LINEAR ALGEBRA

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→ Linear algebra : the study of linear maps on finite dimensional vector spaces.

→ Problems of Type 1 :

$$a_{11}x_1 + a_{12}x_2 \cdots a_{1n}x_n = b_1$$
$$\vdots$$
$$a_{n1}x_1 + a_{n2}x_2 \cdots a_{nn}x_n = b_n$$

Find a list of numbers ($s_1, s_2 \ldots s_n$) that satisfy this system of equations.

n-tuple

- Gaussian elimination

→ Get the echelon form of the equations i.e.

$$c_{11}x_1 + c_{12}x_2 + c_{13}x_3 + c_{14}x_4 = b_1$$
$$c_{22}x_2 + c_{23}x_3 + c_{24}x_4 = b_2$$
$$c_{33}x_3 + c_{34}x_4 = b_3$$
$$c_{44}x_4 = b_4$$

$m = 4, n = 4$ (m → columns, n → rows)

Thus, going bottom to top, we can get values of x_1, x_2, x_3, x_4

→ solutions can also be derived graphically i.e.

each equation is converted to a coordinate structure, and solutions are found at the places where these structures intersect

(A) 2-variable equations = lines that are not parallel to x / y axis
(B) 1-variable equations: line parallel to axes
(C) 3-variable equations = plane
⋮

$\boxed{\text{✦ Linear Algebra: Hoffman - Kunze}}$

→ Some notations :

matrix of dimension $m \times n$: $\begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \\ a_{m1} & \cdots & a_{mn} \end{bmatrix}$

vector of dimension n : $\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_n \end{bmatrix}$

→ Linear combination of vectors:

$a_1 V_1 + a_2 V_2 + a_3 V_3 \ldots a_n V_n = b$

V_1 ↓ vector

→ Problems of Type - II:

"Given $\vec{V_1}, \vec{V_2} \ldots \vec{V_n}$, is $\vec{b}$ a linear combination of these vectors?

→ terms from the definition:

(i) Linear Map: literally just a fn

(ii) [redacted] finite dimension: a dimension to which more elements can be added

→ vector operation properties:

1) $\forall\, V_1, V_2 \in V$ $\lambda_1 V_1 + \lambda_2 V_2 \in V$, $\lambda_1, \lambda_2 \in \mathbb{R}$

V ↓ vectors

2) $V_1 + V_2 = V_2 + V_1$ (commutative)

$V_1 + (V_2 + V_3) = (V_1 + V_2) + V_3$ (associative)

3) $0 + \vec{V} = \vec{V} = \vec{V} + 0, \ \forall\, \vec{V} \in V$

4) $\vec{V} + (-\vec{V}) = 0$

5) $\lambda(\vec{V_1} + \vec{V_2}) = \lambda \vec{V_1} + \lambda \vec{V_2}$

} forms a vector field

—————— X ——————

07/01/25

* Assignments: ~20%
* Quiz 1: 10%
* Mid-Sem: 15-20%
* Tutorial quizzes: ~%

50%

• Fields

→ A set F, with two binary operations, addition (+) and multiplication (·) satisfying the following rules:

i) addition is commutative i.e. $a+b = b+a \ \forall\, a, b \in F$

ii) addition is associative $a + (b+c) = (a+b) + c \ \forall\, a, b, c \in F$

↳ closure is inherent and present by definition as binary operations cannot give results outside the set.

iii) $\exists$ a unique element '0' s.t. $a + 0 = 0$. $\forall a \in F$

→ not the number 0

↳ additive identity

iv) $\forall\, a \in F$, $\exists\, (-a)$ s.t. $a + (-a) = 0$.

→ additive inverse.

↳ additive identity

} 4 rules of addition

v) Multiplication is commutative

$a \cdot b = b \cdot a \quad \forall\, a, b \in F$

vi) Multiplication is associative

$a(b \cdot c) = (a \cdot b) \cdot c \quad \forall\, a, b, c \in F$

vii) $\exists$ a unique, non-zero element, represented by '1', s.t. (multiplicative identity)

$a \cdot 1 = a \quad \forall\, a \in F$

viii) $\forall\, x \in F, x \neq 0, \exists\ x'$ s.t. $x \cdot x' = 1$.

↳ multiplicative inverse

ix) multiplication is distributive over addition:

$a \cdot (b + c) = a \cdot b + a \cdot c, \quad \forall\, a, b, c \in F$

5 rules of multiplication

i.e. the Field $= (F, +, \cdot)$, if the set F follows the above rules w.r.t. the binary operators

if F is an empty set, it cannot be a field as points 3 and 7 state that the set must contain a particular element.

∴ The set must contain at least 2 elements.

→ can the set contain exactly 2 elements?

+	0	1
0	0	1
1	1	0

·	0	1
0	0	0
1	0	1

$\{0, 1\}$ satisfies all the rules w.r.t + and ·

∴ $(\{0,1\}, +, \cdot)$ is a field

∴ a field can contain exactly two elements.

→ prf: $(-1) \cdot x = -x$

$A: x + (-1)x$

$= 1 \cdot x + (-1) \cdot x$ (multiplicative identity)

$x(1 + (-1))$ (distributive)

$= x \cdot 0 = x \cdot 0 + 0 = x \cdot 0 + (x + (-x)) = (x \cdot 0 + x) + (-x)$

$x(0+1) = x \cdot 1 = x$

$= x + (-x)$

$= 0$

$= x + (-x)$

$= -x = (-1)x$

★ Q1] For a field,
$a, b, c \in F$,
prove:
a) if $ab = bc$, then $a = c$.
b) $a + b = b + c$,
then $a = c$
↳ deadline:
Sunday midnight.

10/01/25.

• **Subfield**:

A set S is a subfield of a field $(F, +, \cdot)$ is $S \subset F$ and $(S, +, \cdot)$ is a field.

Eg. Real numbers are subfield of complex numbers, w.r.t. + and ·

★ Q2] Any subfield of a complex field must contain every rational number. Prove.

• **System of linear equations**
→ unknown scalars have degree '1'.

→ linear equations = coefficients + unknown scalars
↳ all coeffs must be from same field as each other and scalar.
↳ $x_1, x_2 \ldots x_n$

i.e. consider

$$\left.\begin{aligned} A_{11}x_1 + A_{12}x_2 + A_{13}x_3 \cdots A_{1n}x_n &= y_1 \\ A_{21}x_1 + A_{22}x_2 + A_{23}x_3 \cdots A_{2n}x_n &= y_2 \\ &\vdots \\ A_{m1}x_1 + A_{m2}x_2 + A_{m3}x_3 \cdots A_{mn}x_n &= y_m \end{aligned}\right\} \text{system of linear eqs.}$$

→ linear eq

A system in which $y_i = 0$ is called a homogenous. A system with even 1 $y_i \neq 0$ is called a non-homogenous.

Ex. $2x_1 + 3x_2 - 4x_3 = 0$ → 2 equations, 3 unknowns

$x_1 + x_2 + x_3 = 0$

$\times(-2)$ → $-2x_1 - 2x_2 - 2x_3$ +

$0 + x_2 - 6x_3 = 0$

$x_2 = 6x_3$ ∴ $(x_1, x_2, x_3) = (-7x_3, 6x_3, x_3)$

$x_1 = -7x_3$

multiply the system w/ $C_1, C_2 \dots C_m$

∴ $(C_1A_{11} + C_2A_{21} + C_3A_{31} \dots C_mA_{m1})x_1 +$

⋮

$(C_1A_{1m} + C_2A_{2m} + \dots C_nA_{nn})$

$= C_1y_1 + C_2y_2 + C_3y_3 \dots C_ny_n$.

$C_1A_{11}x_1 + C_1A_{12}x_2 + \dots + C_1A_{1n}x_n = C_1y_1$

$C_2A_{21}x_2 + C_2A_{22}x_2 + \dots + C_2A_{2n}x_n = C_2y_2$

⋮

$C_mA_{m1}x_1 + C_mA_{m2}x_2 + \dots = C_my_m$

↳ all solutions of original system = solution of this system but not all feasible solutions of this system = solution of original system.

→ How many linear equations can we form from a system of linear equation?:

by changing the value of the coefficient, we can form infinite linear equations.

i.e. Let us form:

$B_{11}x_1 + B_{12}x_2 + B_{13}x_3 \dots B_{1n}x_n = z_1$

$B_{21}x_1 + B_{22}x_2 + B_{23}x_3 \dots B_{2n}x_n = z_2$

⋮

$B_{m1}x_1 + B_{m2}x_2 + B_{m3}x_3 \dots B_{mn}x_n = z_m$

↳ All the solutions of the original system are solutions of this system, but not vice-versa.

- Matrices and elementary row operations

→ original system of eq. can be written as: $AX = Y$,

where $A = \begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ \vdots & & & \vdots \\ A_{m1} & \cdots & & A_{mn} \end{bmatrix}$, $X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$, $Y = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{bmatrix}$.

→ A matrix is always defined over a field i.e. the entries in a matrix must come from the same field.

A matrix is a function that maps pairs of integers (or any other countable) to a scalar.

i.e. $A(i,j) \in F$

$1 \le i \le m,\ 1 \le j \le n$.

→ To solve $AX = Y$, we will try to reduce the matrices such that they form a system of equations where each equation contains only one unknown scalar.

→ 3 elementary row operations:

1) multiplication of row by a non-zero scalar $(c \in R, c \neq 0)$
2) Replacing a row (say 'r') with a row 'R' that is of the form '$r + c \cdot s$' i.e. row + scalar × another row s
3) Interchanging two rows. $r \underset{\text{and } s}{\rule{0pt}{0pt}} r$

i.e. $M = [M_{ij}]$

1) $e(M) = \begin{bmatrix} \overset{\neq 0}{c} \cdot M_{rj}, & \text{if } i = r \\ M_{ij}, & \text{if } i \neq r \end{bmatrix}$ → doing operations only on row 'r'

elementary function

2) $e(M) = \begin{bmatrix} M_{rj} + c \cdot M_{sj}, & \text{if } i = r \\ M_{ij}, & \text{if } i \neq r \end{bmatrix}$

3) $e(M) = \begin{bmatrix} M_{sj}, & \text{if } i = r \\ M_{rj}, & \text{if } i = s \\ M_{ij}, & \text{if } i \neq r \text{ and } i \neq s \end{bmatrix}$.

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Th/. For each elementary row operation e', there exists a corresponding elementary row operation 'e_1', s.t.

$$e_1(e(A)) = e(e_1(A)) = A,$$

and, e_1 is of the same type as e.

↳ Let A and B be two $m \times n$ matrices defined over a field F. A is row equivalent to B if A can be obtained by performing a finite sequence of elementary row operations on B.

↳ if A is row equivalent to B, B is row equivalent to A.

i.e. $A = e_1(e_2(\cdots e_n(B)))$

Th/. if A and B are two row-equivalent $m \times n$ matrices over F, then, homogenous systems $Ax = 0$ and $Bx = 0$ have the same solutions.

you have pending assignments :)

Prf. Elementary row operations of matrix result in linear combination of that matrix

i.e. since A and B are row equivalent,

A is a linear combination of B, and

B is a linear combination of A

$\therefore$ They are equivalent systems, and have the same set of solutions.

> Q. Prove that row equivalence is an equivalence relation
> ↳ i) define eq. relation + give properties of

→ An $m \times n$ matrix A over F is called a row-reduced matrix if:

i) the first non-zero entry of each non-zero row is 1.

ii) Each column of A which contains leading non-zero entry of a row has all other entries '0'.

→ An $m \times n$ matrix R is row-reduced echelon matrix if:

i) R is row-reduced

ii) All the non-zero rows occur together before all the zero rows

i.e. $\begin{bmatrix} \text{———} \\ \text{———} \\ \text{———} \\ \\ \end{bmatrix}$ }→ Non-zero rows }→ Zero rows

iii) If the non-zero rows are rows $1, 2, \dots r$, where the leading entry of row 'i' occurs in column K_i,

$$K_1 < K_2 < \dots < K_r.$$

> Q. Every $m \times n$ matrix over field F is row equivalent to a row-reduced matrix. Prove.
> → Also row-reduced echelon matrix.

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→ Thm/. Every $m \times n$ matrix is row equivalent to a row-reduced echelon matrix.

Prf: Every $m \times n$ matrix is row equivalent to a row-reduced matrix, which is row equivalent to a row reduced echelon matrix.

→ Homogenous systems always have a solution → trivial solution.

↳ check by making all scalars of X 0.

Ex) $\underset{m \times n}{A} \underset{n \times 1}{X} = \underset{m \times 1}{0}$

$(x_1, x_2 \dots x_n)$

consider $\underset{m \times n}{R} \underset{n \times 1}{X} = \underset{m \times 1}{0}$, R → row-reduced echelon, r non-zero rows

There will be m linear equations, with n variables

$\therefore$ there will be m-r trivial equations
↳ $0 = 0$ form

and r non-trivial equations.

Let the leading non-zero entry of a non-zero row 'i' be K_i
(i is from 1 to r)

$\therefore x_{K_i}$ is a non-zero scalar with coefficient '1' occurring
only in the 'i'th linear equation.

↳ this is because acc. to the conditions of a matrix being row-reduced echelon, only the i^{th} row (corresponding to i^{th} equation) will have a non-zero number in the K_i^{th} column

$\therefore$ after matrix multiplication, only the i^{th} eq contains a non-zero coeff. for the K_i^{th} scalar.

Each of the r equations contains a unique x_{K_i}.

$\therefore$ the remaining n-r scalars are present in any combination in the linear equations
(i.e. there are no constraints on them).

$\therefore$ the equations are of the form:

$$x_{K_1} + \sum_{j=1}^{n-r} C_{1j} \times U_j = 0$$

↓ coefficients, take values from R

↳ free n-r unknown scalars x

$$\vdots$$

$$x_{K_r} + \sum_{j=1}^{n-r} C_{rj} + U_j = 0$$

$\therefore$ if $r < m$, $\exists$ trivial equations,

if $r < n$, $\exists$ at least one non-trivial soln

↳ If there were no free scalars $(n-r=0)$, all the r linear eqs would be $x_{K_i} = 0$ form i.e. the value of every scalar must be 0. i.e. a non-trivial soln. cannot exist.

→ Non-homogenous systems: $AX = B$ form → need not have a solution

we can find solutions using elementary row operations → perform on both sides!!

$$A' = \left[A_{m\times n} \mid Y_{m\times 1} \right]_{m\times(n+1)}$$

↓ after performing elementary row operations

$$R' = \left[R_{m\times n} \mid Z_{m\times 1} \right]_{m\times(n+1)}$$

↓

row reduced echelon matrix

consider $R_{m\times n}$ has 'r' non-zero rows

∴ it has $m-r$ zero rows

⇓

we can cross-check this w/ the last $m-r$ rows of Z.

i.e. all coeffs of all scalar in that eq is 0

∴ $0 = Z_i$

i.e. Z_i must be 0 for solution to exist.

if they are not all zero, eq is not consistent and there is no solution.

if they are all zero, eq is consistent and there is a solution.

> ✦ Prove: If A is an $m\times n$ matrix, $m<n$, Then, $AX = 0$ always has a non-trivial solution.
>
> Q. A is a square matrix ($n\times n$). $AX = 0$ has only a trivial solution iff A is row-equivalent to an identity matrix.

Ex) $A = \begin{bmatrix} 1 & -2 & 1 \\ 2 & 1 & 1 \\ 0 & 5 & -1 \end{bmatrix}$ $Y = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$; $AX = Y$.

$$A' = \left[\begin{array}{ccc|c} 1 & 0 & 3/5 & 1/5(y_1+2y_2) \\ 0 & 1 & -1/5 & y_2+2y_1 \\ 0 & 0 & 0 & y_3+2y_1-y_2 \end{array}\right]$$

∴ soln. exists if $y_3+2y_1-y_2 = 0$.

eq: $x_1 + \frac{3}{5}x_3 = \frac{1}{5}(y_1+2y_2)$ → free scalar, can be given any value.

$x_2 - \frac{x_3}{5} = y_2 - 2y_1$

$0 = y_3 + 2y_1 - y_2$

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- Matrix multiplication:

let $C = AB$

$B = \begin{bmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_n \end{bmatrix}$ the i^{th} row of C would be:

$\gamma_i = A_{i1}\beta_1 + A_{i2}\beta_2 \cdots + A_{in}\beta_n$

$$\boxed{\gamma_r = \sum_{j=1}^{n} A_{rj} B_j}$$

$$\boxed{C_{ij} = \sum_{r=1}^{n} A_{ir} B_{rj}}$$

Ex: $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} = \begin{bmatrix} a+4b & 2a+5b & 3a+6b \\ c+4d & 2c+5d & 3c+6d \end{bmatrix}$

→ Exercise, not assignment

Q) $B = [B_1 \ B_2 \ \ldots \ B_p]$

$B_i = \begin{bmatrix} B_{1i} \\ B_{2i} \\ \vdots \\ B_{ni} \end{bmatrix}$ prove that $AB = [AB_1 \ AB_2 \ \ldots \ AB_p]$

n×p matrix

→ Th: A, B, C are matrices over F. AB and (AB)C are defined. Then, BC is defined and A(BC) = (AB)C.

Prf. (AB)C is defined, ∴ no. of columns of AB = no. of rows of C.

Since no. of columns of AB is determined by no. of columns of B,

we can say no. of columns of B = no. of rows of C

∴ BC is defined.

to prove (AB)C = A(BC)

$$[A(BC)]_{ij} = \sum_{r=1}^{n} A_{ir}(BC)_{rj}$$

$$= \sum_{r=1}^{n} A_{ir} \sum_{k=1}^{n} B_{rk} C_{kj}$$

$$[A(BC)]_{ij} = \sum_{r=1}^{n} \sum_{k=1}^{n} A_{ir} B_{rk} C_{kj}$$

we can do this since they are all scalars, so, the summations are associative;

$$= \sum_{r=1}^{n} \left(\sum_{k=1}^{n} A_{ir} B_{rk} \right) C_{kj}$$

also, since A is independent of k, it can be taken into the $\sum_{k=1}^{n}$ summation

$$= \sum_{r=1}^{n} (AB)_{ik} C_{kj}$$

$$= [(AB)C]_{ij}$$

doodle :)

$\therefore [A(BC)]_{ij} = [(AB)C]_{ij} \quad \forall \; i, j$

$\therefore$ we can say $A(BC) = (AB)C$

$\therefore$ Hence proved.

→ A matrix can only be multipied with itself when it is a square matrix

i.e. A^n is well defined only if A is a square matrix.

i.e. $A^p A^q A^r = A^b A^a A^c$ implies $a+b+c = p+q+r$.

Elementary matrix

→ A square matrix A (m×m) is an elementary matrix if it can be obtained by performing a single elementary row operation on an identity matrix

i.e. $A = e(I)$

$\boxed{\text{Prove that theorem} \downarrow}$

↳ do case-by-case

→ Th. Let e be an elementary row operation and E be an elementary matrix s.t. $\underset{m\times m}{E} = e(I)$ ↳ m×m. Then $e(A) = EA$. ↳ m×n

⇒ Corollary: consider ~~equivalent~~ matrices A and B, of dimensions m×n

A and B are row equivalent iff

$B = PA$, where P is the product of elementary matrices.

Prf. $B = e_n(\dots e_2(e_1(A)))$

i.e. $B = e_n(\dots e_2(E_1 A)) \dots \quad B = E_n \cdots E_2 E_1 A$

let $P = E_n \cdots E_2 E_1$

$\therefore B = PA.$

similarly we can prove the reverse, using $E_i A = e_i(A)$

$\therefore$ Hence proved

→ inverTible matrices: A square matrix $A_{m\times m}$ is called a invertible matrix if $\exists$ P and Q s.t

(we will only consider square matrices)

$PA = I_{m\times m}$
↳ called left inverse of A

and

$AQ = I_{m\times m}$
↳ called right inverse of A

i.e. if both left and right inverse exists for the matrix.

Th/ if A is an invertible square matrix and $PA = I = AQ$, then $P = Q$.

i.e left and right inverses of an invertible square matrix are same.

Prf. we know $AQ = I$

and $P \cdot AQ = P \cdot I$

i.e. $(PA)Q = PI$

$IQ = PI$

$\therefore Q = P$

$\therefore$ Hence proved.

we call $P = Q = A^{-1}$

Th/ A and B are $m\times m$ matrices over the same field F.

(a) if A is invertible, A^{-1} is invertible, and $(A^{-1})^{-1} = A$.

(b) if AB exists (i.e. is defined) and A and B are invertible, then AB is also invertible, and $(AB)^{-1} = B^{-1}A^{-1}$.

Prf. a) $AA^{-1} = I$ and $A^{-1}A = I$

$\therefore$ A is the left + right inverse of A^{-1}

$\therefore (A^{-1})^{-1} = A$

b) Let $A\bar{A}^1 = I$

$B\bar{B}^1 = I$

$\therefore$ Let $X\,AB = I$

$XAB \cdot \bar{B}^1 = \bar{B}^1$

$XA\,I = \bar{B}^1$

$XA\,\bar{A}^1 = \bar{B}^1\bar{A}^1$

$\therefore X = \bar{B}^1\bar{A}^1$

$\therefore$ if $ABX = I$, similarly, we get $X = \bar{B}^1\bar{A}^1$

$\therefore \bar{B}^1\bar{A}^1$ is left + right inverse of AB

$\therefore AB$ is invertible and $(AB)^{-1} = \bar{B}^1\bar{A}^1$.

Th, For a square matrix $A_{n\times n}$, the following are equivalent:

i) A is invertible → row eq. to I ✧✧

ii) Homogenous system $AX=0$ has only trivial solution

iii) Non-homogenous system $AX=Y$ has a solution X for every $Y_{n\times 1}$.

↳ use (i) to prove (ii) and (iii)

✦ Prove this ←

28/01/25

- Vector spaces → learn on your own as well!

→ A vector space, also called a linear space, consists of the following:

i) A field F of scalars

ii) A set V of objects called vectors

iii) A rule called vector addition that associates a vector for any pair of vectors $\vec{\alpha}, \vec{\beta} \in V$, $\vec{\alpha} \neq \vec{\beta}$ s.t:

⇒ add^n is commutative:

$\vec{\alpha} + \vec{\beta} = \vec{\beta} + \vec{\alpha} \quad \forall\, \vec{\alpha}, \vec{\beta} \in V$

⇒ add^n is associative:

$\forall\, \vec{\alpha}, \vec{\beta}, \vec{\gamma} \in V$,

$(\vec{\alpha} + \vec{\beta}) + \vec{\gamma} = \vec{\alpha} + (\vec{\beta} + \vec{\gamma})$.

⇒ There exists a unique vector called zero vector s.t.

$\forall\, \vec{\alpha} \in V$

$\vec{\alpha} + \vec{0} = \vec{\alpha}$ → additive identity

⇒ for each $\vec{\alpha} \in V$, ∃ a unique $-\vec{\alpha} \in V$ s.t.

$\vec{\alpha} + (-\vec{\alpha}) = \vec{0}$ → additive inverse i.e. vector addⁿ is analogous to scalar addⁿ.

(iv) $\exists$ a rule called scalar multiplication that maps every pair of a scalar $c \in F$ of a vector $\vec{\alpha} \in V$, a vector $c\vec{\alpha} \in V$ s.t.

$\Rightarrow$ $1 \cdot \vec{\alpha} = \vec{\alpha}, \ \forall \vec{\alpha} \in V$

$\Rightarrow$ $c_1(c_2\vec{\alpha}) = c_2(c_1\vec{\alpha})$

$\Rightarrow$ $c(\vec{\alpha}+\vec{\beta}) = c\vec{\alpha} + c\vec{\beta} \ \forall \ c \in F$ and $\forall \ \vec{\alpha}, \vec{\beta} \in V$

$\Rightarrow$ $(c_1+c_2)\vec{\alpha} = c_1\vec{\alpha} + c_2\vec{\alpha} \ \forall \ c_1, c_2 \in F, \vec{\alpha} \in V$

→ A vector space cannot be empty. it must contain at least one element, the $\vec{0}$ vector.

~~→ Th/ $c(\vec{0}) = \vec{0}$~~

~~$c(\vec{0}) = c(\vec{0}+\vec{0}+\vec{0})$~~

~~i.e. $c(\vec{0}) = c(\theta + \theta + \theta)$~~

~~Th/ : Prf:~~

→ Th/ $c(\vec{0}) = \vec{0}$

Prf:

$c(\vec{0}) = c(\vec{0}+\vec{0})$

$c(\vec{0}) = c(\vec{0}) + c(\vec{0})$

$\therefore c(\vec{0}) = \vec{0}$.

→ Th/ $-\vec{\alpha} = (-1)\vec{\alpha}$

Prf: $\vec{0} \cdot \vec{\alpha} = \vec{0}$

$(1-1)\vec{\alpha} = \vec{0}$

$\vec{\alpha} + (-1)\vec{\alpha} = \vec{0}$

$\therefore (-1)\vec{\alpha} = (-\vec{\alpha})$.

$\therefore$ Hence proved.

→ Th/ if $c\vec{\alpha} = \vec{0}$, either $c = 0$ or $\vec{\alpha}$ is 0

Prf: $c\vec{\alpha} = \vec{0}$ prf somewhere :D

→ Examples of vectors + vector spaces : → any vector space contains a zero vector

1) set of complex numbers, over $\mathbb{R}$ → field

↳ vector space

→ linear combination of vectors :

A vector $\bar{\alpha} \in V$ is called a linear combination of vector $\beta_1, \beta_2 \dots \beta_n \in V$ if

$$\vec{\alpha} = c_1\vec{\beta_1} + c_2\vec{\beta_2} \dots c_n\vec{\beta_n},$$

for some $c_1, \dots c_n \in F$ → any field

04/02/25

→ Vector subspace (or just subspace):

$S \subset V$ is a vector subspace of V if S is also a vector space over the same field as V and has the same addition & multiplication rules as V.

Ex. $\{(0, x_2, x_3)\} \subset \{(x_1, x_2, x_3)\}$ over the field $\mathbb{R}$

→ $\bar{x} + \bar{y} = \{(x_1+y_1, x_2+y_2 \dots x_n+y_n)\}$

$c\bar{x} = \{(cx_1, cx_2 \dots cx_n)\}$

~~Q. A non empty subset $W \subset V$ is a subspace of V iff $\forall$ scalars $c \in F$~~

Q. Solve this ←

Th, A non-empty subset $W \subset V$ is a subspace (→ vector space) iff:

a) for all scalars $c \in F$ and for all scalars and

b) for each pair of vectors $\bar{\alpha}, \bar{\beta} \in W$,

the vector $c\bar{\alpha} + \bar{\beta} \in W$

Th/ Let V be a vector space over field F. The intersection of any collection of subspaces of V is a subspace of V.

Q. An $m \times m$ matrix A is Hermitian if $A_{ij} = (A_{ji})^c$, over complex. Any 2×2 Hermitian matrix is of the form,

$$\begin{bmatrix} z & x+iy \\ x-iy & \omega \end{bmatrix}, \quad x, y, z, \omega \in \mathbb{R}.$$

Prf: Let W_α be a collection of subspaces

∴ to prove: $\bigcap_{\alpha=1}^{n} W_\alpha$ = subspace of V.

∴ $\forall\ \bar{\gamma}, \bar{\beta} \in \bigcap_{\alpha=1}^{n} W_\alpha$ and $\forall\ c \in F$,

$c\bar{\gamma} + \bar{\beta} \in \bigcap_{\alpha=1}^{n} W_\alpha$

→ i.e. $\bar{\gamma}, \bar{\beta}$ also $\in W_\alpha$

∴ $c\bar{\gamma} + \bar{\beta} \in W_\alpha\ \forall\ \alpha$

∴ Hence proved.

→ consider $S = \{\bar{\alpha}_1, \bar{\alpha}_2, \bar{\alpha}_3 \cdots \bar{\alpha}_n\} \subset V$.

∴ There will be a subspace that contain the vectors. ∴ we cannot always find the smallest subspace.

↳ subspaces cannot always be compared (they may different data in them).

→ Let S be a set of vectors in a vector space V. The subspace spanned by S is the intersection of all the subspaces of V contain S.

$\boxed{\text{Q. The subset spanned by a non-empty subset } S \subset V \text{ is the set of all linear combinations of vectors in } S}$

~~Q. The subspace spanned by a non-empty subset~~

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→ If $S_1, S_2 \ldots S_K$ are subsets of a vector space V, then the set of all sums of $\vec{\alpha}_1 + \vec{\alpha}_2 + \cdots + \vec{\alpha}_K$ st $\vec{\alpha}_i \in S_i$ is the called the sum of subsets S_i and is denoted by : $\sum_{i=1}^{K} S_i$.

If $W_1, W_2 \ldots W_K$ are subspaces of V, $\underbrace{W_1 + W_2 + W_3 + \cdots + W_K}$ is also a subspace of V.
↳ spanned by $W_1 \cup W_2 \cdots \cup W_K$.

The sum of all subspace of a vector space is the vector space.

$\boxed{\text{✦ 1. Prove / disprove that } W_1 \cup W_2 \cup \cdots \cup W_K \text{ is a subspace of } V\text{, and that it spans } W_1 + W_2 + \cdots + W_K}$

→ In $AX = 0$,

the vector space containing x is the one containing all $n \times 1$ matrices, and all of its solutions form a subspace.

↳ take $\vec{\alpha} = Y$, $\vec{\beta} = Z$

i.e $AY = 0$, $AZ = 0$

∴ take $c\vec{\alpha} + \vec{\beta} = cY + Z$

if $A(CY+Z)=0$, that means that $CY+Z$ = solution of X

i.e if $\vec{\alpha}, \vec{\beta} \in$ solution set, $c\vec{\alpha}+\vec{\beta} \in$ solution set

∴ solution set = subspace

$\because C \cdot AY + AZ = C(0)+0 = 0$

↳ Have to prove $A(CY) = C(AY)$

$\therefore A(CY+Z)=0$

∴ Hence proved.

∘ Basis

→ A set of vectors 'S', where $S \subset$ vector space V, $\vec{\alpha}_1, \vec{\alpha}_2 \dots \vec{\alpha}_r$ is linearly dependent if $\exists$ scalars $c_1, c_2, \dots c_n$ in the field such that not all $c_i = 0$ and $c_1\vec{\alpha}_1 + c_2\vec{\alpha}_2 \dots c_n\vec{\alpha}_n = \vec{0}$

↳ i.e. at least one non-zero coeff.

If S is not linearly dependent, it is linearly independent.

Ex) $\{(0,0,1), (0,0,2)\} \subset F^3$ → linearly independent because we can choose -ve c_1 and c_2, and zero c_3.

i.e. c_1, c_2 = non-zero

c_3 = zero.

$\{(0,0,1), (0,1,0)\}$ → c_1 non-zero ∴ c_2, c_3 = zero vector

∴ $\underbrace{c_1}_{\text{non-zero}}; \underbrace{c_2, c_3}_{\text{zero}}$

① ∴ $\vec{0}$ is linearly dependent

② any subset of linearly independent set of vector is linearly independent.

③ Any set with linearly dependant subset is linearly dependent

④ A set S is linearly independent iff each finite subset of S is linearly independent.

→ A basis B is a subset of vector space V that is linearly dependent and spans V. ↳ if B is a finite set, then V has finite dimensions

There can be multiple basis sets?

Ex) F^3 defined over $\mathbb{C}$ complex no.s:

$$S = \{ \vec{\alpha_1} = (3, 0, -3), \quad \vec{\alpha_2} = (-1, 1, 2), \quad \vec{\alpha_3} = (4, 2, -2), \quad \vec{\alpha_4} = (2, 1, 1)$$

Here S is ~~dependently~~ linearly dependeant.

~~[illegible]~~ ~~[illegible]~~?

✧ Q. How many ~~linearly independent vectors are in~~ elements are in the basis set of X, the solution space of $A_{m \times n} X = 0$?

Cross check this

↳ given, there are r non-zero rows in the row-reduced échelon form of A.

→ Th, Let V be a vector space, spanned by a finite set of vectors (m vectors): $\{\vec{\beta_1}, \vec{\beta_2} \ldots \vec{\beta_m}\}$. Then, any linearly independent set of vectors in V is finite and contains no more than m elements.

✧ Prove this ↑

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Corollary: All bases of a vector space contain the same number of elements/vectors

→ The dimension of a vector space is the cardinality of any basis of the vector space.

Corollary (a) Any subset of a vector space that contains $>n$ vectors is linearly dependent

(b) no subset of the vector space V containing $<n$ elements/vectors can span V

Lemma. Let S be a linearly independent subset of V. Suppose $\vec{\beta} \in V$ and is not in the subspace spanned by S. Then, $S \cup \{\vec{\beta}\}$. i.e. subset of vectors formed by adjoining $\vec{\beta}$ to S, is linearly independent.

→ i.e dimension of $V \geq n$

Prf: $S = \{\vec{\alpha}_1, \vec{\alpha}_2, \vec{\alpha}_3 \cdots \vec{\alpha}_n\} \subset V$,

$\vec{\beta} \notin \text{span}(S)$, $\vec{\beta} \in V$

i.e. $c_1\vec{\alpha}_1 + c_2\vec{\alpha}_2 \cdots\cdots c_n\vec{\alpha}_n + b\vec{\beta} = \vec{0}$

$$\vec{\beta} = \sum_{i=1}^{n} -(c_i/b)\vec{\alpha}_i$$

→ if $\vec{\beta}$ can be written in this form, that would mean $\vec{\beta} \in \text{span}(S)$, which is not possible.

if $\nexists$ a single c_i that is non-zero, $\vec{\beta}$ will not exist in given form

i.e. if even one $c_i \neq 0$, $\vec{\beta} \in \text{span}(S)$.

$\therefore$ $c_i \; \forall \; i \in [1, n]$ is 0

→ $\vec{\beta}$ cannot be $\vec{0}$ either.

Corollary:
if W is a proper subspace of V and dimension of V is finite., then, dimension of W is finite and less than dimension of V.

will give artist details
- Hands
- Hey Lover!
- make you mine
- maneskin? idk if you'll like it :/
- Blue - Yung Kai
- That's my life (?) - (?)
- Superposition - Young the Giant
- Devil Town - Cavetown (?)
- Struck by lightning
- heart b/w my teeth

pretty popular, so you've prolly heard it but wtv

will give more details :thumbsup:

absolutely not the name of the song, I'll give details later.

also will give details.

Q. → Standard basis of a field F^n:

$$\begin{bmatrix} \{1,0,0 \dots\} \\ \{0,1,0 \dots\} \\ \{0,0,1 \dots 0\} \\ \{0,0 \dots \dots 1\} \end{bmatrix}$$

* Let $A_{n \times n}$ be defined over field F. Consider that the rows of A are linearly independent set of vectors in F. Then, show that A is invertible

↳ $A_{n \times n} = \begin{bmatrix} \vec{\alpha_1} \\ \vdots \\ \vec{\alpha_n} \end{bmatrix}$

collection of n vectors

Th, W_1 and W_2 are finite dimensional spaces of V. Then, $(W_1 + W_2)$ is a finite dimensional space, and:

a) $\dim(W_1) + \dim(W_2) = \dim(W_1 \cap W_2) + \dim(W_1 + W_2)$

* Solve this.

- Coordinates

→ V is a vector plane, B is the ordered basis $= \{\vec{\alpha_1}, \vec{\alpha_2} \dots \vec{\alpha_n}\}$,

where any $\vec{\alpha} \in V = \sum_{i=1}^{n} x_i \vec{\alpha_i}$.

root coordinates do not make sense if basis is not ordered.

→ An ordered basis of a vector space is a sequence of linearly independent vectors that spans the space.

→ Both coordinates of a basis cannot be distinct i.e.

$x_i = y_i \;\; \forall$ bases.

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→ consider an ordered basis $B = \{\bar{\alpha_1}, \bar{\alpha_2} \cdots \bar{\alpha_n}\}$ that spans V

∴ for any $\bar{\alpha} \in V$, $\bar{\alpha}$ can be represented as $\bar{\alpha} = \sum_{i=1}^{n} x_i \bar{\alpha_i}$

the n tuple $(x_1, x_2 \cdots x_n)$ are called the coordinates of $\bar{\alpha}$, and is found in F^n.

it is denoted as $[\bar{\alpha}]_\beta$

Ex) for $\vec{0}$, $x_i = 0 \ \forall i$.

if the basis is changed i.e. basis is now $B' = \{\overline{\alpha_1'}, \ldots \overline{\alpha_n'}\}$,

$$\bar{\alpha} = \sum_{i=1}^{n} x_i \bar{\alpha_i} = \sum_{i=1}^{n} x_i' \overline{\alpha_i'} \quad \text{— (i)}$$

consider $\bar{\alpha} = (x_1, x_2 \ldots x_n)$
$\bar{\beta} = (y_1, y_2 \cdots y_n)$

∴ coordinates of $\bar{\alpha} + \bar{\beta} = \sum_{i=1}^{n} (x_i + y_i)$

i.e. they get added

consider: $[c\bar{\alpha}]_\beta = c[\bar{\alpha}]_\beta$

we can say: ∃ scalars P_{ij} s.t.:

$$\overline{\alpha_j'} = \sum_{i=1}^{n} P_{ij} \bar{\alpha_i} \quad \forall i,j \in \{1,2 \ldots n\}$$

from (i),

$$\bar{\alpha} = \sum_{j=1}^{n} x_j' \overline{\alpha_j'}$$

$$= \bar{\alpha} = \sum_{j=1}^{n} x_j' \left(\sum_{i=1}^{n} P_{ij} \bar{\alpha_i} \right)$$

$$\bar{\alpha} = \sum_{i=1}^{n} \left(\sum_{j=1}^{n} P_{ij} x_j' \right) \bar{\alpha_i} \longrightarrow Px = x'$$

↳ is an $n \times n$ square matrix and is invertible

$$\therefore [\bar{\alpha}]_B = P[\bar{\alpha'}]_{B'}$$

since P is invertible,

$$\boxed{[\bar{\alpha}]_{B'} = P^{-1} [\alpha]_B}$$

Th/ V is a vector space over F, and its dimensions are n. Let B and B' be two ordered basis. Then, $\exists$ a unique $n \times n$ matrix P over F, and P is necessarily invertible, s.t.:

(1) $[\bar{\alpha}]_B = P[\bar{\alpha}]_{B'}$

(2) $P^{-1}[\bar{\alpha}]_{B} = [\bar{\alpha}]_{B'}$,

and columns $P_j = [\bar{\alpha}_j']_B \ \forall j$

Ex) Consider P over $\mathbb{R}^2$,

let $P = \begin{bmatrix} \cos\theta & -\sin\theta \\ +\sin\theta & \cos\theta \end{bmatrix}$

$P^{-1} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}$

• Linear transformations

→ Consider vector spaces V and W over F. A function $T: V \longrightarrow W$ is a linear tranformations if it maps each $\bar{\alpha} \in V$

$\forall c \in F$, and $\forall \alpha, \beta \in V$,

$T(\bar{\alpha} + \bar{\beta}) = T\bar{\alpha} + T\bar{\beta}$

$\therefore T(\bar{0} + \bar{0}) = T(\bar{0}) + T(\bar{0}) = T(\bar{0}) = \bar{0}$

→ no matter which vector spaces you choose, $\exists$ a linear transformation between them.

$$\therefore T\left(\sum_{i=1}^{n} x_i \bar{\alpha_i}\right) = \sum_{i=1}^{n} x_i \left(T\bar{\alpha_i}\right)$$

Th/ Consider a vector space V, W of dimension m over the field F

Let $\{\alpha_1, \alpha_2 \dots \alpha_n\}$ be an ordered basis of V.
↳ linearly independent

Let $\bar{\beta}_1, \bar{\beta}_2 \dots \bar{\beta}_n \in W$ ← these can be linearly dependent or independent. Then, there is precisely one linear transformation $T \cdot V \to W$ s.t.

$$T\bar{\alpha_i} = \bar{\beta_i} \;\forall\; i = \{1, 2 \dots n\}.$$

Ex) Define ($T(x_1\bar{\alpha}_1 + x_2\bar{\alpha}_2 \dots\dots x_n\bar{\alpha}_n)$)

$$\sum_{i=1}^{n} x_i \beta_i$$

go

Q. Thm/ Consider V, a vector space with finite dimensions. The, rank of T + nullity of T = dim(V).
↳ show that null space is subspace
+ image of T
- subspace.

→ image of T, lies in W. ← or range of T, it is a subspace.

→ To know T, it is enough to know image of basis elements of V, rather than the image of V.

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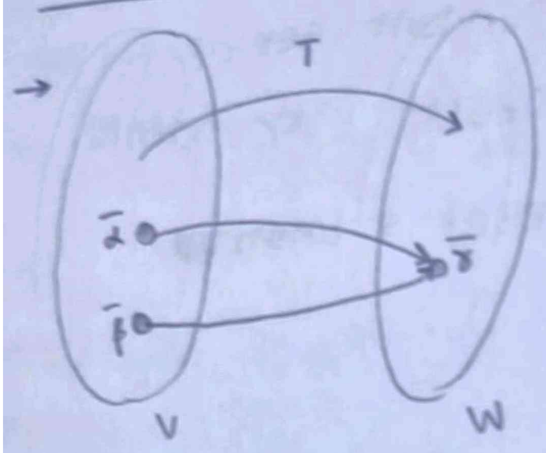


Range(T) = Image(T)

→ T is one-to-one iff if $T\bar{\alpha} = \bar{0}_w$, $\bar{\alpha} = 0_v$

Prf: Let $T\bar{\alpha} = \bar{\gamma}$ and $T\bar{\beta} = \bar{\gamma}$

$$\therefore T\bar{\alpha} = T\bar{\beta}$$

i.e. $T(\bar{\alpha}-\bar{\beta}) = \bar{0}_\omega$, since T is linear

if $\bar{\alpha}$ and $\bar{\beta}$ are distinct, T is many-one.

if T is one-one, $\bar{\alpha}=\bar{\beta}$

i.e. $T(\bar{\alpha}-\bar{\beta}) = \bar{0}_\omega$ means $\bar{\alpha}-\bar{\beta} = \bar{0}_v$

i.e. if $T\bar{\alpha} = \bar{0}_\omega$, $\bar{\alpha} = 0_v$

If $\forall\ T\bar{\alpha} = \bar{0}_\omega$, $\bar{\alpha} = \bar{0}_v$, null space of T is $\{\bar{0}\}$ i.e. T is one-one

Definition:

i.e. null space is $\bar{0}$

→ A linear transformation $T: V \to W$ is non-singular if $T\bar{\alpha} = \bar{0} \Rightarrow \bar{\alpha} = \bar{0}$

Th/ A linear transformation $T: V \to W$ is non-singular iff T carries each linearly independent subset from V to a linearly independent subset in W.

prove each statement separately

Prf:

i) Let T be non-singular, and $\{\alpha_1, \alpha_2 \ldots, \alpha_n\}$ are n linearly independent vectors in V.

$\{\bar{0}\}$: zero vector is always linearly dependent

$$C_1T\bar{\alpha}_1 + C_2T\bar{\alpha}_2 \cdots + C_nT\bar{\alpha}_n = T(C_1\bar{\alpha}_1 + C_2\bar{\alpha}_2 + \cdots + C_n\bar{\alpha}_n) = \bar{0}$$

since T = non-singular,

$$C_1\bar{\alpha}_1 + C_2\bar{\alpha}_2 + \cdots + C_n\bar{\alpha}_n = 0$$

since $\bar{\alpha}_1, \bar{\alpha}_2 \cdots$ are linearly independent, $C_i = 0\ \forall\ i$

∴ Hence proved

arbitrary vector

(ii) Let $\{\bar{\alpha}\}$ be a linearly independent, $\neq \emptyset$ set of V

it is mapped to $\{T\bar{\alpha}\}$ in W

if $\{T\bar{\alpha}\}$ is a linearly independent set in W, $T\bar{\alpha} \neq \bar{0}$

i.e. T is non singular.

∴ Hence proved

Th/ A vector space V over F s.t. dim(v) = n is isomorphic to F^n.

↳ alt. let ∃ $T: V \to F^n$ ↳ 1. Fix a basis for V.

dim(v) = dim(F^n)

∴ if T one-one, T invertible.

∴ V and F^n isomorphic.

— x —

04/03/25

* 2 assignments
2 tut quizzes

A1 → tut quiz 1 → quiz 2 ↓

endsem ← tut quiz 2 ← A2

→ properties of linear transformations (recap).

1. $T(U+V) = T(U) + T(V)$
2. $T(cU) = cT(U)$, where $T: V_n \to W_m$

T can be represented as an m×n matrix A,

s.t. $\underset{m\times n}{A}\underset{n\times 1}{V} = \underset{m\times 1}{W}$

→ properties of matrices: → A rectangular array of matrices,

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ \vdots & & & \\ a_{m1} & \dots & \dots & a_{mn} \end{bmatrix}_{m\times n}$$

$A = [a_{ij}]_{m\times n}$ (m rows, n columns)

a_{ij} ⇓ element in i^{th} row and j^{th} row

1. two matrices are said to be equal if they have the same dimensions, and $a_{ij} = b_{ij}$ ∀ i,j within the bounds of the matrix.

2. Column vector: A matrix with one column
Row vector: A matrix with one row

multiplying row vector × column vector is the equivalent of finding dot product of vectors represented by the matrices.

3. Square matrix: a matrix where the number of rows = number of columns
4. diagonal matrix: a square matrix where $a_{ij} = 0 \ \forall i \neq j$
5. scalar matrix: a diagonal matrix where $a_{ii} = K \ \forall i$
6. identity matrix: a scalar matrix where K=1.

7. zero matrix: every entry of the matrix is zero.

8. triangular matrix:

↳ upper: A square matrix will be called as an upper triangular

$$a_{ij} = 0 \ \forall \ i > j$$

↳ lower: A square matrix will be called as a lower triangular

$$a_{ij} = 0 \ \forall \ i < j$$

→ Matrix Operations:

1. Additional of matrices
2. Multiplication of matrices
3. Transpose of a matrix.

Both matrices must have the same dimensions

if $C = A + B$,

1. $c_{ij} = a_{ij} + b_{ij} = b_{ij} + a_{ij}$ → i.e. addition is commutative
   (i.e. $A+B = B+C$)
2. $(A+B)+C = A+(B+C)$ → addition is associative

if $A+B = A \longrightarrow B = 0$ matrix

$A+B = 0 \longrightarrow B = -A$ (additive inverse)

↳ def: Let A be an $m \times n$ matrix, then $\exists$ a matrix B with $A+B=0$, then B is called the additive inverse of A

$K, \ell \in F \Rightarrow$ scalars

3. $K(\ell A) = (K\ell)A$
4. $(K+\ell)A = KA + \ell A$

$C = \overset{m\times n}{A} \times B \rightarrow n \times p$

i.e. $c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}$

For AB to exist, no. of columns of A = no. of rows of B

if AB exists, BA may not exists → matrix multiplication is not commutative

even if BA exists,

$AB = BA$ is not necessary.

Q. If A and B are square matrices of same dimensions:
is $(A+B)^2 = A^2 + 2AB + B^2$ always true?

A. No, only if $AB = BA$, else $(A+B)^2 = A^2 + AB + BA + B^2$

properties: Let A, B, C be matrices and k be a scalar, such that the following operations are possible.

i) $(AB)C = A(BC)$

ii) $A(B+C) = AB + AC$

iii) $k(AB) = (kA)B = A(kB)$

iv) $AI_m = I_mA = A$ (Here, $A = m \times m$ matrix)

→ If A is a $n \times n$ matrix an inverse of A is an $n \times n$ matrix A' such that $AA' = A'A = I_n$, and A is said to be invertible.

Take an equation: $\underset{\text{invertible}}{A}X = B$,

then, $X = A^{-1}B$ is a unique solution

consider y is a solution other than $A^{-1}B$

i.e. $Ay = B \Rightarrow A^{-1}(Ay) = A^{-1}B = y = A^{-1}B = X$

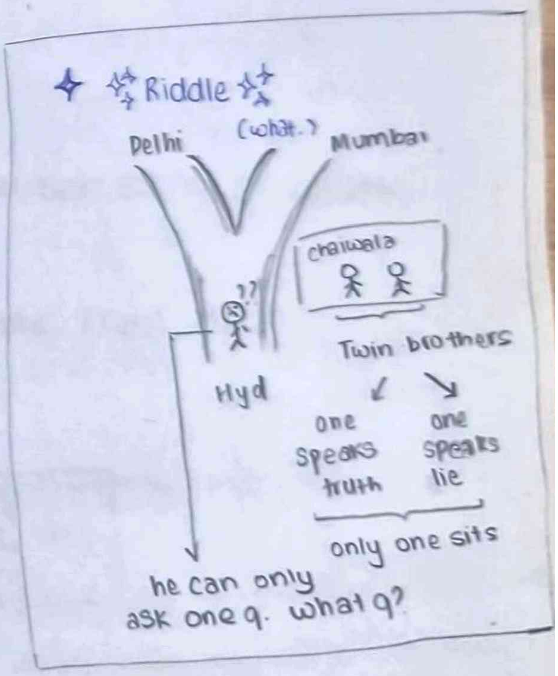
i.e. X is a unique solution.

→ scalar multiplication:

$kA = [ka_{ij}]_{m \times n}$, where $A = [a_{ij}]_{m \times n}$

Q. Is $(M_n(\mathbb{R}), +)$ a vector space under the field $(\mathbb{R}, +, \cdot)$?

yes.



07/03/25

→ Inverse of a matrix

If A is a $n \times n$ matrix, the inverse of A is a $n \times n$ matrix A' such that

$$\boxed{AA' = A'A = I}$$

a zero matrix does not have an inverse

⇒ Th/ If $A_{n\times n}$ is an invertible matrix, then the inverse is unique

Prf: Let A' and A'' be two inverses of A

$$AA' = A'A = I$$
$$AA'' = A''A = I$$

$$\therefore \quad A' = A'I = A'(AA'') = (A'A)A'' = IA'' = A''$$

$$\therefore A' = A''$$

∴ the inverse of the matrix is unique

⇒ if $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $A^{-1} = \dfrac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

A^{-1} exists if $ad-bc \neq 0$

i.e. A is invertible if $ad-bc \neq 0$

$(A)A^{-1} = A^{-1}(A) = I \Rightarrow (A^{-1})A = A(A^{-1}) = I$

$\therefore A^{-1}$ invertible. $A = (A^{-1})^{-1}$

⇒ Properties:

1. If A is an invertible matrix, A^{-1} is also invertible, then $(A^{-1})^{-1} = A$
2. If A is an invertible matrix, c is a non-zero scalar, then

$$(cA)^{-1} = \frac{1}{c}A^{-1}$$

↳ let $\lambda = (cA)^{-1}$

$\lambda(cA) = I$

$c(\lambda A) = I$

$\lambda A = \frac{1}{c}$

$\lambda = \frac{1}{c}A^{-1}$ → do same for $((cA)\lambda = I)$.

3. If A, B are invertible matrices of the same dimensions, AB is invertible, and $(AB)^{-1} = B^{-1}A^{-1}$

↳ Let $\lambda = (AB)^{-1}$

$\lambda(AB) = I$ → $\lambda A = B^{-1}$

$\lambda = B^{-1}A^{-1}$

(do $(AB)\lambda$ also similarly)

4. If A is invertible, then A^n is invertible for all non-negative integers n. ↳ product of invertible matrices are invertible

$(A \cdot A)^{-1} = (A^{-1})(A^{-1})$ $\quad (A^2)^{-1} = (A^{-1})^2$ ↳ By induction, true for A^n as well.

→ Elementary matrices

An elementary matrix is a matrix that can be obtained by performing elementary row operations on an identity matrix.

⇒ $E_1 = \begin{bmatrix} c & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ → multiplying row w/ scalar

$E_2 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ ↳ swapping rows

$E_3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & -2 & 0 & 1 \end{bmatrix}$ ↳ $R_4 = R_4 - 2R_2$ $(R_i = R_i + cR_j)$

Ex) Let $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \\ a_{41} & a_{42} & a_{43} \end{bmatrix}$

$\therefore E_1A = \begin{bmatrix} ca_{11} & ca_{12} & ca_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \\ a_{41} & a_{42} & a_{43} \end{bmatrix}$ $\quad E_2A = \begin{bmatrix} a_{21} & a_{22} & a_{23} \\ a_{11} & a_{12} & a_{13} \\ a_{31} & a_{32} & a_{33} \\ a_{41} & a_{42} & a_{43} \end{bmatrix}$

$E_3A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \\ a_{41} - 2a_{21} & a_{42} - 2a_{22} & a_{43} - 2a_{23} \end{bmatrix}$

→ Let E be an elementary matrix obtained by performing elementary row operations on I_n. If the same elementary row operation is done on a $n \times n$ matrix A, the resultant is same as EA.

→ Fundamental theory of invertible matrices

a) A is a invertible matrix
b) $AX = B$ has a unique solution for any $B \in \mathbb{R}^n$
c) $AX = 0$ has only trivial solution
d) The reduced row echelon form of A is I_n
e) A is a product of elementary matrices

} equivalent statements

Q. Express A as product of elementary matrices.

$$A = \begin{bmatrix} 2 & 3 \\ 1 & 3 \end{bmatrix}$$

A. $\begin{bmatrix} 2 & 3 \\ 1 & 3 \end{bmatrix} \xrightarrow{R_1 = R_1 - R_2} \begin{bmatrix} 1 & 0 \\ 1 & 3 \end{bmatrix} \xrightarrow{R_2 = R_2 - R_1} \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} \xrightarrow{R_2 = R_2 \times 1/3} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$\therefore$ 1. $R_2 = R_2 \times 3 = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$

$\boxed{\text{4}^{\text{th}} \text{ "Tut." quiz on } 19^{\text{th}}}$

2. $R_2 = R_2 + R_1 \quad \begin{bmatrix} 1 & 0 \\ 1 & 3 \end{bmatrix}$

3. $R_1 = R_1 + R_2 = \begin{bmatrix} 2 & 3 \\ 1 & 3 \end{bmatrix}$

$$\therefore \boxed{A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}.}$$

→ Transpose of a matrix

The transpose of a matrix $A = [a_{ij}]_{m \times n}$ is a matrix $B = [b_{ij}]_{n \times m}$ s.t.

$$b_{ij} = a_{ji}$$

⇒ Properties:

1. $(A^T)^T = A$
2. $(kA)^T = kA^T$
3. $(A+B)^T = A^T + B^T$
4. $(AB)^T = B^T A^T$

———— X ————

11/03/25

→ determinant of a matrix:

if $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$, $\det(A)$ or $|A| = a_{11}a_{22} - a_{12}a_{21}$

$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$. $|A| = a_{11}(a_{22}a_{33} - a_{23}a_{32}) - a_{12}(a_{21}a_{33} - a_{31}a_{23}) + a_{13}(a_{21}a_{32} - a_{22}a_{31})$.

∴ $\boxed{\det A = \sum_{j=1}^{n} (-1)^{i+j} a_{ij} \det(A_{ij})}$ → matrix A without the i^{th} row and j^{th} column

→ Laplace expansion: If $A = [a_{ij}]_{m \times n}$ matrix, the determinant of A can be computed as:

$$\det A = a_{i1}C_{i1} + a_{i2}C_{i2} \cdots a_{in}C_{in}$$

$$= \sum_{j=1}^{n} a_{ij}C_{ij}$$ → cofactor expansion along i^{th} row

$$C_{ij} = (-1)^{i+j} \det A_{ij}$$

$\sum_{i=1}^{n} a_{ij} C_{ij}$ → cofactor expansion along j^{th} column

→ Let A be an upper / lower triangular matrix. Then, $\det A$ = product of diagonal elements.

i.e. if $A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{n} \\ 0 & a_{22} & \cdot & \cdot & \vdots \\ 0 & 0 & \cdot & \cdot & \vdots \\ \vdots & 0 & 0 & \cdot & \vdots \\ 0 & \cdots & \cdots & \cdots & a_{nn} \end{bmatrix}$ $\boxed{\det A = \sum_{i=1}^{n} a_{ii}}$

→ properties:

Let $A = [a_{ij}]_{m \times n}$

1. If A has a zero row (column) then $\det A = 0$.

2. If B can be obtained by interchanging two rows / two columns of A, then $\det B = -\det A$ → swap rows, $\det A' = -\det A$ but $A' = A$ so $\det A = \det A' = 0$

3. If A has two identical rows / columns, then $\det A = 0$

4. If B is obtained by multiplying a row/ column of A by a scalar K, then → find determinant along modified row

$\det B = K \det A$

5. If A, B and C are identical except for the i^{th} row, which is different for all three matrices s.t. i^{th} row of C is the sum of i^{th} row of A and i^{th} row of B, then:

$$\det C = \det A + \det B.$$

→ find determinant along i^{th} row.

> ✦ 19th March syllabus: up to class before test

→ determinants of elementary matrices:

1. $E_1 \rightarrow$ swapping rows:

$\det E_1 = -1.$ → $-(\det I) = -1$

2. $E_2 \rightarrow$ multiply K (non-zero scalar) to a row of identity matrix

$\det E_2 = K$ → $K\det(I) = K$

3. $E_3 \rightarrow row1 = row1 + K \cdot row2.$

$\det E_3 = 1.$ → $\det I + K\det B$ (two rows identical) $= \det I = 1.$

⇒ Let B be an $n \times n$ matrix and E be an $n \times n$ elementary matrix, then.

$$\det(EB) = \det(E) \cdot \det(B)$$

→ A square matrix is invertible iff $\det A \neq 0$.

Prf. Let A be an $n \times n$ matrix matrix, and R is the row reduced echelon form of A

Let $E_1, E_2 \dots E_n$ be the elementary matrices corresponding to the row operations applied to A to obtain R i.e. $R = E_n E_{n-1} \cdots E_2 E_1 A$

$$\therefore \det(R) = \underbrace{\det(E_n) \cdot \det(E_{n-1}) \cdots \det(E_2) \cdot \det(E_1)}_{\substack{\text{only has values} \\ -1,\, k,\, +1 \\ \therefore \neq 0}} \det(A)$$

$\therefore \det A \neq 0$ iff $\det R \neq 0$

$\therefore$ if $\det R = 0$, R has a zero row.

$\therefore$ There cannot exist an R^{-1} s.t.

$$R^{-1}R = I \text{ or } RR^{-1} = I$$

$\therefore$ if $\det R = 0$, R is not invertible

$\therefore$ if $\det A = 0$, A is not invertible

$\therefore$ statement proved.

✧ eight 8s to create a thousand :)

——— x ———

18/03/25

→ Cramer's rule

Let A be an invertible n×n matrix and let $b \in \mathbb{R}^n$. Then the unique solution x of the system of equation $Ax = B$ is given by:

$$x_i = \frac{\det(\overset{\text{i}^{\text{th}} \text{ column of A is replaced by column matrix B}}{A_i(B)})}{\det(A)} \quad \text{for } i = 1, 2, 3 \ldots n.$$

(this formula is only applicable if A is invertible i.e. $\det(A) \neq 0$).

Q) $x_1 + 2x_2 = 2$
$-x_1 + 4x_2 = 1$

A.)

$A = \begin{bmatrix} 1 & 2 \\ -1 & 4 \end{bmatrix}$ $B = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ $X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$\det A = 6$
$\det A_1 = 6$
$\det A_2 = 3$

$A_1 = \begin{bmatrix} 2 & 2 \\ 1 & 4 \end{bmatrix}$ $A_2 = \begin{bmatrix} 1 & 2 \\ -1 & 1 \end{bmatrix}$

$\therefore \boxed{x_1 = 1 \quad x_2 = 1/2}$

Q) $A = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 2 & 4 \\ 1 & 3 & -3 \end{bmatrix}$ find ~~adjoint~~ adj(A) and A^{-1}.

A. $cof(A) = \begin{bmatrix} -18 & 10 & 4 \\ 3 & -2 & -1 \\ 10 & -6 & -2 \end{bmatrix}$

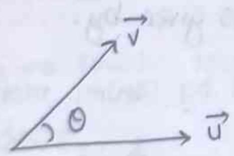
$\det(A) = 1(-18) + 2(10) - 4$
$= -2$

$$\boxed{adj(A) = \begin{bmatrix} -18 & 3 & 10 \\ 10 & -2 & -6 \\ 4 & -1 & -2 \end{bmatrix}}$$

$$\boxed{A^{-1} = \begin{bmatrix} 9 & -3/2 & -5 \\ -5 & 1 & 3 \\ -2 & 1/2 & 1 \end{bmatrix}}$$

→ Inner Product Space

$\vec{u} \cdot \vec{v} = |\vec{u}| \cdot |\vec{v}| \cdot \cos\theta$ is not a closed operation in vector spaces, as the result is a scalar.
↳ scalar / dot product



some closed operations are: cross/vector product, vector addition, etc.

→ An inner product on a vector space V is an ~~op~~ operation that assigns to every pair of vectors in $\vec{\alpha}$ and $\vec{\beta}$ in V a scalar, represented as (α, β).

→ An inner product space is the vector space V over the field F together with an inner product (map):

$$\langle \cdot, \cdot \rangle : V \times V \longrightarrow F$$

that satisfies the following properties:

1. $(\alpha, \beta) = \overline{(\beta, \alpha)}$ [conjugate symmetry]
   ↳ i.e. $\alpha + i\beta \Rightarrow \alpha - i\beta$ conjugate

2. $\langle a\alpha + b\beta, \gamma \rangle = a\langle \alpha, \gamma \rangle + b\langle \beta, \gamma \rangle$ [linearity in first argument]

↳ a, b are scalars $\in F$,

$\alpha, \beta, \gamma \in V$

3. $\langle \alpha, \alpha \rangle$ is always ≥ 0, when $\alpha = 0$ $\langle \alpha, \alpha \rangle = 0$.

[i.e. inner product of a vector with itself is ≥ 0]

(OR)

[positive definiteness].

→ If u and v are two vectors in a inner product space V:

(i) $\underbrace{\text{length of } v}_{\text{norm}}$ is $\|v\| = \sqrt{(v, v)}$

↳ $\| \ \|$ is notation of norm.

(ii) distance between u and v,

$$d(u, v) = \|u - v\| = \sqrt{(u-v), (u-v)}$$

Q) Are $v_1 = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$, $v_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$ and $v_3 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$ orthogonal?

A. $v_1 \cdot v_2 = 2 \cdot 0 + 1 \cdot 1 - 1 \cdot 1 = 0$

$v_1 \cdot v_3 = 2 - 1 \cdot 1 - 1 \cdot 1 = 0$

$v_2 \cdot v_3 = 0 \cdot 1 + 1 \cdot (-1) + 1 \cdot 1 = 0$

$\therefore$ since every $v_i \cdot v_j$ in $\{v_1, v_2, v_3\} = 0$, the three vectors are orthogonal.

Q. Are they linearly independent?

$2x + y - z = 0$

$y + z = 0$

$x - y + z = 0$

the solution for this system is $(0,0,0)$ $\therefore$ the three vectors are linearly independent.

→ If $\{v_1, v_2, \ldots, v_n\}$ is an orthogonal set of non zero vectors in $\mathbb{R}^n$, then, they are linearly independent.

Prf. Let $c_1, c_2 \ldots c_k$ be scalars s.t.:

$$c_1v_1 + c_2v_2 + \cdots + c_kv_k = 0$$

$$\therefore (c_1v_1 + c_2v_2 + \cdots + c_kv_k)\cdot v_j = 0 \cdot v_i = 0$$

$$\therefore c_iv_iv_j = 0$$

$\therefore$ $c_i = 0$, as the vectors are non-zero.

$\therefore$ they are linearly independent.

Q. fn: $\langle u, v\rangle = 2u_1v_1 + 3u_2v_2 \; : (\mathbb{R}^2, +) \longrightarrow (\mathbb{R}, +, \cdot)$.

Is this an inner product? $\left(u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, \; v = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} \right)$

A. $\langle u, v\rangle = 2u_1v_1 + 3u_2v_2$

$\langle v, u\rangle = 2v_1u_1 + 3v_2u_2$

i) $\therefore \langle \overline{v, u}\rangle = 2v_1u_1 + 3v_2u_2 = 2u_1v_1 + 3u_2v_2$

$\langle u, v\rangle = \langle \overline{v, u}\rangle$ $\therefore$ property 1 is satisfied.

ii) Let $u = au' + bu''$

$$\therefore u_1 = au_1' + bu_1''$$
$$u_2 = au_2' + bu_2''$$

$$\therefore \langle u, v\rangle = \langle \underset{\to u_1}{au'} + \underset{\to u_2}{bu''}, v\rangle$$

$$= 2(au_1' + bu_1'')v_1 + 3(au_2' + bu_2'')v_2$$

$$= 2au_1'v_1 + 2bu_1''v_1 + 3au_2'v_2 + 3bu_2''v_2$$

$$= 2au_1'v_1 + 3au_2'v_2 + 2bu_1''v_1 + 3bu_2''v_2$$

$$= a(au_1'v_1 + 3u_2'v_2) + b(2u_1''v_1 + 3u_2''v_2)$$

$$= a\langle u', v_1\rangle + b\langle u'', v_2\rangle$$

$\therefore \langle u, v\rangle = a\langle u' \cdot v_1\rangle + b\langle u'', v_2\rangle$ if

$u = au' + bu''$

$\therefore$ property 2 is satisfied.

(iii) $\langle u,u\rangle = 2u_1^2 + 3u_2^2$ which is always ≥ 0

if $u=0$ i.e. $u_1=0$ and $u_2=0$

$\langle u,u\rangle = 2(0)+3(0)=0$

$\therefore$ property 3 is satisfied.

$\therefore$ $\langle u,v\rangle$ is an inner product, as it satisfies all 3 properties.

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Q) Let $u=\begin{bmatrix} i \\ 1 \end{bmatrix}$, $v=\begin{bmatrix} 2-3i \\ 1+5i \end{bmatrix}$ find i) $u\cdot v$ ii) $\|u\|$ iii) $\|v\|$ iv) $d(u,v)$

[$d(u,u)=\sqrt{\langle u-v, u-v\rangle}$, inner product = dot product]

defined in this qns.

$u\cdot v = \bar{u}_1 v_1 + \bar{u}_2 v_2 + \bar{u}_3 v_3 \ldots \bar{u}_n v_n$

A). $u\cdot v = (-i)(2-3i) + 1(1+5i)$

$= -2i + 3(i^2) + 1 + 5i$

i) $\boxed{u\cdot v = 3i-2}$

ii) $\|u\| = \sqrt{\langle u,u\rangle}$

$\therefore \langle u,u\rangle = (-i)i + 1(1)$

$-i^2+1 = 2$

$\boxed{\therefore \|u\| = \sqrt{2}}$

iii) $\|v\| = \sqrt{\langle v,v\rangle}$

$\langle v,v\rangle = (2+3i)(2-3i) + (1-5i)(1+5i)$

$4-9i^2 + 1-25i^2$

$= 5+25+9 = 39$

$\boxed{\|v\| = \sqrt{39}}$

iv) $d(u,v) = \sqrt{\langle u-v, u-v\rangle}$

$\therefore u-v = \begin{bmatrix} 4i-2 \\ -5i \end{bmatrix}$

$\therefore \langle u-v, u-v\rangle = (-2-4i)(-2+4i) + (5i)(-5i)$

$= 4-16i^2-25i^2 = 45$ $\therefore$ $\boxed{d(u,v) = \sqrt{45}}$

→ consider inner product:

$u \cdot v = \bar{u}_1 v_1 + \bar{u}_2 v_2 + \cdots + \bar{u}_n v_n$

→ properties for all u, v, w in $\mathbb{C}^n$:

i) $u \cdot v = \overline{v \cdot u}$

ii) $u \cdot (v + w) = u \cdot v + u \cdot w$

iii) $(cu) \cdot v = \bar{c}(u \cdot v)$ (or) $u(cv) = c(uv)$

iv) $u \cdot u \geq 0$ and $u \cdot u = 0$ iff $u = 0$

→ If A be a complex matrix, then the conjugate transpose of A given by

$A^* = (\bar{A})^T$

↳ $\bar{A}$ is the matrix obtained from A, where

$\bar{A}[i][j] = [\overline{a_{ij}}]$

∴ if $A = \begin{bmatrix} 2+3i & i & 9 \\ 3i & 4i & 0 \\ 1-i & 5i & 2+i \end{bmatrix}$ find $\bar{A}, A^*$

$\bar{A} = \begin{bmatrix} 2-3i & -i & 9 \\ -3i & -4i & 0 \\ 1+i & -5i & 2-i \end{bmatrix}$, $A^* = \begin{bmatrix} 2-3i & -3i & 1+i \\ -i & -4i & -5i \\ 9 & 0 & 2-i \end{bmatrix}$

→ properties of:

a) $\bar{A}$:

i) $(\bar{\bar{A}}) = A$

ii) $(\overline{cA}) = \bar{c}\bar{A}$

iii) $(\bar{A})^T = (\overline{A^T})$

iv) $(\overline{A+B}) = \bar{A} + \bar{B}$

v) $\overline{AB} = \bar{A}\bar{B}$

b) A^*:

i) $(A^*)^* = A$

ii) $(cA)^* = \bar{c}A^*$

iii) $(A+B)^* = A^* + B^*$

iv) $(AB)^* = B^* A^*$

Q) If $u = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}$, $v = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$ are vectors in $\mathbb{R}^n$, then:

$\langle u, v \rangle = \sum w_i u_i v_i$

↳ where $w_1, w_2 \ldots w_n$ are positive scalar.

A) property 1) $\langle \alpha, \beta \rangle = \overline{\langle \beta, \alpha \rangle}$

$\langle u, v \rangle = \omega_1 u_1 v_1 + \omega_2 u_2 v_2 \cdots \omega_n u_n v_n$

$\langle v, u \rangle = \omega_1 v_1 u_1 + \omega_2 v_2 u_2 \cdots \omega_n v_n u_n$

$\overline{\langle v, u \rangle} = \overline{\omega_1 v_1 u_1 + \omega_2 v_2 u_2 \cdots \omega_n v_n u_n}$

$= \overline{\omega_1 v_1 u_1} + \overline{\omega_2 v_2 u_2} \cdots \overline{\omega_n v_n u_n}$

$= \omega_1 v_1 u_1 + \omega_2 v_2 u_2 \cdots \omega_n v_n u_n = \omega_1 u_1 v_1 + \omega_2 u_2 v_2 + \cdots \omega_n u_n v_n$

$\langle u, v \rangle = \overline{\langle v, u \rangle}$ $\qquad \langle \alpha, \alpha \rangle \geq 0,\ \langle \alpha, \alpha \rangle = 0$
$\alpha = 0$

property 2) Let $u = au' + bu''$

$= \langle u, v \rangle = \langle au' + bu'', v \rangle$

$\langle au' + bu'', v \rangle = \sum \omega_i (au_i' + bu_i'') v_i$

$= \sum \omega_i a u_i' v_i + \sum \omega_i b u_i'' v_i$

$= a \sum \omega_i u_i' v_i + b \sum \omega_i u_i'' v_i$

$= a \langle u', v \rangle + b \langle u'', v \rangle$

$\therefore \langle au' + bu'', v \rangle = a \langle u', v \rangle + b \langle u'', v \rangle$

property 3) $\langle u, u \rangle = \sum \omega_i u_i^2 \geq 0$

if $u = 0$, $u_i = 0 \ \forall i$

$\therefore \sum \omega_i u_i^2 = 0$ i.e. $\langle u, u \rangle = 0$ iff $u = 0$.

$\therefore \langle u, v \rangle$ is an inner product.

→ ~~Orthogonal set of vectors~~

→ ortho~~gonal~~normal set of vectors

A set of vectors is an ortho~~gonal~~normal set of vectors if it is an orthogonal set of unit vectors.

Take $= v_1 = \begin{bmatrix} 1/\sqrt{3} \\ -1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}$, $v_2 = \begin{bmatrix} 1/\sqrt{6} \\ 2/\sqrt{6} \\ 1/\sqrt{6} \end{bmatrix}$ $\quad v_1 \times v_2 = \begin{bmatrix} -1/\sqrt{2} \\ 0 \\ 1/\sqrt{2} \end{bmatrix}$,

$S = \{ v_1, v_2, v_1 \times v_2 \}$ is an orthonormal set.

Th/ Let $\{u_1, u_2, u_3, u_4 \dots u_n\}$ be an orthogonal basis for a subspace W of $\mathbb{R}^n$ and ω be any vector in W. Then, there are unique scalars $c_1, c_2 \dots c_k$ such that.

$$\omega = c_1u_1 + c_2u_2 + \dots + c_ku_k$$

where $\boxed{c_i = \frac{\langle \omega \cdot u_i \rangle}{\langle u_i \cdot u_i \rangle}}$ for $i = 1, 2 \dots k$.

dot product

dot product

Prf. since $\{u_1, u_2 \dots u_n\}$ is a basis of W,

$\exists\ \omega \in W$ s.t.

$$\omega = c_1u_1 + c_2u_2 + \dots + c_ku_k$$

$$\langle \omega \cdot u_i \rangle = c_1u_1 \cdot u_i + c_2u_2 \cdot u_i \dots c_ku_k \cdot u_i$$

$$= c_i(\langle u_i \cdot u_i \rangle)$$

$$c_i = \frac{\langle \omega \cdot u_i \rangle}{\langle u_i \cdot u_i \rangle}$$

∴ Hence product

Q) Let $\omega = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, $v_1 = \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix}$, $v_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$, $v_3 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$. Find coordinates of ω wrt v_1, v_2 and v_3.

A)
$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = c_1\begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} + c_2\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + c_3\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

$$c_i = \frac{\langle \omega \cdot u_i \rangle}{\langle u_i \cdot u_i \rangle}$$

$$c_1 = \left[\frac{2+2-3}{4+1+1}\right] = 1/6 \qquad c_3 = \left(\frac{1-2+3}{1+1+1}\right) = 2/3$$

$$c_2 = \left(\frac{0+2+3}{0+1+1}\right) = 5/2$$

∴ $\boxed{\omega = 1/6\, v_1 + 5/2\, v_2 + 2/3\, v_3}$

→ Let W be a subspace of $\mathbb{R}^n$ and let $\{u_1, u_2, u_3, \dots, u_k\}$ be an orthogonal basis of W.

For any vector $v \in \mathbb{R}^n$, the orthogonal projection of v onto ω is

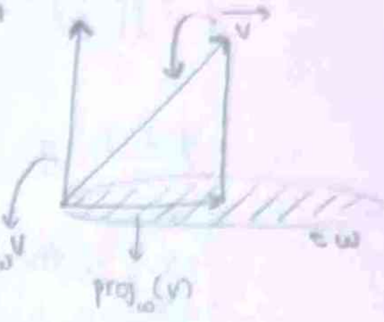
$$\text{proj}_\omega(v) = \frac{\langle u_1 \cdot v\rangle}{\langle u_1 \cdot u_1\rangle} u_1 + \frac{\langle u_2 \cdot v\rangle}{\langle u_2 \cdot u_2\rangle} u_2 + \dots + \frac{\langle u_k \cdot v\rangle}{\langle u_k \cdot u_k\rangle} u_k$$

$$\text{proj}_\omega(v) = \sum_{i=1}^{k} \frac{\langle u_i \cdot v\rangle}{\langle u_i \cdot u_i\rangle} u_i$$

→ dot product

↳ dot product

$$\boxed{\text{perp}_\omega(v) = v - \text{proj}_\omega(v)}$$



—— x ——

25/03/25

Q Find an orthogonal basis for the subspace W of $\mathbb{R}^3$ given by:

$$W = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix}, x - y + 2z = 0 \right\}$$

$\therefore x = y - 2z$

$$\therefore W = \begin{bmatrix} y-2z \\ y \\ z \end{bmatrix} = y\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + z\begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$$

i.e W is a linear combination of:

$$\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \text{ and } \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}.$$

$\therefore W$ is part of span of $\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} \right\}$

$$c_1\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + c_2\begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} = 0$$

$$\therefore \begin{bmatrix} c_1 - 2c_2 \\ c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$\therefore c_1 = c_2 = 0$

since they are linearly independent, this is the basis set.

However, the basis is not orthogonal.

Let $u = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, and let w in W be orthogonal to u

$\therefore \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ s.t. $x - y + 2z = 0$

and

$\begin{bmatrix} x \\ y \\ z \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = 0$ i.e. $x + y = 0$

$\therefore y = z,\ x = -y$

$\therefore \begin{bmatrix} x \\ -x \\ -x \end{bmatrix}$ is the general representation of a vector orthogonal to u

Let us take $\begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}$ (i.e. $x = 0$)

$\therefore \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix} \right\}$ is an orthogonal basis of W.

However, this is trial and error, so, we adopt a proper method:

valid for any inner product space.

→ Gram-Schmidt orthogonalization process:

(to find out an orthogonal basis for subspace W of $\mathbb{R}^n$)

We begin with an arbitrary basis:

$\{x_1, x_2 \ldots x_n\}$, and attempt to orthoganalize it.

Ex. $W = \text{span}\{x_1, x_2\}$,

$x_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, $x_2 = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$

$\therefore v_1 = x_1$

$v_2 = x_2 - \text{proj}_{x_1}(x_2) = \text{perp}_{x_1}(x_2)$

i.e. $v_2 = x_2 - \dfrac{\overbrace{\langle x_1 \cdot x_2 \rangle}^{\text{dot product}}}{\underbrace{\langle x_1 \cdot x_1 \rangle}_{\text{dot product}}} x_1$

$\therefore v_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$

$$v_2 = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} - \frac{(-2)}{(2)} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$$

now, we can see that $\{v_1, v_2\}$ forms an orthogonal basis of W.

$\Rightarrow$ to generalise to K vectors; given $\{x_1, x_2 \dots x_K\}$ as basis:

Step 1. $v_1 = x_1$

Step 2. $v_2 = \text{perp}_{x_1}(x_2) = x_2 - \dfrac{\langle v_1 \cdot x_2 \rangle}{\langle v_1 \cdot v_1 \rangle} v_1$

$$v_3 = \underset{\{v_1, v_2\}}{\text{perp}} (x_3) = x_3 - \underset{\{v_1, v_2\}}{\text{proj}} (x_3)$$

$$v_3 = x_3 - \frac{\langle v_1 \cdot x_3 \rangle}{\langle v_1 \cdot v_1 \rangle} v_1 - \frac{\langle v_2 \cdot x_3 \rangle}{\langle v_2 \cdot v_2 \rangle} v_2$$

$\vdots$

$$v_K = x_K - \left[\sum_{i=1}^{K-1} \frac{\langle v_i \cdot x_K \rangle}{\langle v_i \cdot v_i \rangle} v_i \right]$$

thus, $\{v_1, v_2 \dots v_K\}$ is inherently orthogonal, $\therefore$ also linearly independent

$\therefore \{v_1, v_2 \dots v_K\}$ is an orthogonal basis.

Q. Apply Gram-Schmidt process to find orthonormal basis of W, which is a subspace which is:

$$\text{span}\left\{ x_1 = \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}, x_2 = \begin{bmatrix} 2 \\ 1 \\ 0 \\ 1 \end{bmatrix}, x_3 = \begin{bmatrix} 2 \\ 2 \\ 1 \\ 2 \end{bmatrix} \right\}$$

A. $v_1 = \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}$ $\quad v_2 = \begin{bmatrix} 2 \\ 1 \\ 0 \\ 1 \end{bmatrix} - \dfrac{(2-1+1)}{(1+1+1+1)} \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 0 \\ 1 \end{bmatrix} - \begin{bmatrix} 1/2 \\ -1/2 \\ -1/2 \\ 1/2 \end{bmatrix} = \begin{bmatrix} 1.5 \\ 1.5 \\ 0.5 \\ 0.5 \end{bmatrix}$

i.e $v_2 = \begin{bmatrix} 1.5 \\ 1.5 \\ 0.5 \\ 0.5 \end{bmatrix}$

$$v_3 = \begin{bmatrix} 2 \\ 2 \\ 1 \\ 2 \end{bmatrix} - \frac{(2-2-1+2)}{4}\begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix} - \frac{(3+3+0.5+1)}{5}\begin{bmatrix} 1.5 \\ 1.5 \\ 0.5 \\ 0.5 \end{bmatrix}$$

$$= \begin{bmatrix} 2 \\ 2 \\ 1 \\ 2 \end{bmatrix} - \begin{bmatrix} 0.25 \\ -0.25 \\ -0.25 \\ 0.25 \end{bmatrix} - \begin{bmatrix} 2.25 \\ 2.25 \\ 0.75 \\ 0.75 \end{bmatrix}$$

$$\begin{bmatrix} 2 \\ 2 \\ 1 \\ 2 \end{bmatrix} - \begin{bmatrix} 2.5 \\ 2 \\ 0.5 \\ 1 \end{bmatrix} = \begin{bmatrix} -0.5 \\ 0 \\ 0.5 \\ 1 \end{bmatrix}$$

i.e. $v_3 = \begin{bmatrix} -1/2 \\ 0 \\ 1/2 \\ 1 \end{bmatrix}, v_2 = \begin{bmatrix} 3/2 \\ 3/2 \\ 1/2 \\ 1/2 \end{bmatrix}, v_1 = \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}$

i.e. orthogonal basis is:

$$\left\{ \underbrace{\begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix}}_{v_1}, \underbrace{\begin{bmatrix} 3/2 \\ 3/2 \\ 1/2 \\ 1/2 \end{bmatrix}}_{v_2}, \underbrace{\begin{bmatrix} -1/2 \\ 0 \\ 1/2 \\ 1 \end{bmatrix}}_{v_3} \right\}$$

to get orthonormal basis, we convert this to a set of unit vectors.

i.e., divide each vector by its norm (i.e. length of vector)

$$\|v_1\| = \sqrt{1+1+1+1} = \sqrt{4} = 2$$

$$\|v_2\| = \sqrt{9/4+9/4+1/4+1/4} = \sqrt{5}$$

$$\|v_3\| = \sqrt{1/4+1/4+1} = \sqrt{3/2}$$

$\therefore v_1' = \begin{bmatrix} 1/2 \\ -1/2 \\ -1/2 \\ 1/2 \end{bmatrix}, \quad v_2' = \begin{bmatrix} 3/2\sqrt{5} \\ 3/2\sqrt{5} \\ 1/2\sqrt{5} \\ 1/2\sqrt{5} \end{bmatrix}, \quad v_3' = \begin{bmatrix} -1/\sqrt{6} \\ 0 \\ 1/\sqrt{6} \\ \sqrt{2}/\sqrt{3} \end{bmatrix}$

$$\therefore \boxed{\text{orthonormal basis} = \left\{ \begin{bmatrix} 1/2 \\ -1/2 \\ -1/2 \\ 1/2 \end{bmatrix}, \begin{bmatrix} 3/2\sqrt{5} \\ 3/2\sqrt{5} \\ 1/2\sqrt{5} \\ 1/2\sqrt{5} \end{bmatrix}, \begin{bmatrix} -1/\sqrt{6} \\ 0 \\ 1/\sqrt{6} \\ \sqrt{2}/\sqrt{3} \end{bmatrix} \right\}}$$

→ set of polynomials of degree 2.

Q. Construct an orthonormal basis for $\mathbb{P}_2$ w.r.t. to the inner product:

$$\langle f, g \rangle = \int_{-1}^{1} f(x) g(x)\, dx$$

by applying GS process to the basis $\{1, x, x^2\}$.

A. $v_1 = 1$

$$v_2 = x - \frac{\langle v_1, x \rangle}{\langle v_1, v_1 \rangle} v_1$$

$$= x - \frac{\int_{-1}^{1} 1 \cdot x\, dx}{\int_{-1}^{1} 1 \cdot 1\, dx} \times 1 = x - \frac{[x^2/2]_{-1}^{1}}{[x]_{-1}^{1}} = x \Rightarrow v_2 = x$$

$$v_3 = x^2 - \frac{\langle v_1, x^2 \rangle}{\langle v_1, v_1 \rangle} v_1 - \frac{\langle v_2, x^2 \rangle}{\langle v_2, v_2 \rangle} v_2$$

$$x^2 - \frac{\int_{-1}^{1} 1 \cdot x^2\, dx}{\int_{-1}^{1} 1 \cdot 1\, dx} \times 1 - \frac{\int_{-1}^{1} x \cdot x^2\, dx}{\int_{-1}^{1} x \cdot x\, dx} \times x$$

$$x^2 - \frac{[x^3/3]_{-1}^{1}}{[x]_{-1}^{1}} - \frac{[x^4/4]_{-1}^{1}}{[x^3/3]_{-1}^{1}} \cdot x = x^2 - 1/3 \Rightarrow v_3 = x^2 - 1/3$$

orthogonal basis: $\{1, x, x^2 - 1/3\}$ → these are already normalised, so this is also the orthonormal basis.

→ Cauchy-Schwartz inequality:

Let u and v be vectors in the inner product space V. Then,

$\boxed{|\langle u,v\rangle| \leq \|u\|\cdot\|v\|}$ holds iff u and v are scalar multiples of each other.

⇒ If $u=0$, $|\langle 0, v\rangle| = 0 = \|0\|\cdot\|v\|$.

⇒ If $u \neq 0$, let W be the subspace of V spanned by u,

$$\text{proj}_W(v) = \frac{\langle u,v\rangle}{\langle u,u\rangle}\cdot u$$

$$\text{perp}_W(v) = v - \frac{\langle u,v\rangle}{\langle u,u\rangle} u$$

$$v = \text{proj}_W(v) + \text{perp}_W(v)$$

$$\|v\|^2 = \|\text{proj}_W(v) + \text{perp}_W(v)\|^2$$

$$= \langle (\text{proj}_W(v) + \text{perp}_W(v)), (\text{proj}_W(v) + \text{perp}_W(v)) \rangle$$

$$= \|proj_w(v)\|^2 + \|perp_w(v)\|^2$$ since $\langle proj_w(v), perp_w(v)\rangle = 0$, $\|proj_w(v) + perp_w(v)\|^2 =$

$$\therefore \|v\|^2 = \|proj_w(v)\|^2 + \|perp_w(v)\|^2 \quad -(1)$$

we take:

$$\|proj_w(v)\|^2 \le \|v\|^2$$

$proj_w(v)$ → $\|a\| = \sqrt{\langle a,a\rangle}$

$\therefore \|a\|^2 = \langle a,a\rangle$

$$\langle \frac{\langle u,v\rangle}{\langle u,u\rangle}u, \frac{\langle u,v\rangle}{\langle u,u\rangle}u\rangle$$

$$= \left(\frac{\langle u,v\rangle}{\langle u,u\rangle}\right)^2 \langle u,u\rangle = \frac{(\langle u,v\rangle)^2}{\langle u,u\rangle} = \frac{\langle u,v\rangle^2}{\|u\|^2}$$

$$\therefore \frac{\langle u,v\rangle^2}{\|u\|^2} \le \|v\|^2$$

$$\therefore \langle u,v\rangle^2 \le \|v\|^2 \cdot \|u\|^2$$

i.e. $\boxed{\langle u,v\rangle^2 \le \|u\|^2 \cdot \|v\|^2}$

$\therefore$ Hence proved.

→ triangle inequality:

Let u and v be vectors in an inner product space V, then:

$$\|u+v\| \le \|u\| + \|v\|$$

Prf: $\|u+v\|^2 = \|u\|^2 + \|v\|^2 + 2|\langle u,v\rangle| \le \|u\|^2 + 2\|u\|\|v\| + \|v\|^2$

$$\le (\|u\| + \|v\|)^2$$

$$\therefore \|u+v\|^2 \le (\|u\| + \|v\|)^2$$

$$\therefore \boxed{\|u+v\| \le \|u\| + \|v\|}$$

$\therefore$ Hence proved

28/03/25

→ Orthogonal matrix

The orthogonal decomposition theorem:

Let W be a subspace of $\mathbb{R}^n$ and let u be a vector in $\mathbb{R}^n$. Then, there are unique vectors $w \in W$ and $w^{\perp} \in W^{\perp}$ such that

$W^{\perp}$ → orthogonal complement

$$u = w + w^{\perp}$$
$$= \text{proj}_w(u) + \text{perp}_w(u)$$

⇒ orthogonal complement:

Let W be a subspace of $\mathbb{R}^n$. We say a vector $v \in \mathbb{R}^n$ is orthogonal to W if the vector v is orthogonal to every vector $w \in W$.

The set of all vectors orthogonal to W is called the orthogonal complement of W, represented by W^T.

i.e. $W^T = \{v \in \mathbb{R}^n;\ v \cdot w = 0 \ \forall\ w \in W\}$

Q. Let W be a subspace of $\mathbb{R}^n$, then show that $W^{\perp}$ is also a subspace of $\mathbb{R}^n$.

A. $W^{\perp}$ is a non-empty set. — (I)

$0 \cdot w = 0 \quad \therefore 0 \in W^{\perp}$ — (II)

Let $u, v \in W^{\perp}$,

i.e. $u \cdot w = v \cdot w = 0 \ \forall\ w \in W$

$\therefore (u+v) \cdot w = u \cdot w + v \cdot w = 0 + 0 = 0 \quad \forall\ w \in W$

$\therefore (u+v) \in W^{\perp}$ — (III)

$(cu) \cdot w = c(u \cdot w) = c(0) = 0$

$\therefore (cu) \in W^{\perp}$ — (IV)

$\therefore W^{\perp}$ is a subspace of $\mathbb{R}^n$

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Q. Show $(W^{\perp})^{\perp} = W$.

A. Let $u \in W^{\perp}$

i.e. $u \cdot w = 0 \;\; \forall \; w \in W, \; \forall \; u \in W^{\perp}$

$\therefore \; w \cdot u = 0 \;\; \forall \; w \in W \;\; \forall \; u \in W^{\perp}$

i.e. W is a set of elements that is orthogonal to every element in $W^{\perp}$

$\therefore \; W = (W^{\perp})^{\perp}$

→ Orthogonal matrices

The columns of Q, an $m \times n$ matrix forms an orthonormal set iff:

$Q^T Q = I_n$.

Prf. To show: $Q^T Q_{ij} = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{if } i \neq j \end{cases}$

Let q_i be the i^{th} column of Q. / i^{th} row of Q^T.

$Q^T Q_{ij} = (i^{th}$ row of $Q^T) \cdot (j^{th}$ column of $Q)$

$= (i^{th}$ column of $Q)(j^{th}$ column of $Q)$

i.e. $q_i \cdot q_j$

$= 0$ if $i \neq j$, as columns of Q are orthonormal

1 if $i = j$.

$\therefore \; Q^T Q_{ij} = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases}$

An orthogonal matrix is an $n \times n$ matrix Q whose columns form an orthogonal set.

~~i.e. a matrix for which $Q^TQ = I_n$~~

~~[illegible]~~

Ex) check if $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$ is an orthogonal matrix.

i.e. columns:

$$q_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \quad q_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad q_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

these three vectors are orthonormal,

$\therefore$ matrix is orthogonal.

$\|q_1\| = \|q_2\| = \|q_3\| = 1$

$\therefore$ since they are orthogonal, they are orthonormal.

Q. Take $Q = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{3} & x_1 \\ 0 & 1/\sqrt{3} & x_2 \\ -1/\sqrt{2} & 1/\sqrt{3} & x_3 \end{bmatrix}$. For what values of the missing entries is Q an orthogonal matrix?

A. $x_1^2 + x_2^2 + x_3^2 = 1$

$\frac{1}{\sqrt{2}} x_1 - \frac{1}{\sqrt{2}} x_3 = 0$

$\Rightarrow x_1 = x_3 = K$ (let)

$\frac{1}{\sqrt{3}}(x_1 + x_2 + x_3) = 0$

$\Rightarrow x_1 + x_2 + x_3 = 0,$

$\therefore 2K + x_2 = 0$

$x_2 = -2K$

$\therefore K^2 + (-2K)^2 + K^2 =$

$K = 1/\sqrt{6}$

$$\therefore \boxed{x_1 = 1/\sqrt{6}, \; x_2 = -2/\sqrt{6}, \; x_3 = 1/\sqrt{6}}$$

→ A square matrix Q is orthogonal if $Q^{-1} = Q^T$.

(as $Q^TQ = I_n$, $\therefore$ $Q^T = Q^{-1}$).

→ QR factorization :

Let A be a $m \times n$ matrix with linearly independent columns. Then, A can be factored as $A = QR$, where Q is an $m \times n$ matrix with orthonormal columns, and R is an invertible upper triangular matrix.

Prf. Let $a_1, a_2 \dots a_n$ be the linearly independent columns of A, and let $q_1, q_2, \dots q_n$ be the orthonormal vectors obtained by applying Gram-Schmidt process to $\{a_1, a_2 \dots a_n\}$ with normalization.

$$W_i = \text{span}\{a_1, a_2, \dots a_i\} = \text{span}\{q_1, q_2 \dots q_i\}$$

$$a_1 = r_{11}q_1$$

$$a_2 = r_{12}q_1 + r_{22}q_2$$

$$\vdots$$

$$a_n = r_{1n}q_1 + r_{2n}q_2 \cdots r_{nn}q_n$$

i.e. for $a_i = r_{1i}q_1 + r_{2i}q_2 \cdots r_{ii}q_i \quad \forall\, i$ in $1, 2, 3 \dots n$.,

where r_{ij} is a scalar.

i.e. $$[a_1, a_2 \dots a_n] = [q_1, q_2 \dots q_n] \begin{bmatrix} r_{11} & r_{12} & \cdots & r_{1n} \\ 0 & \ddots & & \vdots \\ 0 & 0 & \ddots & \vdots \\ \vdots & & \cdots & r_{nn} \end{bmatrix}$$

(or)

$$A = QR$$

Q. Work out QR factorization for

$$A = \begin{bmatrix} 1 & 2 & 2 \\ -1 & 1 & 2 \\ -1 & 0 & 1 \\ 1 & 1 & 2 \end{bmatrix}$$

A. $q_1 = \begin{bmatrix} 1/2 \\ -1/2 \\ -1/2 \\ 1/2 \end{bmatrix}$, $q_2 = \begin{bmatrix} 3/2\sqrt{5} \\ 3/2\sqrt{5} \\ 1/2\sqrt{5} \\ 1/2\sqrt{5} \end{bmatrix}$, $q_3 = \begin{bmatrix} -1/\sqrt{6} \\ 0 \\ 1/\sqrt{6} \\ \sqrt{2}/\sqrt{3} \end{bmatrix}$

$a_1 = r_{11} q_1$

$\therefore \boxed{r_{11} = 2}$

$a_2 = r_{12} q_1 + r_{22} q_2$

$$\therefore \begin{bmatrix} 2 \\ 1 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} r_{12}/2 \\ -r_{12}/2 \\ -r_{12}/2 \\ r_{12}/2 \end{bmatrix} + \begin{bmatrix} 3r_{22}/2\sqrt{5} \\ 3r_{22}/2\sqrt{5} \\ r_{22}/2\sqrt{5} \\ r_{22}/2\sqrt{5} \end{bmatrix}$$

$\therefore \frac{r_{12}}{2} + \frac{3r_{22}}{2\sqrt{5}} = 2$

$\frac{r_{12}}{2} - \frac{3r_{22}}{2\sqrt{5}} = -1$

$\therefore \frac{2r_{12}}{2} = 1 \Rightarrow \boxed{r_{12} = 1}$

$\frac{1}{2} + \frac{3r_{22}}{2\sqrt{5}} = 2$

$\boxed{r_{22} = \sqrt{5}}$

$$\begin{bmatrix} 2 \\ 2 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} r_{13}/2 + 3r_{23}/2\sqrt{5} - r_{33}/\sqrt{6} \\ -r_{13}/2 + 3r_{23}/2\sqrt{5} + 0 \\ -r_{13}/2 + r_{23}/2\sqrt{5} + r_{33}/\sqrt{6} \\ r_{13}/2 + r_{23}/2\sqrt{5} + \sqrt{2}r_{33}/\sqrt{3} \end{bmatrix}$$

$\frac{\overset{2}{\cancel{4}} r_{23}}{\cancel{2}\sqrt{5}} = 3$

$\boxed{r_{23} = \frac{3\sqrt{5}}{2}}$

$\frac{3r_{23}}{2\sqrt{5}} - \frac{r_{13}}{2} = 2$

$\frac{3 \times 3\cancel{\sqrt{5}}}{2 \times 2\cancel{\sqrt{5}}} - \frac{8}{4} = \frac{r_{13}}{2} \Rightarrow \boxed{r_{13} = 1/2}$

$\left(\frac{1}{2}\right)\left(\frac{1}{2}\right) + \frac{3\cancel{\sqrt{5}}}{2 \times 2\cancel{\sqrt{5}}} + \frac{\sqrt{2}}{\sqrt{3}} r_{33} = 2$

$\frac{1}{4} + \frac{3}{4} - \frac{8}{4} = -\sqrt{\frac{2}{3}} r_{33}$

$-1 = -\sqrt{\frac{2}{3}} r_{33}$

$\boxed{r_{33} = \sqrt{\frac{3}{2}}}$

$\therefore R = \begin{bmatrix} 2 & 1 & 1/2 \\ 0 & \sqrt{5} & 3\sqrt{5}/2 \\ 0 & 0 & \sqrt{3/2} \end{bmatrix}$

→ Eigenvalues and eigenvectors
↳ represent fundamentals of physical system.

Let us consider an $n \times n$ matrix A. A scalar λ is called an eigenvalue of A if there is a non zero vector x such that:

$$\underbrace{Ax = \lambda}_{\text{eigenvalue}} \; x \longrightarrow \text{eigenvector.}$$

01/01/25

• Eigenvalues and eigenvectors

→ For any $n \times n$ matrix A that can be written as $Ax = \lambda x$, x is the eigenvector, and λ is the eigenvalue of A.

Q. Show that $x = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is an eigenvector of $A = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$, and find the corresponding eigenvalue.

A. $Ax = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \end{bmatrix} = 4\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \lambda x$, where $\lambda = 4$

i.e $Ax = \lambda x$, with eigenvalue 4

$\therefore x =$ eigenvector of A,
$\lambda = 4 =$ eigenvalue of A

→ Let A be an $n \times n$ matrix, and let λ be the eigenvalue of A. The collection of all eigenvectors corresponding to λ, together with the zero vector forms a vector space called the eigen space, denoted by E_λ.

Take $Ax = \lambda x$

$\therefore Ax = \lambda I x$

$\therefore (A - \lambda I)x = 0$

$\therefore$ If $A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{n1} & \cdots & a_{nn} \end{bmatrix}$

$\therefore A - \lambda I = \begin{bmatrix} a_{11}-\lambda & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn}-\lambda \end{bmatrix}$

$$\therefore (A-\lambda I)x = \begin{bmatrix} a_{11}-\lambda & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn}-\lambda \end{bmatrix} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = 0$$

Since det(x) cannot be found / isn't 0,

$\boxed{\det(A-\lambda I) = 0}$ $\longrightarrow$ characteristic equation.

the eigenvalues of the matrix are the solutions of the system of equations:

$(a_{11}-\lambda)x_1 + \cdots + a_{1n}x_n = 0$

$\vdots$

$a_{n1}x_1 + \cdots (a_{nn}-\lambda)x_n = 0.$

$\Rightarrow$ steps to find eigenvectors corresponding to λ:

1. Compute the characteristic polynomial $\det(A-\lambda I) = 0$ of A
2. Find the eigen values of A by solving this equation.
3. For each λ, calculate the null space of $A - \lambda I$. This is also the eigen space of A.

All the non-zero vectors in this space are the eigenvectors of A corresponding to λ.

⇒ Algebraic multiplicity : number of times a certain eigenvalue repeats itself.

Geometric multiplicity : dimension of the eigen space corresponding to a certain λ.

Q. Find the eigenvalues and corresponding eigenspaces of

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & -5 & 4 \end{bmatrix}$$

A. $\det(A-\lambda I) = 0$.
$$\begin{vmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 1 \\ 2 & -5 & 4-\lambda \end{vmatrix} = 0$$

$$-\lambda(+\lambda(\lambda-4)+5) - 1(-2) = 0$$

$$-\lambda(\lambda^2-4\lambda+5)+2=0$$

$$-\lambda^3+4\lambda^2-5\lambda+2=0$$

$$\lambda^3-4\lambda^2+5\lambda^2-2=0$$

$\therefore \lambda = 1, 1, 2 \Rightarrow$ distinct eigenvalues $= 1, 2$.

i.e. algebraic multiplicity of $\lambda=1 = 2$
$\lambda = 2 = 1$

for $\lambda = 1$,

$(A-I)x = 0$

$$\therefore \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 2 & -5 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$x_2 - x_1 = 0$
$x_3 - x_2 = 0$ $\Rightarrow x_1 = x_2 = x_3$
$\therefore 2x_1 - 5x_2 + 3x_3 = 0$

eigenspace for $\lambda=1$

$$\therefore \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = k\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \Rightarrow E_1 = \left\{ t\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, t \in \mathbb{R} \right\}$$

i.e. E_1 is a span of $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$

$\therefore$ geometric multiplicity of $\lambda = 1 = 1$.

$(A-2\lambda)x=0$

$$\begin{bmatrix} -2 & 1 & 0 \\ 0 & -2 & 1 \\ 2 & -5 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$x_2 - 2x_1 = 0 \Rightarrow x_2 = 2x_1$

$x_3 - 2x_2 = 0 \Rightarrow x_3 = 4x_1$

$\therefore 2x_1 - 5x_2 + 2x_3 = 0$

$\hookrightarrow 2x_1 - 5(2x_1) + 2(4x_1) = 0$

$0 = 0$

eigenspace of $\lambda = 2$

$$\therefore \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = k\begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix} \Rightarrow E_2 = \left\{ k\begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}, k \in \mathbb{R} \right\} \Rightarrow \text{span}\left(\begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}\right)$$

geometric multiplicity of $\lambda_2 = 2 = 1$.

Q. $A = \begin{bmatrix} -1 & 0 & 1 \\ 3 & 0 & -3 \\ 1 & 0 & -1 \end{bmatrix}$, find eigenvalues and eigenspaces.

A. $\det(A-\lambda I) = 0$

$$\begin{vmatrix} -1-\lambda & 0 & 1 \\ 3 & -\lambda & -3 \\ 1 & 0 & -1-\lambda \end{vmatrix} = 0$$

$(-1-\lambda)(-\lambda(-1-\lambda)) + 1(\lambda) = 0$

$-(1+\lambda)(\lambda+\lambda^2) + \lambda = 0$

$-[\lambda + \lambda^2 + \lambda^2 + \lambda^3] + \lambda = 0$

$2\lambda^2 + \lambda^3 = 0$

$\therefore 0, 0, -2$ algebraic multiplicity of $\lambda = 0, = 2$
$\lambda = -2, = 1$.

$\lambda = 0$

eigenspace of $\lambda = 0$ $\Rightarrow g.m = 2$

$$(A)x = 0 : \begin{bmatrix} -1 & 0 & 1 \\ 3 & 0 & -3 \\ 1 & 0 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$x_1 = x_3$

$x_2 \rightarrow$ anything $\therefore E_1 = \left\{ k_1\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + k_2\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, k_1, k_2 \in \mathbb{R} \right.$

$\lambda = -2$

$(A+2I)x = 0$

$$\begin{bmatrix} 1 & 0 & 1 \\ 3 & 2 & -3 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$$

$x_1 = -x_3$

$3x_1 + 2x_2 - 3x_3 = 0$

$2x_2 - 6x_3 = 0$

$x_2 = +3x_3$

eigen space of $\lambda = -2$

g.m = 1.

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = K \begin{bmatrix} -3 \\ -1 \\ 1 \end{bmatrix} \Rightarrow E_2 = \left\{ K \begin{bmatrix} -1 \\ 3 \\ 1 \end{bmatrix}, K \in \mathbb{R} \right\}$$

Q. Show that a square matrix A is invertible, iff 0 is not an eigen value of A.

A. if 0 is an eigenvalue, then we can substitute in characteristic equation as:

$$\det(A - 0 \cdot I) = 0 \Rightarrow \det(A) = 0$$

i.e. A is not invertible if 0 is an eigen value of A

$\Rightarrow$ A is invertible if 0 is not an eigenvalue of A

If A is invertible,

$\det A \neq 0$

$\therefore \det(A - 0 \cdot I) \neq 0$, which goes against characteristic eq.

$\therefore$ 0 is not an eigenvalue.

$\therefore$ Hence proved.

→ Th/: Let A be a square matrix with eigenvalue λ and the corresponding eigenvector x.

(i) For any positive integer n, λ^n is the eigenvalue of A^n, with corresponding eigenvector x.

(ii) If A is invertible, then $1/\lambda$ is the eigen value of A^{-1} with corresponding eigenvector x.

Prf. (i) using induction,

→ true for $n=1$.

→ Let us assume true for $n=K$

i.e. $A^K x = \lambda^K x$

→ Now, prove for $n = k+1$

$$A^{k+1}x = A(A^k x) = A(\lambda^k x) = \lambda^k (Ax) = \lambda^k(\lambda x) = \lambda^{k+1} x$$

i.e. $A^{k+1}x = \lambda^{k+1}x$

$\therefore$ Hence proved by induction.

(ii) $Ax = \lambda x$

$A^{-1}Ax = A^{-1}(\lambda x)$

$\therefore Ix = \lambda(A^{-1}x)$

$\frac{1}{\lambda}(Ix) = A^{-1}x$

$\therefore \frac{1}{\lambda}x = A^{-1}x$

$\therefore$ Hence proved.

→ not just $n > 0$.

Th, (iii) If A is invertible, then for $\boxed{\text{any integer } n}$, λ^n is the eigenvalue of A^n with corresponding eigenvector x.

Q. Compute $\begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix}^{10} \begin{bmatrix} 5 \\ 1 \end{bmatrix}$

A. Let us find eigenvectors of $\begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix}$

$\therefore \det(A - \lambda I) = 0$

$\det\left(\begin{bmatrix} -\lambda & 1 \\ 2 & 1-\lambda \end{bmatrix}\right) = 0$

$\therefore -\lambda(1-\lambda) - 2 = 0$

$\lambda^2 - \lambda - 2 = 0$

$(\lambda-2)(\lambda+1) = 0$

$\therefore \lambda = 2, -1$

for $\lambda = 2$,

$(A - 2I)x = 0$

$\begin{bmatrix} -2 & 1 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$

$x_2 - 2x_1 = 0 \Rightarrow x_2 = 2x_1$

$2x_1 - x_2 = 0$

$\therefore E_1 = \left\{ k_1\begin{bmatrix} 1 \\ 2 \end{bmatrix}, k \in \mathbb{R} \right\}$

for $\lambda = -1$

$(A+1)x = 0$

$$\begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$x_1 = -x_2$

$\therefore E_2 = \left\{ k_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix}, k_2 \in \mathbb{R} \right\}$

$\therefore k_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + k_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 5 \\ 1 \end{bmatrix}$

$\therefore k_1 + k_2 = 5$

$2k_1 - k_2 = 1$

$k_1 = 2$

$k_2 = 3$

$\therefore \begin{bmatrix} 5 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix} + \begin{bmatrix} 3 \\ -3 \end{bmatrix}$, both of which are eigenvectors of $\begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix}$, with $\lambda = 2$ and -1 respectively

$$\therefore \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix}^{10} \begin{bmatrix} 5 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix}^{10} \begin{bmatrix} 2 \\ 4 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix}^{10} \begin{bmatrix} 3 \\ -3 \end{bmatrix}$$

$$= 2^{10} \begin{bmatrix} 2 \\ 4 \end{bmatrix} + (-1)^{10} \begin{bmatrix} 3 \\ -3 \end{bmatrix}$$

$$= 1024 \begin{bmatrix} 2 \\ 4 \end{bmatrix} + \begin{bmatrix} 3 \\ -3 \end{bmatrix}$$

$$= \begin{bmatrix} 2051 \\ 4093 \end{bmatrix}$$

i.e. $\begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix}^{10} \begin{bmatrix} 5 \\ 1 \end{bmatrix} = \begin{bmatrix} 2051 \\ 4093 \end{bmatrix}$

→ Th, Suppose the $n \times n$ matrix A has eigenvectors $v_1, v_2, v_3 \dots v_m$ corresponding to the eigenvalues $\lambda_1, \lambda_2 \dots \lambda_m$. If x is a vector in $\mathbb{R}^n$ that can be expressed as a linear combination of the eigenvectors :

i.e. $x = c_1 v_1 + c_2 v_2 + \dots + c_m v_m$,

$\therefore$ for any integer k,

$$\boxed{A^k x = c_1 \lambda_1^k v_1 + c_2 \lambda_2^k v_2 \dots c_m \lambda_m^k v_m.}$$

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$+\vec{0}$ cannot be an eigenvector.

Th/. Let A be an $n \times n$ matrix and let $\lambda_1, \lambda_2 \dots \lambda_n$ be distinct eigenvalues of A with corresponding eigenvectors $v_1, v_2 \dots v_n$. Then, v_1, v_2, v_m are linearly independent.

Prf. Let $v_1, v_2 \dots v_m$ be linearly dependent., and let v_{k+1} be the first vector which is linearly dependent on $\{v_1, v_2 \dots v_k\}$

$$\therefore v_{k+1} = c_1 v_1 + c_2 v_2 \dots + c_k v_k \quad (1)$$

multiply both sides of (1) with A,

$$A_1 v_{k+1} = A c_1 v_1 + A c_2 v_2 + \dots + A c_k v_k.$$

$$\therefore \lambda_{k+1} v_{k+1} = c_1 \lambda_1^{v_1} + c_2 \lambda_2^{v_2} + \dots + c_k \lambda_k^{v_k} \quad (2)$$

multiply both sides of (1) with λ_{k+1},

$$\lambda_{k+1} v_{k+1} = \lambda_{k+1} c_1 v_1 + \lambda_{k+1} c_2 v_2 + \dots + \lambda_{k+1} c_k v_k \quad (3)$$

$\therefore$ (3) - (2)

$$0 = c_1(\lambda_{k+1} - \lambda_1)v_1 + c_2(\lambda_{k+1} - \lambda_2)v_2 + \dots + c_k(\lambda_{k+1} - \lambda_k)v_k$$

i.e. $$\sum_{i=1}^{k} c_i(\lambda_{k+1} - \lambda_i)v_i = 0$$

we know that from $i=0$ to $i=k$, v_i are linearly independent

$\therefore c_i(\lambda_{k+1} - \lambda_i) = 0 \ \forall \ i \in \{1, 2 \dots k\}$

$\lambda_{k+1} - \lambda_i \neq 0$, since eigenvalues are distinct

$\therefore c_i = 0 \ \forall \ i$

i.e. substituting in (1),

$v_{n+1} = \vec{0}$, which is not possible since eigenvectors cannot be $\vec{0}$

$\therefore$ this is a contradiction, and $v_1, v_2 \dots v_m$ is linearly independent.

- Similar matrices

→ Let A and B be similar matrices. A is similar to B if there is an invertible matrix P of size $n \times n$ s.t.

$$P^{-1}AP = B,$$

represented by $A \sim B$.

$P^{-1}AP = B$

$PP^{-1}APP^{-1} = PBP^{-1} \Rightarrow \quad A = PBP^{-1}$ also means A is similar to B.

$\Rightarrow AP = PB$

Th,

i) $A \sim A$

ii) if $A \sim B$, then $B \sim A$

iii) if $A \sim B$, $B \sim C$ then $A \sim C$.

Prf.

a) Let $P = I$

$P^{-1} = I^{-1} = I$

$\therefore IAI^{-1} = A \quad \therefore A \sim A$

b) if $A \sim B$, $PAP^{-1} = B$

$\therefore A = P^{-1}AP,$

Let $P' = P^{-1}$

$\therefore P'B(P')^{-1} = A$

i.e. $B \sim A$ as well.

c) $A \sim B \Rightarrow QAQ^{-1} = B$

$B \sim C \Rightarrow PBP^{-1} = C$

$\therefore$

$P(QAQ^{-1})P^{-1} = C$

$(PQ)A(Q^{-1}P^{-1}) = C$

Let $R = PQ$

$\therefore R^{-1} = Q^{-1}P^{-1}$

$\therefore RAR^{-1} = C$ i.e. $A \sim C$ $\therefore C \sim A$

$\therefore$ Hence proved.

Q) $A = \begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 \\ -2 & -1 \end{bmatrix}$. Find P s.t. $A \sim B$

A) $P = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$,

$AP = PB$

$$\begin{bmatrix} 1 & 2 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -2 & -1 \end{bmatrix}$$

$$\begin{bmatrix} a+2c & b+2d \\ -c & -d \end{bmatrix} = \begin{bmatrix} a-2b & -b \\ c-2d & -d \end{bmatrix}$$

$2d+2b=0 \Rightarrow b=-d$

$b=-c$

$c=d$

$-c = c-2d$

$c = d$.

$b=-c$

$c=c$

$d=c$

$\therefore P = \begin{bmatrix} a & -c \\ c & c \end{bmatrix}$ form.

→ If $A \sim B$,

i) $\det A = \det B$

ii) A and B have same characteristic polynomial.

Prf:

i) $A \sim B \Rightarrow PAP^{-1} = B$

$\det(PAP^{-1}) = \det B$

$\det(P)\det(A)\det(P^{-1}) = \det B$

$\cancel{\det}(P)\det(A) \cdot \frac{1}{\cancel{\det}(P)} = \det B$

$\det(A) = \det(B)$

(iii) characteristic eq:

$\det(A-\lambda I)=0$ and $\det(B-\lambda I)$ is same.

$$\det(B-\lambda I) = \det(PAP^{-1} - \lambda P I P^{-1})$$

$$= \det(P(A-\lambda I)P^{-1})$$

$\because$ since $B = PAP^{-1}$, $= \det(A-\lambda I)$

$\therefore$ characteristic eq is same.

$\therefore$ eigenvalues of B and A are same.

- Diagonalization of matrix

→ An nxn matrix A is said to be diagonizable if there is a diagonal matrix D such that A is similar to D.

Here, D turns out to be the diagonal matrix of eigenvalues of A, and column vectors of P are the eigenvectors.

Th/ Let A be an nxn matrix. Then, A is diagonizable iff A has n linearly independent eigenvectors.

Prf: (forward proof, backward proof is similar)

Let A be diagonalizable,

$$P^{-1}AP = D$$

$$AP = PD$$

Let $p_1, p_2 \dots p_n$ be column vectors of P and $\lambda_1, \lambda_2 \dots \lambda_n$ be entries of D (eigenvalues)

$$\therefore A[p_1\ p_2 \dots p_n] = [p_1\ p_2 \dots p_n]\begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & \lambda_2 & 0 \\ 0 & 0 & \lambda_n \end{bmatrix}$$

i.e. $[Ap_1\ Ap_2\ Ap_3 \dots Ap_n] = [p_1\lambda_1\ p_2\lambda_2 \dots p_n\lambda_n]$

i.e $p_1, p_2 \dots p_n$ are eigenvectors $\therefore$ they are linearly independent.

can be proved using fundamental th, of invertible matrices

* since P is invertible its columns are linearly independent

↳ hence eigenvectors of A are LI.

Let $C = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_n \end{bmatrix}$, then we know $AC = 0$ has only trivial soln.

$\therefore$ column vectors of A are linearly independent.

Q. If possible, find the matrix D that diagonalizes

$$A = \begin{bmatrix} -1 & 0 & 1 \\ 3 & 0 & -3 \\ 1 & 0 & -1 \end{bmatrix}$$

eigenvectors: $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 3 \\ 1 \end{bmatrix}$

$P = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 3 \\ 1 & 0 & 1 \end{bmatrix} \Rightarrow \begin{matrix} a - c = 0 \\ b + 3c = 0 \\ a + c = 0 \end{matrix} \quad \begin{matrix} a, c = 0 \\ b = 0 \end{matrix}$

$\therefore$ P → linearly indept.

$AP = PD$, $D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{bmatrix}$ (eigenvalues)

$$AP = \begin{bmatrix} -1 & 0 & 1 \\ 3 & 0 & -3 \\ 1 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 3 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 2 \\ 0 & 0 & -6 \\ 0 & 0 & -2 \end{bmatrix}$$

$$PD = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 3 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 2 \\ 0 & 0 & -6 \\ 0 & 0 & -2 \end{bmatrix}$$

$\therefore$ A is diagonalization,

$$D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

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Q. If possible find a matrix P that diagonalises

a) $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & -5 & 4 \end{bmatrix}$ b) $A = \begin{bmatrix} -1 & 0 & 1 \\ 3 & 0 & -3 \\ 1 & 0 & -1 \end{bmatrix}$

A. b) $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -2 \end{bmatrix} = D,$

$P = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 3 \\ 1 & 0 & 1 \end{bmatrix}$

a) $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & -5 & 4 \end{bmatrix}$

$\det(A - \lambda I) = 0$

$$\begin{vmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 1 \\ 2 & -5 & 4-\lambda \end{vmatrix} = 0$$

$-\lambda(-\lambda(4-\lambda)+5) - 1(-2) = 0$

$\lambda^2(4-\lambda) - 5\lambda + 2 = 0$

$-\lambda^3 + 4\lambda^2 - 5\lambda + 2 = 0$

$\lambda = 1, 1, 2$

$\therefore D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

$\lambda = 1 \rightarrow (A - \lambda I)x = 0$

$\begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 2 & -5 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \Rightarrow$ $x_2 - x_1 = 0$, $x_3 - x_2 = 0$ $\Rightarrow x_1 = x_2 = x_3 =$

$E_1 = \left\{ k\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, k \in \mathbb{R} \right\}$

$= \begin{bmatrix} -2 & 1 & 0 \\ 0 & -2 & 1 \\ 2 & -5 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0$

$E_2 = \left\{ k\begin{bmatrix} 1 \\ 2 \\ 4 \end{bmatrix}, k \in \mathbb{R} \right\}$

$\therefore$ A is not diagonalizable.

→ if eigenvalues of a matrix are not linearly dependent, the matrix is not diagonalizable.

If they are, find P s.t. $AP = PD$.

$\vec{Q}$ if $A \sim D$, $A^n \sim D^n$, i.e. $A = PDP^{-1} \Rightarrow A^n = PD^nP^{-1}$

Q. Compute A^{10}, $A = \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix}$

A) for $A = \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix}$,

$\det(A - \lambda I) = 0$

is $\begin{vmatrix} -\lambda & 1 \\ 2 & 1-\lambda \end{vmatrix} = 0$

$-\lambda(1-\lambda) - 2 = 0$

$\lambda^2 - \lambda - 2 = 0$

$(\lambda - 2)(\lambda + 1) = 0$

$\lambda = 2, -1$

$\therefore D = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}$, $P = \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix}$

$D^n = \begin{bmatrix} 2^n & 0 \\ 0 & (-1)^n \end{bmatrix} \Rightarrow D^{10} = \begin{bmatrix} 1024 & 0 \\ 0 & 1 \end{bmatrix}$

$\therefore A^{10} = \begin{bmatrix} 1 & 1 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 1024 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1/3 & 1/3 \\ 2/3 & -1/3 \end{bmatrix}$

$= \begin{bmatrix} 1024 & 1 \\ 2048 & -1 \end{bmatrix} \begin{bmatrix} 1/3 & 1/3 \\ 2/3 & -1/3 \end{bmatrix}$

$= \begin{bmatrix} 1026/3 & 1023/3 \\ 2046/3 & 2049/3 \end{bmatrix} = \begin{bmatrix} 342 & 341 \\ 682 & 683 \end{bmatrix}$

Orthogonal diagonalization of symmetric matrices

→ A square matrix A is said to be orthogonally diagonalizable iff there exists an orthogonal matrix Q and a diagonal matrix D such that, $Q^T A Q = D$ $\quad (Q^T = Q^{-1})$

Q. Find Q such that $A = \begin{bmatrix} 1 & 2 \\ 2 & -2 \end{bmatrix}$ is diagonalizable.

A. $\det(A - \lambda I) = 0$

$\therefore \begin{vmatrix} 1-\lambda & 2 \\ 2 & -2-\lambda \end{vmatrix} = 0$

$-(1-\lambda)(2+\lambda) - 4 = 0$

$(\lambda - 1)(2+\lambda) - 4 = 0$

$2\lambda + \lambda^2 - 2 - \lambda - 4 = 0$

$\lambda^2 + \lambda - 6 = 0$

$(\lambda + 3)(\lambda - 2) = 0$

$\lambda = -3, 2.$

$E_1 : \begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$E_1 = \left\{ k \begin{bmatrix} 1 \\ -2 \end{bmatrix}, k \in \mathbb{R} \right\}$

$E_2 = \begin{bmatrix} -1 & 2 \\ 2 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$\therefore E_2 = \left\{ k \begin{bmatrix} 1 \\ +1/2 \end{bmatrix}, k \in \mathbb{R} \right\}$

$\therefore \left\{ \begin{bmatrix} 1 \\ -2 \end{bmatrix}, \begin{bmatrix} 1 \\ +1/2 \end{bmatrix} \right\}$ → not orthonormal

$\therefore$ orthonormal $= \left\{ \begin{bmatrix} 1/\sqrt{5} \\ -2/\sqrt{5} \end{bmatrix}, \begin{bmatrix} 2/\sqrt{5} \\ 1/\sqrt{5} \end{bmatrix} \right\}$

$\therefore Q = \begin{bmatrix} 1/\sqrt{5} & 2/\sqrt{5} \\ -2/\sqrt{5} & 1/\sqrt{5} \end{bmatrix}$

Th/. if A is orthogonally diagonizable then A is symmetric.

Prf. A is orthogonally diagonizable

$$Q^TAQ = D$$

$$QQ^TAQQ^T = QDQ^T$$

$$IAI = QDQ^T$$

$$A = QDQ^T$$

$\therefore A^T = (QDQ^T)^T$

$$A^T = (Q^T)^T D^T Q^T$$

$$A^T = QDQ^T = A$$

$\therefore A = A^T$ i.e. A is symmetric.

$\therefore$ Hence proved

Th. If A is a real symmetric matrix then eigen-values of A are real.

Prf. Let λ be the eigen value of A corresponding to the eigen vector v

$$AV = \lambda V \quad — (1).$$

Taking conjugate on both sides,

$$(\overline{AV}) = (\overline{\lambda V}) \quad — (2)$$

$$(\overline{AV}) = \bar{A}\bar{V} = A\bar{V} = \bar{\lambda}\bar{V} \quad — (3)$$

$$\bar{V}^T A = \bar{V}^T A^T = (A\bar{V})^T = (\bar{\lambda}\bar{V})^T = \bar{\lambda}(\bar{V})^T \quad — (4)$$

$$\because \lambda(\bar{V}^T V) = \bar{V}^T(\lambda V) = \bar{V}^T(AV) = (\bar{V}^T A)V = (\bar{\lambda}(\bar{V})^T)V = \bar{\lambda}(\bar{V}^T V)$$

(from 4)

$\therefore \lambda(\bar{V}^T V) = \bar{\lambda}(\bar{V}^T V)$

$\therefore \lambda = \bar{\lambda} \Rightarrow \lambda$ is real

$\therefore$ eigenvalues are real.

Th, If A is a symmetric matrix, then any two eigenvectors corresponding to distinct eigenvalues are orthogonal.

Prf. Let v_1 and v_2 be two eigen-vectors correspondings to two distinct eigenvalues λ_1 and λ_2

$$\left.\begin{aligned} Av_1 &= \lambda_1 v_1 \\ Av_2 &= \lambda_2 v_2 \end{aligned}\right\} \quad \lambda_1(v_1 \cdot v_2) = (\lambda_1 v_1)\cdot v_2 = (Av_1)\cdot v_2$$

dot product $\downarrow$ matrix multiplication

$$\Downarrow$$

$$[Av_1]^T v_2$$

$$= (v_1^T A^T) v_2 = v_1^T (A v_2) = v_1^T (\lambda_2 v_2)$$

$$= \lambda_2 (v_1^T v_2) \quad \text{matrix mult}$$

$$\Downarrow \qquad\qquad \Downarrow$$

$$\lambda_2 (v_1 \cdot v_2) \quad \text{dot product}$$

$\therefore \lambda_1 (v_1 \cdot v_2) = \lambda_2 (v_1 \cdot v_2)$

since $\lambda_1 \neq \lambda_2$, $v_1 \cdot v_2 = 0$.

$\therefore$ Hence proved.

Q. verify above th, for $A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$

A. $\det(A - \lambda I) = 0$

$$\begin{vmatrix} 2-\lambda & 1 & 1 \\ 1 & 2-\lambda & 1 \\ 1 & 1 & 2-\lambda \end{vmatrix} = 0$$

$$2-\lambda\left((2-\lambda)^2 - 1\right) - 1\left(2-\lambda-1\right) + 1\left(1-(2-\lambda)\right) = 0$$

$$(2-\lambda)^3 - (2-\lambda) - (1-\lambda) + 1(\lambda - 1) = 0$$

$$(2-\lambda)^3 - (2-\lambda) + 2(\lambda - 1) = 0$$

$$8 - \lambda^3 - 12\lambda + 6\lambda^2 - 2 + \lambda + 2\lambda - 2 = 0$$

$$4 - \lambda^3 - 9\lambda + 6\lambda^2 = 0 \Rightarrow \lambda^3 - 6\lambda^2 + 9\lambda - 4$$

$$\lambda = 1, 1, 4$$

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- The spectral theorem

→ Let A be a $n \times n$ matrix. Then A is symmetric iff. it is orthogonally diagonalizable.

→ spectral decomposition of a symmetric matrix A:

$$A \cdot \begin{bmatrix} q_1 & q_2 & \dots & q_n \end{bmatrix} \begin{bmatrix} \lambda_1 & 0 & \dots & \\ 0 & \lambda_2 & & \\ \vdots & & \ddots & \\ & & & \lambda_n \end{bmatrix} \begin{bmatrix} q_1^T \\ q_2^T \\ \vdots \\ q_n^T \end{bmatrix}$$

spectral decomposition:

$$\lambda_1 q_1 q_1^T + \lambda_2 q_2 q_2^T \dots \lambda_n q_n q_n^T$$

Ex. $A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$

$\lambda = 4, 1, 1$

$$E_4 = \left\{ k \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, k \in \mathbb{R} \right\}$$

$$E_1 = \left\{ k_1 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} + k_2 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, k_1, k_2 \in \mathbb{R} \right\}$$

→ problem given last class continuation:

$\lambda = 1, 4 \Rightarrow$ distinct eigenvalues

$$E_1 = \left\{ k_1 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} + k_2 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, k_1, k_2 \in \mathbb{R} \right\} \Rightarrow \text{eigenvectors: } v_1: \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \quad v_2: \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

$$E_4 = \left\{ k \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, k \in \mathbb{R} \right\} \Rightarrow \text{eigenvectors: } v_3: \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

we can see that:

$V_1 \cdot V_3 = 0$ and $V_2 \cdot V_3 = 0$

$\therefore \langle v_1, v_3 \rangle = 0$ and $\langle v_2, v_3 \rangle = 0$ $\therefore$ they are orthogonal

$\therefore$ Hence proved.

$$Q = \begin{bmatrix} 1/\sqrt{3} & -1/\sqrt{2} & -1/\sqrt{6} \\ 1/\sqrt{3} & 0 & 2/\sqrt{6} \\ 1/\sqrt{3} & 1/\sqrt{2} & -1/\sqrt{6} \end{bmatrix} \quad \therefore Q^T = Q^{-1} = \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \\ -1/\sqrt{2} & 0 & 1/\sqrt{2} \\ -1/\sqrt{6} & 2/\sqrt{6} & -1/\sqrt{6} \end{bmatrix}$$

D for A would be:

$$\begin{bmatrix} 4 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\therefore$ if $A = QDQ^T$, it is orthogonally diagonalizable:

i.e. if $$\begin{bmatrix} 1/\sqrt{3} & -1/\sqrt{2} & -1/\sqrt{6} \\ 1/\sqrt{3} & 0 & 2/\sqrt{6} \\ 1/\sqrt{3} & 1/\sqrt{2} & -1/\sqrt{6} \end{bmatrix} \begin{bmatrix} 4 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \\ -1/\sqrt{2} & 0 & 1/\sqrt{2} \\ -1/\sqrt{6} & 2/\sqrt{6} & -1/\sqrt{6} \end{bmatrix} = A$$

$$= \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix} = A \Rightarrow \therefore \text{th, is proved.}$$

spectral decomposition:

$$A = \underbrace{4 \begin{bmatrix} 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \end{bmatrix}}_{\lambda_1 \cdot q_1 \cdot q_1^T} + \underbrace{1 \begin{bmatrix} 1/2 & 0 & -1/2 \\ 0 & 0 & 0 \\ -1/2 & 0 & 1/2 \end{bmatrix}}_{\lambda_2 \cdot q_2 \cdot q_2^T} + \underbrace{1 \begin{bmatrix} 1/6 & -1/3 & 1/6 \\ -1/3 & 2/3 & -1/3 \\ 1/6 & -1/3 & 1/6 \end{bmatrix}}_{\lambda_3 \cdot q_3 \cdot q_3^T}.$$

Q. Find a 2D matrix with eigenvalues 3, -2 and corresponding eigenvectors

$$v_1 = \begin{bmatrix} 3 \\ 4 \end{bmatrix} \text{ and } v_2 = \begin{bmatrix} -4 \\ 3 \end{bmatrix}$$

A. $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$A \cdot V = \lambda V$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 9 \\ 12 \end{bmatrix} \qquad \begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} -4 \\ 3 \end{bmatrix} = \begin{bmatrix} -8 \\ -6 \end{bmatrix}$$

$3a + 4b = 9 \quad 3b - 4a = 8$

$12a + 16b = 36$

$-12a + 9b = 24$

$\overset{5}{\cancel{25}}b = \cancel{60}\,12$

$b = 12/5 \quad a = -1/5$

$3c + 4d = 12 \quad 3d - 4c = -6$

$12c + 16d = 48$

$-12c + 9d = -18$

$5\ \cancel{25}d = \cancel{30}\,6$

$d = 6/5 \quad c = 12/5$

$$\therefore \begin{bmatrix} -1/5 & 12/5 \\ 12/5 & 6/5 \end{bmatrix}$$

- Quadratic form

→ An expression of the form $ax^2 + by^2 + cxy$ is called a quadratic form in x, y.

→ $ax^2 + by^2 + cz^2 + dxy + exz + fyz$: quadratic in x, y and z.

→ $ax^2 + cxy + by^2 = [x \ y]\begin{bmatrix} a & c/2 \\ c/2 & b \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$

$$ax^2 + by^2 + cz^2 + dxy + eyz + fxz = [x \ y \ z]\begin{bmatrix} a & d/2 & e/2 \\ d/2 & b & f/2 \\ e/2 & f/2 & c \end{bmatrix}\begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

Q. A quadratic form in n variables is a function $\mathbb{R}^n \to \mathbb{R}$ of the form:

$$f(x) = x^T A x$$ → matrix associated w/ quadratic form

where A is a $n \times n$ symmetric matrix.

i) what is the quadratic form of $A = \begin{bmatrix} 2 & -3 \\ -3 & 5 \end{bmatrix}$?

A. $2x^2 + 5y^2 - 6xy$

ii) what is the associated matrix corresponding to the quadratic form:

$$f(x_1, x_2, x_3) = 2x_1^2 - x_2^2 + 5x_3^2 + 6x_1x_2 - 3x_1x_3$$

A. $\begin{bmatrix} 2 & 3 & -3/2 \\ 3 & -1 & 0 \\ -3/2 & 0 & 5 \end{bmatrix}$

• Principal axes th.

→ Let A be an $n \times n$ symmetric associated with a quadratic form $x^T A x$.

If Q is an orthogonal matrix such that $Q^T A Q = D$ → a diagonal matrix,

$$x = Qy$$

$$\rightarrow x^T A x = (Qy)^T A (Qy)$$

$$= y^T Q^T A Q y$$

$$= y^T (D) y = y^T D y$$

$\therefore x^T A x = y^T D y$ are equivalent.

Q. find a change of variable that transforms the quadratic form

$$f(x_1, x_2) = 5x_1^2 + 4x_1x_2 + 2x_2^2.$$

A.

$$f(x_1,x_2) = [x_1\ x_2]\begin{bmatrix}5 & 2\\ 2 & 2\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}$$

$$A = \begin{bmatrix}5 & 2\\ 2 & 2\end{bmatrix} \rightarrow \lambda = 1, 6 \quad D = \begin{bmatrix}1 & 0\\ 0 & 6\end{bmatrix}$$

$\therefore$ since $x^TAx = y^TDy$

quadratic form in y : $\boxed{y_1^2 + 6y_2^2}$

→ A quadratic form $f(x) = x^TAx$ is classified into categories:

i) $f(x) > 0 \ \forall x \neq 0 \Leftrightarrow$ +ve definite $\Longleftrightarrow$ all eigenvalues are positive

ii) $f(x) \geq 0 \ \forall x \neq 0 \Leftrightarrow$ +ve semi definite $\Longleftrightarrow$ all eigenvalues are non negative

iii) $f(x) < 0 \ \forall x \neq 0 \Longleftrightarrow$ -ve definite $\Longleftrightarrow$ all eigenvalues are negative

iv) $f(x) \leq 0 \ \forall x \neq 0 \Longleftrightarrow$ -ve semi definite $\Longleftrightarrow$ all eigenvalues are non positive.

v) $f(x)$ can be < 0 or $\geq 0 \Leftrightarrow$ indefinite $\Leftrightarrow$ eigenvalues can be positive or negative.

Q. determine:

$$f(x,y,z) = 3x^2 + 3y^2 + 3z^2 - 2xy - 2xz - 2yz$$

A.

$$A = \begin{bmatrix}3 & -1 & -1\\ -1 & 3 & -1\\ -1 & -1 & 3\end{bmatrix}$$

$$\det(A - \lambda I) = 0 \Rightarrow \begin{vmatrix}3-\lambda & -1 & -1\\ -1 & 3-\lambda & -1\\ -1 & -1 & 3-\lambda\end{vmatrix} = 0$$

$(3-\lambda)[(3-\lambda)^2-1]+1[+(3-\lambda)-1]-1[1+3-\lambda]=0$

$(3-\lambda)(9+\lambda^2-1-6\lambda)+1(\lambda-4)-1(4-\lambda)=0$

$(3-\lambda)(\lambda^2-6\lambda+8)+2\lambda-8=0$

$3\lambda^2-18\lambda+24-\lambda^3+6\lambda^2-8\lambda+2\lambda-8=0$

$-\lambda^3+9\lambda^2-24\lambda+16=0$

$\lambda^3-9\lambda^2+24\lambda-16=0$

$(\lambda-1)(\lambda-4)^2=0$

$\lambda=1,4,4 \longrightarrow f(x_1,x_2,x_3)$ is +ve definite.

$\therefore f(y_1,y_2,y_3)=y_1^2+4y_2^2+4y_3^2$.

15/04/25

Singular value of a matrix

→ For any $m \times n$ matrix A, $A^T A$ is a $n \times n$ symmetric matrix. Using the spectral th, we can orthogonally diagonalize the matrix, and all the eigenvalues are real, and non-negative.

Prf:

Let λ be the eigen value of $A^T A$, with corresponding unit eigenvector v

$$0 \leq \|Av\|^2 = (Av) \cdot (Av)$$

$$0 \leq (Av)^T (Av)$$

$$0 \leq v^T A^T A v$$

i.e. $f(v)$ [quadratic form of $A^T A$] is ≥ 0

∴ eigenvalues are non-negative.

$$v^T (A^T A v) = \lambda v^T v = \lambda (v \cdot v) = \lambda \|v\|^2 = \lambda$$

$$\therefore \|Av\|^2 = \lambda$$

$\boxed{\lambda = \|Av\| = \sigma}$ → singular value

Q. Find singular value of $A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$

A. $A^T = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$

$K = A^TA = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$

$\therefore \det(K - \lambda I) = 0$

$$\begin{vmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{vmatrix} = 0$$

$(2-\lambda)^2 - 1 = 0$

$4 + \lambda^2 - 4\lambda - 1 = 0$

$\lambda^2 - 4\lambda + 3 = 0$

$(\lambda - 3)(\lambda - 1) = 0$

$\boxed{\sigma_1 = \sqrt{3}, \quad \sigma_2 = 1}$ $\} \rightarrow \lambda_n \geq \lambda_{n+1}$ (always label it this way)

• Singular value decomposition theorem

→ Let A be an $m \times n$ matrix with singular values.

$\sigma_1 \geq \sigma_2 \geq \sigma_3 \dots \geq \sigma_r > 0$ and $\sigma_r = \sigma_{r+1} = \dots = \sigma_n = 0$.

Then, $\exists$ an orthogonal matrix U of size $m \times m$ and an orthogonal matrix V of size $n \times n$ and a matrix Σ of size $m \times n$ where:

$$A = U \Sigma V.$$

where,

$$\begin{array}{c|cc} & r & n-r \\ \hline r & & \\ m-r & & \end{array} \quad \Sigma = \left[\begin{array}{c|c} D & 0 \\ \hline 0 & 0 \end{array}\right], \text{ where } D = \begin{bmatrix} \sigma_1 & \cdots & 0 \\ \vdots & \sigma_2 & \vdots \\ 0 & \cdots & \sigma_r \end{bmatrix}$$

$\hookrightarrow$ it is crucial that $r < m$

Ex:

Σ can be:

$$\begin{bmatrix} 4 & 0 & 0 \\ 0 & 3 & 0 \end{bmatrix}, \quad \begin{bmatrix} 2 & 0 \\ 0 & 2 \\ 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 8 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad \begin{bmatrix} 5 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Prf: To construct V, we find an orthogonal basis $\{v_1, v_2 \dots v_n\}$ of $\mathbb{R}^n$ consisting of eigenvectors of A^TA.

$V = [v_1 \; v_2 \dots v_n]$ is an $n \times n$ orthogonal matrix.

$\therefore AV = [Av_1 \; Av_2 \dots Av_n]$ is an orthogonal set of vectors

Let us suppose v_i eigenvector of A^TA corresponding to eigenvalue λ_i.

$$(Av_i)(Av_j) = (Av_i)^T \cdot (Av_j) = v_i^T A^T A v_j = v_i^T \lambda_j v_j = \lambda_j v_i^T v_j = \lambda_j (v_i \cdot v_j) = 0$$

and $\sigma_i = \|Av_i\|$

$$\therefore u_i = \frac{1}{\sigma_i} Av_i \;\; \forall \; i \in \{1, 2 \dots r\}$$

Now, $\{u_1, u_2 \dots u_r\}$ forms an orthonormal set in $\mathbb{R}^m$. If $r<m$, then it will not be an orthonormal basis, $\therefore$ we extend it to form:

$\{u_1, u_2 \dots u_m\}$

and $U = [u_1 \; u_2 \dots u_m]$

Now, $A = U\Sigma V^T$

$= U\Sigma V^{-1}$ (since $V^T = V^{-1}$ for an orthogonal matrix)

$$\boxed{AV = U\Sigma}$$

using $Av_i = \sigma_i u_i$ for $i \in \{1, 2 \ldots r\}$.

$Av_i = 0$ for $i \in \{r+1, \ldots n\}$

$\therefore AV = [Av_1 \; Av_2 \ldots Av_r \; 0 \ldots 0]$

$= [\sigma_1 u_1 \; \sigma_2 u_2 \ldots \sigma_r u_r \; 0 \ldots 0]$

$$= [u_1 \; u_2 \ldots u_m] \begin{bmatrix} \sigma_1 & 0 & \cdots & 0 & 0 \\ 0 & \sigma_2 & & \vdots & \vdots \\ \vdots & & \ddots \; \sigma_r & & \vdots \\ 0 & \cdots & & & 0 \end{bmatrix}$$

i.e. $AV = U\Sigma$.

Q) Find SVD of:

a) $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$A^T = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$

$A^TA = \begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$$\begin{vmatrix} 1-\lambda & 1 & 0 \\ 1 & 1-\lambda & 0 \\ 0 & 0 & 1-\lambda \end{vmatrix} = 0$$

$(1-\lambda)((1-\lambda)^2) - 1\,(1-\lambda) = 0$

$(1-\lambda)(1+\lambda^2-2\lambda) + \lambda - 1 = 0 \Rightarrow \cancel{1} + \lambda^2 - 2\lambda - \cancel{\lambda} - \lambda^3 + 2\lambda^2 \cancel{+\lambda} \cancel{-1} = 0$

$$\lambda^3 - 3\lambda^2 + 2\lambda = 0$$

$$\lambda(\lambda-2)(\lambda-1) = 0$$

$$\lambda_1 = 2,\ \lambda_2 = 1,\ \lambda_3 = 0$$

$$\sigma_1 = \sqrt{2},\ \sigma_2 = 1,\ \sigma_3 = 0$$

$$\Sigma = \begin{bmatrix} \sqrt{2} & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \quad -\ (\text{I})$$

$$E_2 = \begin{bmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$x_1 = x_2$$

$$E_2 = \left\{ k \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix},\ k \in \mathbb{R} \right\}$$

$$E_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$E_1 = \left\{ k \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},\ k \in \mathbb{R} \right\}$$

$$E_0 = \left\{ k \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix},\ k \in \mathbb{R} \right\}$$

$$\therefore v_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix},\ v_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix},\ v_3 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$$

$$V = \begin{bmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 0 & 1 & 0 \end{bmatrix} \quad -\ (\text{II})$$

$$u_i = \frac{1}{\sigma_i} A v_i \Rightarrow$$

$$u_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} \sqrt{2} \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$u_2 = 1 \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\therefore U = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad -\ (\text{III})$$

$$\therefore \underbrace{\begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{A} \cdot \underbrace{\begin{bmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 0 & 1 & 0 \end{bmatrix}}_{V} = \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}}_{U} \cdot \underbrace{\begin{bmatrix} \sqrt{2} & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}}_{\Sigma}$$

b) $\begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$

a. $A^T = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$

$A^TA = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$

$\det(A^TA - \lambda I) = 0$

$\lambda_1 = 3, \lambda_2 = 1$

$\sigma_1 = \sqrt{3} \quad \sigma_2 = 1$

$\Sigma = \begin{bmatrix} \sqrt{3} & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$ — (i)

$E_3 = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$E_1 = \left\{ k\begin{bmatrix} 1 \\ 1 \end{bmatrix}, k \in \mathbb{R} \right\} \longrightarrow v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$E_1 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$E_1 = \left\{ k\begin{bmatrix} 1 \\ -1 \end{bmatrix}. k \in \mathbb{R} \right\} \longrightarrow v_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

$\therefore V = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}$ — (ii)

$u_1 = \frac{1}{\sigma_1} A v_1 = \frac{1}{\sqrt{3}} \cdot \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} = \frac{1}{\sqrt{3}}\begin{bmatrix} \sqrt{2} \\ 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} \quad \begin{bmatrix} \sqrt{2}/\sqrt{3} \\ 1/\sqrt{6} \\ 1/\sqrt{6} \end{bmatrix} = u_1$

$u_2 = \frac{1}{\sigma_1} A v_2 = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix} = \begin{bmatrix} 0 \\ 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix} = u_2$

To extend, take a standard vector e_3 that is linearly independent here, $e_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$, and use gram-schmidt orthogonalization:

$u_1 = \begin{bmatrix} 2/\sqrt{6} \\ 1/\sqrt{6} \\ 1/\sqrt{6} \end{bmatrix}$,

$\therefore$ SVD $\Rightarrow$

$$\underset{A}{\begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}} \underset{V}{\begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}} = \underset{U}{\begin{bmatrix} 2/\sqrt{6} & 0 & -1/\sqrt{3} \\ 1/\sqrt{6} & 1/\sqrt{2} & 1/\sqrt{3} \\ 1/\sqrt{6} & -1/\sqrt{2} & 1/\sqrt{3} \end{bmatrix}} \underset{\Sigma}{\begin{bmatrix} \sqrt{3} & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}}$$

$\rightarrow A = \sigma_1 u_1 v_1^T + \sigma_2 u_2 v_2^T + \sigma_3 u_3 v_3^T + \dots + \sigma_r u_r v_r^T \} \rightarrow$ outer product form of A

$u_1, u_2 \dots u_r \rightarrow$ left singular vectors

$v_1, v_2 \dots v_r \rightarrow$ right singular vectors.

———— × ————

17/04/25

→ If A is an $m \times n$ matrix with SVD of:

$$A = U\Sigma V^T = [u_1\ u_2 \cdots u_m]\begin{bmatrix} \sigma_1 & 0 & \cdots & & 0 \\ 0 & \sigma_2 & & & \vdots \\ \vdots & & \ddots & 0 & \\ & \cdots & \cdots & \sigma_r & \\ 0 & \cdots & \cdots & & 0 \end{bmatrix}\begin{bmatrix} v_1^T \\ v_2^T \\ \vdots \\ v_n^T \end{bmatrix}$$

, where $\sigma_1, \sigma_2 \ldots \sigma_r$ are singular values of A

this can be broken down into:

$$[u_1\ u_2 \cdots u_r \,\vdots\, u_{r+1} \cdots u_m]\Big[\ \Sigma\ \Big]\begin{bmatrix} v_1^T \\ \vdots \\ v_r^T \\ \hline v_{r+1}^T \\ \vdots \\ v_m^T \end{bmatrix}$$

$$= [u_1 \cdots u_r]\Big[\ \Sigma\ \Big]\begin{bmatrix} v_1^T \\ v_r^T \end{bmatrix} + \underbrace{[u_{r+1} \cdots u_m]\Big[\ \cancel{\Sigma}\ \Big]\begin{bmatrix} v_{r+1}^T \\ \vdots \\ v_m^T \end{bmatrix}}_{0}$$

$$= [\sigma_1 u_1 \cdots \sigma_r u_r]\begin{bmatrix} v_1^T \\ \vdots \\ v_r^T \end{bmatrix}$$

$$A = \underbrace{\sigma_1 u_1 v_1^T + \sigma_2 u_2 v_2^T + \cdots + \sigma_r u_r v_r^T}_{\text{outer product form}}$$

$[u_1 \cdots u_r]$: left singular vectors
$[v_1 \cdots v_r]$: right singular vectors

Q. Find outer product of:

a) $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

A. SVD of A is: (obtained in previous class).

$$A = \Big[\underset{u_1}{\begin{pmatrix} 1 \\ 0 \end{pmatrix}} \underset{u_2}{\begin{pmatrix} 0 \\ 1 \end{pmatrix}}\Big]\begin{bmatrix} \overset{\sigma_1}{\sqrt{2}} & 0 & 0 \\ 0 & \underset{\sigma_2}{1} & 0 \end{bmatrix}\begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \\ -1/\sqrt{2} & 1/\sqrt{2} & 0 \end{bmatrix}\begin{matrix} \to v_1 \\ \to v_2 \\ \to v_3 \end{matrix}$$

$\therefore \sigma_1 u_1 v_1^T + \sigma_2 u_2 v_2^T$

$$= \sqrt{2}\begin{bmatrix} 1 \\ 0 \end{bmatrix}\begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \end{bmatrix} + 1\begin{bmatrix} 0 \\ 1 \end{bmatrix}\begin{bmatrix} 0 & 0 & 1 \end{bmatrix}$$

$$\sqrt{2}\begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 0 \end{bmatrix} + 1\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = A$$

$\therefore$ Hence proved.

b) $A = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$

SVD (obtained in previous class):

$$A = \begin{array}{c} \begin{bmatrix} 2/\sqrt{6} & 0 & -1/\sqrt{3} \\ 1/\sqrt{6} & 1/\sqrt{2} & 1/\sqrt{3} \\ 1/\sqrt{6} & -1/\sqrt{2} & 1/\sqrt{3} \end{bmatrix} \\ \begin{array}{ccc} u_1 & \quad u_2 & \quad u_3 \end{array} \end{array} \begin{bmatrix} \sqrt{3} & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{array}{c} \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix} \end{array} \begin{array}{l} \to v_1 \\ \to v_2 \end{array}$$

$\sigma_1 = \sqrt{3}$, $\sigma_2 = 1$

$\therefore \sigma_1 u_1 v_1^T + \sigma_2 u_2 v_2^T$

$$= \sqrt{3}\begin{bmatrix} 2/\sqrt{6} \\ 1/\sqrt{6} \\ 1/\sqrt{6} \end{bmatrix}\begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} \end{bmatrix} + 1\begin{bmatrix} 0 \\ 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}\begin{bmatrix} 1/\sqrt{2} & -1/\sqrt{2} \end{bmatrix}$$

$$\sqrt{3}\begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{3} \\ 1/2\sqrt{3} & 1/2\sqrt{3} \\ 1/2\sqrt{3} & 1/2\sqrt{3} \end{bmatrix} + 1\begin{bmatrix} 0 & 0 \\ 1/2 & -1/2 \\ -1/2 & 1/2 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 1 \\ 1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 1/2 & -1/2 \\ -1/2 & 1/2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} = A.$$

◦ Pseudo inverse

→ Let A be an $m \times n$ matrix with SVD :

$$A = U \Sigma V^T \text{ where } \Sigma = \left[\begin{array}{c|c} D & 0 \\ \hline 0 & 0 \end{array}\right]$$ and D is a diagonal matrix of

size r, $D = \begin{bmatrix} \sigma_1 & 0 & \cdots \\ & \ddots & \vdots \\ 0 & \cdots & \sigma_r \end{bmatrix}$ is made up of non zero singular values of matrix A,

$\sigma_1 \geq \sigma_2 \cdots \geq \sigma_r > 0$.

Then, the pseudo inverse of A is given by :

$$A^+ = V \Sigma^+ U^T, \text{ where } \Sigma^+ = \left[\begin{array}{c|c} D^{-1} & 0 \\ \hline 0 & 0 \end{array}\right]$$

Q. Find pseudo inverse of :

a) $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

$$\Sigma^+ = \begin{bmatrix} \begin{bmatrix} \sqrt{2} & 0 \\ 0 & 1 \end{bmatrix}^{-1} \\ 0 \quad 0 \end{bmatrix} = \begin{bmatrix} 1/\sqrt{2} & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$U = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \Rightarrow U^T = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$V^T = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \\ -1/\sqrt{2} & 1/\sqrt{2} & 0 \end{bmatrix} \Rightarrow V = \begin{bmatrix} 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & 1 & 0 \end{bmatrix}$$

$$\therefore A^+ = \begin{bmatrix} 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1/\sqrt{2} & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \underset{3\times 2}{\begin{bmatrix} 1/2 & 0 \\ 1/2 & 0 \\ 0 & 1 \end{bmatrix}} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 1/2 & 0 \\ 1/2 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\therefore A^{+} = \begin{bmatrix} 1/2 & 0 \\ 1/2 & 0 \\ 0 & 1 \end{bmatrix}$$

- **Hermitian matrix**: a square $n \times n$ matrix A is said to be Hermitian iff

$$A^* = (\bar{A})^T = A$$

Ex) $A = \begin{bmatrix} 3 & 1-i \\ 1+i & 4 \end{bmatrix}$

unitary matrix: A square complex matrix is unitary if:

$$U^* = (\bar{U})^T = U^{-1}$$

→ A square complex matrix A is called unitarily diagonalizable if there exists a unitary matrix U and a diagonal matrix D s.t.:

$$U^* A U = D$$

to check:

1. find eigenvalues of matrix A
2. Find basis for each eigenspace
3. Ensure that each eigenspace contains orthonormal vectors.
   (use gram-schmidt with complex dot product)
   ↳ $u \times v = \bar{u}_1 v_1 + \bar{u}_2 v_2 \cdots \bar{u}_n v_n$
4. Find the matrix U whose columns are orthonormal eigen-vectors

Q Find the unitary matrix U and diagonal matrix D s.t. $U^* A U = D$

a) $A = \begin{bmatrix} 2 & i \\ -i & 2 \end{bmatrix}$

A) $\det(A - \lambda I) = 0 \qquad \begin{vmatrix} 2-\lambda & i \\ -i & 2-\lambda \end{vmatrix} = 0 \qquad (2-\lambda)^2 - 1 = 0$

$2 - \lambda = \pm 1 \Rightarrow \lambda = 1, 3$

$\lambda = 1, 3 \Rightarrow D = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$

$E_1 = \begin{bmatrix} 1 & i \\ -i & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$x_1 + i x_2 = 0$

$x_1 = -i x_2$

$x_2 = i x_1$

$E_1 = \left\{ K \begin{bmatrix} 1 \\ i \end{bmatrix}, K \in \mathbb{R} \right\} \Rightarrow v_1 = \begin{bmatrix} 1 \\ i \end{bmatrix}$

→ 22^{nd} notes not there, refer external notes on : principle component analysis.