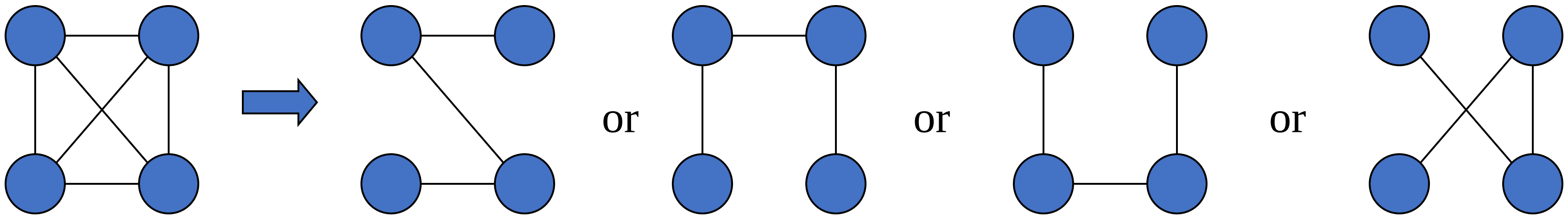
Spanning Trees

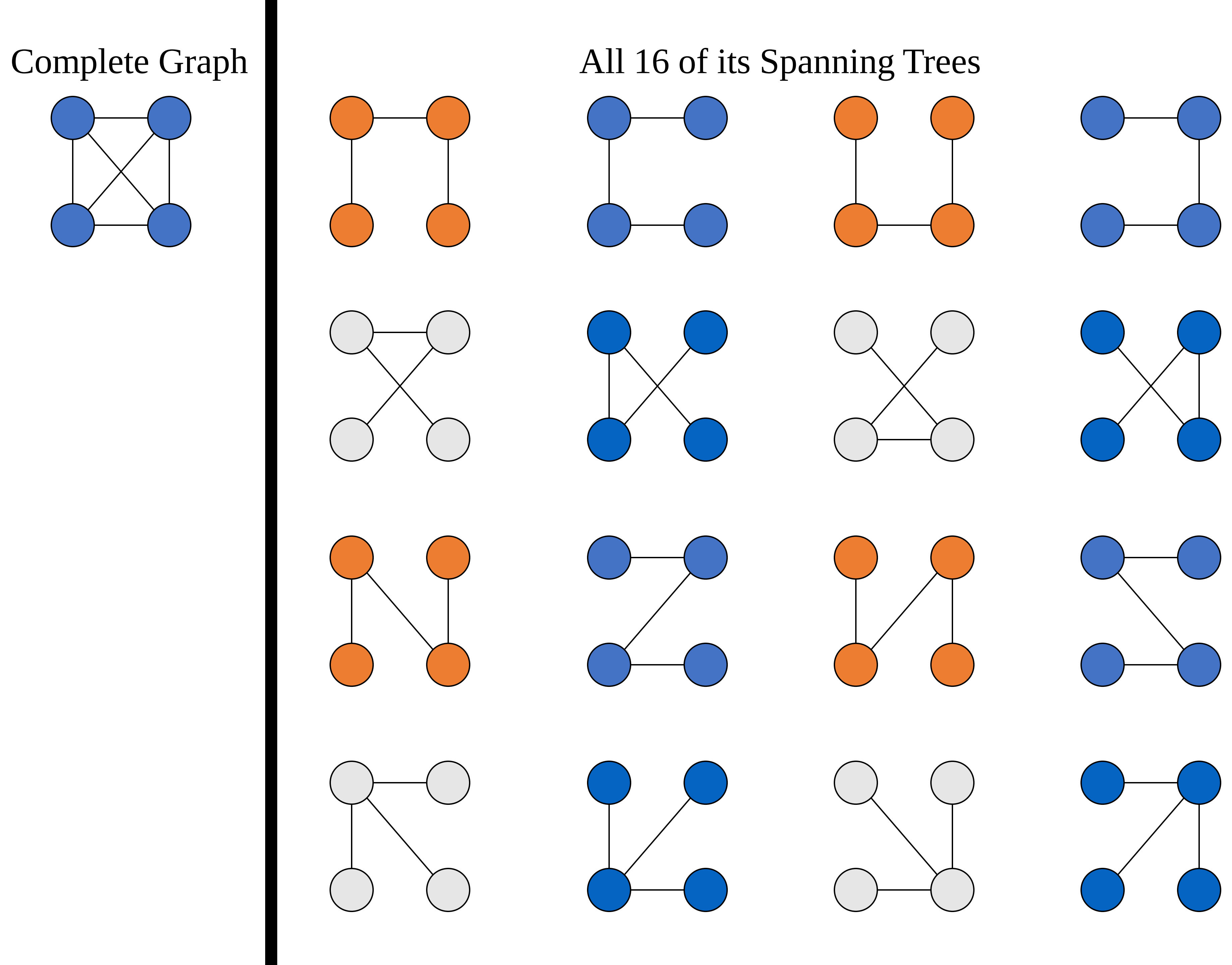
A spanning tree of a graph is just a subgraph that contains all the vertices and is a tree.

A graph may have many spanning trees.

Graph A

Some Spanning Trees from Graph A

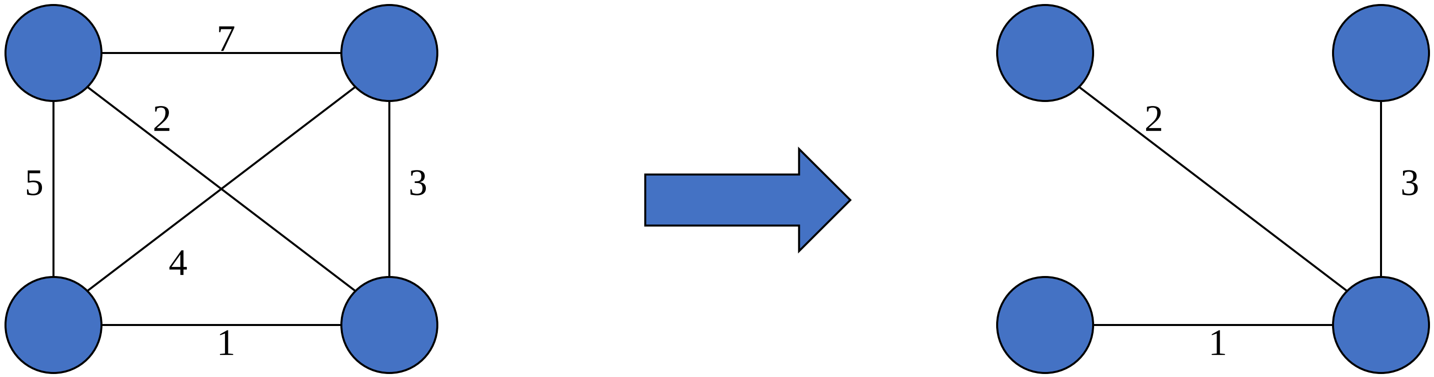



Complete Graph
All 16 of its Spanning Trees

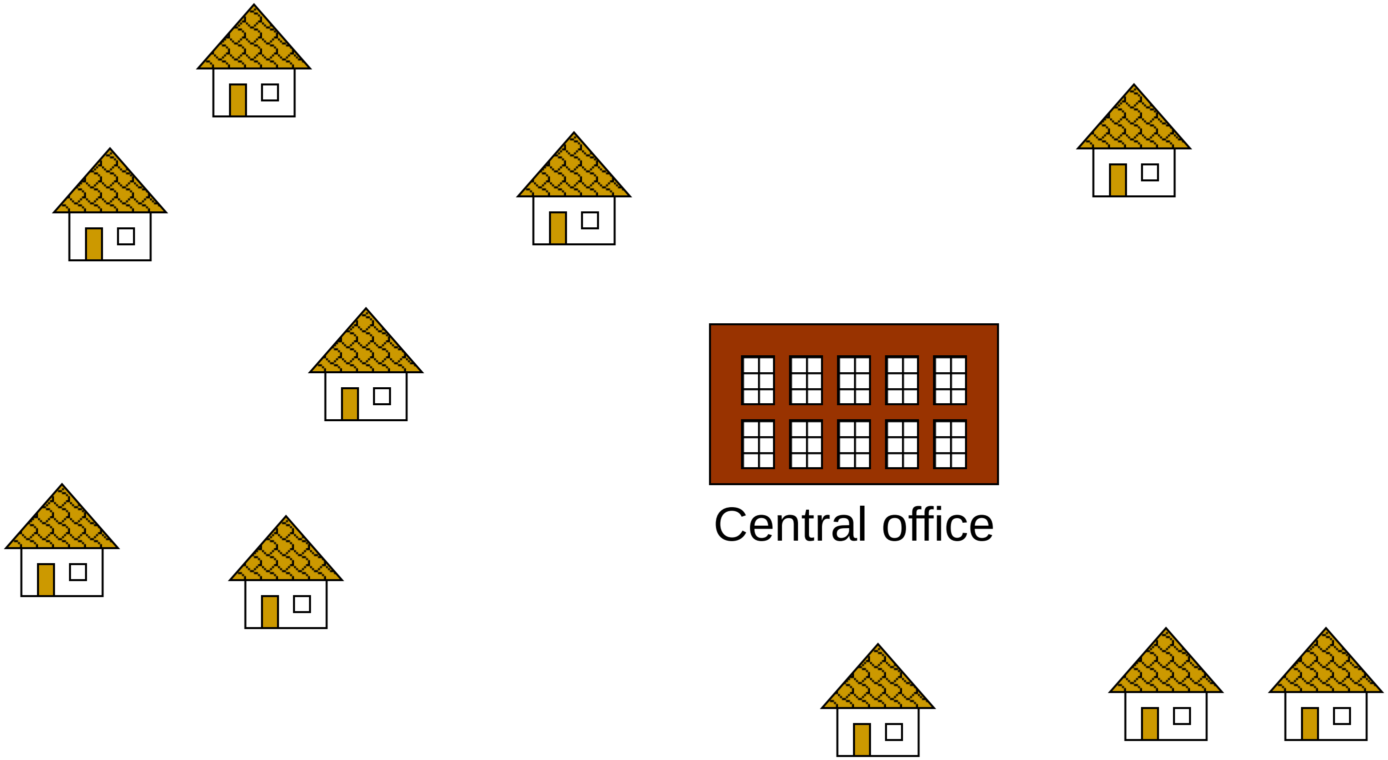
Minimum Spanning Trees

The Minimum Spanning Tree for a given graph is the Spanning Tree of minimum cost for that graph.

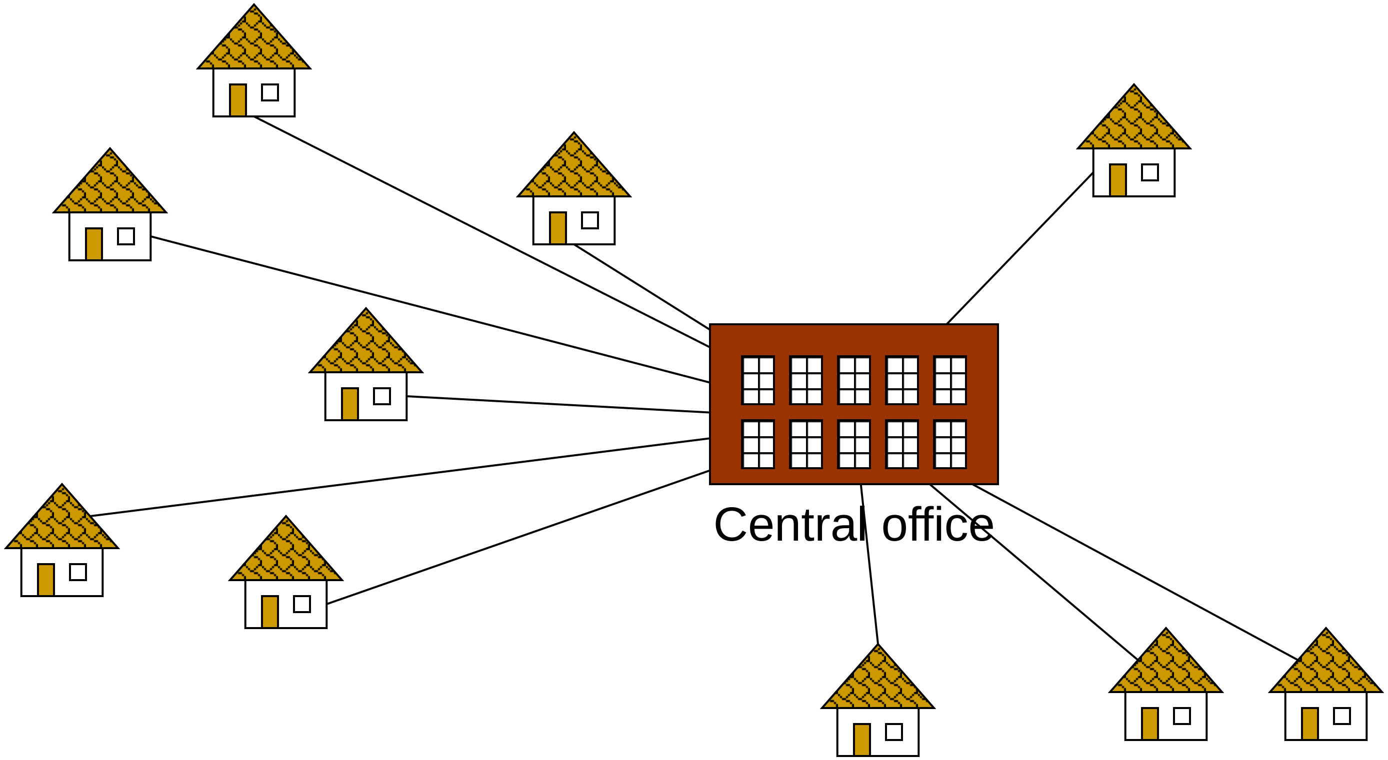
Complete Graph | Minimum Spanning Tree



Problem: Laying Telephone Wire

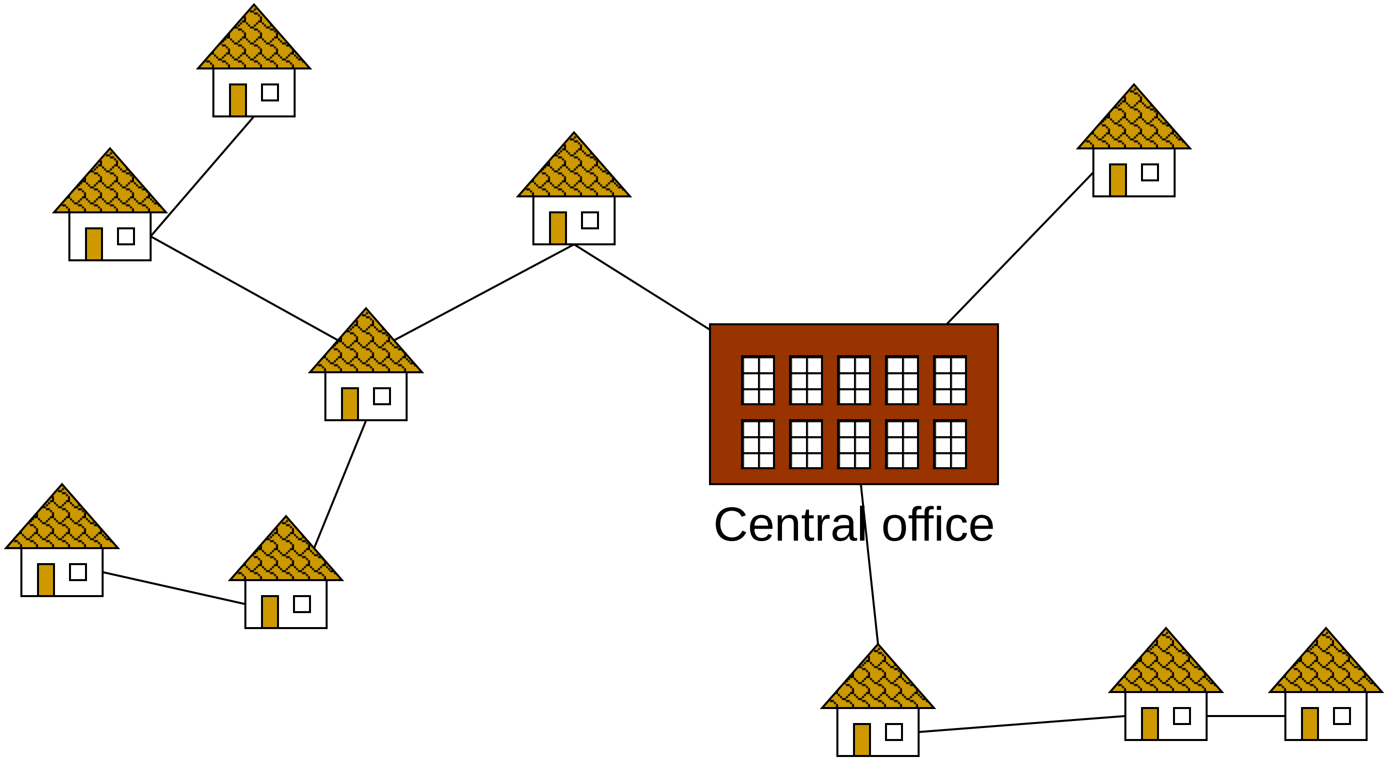


Wiring: Naïve Approach



Expensive!

Wiring: Better Approach



Minimize the total length of wire connecting the customers

Algorithms for Obtaining the Minimum Spanning Tree

- Kruskal's Algorithm
- Prim's Algorithm

Minimum Spanning Tree

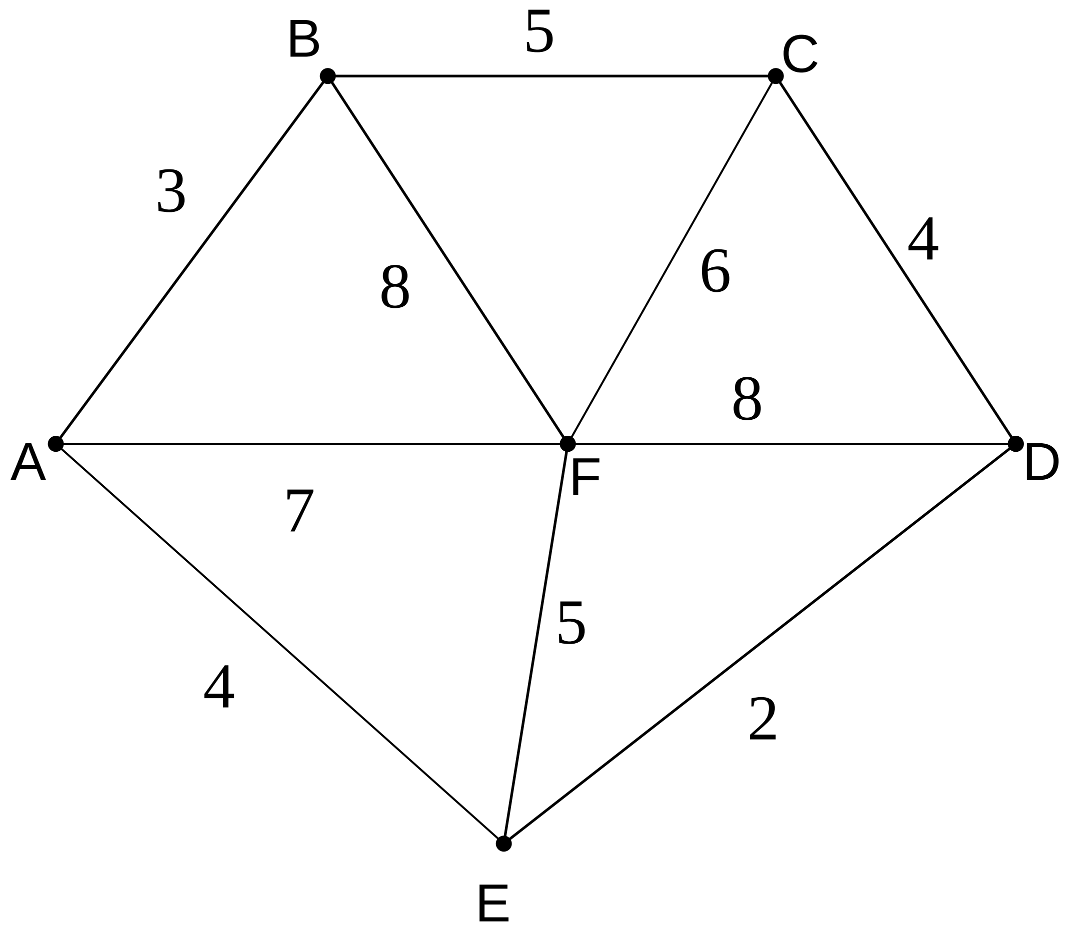
Kruskal's algorithm

1. Select the shortest edge in a network
2. Select the next shortest edge which does not create a cycle
3. Repeat step 2 until all vertices have been connected

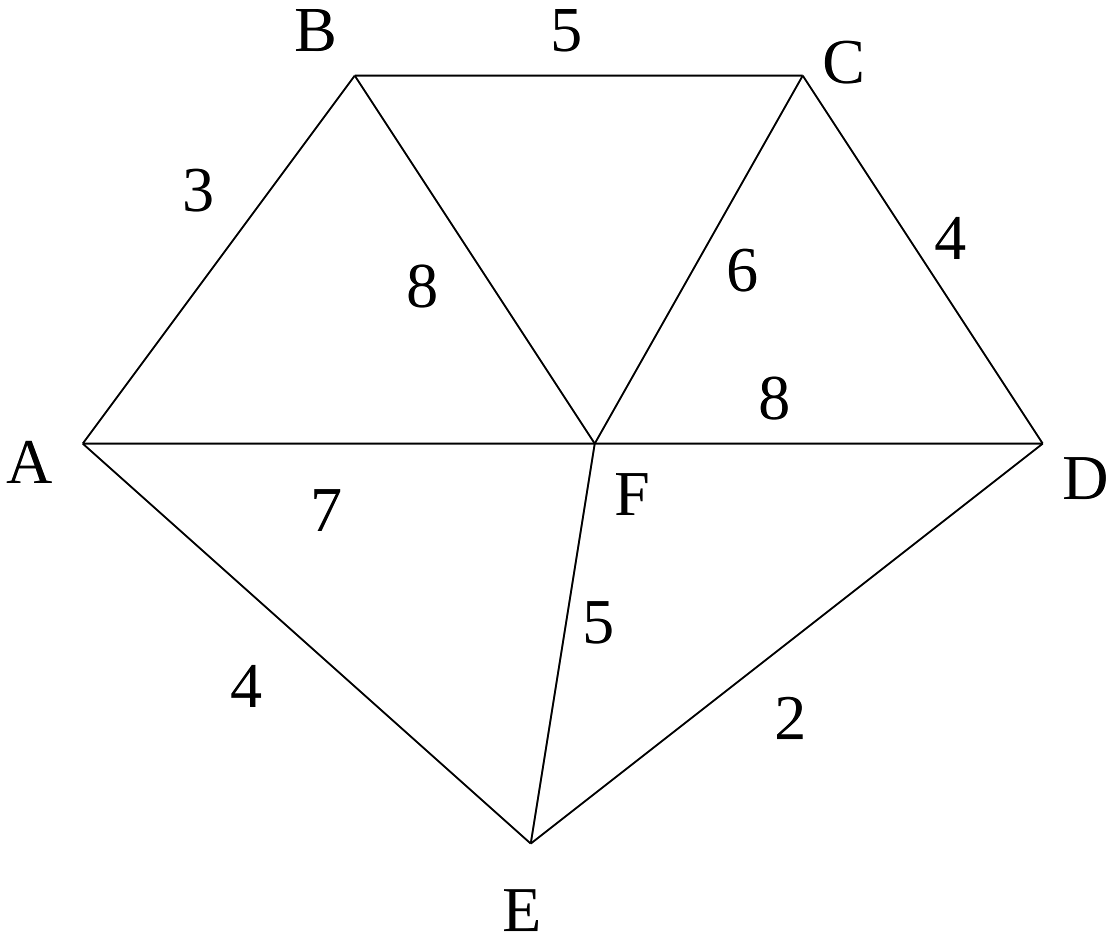
Prim's algorithm

1. Select any vertex
2. Select the shortest edge connected to that vertex
3. Select the shortest edge connected to any vertex already connected
4. Repeat step 3 until all vertices have been connected

Example



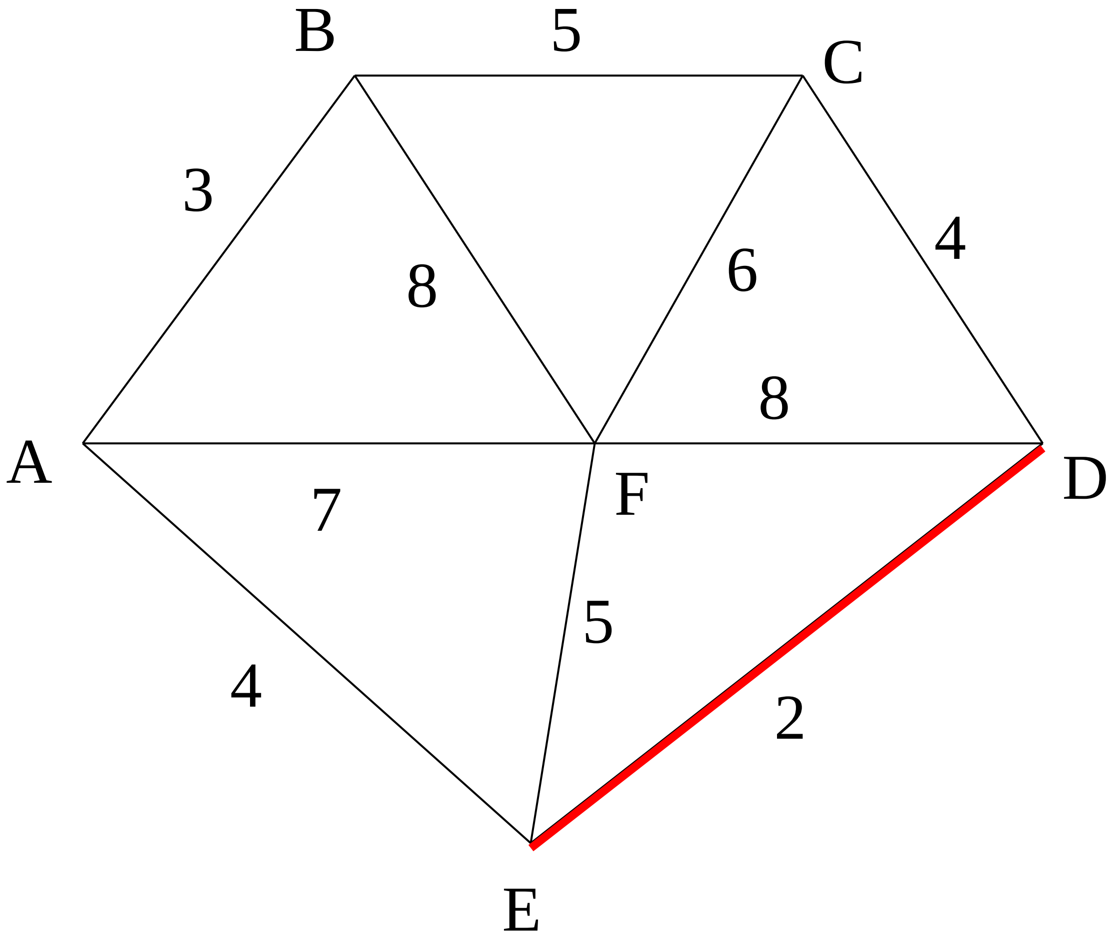
Kruskal's Algorithm



List the edges in order of size:

ED 2
AB 3
AE 4
CD 4
BC 5
EF 5
CF 6
AF 7
BF 8
CF 8

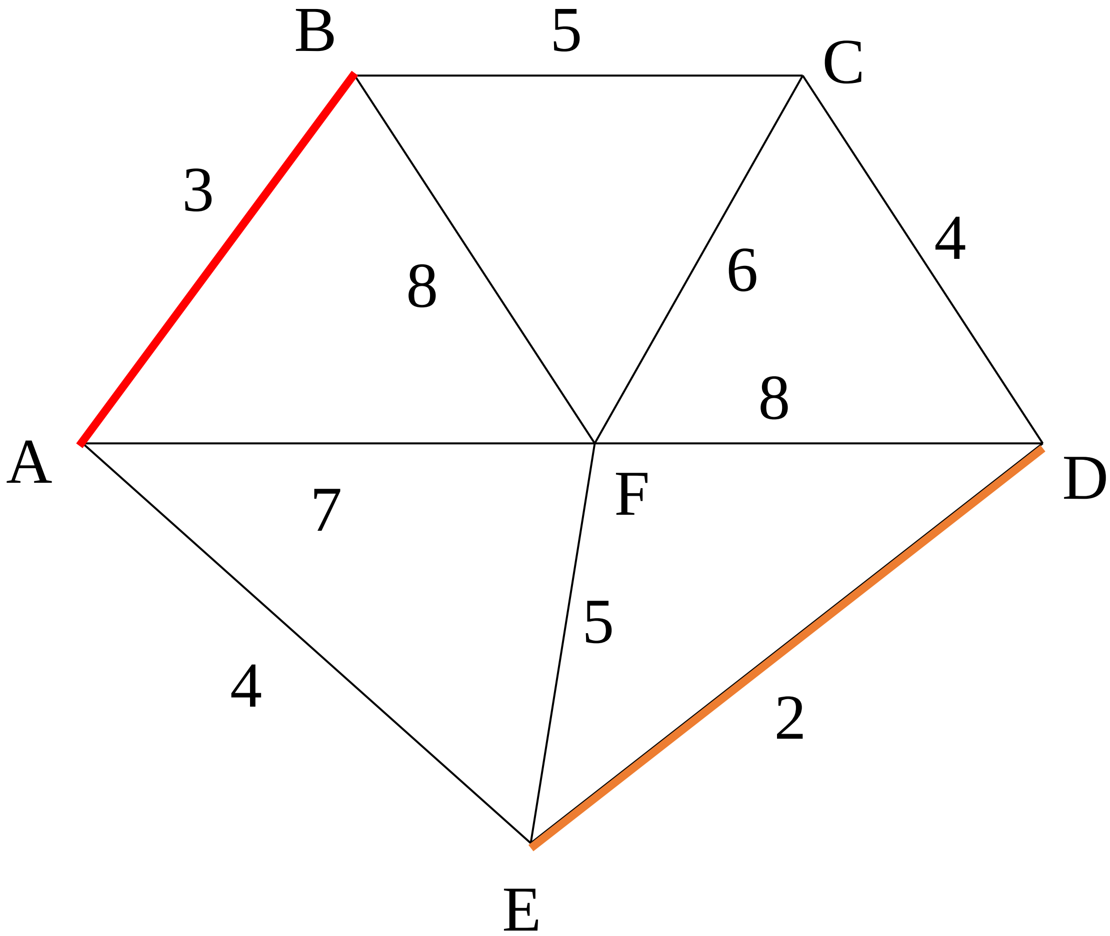
Kruskal’s Algorithm



Select the shortest edge in the network

ED 2

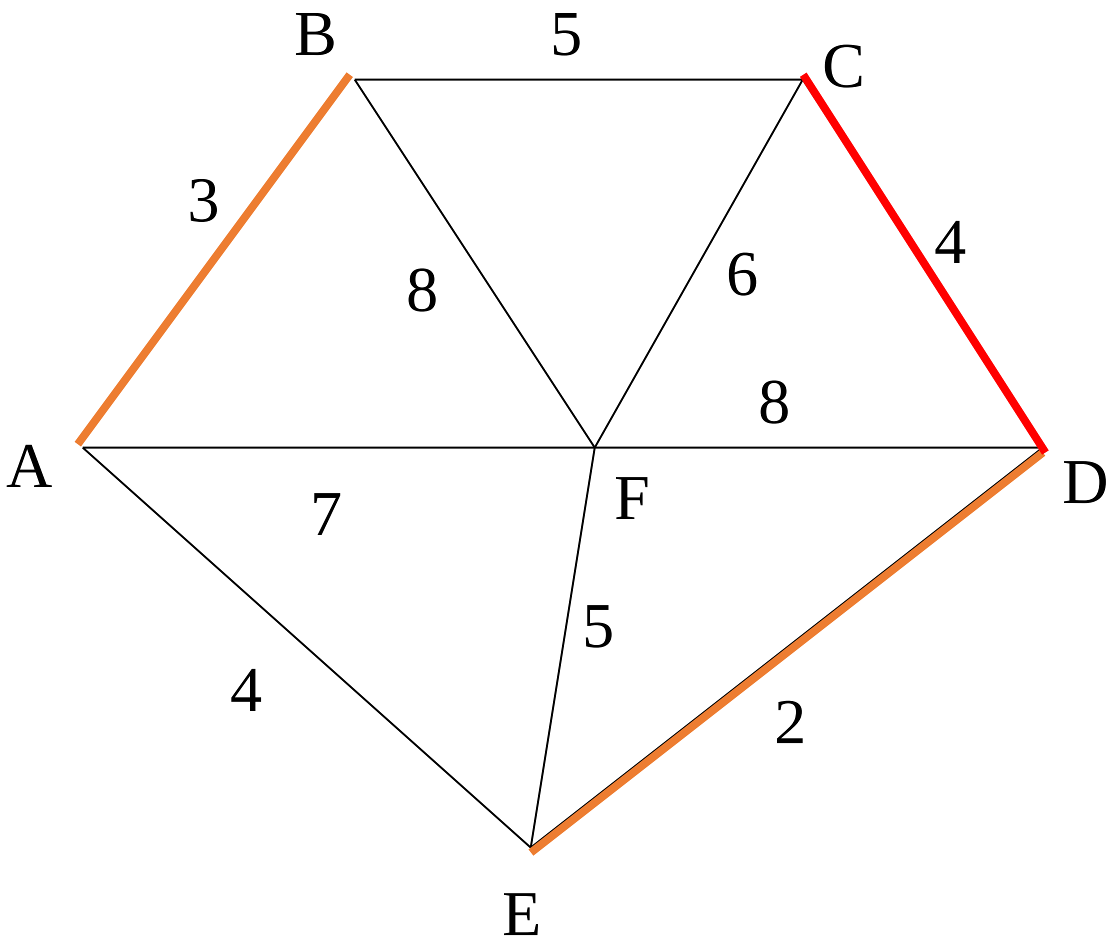
Kruskal's Algorithm



Select the next shortest edge which does not create a cycle

ED 2
AB 3

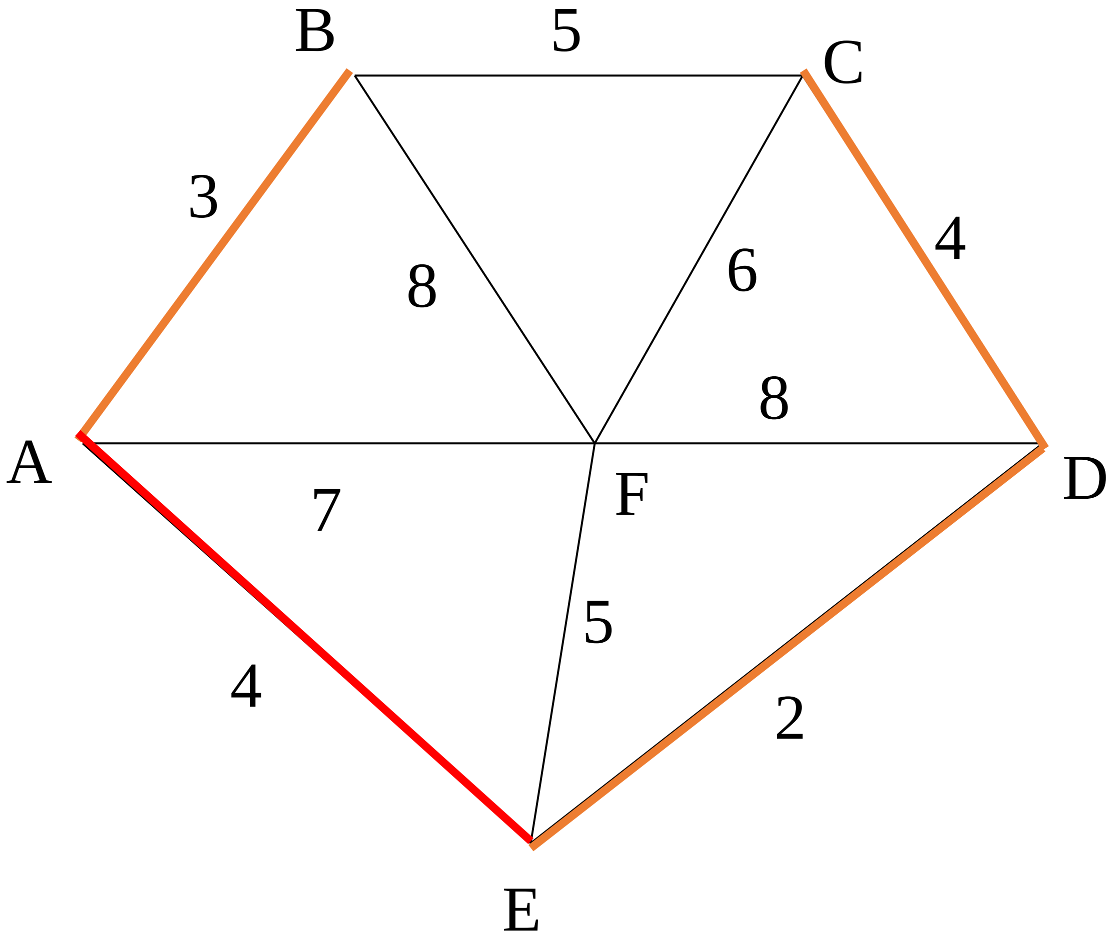
Kruskal’s Algorithm



Select the next shortest edge which does not create a cycle

ED 2
AB 3
CD 4 (or AE 4)

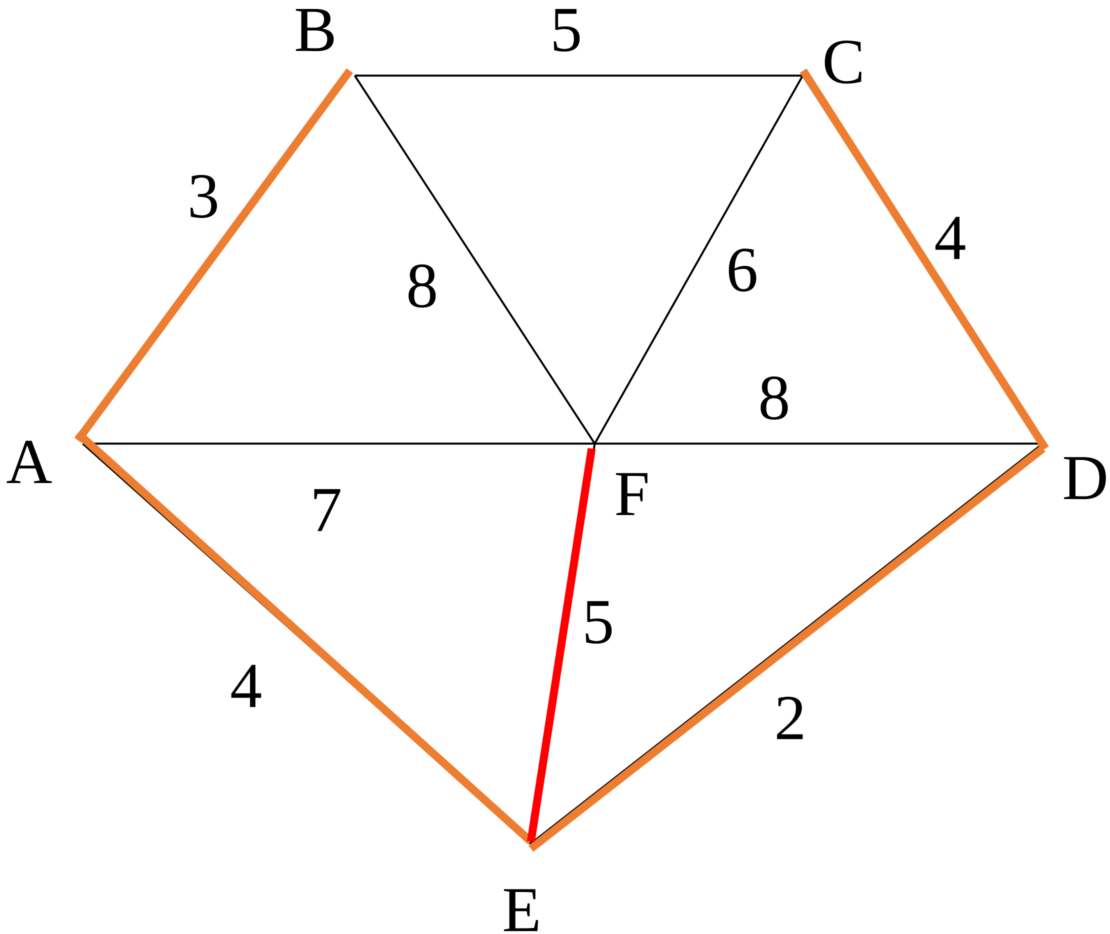
Kruskal’s Algorithm



Select the next shortest edge which does not create a cycle

ED 2
AB 3
CD 4
AE 4

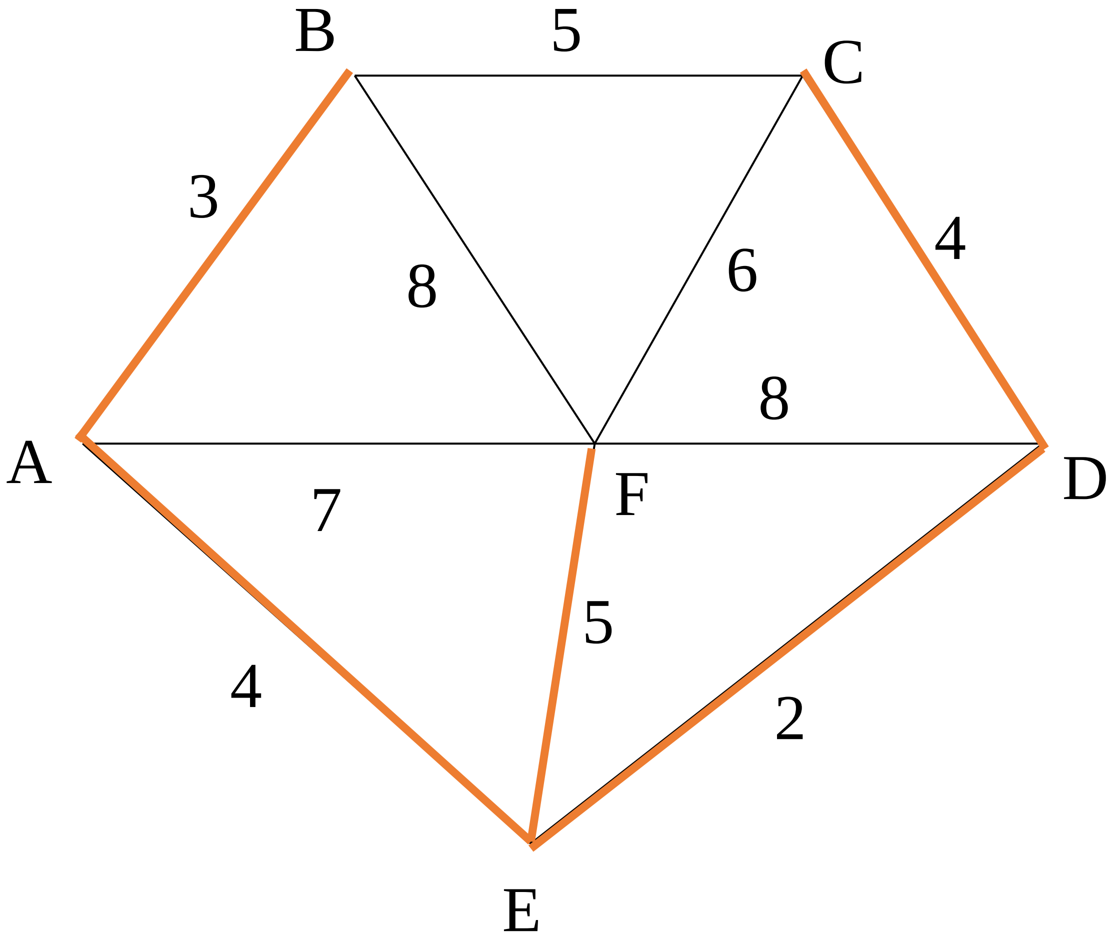
Kruskal's Algorithm



Select the next shortest edge which does not create a cycle

ED 2
AB 3
CD 4
AE 4
BC 5 – forms a cycle
EF 5

Kruskal's Algorithm



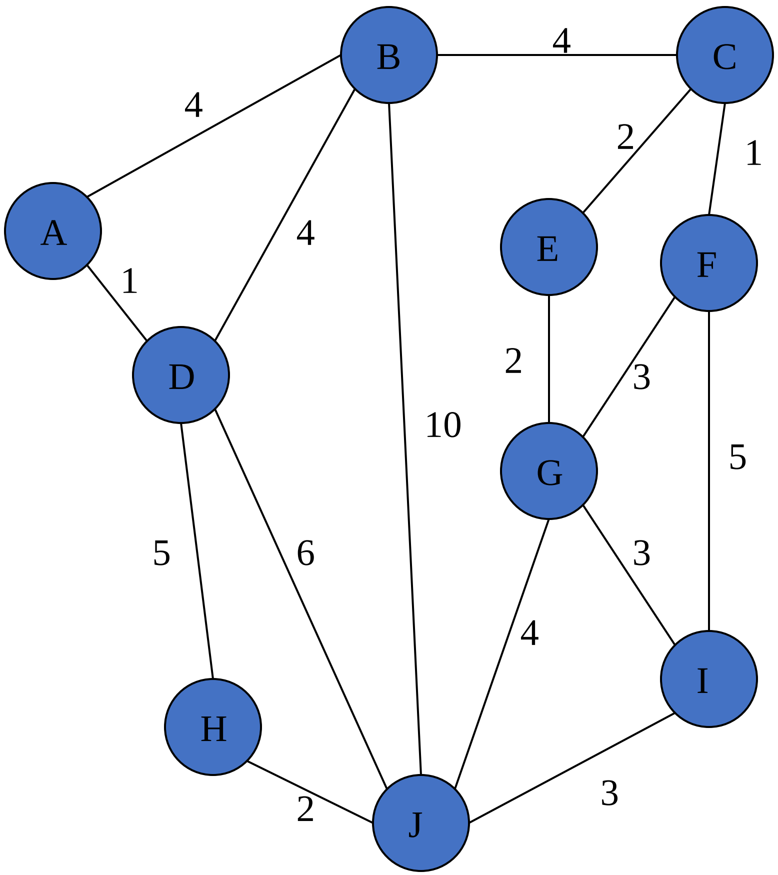
All vertices have been connected.

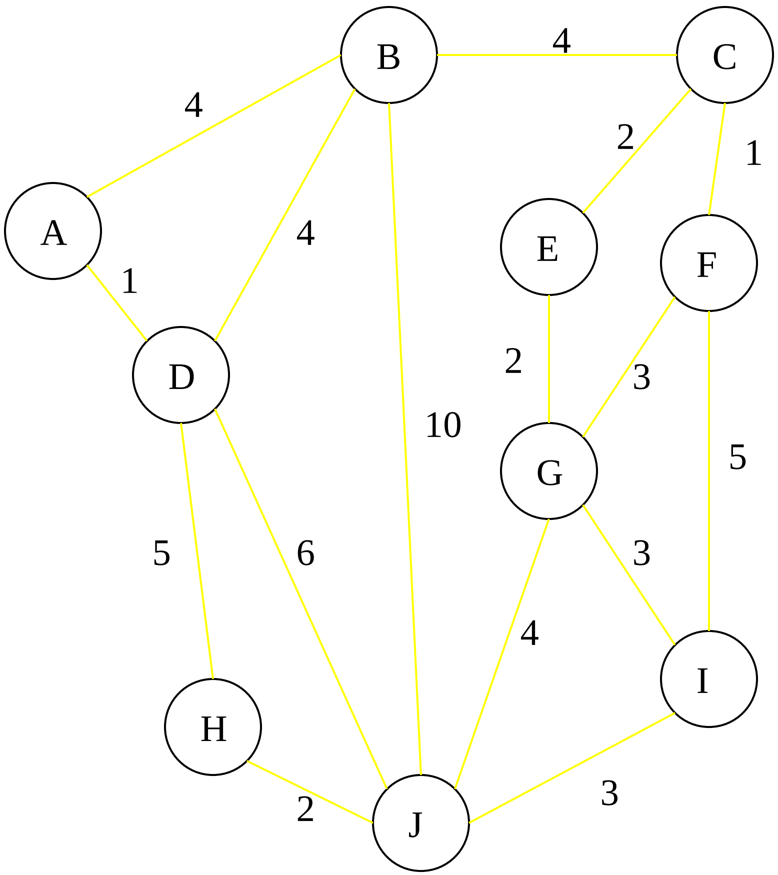
The solution is

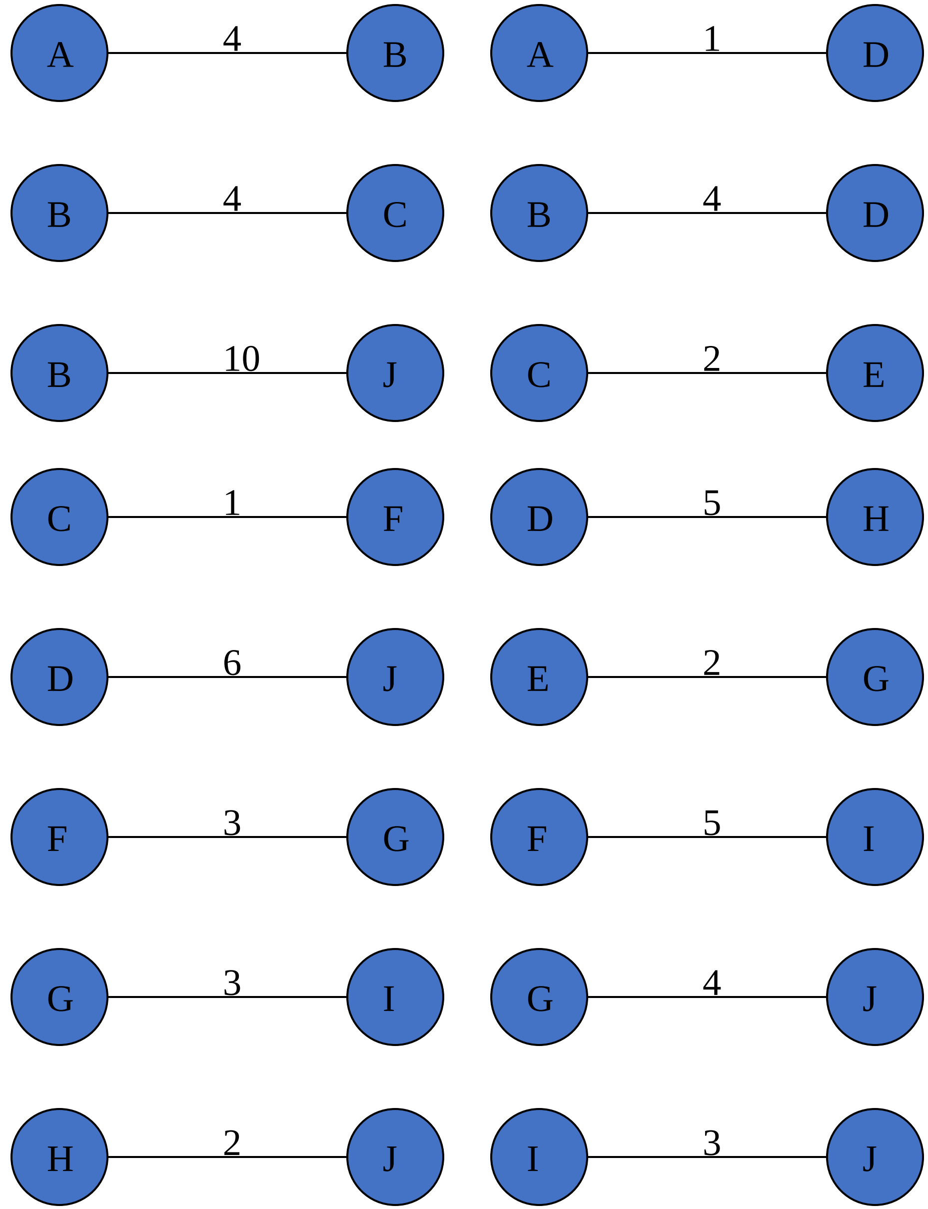
ED 2
AB 3
CD 4
AE 4
EF 5

Total weight of tree: 18

Complete Graph

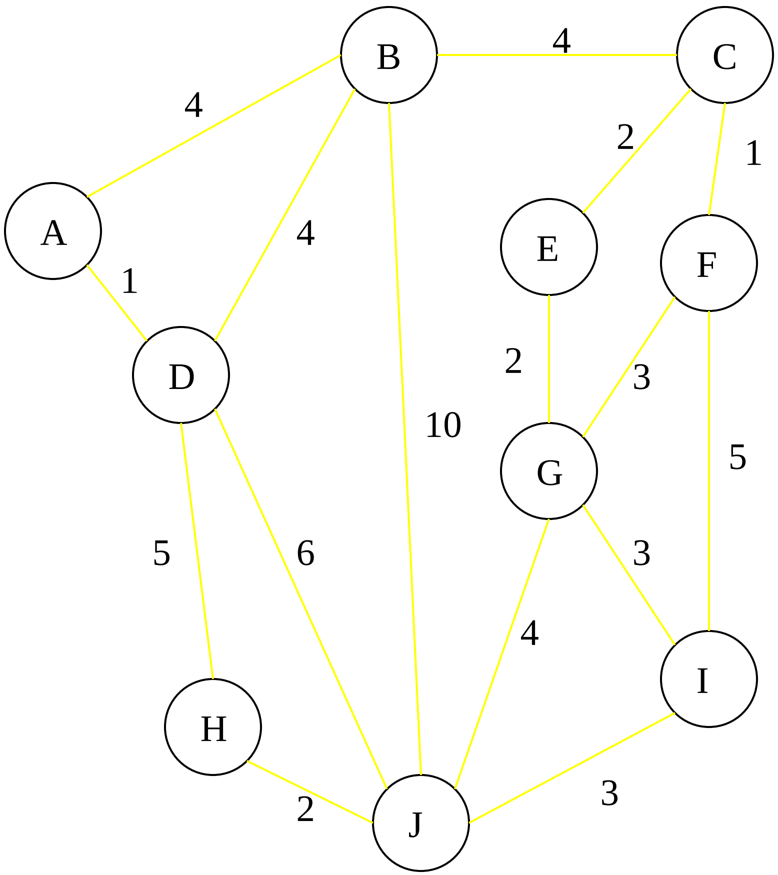


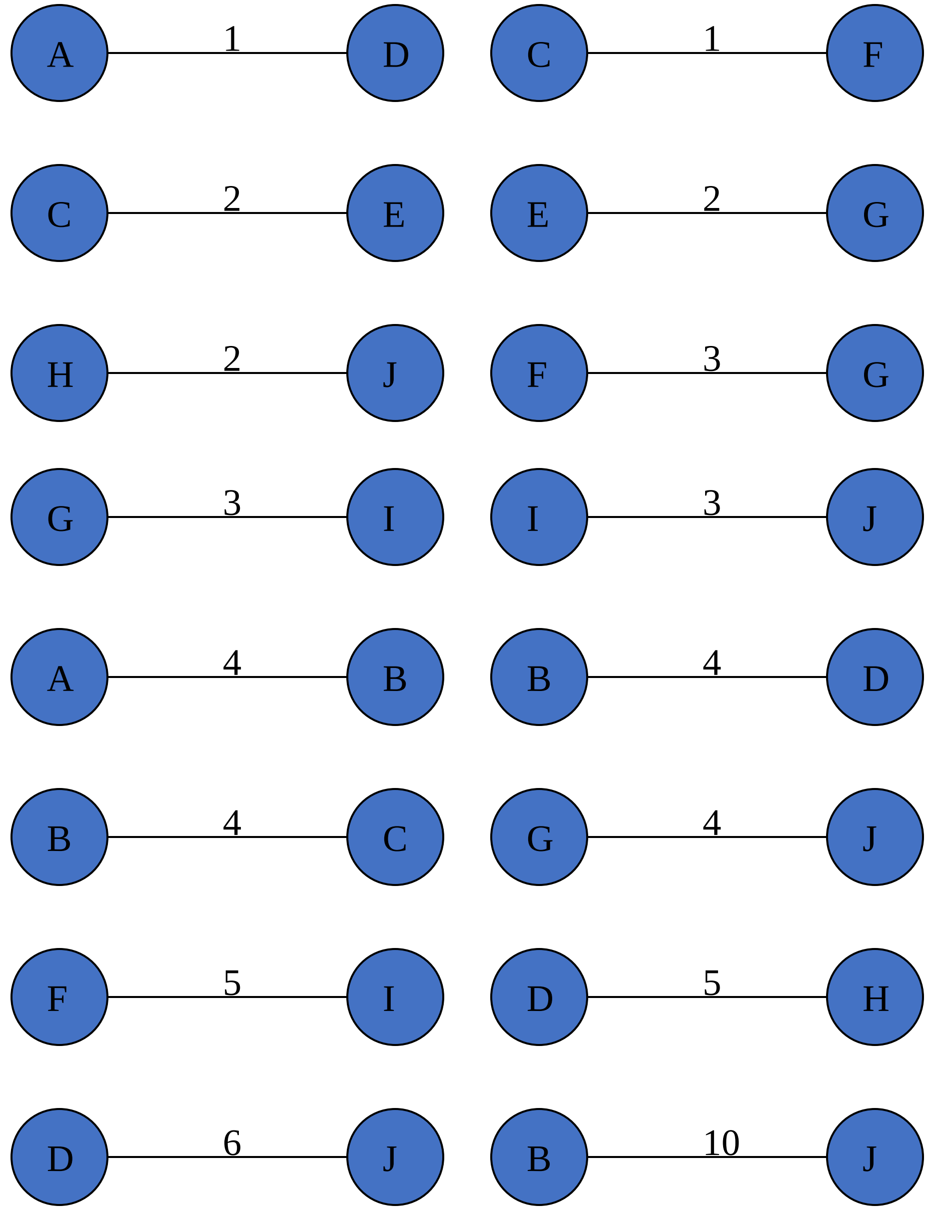

A
B
C
D
E
F
G
H
I
J
4
4
4
1
10
2
1
2
3
5
5
6
3
4
2
3


A 4 B
A 1 D
B 4 C
B 4 D
B 10 J
C 2 E
C 1 F
D 5 H
D 6 J
E 2 G
F 3 G
F 5 I
G 3 I
G 4 J
H 2 J
I 3 J

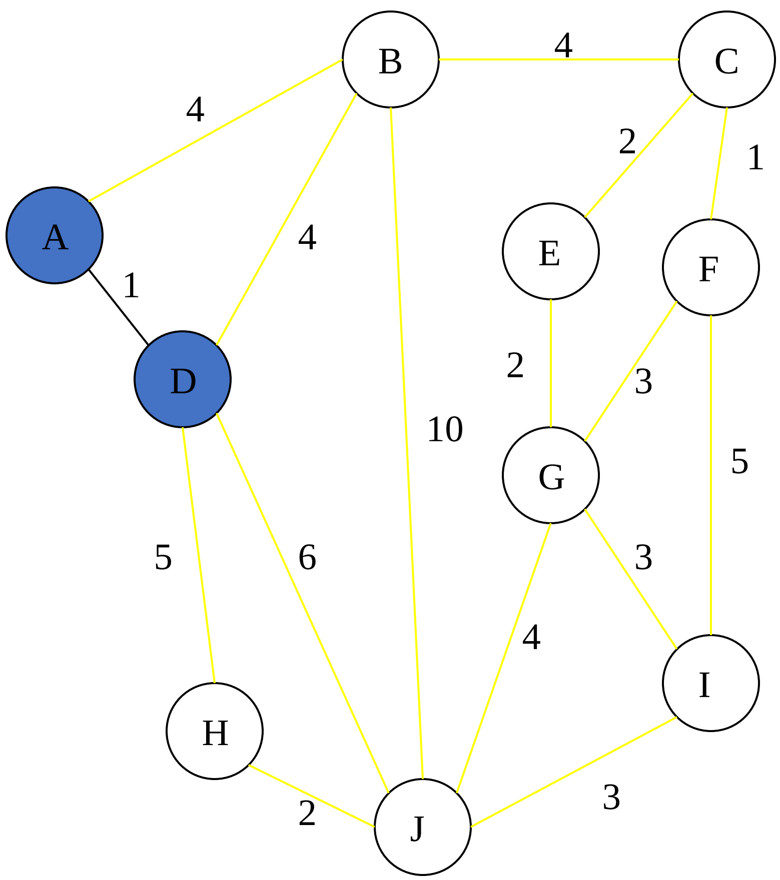
Sort Edges

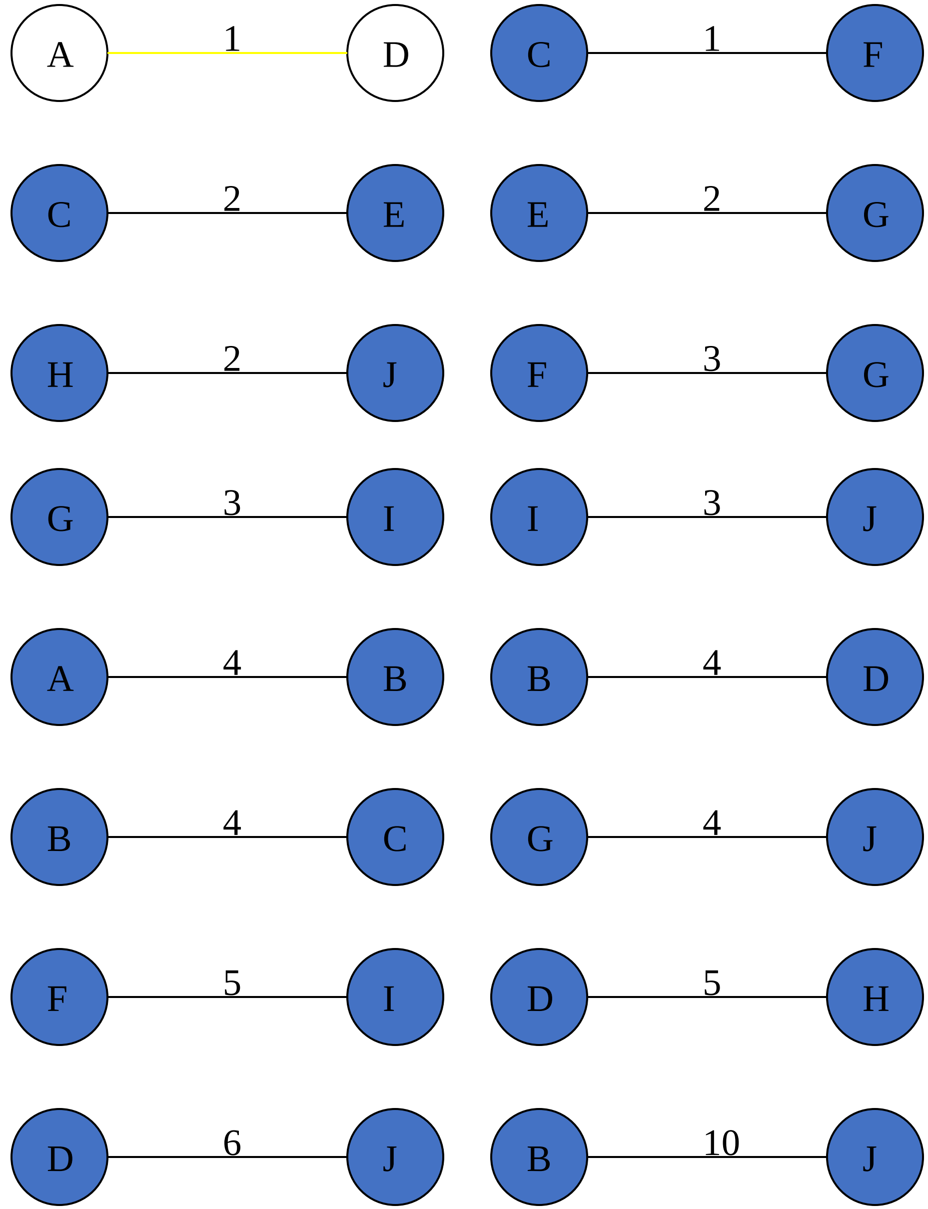
(in reality they are placed in a priority queue - not sorted - but sorting them makes the algorithm easier to visualize)



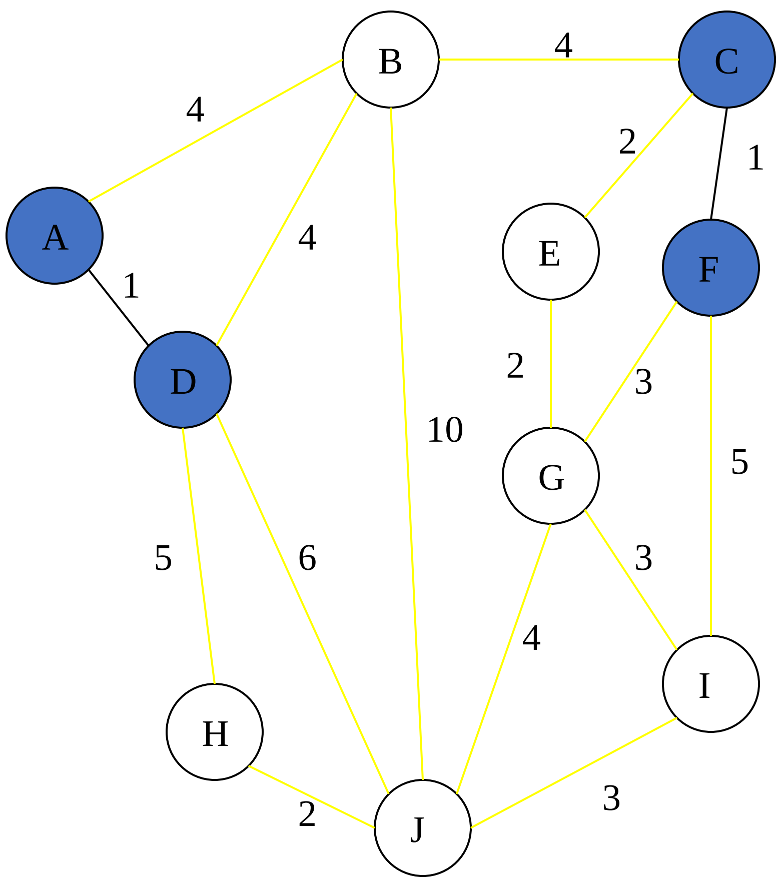


Add Edge

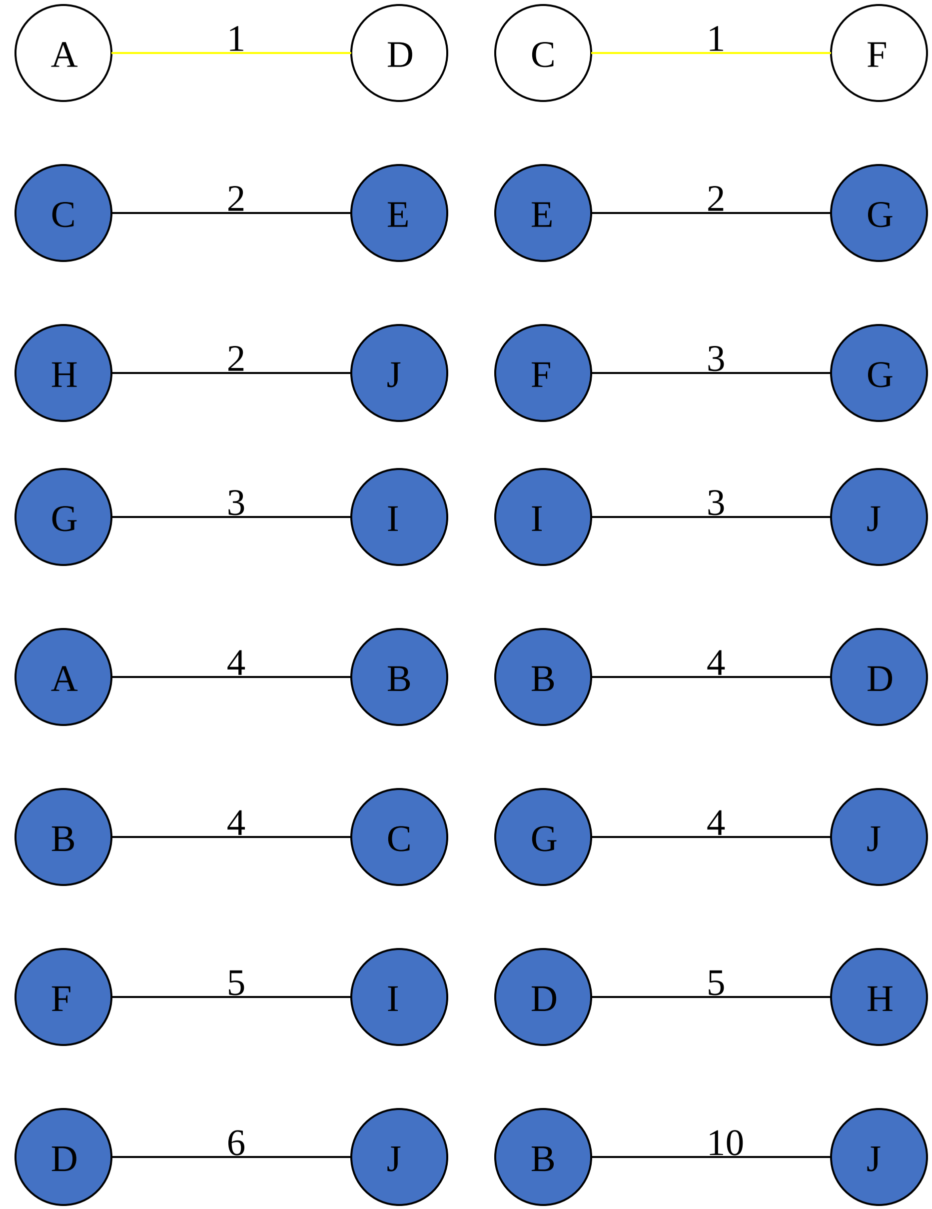

A
B
C
D
E
F
G
H
I
J
4
4
1
4
2
1
2
3
10
5
5
6
3
4
2
3


A 1 D
C 1 F
C 2 E
E 2 G
H 2 J
F 3 G
G 3 I
I 3 J
A 4 B
B 4 D
B 4 C
G 4 J
F 5 I
D 5 H
D 6 J
B 10 J

Add Edge

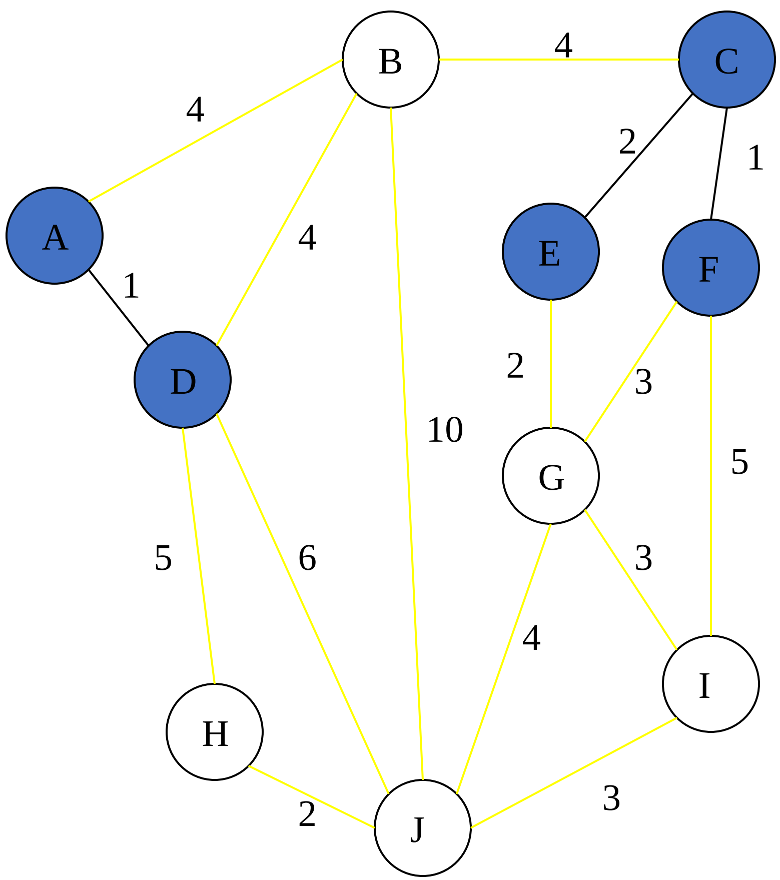


A
B
C
D
E
F
G
H
I
J
4
4
1
4
2
1
2
3
10
5
5
6
3
4
2
3

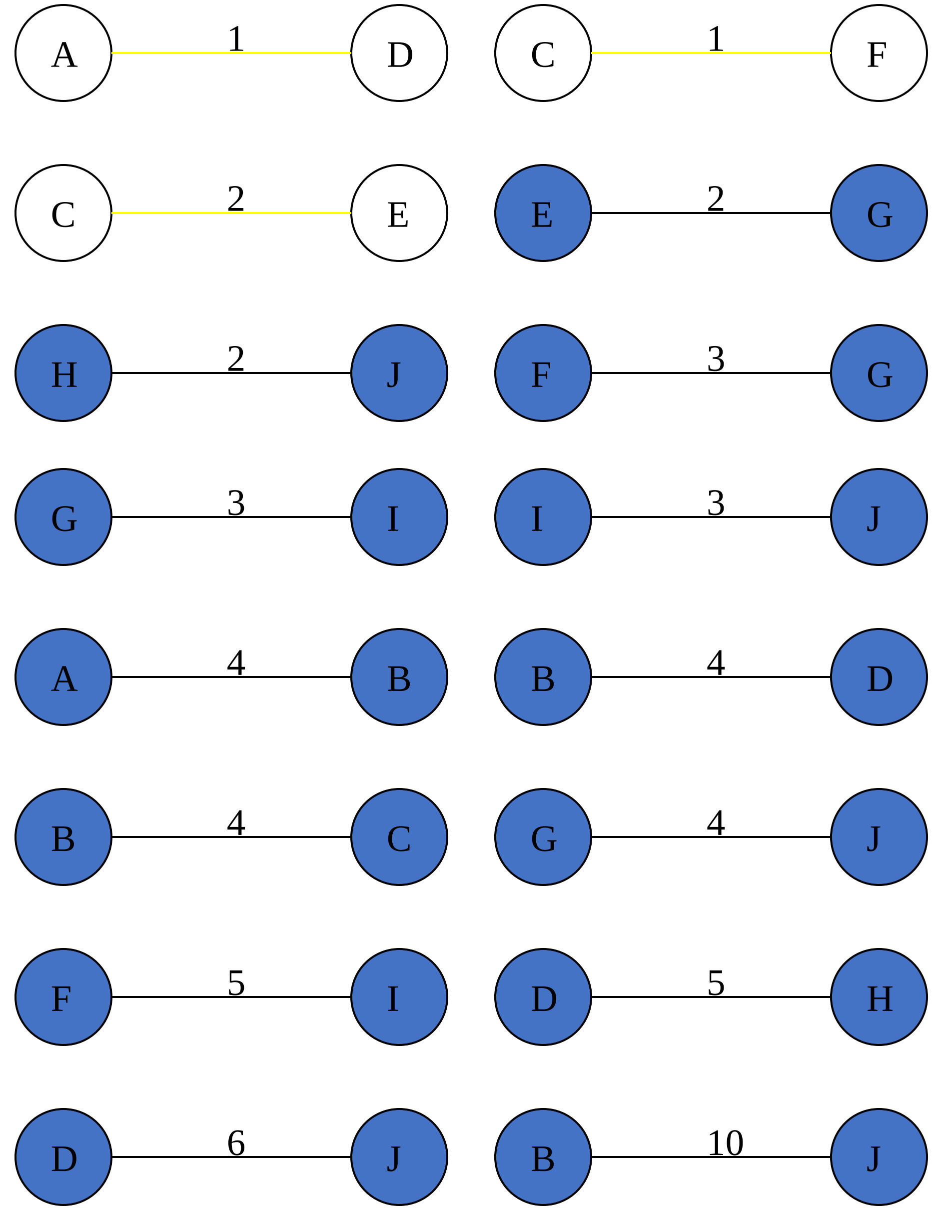


A 1 D
C 1 F
C 2 E
E 2 G
H 2 J
F 3 G
G 3 I
I 3 J
A 4 B
B 4 D
B 4 C
G 4 J
F 5 I
D 5 H
D 6 J
B 10 J

Add Edge

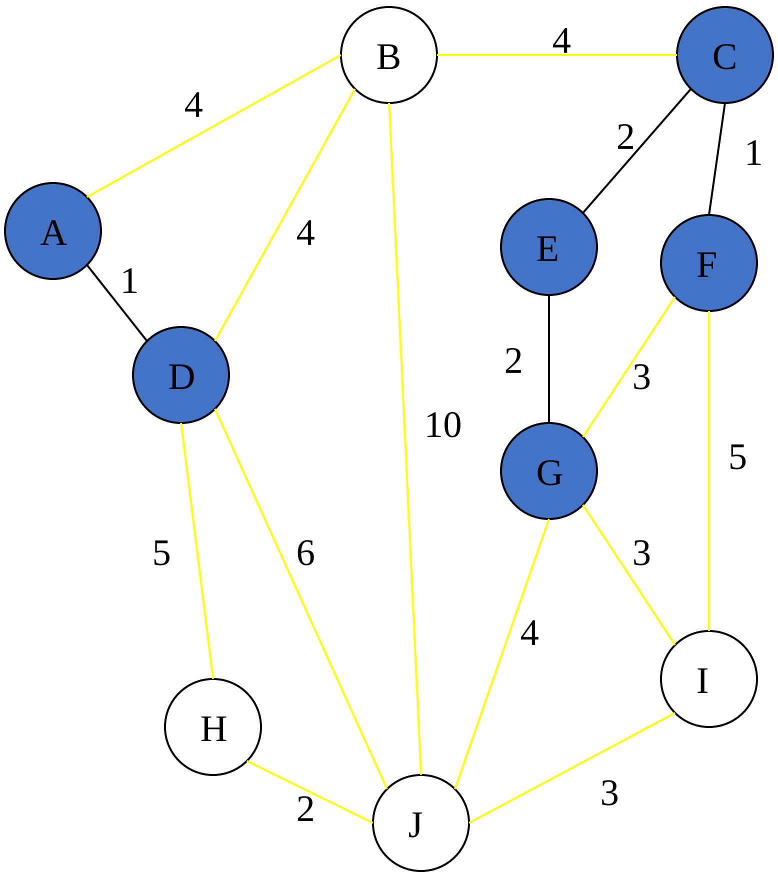


A
B
C
D
E
F
G
H
I
J
4
4
4
1
2
1
2
3
5
10
5
6
4
3
2
3

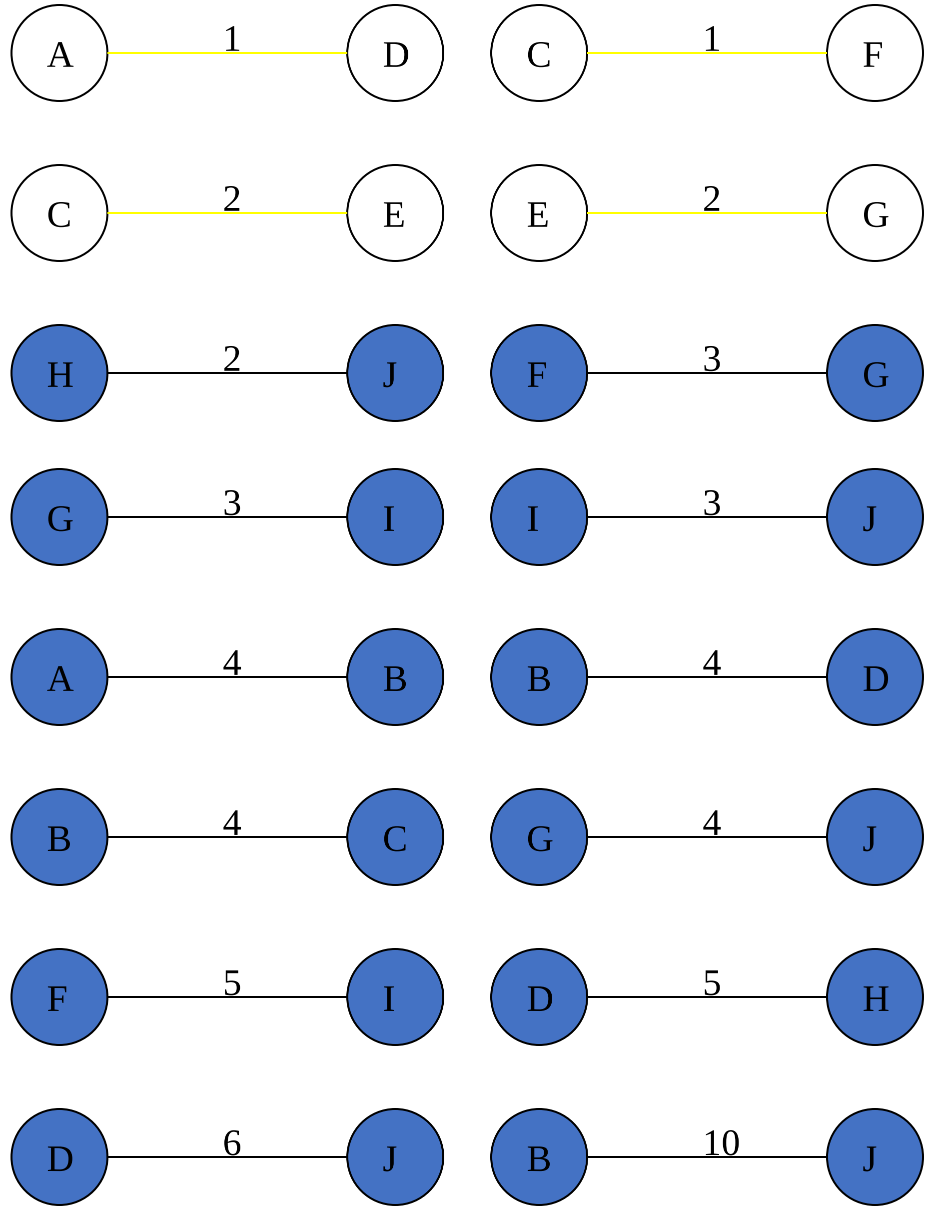


A 1 D
C 1 F
C 2 E
E 2 G
H 2 J
F 3 G
G 3 I
I 3 J
A 4 B
B 4 D
B 4 C
G 4 J
F 5 I
D 5 H
D 6 J
B 10 J

Add Edge

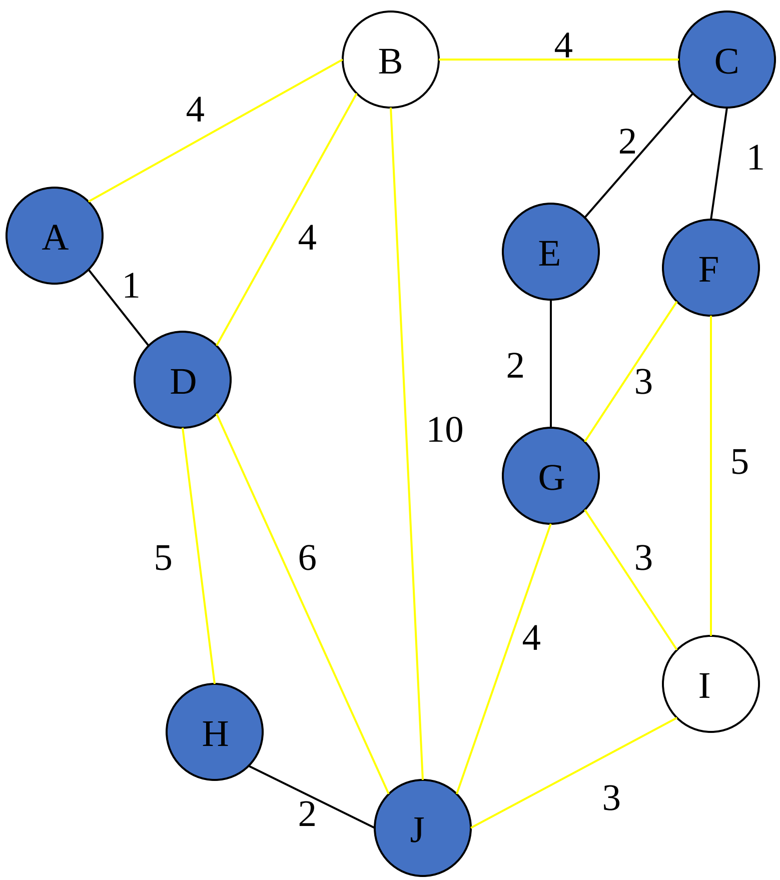


B
4
C
4
2
1
A
4
E
F
1
D
2
3
10
5
G
5
6
3
4
I
H
2
J
3

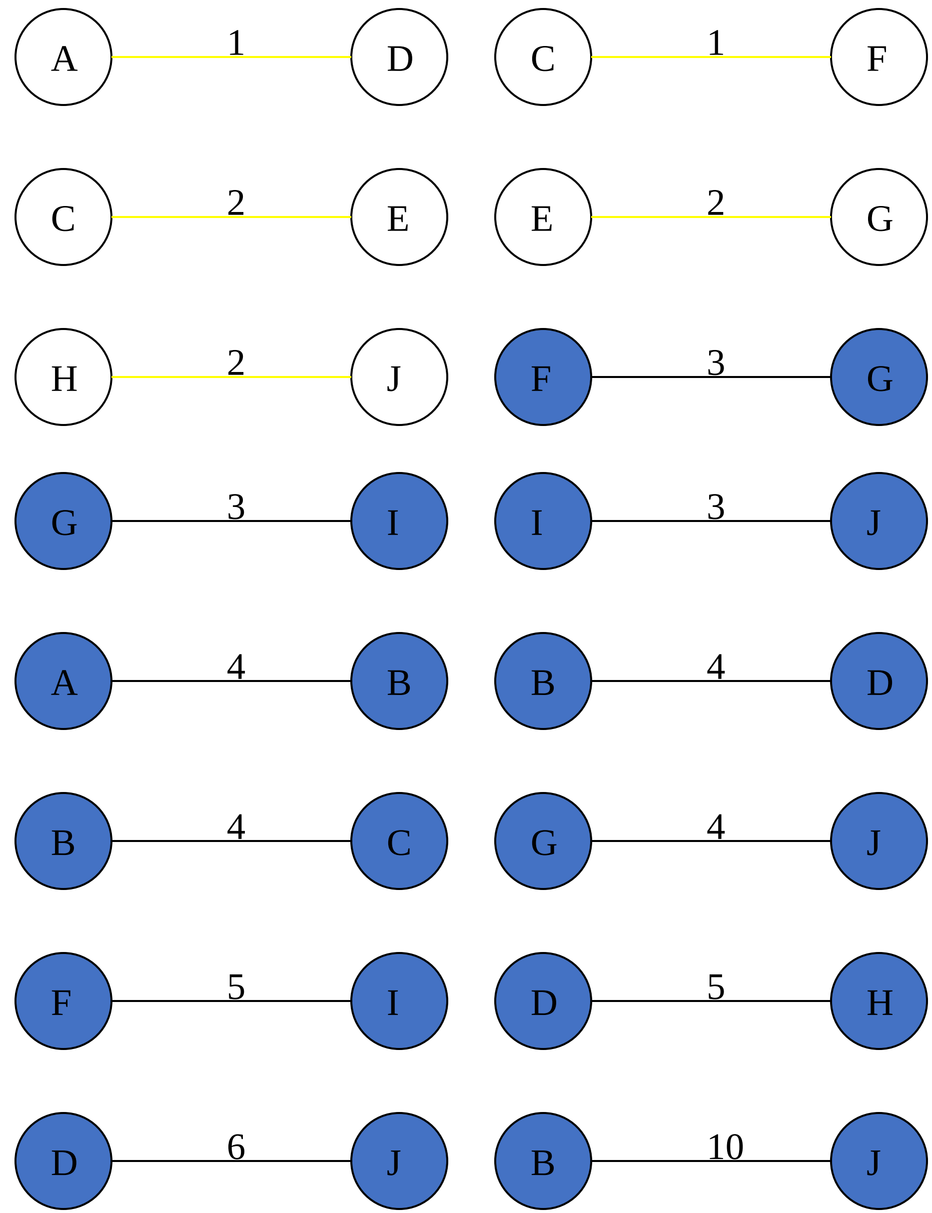


A 1 D
C 1 F
C 2 E
E 2 G
H 2 J
F 3 G
G 3 I
I 3 J
A 4 B
B 4 D
B 4 C
G 4 J
F 5 I
D 5 H
D 6 J
B 10 J

Add Edge



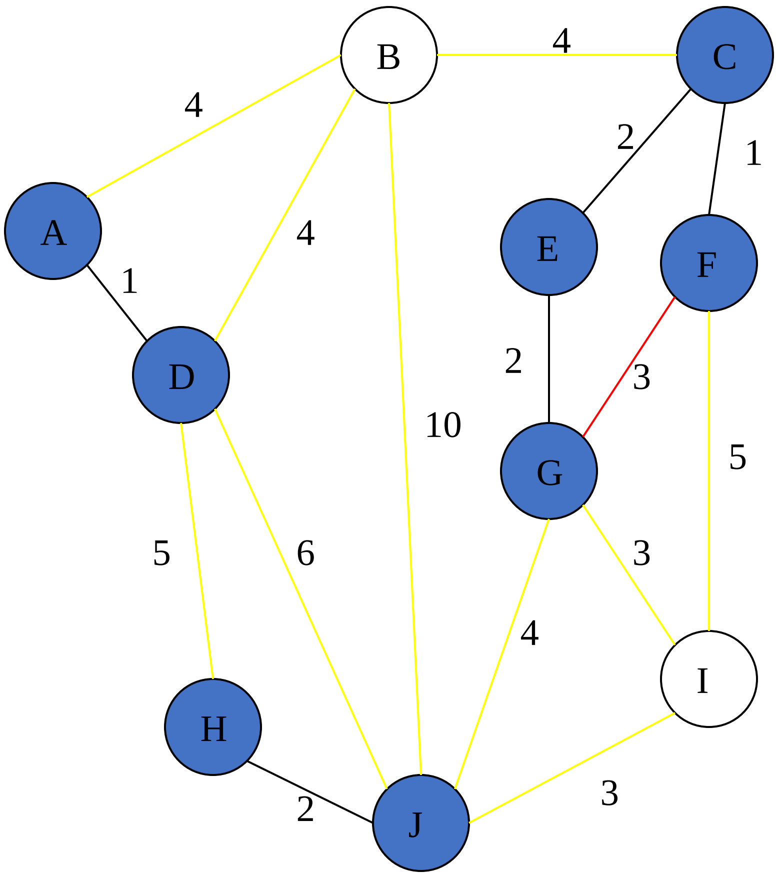
A
B
C
D
E
F
G
H
I
J
4
4
1
4
10
2
1
2
3
5
5
6
3
4
2
3

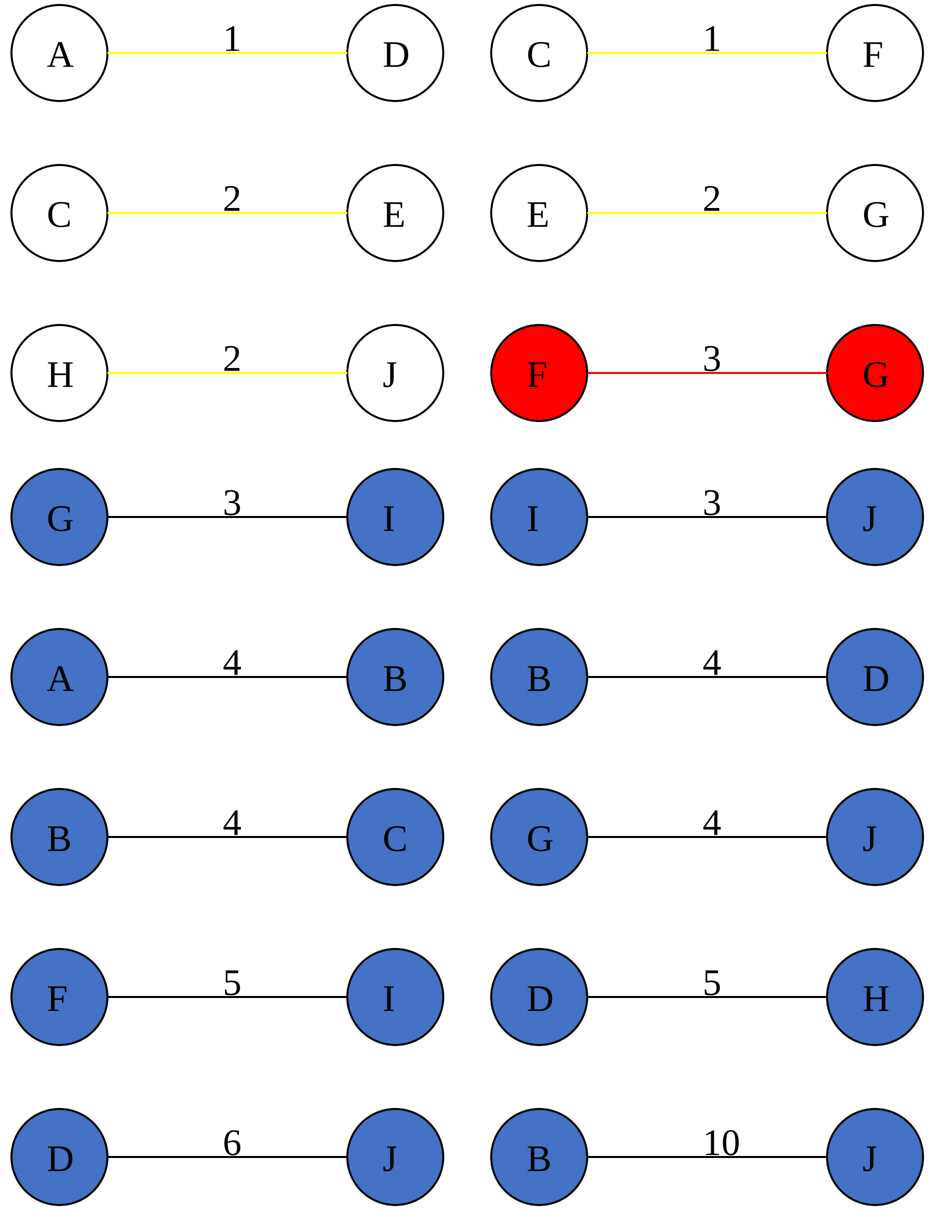


A 1 D
C 1 F
C 2 E
E 2 G
H 2 J
F 3 G
G 3 I
I 3 J
A 4 B
B 4 D
B 4 C
G 4 J
F 5 I
D 5 H
D 6 J
B 10 J

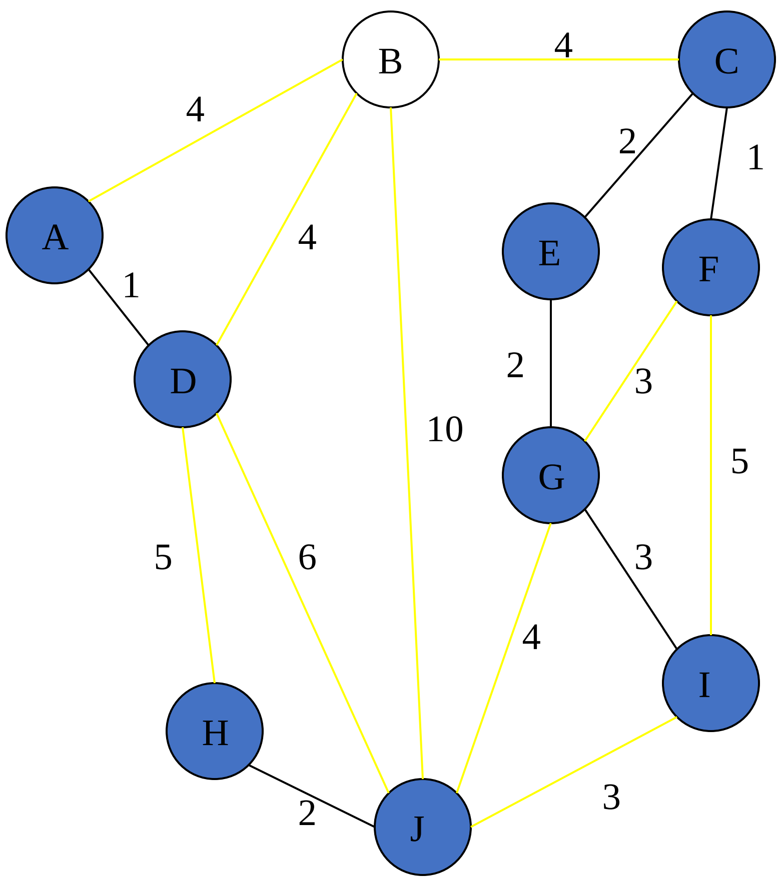
Cycle

Don’t Add Edge

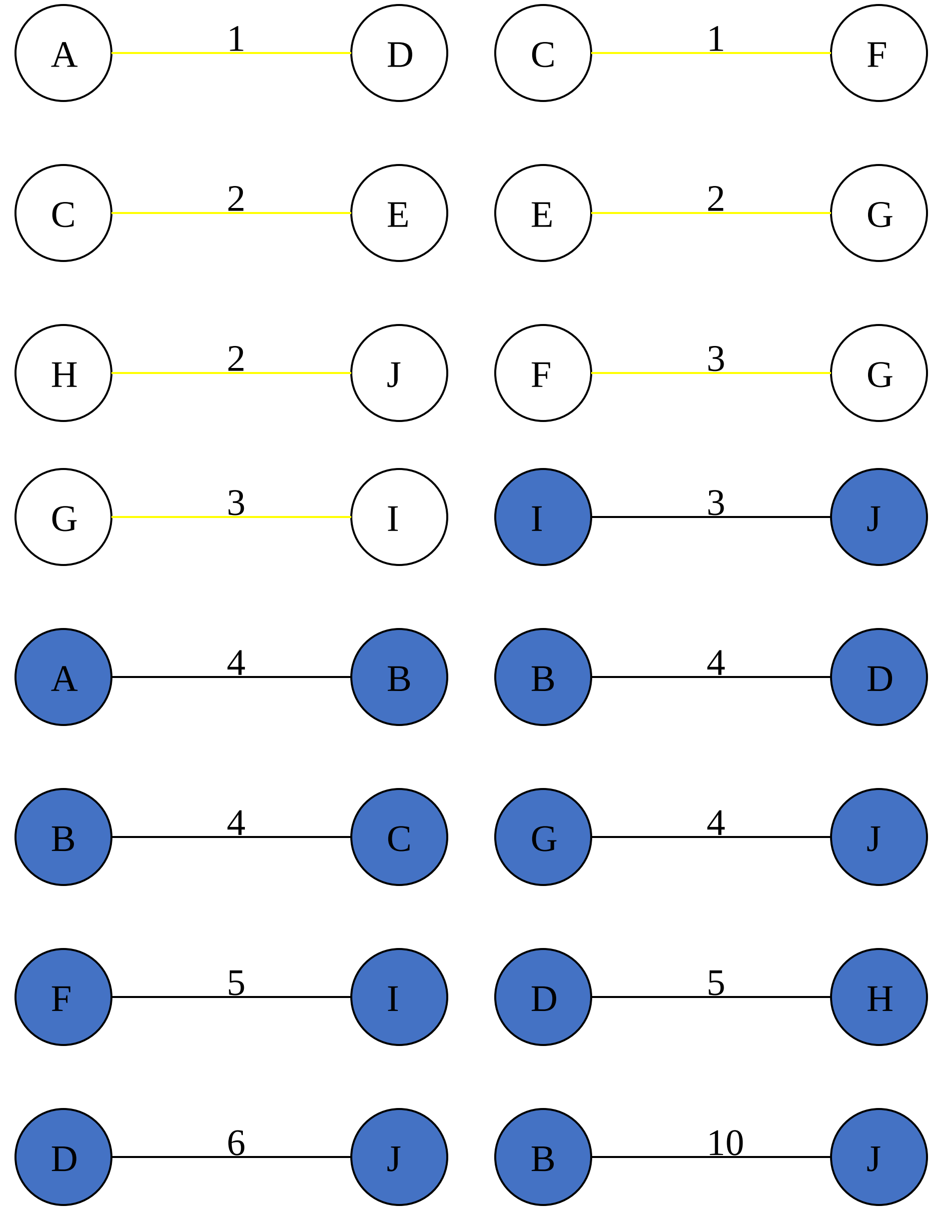




Add Edge

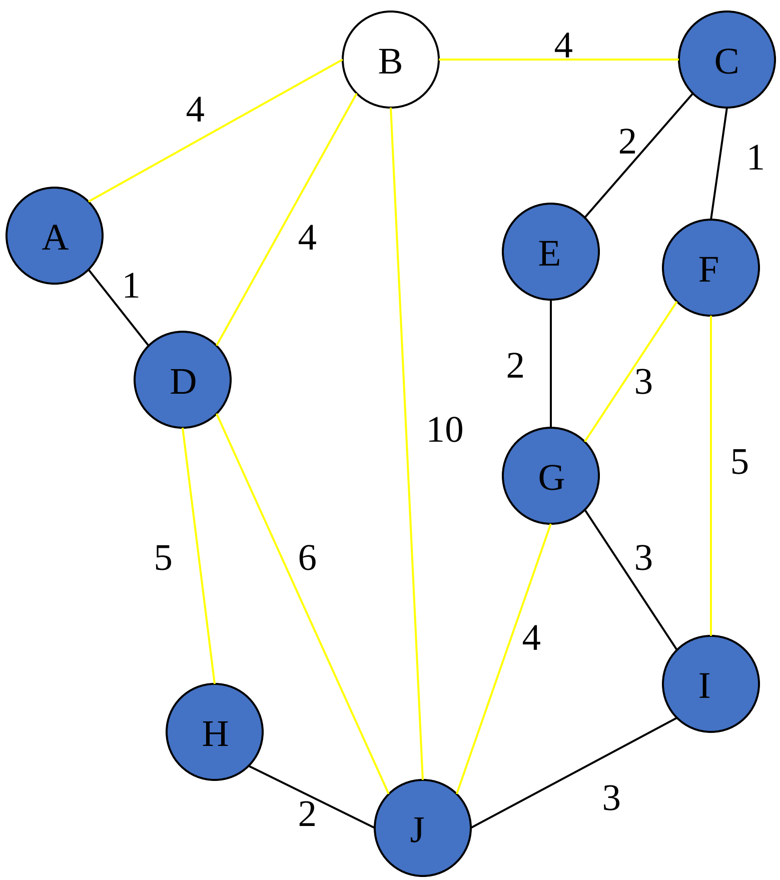


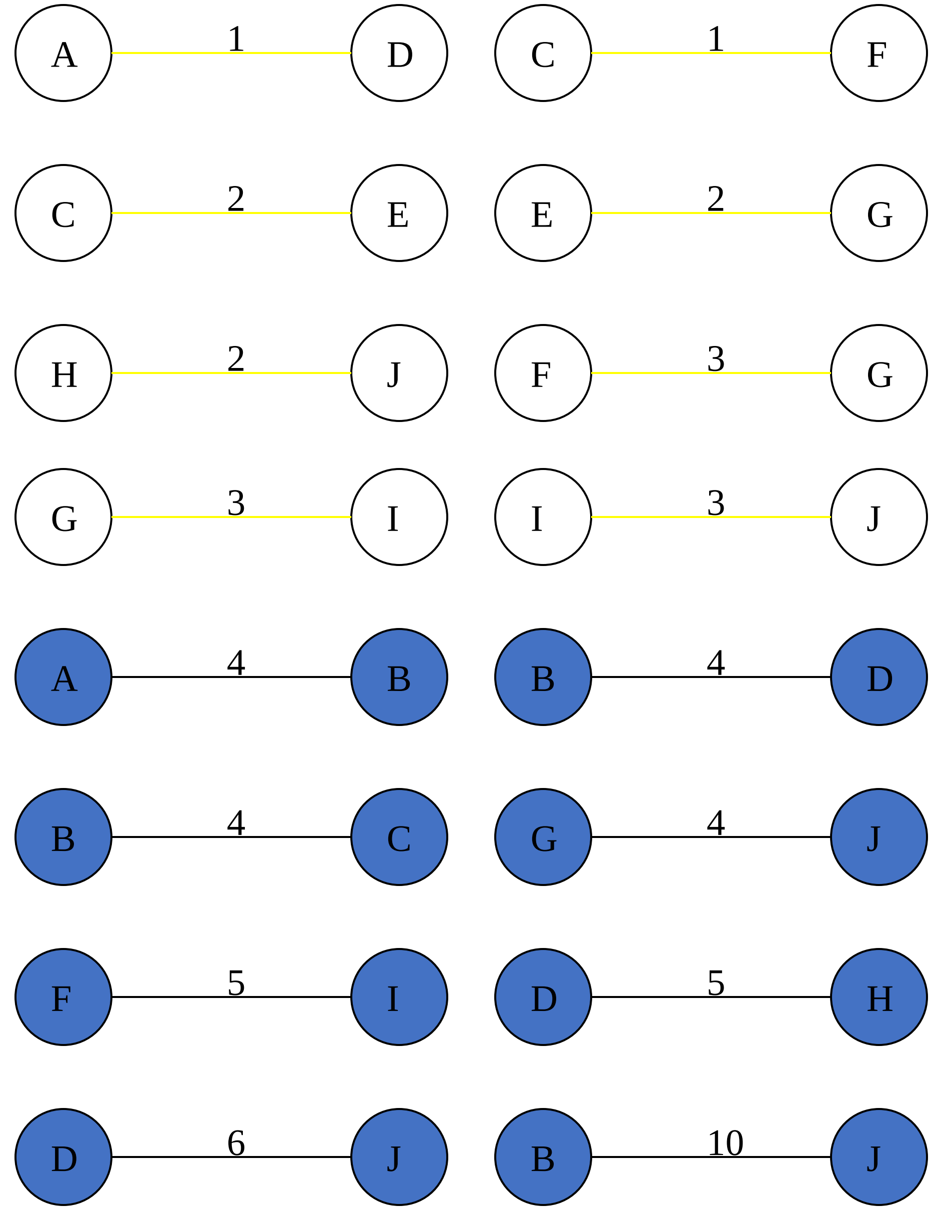
A
B
C
D
E
F
G
H
I
J
4
4
1
4
2
1
10
2
3
5
5
6
3
4
2
3



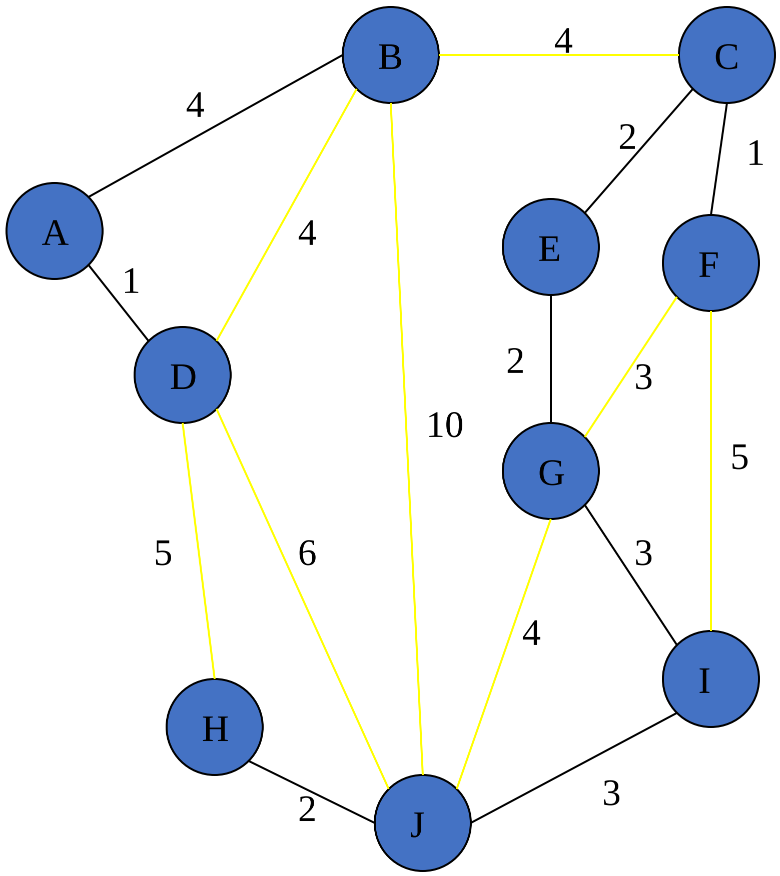
A 1 D
C 1 F
C 2 E
E 2 G
H 2 J
F 3 G
G 3 I
I 3 J
A 4 B
B 4 D
B 4 C
G 4 J
F 5 I
D 5 H
D 6 J
B 10 J

Add Edge

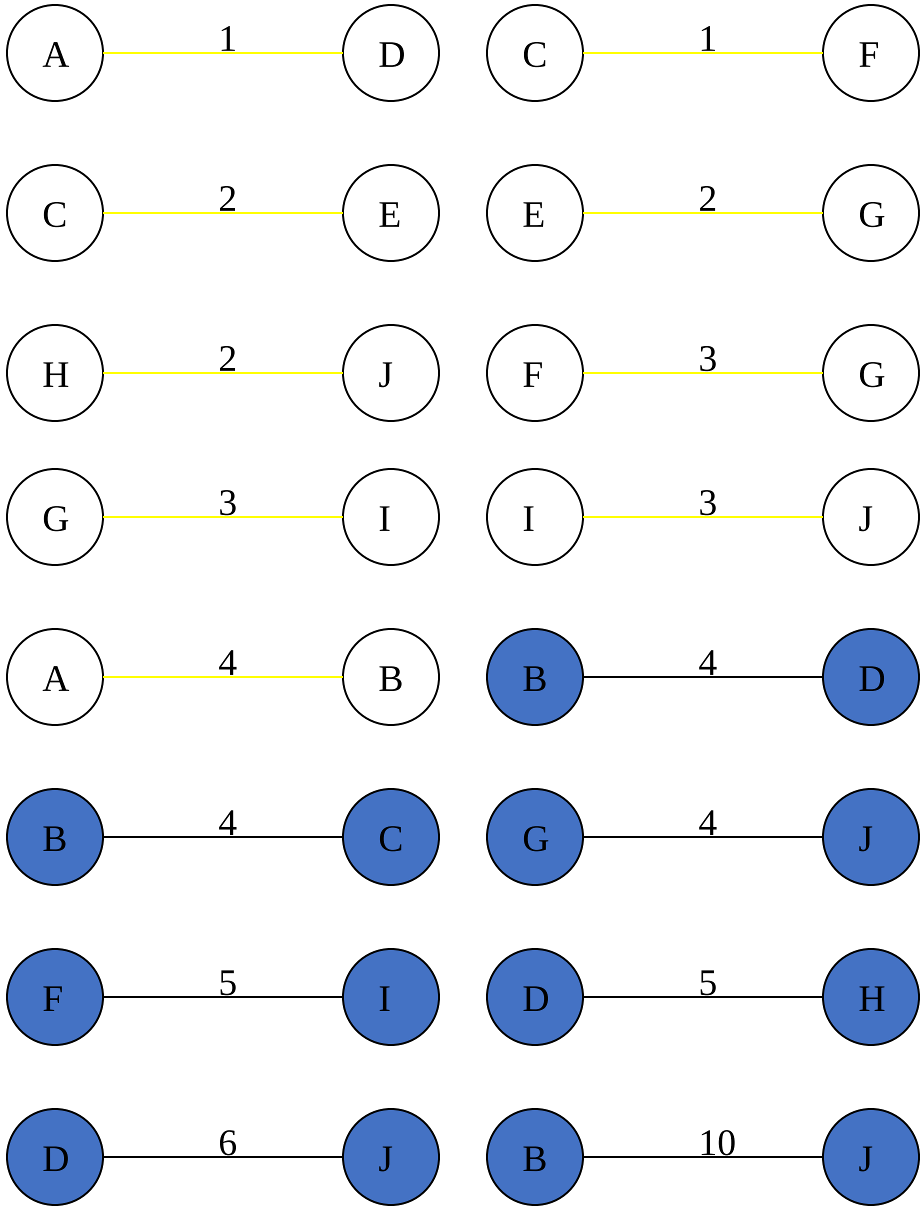

B
4
C
4
A
4
2
1
E
F
1
D
2
3
10
G
5
5
6
3
4
I
H
2
J
3


A 1 D
C 1 F
C 2 E
E 2 G
H 2 J
F 3 G
G 3 I
I 3 J
A 4 B
B 4 D
B 4 C
G 4 J
F 5 I
D 5 H
D 6 J
B 10 J

Add Edge



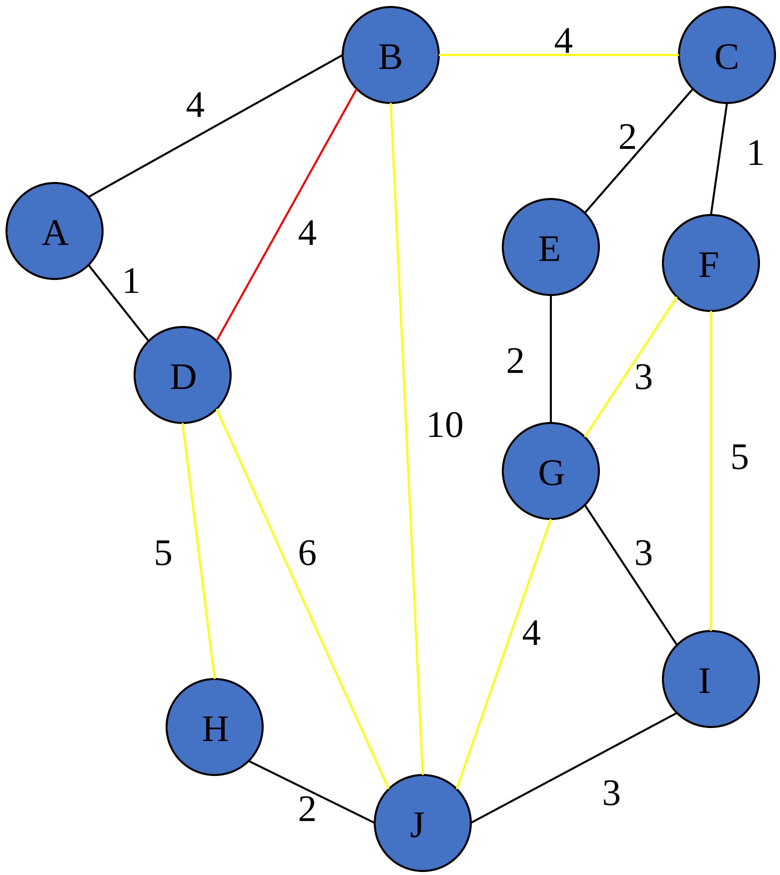
A
B
C
D
E
F
G
H
I
J
4
4
2
1
1
4
2
3
10
5
5
6
3
4
2
3

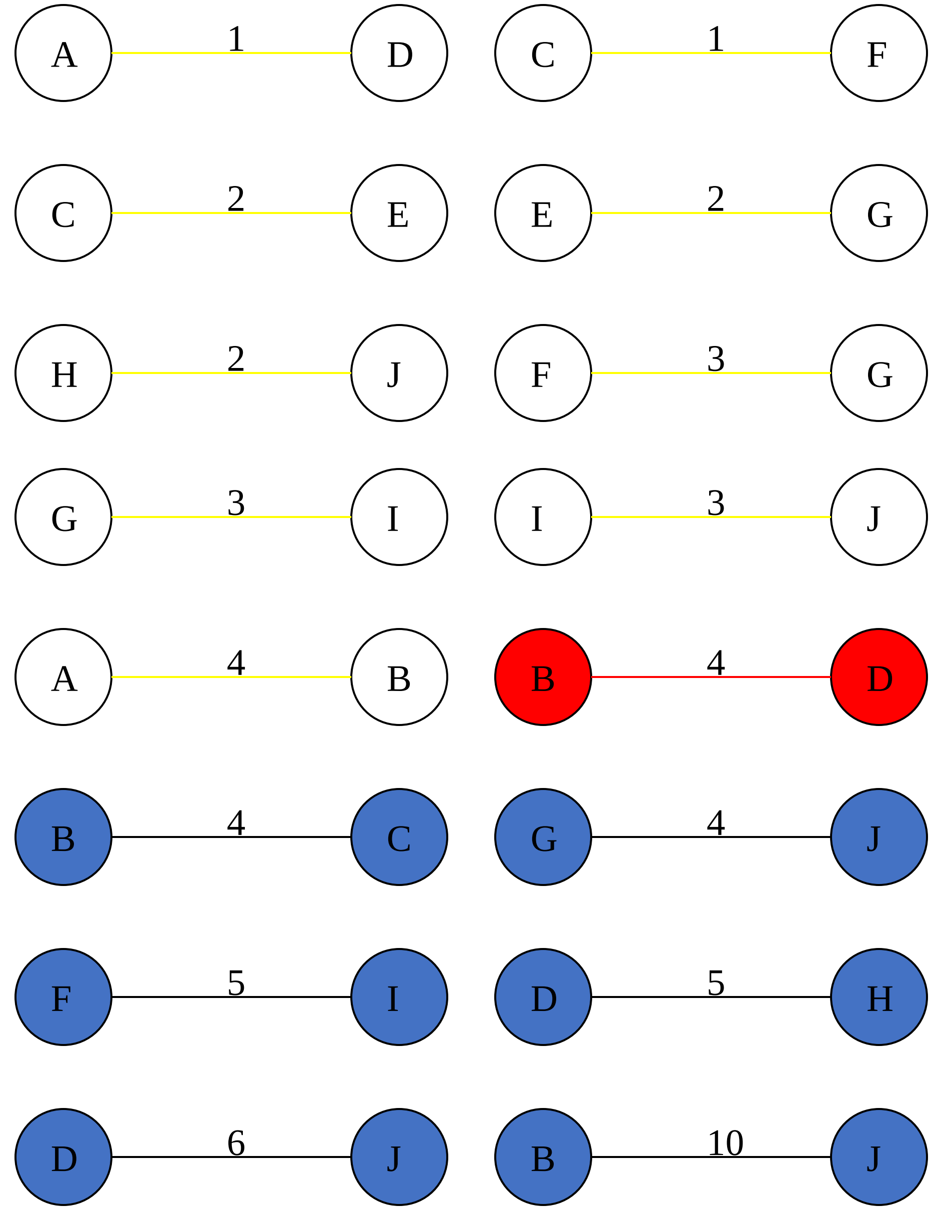


A 1 D
C 1 F
C 2 E
E 2 G
H 2 J
F 3 G
G 3 I
I 3 J
A 4 B
B 4 D
B 4 C
G 4 J
F 5 I
D 5 H
D 6 J
B 10 J

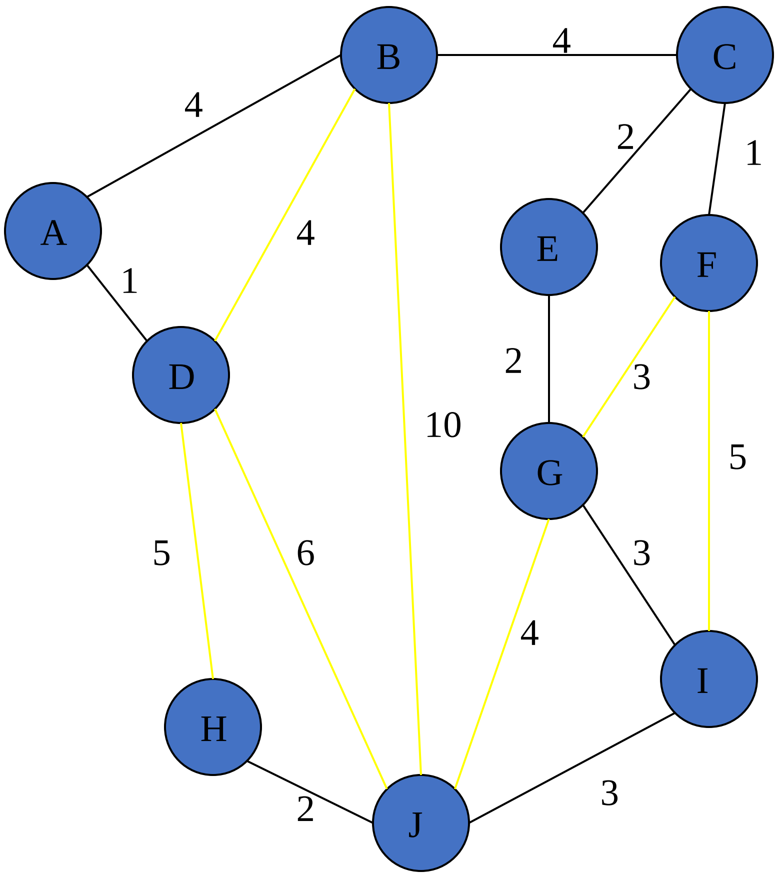
Cycle

Don’t Add Edge

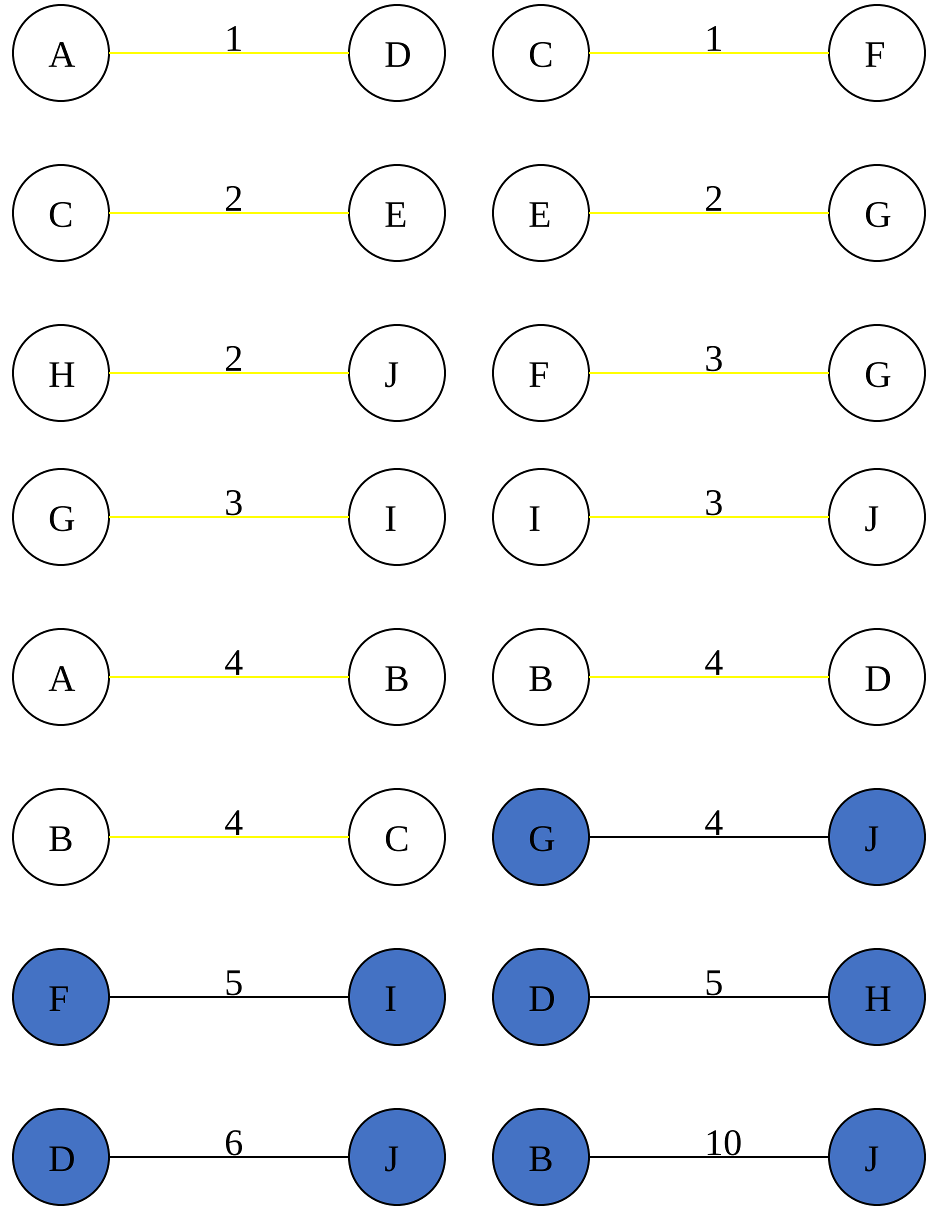




Add Edge

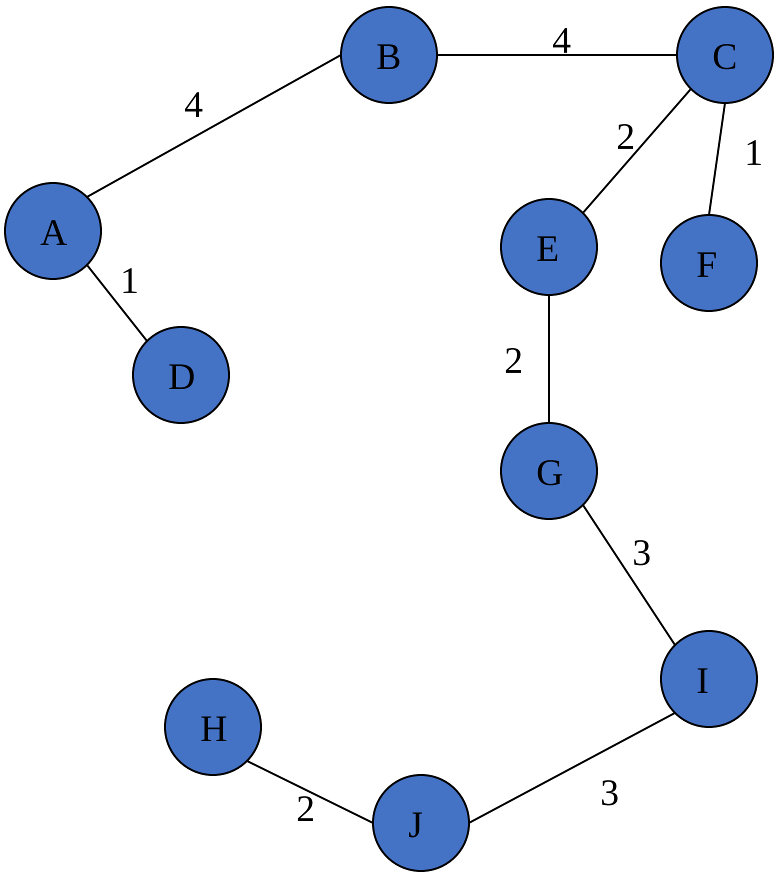


A B C D E F G H I J
4 4 1 4 2 1 2 3 10 5 5 6 3 4 2 3

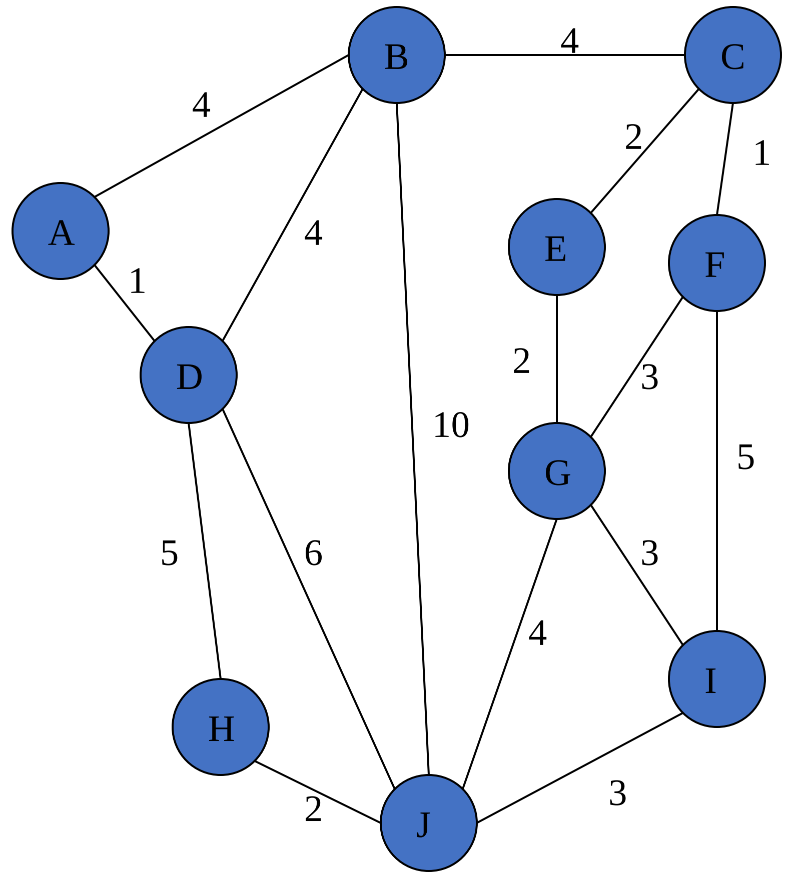


A 1 D
C 1 F
C 2 E
E 2 G
H 2 J
F 3 G
G 3 I
I 3 J
A 4 B
B 4 D
B 4 C
G 4 J
F 5 I
D 5 H
D 6 J
B 10 J

Minimum Spanning Tree



Complete Graph



Analysis of Kruskal's Algorithm

Running Time = O(m log n) (m = edges, n = nodes)

Minimum Spanning Tree

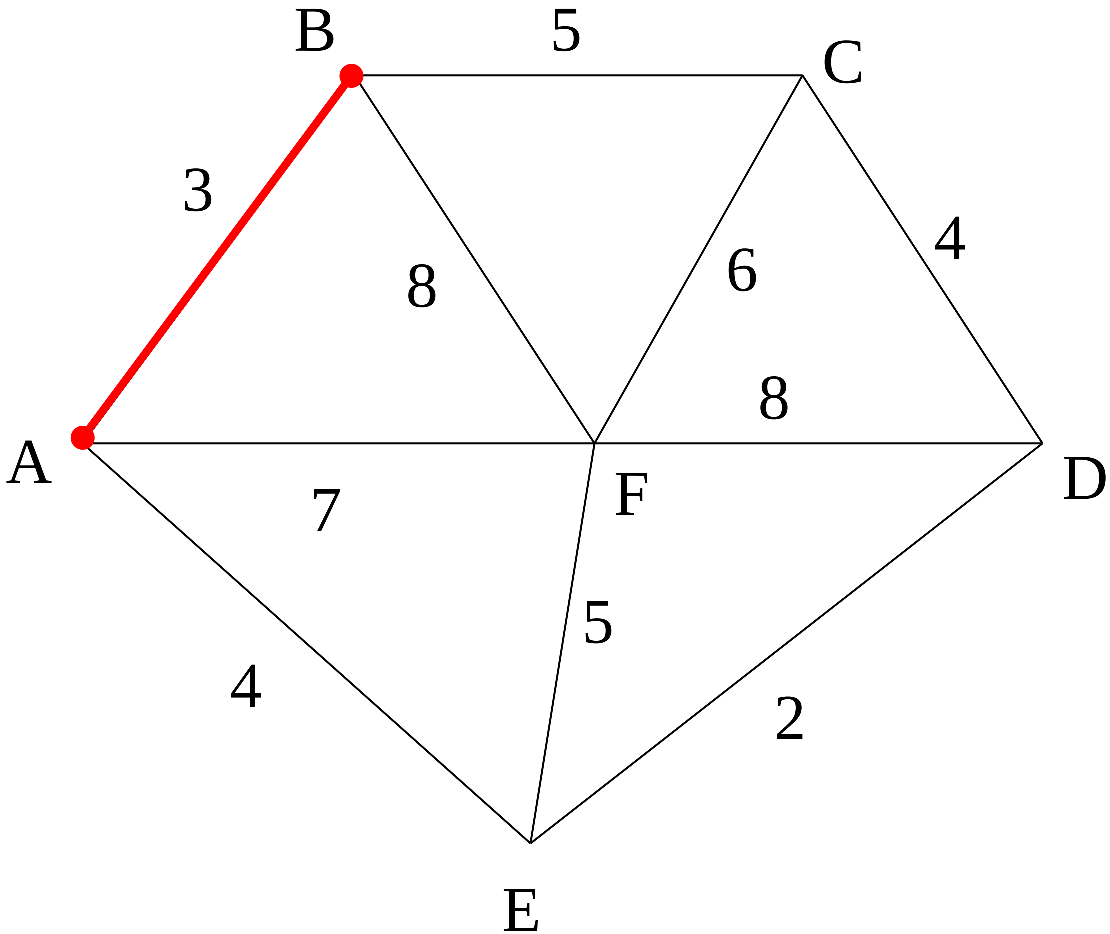
Kruskal's algorithm

1. Select the shortest edge in a network
2. Select the next shortest edge which does not create a cycle
3. Repeat step 2 until all vertices have been connected

Prim's algorithm

1. Select any vertex
2. Select the shortest edge connected to that vertex
3. Select the shortest edge connected to any vertex already connected
4. Repeat step 3 until all vertices have been connected

Prim’s Algorithm



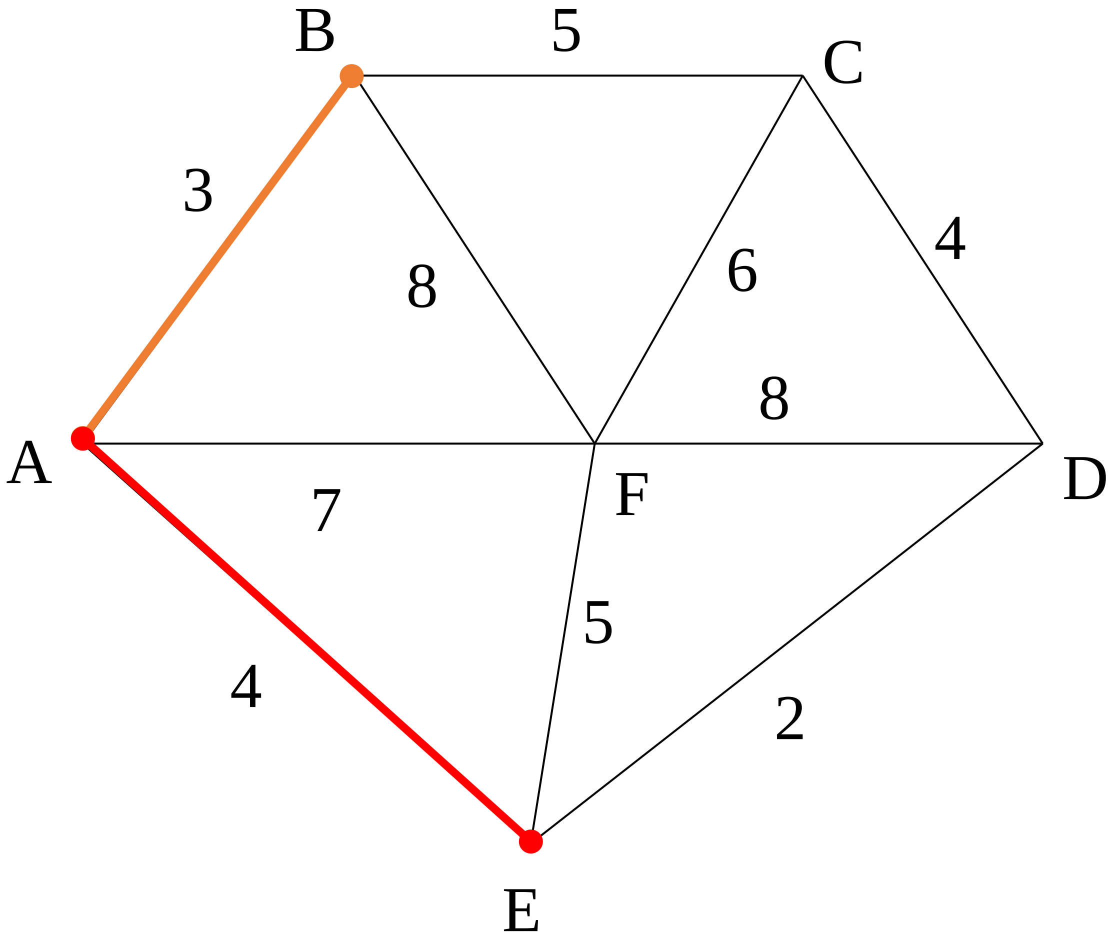
Select any vertex

A

Select the shortest edge connected to that vertex

AB 3

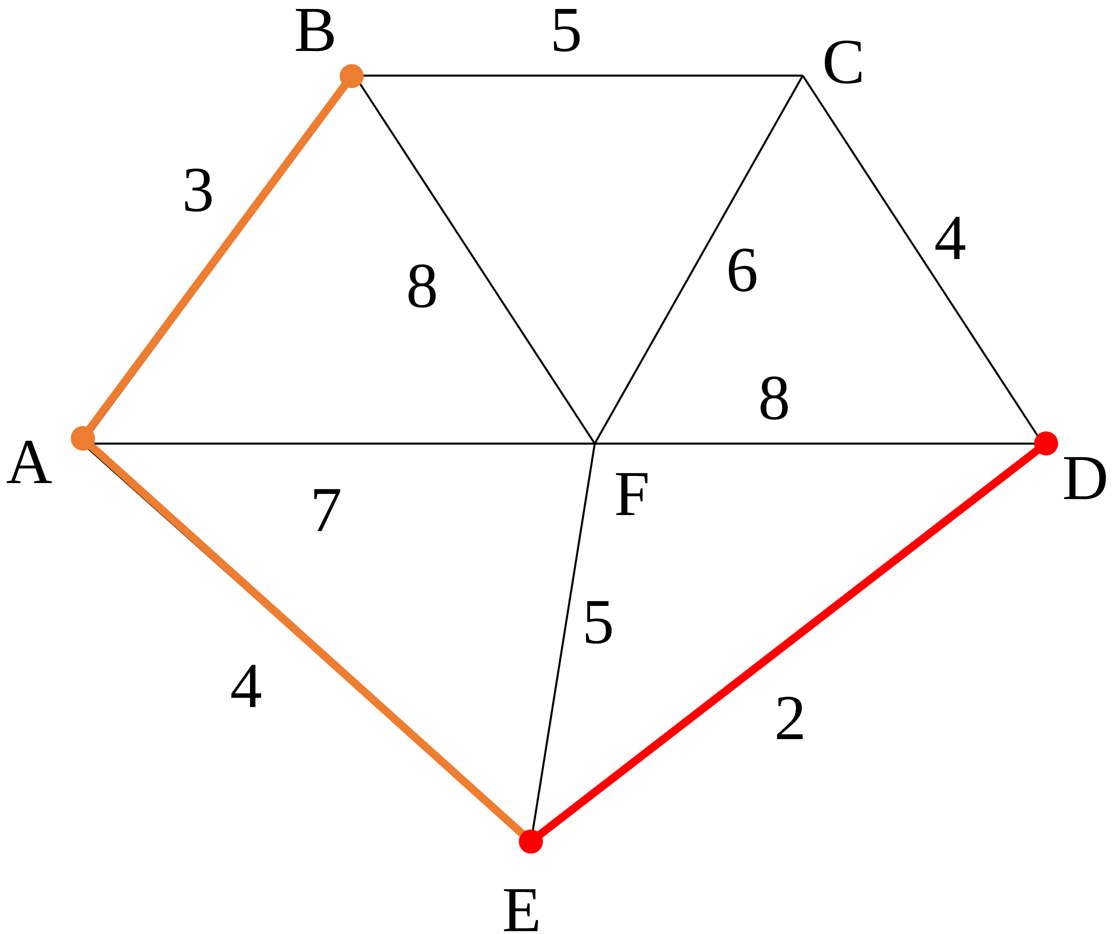
Prim's Algorithm



Select the shortest edge connected to any vertex already connected.

AE 4

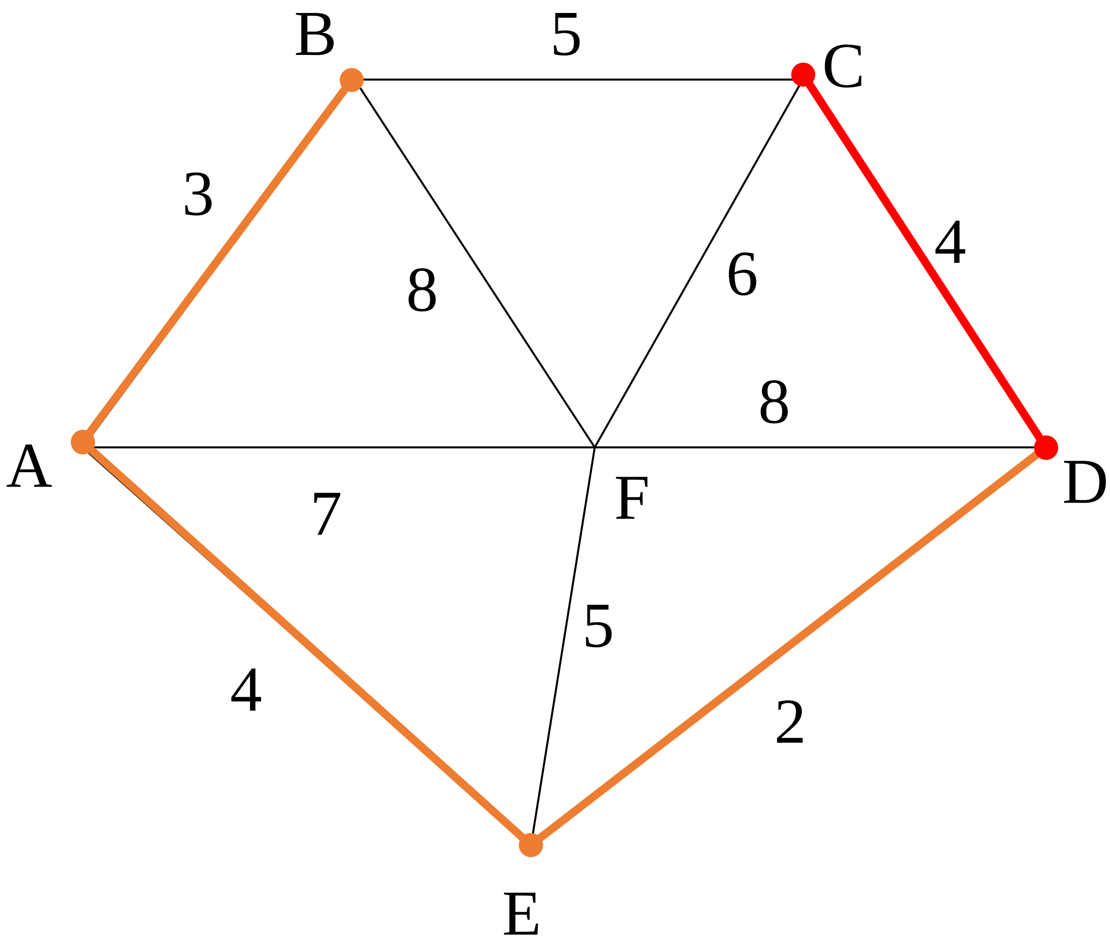
Prim's Algorithm



Select the shortest edge connected to any vertex already connected.

ED 2

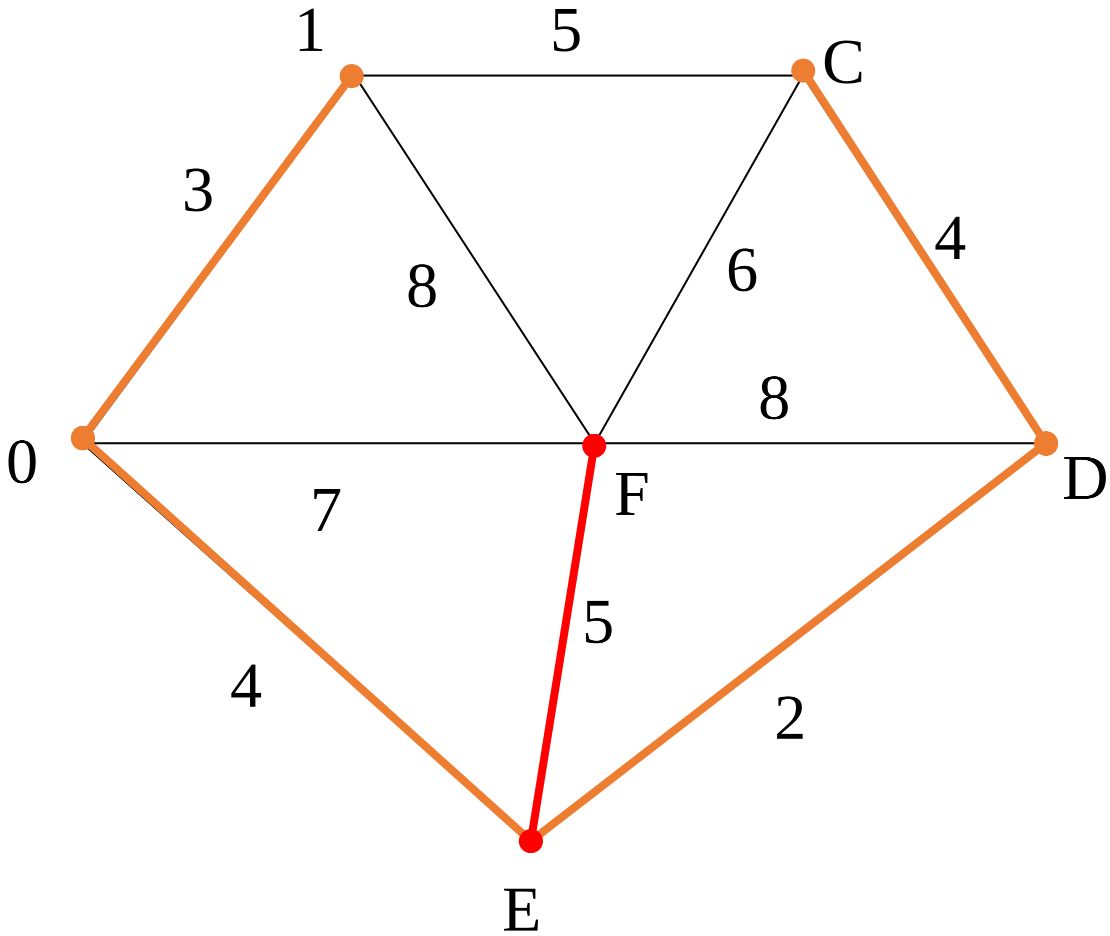
Prim's Algorithm



Select the shortest edge connected to any vertex already connected.

DC 4

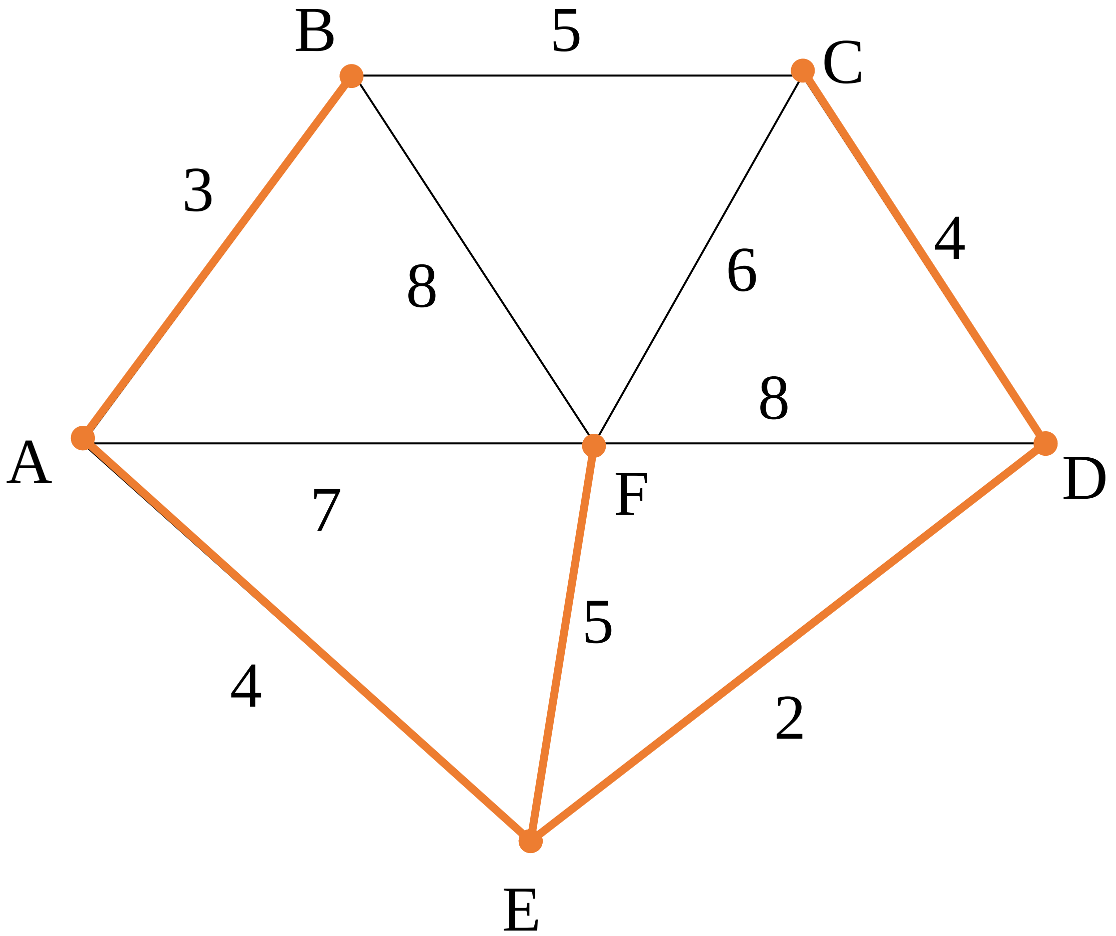
Prim’s Algorithm



Select the shortest edge connected to any vertex already connected.

EF 5

Prim’s Algorithm



All vertices have been connected.

The solution is

AB 3
AE 4
ED 2
DC 4
EF 5

Total weight of tree: 18

Prim's Algorithm

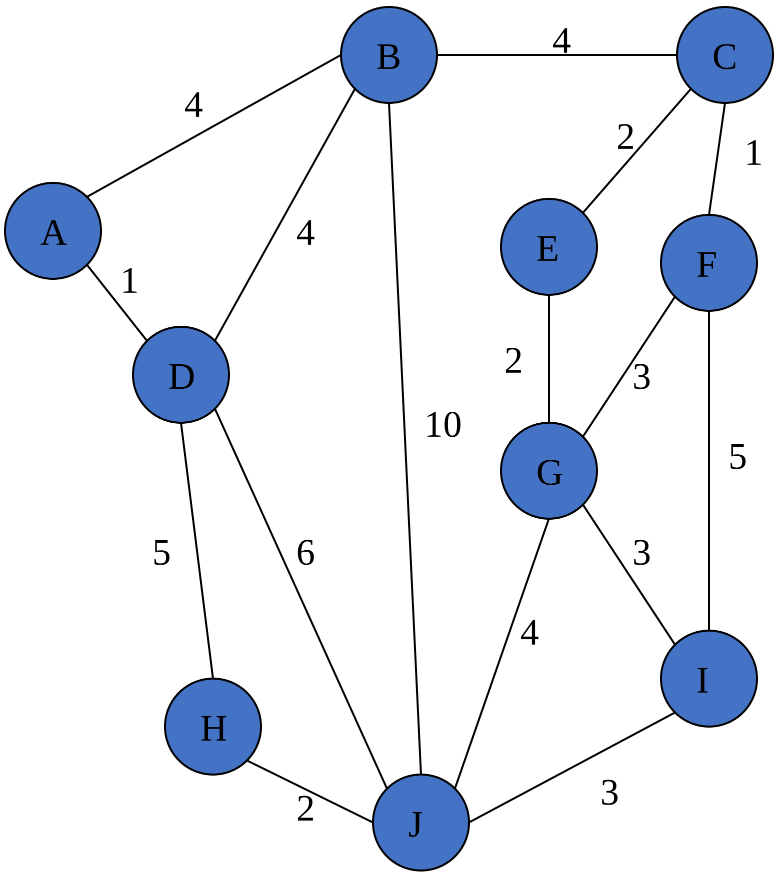
This algorithm starts with one node. It then, one by one, adds a node that is unconnected to the new graph to the new graph, each time selecting the node whose connecting edge has the smallest weight out of the available nodes' connecting edges.

The steps are:

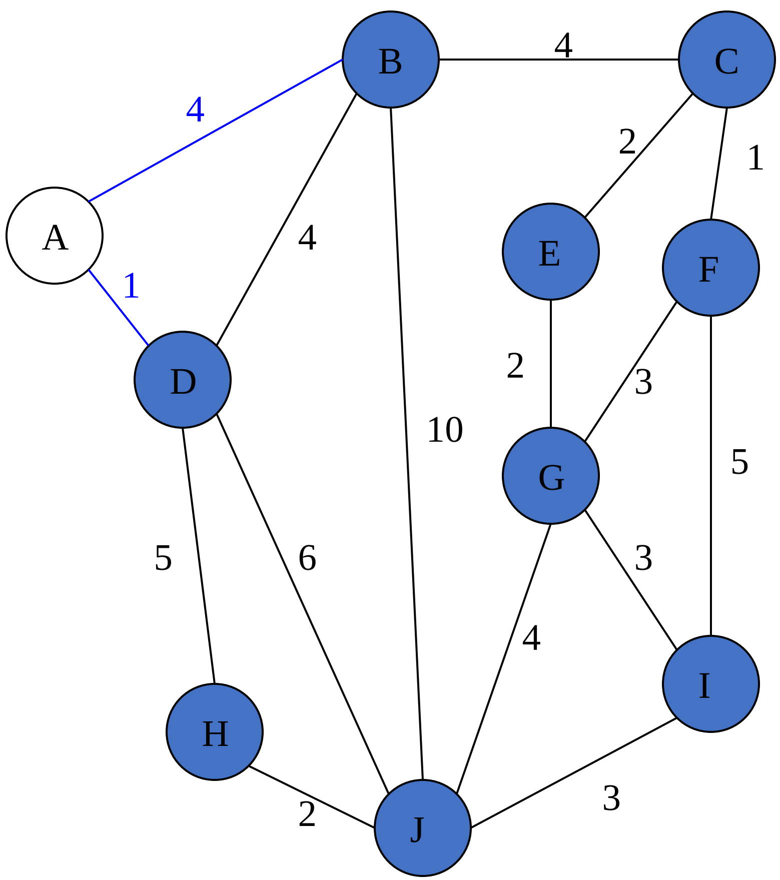
1. The new graph is constructed - with one node from the old graph.
2. While new graph has fewer than n nodes,
    1. Find the node from the old graph with the smallest connecting edge to the new graph,
    2. Add it to the new graph

Every step will have joined one node, so that at the end we will have one graph with all the nodes and it will be a minimum spanning tree of the original graph.

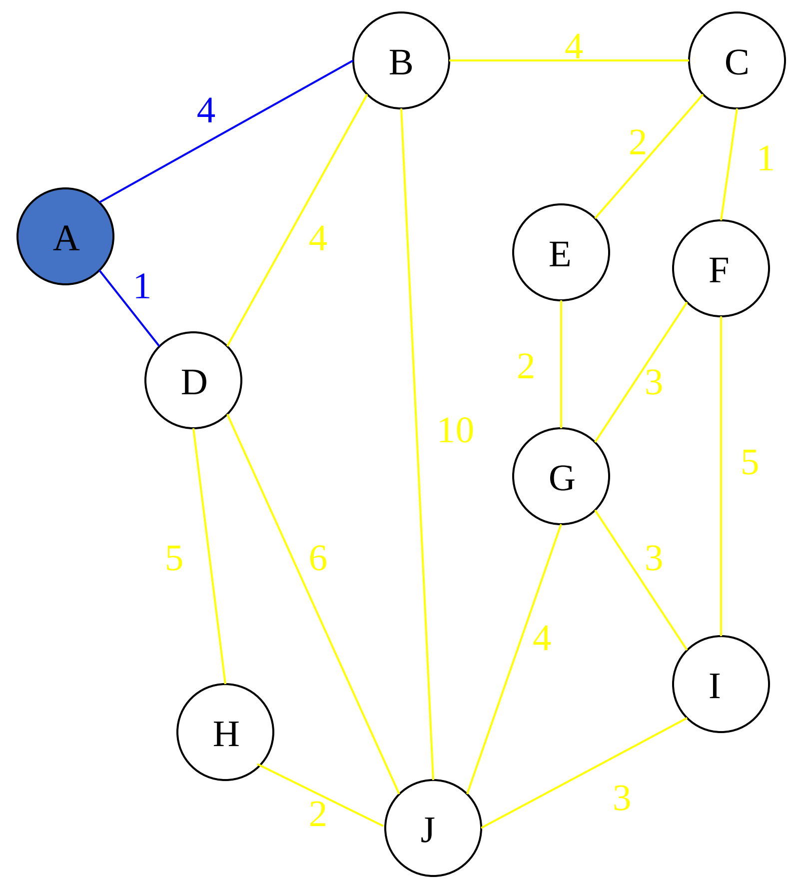
Complete Graph



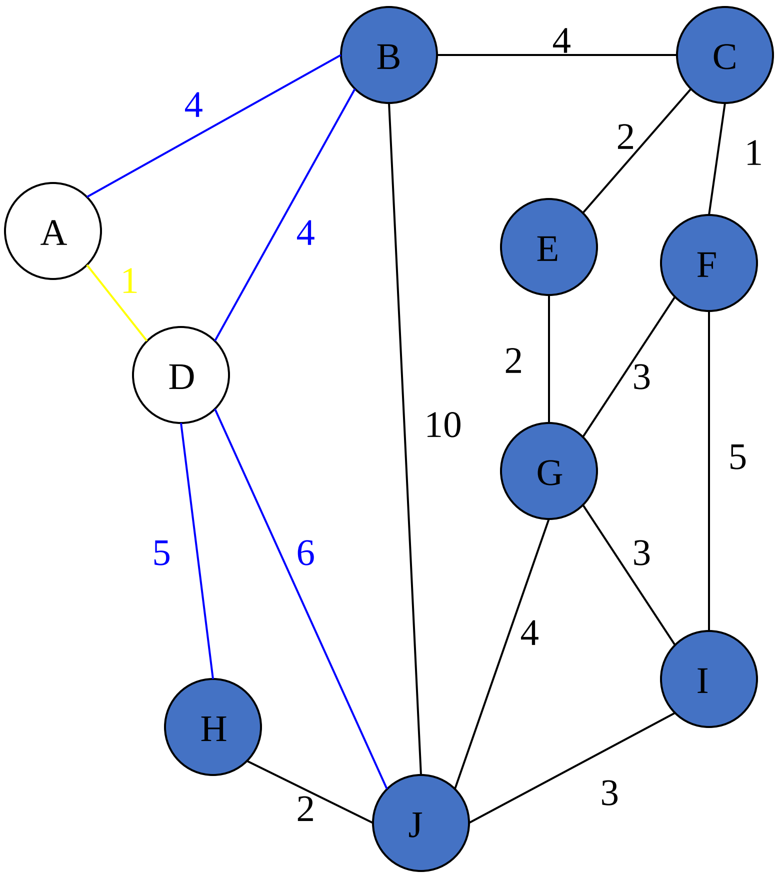
Old Graph



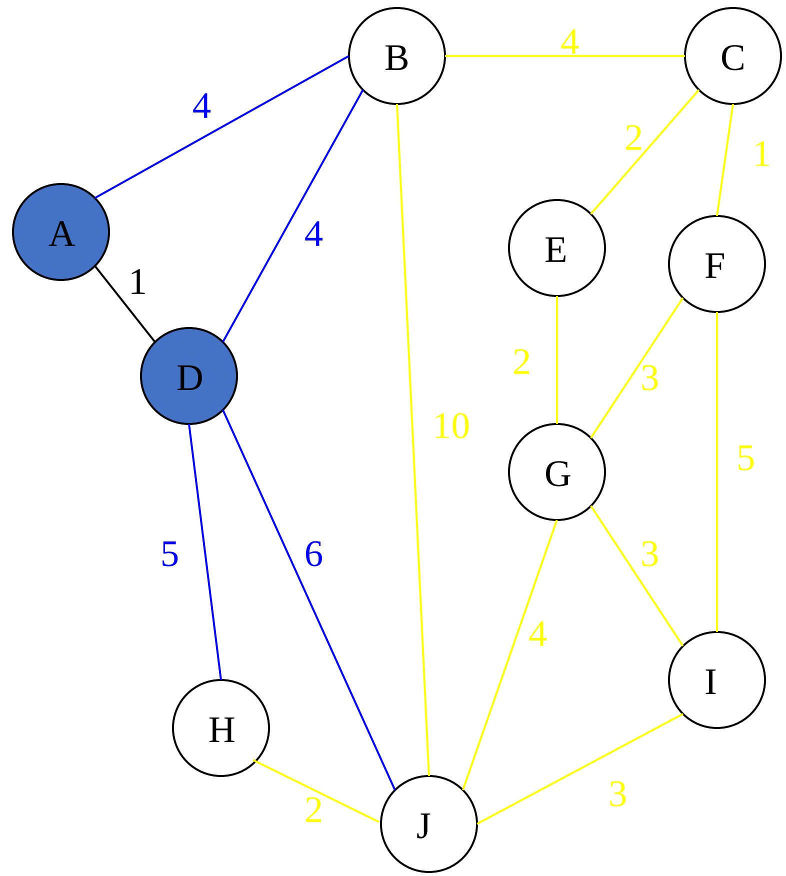
New Graph



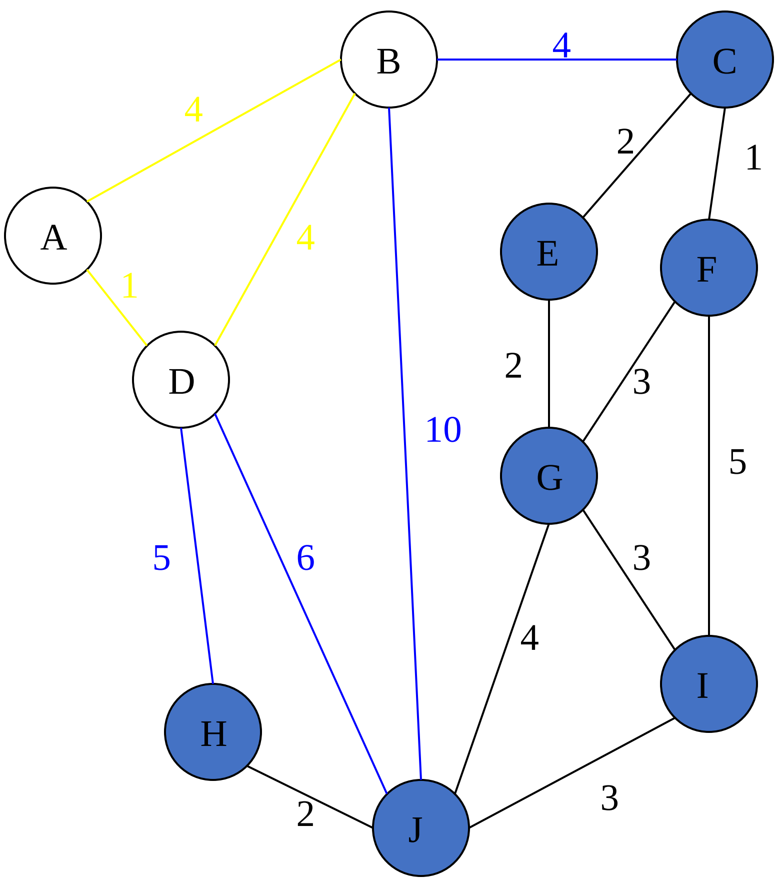
Old Graph



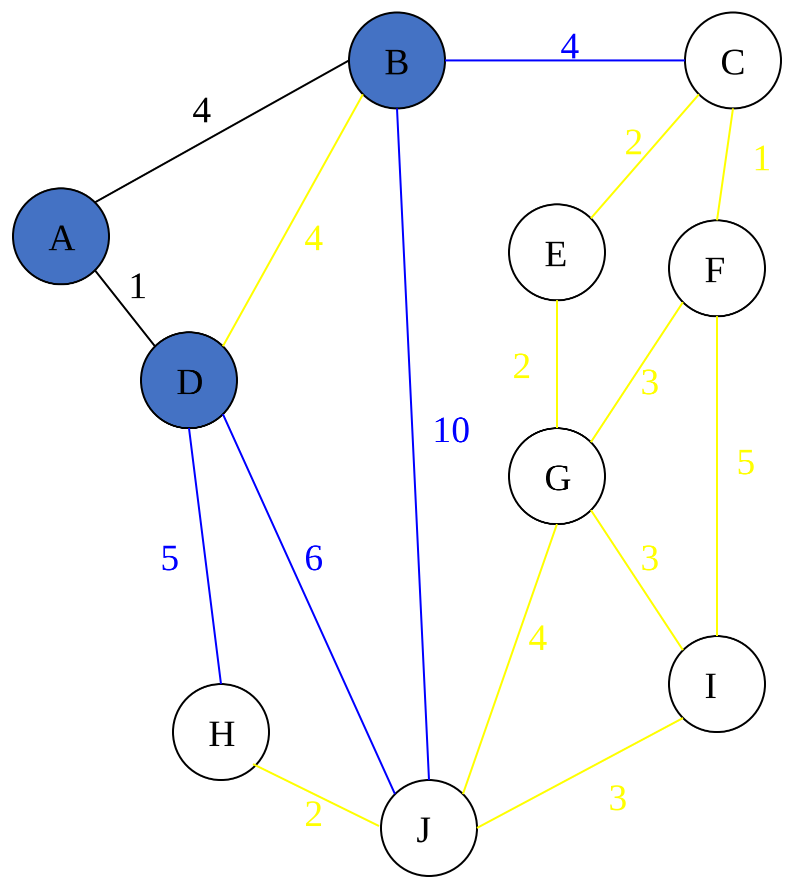
New Graph



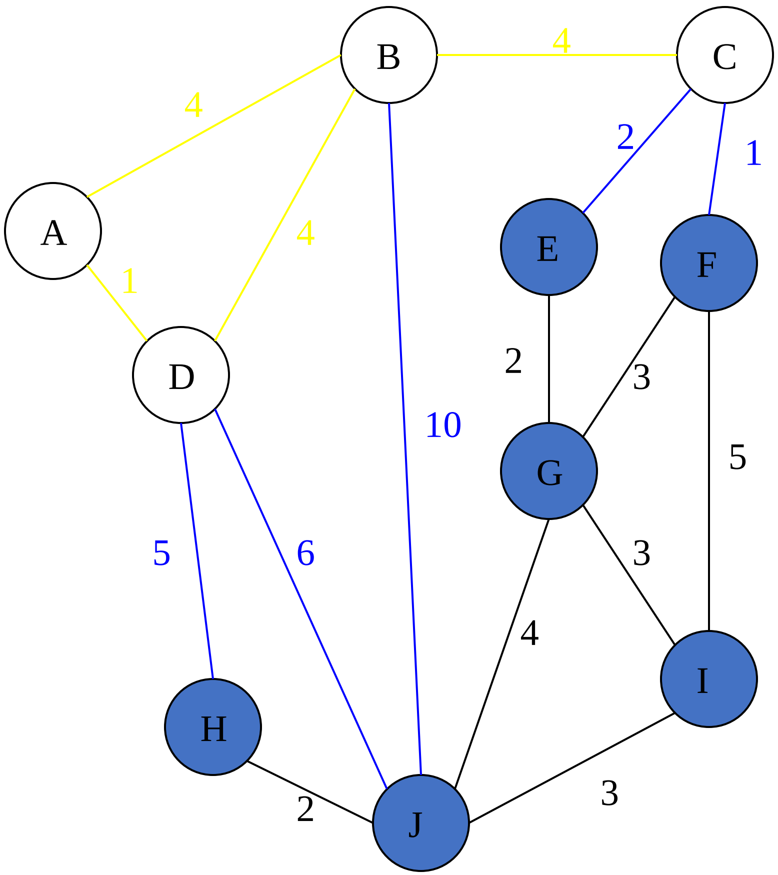
Old Graph



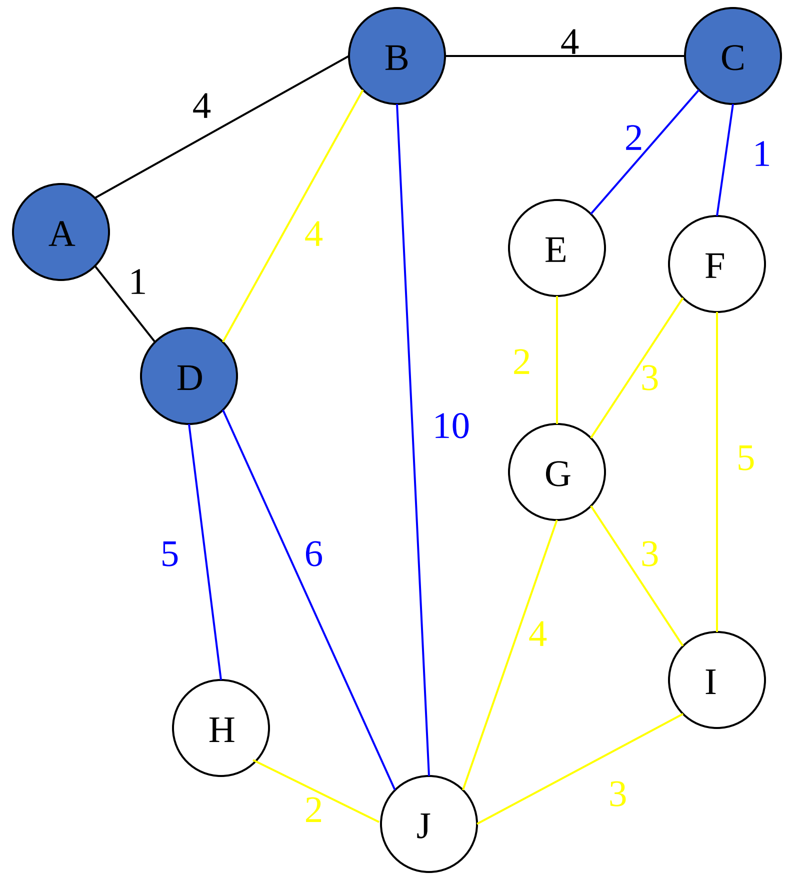
New Graph



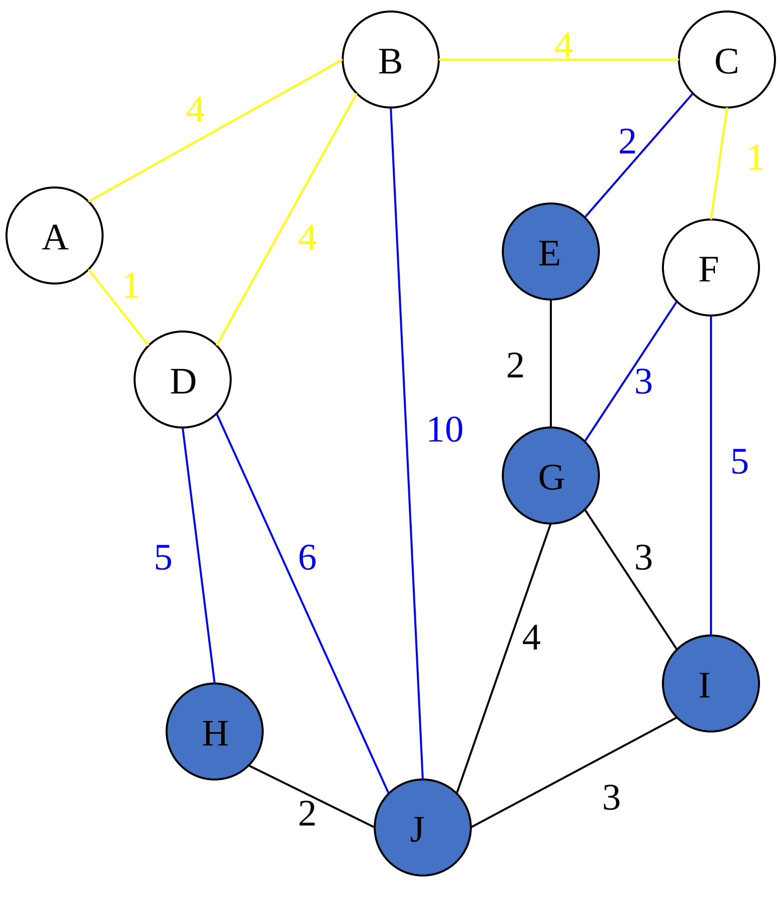
Old Graph



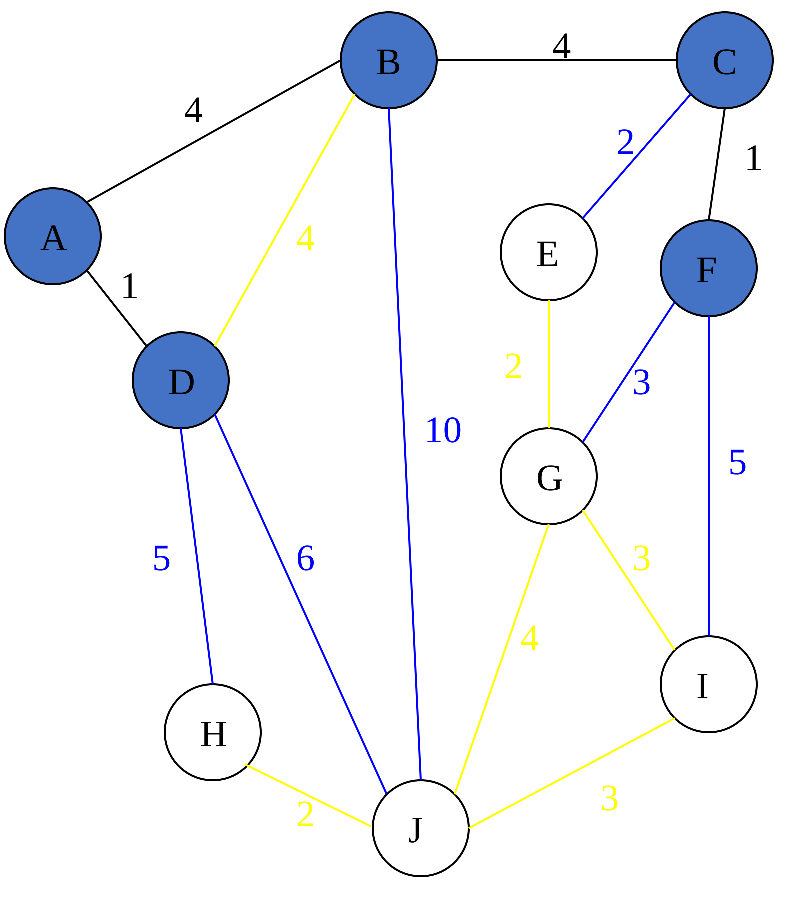
New Graph



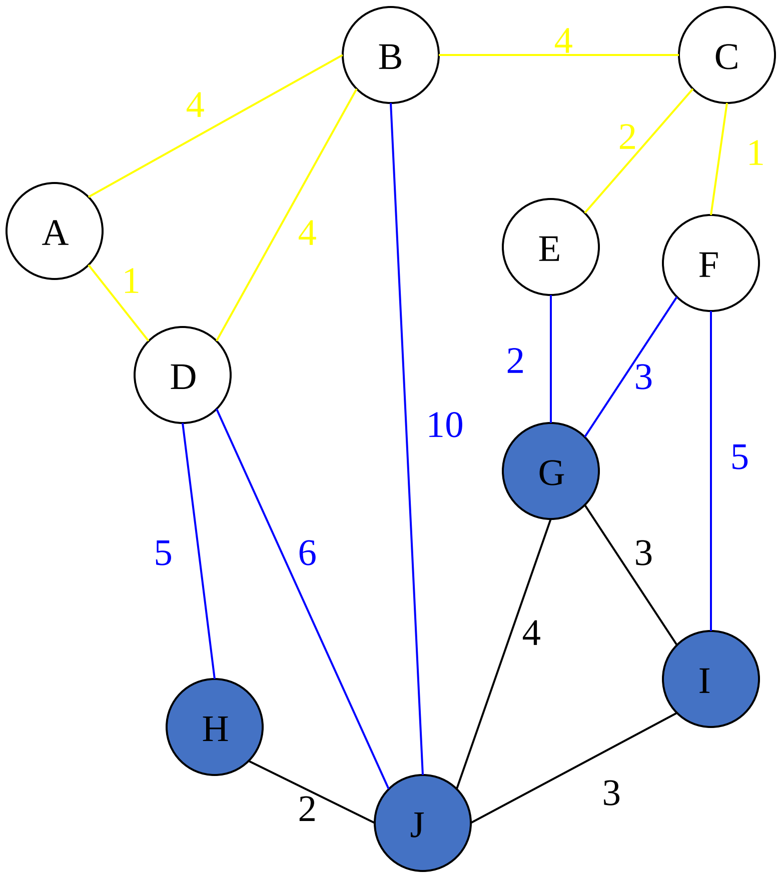
Old Graph



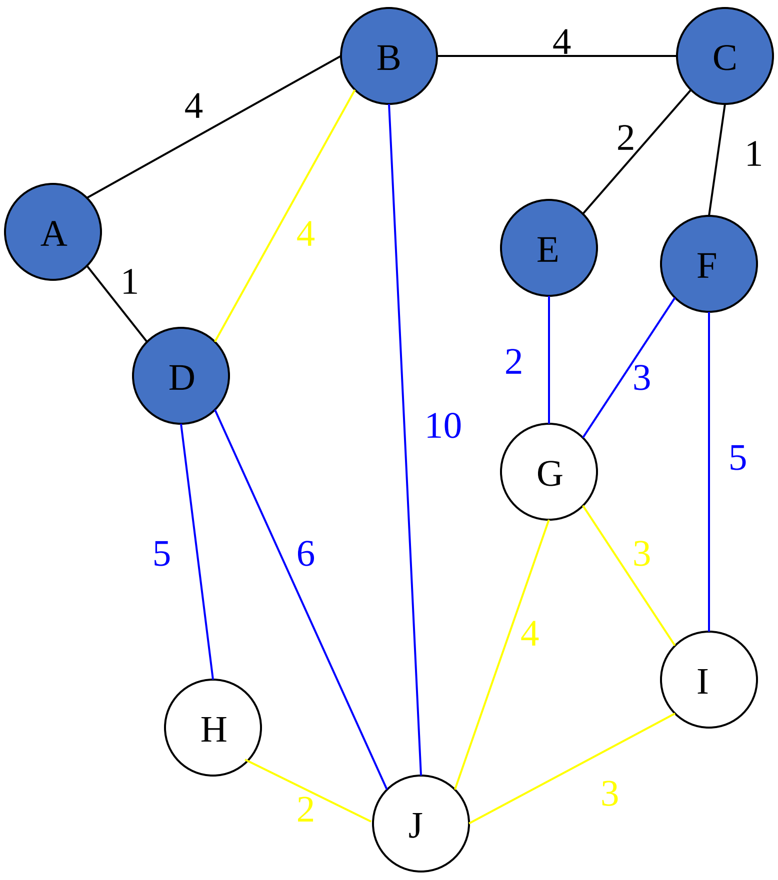
New Graph



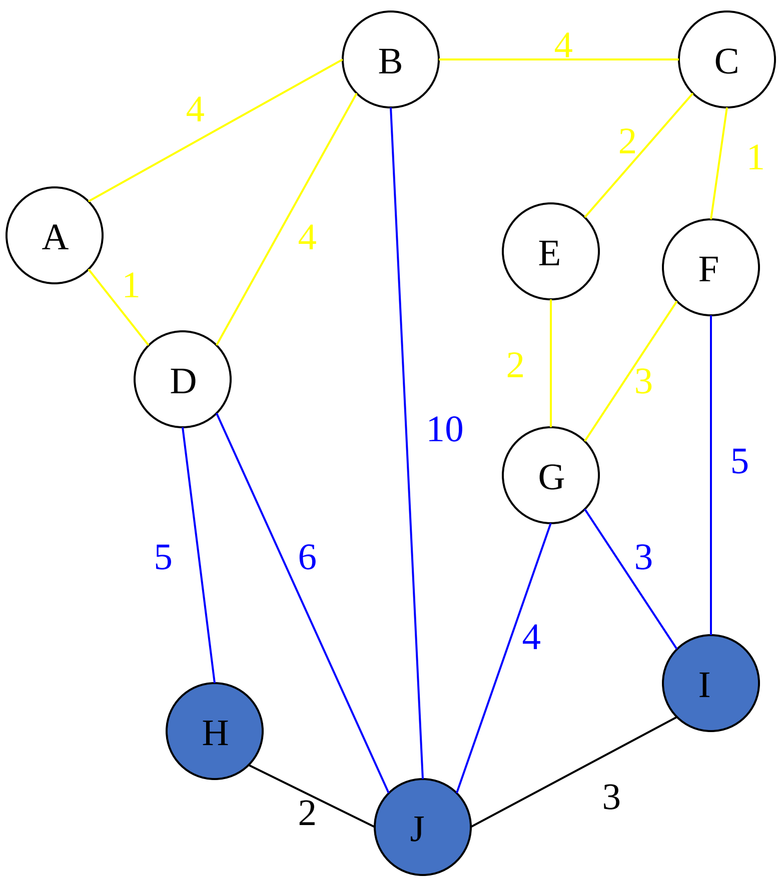
Old Graph



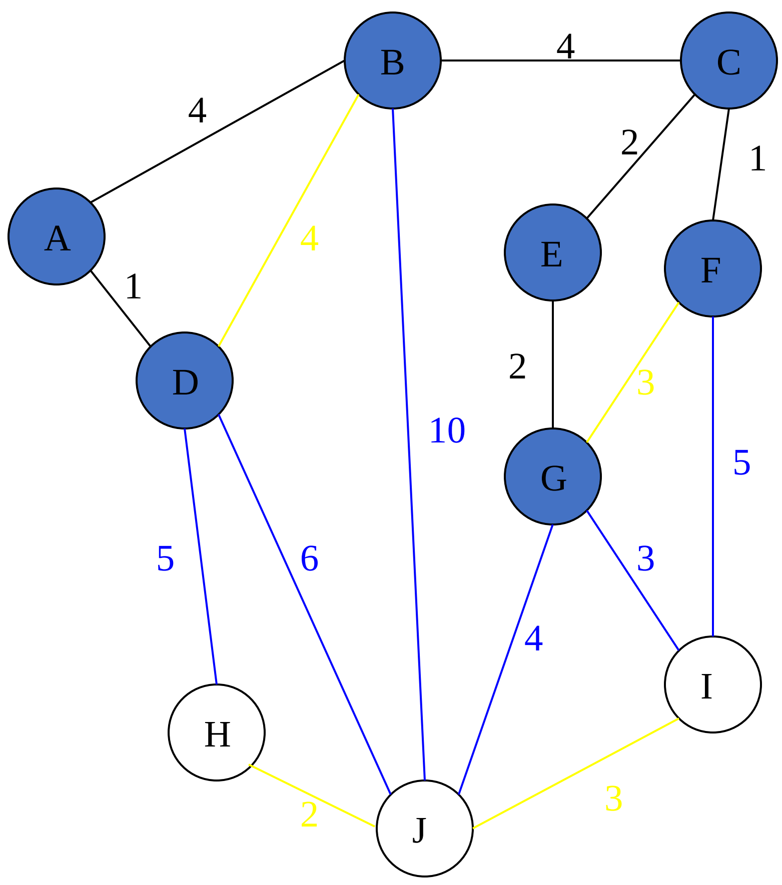
New Graph



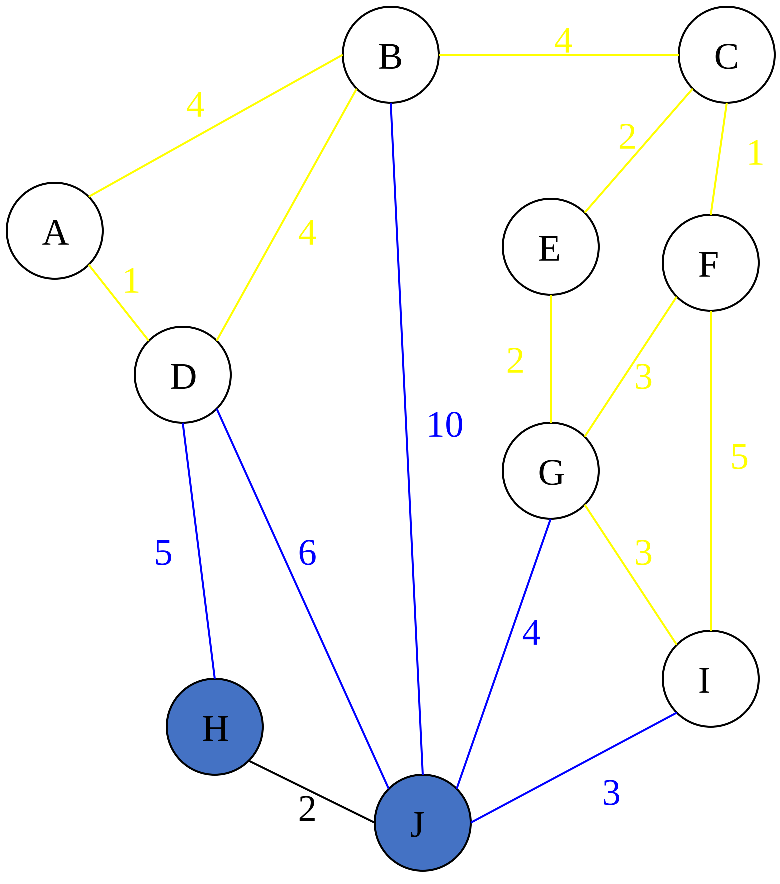
Old Graph



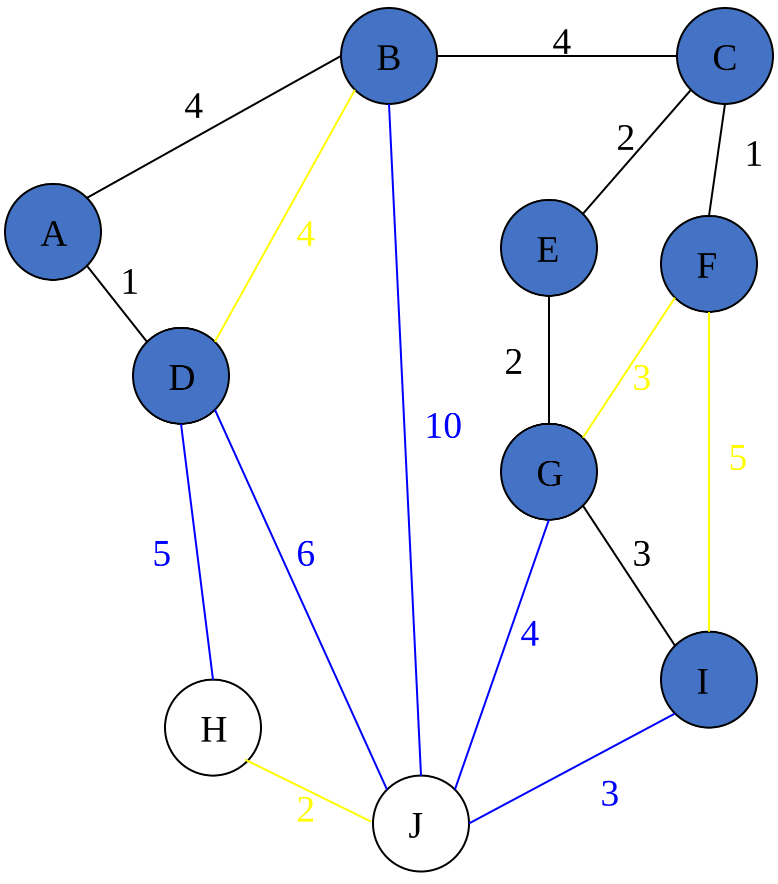
New Graph



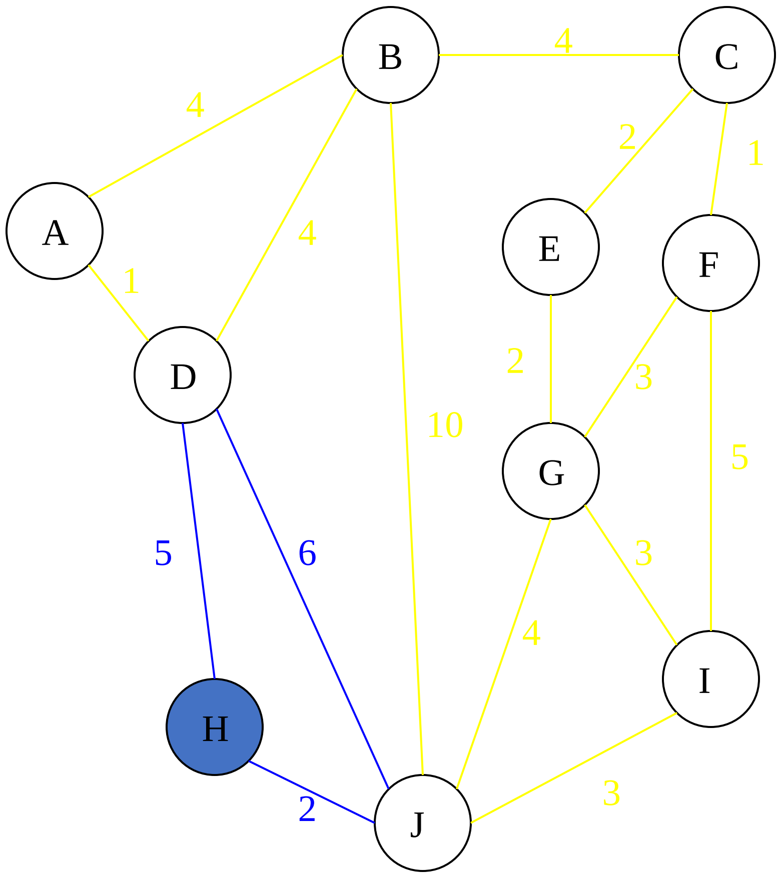
Old Graph



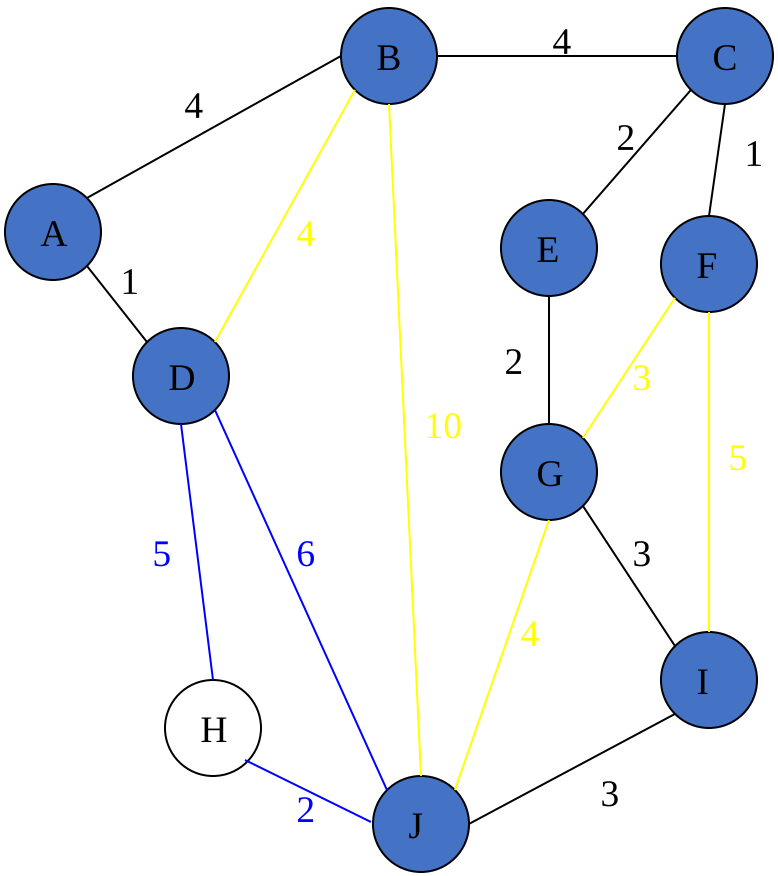
New Graph



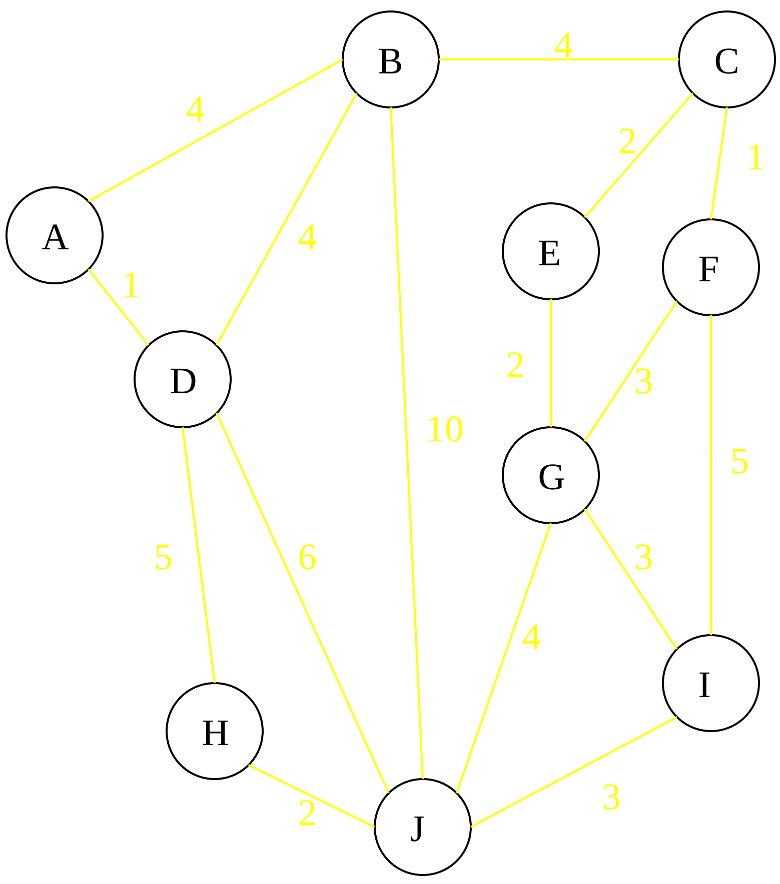
Old Graph



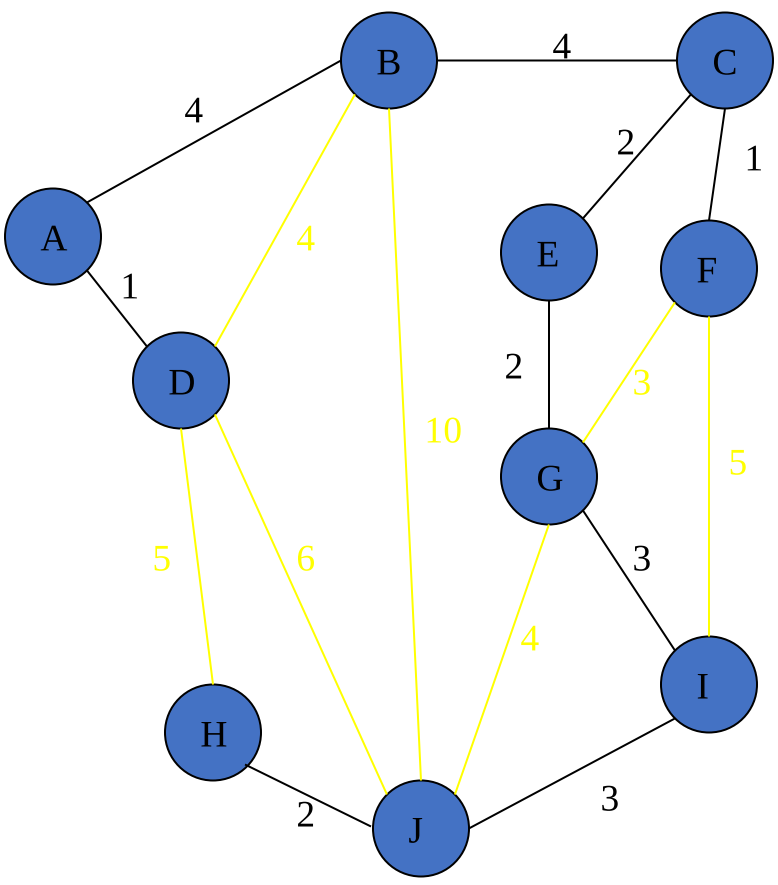
New Graph



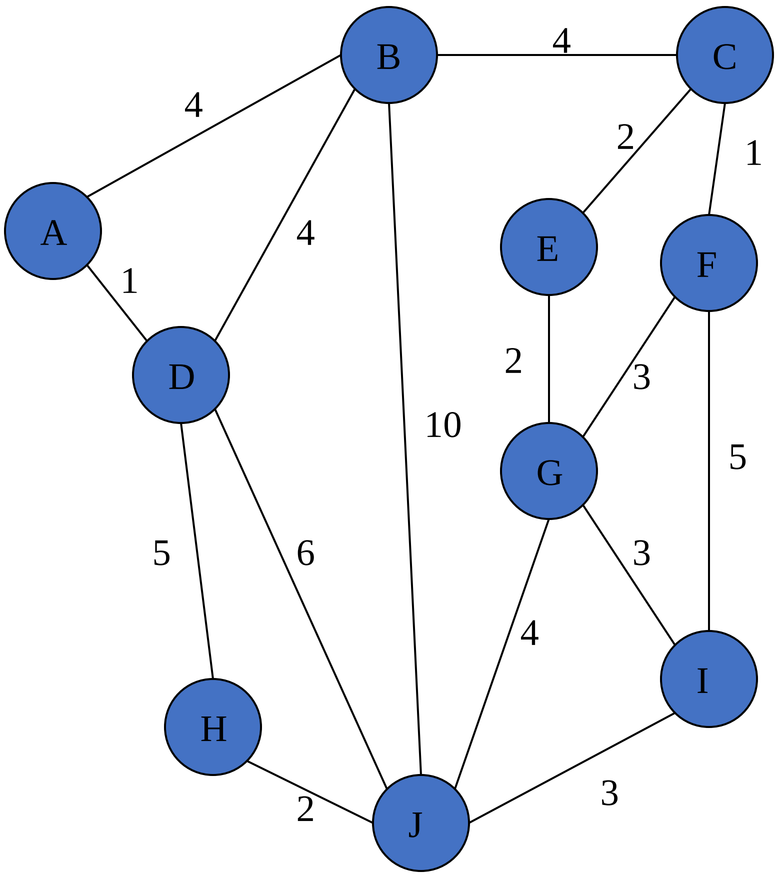
Old Graph



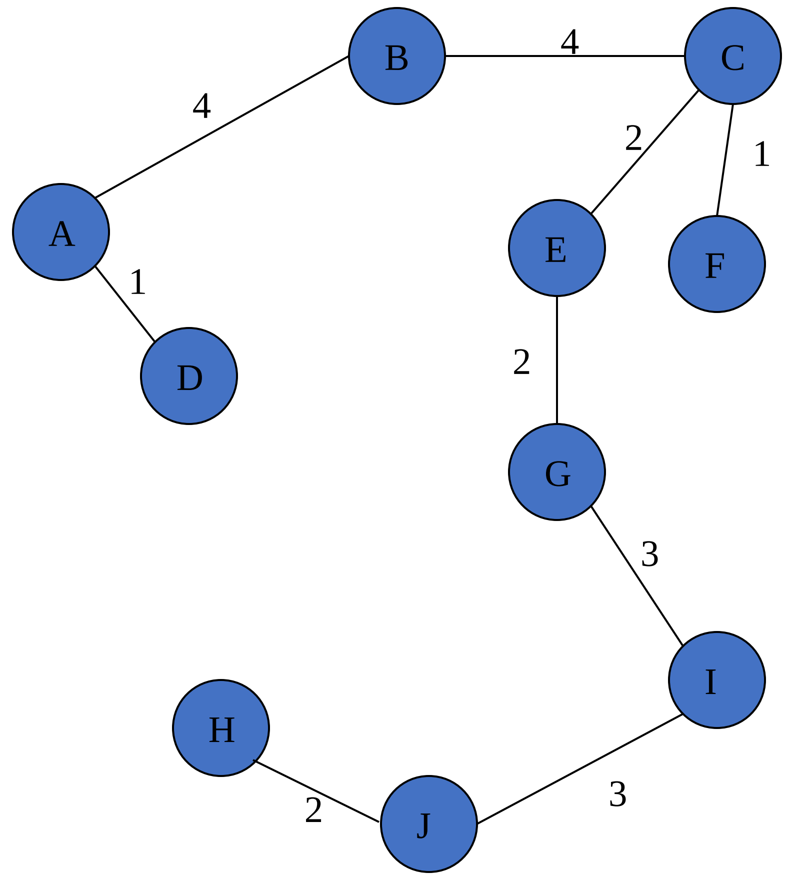
New Graph



Complete Graph



Minimum Spanning Tree



Analysis of Prim's Algorithm

Running Time = $O(m + n \log n)$ (m = edges, n = nodes)

For this algorithm the number of nodes needs to be kept to a minimum in addition to the number of edges. For small graphs, the edges matter more, while for large graphs the number of nodes matters more.