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Networks, Signals & Systems, 2024.

Networks → Aftab

Signals & Systems → Prasad

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What is a signal? → Something which has utility for some specific purpose.

→ The signal itself may be important totally (like AC/DC voltage signal)

or it may carry some information that we seek / can-use. (like telephone signal)

Examples of various signals:

(1) EM waves
(2) Voltage/current signals
(2a) Sequence of pulses sent to a computer
(4) Sound waves
(5) Photos, Videos
(6) Vibrational signals like Seismic signals
(7) Chemical signals - smell/taste.
(8) Volume of a tank over time
(9) Attention level of a student over class-time
(10) No. of people who vote in different age groups.
(11) Temperature variation in different places in Hyderabad.
(12) Many many such examples..

→ These are all classes of signals. Each of these have one or more 'independent' parameter & one or more dependent parameters

→ If we see each signal as a function from a domain to a codomain, the 'independent' parameters are in the 'domain' of the function which represents the signal, the value of the function is the dependent parameter, which is in the co-domain

Eg: (1) Electrical signal (voltage / current) generally represents Amplitude of voltage at any given time.

$\underline{V(t)} \rightarrow$ notation

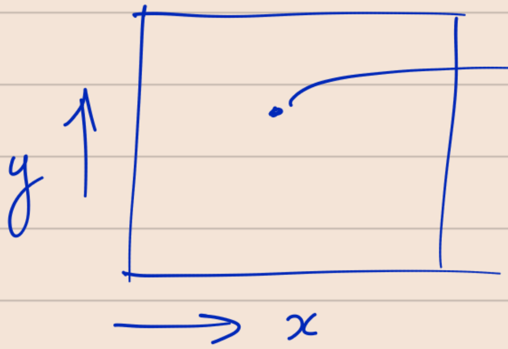
t - denotes time, $V(t) \rightarrow$ value of voltage at time t.

V: Time $\longrightarrow$ Voltage levels
($\in$ Real numbers) ($\in$ Real numbers)

Here, Time t is the independent variable, Voltage is the dependent variable.

E.g 2: Picture/Photo (assume digital photo).

A digital photo on a screen is simply a set of (R,G,B) values for each 'pixel' of the screen



$\underbrace{f(x,y)}_{} = \underline{(r,g,b)}$

(x,y) capturing the position of the picture (independent)

3 values capturing the color. (dependent).

f: 2D-space $\longrightarrow$ 3D color space

this function describes the 'photo' signal.

E.g: 3: Video
(A 'moving' sequence of images.)

So we have video signal: $f(x,y,t) \longrightarrow (r,g,b)$

(extra variable. time)

Note:. Observe that this style of writing a signal as a function captures not one particular signal (like only one electrical signal, one particular photo, etc.) but all signals in a class of signals (i.e. a category of signals) are write-able in this way.

That is, for every electrical signal,
time is independent parameter, while amplitude is dependent parameter.

H.W: Try writing the other signals presented before as functions of independent variables resulting in the values of dependent variables. (Identify what the dep./indep. variables are, & what is the domain / co-domain).

In this course:

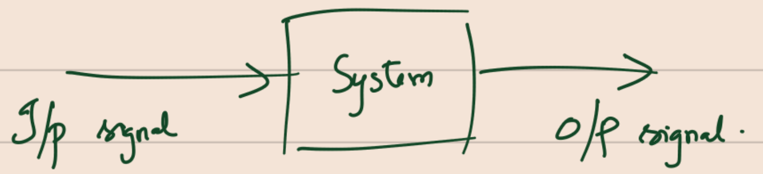
We will deal with signals which have domain is the set of real numbers (denoted by $\mathbb{R}$), representing time. The co-domain can be complex numbers ($\mathbb{C}$) or real numbers ($\mathbb{R}$).

→ Thus, we will generally have that the independent variable is time t.

→ Thus, signals for this course will generally look like $f : \mathbb{R} \to \mathbb{R}$ & written as $f(t)$. Observe that this is also easy for us to plot on 2D plane (like paper, board, etc.)

Systems:

↳ Converts / modifies signals to obtain other signals



→ The input & output signals may not necessarily be of the same type.

→ We might have more than one input / output signals also.

Examples:

(a) $\xrightarrow{\text{electrical signal}}$ System $\rightarrow$ Sound signal or EM wave.

→ Loudspeakers (Sound signal)

→ Radio transmitters (EM wave)

(b) Mechanical signal $\rightarrow$ System $\rightarrow$ Electrical signal

(eg:- Dynamo, piezoelectric).

(c) sound, touch, smell, taste, light $\rightarrow$ human being $\rightarrow$ action

→ In this course, we will look at single-input, single-output system, where the input & output signals are of the same type

Examples:

(1) $x(t) \rightarrow$ Delay system $\rightarrow \underbrace{y(t) = x(t-t_0)}$

This means,

" the value of the function y at time t, is the same as the value of function x, at time t."

(2) $x(t) \rightarrow$ Amplify $\rightarrow A\,x(t)$

(3) Series of systems

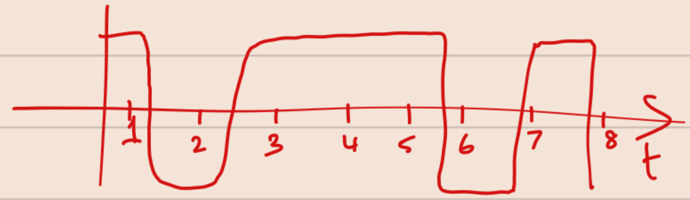
$x(t) \rightarrow$ delay $\rightarrow$ Amplify $\rightarrow A\,x(t-t_0)$

(4) $x(t) \rightarrow$ time scaling $\rightarrow y(t)$

Time scaling means?.

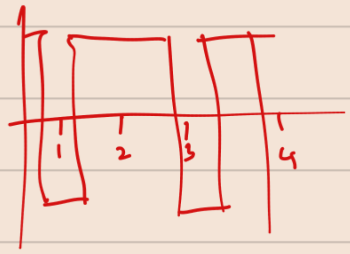
Consider a signal like this

$x(t)$ =

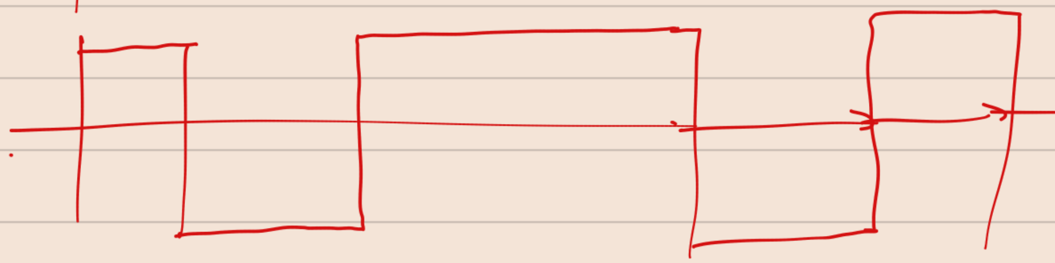


Suppose we want to compress or stretch this signal in time

→ 'Compress'



→ 'expand'



How to write $y(t)$ in these cases?

Compress: $y(t) = x(2t)$ (Value of y at time t is value of x at time $2t$)

Expand: $y(t) = x(t/2)$ (" " " " t " " " " $t/2$)

(5) System that tracks changes in input. (rate of change)

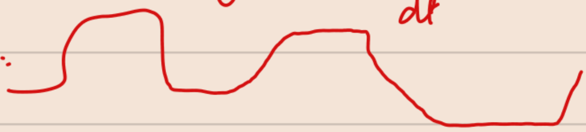
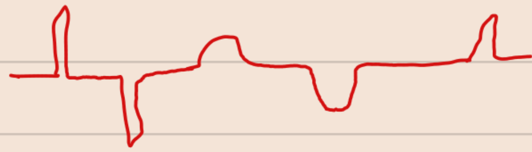
Suppose we want to track changes in $x(t)$

→ If signal $x(t)$ is not changing much, we want to keep $|y(t)|$ small ($|y(t)|$ = absolute value of $y(t)$)

→ Otherwise, if $x(t)$ is experiencing lot of changes, $|y(t)|$ should be large.

$x(t) \longrightarrow$ Differentiator $\longrightarrow y(t)$

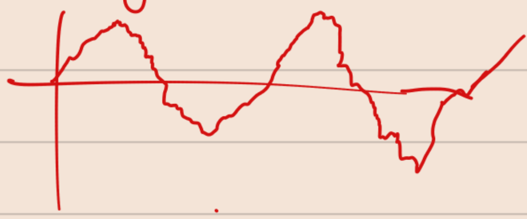
$$y(t) = \frac{d}{dt} x(t).$$

$x(t)$:  , $y(t) =$ 

(6) 'Moving average' system (or) Smoothing system:

Suppose we want to 'smooth' out the fast changes in $x(t)$

$x(t) \longrightarrow$ Moving average / Smoothing $\longrightarrow y(t) = \int_{\tau = t-\varepsilon}^{t+\varepsilon} x(\tau)\, d\tau$



The 'integrator' is another system defined as follows. (It 'accumulates' the effect of $x(t)$ until the specific time instant)

$x(t) \longrightarrow$ Integrator $\longrightarrow y(t) = \int_{\tau = -\infty}^{t} x(\tau)\, d\tau$

Exercise:

Think about the 2D versions of these systems ((ie) if our input was a picture & the output was another picture).

—x—

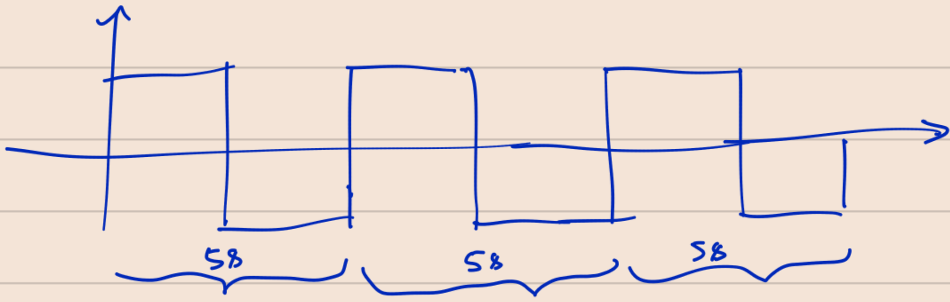
Signals in time variable: Various Classifications:

① ↳ Continuous-time (mostly our focus in this course)

↳ Discrete-time (arises from sampling continuous signals as well as naturally)

② Periodic → signal repeats after some finite time

$$x(t+T) = x(t), \ \forall t$$



$$\left.\begin{array}{l} x(t+10) = x(t) \\ x(t+5) = x(t) \end{array}\right\} \longrightarrow \underbrace{\text{Smallest } T \text{ is } 5.}$$

This smallest T s.t.
$x(t+T) = x(t), \forall t$
is called $T = 5s$.

Non-periodic: (No such T exists)

③ Even & odd signals:

A signal is called 'even' → $x(t) = x(-t), \forall t$

- - - - - 'odd' → $x(t) = -x(-t), \forall t$

Remark: Any signal can be written as the sum of an even signal & an odd signal

$$x(t) = x_e(t) + x_o(t), \text{ where } x_e(t) \triangleq \frac{x(t) + x(-t)}{2}.$$

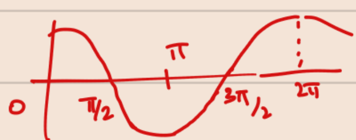
$$\& \quad x_o(t) \triangleq \frac{x(t) - x(-t)}{2}.$$

(Note: " $\triangleq$ " is notation for "is defined as")

Some input signal & signal classes:

①(Real-valued) Sinusoidal signals..

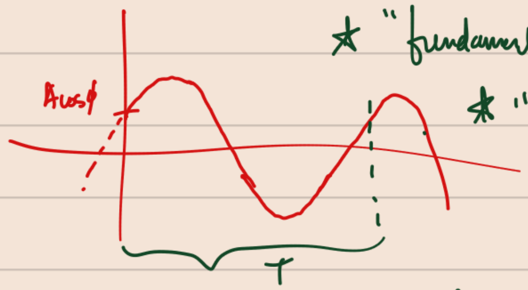
$x(t) = \cos t$



→ (periodic signal with max amplitude = 1, period = 2π (can change), phase shift = 0)

More generally $x(t) = A\cos\Big(\underbrace{\tfrac{2\pi}{T}t + \phi}\Big)$ → " instantaneous phase of the sinusoid "

$\phi \triangleq$ phase shift



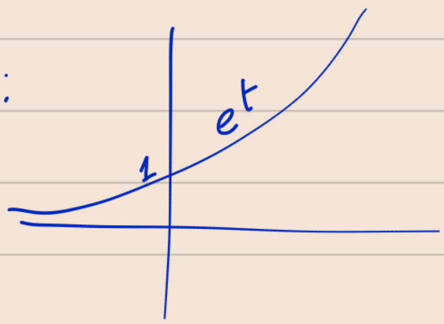
★ " fundamental period " $= T$

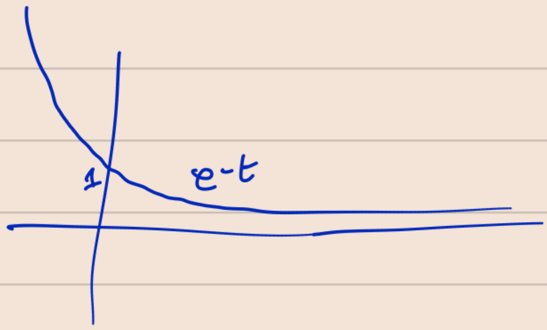
★ " fundamental angular frequency " of the sinusoid $w_0 \triangleq 2\pi/T$.

We then write $x(t) = A\cos(w_0 t + \phi)$.

Third quantity:
" fundamental ordinary freq " $\triangleq 1/T$

Exponential signals:





More generally $x(t) = Ce^{at}$

C, a being real numbers.

$(a \neq 0)$

Complex sinusoid signals:

$$x(t) = e^{jw_0 t} = \cos w_0 t + j\sin w_0 t.$$

(both real & imaginary parts of complex sinusoids are sinusoids)

Complex exponential signals

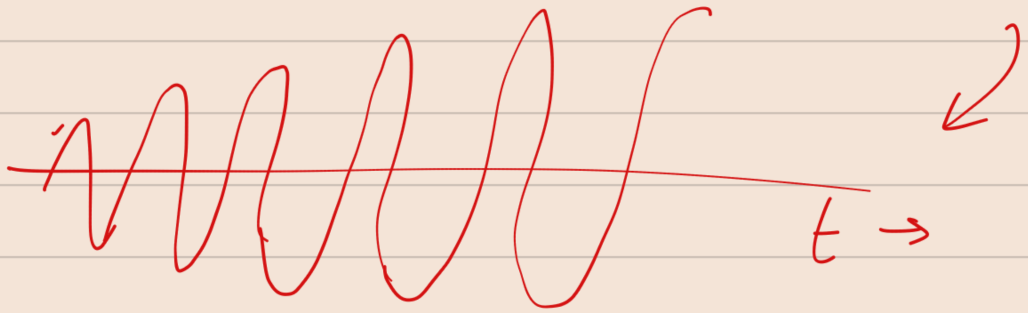
$$x(t) = Ce^{at}, \qquad a, C \in \mathbb{C} \text{ [set of complex numbers]}$$

Let $C = |C|e^{j\theta}$ be the polar representation of C.

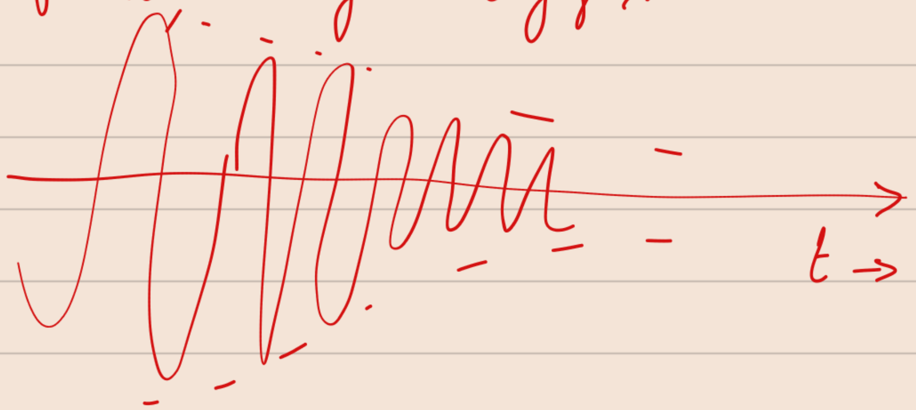
Let $a = r + j\omega_0$ be the rectangular representation of a.

Thus $x(t) = |C|e^{j\theta} \cdot e^{(r+j\omega_0)t} = |C|e^{rt} \cdot e^{j(\omega_0 t + \theta)}$.

Now, if $r > 0$, then the real & imaginary parts of $x(t)$ are growing sinusoids.



If $r < 0$ then they are decaying sinusoids



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Some important quantities associated with a signal.

Energy of a signal : (We are defining a new quantity here).

Energy of signal $x(t)$ in an interval (t_1, t_2)

$$\triangleq \int_{t_1}^{t_2} x^2(t)\,dt$$

(Total) Energy of the signal $\triangleq \int_{-\infty}^{\infty} x^2(t)\,dt$. (denoted by E_∞)

* If $E_\infty < \infty$ (i.e. energy of $x(t)$ is finite), then we call $x(t)$ as a (finite) energy signal.

(Average) Power of the signal $x(t)$

$$P_\infty \triangleq \lim_{T\to\infty} \frac{1}{T} \underbrace{\int_{-T/2}^{T/2} |x(t)|^2 dt}_{\leq E_\infty \text{ for every } T}$$

Thus $P_\infty = 0$, if $E_\infty < \infty$.
$\Rightarrow$ finite energy signal have zero average power

* If $E_\infty = \infty$, but $P_\infty < \infty$ ((i.e) power is finite) then $x(t)$ is called a (finite) power signal
$\rightarrow$ Example: Periodic signals.

HW: Show that, for a signal with period T_0, $E_\infty = \infty$ but

$$P_\infty = \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} x^2(t)\, dt.$$

(i.e for calculating average power it is enough to restrict to one period)

* We could also have signals which have infinite energy & infinite power. Example: Diverging signals.

Special signals of interest:

Impulse (Dirac delta) signal:

Consider a very short 'rectangular' pulse at time $t=0$, of width Δ and area under the signal $=1$.

(i.e) $\delta_\Delta(t) \triangleq \begin{cases} 1/\Delta, & \text{for } t\in[0,\Delta] \\ 0, & \text{otherwise.} \end{cases}$

Observe that $\int_{t=-\infty}^{\infty} \delta_\Delta(t)\, dt = \int_{t=0}^{\Delta} \frac{1}{\Delta}\, dt$

$$= \frac{1}{\Delta} [t]_0^{\Delta} = 1.$$

Now consider the 'limiting' version of the signal $\delta_\Delta(t)$ as $\Delta \to 0$. We get a signal whose area $= 1$, but 'width' $= 0$.

This is our impulse function $\delta(t)$. Formally it is defined as follows.

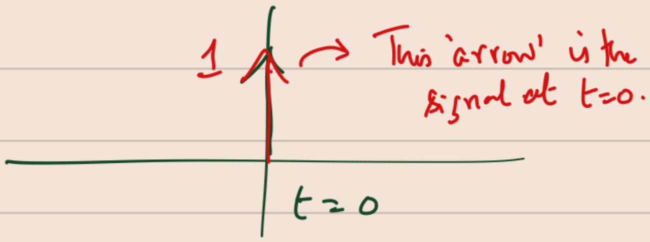
Defn: The impulse function, $\delta(t)$ is that function such that

$$\delta(t) = 0, \forall\, t \neq 0 \quad \text{and} \quad \int_{-\infty}^{\infty} \delta(t)\, dt = 1.$$

(Since the area of the signal is "unity" $(=1)$, this is also called unit impulse).

Pictorially it is represented as

(This is just a representation)



Unit step function:

Now consider a signal $u(t)$, obtained by integrating the impulse from $-\infty$ to t.

$$u(t) \stackrel{\Delta}{=} \int_{-\infty}^{t} \delta(z)\, dz.$$

Observe that for $t < 0$, this signal $u(t) = 0$.

And for $t > 0$, $u(t) = 1$.

At $t = 0$, there is a discontinuity. This signal is called the unit-step function

Formally: $u(t) = \begin{cases} 1, & \forall\, t > 0 \\ 0, & \forall\, t < 0 \end{cases}$

Also know: Scaled, shifted impulse function. } → representation & figure
Shifted unit step (see text / notes from class)

— x —

Some important classes of systems (notes to be added).

Linear System:

→ The idea of a linear system is informally that "sums of input signals" lead to "sums of output signals"

Formally, we have the following definition.

Definition (Linear system):

Consider any two arbitrary input signals $x_1(t)$ and $x_2(t)$, and let $y_1(t)$ and $y_2(t)$ be the outputs of the system corresponding to these two inputs. Let α, β be any two arbitrary constants.

If the output of the system to the input $\alpha x_1(t) + \beta x_2(t)$ is $\alpha y_1(t) + \alpha y_2(t)$, then the system is said to be a linear system.

Note: The property of linearity is also described separately in two properties, as follows.

(1) Homogeneity: If $x(t) \to y(t)$, then $\alpha x(t) \to \alpha y(t)$, $\forall x(t), \forall \alpha$.

(2) Superposition: If $x_1(t) \to y_1(t)$ & $x_2(t) \to y_2(t)$,

then $x_1(t) + x_2(t) \longrightarrow y_1(t) + y_2(t)$, $\forall x_1(t)$
$\forall x_2(t)$.

② <u>Time-invariant systems:</u>

Succinctly, we understand this as follows: "Delay of any fixed time to an input signal causes a similar delay in the output"!

Formally.

For any input $x(t)$, let the output be $y(t)$.
Then for any $t_0 \in \mathbb{R}$ being a constant,

if $x(t-t_0)$ is the input to the system, if the system output is $y(t-t_0)$, then the system is time-invariant.

Figuratively:

If $x(t) \to y(t)$, then $x(t-t_0) \longrightarrow y(t-t_0)$,
$\forall x(t)$, $\forall t_0$.

<u>Linear Time invariant systems (LTI)</u>

Defn: A system is said to LTI system if it is both linear & time-invariant.

Note: LTI systems are easy to analyze compared to general system. Further many real-world systems are approximated as LTI systems

<u>Causal systems:</u> A system is said to be a causal system

if for any input signal $x(t)$, the output at any time instant t depends ONLY on the i/p values at time instant t and before ((ie) for every t, $y(t)$ depends on $x(t')$, for $t' \leq t$ AND NOT ON $x(t')$ for any $t' > t$).

→ For instance: $y(100)$ depends on $x(t')$ for $t' \leq 100$ ONLY and not on $x(100.1)$ or any such value of $x(t)$ for $t > 100$.

Stable systems:

A stable system, or more specifically a bounded-input bounded-output (BIBO) stable system is one in which, for any input signal $x(t)$ such that $|x(t)|$ is bounded by some (finite) constant b_1, the absolute value of the output at every time instant is also bounded by (possibly some other) constant b_2.

(ie) $\forall$ $x(t)$ such that $\underset{\forall t}{|x(t)| \leq b_1}$ (for some finite b_1),

we get an o/p $y(t)$ such that

$$|y(t)| \leq b_2 \quad (\text{for some } b_2), \forall t$$

(The crucial part above is that b_1 & b_2 are NOT functions of time above).

Note: When is a system proved to be not stable?

→ To do this, find a input signal which is bounded, but then the output is NOT bounded.

(For all these concepts given here, look at the Oppenheim textbook, go through examples carefully.)

Fourier Series Representation of Periodic Signals: (FS)

Let $x(t)$ be a (real or complex) periodic with fundamental period T (& thus, fundamental angular frequency $\frac{2\pi}{T} = \omega_0$).

Then, it turns out that $x(t)$ can be represented as a sum of sinusoids of frequencies of the form $k\omega_0$, where $k \in \mathbb{Z}$. To be precise, we give the following as a theorem.

Theorem: There exists constants $a_k \in \mathbb{C}$, for $k \in \mathbb{Z}$ such that, $x(t)$ can be written as

$$x(t) = \sum_{k \in \mathbb{Z}} a_k e^{jk\omega_0 t} \rightarrow (A)$$

Note that this is an infinite sum, in general

Here, the complex sinusoids $a_{\pm k} e^{\pm jk\omega_0 t}$ are called the k^{th} harmonics of $x(t)$.

Caution: this equality between $x(t)$ & the RHS is generally true under some conditions. We are not discussing these here in this course.

Further, the components $a_{\pm 1} e^{\pm j\omega_0 t}$ are ALSO called the fundamental components or the first order harmonics of $x(t)$.

Terminology note: The value 'a_k' in (A) is called the k^{th} Fourier Series (FS) coefficient in the FS expansion of $x(t)$. Observe that a_k is the constant that multiplies with the $k\omega_0$-frequency sinusoid $e^{jk\omega_0 t}$

See that value of a_k depends on $x(t)$ & k, and is given by the following formula.

$$\boxed{a_k = \frac{1}{T} \int_{t \in \text{one period (say } t \in [0,T])} x(t) e^{-jk\omega_0 t} dt.} \rightarrow (1)$$

→ We now verify the expression for a_k in (1), assuming that the fourier series theorem above is true:

Verification that the value of a_k satisfies the RHS of ①:

Consider the integral

$$\int_{t \in \text{One period}} x(t)\, e^{-jk\omega_0 t}\, dt = \int_{t \in \text{one period}} \left(\sum_{k' \in \mathbb{Z}} a_{k'} e^{jk'\omega_0 t} \right) e^{-jk\omega_0 t}\, dt$$

using Ⓐ

Note: we use a different counter k' here, to avoid confusion as the value of k is fixed by LHS

$$= \sum_{k' \in \mathbb{Z}} a_{k'} \left(\int_{t \in \text{one period}} e^{j\omega_0 (k'-k) t}\, dt \right) \rightarrow ②$$

we pull out the sum & push in the integral

Now, if $k' = k$, then

$$e^{j\omega_0 (k'-k) t} = 1,$$

& thus $$\int_{t \in \text{one period}} e^{j\omega_0 (k'-k) t}\, dt = \int_{t \in \text{one period}} 1\, dt = T.$$

If $k' \neq k$, then observe that $e^{j\omega_0 (k-k')t}$ is a sinusoid with frequency $|k'-k|\,\omega_0$. When we integrate this over a period, we get 0, as both the real & imaginary parts ($\cos(k-k')\omega_0 t$ & $\sin(k'-k)\omega_0 t$) go through $(k-k')$ full cycles in one T duration.

Hence, we see that

$$\int_{t \in [0,T]} e^{j\omega_0 (k-k')t}\, dt = \begin{cases} T & \text{if } k = k' \\ 0, & \text{otherwise} \end{cases} \rightarrow ③$$

Thus, using ③ in ②, we see that

$$\int_{t \in [0,T]} x(t)\, e^{-jk\omega_0 t}\, dt = \sum_{k' \in \mathbb{Z}} a_{k'} \left(\int e^{j(k'-k)\omega_0 t}\, dt \right)$$

$$= a_k (T)$$ (since only for $k = k'$, we get a non-zero value from the inner integral).

Thus $$a_k = \frac{1}{T} \int_{t \in [0,T]} x(t)\, e^{-jk\omega_0 t}\, dt.$$

Thus, we have verified ①.

Note: The value of a_k has to be determined for every $k \in \mathbb{Z}$, for us to get the FS representation of $x(t)$.

→ Generally, we write the formula for a_k in ① and compute it. We end up with a simplified expression for RHS of ①, which depends on k

→ Sometimes, it may not be necessary to compute the integral expression in ①. Just by observation of the given signal $x(t)$, we may be able to write $x(t)$ as a FS expansion. The values for a_k can then 'pop' out from this expansion

→ See examples done in class & in the book for both of the above two 'types' of obtaining FS expansions.

Properties of FS: (We only mention some important properties here. For all relevant properties see Table 3.1 in text

① Let $a_k : k \in \mathbb{Z}$ be the FS coefficients of $x(t)$.
We want to determine the FS coefficients of $x^*(t)$ (the conjugate of $x(t)$, which is a different signal than $x(t)$) as a function of the FS coefficients of $x(t)$.
Firstly, observe that $x^*(t)$ has the same fundamental angular frequency w_0 as $x(t)$. Let $b_k : k \in \mathbb{Z}$ be the fourier series coefficients of $x^*(t)$.

Thus $$x^*(t) = \sum_{k \in \mathbb{Z}} b_k e^{jkw_0 t} \rightarrow ①.$$

Now, we already know $x(t) = \sum_{k \in \mathbb{Z}} a_k e^{jkw_0 t}$. Conjugating on both sides,

$$x^*(t) = \sum_{k \in \mathbb{Z}} a_k^* e^{-jkw_0 t}$$

$$= \sum_{k \in \mathbb{Z}} a_{-k}^* e^{jkw_0 t} \rightarrow ②$$

To see these two are equal, just expand them separately and see yourself.

By comparing ① & ②, we see that $b_k = a_{-k}^*$.

Note: The other way to do this is to use b_k's formula as in ① & a_k's formula as in ① & do simple manipulations].

② Consider $x(t)$ & $y(t)$, both with same fundamental frequency w_0 & with FS coefficients, $\{a_k : k \in \mathbb{Z}\}$ & $\{b_k : k \in \mathbb{Z}\}$.
Let $z(t) = x(t)y(t)$.
Let the FS coefficients of $z(t)$ be $\{c_k : k \in \mathbb{Z}\}$. We want to

understand how c_ks are obtainable from a_ks & b_ks.

By given statement, $z(t) = \sum_{k\in\mathbb{Z}} c_k e^{jk\omega_0 t} \rightarrow ①$ (Again observe that $z(t)$ has fundamental frequency ω_0).

Now $z(t) = x(t)y(t) = \left(\sum_{k_1\in\mathbb{Z}} a_{k_1} e^{jk_1\omega_0 t}\right)\left(\sum_{k_2\in\mathbb{Z}} b_{k_2} e^{jk_2\omega_0 t}\right)$

Note the counters here.

$$= \left(\cdots + a_{-2}e^{-j2\omega_0 t} + a_{-1}e^{-j\omega_0 t} + a_0 + a_1 e^{j\omega_0 t} + a_2 e^{j2\omega_0 t} + \cdots\right)$$
$$\times \left(\cdots + b_{-2}e^{-j2\omega_0 t} + b_{-1}e^{-j\omega_0 t} + b_0 + b_1 e^{j\omega_0 t} + b_2 e^{j2\omega_0 t} + \cdots\right)$$

$\rightarrow$ Observe that, in the product above, we again get a sum of weighted exponentials. Specifically, for example, for the term $e^{j2\omega_0 t}$, we get the weight as follows.

$$(\cdots a_{-3}b_5 + a_{-2}b_4 + a_{-1}b_3 + a_0 b_2 + a_1 b_1 + a_2 b_0 + a_3 b_{-1} + a_4 b_{-2} + \cdots\cdots)e^{j2\omega_0 t}$$

Thus, we can write the above equation as

$$= \sum_{k\in\mathbb{Z}} \left(\sum_{k_1\in\mathbb{Z}} a_{k_1} b_{k-k_1}\right) e^{jk\omega_0 t} \longrightarrow ②$$

Thus, we see that by comparing ① & ② above that

$$\underbrace{c_k = \left(\sum_{k_1\in\mathbb{Z}} a_{k_1} b_{k-k_1}\right)}.$$

Observe this expression. We take all pairs of elements from the sequences a_{k_1} & b_{k_2} such that $k_1 + k_2 = k$ & then sum up all such products. Note that the value k is fixed by the subscript of c on the right.

Side note: This type of generating a new sequence c_k, from the two sequences a_{k_1} & b_{k_2} is called a "Convolution" of the two sequences a_{k_1} & b_{k_2}.

FS of multiplication of $x(t)\,y(t)$ $\longrightarrow$ Convolution of FS of $x(t)$ & $y(t)$.

There are many other properties of FS in the text. Do prove all of them yourself! Also go over multiple examples.

FS of real signals:

Suppose $x(t)$ is a real signal, i.e. $x(t) = x^*(t)$

Let $a_k : k \in \mathbb{Z}$ be the FS coefficients of $x(t)$.

Then we know a^*_{-k} are the FS coefficients of $x^*(t)$.

But since $x(t) = x^*(t)$, we must thus have $a_k = a^*_{-k}$, $\forall k$. $\rightarrow$ (1)

$\rightarrow$ This is called Conjugate Symmetry of the FS coefficients, for real signals. Note that, if we apply (1) for $k=0$, we get $a_0 = a^*_{-0}$ (i.e.) $a_0 = a_0^*$. Thus, for real signals, the 0th FS coefficient, is always real. Further $a_1 = a^*_{-1}$, $a_2 = a^*_{-2}$, $a_3 = a^*_{-3}$, & so on.

★ For example, $x(t) = (2-j)e^{-j5\omega_0 t} + 5 + 7e^{-j\omega_0 t} + 7e^{+j\omega_0 t} + (2+j)e^{j5\omega_0 t}$

is surely a real signal. Check this by simplifying the terms on the R.H.S.

Now, using the conjugate symmetry property of the a_ks, we can get different 'real' versions of the FS expression for $x(t)$.

$$x(t) \underset{\substack{\text{(by defn} \\ \text{of FS)}}}{=} \sum_{k \in \mathbb{Z}} a_k e^{jk\omega_0 t} \underset{\text{(rewriting)}}{=} a_0 + \sum_{k=1}^{\infty} \left(a_k e^{jk\omega_0 t} + a_{-k} e^{-jk\omega_0 t}\right)$$

$$\underset{\substack{\text{(using conjugate} \\ \text{symmetry as } x(t) \\ \text{is real)}}}{=} a_0 + \sum_{k=1}^{\infty} \left(a_k e^{jk\omega_0 t} + a_k^* \left(e^{jk\omega_0 t}\right)^*\right)$$

these 2 are conjugates of each other

$$= a_0 + \sum_{k=1}^{\infty} 2\,\mathrm{Re}\left(a_k e^{jk\omega_0 t}\right) \rightarrow (1)$$

Now, recall that a_0 is real. But a_k may be complex.

Using two different ways to express a_k, we get two expressions for $x(t)$.

(i) Using the polar representation for a_k:

Let $a_k = A_k e^{j\phi_k}$, where $A_k = |a_k|$ (note that this is real)

and $\phi_k = \tan^{-1}\left(\frac{\mathrm{Im}(a_k)}{\mathrm{Re}(a_k)}\right)$

Using this, we get from (1) above

$$x(t) = a_0 + 2 \sum_{k=1}^{\infty} A_k \,\mathrm{Re}\left(e^{j(k\omega_0 t + \phi_k)}\right)$$

$$= a_0 + 2 \sum_{k=1}^{\infty} A_k \cos\left(k\omega_0 t + \phi_k\right)$$

(ii) Using the Cartesian representation for a_k:

Let $a_k = B_k + j\, C_k$, where $B_k = Re(a_k)$

$C_k = Im(a_k)$

Then $$x(t) = a_0 + 2\sum_{k=1}^{\infty} Re\left[(B_k + jC_k)(\cos k\omega_0 t + j\sin k\omega_0 t)\right]$$

$$= a_0 + 2\sum_{k=1}^{\infty}\left(B_k \cos k\omega_0 t - C_k \sin k\omega_0 t\right).$$

— x —

Laplace transform:

A generalization of the Fourier Series to general signals (which may or may not be periodic) is the idea of a Laplace transform.

Let $x(t)$ be any signal in time. The Laplace transform of $x(t)$, denoted by $X(s)$, is a function of the complex variable s. (We generally denote the real & imaginary parts of the variable s as σ & ω; thus $s = \sigma + j\omega$ where $\sigma, \omega \in \mathbb{R}$).

The LT $X(s)$ of $x(t)$ is defined as follows (for any $s \in \mathbb{C}$)

$$X(s) \triangleq \int_{t=-\infty}^{\infty} x(t)\, e^{-st}\, dt.$$

Because this integral is defined from $-\infty$ to ∞, it may converge to some finite value or otherwise.

The set of all s values such that $X(s)$ converges to a finite value is called the Region of Convergence (or ROC) of the LT of $x(t)$.

(ie) $$\underset{\text{of } X(s)}{ROC} = \{ s \in \mathbb{C} : X(s) \text{ converges} \}.$$

The definition of $X(s)$ is ALWAYS going along with $X(s)$.
The description of $X(s)$ is incomplete without giving the ROC.

—x—

Examples of LT of signals

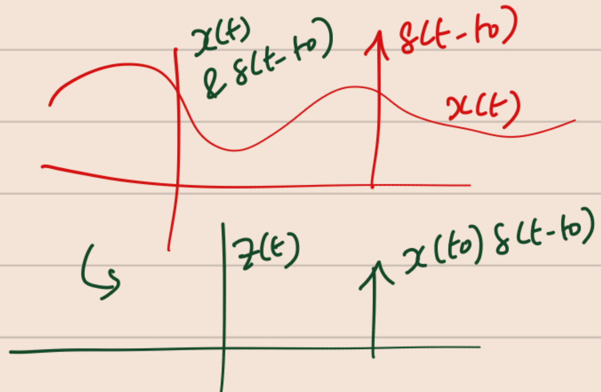
To describe some examples of Laplace transforms, we first prove one important property, called the Sifting property of the unit-impulse function

Consider the impulse function $\delta(t)$. Recall that $\delta(t)$ satisfies

① $\delta(t) = 0, \ \forall t \neq 0$ (value at any $t \neq 0$ is 0).

② & $\int_{t=-\infty}^{\infty} \delta(t) = 1$ (area under $\delta(t) = 1$)

Now, for any signal $x(t)$,
& for some $t_0 \in \mathbb{R}$ (constant), consider the product
$z(t) = x(t)\,\delta(t-t_0)$



By observing their curve, we note that

$$z(t) = \begin{cases} 0, & \forall t \neq t_0 \\ x(t_0)\,\delta(0), & \text{if } t = t_0. \end{cases} \rightarrow ①$$

By the above description, we can therefore write

$$z(t) = \boxed{x(t)\,\delta(t-t_0) = x(t_0)\,\delta(t-t_0)} \rightarrow (2)$$

(Note that the RHS signal has exactly the same definition as (1)).

Note: This itself is called the "Sifting property of the impulse function". This is because somehow the impulse function 'plucks' or 'sifts' out the value $x(t_0)$ from the signal $x(t)$ at the position $\delta(t-t_0)$

Now, consider the convolution of $x(t)$ and $\delta(t)$, defined as follows

$$x(t) * \delta(t) \triangleq \int_{z=-\infty}^{\infty} x(z)\,\delta(t-z)\,dz .$$

By the sifting property defined above, we see that

$$x(z)\,\delta(t-z) = x(z)\,\delta(z-t) = x(t)\,\delta(z-t)$$

(as $\delta(-t') = \delta(t')$ $\forall t'$.) (by sifting property)

Thus

$$\int_{z=-\infty}^{\infty} x(z)\,\delta(t-z)\cdot dz = x(t) \underbrace{\int_{z=-\infty}^{\infty} \delta(z-t)\,dz}_{=1}$$

$$= x(t) . \longrightarrow (A)$$

Therefore we see that the convolution of $x(t)$ with $\delta(t)$ produces $x(t)$ itself.

— x —

Now we return to finding LT of some important signals, which you should know how to calculate & remember also.

① Let $x(t) = \delta(t)$.

Then, for any s, $X(s) = \int_{t=-\infty}^{\infty} \delta(t) e^{-st} dt = \int_{z=-\infty}^{\infty} \delta(z) e^{-sz} dz$

By using Ⓐ above, (plug $x(z) = e^{-sz}$, & $t=0$ in LHS of Ⓐ),

we see that $X(s) = e^{-s(0)} = 1$. . Thus $X(s) = 1, \forall s$.

Note that since $X(s)$ is always converging (bounded here by 1)

thus ROC in this case = entire set of complex numbers = entire s-plane.

② Let $x(t) = e^{-at} u(t)$, where $u(t)$ is the unit step signal and $a \in \mathbb{R}$.

Thus, $X(s) = \int_{t=-\infty}^{\infty} e^{-at} u(t) e^{-st} dt$

$$= \int_{t=0}^{\infty} e^{-(s+a)t} dt = \left[\frac{e^{-(s+a)t}}{-(s+a)} \right]_0^\infty$$

(change limits: From 0 to ∞ as $u(t) = 0, \forall t < 0$)

Let s be denoted as $\sigma + j\omega$. Then we have.

$$\left[\frac{e^{-(s+a)t}}{-(s+a)} \right]_0^\infty = \left[\frac{e^{-(\sigma+a)t} \cdot e^{-j\omega t}}{-(s+a)} \right]_0^\infty$$

Now, observe that $e^{-j\omega t}$ is always a "magnitude 1" complex number. So it is not affected by the value of t.

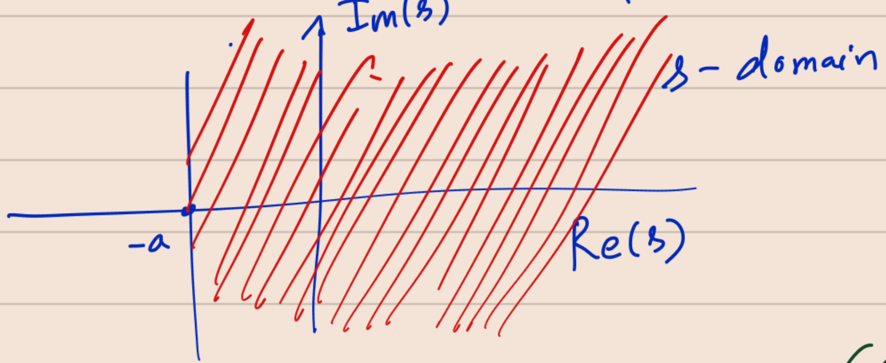
The convergence thus depends only on the value of $(\sigma + a) \in \mathbb{R}$.

If $\sigma + a < 0$, then, the above integral diverges. If $(\sigma + a) > 0$, then it converges. Note that, when $(\sigma + a) > 0$, the upper limit goes to 0.

Thus, we see that

$$X(s) = \left(\frac{1}{s+a}\right), \quad \text{with } ROC = \{s \in \mathbb{C} : Re(s) > -a\}.$$

This is represented pictorially as follows



③ Consider $x(t) = -e^{-at} u(-t)$ (Note that $u(-t) = \begin{cases} 0, & \forall t < 0 \\ 1, & \forall t > 0 \end{cases}$

By a similar calculation as example ②,

we can calculate

$$X(s) = \frac{1}{s+a}, \quad \text{with } ROC = \{s : Re(s) < -a\}$$

Note: Observe that the expression for $X(s)$ is the same for examples ② & ③, but the ROC is different.

"Poles" & "Zeros" of the LT $X(s)$:

In many cases, we can simplify the expression for the LT $X(s)$ and get it in a rational form.

(i.e) $X(s) = \dfrac{A(s)}{B(s)}$, where $A(s)$ & $B(s)$ are polynomials.

→ The set of values of s such that $B(s) = 0$ (i.e, $X(s)$ is not converging) is the set of "poles" of $X(s)$

→ The set of values of s s.t $A(s) = 0$ (i.e $X(s) = 0$) is called the set of "zeros" of $X(s)$.

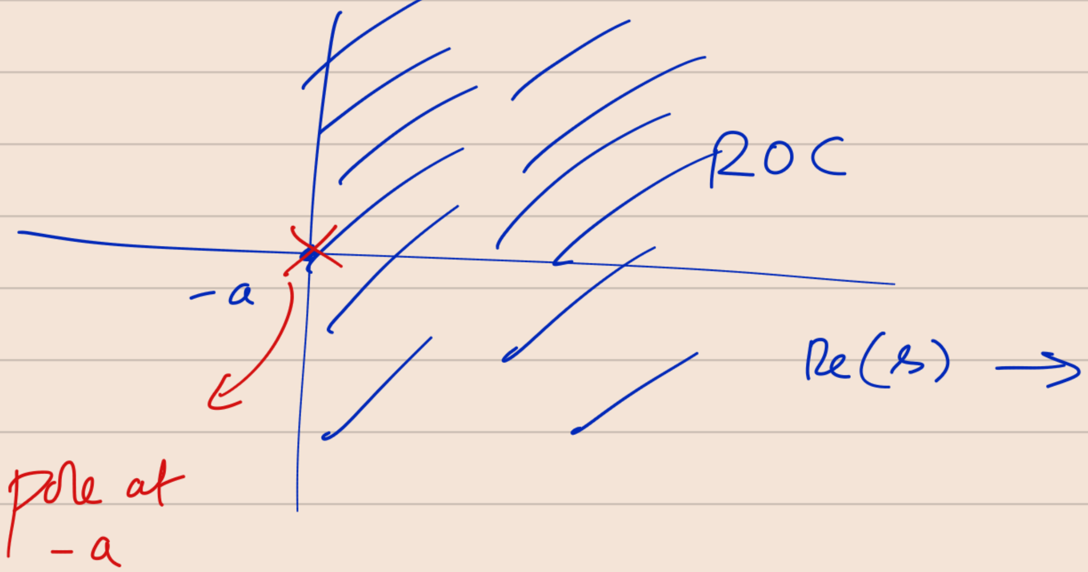
→ The poles & zeros of the LT are quite important for stability analysis of LTI systems and more.

→ For example: $X(s) = \dfrac{1}{s+a}$ has a pole at $s = -a$.

→ What about $X(s) = \dfrac{1}{s+5} - \dfrac{1}{s+1}$? What are its poles and zeros? Calculate. (Express $X(s)$ in rational form first).

In the ROC of $X(s)$, poles are represented by cross-mark "x" & zeros by circles "o".

For example $X(s) = \dfrac{1}{s+a}$, ROC : $\{s : Re(s) > -a\}$.



Properties of Laplace Transforms:

Only some important properties here. For others see text.

(1) Linearity.

If $x(t) \xrightarrow[\text{has LT}]{} X(s)$ with ROC $R_1 \subseteq \mathbb{C}$

$y(t) \longrightarrow Y(s)$ with ROC $R_2 \subseteq \mathbb{C}$

then $\forall\, \alpha, \beta$ constants

$$\alpha x(t) + \beta y(t) \rightarrow \alpha X(s) + \beta Y(s)$$

with ROC containing $R_1 \cap R_2$ (but can be bigger also).

(b) $x(t) \rightarrow X(s)$ with ROC R_1

$y(t) \rightarrow Y(s)$ with ROC R_2

then $$x(t) * y(t) = \int_{z=-\infty}^{\infty} x(z)\, y(t-z)\, dz$$

has LT $X(s)Y(s)$, with ROC containing $R_1 \cap R_2$.

(c) Differentiation: If $x(t) \rightarrow X(s)$, with ROC R_1, then

$$\frac{d}{dt} x(t) \longrightarrow sX(s) \text{ with ROC containing } R_1.$$

— x —

Inverse Laplace transform:

The inverse LT is an operation that helps us to recover the signal $x(t)$ from its LT. $X(s)$

Consider any σ in the ROC of $X(s)$. (fix it as some constant).

Then, we get the following expression for the inverse LT.

$$x(t) = \frac{1}{2\pi j} \int_{s=\sigma+j(-\infty)}^{s=\sigma+j(\infty)} X(s)\, e^{st}\, ds .$$

For example, if $\sigma = 5 \in$ ROC of $X(s)$, then we have

$$x(t) = \frac{1}{2\pi j} \int_{s=5+j(-\infty)}^{5+j\infty} X(s)\, e^{st}\, ds.$$

The limits go only for ω, whereas $\sigma = 5$ is fixed.

— x —

The Fourier transform of $x(t)$:

The FT of $x(t)$ is another operation that represents the signal $x(t)$ in the angular frequency (ω) domain (or, the frequency (f) domain, according to other applications/books).

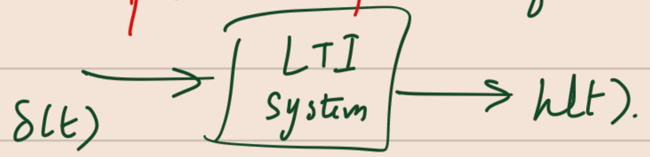
Denoting the FT of $x(t)$ by $\tilde{X}(\omega)$, we have the definition:

$$\tilde{X}(\omega) \triangleq X(s)\big|_{s=j\omega} = X(j\omega)$$

That is, $\tilde{X}(\omega)$ is the value of the LT $X(s)$ evaluated at $s = j\omega$.

Now, we go back to LTI systems. Using the tools we have developed so far (Fourier Series, Impulse functions, Laplace transforms, Convolution) we will analyse & present some really simple descriptions of such systems.

Consider an LTI system. Let $h(t)$ be the response of the system to the input being $\delta(t)$. We call $h(t)$ as the "impulse response" of the system



Claim: The response of the LTI system to any input $x(t)$ is given by

$$x(t) * h(t) = \int_{\tau=-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau$$

$$= \int_{\tau=-\infty}^{\infty} x(t-\tau)\, h(\tau)\, d\tau$$

Proof of claim:

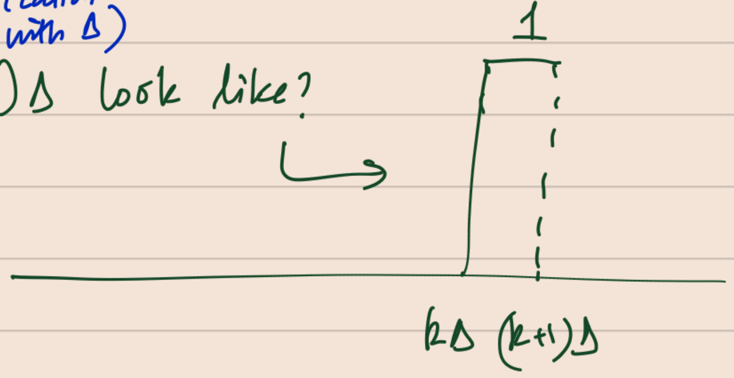
We will first approximate $x(t)$ using a series of rectangular pulses of width Δ, first, as follows.

Consider $\delta_\Delta(t) \triangleq \begin{cases} 1/\Delta & , \text{for } t \in [0, \Delta) \\ 0 & \text{otherwise} \end{cases}$

Thus $\delta_\Delta(t-k\Delta)\cdot\Delta = \begin{cases} 1, & \text{for } t \in [k\Delta, (k+1)\Delta) \\ 0 & \text{otherwise} \end{cases}$

(notice here the multiplication with Δ)

How does $\delta_\Delta(t-k\Delta)\Delta$ look like?



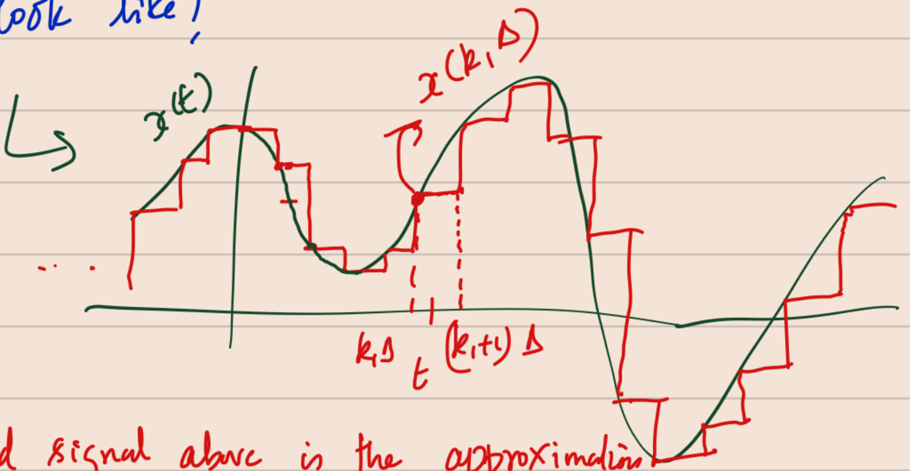
Now, we approximate $x(t)$ in the interval $t \in [k\Delta, (k+1)\Delta)$

as $x(k\Delta)\,\delta_\Delta(t-k\Delta)\,\Delta$.

Thus, for any $t \in \mathbb{R}$, the approximation of $x(t)$, denoted by $x_\Delta(t)$,

looks like : $x_\Delta(t) = x(k_1\Delta)\,\delta_\Delta(t-k_1\Delta)\,\Delta$, for that k_1 such that $t \in [k_1\Delta, (k_1+1)\Delta)$

How does this look like?



The red signal above is the approximation of $x(t)$, denoted by $x_\Delta(t)$

The green signal is $x(t)$.

Now, for any $t \in [k_1\Delta, (k_1+1)\Delta)$, we observe that

$\delta_\Delta(t-k\Delta) = 0, \ \forall\, k \neq k_1$.

Thus we can write

$$\begin{aligned} x_\Delta(t) = \cdots &+ x((k_1-2)\Delta)\,\delta(t-(k_1-2)\Delta)\,\Delta \\ &+ x((k_1-1)\Delta)\,\delta(t-(k_1-1)\Delta)\,\Delta \\ &+ x(k_1\Delta)\,\delta(t-k_1\Delta)\,\Delta \\ &+ x((k_1+1)\Delta)\,\delta(t-(k_1+1)\Delta)\,\Delta \\ &+ x((k_1+2)\Delta)\,\delta(t-(k_1+2)\Delta)\,\Delta \\ &+ \cdots \\ &= \sum_{k=-\infty}^{\infty} x(k\Delta)\,\delta(t-k\Delta)\,\Delta \end{aligned}$$

Here, $[k_1\Delta, (k_1+1)\Delta)$ is that interval in which time instant t is present. Thus, only this red term is non zero (if $x(k_1\Delta) \neq 0$).

All other green terms in the RHS is equal to 0

Note this expression is valid $\forall t$. Also note that $x(k\Delta)$ & Δ do NOT depend on t value. Only $\delta(t-k\Delta)$ does depend on t.

From this expression we can say that $x_\Delta(t)$ is a linear combination of shifted versions of the signal $\delta_\Delta(t)$.

Further, we can observe that

$$x(t) = \lim_{\Delta \to 0} x_\Delta(t) = \lim_{\Delta \to 0} \sum_{k=-\infty}^{\infty} x(k\Delta)\,\delta_\Delta(t-k\Delta)\,\Delta$$

Assume that $h_\Delta(t)$ is the response of the system to the input $\delta_\Delta(t)$. Also observe that

$$\begin{aligned} \lim_{\Delta \to 0} h_\Delta(t) &= \lim_{\Delta \to 0} \left(\text{response of the system to } \delta_\Delta(t)\right) \\ &= \text{response of the system to } \lim_{\Delta \to 0} \delta_\Delta(t) \\ &= \text{response of the system to } \delta(t) \\ &= h(t). \end{aligned}$$

Now, consider the signal $x_\Delta(t)$ as input to the LTI system.

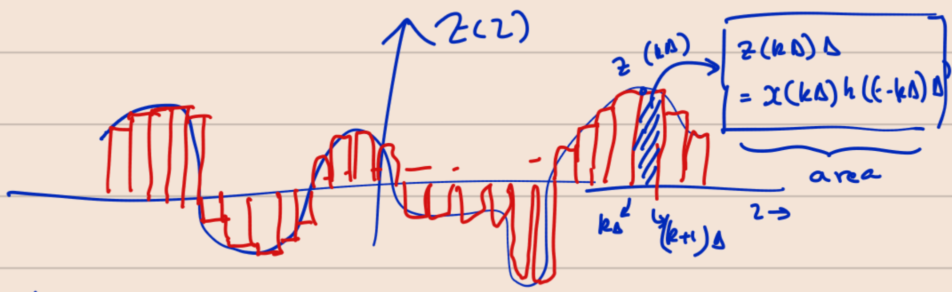
The response $y_\Delta(t)$ will be

$$y_\Delta(t) = \sum_{k=-\infty}^{\infty} x(k\Delta)\, h_\Delta(t-k\Delta)\,\Delta$$

by using the fact that system is LTI

Observe that $y_\Delta(t)$ is the approximation of the area under the curve $z(\tau) = x(\tau)h(t-\tau)$

See figure $\longrightarrow$



Thus $\lim_{\Delta \to 0} y_\Delta(t) =$ Area under the curve $z(\tau)$ above

$$= \int_{\tau=-\infty}^{\infty} z(\tau)\, d\tau$$

$$= \int_{\tau=-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau$$

$$= x(t) * h(t)$$

But $\lim_{\Delta \to 0} y_\Delta(t)$ is precisely the response of the system to input being $\lim_{\Delta \to 0} x_\Delta(t) = x(t)$.

Thus, we have proved the claim.

———— x ————

Causality & Stability of LTI systems via the properties of impulse response $h(t)$:

→ Now, we have shown that the o/p of the LTI system to ANY input signal $x(t)$ is completely dependent only on $x(t)$ & $h(t)$, as $\underbrace{y(t)}_{o/p} = x(t) * h(t)$.

↳ $h(t)$ being the "impulse response" of the system.

→ Thus, we may say that the LTI systems behaviour is completely characterized by the impulse response $h(t)$.

→ So, if we know the response $h(t)$ of the LTI system for unit impulse $\delta(t)$ input, we "know the LTI system".

→ In fact, it turns out that we can characterize the causality & stability of an LTI system purely using the impulse response $h(t)$.

Causality :-

We know that a causal system is one in which the o/p at time t depends only on the i/p at time instants $\tau \leq t$.

Now, for LTI system, we have

$$y(t) = \int_{\tau=-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau$$

If $h(t-\tau) = 0$ for $\tau > t$ (i.e for $(t-\tau) < 0$) → (A)

then, we get

$$y(t) = \int_{\tau=-\infty}^{t} x(\tau)\, h(t-\tau)\, d\tau. \rightarrow (1)$$

Note that, in ①, $y(t)$ depends only on $x(\tau)$ for $\tau \leq t$.
Thus ① is a description of a causal system.

By Ⓐ above, this happens when $h(t-\tau) = 0 \;\; \forall \tau : (t-\tau) < 0$,
Fixing $t' = t-\tau$, this means $h(t') = 0 \;\; \forall t' < 0$.

Thus, for an LTI system, the necessary & sufficient condition

(easy to check this also) (we checked this)

is that $h(t) = 0$, $\forall t < 0$.

$$\boxed{\text{Causal LTI system} \iff \text{Impulse response satisfies } h(t) = 0 \;\; \forall t < 0.}$$

Stability (Bounded input bounded o/p stability):

Now, imagine a bounded input signal $x(t)$, which satisfies $|x(t)| \leq B$, $\forall t$, where B is a constant independent of time t.

The o/p of LTI system to $x(t)$ is

$$y(t) = \int_{\tau=-\infty}^{\infty} x(t-\tau)\, h(\tau)\, d\tau$$

$$\leq \int_{\tau=-\infty}^{\infty} |x(t-\tau)|\,|h(\tau)|\, d\tau$$

$$\leq B \int_{\tau=-\infty}^{\infty} |h(\tau)|\, d\tau$$

$$\leq B \int_{\tau=-\infty}^{\infty} |h(\tau)|\, d\tau \longrightarrow ①$$

Now, if $\int_{\tau=-\infty}^{\infty} |h(\tau)|\, d\tau$ is bounded by some constant, then the RHS of ① is also bounded.

The property that $\int_{t=-\infty}^{\infty} |h(t)|\,dt < \infty$ (bounded) is also referred to as the "absolutely integrable property of $h(t)$". (or) that "$h(t)$ is absolutely integrable".

Thus, if $h(t)$ is absolutely integrable, i.e. $\int_{t=-\infty}^{\infty} |h(t)|\,dt < \infty$, then the system is stable.
Indeed, this is also a necessary condition for the LTI system to be stable. (We can prove this easily by choosing $x(t)=1,\ \forall t$. as our input signal & showing that the output is unbounded).

Thus $\boxed{\text{LTI system is stable} \iff h(t) \text{ is absolutely integrable (or) } \int_{-\infty}^{\infty} |h(t)|\,dt < \infty.}$

— x —

Exponentials as Eigen functions of LTI systems:

There is a surprising fact about LTI systems that we now will understand. Suppose $x(t) = e^{st}$, for some complex value s. is the i/p of the LTI system having $h(t)$ as the impulse response. The o/p of the system is then obtained as follows.

$$y(t) = x(t) * h(t) = \int h(\tau)\, x(t-\tau)\, d\tau$$

$$= \int_{\tau=-\infty}^{\infty} h(\tau)\, e^{s(t-\tau)}\, d\tau$$

$$= e^{st} \left(\int_{\tau=-\infty}^{\infty} h(\tau)\, e^{-s\tau}\, d\tau \right)$$

$$= e^{st} \cdot H(s)$$, where $H(s)$ is the LT of $h(t)$ at s.

$H(s) \to$ the LT of the impulse response – is also called the 'transfer function' of the LTI system.

Thus

$$e^{st} \longrightarrow \boxed{LTI} \longrightarrow H(s)\, e^{st}.$$

Note that $H(s)$ does NOT depend on t. So it is a constant (for given s).

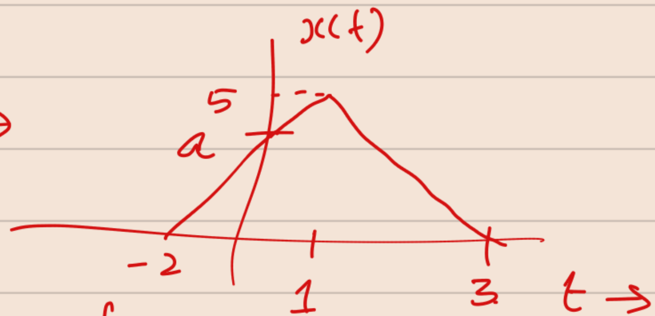
→ This is just like the eigen-value -vector of a matrix. (A square matrix A has eigen-vector $\underline{x}$ & eigen value λ if $A\underline{x} = \lambda \underline{x}$.)

→ Similarly here, the complex exponentials e^{st} form the eigen-functions of an LTI system with eigen-value like role being played by $H(s)$.

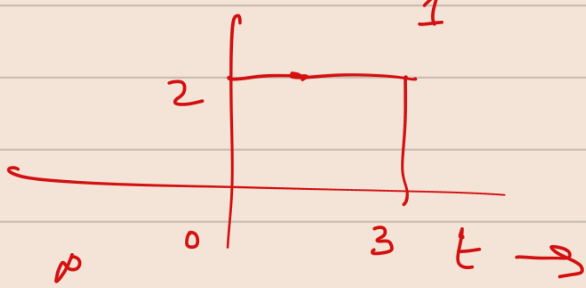
Graphical interpretation of convolution & therefore o/p of LTI system:

We want to understand the convolution of $x(t)$ & $h(t)$ graphically. This helps us solve certain problems & also get a better understanding of convolution operation.

Consider $x(t) \rightarrow$



$h(t) =$



Then consider $y(t) = \int_{\tau=-\infty}^{\infty} x(\tau)\, h(t-\tau)\, d\tau$

Since $y(t)$ looks like the area under the curve $x(\tau)h(t-\tau)$ (as a function of τ), let us draw this out and understand.

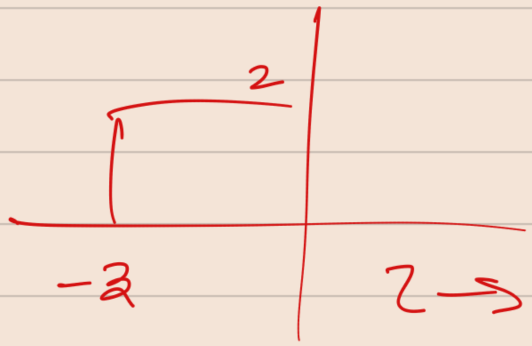
Clearly $x(\tau)$ looks like $x(t)$ (with x-axis replaced by τ)

To picturize $h(t-\tau)$ (as a function of τ), we 'flip' $h(\tau)$ first to get $h(-\tau)$ and then shift it by t (if t is positive, shift to right; if t is $-ve$, shift to left).

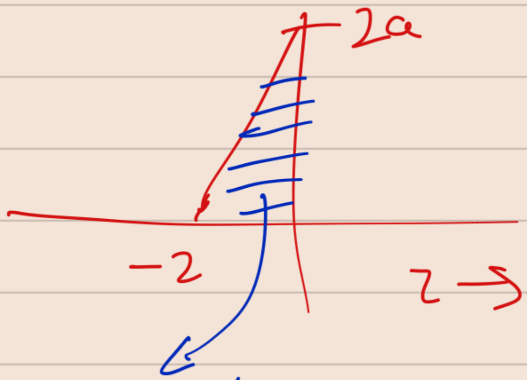
Say $t=0$. Then

$$y(0) = \int_{-\infty}^{\infty} x(\tau)\,h(-\tau)\,d\tau$$

$h(-\tau) =$



Thus $x(\tau)\,h(-\tau)$ looks like

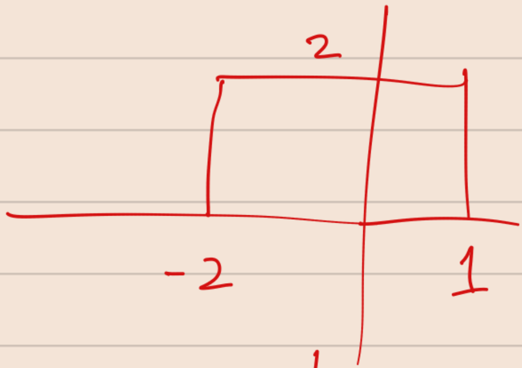


The area under this curve is precisely $y(0)$.

Similarly

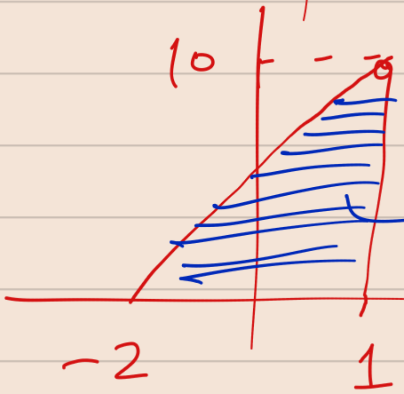
$$y(1) = \int_{\tau=-\infty}^{\infty} x(\tau)\,h(1-\tau)\,d\tau$$

Now $h(1-\tau) \rightarrow$



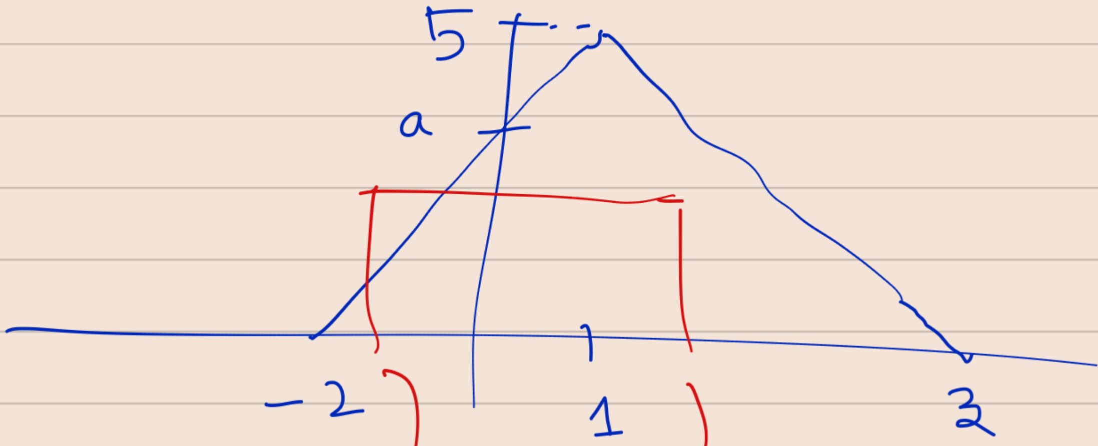
$x(\tau)\, h(1-\tau)$

$\hookrightarrow$



This area is precisely $y(1)$.

Thus we see $x(\tau)$ & $h(t-\tau)$ together



time instant $(t-2)$

time instant t.

→ The area under the product of the blue & red curve gives us the value of $y(t)$.

→ Keep moving t to the right to get the curve $y(t)$.

→ Easy to see that when $t < -2$ and $t > 6$

We get $y(t) = 0$.

The technique in the example above helps us to understand convolution graphically.

— x —