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UE18MA251 Linear Algebra and its Applications

Assignment

① Find the Equation of the Parabola $y = A + Bx + Cx^2$ that passes through 3 Points $(1,1)$, $(2,-1)$ and $(3,1)$ using gaussian elimination.

Sol:- $y = A + Bx + Cx^2$

Passes through

$(1,1)$ $(2,-1)$ $(3,1)$

∴ We have

$A + B + C = 1$

$A + 2B + 4C = -1$

$A + 3B + 9C = 1$

matrix $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \end{bmatrix}$ $x = \begin{bmatrix} A \\ B \\ C \end{bmatrix}$ $b = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$

Augmented matrix $[A\ b] = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & 4 & -1 \\ 1 & 3 & 9 & 1 \end{bmatrix}$

Using gaussian Elimination:

$R_2 \leftarrow R_2 - R_1$

$R_3 \leftarrow R_3 - R_1$

$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & -2 \\ 0 & 2 & 8 & 0 \end{bmatrix}$

$R_3 \leftarrow R_3 - 2R_2$

$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 3 & -2 \\ 0 & 0 & 2 & 4 \end{bmatrix}$

∴ $2C = 4 \Rightarrow \boxed{C = 2}$

$B + 3C = -2$

$B + 3(2) = -2$

$\boxed{B = -8}$

$A + B + C = 1$

$A - 8 + 2 = 1$ $\boxed{A = 7}$

$\therefore A = 7, \quad B = -8, \quad C = 2.$

$\therefore$ eq of parabola

$$y = 7 - 8x + 2x^2$$

2) Find the LU decomposition for the matrix

$$A \begin{bmatrix} 2 & 5 & 2 & -5 \\ 4 & 12 & 3 & -14 \\ -10 & -29 & -5 & 38 \\ 10 & 21 & 21 & -6 \end{bmatrix}$$

$R_2 \leftarrow R_2 - 2R_1$

multiplier $\cdot$ 2 (L_{21})

$$E_{21} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -2 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$R_3 \leftarrow R_3 - (-5)R_1$

multiplier $= -5$ (L_{31})

$$E_{31} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 5 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$R_4 \leftarrow R_4 - 5R_1$

Multiplier $= 5$ (L_{41})

$$E_{41} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -5 & 0 & 0 & 1 \end{bmatrix}$$

$$E_{41} E_{31} E_{21} A = \begin{bmatrix} 2 & 5 & 2 & -5 \\ 0 & 2 & -1 & -4 \\ 0 & -4 & 5 & 13 \\ 0 & -4 & 11 & 19 \end{bmatrix}$$

$R_3 \leftarrow R_3 - (2) R_2$

$\therefore$ Mult $= -2 (L_{32})$

$$E_{32} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$R_4 \leftarrow R_4 - (-2) R_2$

Mult $= -2 (L_{42})$

$$E_{42} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 2 & 0 & 1 \end{bmatrix}$$

$$E_{42} E_{32} E_{41} E_{31} E_{21} A = \begin{bmatrix} 2 & 5 & 2 & -5 \\ 0 & 2 & 1 & 4 \\ 0 & 0 & 3 & 5 \\ 0 & 0 & 9 & 11 \end{bmatrix}$$

$R_4 \leftarrow R_1 - 3R_3$

$\therefore$ Multi $= 3 (L_{43})$

$$E_{43} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & -3 & 1 \end{bmatrix}$$

$$E_{43} E_{42} E_{32} E_{41} E_{31} E_{21} A = \begin{bmatrix} 2 & 5 & 2 & -5 \\ 0 & 2 & -1 & -4 \\ 0 & 0 & 3 & 5 \\ 0 & 0 & 0 & -4 \end{bmatrix} = U$$

$LU = A$

$\Rightarrow L = E_{21}^{-1} E_{31}^{-1} E_{41}^{-1} E_{32}^{-1} E_{42}^{-1} E_{43}^{-1}.$

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$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ -5 & -2 & 1 & 0 \\ 5 & -2 & 3 & 1 \end{bmatrix}$$

$A = LU$

$$\Rightarrow \underset{A}{\begin{bmatrix} 2 & 5 & 2 & -5 \\ 4 & 12 & 3 & -14 \\ -10 & -29 & -5 & 38 \\ 11 & 21 & 21 & -6 \end{bmatrix}} = \underset{L}{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 \\ -5 & -2 & 1 & 0 \\ 5 & -2 & 3 & 1 \end{bmatrix}} \underset{U}{\begin{bmatrix} 2 & 5 & 2 & -5 \\ 0 & 2 & -1 & -4 \\ 0 & 0 & 3 & 5 \\ 0 & 0 & 0 & -4 \end{bmatrix}}$$

3) Let $T: R^3 \to R^3$ defined by

$$T(x,y,z) = (x+2y-z,\ y+z,\ x+y-2z)$$

(i) Find the matrix T relative to the standard basis of R^3

(ii) Find the basis for 4 fundamental subspaces of T

(iii) Find the eigen values and eigen vectors of T.

(iv) Decompose $T = QR$.

$T: R^3 \to R^3$

$T(x,y,z) = (x+2y-z,\ y+z,\ x+y-2z)$

(i) Stan bases of R^3 are $(1,0,0)$, $(0,1,0)$ & $(0,0,1)$ using transformation given

$T(1,0,0) = (1,0,1)$

$$= 1\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + 0\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + 1\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$T(0,1,0) = (2,1,1)$

$$= 2\begin{bmatrix}1\\0\\0\end{bmatrix} + 1\begin{bmatrix}0\\1\\0\end{bmatrix} + 1\begin{bmatrix}0\\0\\1\end{bmatrix}$$

$T(0,0,1) = (-1,1,-2)$

$$= -1\begin{bmatrix}1\\0\\0\end{bmatrix} + 1\begin{bmatrix}0\\1\\0\end{bmatrix} + (-2)\begin{bmatrix}0\\0\\1\end{bmatrix}$$

$$T = \begin{bmatrix}1 & 2 & -1\\0 & 1 & 1\\1 & 1 & -2\end{bmatrix}$$

(ii) for column space

$$T = \begin{bmatrix}1 & 2 & -1\\0 & 1 & 1\\1 & 1 & -2\end{bmatrix} \Rightarrow \text{Perf GE}$$

$R_3 \leftarrow R_3 - R_1$ $\quad$ $R_3 \rightarrow R_3 + R_2$

$$\begin{bmatrix}1 & 2 & -1\\0 & 1 & 1\\0 & -1 & -1\end{bmatrix} \quad \begin{bmatrix}1 & 2 & -1\\0 & 1 & 1\\0 & 0 & 0\end{bmatrix}$$

Basis for column space =

$$C(T) = \{(1,0,1), (2,1,1)\}$$

for null space:

solving for $Tx = 0$

$$\begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 1 \\ 1 & 1 & -2 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

reducing to row reduced form

$$\begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

$R_1 \leftarrow R_1 - 2R_2$

$$\begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

let $z = z$, some real val.

$y + z = 0$

$y = -z$

$x - 3z = 0 \Rightarrow x = 3z$

$$x \begin{bmatrix} x \\ y \\ z \end{bmatrix} = z \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix}$$

Basis for $N(T) = \{(3,-1,1)\}$

for row space

$$(T)^T = \begin{bmatrix} 1 & 0 & 1 \\ 2 & 1 & 1 \\ -1 & 1 & -2 \end{bmatrix}$$

$R_2 \leftarrow R_2 - 2R_1$

$R_3 \leftarrow R_3 + R_1$

$R_3 \leftarrow R_3 - R_2$

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$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 1 & -1 \end{bmatrix} \quad \Bigg| \quad \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

Basis for $C(T^T) = \{(1,2,-1)\ (0,1,1)\}$.

- for left null space

$(T)^T x = 0$

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$. In row red form.

let $z = Z$, some real val.

$y - z = 0 \Rightarrow y = z$

$x = z = 0 \Rightarrow x = -Z$

$$x = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = Z \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$$

Basis for $N(T^T) = \{(-1,1,1)\}$.

(iii) $T = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 1 \\ 1 & 1 & -2 \end{bmatrix}$

finding eigen val:

$|T - \lambda I| = 0$

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$$\left|\begin{bmatrix} 1-\lambda & 2 & -1 \\ 0 & 1-\lambda & 1 \\ 1 & 1 & -2-\lambda \end{bmatrix}\right| = 0$$

$= (1-\lambda)((1-\lambda)(-2-\lambda)-1) + 1(2+1-\lambda) = 0$

$= -\lambda^3 + 3\lambda = 0$

$\Rightarrow \lambda = 0, \sqrt{3}, -\sqrt{3}$

(eigen val)

finding Eigen vectors.

for $\lambda_1 = 0$, Solve $(T - \lambda_1 I)x = 0$.

$$\begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

$$x = z \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix}$$

$$\lambda_1 = \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix}$$

for $\lambda_2 = \sqrt{3}$

$(T = \lambda_2)x = 0$

$$\begin{bmatrix} 1-\sqrt{3} & 2 & -1 \\ 0 & 1-\sqrt{3} & 1 \\ 1 & 1 & -2-\sqrt{3} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

$R_3 \leftarrow R_3 - \frac{1}{1-\sqrt{3}} R_1$

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$$\begin{bmatrix} 1-\sqrt{3} & 2 & 1 \\ 0 & 1-\sqrt{3} & 1 \\ 0 & 1-\frac{2}{1-\sqrt{3}} & \frac{-2\sqrt{3}-1}{1-\sqrt{3}} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

$$R_3 \leftarrow R_3 + \frac{\sqrt{3}+1}{(1-\sqrt{3})^2} R_2$$

$$\begin{bmatrix} 1-\sqrt{3} & 2 & 1 \\ 0 & 1-\sqrt{3} & 1 \\ 0 & 0 & \underbrace{-2-\sqrt{3}+\frac{1}{1-\sqrt{3}}+\frac{\sqrt{3}+1}{(1-\sqrt{3})^2}}_{0} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

z = some real val

$(1-\sqrt{3})y + z = 0$

$\Rightarrow y = \frac{-1}{1-\sqrt{3}} z$

$(1-\sqrt{3})x + 2\left(\frac{-1}{1-\sqrt{3}} z\right) - z = 0$

$\Rightarrow x = \frac{z(-\sqrt{3})}{1-\sqrt{3}}$

$$\Rightarrow x_2 = \begin{bmatrix} -\sqrt{3}/(1-\sqrt{3}) \\ -1/(1-\sqrt{3}) \\ 1 \end{bmatrix}$$

~~$R_3 \leftarrow R_3 - \frac{1}{1+\sqrt{3}} R_1$.~~

for $\lambda_3 = -\sqrt{3}$

$\{(T - \lambda_3 I)x = 0\}$.

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$$\begin{bmatrix} 1+\sqrt{3} & 2 & -1 \\ 0 & 1+\sqrt{3} & 1 \\ 1 & 1 & -2+\sqrt{3} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0.$$

$$R_3 \leftarrow R_3 - \frac{1}{1+\sqrt{3}} R_1$$

$$\begin{bmatrix} 1+\sqrt{3} & 2 & -1 \\ 0 & 1+\sqrt{3} & 1 \\ 0 & 1-\frac{2}{1+\sqrt{3}} & -2+\sqrt{3}+\frac{1}{1+\sqrt{3}} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

$$R_3 \leftarrow R_3 - \frac{(\sqrt{3}-1)}{(1+\sqrt{3})^2} R_2$$

$$\begin{bmatrix} 1+\sqrt{3} & 2 & -1 \\ 0 & 1+\sqrt{3} & 1 \\ 0 & 0 & -2+\sqrt{3}+\frac{1}{1+\sqrt{3}} - \frac{(\sqrt{3}-1)}{(1+\sqrt{3})^2} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

let $z = z$, some real val.

$$(1+\sqrt{3})y + z = 0$$

$$y = \frac{-1}{1+\sqrt{3}} z$$

$$(1-\sqrt{3})x + 2\left(\frac{-1}{1+\sqrt{3}} z\right) - z = 0$$

$$x = \frac{\sqrt{3}}{1+\sqrt{3}} z$$

$$x = z \begin{bmatrix} \sqrt{3}/(1+\sqrt{3}) \\ -1/(1+\sqrt{3}) \\ 1 \end{bmatrix}$$

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$$x_3 = \begin{bmatrix} \sqrt{3}/(1+\sqrt{3}) \\ -1/(1+\sqrt{3}) \\ 1 \end{bmatrix}$$

$\therefore$ Eigen Val are

$$x_1 \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix}, \quad x_2 = \begin{bmatrix} -\sqrt{3}/(1-\sqrt{3}) \\ -1/(1-\sqrt{3}) \\ 1 \end{bmatrix}$$

$$x_3 = \begin{bmatrix} \sqrt{3}/(1+\sqrt{3}) \\ -1/(1+\sqrt{3}) \\ 1 \end{bmatrix}$$

We get the eigen vectors by rationalising the denominators

$$x_1 = \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix}, \quad x_2 \begin{bmatrix} \frac{\sqrt{3}+3}{2} \\ \frac{\sqrt{3}+1}{2} \\ 1 \end{bmatrix}, \quad x_3 = \begin{bmatrix} \frac{3-\sqrt{3}}{2} \\ \frac{1-\sqrt{3}}{2} \\ 1 \end{bmatrix}$$

$\lambda = 0$ $\quad \lambda_2 = \sqrt{3}$ $\quad \lambda_3 = -\sqrt{3}$.

(ii) $T = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 1 \\ 1 & 1 & -2 \end{bmatrix}$ $\quad a = (1,0,1)$, $b = (2,1,1)$, $c = (-1,1,-2)$

The Gram-Schmidt process.

$$q_1 = \frac{a}{\|a\|} = \frac{1}{\sqrt{2}}(1,0,1)$$

$$\|a\| = \sqrt{1^2+0^2+1^2} = \sqrt{2}$$

$q_2 = \frac{B}{\|B\|}$, Where $B = b - (q_1^T b) q_1$.

$q_1^T b = \frac{1}{\sqrt{2}} [1\ 0\ 1] \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} \cdot \frac{3}{\sqrt{2}}$

$B = b - (q_1^T b) q_1 = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} - 3/\sqrt{2} \begin{bmatrix} 1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix}$

$= \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 1/2 \\ 1 \\ -1/2 \end{bmatrix}$

$q_2 = \frac{B}{\|B\|} = \frac{1}{\sqrt{6}} (1, 2, -1)$

$q_3 = \frac{C}{\|C\|}$, Whae $C = c - (q_2^T c) q_2 - (q_1^T c) q_1$

$q_1^T c = \frac{1}{\sqrt{2}} [1\ 0\ 1] \begin{bmatrix} -1 \\ 1 \\ -2 \end{bmatrix} = \frac{-3}{\sqrt{2}}$

$q_2^T c = \frac{1}{\sqrt{6}} [1\ 2\ -1] \begin{bmatrix} -1 \\ 1 \\ -2 \end{bmatrix} = \frac{3}{\sqrt{6}}$

$C = (-1, 1, -2) - \frac{3}{\sqrt{6}} \times \frac{1}{\sqrt{6}} (1, 2, -1)$

$- \left(\frac{-3}{\sqrt{2}}\right)\left(\frac{1}{\sqrt{2}}\right)(1, 0, 1)$

$C = (-1, 1, 2) + \left(\frac{-1}{\sqrt{2}}, -1, \frac{1}{2}\right) + \left(\frac{3}{2}, 0, \frac{3}{\sqrt{2}}\right) = (0, 0, 0)$

$q_3 = (0, 0, 0)$

QR - factorisation

$$R = \begin{bmatrix} q_1^T a & q_1^T b & q_1^T c \\ 0 & q_2^T b & q_2^T c \\ 0 & 0 & q_3^T C \end{bmatrix}$$

$$q_1^T a = \frac{1}{\sqrt{2}}[1\ 0\ 1]\begin{bmatrix}1\\0\\1\end{bmatrix} = \sqrt{2}$$

$$q_1^T b = \frac{1}{\sqrt{6}}[1\ 2\ -1]\begin{bmatrix}2\\1\\1\end{bmatrix} = \frac{3}{\sqrt{6}}$$

$$q_3^T C = 0$$

$$R = \frac{1}{\sqrt{6}}\begin{bmatrix} \sqrt{12} & 3\sqrt{3} & -3\sqrt{3} \\ 0 & 3 & 3 \\ 0 & 0 & 0 \end{bmatrix}$$

$$q_1^T b = \frac{3}{\sqrt{2}} = \frac{3\sqrt{3}}{\sqrt{6}}$$

$$q_3^T c = \frac{-3}{\sqrt{2}} = \frac{-3\sqrt{3}}{\sqrt{6}}$$

$$q_2^T C = \frac{3}{\sqrt{6}}.$$

Q is $[q_1\ q_2\ q_3]$

$\Rightarrow T = QR.$

$$= \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{6} & 0 \\ 0 & 2/\sqrt{6} & 0 \\ 1/\sqrt{2} & -1/\sqrt{6} & 0 \end{bmatrix}\begin{bmatrix} \sqrt{2} & 3/\sqrt{2} & -3\sqrt{2} \\ 0 & 3/\sqrt{6} & 3/\sqrt{6} \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 2 & -1 \\ 0 & 1 & 1 \\ 1 & 1 & -2 \end{bmatrix}$$

4) $y = c + dx$

$\therefore$ According to given

4) Fit a best straight line $y = c + dx$ for the following data, using ~~least~~ least square principles.

x	-4	1	2	3
y	4	6	10	8

$y = c + dx$.

$\therefore$ According to given info

$c - 4d = 4$

$c + d = 6$

$c + 2d = 10$

$c + 3d = 8$

$$Ax = b \Rightarrow \begin{bmatrix} 1 & -4 \\ 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} 4 \\ 6 \\ 10 \\ 8 \end{bmatrix}$$

$$A^T A \hat{x} = A^T b \Rightarrow \hat{x} = (A^T A)^{-1} A^T b$$

$$A^T A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ -4 & 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1 & -4 \\ 1 & 1 \\ 1 & 2 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 4 & 2 \\ 2 & 30 \end{bmatrix}$$

$$(A^T A)^{-1} = \frac{1}{116} \begin{bmatrix} 30 & -2 \\ -2 & 4 \end{bmatrix}$$

$$(A^T A)^{-1} A^T = \frac{1}{116} \begin{bmatrix} 30 & -2 \\ -2 & 4 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 & 1 \\ -4 & 1 & 2 & 3 \end{bmatrix}$$

$$= \frac{1}{116}\begin{bmatrix} 38 & 28 & 26 & -24 \\ -18 & 2 & 6 & 10 \end{bmatrix}$$

$\therefore$

$$\hat{x} = \begin{bmatrix} c \\ d \end{bmatrix} = (A^TA)^{-1}A^Tb$$

$$= \frac{1}{116}\begin{bmatrix} 38 & 28 & 26 & 24 \\ -18 & 2 & 6 & 10 \end{bmatrix}\begin{bmatrix} 4 \\ 6 \\ 10 \\ 8 \end{bmatrix}$$

$$\frac{1}{116}\begin{bmatrix} 772 \\ 80 \end{bmatrix} = \begin{bmatrix} 193/29 \\ 20/29 \end{bmatrix}$$

$\therefore$ The eqn is

$$y = \frac{193}{29} + \frac{20}{29}x$$

(5) Find the Projection matrices P & Q onto the Plane $x_1 + x_2 + 3x_3 + 4x_4 = 0$ and its orthogonal complement respectively.

Soln: $x_1 + x_2 + 3x_3 + 4x_4 = 0$

$$\begin{bmatrix} 1 & 1 & 3 & 4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = 0$$

$x_1 = -x_2 - 3x_3 - 4x_4$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = x_2 \begin{bmatrix} -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -3 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} -4 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$A = \begin{bmatrix} -1 & -3 & -4 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Projection $P_J = A(A^TA)^{-1}A^T$ & $Q = I - P$

$\downarrow$

Id matrix.

$$A^TA = \begin{bmatrix} -1 & 1 & 0 & 0 \\ -3 & 0 & 1 & 0 \\ -4 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & -3 & -4 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 3 & 4 \\ 3 & 10 & 12 \\ 4 & 12 & 17 \end{bmatrix}$$

$$(A^TA)^{-1} = \begin{bmatrix} 26/27 & -1/9 & -4/27 \\ -1/9 & 2/3 & -4/9 \\ -4/27 & -4/9 & 11/29 \end{bmatrix}$$

$$P = A(A^TA)^{-1}A^T = \begin{bmatrix} 26/27 & -1/27 & -1/9 & -4/27 \\ -1/27 & 26/27 & -1/9 & -4/27 \\ -1/9 & -3/27 & 6/9 & -12/27 \\ -4/27 & -4/27 & -4/9 & 11/27 \end{bmatrix}$$

$$= \frac{1}{29}\begin{bmatrix} 26 & -1 & -3 & -4 \\ -1 & 26 & -3 & -4 \\ -3 & -3 & 18 & -12 \\ -4 & -4 & -12 & 11 \end{bmatrix}$$

$$Q = I - P = \frac{1}{27}\begin{bmatrix} 1 & 1 & 3 & 4 \\ 1 & 1 & 3 & 4 \\ -3 & 3 & 9 & 12 \\ 4 & 4 & 12 & 16 \end{bmatrix} //$$

(6) $A =$

for which range of number 'a' the matrix A is positive definite?

$$A = \begin{bmatrix} a & 2 & 2 \\ 2 & a & 2 \\ 2 & 2 & a \end{bmatrix}$$

Which 3x3 matrix (symmetric) produces these function $f = x^TAx$?

When $f = 2(x_1^2 + x_2^2 + x_3^2 - x_1x_2 - x_2x_3)$

Sol

$$A = \begin{bmatrix} a & 2 & 2 \\ 2 & a & 2 \\ 2 & 2 & a \end{bmatrix}$$

- Will be +ve def if :
- Pivots are +ve

red to Echelon form

$R_2 \leftarrow R_2 - 2/a\, R_1$

$R_3 \leftarrow R_3 - 2/a\, R_1$

$$\begin{bmatrix} a & 2 & 2 \\ 0 & a-4/a & 2-4/a \\ 0 & 2-4/a & a-4/a \end{bmatrix}$$

$R_3 \leftarrow R_3 - \left(\frac{2a-4}{a^2-4}\right) R_2$

$$\begin{bmatrix} a & 2 & 2 \\ 0 & \frac{a^2-4}{a} & \frac{2a-4}{a} \\ 0 & 0 & \frac{a^2-4}{a} - \frac{(2a-4)(2a-4)}{a(a^2-4)} \end{bmatrix}$$

- $a > 0$
- $\frac{a^2-4}{a} > 0$

$\Rightarrow a^2 - 4 > 0$

$\Rightarrow (a-2)(a+2) > 0$

$\Rightarrow (-\infty, -2) \cup (2, \infty)$

$\times$

as $a > 0$

- $\frac{a^2-4}{a} - \frac{(2a-4)(2a-4)}{a(a^2-4)}$

$\Rightarrow (a^2-4)^2 - (2a-4)^2 > 0$

$\Rightarrow (a^2-4-2a+4)(a^2-4+2a-4) > 0$

$\Rightarrow (a^2-2a)(a^2+2a-8) > 0$

$\therefore (a^2-2a) > 0$ and $(a^2+2a-8) > 0$

$a(a-2) > 0 \quad a^2+4a-2a-8 > 0$

$a(a+4) - 2(a+4) > 0$

$(a+4)(a-2) > 0$

$\therefore$ We get $a > 0 \cup a > 2 \cup a > -4 \cup a > 2$

Range of a is $(2, \infty)$

$f = x^T A x$

$f = 2x_1^2 + 2x_2^2 + 2x_3^2 - 2x_1x_2 - 2x_2x_3$

Let $x = (x_1, x_2, x_3)$

A = req 3×3 matrix

$$= \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{13} & a_{23} & a_{33} \end{bmatrix}$$

$$x^T A x = [x\ y\ z] \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{13} & a_{23} & a_{33} \end{bmatrix}$$

$$x^T A x = [x\ y\ z] \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{13} & a_{23} & a_{33} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

$$= a_{11}x^2 + a_{22}y^2 + a_{33}z^2 + 2a_{12}xy + 2a_{13}xz + 2a_{23}yz$$

Comparing with i)

$$(x = x_1, y = x_2, z = x_3)$$

$\Rightarrow a_{11} = 2 \quad a_{22} = 2 \quad a_{33} = 2$

$a_{12} = -1 \quad a_{13} = 0 \quad a_{23} = -1$

Req 3×3 symm - matrix $= \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix}$

(7) Find the SVD of A, $U\Sigma V^T$ Whor

$$A = \begin{bmatrix} -3 & 1 \\ 6 & -2 \\ 6 & -2 \end{bmatrix}$$

(7) $A = \begin{bmatrix} -3 & 1 \\ 6 & -2 \\ 6 & -2 \end{bmatrix}_{3\times2}$ (Tall matrix of order 3×2
$A_{3\times2} = U_{3\times3}\ \Sigma_{3\times2}\ V^T_{2\times2}$
SVD of A?)

$$A^TA = \begin{bmatrix} -3 & 6 & 6 \\ 1 & -2 & -2 \end{bmatrix} \begin{bmatrix} -3 & 1 \\ 6 & -2 \\ 6 & -2 \end{bmatrix} = \begin{bmatrix} 81 & -27 \\ -27 & 9 \end{bmatrix}_{2\times2}$$

To obtain eigen val, solve for $|A-\lambda I| = 0$.

$$\begin{vmatrix} 81-\lambda & -27 \\ -27 & 9-\lambda \end{vmatrix} = 0 \Rightarrow (81-\lambda)(9-\lambda) - (-27)(-27) = 0$$

$\Rightarrow (729 - 81\lambda - 9\lambda + \lambda^2 - 729) = 0$

$\Rightarrow -90\lambda - \lambda^2 = 0$

$\Rightarrow \lambda(\lambda - 90) = 0 \Rightarrow \lambda = 90, 0.$

$\lambda_1 = 90,\ \lambda_2 = 0$

To obtain eigen vectors of A^TA

$\rightarrow$ from $\lambda_1 = 90$

$$\begin{bmatrix} -9 & -27 \\ -27 & -81 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 0$$

$R_2 \leftarrow R_2 - 3R_1$

$$\begin{bmatrix} -9 & -27 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = 0$$

$x = -3y$

$x = \begin{bmatrix} -3 \\ 1 \end{bmatrix}$

$\sqrt{(-3)^2+(1)^2} = \sqrt{10}$

→ from $\lambda_2 = 0$

$$\begin{bmatrix} 81 & -27 \\ -27 & 9 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = 0$$

$R_2 \leftarrow R_2 + 1/3 R_1$

$$\begin{bmatrix} 81 & -27 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = 0$$

$81x - 27y = 0$

$x = 1/3\, y$

$$x_2 = \begin{bmatrix} 1/3 \\ 1 \end{bmatrix} \qquad \sqrt{(1/3)^2+(1)^2} = \sqrt{10}/3$$

Normalsing x_1 & x_2.

$$V_1 = \begin{bmatrix} -3/\sqrt{10} \\ 1/\sqrt{10} \end{bmatrix} \qquad V_2 = \begin{bmatrix} 1/\sqrt{10} \\ 3/\sqrt{10} \end{bmatrix}$$

$$\text{matrix } V = \begin{bmatrix} -3/\sqrt{10} & 1/\sqrt{10} \\ 1/\sqrt{10} & 3/\sqrt{10} \end{bmatrix}$$

Singular val of A are $\sigma_1 = \sqrt{90}$ & $\sigma_2 = 0$

Eigen val of AA^T (order 3×3) are 90, 0, 0

$$u_i = \frac{AV_i}{\sigma_1} = \frac{\begin{bmatrix} -3 & 1 \\ 6 & -2 \\ 6 & -2 \end{bmatrix}\begin{bmatrix} -3/\sqrt{10} \\ 1/\sqrt{10} \end{bmatrix}}{\sqrt{10}} = \begin{bmatrix} 1/\sqrt{3} \\ -2/3 \\ -2/3 \end{bmatrix}$$

The other colums of U are found by finding vector soln to $u_1^T x = 0$ as u_2 & u_3 are orthogonal to u_1

$$u_1^T x = 0 \Rightarrow \frac{x_1}{3} - \frac{2x_2}{3} - \frac{2x_3}{3} = 0$$

$$\Rightarrow x_1 - 2x_2 - 2x_3 = 0$$

$$\Rightarrow \begin{bmatrix} 1 & -2 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$$\Rightarrow x_1 = 2x_2 + 2x_3$$

Bases $\Rightarrow$ $x_2 \underbrace{\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}}_{a_2} + x_3 \underbrace{\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}}_{a_3}$

Applying Gram-schmidt process to a_2, a_3 to obtain u_2 & u_3 orthogonal vectors

The vectors $(2, 1, 0)$ & $(2, 0, 1)$

$$u_2 = \frac{a_2}{\|a_2\|} = \begin{bmatrix} \frac{2}{\sqrt{3}}, & \frac{1}{\sqrt{5}}, & 0 \end{bmatrix}$$

$$e = a_2 - (u_2^T a_2) u_2$$

$$= \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} - \left(\begin{bmatrix} 2/\sqrt{5} & 1/\sqrt{5} & 0 \end{bmatrix} \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} \right) \begin{bmatrix} 2/\sqrt{5} \\ 1/\sqrt{5} \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} - \begin{bmatrix} 8/5 \\ 4/5 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 2/5 \\ 0 \\ 1 \end{bmatrix} - \begin{bmatrix} 4/5 \\ 4/5 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} 2/5 \\ -4/5 \\ 1 \end{bmatrix} \quad \|c\| = \sqrt{\frac{4}{25} + \frac{16}{25} + 1} = \sqrt{\frac{45}{5}} = \frac{3}{\sqrt{5}}$$

$$u_3 = \frac{c}{\|c\|} = \left(\frac{2}{3\sqrt{5}}, \frac{-4}{3\sqrt{5}}, \frac{\sqrt{5}}{3}\right) (or)$$

$$\left(\frac{2}{\sqrt{45}}, \frac{-4}{\sqrt{45}}, \frac{5}{\sqrt{45}}\right)$$

$$\Sigma = \begin{bmatrix} 90 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

$$A = U\Sigma V^T$$

$$\begin{bmatrix} -3 & 1 \\ 6 & -2 \\ 6 & -2 \end{bmatrix} = \begin{bmatrix} 1/3 & 2/\sqrt{3} & 2/\sqrt{45} \\ -2/3 & 1/\sqrt{3} & -4/\sqrt{45} \\ -2/\sqrt{3} & 0 & 5/\sqrt{45} \end{bmatrix} \begin{bmatrix} \sqrt{90} & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} -3/\sqrt{10} & 1/\sqrt{10} \\ 1/\sqrt{10} & 3/\sqrt{10} \end{bmatrix}$$

$$A = U \Sigma V^T$$