quad I(A;B) = H(A) - H(A|B)$$
$$\leq H(A) + |H(A|B)|$$
$$\leq 2\log \dim(H_A) \Rightarrow \boxed{2\log d_A \geq I(A:B)}$$
as $H(A) \leq \log \dim(H_A)$

similarly $\Rightarrow$ $I(A;B) \leq 2\log \dim(H_B)$
$$I(A:B) \leq 2\log d_B$$
$$\therefore \quad I(A;B) \leq \min(2\log d_A, 2\log d_B)$$
$$\boxed{I(A;B) \leq 2\min(\log d_A, \log d_B)}$$

11.6.5 $\Rightarrow$ To prove $\Rightarrow$ $\boxed{I(R:A)_\psi + I(R:S)_\psi = I(R:B)_\phi + I(R:SE)_\phi}$

$$U^{A\to BE} \; |\psi\rangle^{SRA} \Rightarrow |\phi\rangle^{SRBE}$$

$$I(R:A) = H(R) + H(A) - H(RA) \quad H(RA) \to H(S)$$
$$I(R:S) = H(R) + H(S) - H(RS) \quad H(RS) \to H(A)$$

$$\boxed{I(R:A) + I(R:S) = 2H(R)}$$

$$I(R:B) = H(R) + H(B) - H(RB) \quad H(RB) = H(SE)$$
$$I(R:SE) = H(R) + H(SE) - H(RSE) \quad H(RSE) \to H(B)$$

$$\therefore \boxed{I(R:B) + I(R:SE) = 2H(R)}$$

(H)

$$\boxed{\therefore I(R:A) + I(R:S) = I(R:B) + I(R:SE)}$$

__/__/____

$\boxed{11.7.6}$ $\Rightarrow$ $\boxed{H(A|B)_\rho \geq H(A|BC)_\rho}$ $\Rightarrow$ conditioning does not improve entropy $\Rightarrow$

$$I(A;B|C) = H(A|C) + H(B|C) - H(AB|C)$$
$$= H(AC) - H(C) + H(BC) - H(C) - H(ABC) + H(C)$$

$$= H(AC) + H(BC) - H(C) - H(ABC)$$
$$= H(B|C) - H(B|AC)$$

$\therefore I(A;B|C) \geq 0 \Rightarrow H(B|C) \geq H(B|AC)$

or $I(A;B|C) \geq 0 \Rightarrow \boxed{H(AC) + H(BC) \geq H(C) + H(ABC)}$

$\boxed{11.8.5}$ $\Rightarrow$ To prove $\Rightarrow$ $D(\rho_1 \otimes \rho_2 \| \sigma_1 \otimes \sigma_2) = D(\rho_1\|\sigma_1) + D(\rho_2\|\sigma_2)$

LHS $\Rightarrow$ $D(\rho_1 \otimes \rho_2 \| \sigma_1 \otimes \sigma_2) = \mathrm{Tr}\big((\rho_1 \otimes \rho_2 \log(\rho_1 \otimes \rho_2)$
$$- \rho_1 \otimes \rho_2 \log(\sigma_1 \otimes \sigma_2)\big)$$
$$= \mathrm{Tr}\,(\rho_1 \otimes \rho_2 \log \rho_1 - \rho_1 \otimes \rho_2 \log \sigma_1$$
$$+ \rho_1 \otimes \rho_2 \log \rho_2 - \rho_1 \otimes \rho_2 \log \sigma_2)$$
$$= D(\rho_1\|\sigma_1) + D(\rho_2\|\sigma_2)$$ $\Rightarrow$ Hence proved.