Assignment - 4

1) To prove: If $A \in F^{n \times n}$ be such that the rows of A form a linearly independent set of vectors in F^n. Show that A is invertible.

So, rows of A are linearly independent vectors in F^n.

So,
r_i be a row vector of matrix. (all vectors are linearly independent)

s.t since, every row vector
$c_1\bar{r}_1 + c_2\bar{r}_2 + \dots + c_n\bar{r}_n = \bar{0}$ — ①

only if $c_1 = c_2 = \dots = c_n = 0$, $c_i \in F$

consider Homogeneous system of linear equations
$AX = 0$, where $X \in F^n$

$\vec{\alpha_i} X = \vec{0}\ (r_i)\ X = 0 \Rightarrow$ for all $i = 1, 2, \dots, n$.

Since the rows are linearly independent, The only solution of $AX = 0$ is $X = 0$
because, The $\bar{0}$ cannot be represented as Linear combination of rows (row vectors) until $(x_1, x_2, \dots x_n) = \{0\}$ $X = 0$.

from ① s.t no Zero rows are formed by Linear combination these vectors.

Since, $AX = 0$ has only trivial solution so, A can be row-reduced to $I_{n \times n}$ s.t A is invertible.

Since row of A are linearly independent, $AX = 0$ has only trivial solⁿ. Thus A is invertible.

② To prove:- If W_1 and W_2 are finite dimensional Subspaces of a vector space V, then (I) W_1+W_2 is finite dimensional and

(II) $\dim(W_1)+\dim(W_2)=\dim(W_1\cap W_2)+\dim(W_1+W_2)$

(I) Let
$\dim(W_1)=m$ and $\dim(W_2)=n$.

basis of $W_1=\{\bar{\alpha}_1,\bar{\alpha}_2,\ldots\bar{\alpha}_r,\bar{y}_1,\ldots\bar{y}_{m-r}\}$

$W_2=\{\bar{\alpha}_1,\bar{\alpha}_2,\ldots\bar{\alpha}_r,\bar{k}_1,\ldots,k_{n-r}\}$

Let, The vectors in W_1+W_2 Consists in the form of

$w_1\in W_1$, $w_2\in W_2$ w_1+w_2.

w_1 can be expressed as Linear combination of basis vectors in W_1
w_2 " " " " " " of basis vectors in W_2

s.t w_1+w_2 can be expressed as Linear Combination of union of basis vectors from W_1 and W_2

$(w_1+w_2)\in W_1+W_2$

Thus W_1+W_2 is spanned by a finite set of vectors (The union of the bases W_1 and W_2).

So it is finite dimensional.

(11)

Let $\dim(W_1 \cap W_2) = k$,

$\text{basis}(W_1 \cap W_2) = \{v_1, v_2, \ldots, v_k\}$

$W_1 \cap W_2$ subspace of W_1 & W_2

By Lemma:-

We can Extend basis of $W_1 \cap W_2$

to (W_1) basis

Let, $\vec{\beta} \notin W_1 \cap W_2$ s.t basis $(W_1 \cap W_2) \cup \vec{\beta}$

spans W' (bcz $c_1v_1 + c_2v_2 + \cdots + c_nv_n + b\vec{\beta} = \vec{0}$) [if only $c_1, \ldots, c_n = \{0\}$ and $b = \{0\}$]

by following this until cardinality of basis

becomes until m (basis of W_1)

since Extended basis be

basis of $W_1 = \{v_1, v_2, v_3, \ldots, v_k, w_1, w_2, \ldots, w_{m-k}\}$

where $\dim(W_1) = m$

Similarly

$W_1 = \{v_1, v_2, \ldots v_k, w_1, w_2, \ldots, w_{m-k}\}$

$W_2 = \{v_1, v_2, \ldots v_k, u_1, u_2, u_3, \ldots, u_{n-k}\}$

$$\text{Basis of } (W_1 + W_2) = \{\overbrace{v_1, v_2, \ldots, v_k}^{k}, \overbrace{w_1, w_2, \ldots, w_{m-k}}^{m-k}, \overbrace{u_1, u_2, \ldots, u_{n-k}}^{n-k}\}$$

Spans $W_1 + W_2$

Basis of $W_1 + W_2$ is union of basis(W_1), basis(W_2)

s.t still Basis $(W_1 + W_2)$ is linearly dependent.

$\dim(W_1+W_2) = k + (m-k) + (n-k) = m+n-k.$

WKT,

$\dim(W_1) = m$

$\dim(W_2) = n$

$\dim(W_1) + \dim(W_2) = m+n$

$\dim(W_1 \cap W_2) + \dim(W_1+W_2) = k + (m+n-k)$

$= m+n$

$\therefore$ Hence,

$\dim(W_1) + \dim(W_2) = \dim(W_1 \cap W_2) + \dim(W_1+W_2)$