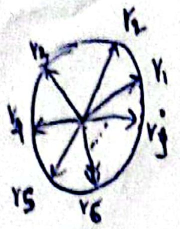
Q① 4.1.8 ⇒ Let the ensemble be $\{P(j), |\psi_j\rangle\}$

$$\boxed{\rho = \sum_j P(j)|\psi_j\rangle\langle\psi_j|} \Rightarrow \text{Mixed state}$$

$$\boxed{\rho_j = |\psi_j\rangle\langle\psi_j|} \Rightarrow \text{Pure state}$$

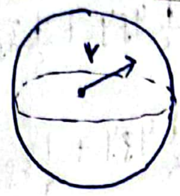


$$\therefore \boxed{\rho = \sum_j P(j)\,\rho_j} \quad - (A)$$

⇓

Bloch vectors of pure state ⇒ $\boxed{r_1 \text{ to } r_j}$

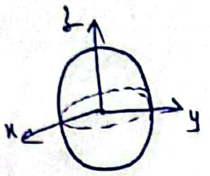
Length of Bloch vector = 1



Bloch vector of mixed state resulting from ensemble

Length of Bloch vector < 1

$|\bar{r}| < 1$



We know that,

ρ can be written as, in Pauli Basis

$$\boxed{\rho = \frac{1}{2}\left[I + r_x X + r_y Y + r_z Z\right]}$$

For C_2 → Hilbert space of dimension 2

where X, Y, Z are Pauli Matrices & r_x, r_y, r_z are Bloch vectors in x, y, z directⁿ

∴ (A) can be written in Pauli Basis as,

$$1/2\left[I + r_x X + r_y Y + r_z Z\right] = \sum_j P(j)\left[\frac{1}{2}\left[I + r_{x_j} X + r_{y_j} Y + r_{z_j} Z\right]\right]$$

$$\Rightarrow \frac{1}{2}\begin{bmatrix} 1+r_z & r_x - i r_y \\ r_x + i r_y & 1 - r_z \end{bmatrix} = \sum_j \frac{1}{2}\begin{bmatrix} P(j)(1+r_{z_j}) & P(j)(r_{x_j} - i r_{y_j}) \\ P(j)(r_{x_j} + i r_{y_j}) & P(j)(1 - r_{z_j}) \end{bmatrix}$$

⇒ Comparing the two,

$$\boxed{r_z = \sum_j P(j)\, r_{z_j}, \quad r_x = \sum_j P(j)\, r_{x_j}, \quad r_y = \sum_j P(j)\, r_{y_j}}$$

$$\therefore \bar{r} = r_x \hat{i} + r_y \hat{j} + r_z \hat{k}$$

where $\hat{i}, \hat{j}, \hat{k}$ are unit vectors in x, y, z directⁿ

$$\therefore \bar{r} = r_x \hat{i} + r_y \hat{j} + r_z \hat{k} = \sum_j P(j)\,\underbrace{(r_{x_j}\hat{i} + r_{y_j}\hat{j} + r_{z_j}\hat{k})}_{\bar{r}_j}$$

$$\therefore \boxed{\bar{r} = \sum_j P(j)\,\bar{r}_j}$$

i.e. $\boxed{r = \sum_j P(j)\, r_j}$ ⇒ Hence proved

$\boxed{4.2.2} \Rightarrow$ Expected success probability $(\epsilon) \Rightarrow$ ~~[illegible]~~ $\sum_x p(x) \mathrm{Tr}\{\rho_x \Lambda_x\}$

For finding upper bound, take POVM for all x 's same as pure state $\underline{\rho_x}$

i.e $\boxed{\Lambda_x = \rho_x \quad \forall x} \Rightarrow$ upper bound condn

$\sum_x p(x) \mathrm{Tr}\{\rho_x^2\}$

Pure state $\Rightarrow$ $\underline{\rho_x^2 = \rho_x}$

$\underline{E_{max}} = \boxed{\sum_x p(x) \mathrm{Tr}\{\rho_x\}} \Rightarrow \sum_x \underline{\mathrm{Tr}\{p(x).\rho_x\}}$, Now $\underline{\tau \geq p(x)\rho_x}$

$\therefore \boxed{E_{max} = \boxed{\sum_x \mathrm{Tr}\{\tau\}}} \Rightarrow$ Hence proved

Now, $\underline{n \text{ bits}} \longrightarrow \underline{d \text{ dimensional subspace}}$

$\{0,1\}^n \longrightarrow \underline{|a_1 a_2 \dots a_m\rangle}$, $\rho_i = |a_1 a_2 \dots a_m\rangle\langle a_1 a_2 \dots a_m|$

o $\underline{ex} \Rightarrow$ $\underline{n=2} \Rightarrow$ $00 \longrightarrow |00\rangle$

$01 \rightarrow |01\rangle$

~~enco~~ encoding $\Leftarrow$ $10 \rightarrow |10\rangle$ $\Rightarrow \left\{\frac{1}{2^2}, \rho_i\right\}_{i \in \{0,1\}^n}$

$11 \rightarrow |11\rangle$

$\underbrace{}_{\text{classical bit}}$ $\underbrace{}_{\text{Quantum bits}}$

$\boxed{d = 4}$

$\rho_0 \Rightarrow |00\rangle\langle 00|$

$\rho_1 \Rightarrow |01\rangle\langle 01|$

$\rho_2 \Rightarrow |10\rangle\langle 10|$

$\rho_3 = |11\rangle\langle 11|$

$\therefore \rho = \frac{1}{4}\left(|00\rangle\langle 00| + |01\rangle\langle 01| + |10\rangle\langle 10| + |11\rangle\langle 11|\right)$

Max Success probability $= \sum_x \mathrm{Tr}\{p(x)\rho_x\} = \frac{1}{4}\sum_d \mathrm{Tr}\{\rho_d\}$

$= \frac{1}{4} \times 4 = \boxed{1}$ i.e ~~[illegible]~~

$\frac{1}{4} \downarrow 1/2^n$, $4 \downarrow d$

(ii) $\underline{\text{In general}} \Rightarrow \boxed{\{0,1\}^n \rightarrow |a_1 a_2 \dots a_m\rangle}$

$\{0,1\}^n \downarrow$ n bits , $\underbrace{|a_1 a_2 \dots a_m\rangle}_{\text{d dimensional subspace}} \Rightarrow \boxed{2^m = d}$

$\Rightarrow \left\{\frac{1}{2^n}, \rho_i\right\}_{i \in \{0,1\}^n}$

$\boxed{\text{Max Success probability}} \Rightarrow \sum_d \mathrm{Tr}\{p(x)\rho_d\} = \underline{\frac{1}{2^n}\sum_d \mathrm{Tr}(\rho_d)} = \boxed{\frac{1}{2^n} \times d}$

4.4.7 $\Rightarrow$

Depolarising channel $\Rightarrow$ $\rho \to (1-p)\rho + \frac{pI}{2}$ — (1)

We want to prove,

$$\rho \to (1-3p/4)\rho + p\left(\frac{1}{4}X\rho X + \frac{1}{4}Y\rho Y + \frac{1}{4}Z\rho Z\right)$$

$$= (1-p)\rho + \frac{p}{4}\rho + p\left(\frac{1}{4}X\rho X + \frac{1}{4}Y\rho Y + \frac{1}{4}Z\rho Z\right)$$

$$= \left(1 - \frac{3p}{4}\right)\rho + p\left(\frac{1}{4}\rho + \frac{1}{4}X\rho X + \frac{1}{4}Y\rho Y + \frac{1}{4}Z\rho Z\right) \Rightarrow \text{Comparing with (1)}$$

We have prove $\Rightarrow$

$$\boxed{\frac{I}{2} = \frac{1}{4}\rho + \frac{1}{4}X\rho X + \frac{1}{4}Y\rho Y + \frac{1}{4}Z\rho Z} \Rightarrow$$

$$\rho = \frac{I + r_x X + r_y Y + r_z Z}{2}$$

$$\frac{I}{2} = \frac{1}{4}\left(\frac{I + r_x X + r_y Y + r_z Z}{2}\right) + \underbrace{\frac{1}{4}X\left(\frac{I + r_x X + r_y Y + r_z Z}{2}\right)X}_{\text{I}}$$
$$+ \underbrace{\frac{1}{4}Y\left(\frac{I + r_x X + r_y Y + r_z Z}{2}\right)Y}_{\text{II}} + \underbrace{\frac{1}{4}Z\left(\frac{I + r_x X + r_y Y + r_z Z}{2}\right)Z}_{\text{III}} \Rightarrow (A)$$

Terms of (I)

(1) $XIX = X^2 = I = \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$

(2) $X^3 = X = \begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}$

(3) $XYX = \begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}\begin{bmatrix}0 & -i\\i & 0\end{bmatrix}\begin{bmatrix}0 & 1\\1 & 0\end{bmatrix} = \begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}\begin{bmatrix}-i & 0\\0 & i\end{bmatrix} = \begin{bmatrix}0 & i\\-i & 0\end{bmatrix} = -Y$

(4) $XZX = \begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}\begin{bmatrix}1 & 0\\0 & -1\end{bmatrix}\begin{bmatrix}0 & 1\\1 & 0\end{bmatrix} = \begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}\begin{bmatrix}0 & 1\\-1 & 0\end{bmatrix} = \begin{bmatrix}-1 & 0\\0 & 1\end{bmatrix} = -Z$

Terms of (II)

$YIY = Y^2 = I = \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$

$YXY = \begin{bmatrix}0 & -i\\i & 0\end{bmatrix}\begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}\begin{bmatrix}0 & -i\\i & 0\end{bmatrix} = \begin{bmatrix}0 & -i\\i & 0\end{bmatrix}\begin{bmatrix}i & 0\\0 & -i\end{bmatrix} = \begin{bmatrix}0 & -1\\-1 & 0\end{bmatrix} = -X$

$Y^3 = Y = \begin{bmatrix}0 & -i\\i & 0\end{bmatrix}$

$YZY = \begin{bmatrix}0 & -i\\i & 0\end{bmatrix}\begin{bmatrix}1 & 0\\0 & -1\end{bmatrix}\begin{bmatrix}0 & -i\\i & 0\end{bmatrix} = \begin{bmatrix}0 & -i\\i & 0\end{bmatrix}\begin{bmatrix}0 & -i\\-i & 0\end{bmatrix} = \begin{bmatrix}-1 & 0\\0 & 1\end{bmatrix} = -Z$

$$ZIZ = Z^2 = \begin{bmatrix}1&0\\0&1\end{bmatrix}$$

Term of III

$$ZXZ = \begin{bmatrix}1&0\\0&-1\end{bmatrix}\begin{bmatrix}0&1\\1&0\end{bmatrix}\begin{bmatrix}1&0\\0&-1\end{bmatrix} = \begin{bmatrix}1&0\\0&-1\end{bmatrix}\begin{bmatrix}0&-1\\1&0\end{bmatrix} = \begin{bmatrix}0&-1\\-1&0\end{bmatrix} = \underline{\underline{-X}}$$

$$Z^3 = Z = \begin{bmatrix}1&0\\0&-1\end{bmatrix}$$

$$ZYZ = \begin{bmatrix}1&0\\0&-1\end{bmatrix}\begin{bmatrix}0&-i\\i&0\end{bmatrix}\begin{bmatrix}1&0\\0&-1\end{bmatrix} = \begin{bmatrix}1&0\\0&-1\end{bmatrix}\begin{bmatrix}0&i\\i&0\end{bmatrix} = \begin{bmatrix}0&i\\-i&0\end{bmatrix} = \underline{\underline{-Y}}$$

eq^n (A) becomes,

$$\therefore \Rightarrow \frac{1}{4}\left(\frac{I + r_x X + r_y Y + r_z Z}{2}\right) + \frac{1}{4}\left(\frac{I + r_x X - r_y Y - r_z Z}{2}\right)$$
$$+ \frac{1}{4}\left(\frac{I + r_x(-X) + r_y Y - r_z Z}{2}\right) + \frac{1}{4}\left(\frac{I - r_x X - r_y Y + r_z Z}{2}\right)$$

$$\Rightarrow \boxed{\frac{I}{2}}$$

Hence proved $\Rightarrow$ $$\boxed{\rho \to \left(1 - \frac{3p}{4}\right)\rho + p\left(\frac{1}{4}X\rho X + \frac{1}{4}Y\rho X + \frac{1}{4}Z\rho Z\right)}$$

4.4.10 $$\underline{\underline{\rho}} = \begin{bmatrix}1-p & N\\ N^* & p\end{bmatrix} \qquad \boxed{0 \le p \le 1 \ , \ N}$$

$$\begin{cases} A_0 = \sqrt{r}\,|0\rangle\langle 1| = \sqrt{r}\begin{bmatrix}1\\0\end{bmatrix}\begin{bmatrix}0&1\end{bmatrix} = \sqrt{r}\begin{bmatrix}0&1\\0&0\end{bmatrix} \\ A_1 = |0\rangle\langle 0| + \sqrt{1-r}\,|1\rangle\langle 1| = \begin{bmatrix}1&0\\0&\sqrt{1-r}\end{bmatrix}\end{cases}$$

Under the act^n of amplitude damping channel,

$$\rho \longrightarrow A_0 \rho A_0^+ + A_1 \rho A_1^+$$

$$\Rightarrow \sqrt{r}\sqrt{r}\begin{bmatrix}0&1\\0&0\end{bmatrix}\begin{bmatrix}1-p & N\\ N^+ & p\end{bmatrix}\begin{bmatrix}0&0\\1&0\end{bmatrix} + \begin{bmatrix}1&0\\0&\sqrt{1-r}\end{bmatrix}\begin{bmatrix}1-p & N\\ N^+ & p\end{bmatrix}\begin{bmatrix}1&0\\0&\sqrt{1-r}\end{bmatrix}$$

$$\Rightarrow r\begin{bmatrix}N^* & p\\0&0\end{bmatrix}\begin{bmatrix}0&0\\1&0\end{bmatrix} + \begin{bmatrix}1-p & N\\ N^*\sqrt{1-r} & p\sqrt{1-r}\end{bmatrix}\begin{bmatrix}1&0\\0&\sqrt{1-r}\end{bmatrix}$$

$$= r\begin{bmatrix}p&0\\0&0\end{bmatrix} + \begin{bmatrix}1-p & N\sqrt{1-r}\\ N^+\sqrt{1-r} & p(1-r)\end{bmatrix} = \underline{\underline{\begin{bmatrix}1-p+rp & N\sqrt{1-r}\\ N^*\sqrt{1-r} & p(1-r)\end{bmatrix}}}$$

$\boxed{4.4.11} \Rightarrow \mathcal{N} = \mathcal{N}_2 \circ \mathcal{N}_1 \Rightarrow$ compound channel '$\mathcal{N}$'

Method

i.e $\mathcal{N}(\rho) = \mathcal{N}_2(\mathcal{N}_1 \rho)$

$\mathcal{N}_1 \rho \rightarrow A_0 \rho A_0^\dagger + A_1 \rho A_1^\dagger$

$\mathcal{N}_2(\mathcal{N}_1 \rho) \rightarrow A_3(A_0 \rho A_0^\dagger + A_1 \rho A_1^\dagger) A_3^\dagger + A_4(A_0 \rho A_0^\dagger + A_1 \rho A_1^\dagger) A_4^\dagger$

→ where A_0 & A_1 are kraus-operators for $\mathcal{N}_1$ channel
A_3 & A_4 are kraus " for $\mathcal{N}_2$ channel.

$A_0 = \sqrt{r_1}\,|0\rangle\langle 1|$, $A_1 = |0\rangle\langle 0| + \sqrt{1-r_1}\,|1\rangle\langle 1|$

$A_3 = \sqrt{r_2}\,|0\rangle\langle 1|$, $A_4 = |0\rangle\langle 0| + \sqrt{1-r_2}\,|1\rangle\langle 1|$

$\rho' = \mathcal{N}_2(\mathcal{N}_1 \rho) \rightarrow A_3 A_0 \rho A_0^\dagger A_3^\dagger + A_3 A_1 \rho A_1^\dagger A_3^\dagger + A_4 A_0 \rho A_0^\dagger A_4^\dagger + A_4 A_1 \rho A_1^\dagger A_4^\dagger$

$A_3 A_0 = \sqrt{r_2 r_1}\,\underbrace{|0\rangle\langle 1|0\rangle\langle 1|}_{0} = 0$

$A_0^\dagger A_3^\dagger = \sqrt{r_1 r_2}\,|1\rangle\langle 0|1\rangle\langle 0| = 0$

$A_3 A_1 = \sqrt{r_2}\,(\underbrace{|0\rangle\langle 1|0\rangle\langle 0|}_{0} + \sqrt{(1-r_1)}\,|0\rangle\langle 1|) = \sqrt{r_2(1-r_1)}\,|0\rangle\langle 1|$

$A_1^\dagger A_3^\dagger = (|0\rangle\langle 0| + \sqrt{1-r_1}\,|1\rangle\langle 1|)(\sqrt{r_2}\,|1\rangle\langle 0|)$

$= 0 + (\sqrt{1-r_1})\sqrt{r_2}\,|1\rangle\langle 0| = \sqrt{r_2(1-r_1)}\,|1\rangle\langle 0|$

$A_4 A_0 = (|0\rangle\langle 0| + \sqrt{1-r_2}\,|1\rangle\langle 1|)(\sqrt{r_1}\,|0\rangle\langle 1|)$

$= \sqrt{r_1}\,|0\rangle\langle 0|0\rangle\langle 1| + \sqrt{r_1(1-r_2)}\,0 = \sqrt{r_1}\,|0\rangle\langle 1|$

$A_0^\dagger A_4^\dagger = (\sqrt{r_1}\,|1\rangle\langle 0|)(|0\rangle\langle 0| + \sqrt{1-r_2}\,|1\rangle\langle 1|) = \sqrt{r_1}\,|1\rangle\langle 0| + \sqrt{r_1(1-r_2)} \times 0$
$= \sqrt{r_1}\,|1\rangle\langle 0|$

$A_4 A_1 = (|0\rangle\langle 0| + \sqrt{1-r_2}\,|1\rangle\langle 1|)(|0\rangle\langle 0| + \sqrt{1-r_1}\,|1\rangle\langle 1|)$

$A_4 A_1 = |0\rangle\langle 0| + 0 + 0 + \sqrt{(1-r_1)(1-r_2)}\,|1\rangle\langle 1| = |0\rangle\langle 0| + \sqrt{1-r_2-r_1+r_1 r_2}\,|1\rangle\langle 1|$

$(A_1^\dagger A_4^\dagger) = |0\rangle\langle 0| + \sqrt{(1-r_1)(1-r_2)}\,|1\rangle\langle 1|$

$\therefore \mathcal{N}_2(\mathcal{N}_1 \rho) \rightarrow 0 + r_2(1-r_1)\,|0\rangle\langle 1|\rho|1\rangle\langle 0| + r_1\,|0\rangle\langle 1|\rho|1\rangle\langle 0|$
$+ (|0\rangle\langle 0| + (\sqrt{(1-r_1)(1-r_2)})\,|1\rangle\langle 1|)\,\rho\,(|0\rangle\langle 0| + \sqrt{(1-r_1)(1-r_2)}\,|1\rangle\langle 1|)$

$\Rightarrow r_2(1-r_1)\,|0\rangle\langle 1|\rho|1\rangle\langle 0| + r_1\,|0\rangle\langle 1|\rho|1\rangle\langle 0| + |0\rangle\langle 0|\rho|0\rangle\langle 0|$
$+ \sqrt{(1-r_1)(1-r_2)}\,|1\rangle\langle 1|\rho|0\rangle\langle 0| + (|0\rangle\langle 0|\rho|\sqrt{(1-r_1)(1-r_2)}\,|1\rangle\langle 1|)$
$+ (1-r_1)(1-r_2)\,|1\rangle\langle 1|\rho|1\rangle\langle 1|$

In matrix form

$$\rho' = N_2(N_1\rho) = \begin{array}{cc} |0\rangle\langle 0| & |0\rangle\langle 1| \\ \left[\begin{matrix} r_2(1-r_1)\langle 1|\rho|1\rangle + r_1\langle 1|\rho|1\rangle + \langle 0|\rho|0\rangle & \sqrt{(1-r_1)(1-r_2)}\,\langle 0|\rho|1\rangle \\ \sqrt{(1-r_1)(1-r_2)}\,\langle 1|\rho|0\rangle & (1-r_1)(1-r_2)\,\langle 1|\rho|1\rangle \end{matrix}\right] & \\ |1\rangle\langle 0| & |1\rangle\langle 1| \end{array}$$

$$= \begin{bmatrix} \rho_{11}' & \rho_{12}' \\ \rho_{21}' & \rho_{22}' \end{bmatrix}$$

Usually $\Rightarrow$ element ρ_{22}' of ρ' = (transmission parameter) x $\underbrace{\text{const}}_{\langle 1|\rho|1\rangle}$ in this case

$$\boxed{\text{Transmission parameter} = (1-r_1)(1-r_2)}$$

Usually $\Rightarrow$ ρ_{11}' = (Damping parameter) x (const)$_1$ + (const)$_2$

$= (1 - \text{Transmission parameter}) \times \underbrace{(\text{const})_1}_{\langle 1|\rho|1\rangle} + \underbrace{(\text{const})_2}_{\langle 0|\rho|0\rangle}$

$= (r_2(1-r_1) + r_1)\,\langle 1|\rho|1\rangle$

$= \underbrace{(1-(1-r_1)(1-r_2))}_{\text{Damping parameter}}\langle 1|\rho|1\rangle$

Method 2 $\Rightarrow$ Matrix method

Let $\rho = \begin{bmatrix} \rho_{11} & \rho_{12} \\ \rho_{21} & \rho_{22} \end{bmatrix}$

$N_1(\rho) \to A_0\rho A_0^\dagger + A_1\rho A_1^\dagger = \sum_i A_i\rho A_i^\dagger$

$\to \begin{bmatrix} 0 & \sqrt{r_1} \\ 0 & 0 \end{bmatrix}\begin{bmatrix} \rho_{11} & \rho_{12} \\ \rho_{21} & \rho_{22} \end{bmatrix}\begin{bmatrix} 0 & 0 \\ \sqrt{r_1} & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & \sqrt{1-r_1} \end{bmatrix}\begin{bmatrix} \rho_{11} & \rho_{12} \\ \rho_{21} & \rho_{22} \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & \sqrt{1-r_1} \end{bmatrix}$

$\Rightarrow \begin{bmatrix} 0 & \sqrt{r_1} \\ 0 & 0 \end{bmatrix}\begin{bmatrix} \rho_{12}\sqrt{r_1} & 0 \\ \rho_{22}\sqrt{r_1} & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & \sqrt{1-r_1} \end{bmatrix}\begin{bmatrix} \rho_{11} & \rho_{12}\sqrt{1-r_1} \\ \rho_{21} & \rho_{22}\sqrt{1-r_1} \end{bmatrix}$

$\Rightarrow \begin{bmatrix} \rho_{22}r_1 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} \rho_{11} & \rho_{12}\sqrt{1-r_1} \\ \sqrt{1-r_1}\,\rho_{21} & (\sqrt{1-r_1})\,\rho_{22}\sqrt{1-r_1} \end{bmatrix}$

$= \begin{bmatrix} \rho_{11} + \rho_{22}r_1 & \rho_{12}\sqrt{1-r_1} \\ (\sqrt{1-r_1})\,\rho_{21} & (1-r_1)(\rho_{22}) \end{bmatrix}$

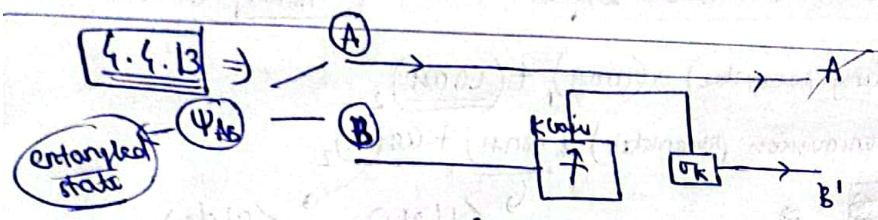
$N_2(N_1\rho) = \begin{bmatrix} 0 & \sqrt{r_2} \\ 0 & 0 \end{bmatrix}\begin{bmatrix} \rho_{11}+\rho_{22}r_1 & \rho_{12}(\sqrt{1-r_1}) \\ (\sqrt{1-r_1})\rho_{21} & (1-r_1)\rho_{22} \end{bmatrix}\begin{bmatrix} 0 & 0 \\ \sqrt{r_2} & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & \sqrt{1-r_2} \end{bmatrix}\begin{bmatrix} \rho_{11}+\rho_{22}r_1 & \rho_{12}(\sqrt{1-r_1}) \\ (\sqrt{1-r_1})\rho_{21} & (1-r_1)\rho_{22} \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & \sqrt{1-r_2} \end{bmatrix}$

$$= \begin{bmatrix} \sqrt{r_2(1-r_1)}\,\rho_{21} & \sqrt{r_2(1-r_1)}\,\rho_{22} \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ \sqrt{r_2} & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & \sqrt{1-r_2} \end{bmatrix} \begin{bmatrix} \rho_{11}+\rho_{22}r_1 & \rho_{12}\sqrt{(1-r_1)(1-r_2)} \\ \sqrt{1-r_1}\,\rho_{21} & (1-r_1)\sqrt{1-r_2}\,\rho_{22} \end{bmatrix}$$

$$= \begin{bmatrix} r_2(1-r_1)\rho_{22} & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} \rho_{11}+\rho_{22}r_1 & \rho_{12}\sqrt{(1-r_1)(1-r_2)} \\ \sqrt{(1-r_1)(1-r_2)}\,\rho_{21} & (1-r_1)(1-r_2)\rho_{22} \end{bmatrix}$$

$$N_2(N_1(\rho)) = \rho' = \boxed{\begin{bmatrix} \rho_{11} + \rho_{22}r_1 + r_2\rho_{22} - r_1r_2\rho_{22} & \rho_{12}\sqrt{(1-r_1)(1-r_2)} \\ \sqrt{(1-r_1)(1-r_2)}\,\rho_{21} & \underbrace{(1-r_1)(1-r_2)}_{\text{Transmission parameter}}\rho_{22} \end{bmatrix}}$$

$\boxed{4.4.13}$ ⇒



Classical Quantum channel

To prove ⇒

Classical Quantum channel is entanglement breaking channel

$$\boxed{N(\rho) = \sum_k \langle k|\rho|k\rangle \sigma_k} \Rightarrow \text{classical Quantum channel} \quad (A)$$

$$N_{EB}^{B\to B'}(\phi^{AB}) = \sum_z P_Z(z)\,|\phi_z\rangle\langle\phi_z|^A \otimes |\psi_z\rangle\langle\psi_z|^{B'}$$

Since Ψ_{AB} is entangled state ⇒ Consider Measurement operator as $\boxed{I_A \otimes M_B}$

Proof by example ⇒ Consider a Bell state ⇒ $|\psi^+\rangle_{AB} = \dfrac{|00\rangle + |11\rangle}{\sqrt{2}}$

$$\rho_{AB} = |\psi\rangle\langle\psi| = \boxed{\frac{1}{2}\left(|00\rangle\langle 00| + |00\rangle\langle 11| + |11\rangle\langle 00| + |11\rangle\langle 11|\right)} \quad - (B)$$

Now, we are doing measurement in orthonormal basis for qubit B of the Bell state (let's say ⇒ $|0\rangle$ & $|1\rangle$ basis ⇒ $\{|k\rangle\} \Rightarrow \{|0\rangle, |1\rangle\}$ ⇒ But ρ_{AB} is entangled state $|k\rangle = \{I_A \otimes |0_B\rangle, I_A \otimes |1_B\rangle\}$

If output is $\boxed{|0\rangle_B}$ state then we corelate σ_0 to post measurement state

& o/p is $\boxed{|1_B\rangle}$ " then " " σ_1 to " " "

σ_0 & σ_1 are density operators

Post Measurement state corelate with σ_0 & σ_1 ⇒ $(I_A \otimes M_B)\rho_{AB}$

$$M_B = |0\rangle_B\langle 0|_B + |1\rangle_B\langle 1|_B = M_{B_0} + M_{B_1}$$

~~$(I_A \otimes M_B)\rho_{AB}$~~

So basically ⇒ Putting (B) in (A) , $|k_0\rangle = |I_A \otimes 0_B\rangle$, $|k_1\rangle = |I_A \otimes 1_B\rangle$

(Here I_A is not 2x2 Matrix ⇒ It just denote Identity operation on (A)) as $|\Psi_{AB}\rangle$ is entangled state

Method 1

$$N(\rho_{AB}) = \langle I_A \otimes 0_B | \rho_{AB} | I_A \otimes 0_B \rangle \otimes \sigma_0 + \langle I_A \otimes 1_B | \rho_{AB} | I_A \otimes 1_B \rangle \otimes \sigma_1$$

$$= \frac{1}{2}\left(|0\rangle_A\langle 0|_A \otimes \underset{B'}{\sigma_0} + |1\rangle_A\langle 1|_A \otimes \underset{B'}{\sigma_1} \right)$$

Hence, we can write,

$$\boxed{N(\rho_{AB}) = \sum_z P_z(z) |\phi_z\rangle\langle\phi_z|^A \otimes \sigma_z^{B'}}$$

Hence $N(\rho_{AB})$ is an entanglement breaking channel

Method 2 (OR) ⇒ Post measurement state on density operator is for ensemble

$$\rho' = \sum_i p_i \frac{M_{B_i}^{\dagger} \rho M_{B_i}}{p_i}$$

$\{p_i, M_{B_i}^{\dagger} \rho M_{B_i}\}$

$$= p_0 \times \frac{M_{B_0}^{\dagger} \rho M_{B_0}}{p_0} + p_1 \times \frac{M_{B_1}^{\dagger} \rho M_{B_1}}{p_1}$$

Here I_A is 2x2 Identity Matrix

$$M'_{B_0} = I_A \otimes M_{B_0} \Rightarrow I_A \otimes |0\rangle_B\langle 0|_B$$

$$M'_{B_1} = I_A \otimes M_{B_1} = I_A \otimes |1\rangle_B\langle 1|_B$$

Now, we are corelating σ_0 & σ_1 dependent on ~~post Measu~~ Measurement outcome

Resulting mixed state ⇒

$$\boxed{\rho'_{AB} = p_0 \times \frac{M_{B_0}'^{\dagger} \rho M'_{B_0}}{p_0} \otimes \sigma_0 + p_1 \times \frac{M_{B_1}'^{\dagger} \rho M'_{B_1}}{p_1} \otimes \sigma_1}$$

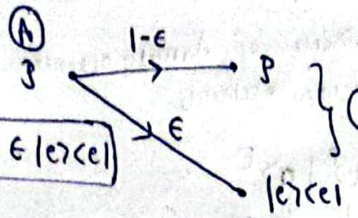
(A) . Tracing out the B system we get,

$$\boxed{\rho'_{AB'} = N(\rho_{AB}) = \frac{1}{2}\left(\underbrace{|0\rangle_A\langle 0|_A}_{A} \otimes \underset{B'}{\sigma_0} + \underbrace{|1\rangle_A\langle 1|_A}_{A} \otimes \underset{B'}{\sigma_1} \right)}$$

↳ Separable state

② ⓐ Erasure channel   In I/p A , O/p B

for a qubit ⇒



$\therefore \boxed{\rho \rightarrow (1-\epsilon)\rho + \epsilon |e\rangle\langle e|}$

$\therefore \boxed{A_i\,|i/p\rangle = \sqrt{p_i}\,|o/p\rangle}$ ⓗ

Kraus operator

If error occurs then, if i/p state is $|0\rangle^A$ then O/p state is $|e\rangle^B$ ⇒ this happens with probability ϵ

$\therefore$ Kraus operator is written in standard form as, for a channel A

$$\boxed{A_i = \sqrt{p_i}\,|o/p\ \text{state}\rangle\langle I/p\ \text{state}|}$$

$\therefore$ Where $\sqrt{p_i}$ = probability amplitude with which state goes from I/p state to o/p state over Quantum channel

$$\therefore \boxed{A_1 = \sqrt{\epsilon}\,|e\rangle^B\langle 0|^A}$$

Again if input state is $|1\rangle^A$ then @ o/p error can occur with probability ϵ & state @ o/p $|e\rangle^B$

$$\therefore \boxed{A_2 = \sqrt{\epsilon}\,|e\rangle^B\langle 1|^A}$$

Now, if I/p is $|0\rangle^A$ then o/p will be $|0\rangle^A$ with probability $1-\epsilon$

" " " $|1\rangle^A$ " $|1\rangle^A$ " " "

$$\therefore \boxed{A_3 = \sqrt{1-\epsilon}\left(|0\rangle^B\langle 0|^A + |1\rangle^B\langle 1|^A\right)}$$

Now, we know, $\sum_k A_k^{\dagger} A_k = I$ ⇒ Validates that above defined Kraus operators are correct

$$\left(\sqrt{\epsilon}\,|0\rangle^A\langle e|^B\,\sqrt{\epsilon}\,|e\rangle^B\langle 0|^A\right) + \sqrt{\epsilon}\left(|1\rangle^A\langle e|^B\right)\sqrt{\epsilon}\left(|e\rangle^B\langle 1|^A\right)$$
$$+\ 1-\epsilon\left(|0\rangle^A\langle 0|^B + |1\rangle^A\langle 1|^B\right)\left(|0\rangle^B\langle 0|^A + |1\rangle^B\langle 1|^A\right)$$
$$= \epsilon\,|0\rangle^A\langle 0|^A + \epsilon\,|1\rangle^A\langle 1|^A + 1-\epsilon\left(|0\rangle^A\langle 0|^A + |1\rangle^A\langle 1|^A\right)$$
$$= \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} = I$$

Note ⇒ $\boxed{\text{O/p Hilbert space} = 1 + \text{I/p Hilbert space}}$ ⇒ as we can't erase a particular state, its I/p will be store in some other state $|e\rangle$

$|0\rangle\langle 0|$ $\therefore$ It is better to write Kraus operators as 2x3 matrix form

$$A_1 = \begin{bmatrix}\sqrt{1-\epsilon} & 0 & 0\\ 0 & \sqrt{1-\epsilon} & 0\end{bmatrix},\quad A_2 = \begin{bmatrix}0 & 0 & \sqrt{\epsilon}\\ 0 & 0 & 0\end{bmatrix},\quad A_3 = \begin{bmatrix}0 & 0 & 0\\ 0 & 0 & \sqrt{\epsilon}\end{bmatrix}$$

$|1\rangle\langle 1|$ $|e\rangle\langle 0|$ $|e\rangle\langle 1|$

⑥ $\boxed{\text{Generalized dephasing channel}} \Rightarrow$

Keep the diagonal elements of density operator in some orthonormal basis but adds phase to off diagonal elements

$$A^{A' \to BE} |x\rangle^{A'} = |x\rangle^{B} |a_x\rangle^{E}$$

A ↓ Kraus operator for general dephasing channel; $|x\rangle^{A'}$ ↓ I/p state; $|x\rangle^B$ ↓ O/p state; $|a_x\rangle^E$ ↳ environment state (Need not be orthogonal) but is normalized

$$\therefore \boxed{A^{A' \to BE} = \sum_x |x\rangle^B |a_x\rangle^E \langle x|^{A'}}$$

Action of kraus operator on density Matrix.

Channel + env $\Rightarrow$
$$\boxed{\sum_i A_i \rho A_i^{\dagger} = \sum_{x, x'} \langle x|\rho|x'\rangle |x\rangle\langle x'|^B \otimes {}^E|a_x\rangle\langle a_{x'}|^E}$$

Tracing o/p the environment we get actⁿ of kraus operator @ channel on $\underline{\rho}$ (input state)

$$\boxed{N_D(\rho) = \sum A_i \rho A_i^{\dagger} = \sum_{x, x'} \langle x|\rho|x'\rangle |x\rangle\langle x'|^B \underbrace{\langle a_{x'}|a_x\rangle^E}_{\text{Need not be orthogonal}}}$$

↳ only channel

d diagonal elements same $\Leftarrow$ if $x = x' \Rightarrow$ diagonal elements of ρ are present. $\Rightarrow \langle a_x|a_x\rangle = \underline{\underline{1}} \Rightarrow$ diag

$d(d-1)$ $\Leftarrow$ off~~e~~-diagonal element: $x \neq x' \Rightarrow \langle a_{x'}|a_x\rangle \neq 1 \Rightarrow$ off diagonal elements Multiplied by a phase $\langle a_{x'}|a_x\rangle$

In Matrix form
$$\sum_i A_i \rho A_i^{\dagger} = \begin{bmatrix} \langle 0|\rho|0\rangle & \langle a_0|a_1\rangle\langle 0|\rho|1\rangle & \cdots & \langle a_0|a_d\rangle\langle 0|\rho|d\rangle \\ & \langle 1|\rho|1\rangle & & \vdots \\ \vdots & & \ddots & \\ \langle a_d|a_0\rangle^{*}\langle d|\rho|0\rangle & \cdots & & \langle d|\rho|d\rangle \end{bmatrix} \quad \boxed{1 \leq i \leq d^2} \quad -Ⓐ$$

Let $1-\epsilon$ be probability that off diagonal elements are preserved i.e $\rho \to \rho$, & $\boxed{\epsilon \Rightarrow \text{off-diagonal elements are not preserved}}$

1st kraus operator $A_1 \Rightarrow$ preserves the off diagonal elements

$$\therefore A_1 = \begin{bmatrix} \sqrt{1-\epsilon} & 0 & \cdots & 0 \\ 0 & \sqrt{1-\epsilon} & & 0 \\ \vdots & 0 & \sqrt{1-\epsilon} & 0 \\ & & & \sqrt{1-\epsilon} \end{bmatrix}_{d\times d} \Rightarrow \sqrt{1-\epsilon}\, I_d \Rightarrow \boxed{\sum_{m=0}^{d} \sqrt{1-\epsilon}\, |m\rangle\langle m| = A_1}$$

$$A_2 = \begin{bmatrix} \sqrt{\epsilon} & 0 & 0 & \cdots \\ & 0 & & \\ & & \ddots & \\ & 0 & & 0 \end{bmatrix}, \quad A_3 = \begin{bmatrix} 0 & 0 & \cdots & \\ 0 & \sqrt{\epsilon} & & 0 \\ \vdots & & \ddots & \\ & 0 & & 0 \end{bmatrix}$$

$\boxed{A_2 = |0\rangle\langle 0| \sqrt{\epsilon}}$

diagonal elements measured

$$A_{d+1} = \begin{bmatrix} 0 & & \\ & 0 & 0 \\ & & \ddots \\ 0 & & \sqrt{\epsilon} \end{bmatrix}$$

d+1 Kraus operator

for qubit dephasing channel $\Rightarrow$

$\Rightarrow$ $N(\rho) = A_1 \rho A_1^{\dagger} + A_2 \rho A_2^{\dagger} + A_3 \rho A_3^{\dagger}$

$$A_1 = \begin{bmatrix} \sqrt{1-\epsilon} & 0 \\ 0 & \sqrt{1-\epsilon} \end{bmatrix}, \quad A_2 = \begin{bmatrix} \sqrt{\epsilon} & 0 \\ 0 & 0 \end{bmatrix}, \quad A_3 = \begin{bmatrix} 0 & 0 \\ 0 & \sqrt{\epsilon} \end{bmatrix}$$

$$N(\rho) = (1-\epsilon)\rho + \epsilon \rho_{11} |0\rangle\langle 0| + \epsilon \rho_{22} |1\rangle\langle 1|$$

$$= \begin{bmatrix} (1-\epsilon)\rho_{11} + \epsilon\rho_{11} & \rho_{12} \\ \rho_{22} & (1-\epsilon)\rho_{22} + \epsilon\rho_{22} \end{bmatrix} = \begin{bmatrix} \rho_{11} & \rho_{12} \\ \rho_{21} & \rho_{22} \end{bmatrix}$$

Example of qubit dephasing channel $\Rightarrow$ $\boxed{\rho \to (1-\epsilon)\rho + \epsilon Z \rho Z}$

i.e $\begin{bmatrix} \rho_{11} & \rho_{12} \\ \rho_{21} & \rho_{22} \end{bmatrix} \to \begin{bmatrix} \rho_{11} & -\rho_{12} \\ -\rho_{21} & \rho_{22} \end{bmatrix}$

i.e $\boxed{\langle a_0 | a_1 \rangle = \langle a_1 | a_0 \rangle = -1}$ when comparing with (A)

(8-21) $\Rightarrow$ ① $U = \exp(-iH\Delta t)$, $H = \chi(a^\dagger b + b^\dagger a)$

$E_k = \langle k_b | U | 0_b \rangle$

To prove $\Rightarrow$ $E_k = \sum_n \sqrt{\binom{n}{k}} \sqrt{(1-r)^{n-k} r^k} |n-k\rangle\langle n|$

$b^\dagger b |k_b\rangle = \lambda_b |k_b\rangle$, $a^\dagger a |n\rangle = \lambda_a |n\rangle$

$r = 1 - \cos^2(\chi \Delta t)$

Let $|n, m\rangle$ be basis for a & b resp.

$\swarrow$ H.O annihilation operator $\searrow$ env. annihilation operator

$H \Rightarrow$

$$H|n,m\rangle = \chi(a^\dagger b + b^\dagger a)|n,m\rangle$$

$$= \chi(\overbrace{c|n+1\rangle|m-1\rangle + d|n-1\rangle|m+1\rangle}^{\text{constants}})$$

i.e $\boxed{n_{new} + m_{new} = n + m}$ $\Rightarrow$ i.e H preserves $n+m$

$\therefore$ Consider,

$$\boxed{E_k = \langle k_b | U | 0_b \rangle = \sum_n \langle n-k_a, k_b | U | n, 0\rangle |n-k_a\rangle\langle n_a|}$$ $\to$ (I)

Now,

$$\langle n-k_a, k_b | U | n, 0\rangle = \langle 0_a, 0_b | \frac{a^{n-k}}{\sqrt{(n-k)!}} \frac{b^k}{\sqrt{k!}} \frac{U a^{\dagger n}}{\sqrt{n!}} | 0_a, 0_b\rangle$$

$U^\dagger U = I \Rightarrow$

$$\frac{\sqrt{\frac{n!}{(n-k)!\,k!}}}{n!} \langle 0_a, 0_b | a^{n-k} b^k U a^{\dagger n} U^\dagger U |0_a, 0_b\rangle$$

$$\Rightarrow \frac{\sqrt{{}^nC_k}}{n!} \langle 0_a, 0_b | a^{n-k} b^k U a^{\dagger n} U^\dagger U |0_a, 0_b\rangle$$

$\boxed{U|0_a, 0_b\rangle = |0_a, 0_b\rangle}$

as $\exp(-iH\Delta t)|0_a, 0_b\rangle = e^{(-i\chi a^\dagger b \Delta t - i\chi b^\dagger a \Delta t)}|0_a, 0_b\rangle$

$$= \left(1 - i\chi a^\dagger b \Delta t + \frac{(\chi a^\dagger b \Delta t)^2}{2!} \cdots\right)\left(1 - i\chi b^\dagger a \Delta t + \frac{(\chi b^\dagger a \Delta t)^2}{2!}\right)|0_a, 0_b\rangle$$

(#) $\boxed{a|0_a\rangle = 0,\ b|0_b\rangle = 0}$

$\therefore$ $U|0_a, 0_b\rangle \Rightarrow |0_a, 0_b\rangle$

$$= \boxed{\frac{\sqrt{{}^nC_k}}{n!} \langle 0_a, 0_b | a^{n-k} b^k U a^{\dagger n} U^\dagger | 0_a, 0_b \rangle} - (A)$$

$\Rightarrow$ Now, $\underline{U a^\dagger U^\dagger} \Rightarrow \left(e^{-iH\Delta t} a^\dagger e^{iH\Delta t}\right)$

By Campbell Hausdorff formula,

$$U a^\dagger U^\dagger \left(e^{-iH\Delta t} a^\dagger e^{iH\Delta t}\right) = \sum_{n=0}^{\infty} \frac{[(iH\Delta t)^n, a^\dagger]}{n!}$$

$$= a^\dagger + [iH\Delta t, a^\dagger] + \frac{1}{2!}[iH\Delta t, [iH\Delta t, a^\dagger]] + \cdots$$

$$[H, a^\dagger] = \chi(a^\dagger b + b^\dagger a) a^\dagger - a^\dagger \chi (a^\dagger b + b^\dagger a)$$

$$= \chi(a^\dagger b a^\dagger + b^\dagger a a^\dagger) - \chi a^\dagger a^\dagger b - \chi a^\dagger b^\dagger a$$

similarly find this.

Simplifying we get,

$$U a^\dagger U^\dagger = a^\dagger \left(1 + \frac{\chi^2 \Delta t^2}{2!} + \frac{\chi^4 \Delta t^4}{4!} + \cdots\right) - i b^\dagger \left(\chi \Delta t - \frac{(\chi \Delta t)^3}{3!} + \cdots\right)$$

$$\boxed{U a^\dagger U^\dagger = a^\dagger \cos(\chi \Delta t) - i b^\dagger \sin(\chi \Delta t)}$$

Now, $U (a^\dagger)^n U^\dagger = \underbrace{(U a^\dagger U^\dagger)(U a^\dagger U^\dagger) \cdots (U a^\dagger U^\dagger)}_{n \text{ times}}$

$$= \left(a^\dagger \cos(\chi \Delta t) - i b^\dagger \sin(\chi \Delta t)\right)^n$$

From (A)

$$= \frac{\sqrt{{}^nC_k}}{n!} \langle 0_a, 0_b | a^{n-k} b^k \left(a^\dagger \cos(\chi \Delta t) - i b^\dagger \sin(\chi \Delta t)\right)^n | 0_a, 0_b \rangle$$

$$= \sum_{r=0}^{r=n} \frac{\sqrt{\binom{n}{k}}}{n!} \binom{n}{r} (-i \sin(\chi \Delta t))^r (\cos \chi \Delta t)^{n-r} \langle 0_a, 0_b | a^{n-k} b^k (a^\dagger)^{n-r} (b^\dagger)^r | 0_a, 0_b \rangle$$

Terms with $\underline{r = k}$ survives, as total operators must be such that there is no change in no. states for inner product to be non-zero

$$= \frac{\sqrt{\binom{n}{k}}}{n!} \binom{n}{k} (-i \sin(\chi \Delta t))^k (\cos(\chi \Delta t))^{n-k} \langle 0_a, 0_b | a^{n-k} b^k (a^\dagger)^{n-k} (b^\dagger)^k | 0_a, 0_b \rangle$$

$$= \frac{\sqrt{\binom{n}{k}}}{n!} \binom{n}{k} \left(-i\sin(\chi\Delta t)\right)^k \left(\cos(\chi\Delta t)\right)^{n-k} \sqrt{(n-k)!\,(k!)(k!)(n-k)!}\ \langle 0_a, 0_b | a, 0_b \rangle$$

Let $\Rightarrow$ $r = 1-\cos^2 r\Delta t$, putting above result in eq^n (I)

$$E_k = \langle k_b | U | 0_b \rangle = \underbrace{(-i)^k}_{\text{Global phase. Ignore}} \sum_n \sqrt{\binom{n}{k}} \sqrt{(1-r)^{n-k} r^k}\, |n-k_a\rangle\langle k_a|$$

(II)
$$\boxed{E_k = \sum_n \sqrt{\binom{n}{k}} \sqrt{(1-r)^{n-k} r^k}\ |n-k_a\rangle\langle k_a|} \Rightarrow \text{Hence proved}$$

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(2) We know that Unitary operators are trace preserving operat^{n}s.

$\because \sum_k E_k^{+} E_k = I \Rightarrow$ then operation E_k is trace preserving

$$E_k = \langle k_b | U | 0_b \rangle$$
$$E_k^{+} = \langle 0_b | U^{+} | k_b \rangle$$

~~$\langle k_b|U|0_b\rangle\langle 0_b|U^+|k_b\rangle$~~

$$\sum_k E_k^{+} E_k = \langle 0_b | U^{+} | k_b \rangle \langle k_b | U | 0_b \rangle$$
$$= \sum_k \underbrace{\langle 0_b | 0_b \rangle}_{1}\ \underbrace{| U^{+} U}_{I}\ |k_b\rangle\langle k_b|$$

as environment states for complete Basis

$$\sum_k E_k^{+} E_k = \boxed{\sum_k |k_b\rangle\langle k_b| = I} \Rightarrow \boxed{\text{Trace preserving quantum operat}^{n}}$$

③ ~~$\boxed{8.24}$~~ ①

③ $\boxed{8.23}$ $\Rightarrow$ $\rho = |\psi\rangle\langle\psi| = (a|01\rangle + b|10\rangle)(a^*\langle 01| + b^*\langle 10|)$

$$= (aa^*|01\rangle\langle 01| + ab^*|01\rangle\langle 10| + ba^*|10\rangle\langle 01| + bb^*|10\rangle\langle 10|)$$

$$\mathcal{E}_{AD}(\rho) = E_0\rho E_0^\dagger + E_1\rho E_1^\dagger$$

$$E_0 = \begin{bmatrix} 1 & 0 \\ 0 & \sqrt{1-r} \end{bmatrix}, \quad E_1 = \begin{bmatrix} 0 & \sqrt{r} \\ 0 & 0 \end{bmatrix}, \quad E_0 = |0\rangle\langle 0| + \sqrt{1-r}\,|1\rangle\langle 1|, \quad E_1 = \sqrt{r}\,|0\rangle\langle 1|$$

$$\mathcal{E}_{AD}(\rho) = \begin{bmatrix} 1 & 0 \\ 0 & \sqrt{1-r} \end{bmatrix} \rho \begin{bmatrix} 1 & 0 \\ 0 & \sqrt{1-r} \end{bmatrix} + \begin{bmatrix} 0 & \sqrt{r} \\ 0 & 0 \end{bmatrix} \rho \begin{bmatrix} 0 & 0 \\ \sqrt{r} & 0 \end{bmatrix}$$

$$(\mathcal{E}_{AD} \otimes \mathcal{E}_{AD})(\rho) = \begin{bmatrix} 1 & 0 \\ 0 & \sqrt{1-r} \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ 0 & \sqrt{1-r} \end{bmatrix} \rho \begin{bmatrix} 1 & 0 \\ 0 & \sqrt{1-r} \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ 0 & \sqrt{1-r} \end{bmatrix}$$

$\boxed{E_0 \otimes (E_0\rho E_0^\dagger + E_1\rho E_1^\dagger)E_0^\dagger + E_1 \otimes (E_0\rho E_0^\dagger + E_1\rho E_1^\dagger)E_1^\dagger}$

$$+ \begin{bmatrix} 1 & 0 \\ 0 & \sqrt{1-r} \end{bmatrix} \otimes \begin{bmatrix} 0 & \sqrt{r} \\ 0 & 0 \end{bmatrix} \rho \begin{bmatrix} 0 & 0 \\ \sqrt{r} & 0 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ 0 & \sqrt{1-r} \end{bmatrix}$$

$$+ \begin{bmatrix} 0 & \sqrt{r} \\ 0 & 0 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ 0 & \sqrt{1-r} \end{bmatrix} \rho \begin{bmatrix} 1 & 0 \\ 0 & \sqrt{1-r} \end{bmatrix} \otimes \begin{bmatrix} 0 & 0 \\ \sqrt{r} & 0 \end{bmatrix}$$

$$+ \begin{bmatrix} 0 & \sqrt{r} \\ 0 & 0 \end{bmatrix} \otimes \begin{bmatrix} 0 & \sqrt{r} \\ 0 & 0 \end{bmatrix} \rho \begin{bmatrix} 0 & 0 \\ \sqrt{r} & 0 \end{bmatrix} \otimes \begin{bmatrix} 0 & 0 \\ \sqrt{r} & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \sqrt{1-r} & 0 & 0 \\ 0 & 0 & \sqrt{1-r} & 0 \\ 0 & 0 & 0 & 1-r \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & aa^* & ab^* & 0 \\ 0 & ba^* & bb^* & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \sqrt{1-r} & 0 & 0 \\ 0 & 0 & \sqrt{1-r} & 0 \\ 0 & 0 & 0 & 1-r \end{bmatrix}$$

$$+ \begin{bmatrix} 0 & \sqrt{r} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \sqrt{r(1-r)} \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & aa^* & ab^* & 0 \\ 0 & ba^* & bb^* & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \sqrt{r} & 0 & 0 & 0 \\ 0 & \sqrt{r-r^2} & 0 & 0 \end{bmatrix}$$

+ Continued

$$+\begin{bmatrix}0&0&\sqrt{r}&0\\0&0&0&\sqrt{r(1-r)}\\0&0&0&0\\0&0&0&0\end{bmatrix}\begin{bmatrix}0&0&0&0\\0&aa^*&ab^*&0\\0&ba^*&bb^*&0\\0&0&0&0\end{bmatrix}\begin{bmatrix}0&0&0&0\\\sqrt{r}&0&0&0\\0&0&0&0\\0&0&\sqrt{r-r^2}&0\end{bmatrix}$$

$$+\begin{bmatrix}0&0&0&r\\0&0&0&0\\0&0&0&0\\0&0&0&0\end{bmatrix}\begin{bmatrix}0&0&0&0\\0&aa^*&ab^*&0\\0&ba^*&bb^*&0\\0&0&0&0\end{bmatrix}\begin{bmatrix}0&0&0&0\\0&0&0&0\\0&0&0&0\\r&0&0&0\end{bmatrix}$$

$$=\begin{bmatrix}r(ab^*+ba^*+bb^*+aa^*)&0&0&0\\0&(1-r)aa^*&(1-r)ab^*&0\\0&(1-r)ba^*&(1-r)bb^*&0\\0&0&0&0\end{bmatrix}\quad -\ (A)$$

Now, consider $\Rightarrow$ $E_0^{dr}=\sqrt{1-r}\,I_4$, $E_1^{dr}=\sqrt{r}\left[|00\rangle\langle 01|+|00\rangle\langle 10|\right]$

$$\mathcal{E}(\rho)=\underline{E_0^{dr}\rho E_0^{dr\dagger}}+\underline{E_1^{dr}\rho E_1^{dr\dagger}}$$

$$=(1-r)\rho+r\left[\left[|00\rangle\langle 01|+|00\rangle\langle 10|\right]\rho\left[|01\rangle\langle 00|+|10\rangle\langle 0\right.\right.$$

$$=(1-r)\rho+r\Big[\left[|00\rangle\langle 01|+|00\rangle\langle 10|\right]aa^*|01\rangle\langle 01|+ab^*|01\rangle\langle 10|+ba^*|10\rangle\langle 01|+bb^*|10\rangle\langle 10|\left[|01\rangle\langle 00|+|10\rangle\langle 0\right.$$

$$=(1-r)\rho+r\Big[\big(aa^*|00\rangle\langle 01|+ab^*|00\rangle\langle 10|+ba^*|00\rangle\langle 01|+bb^*|00\rangle\langle 10|\big)\big(|01\rangle\langle 00|+|10\rangle\langle 00|\big)\Big]$$

$$=(1-r)\rho+r\big(aa^*|00\rangle\langle 00|+ab^*|00\rangle\langle 00|+ba^*|00\rangle\langle 00|+bb^*|00\rangle\langle 00|\big)$$

$$=\begin{bmatrix}r(aa^*+ab^*+ba^*+bb^*)&0&0&0\\0&(1-r)aa^*&(1-r)ab^*&0\\0&(1-r)ba^*&(1-r)bb^*&0\\0&0&0&0\end{bmatrix}-(B)$$

Hence from (A) + (B) proved