Cantors Intersection Theorem & Baire category Theorem

Cantors Intersection Theorem

Let (X,d) be a **complete** metric space. Let {Fn} be a **decreasing** sequence of **closed, non-empty** sets such that **dia(Fn) → 0 as n → ∞**.

Then the intersection of all Fn's consists of a single element.

// The diameter of a set is the maximum distance between any two points in the set.

// "Decreasing" means that F1 ⊇ F2 ⊇ F3 ⊇ ... and so on.

Consider F as the intersection of all Fn.

F is a subset of each Fn.

dia(F) ≤ dia(Fn)

As dia(Fn) → 0,

dia(F) = 0

|F| is at most 1, because if there were 2 or more elements, there would exist a positive distance between those points.

Therefore, |F| is either 0 or 1.

Let xn ∈ Fn (non-empty)

x1 ∈ F1, x2 ∈ F2

xn+1 ∈ Fn+1 ⊆ Fn (decreasing family)

So xn+1 ∈ Fn

We can generalize: xm ∈ Fn whenever m ≥ n

Claim: {xn} is a Cauchy sequence

d(xm, xn) ≤ dia(Fn)

As n → ∞, dia(Fn) → 0

So $d(x_m, x_n) \to 0$

Therefore, $\{x_n\}$ is a Cauchy sequence.

Since X is a complete metric space, $\{x_n\}$ converges to some point x in X.

The subsequence $\{x_{n+1}, x_{n+2}, x_{n+3}, ...\}$ also converges to x.

All elements of this subsequence are in F_n. Since F_n is closed, x must also belong to F_n.

Consequently, x belongs to the intersection of all F_n's.

This proves that F contains at least one element.

As we've already shown that F contains at most one element,

we can conclude that F contains exactly one element.

QED

A set is nowhere dense if the interior of its closure is empty.

A set A is a dense subset of X if the closure of A equals X.\

Consider the set of natural numbers N = {1, 2, 3, 4, ...}

The closure of a set A is defined as the union of A and all its limit points.

The closure of N is N itself.

The interior of the closure of N is empty (ø).