Q(1). wkT,

$A = USV^T$

~~AA^T~~

Toprove:- AA^T has the columns of U as eigenvectors with associated eigenvalues s^2.

wkT, $AA^T = (USV^T)(USV^T)^T = USV^TVS^TU^T$

V is orthogonal ($V^TV = I$) $S^T = S$ (S is diagonal matrix)

$\boxed{Y = diag[\sigma_1^2, \sigma_2^2 \dots \sigma_r^2, 0, 0, 0..]}$ $AA^T = USS^TU^T$.

only if S is square matrix

$S \cdot S^T = Y$

$$S \cdot S^T = \begin{bmatrix} \sigma_1 & 0 & 0 & \dots & 0 \\ 0 & \sigma_2 & \dots & & \\ 0 & 0 & \sigma_3 & \dots & 0 \end{bmatrix}$$

Y is a diagonal (symmetric matrix)

$AA^T = UYU^T$ compar (QDQ^T)

The eigen values AA^T are the diagonal entries of Y and U is orthogonal matrix.

$U \cdot U^T = I$ (property of SVD)

and columns of U are eigen vectors of $A \cdot A^T$ $Y = S^2$ has $[\sigma_1^2, \sigma_2^2, \dots \sigma_n^2]$ diagonally.

because square diagonal matrix is square of entries

So, $\therefore$

The coloumns of are the eigen vectors of AA^T

The eigen values of AA^T are the squares of the ~~the~~ singular values of A (i.e σ_i^2)

~~...~~

u_i be i-th coloumn of U

$$\rightarrow AA^T u_i = UDU^T u_i$$

$$\rightarrow De_i = \sigma_i^2 e_i, =$$

$$\rightarrow = \sigma_i^2 u e_i = \sigma_i^2 u_i$$

confirms that u_i is an eigen vector of AA^T with eigen value σ_i^2.

Q2) $A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$ WKT, (SVD) $A = U \Sigma V^T$

$$A^TA = \begin{pmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 2 & 2 & 2 \\ 2 & 2 & 2 \\ 2 & 2 & 2 \end{pmatrix}$$

$$AA^T = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 3 & 3 \\ 3 & 3 \end{pmatrix}$$

$$|A^TA - \lambda I| = 0$$

$$\cancel{\frac{1-\lambda}{2}\left(\left(\frac{1-\lambda}{2}\right)^2 - 1\right)}$$

$$(2-\lambda)((2-\lambda)^2 - 4) - 2((2-\lambda)2 - 4)$$

$$2(4 - (2-\lambda)2)$$

$$(2-\lambda)((2-\lambda)^2 - 4) - 4((2-\lambda)2 - 4)$$

$$(2-\lambda)((2-\lambda)^2-2)((2-\lambda)+2)-8(2-\lambda-2)$$

$$-\lambda((2-\lambda)(4-\lambda)-8)=0.$$

$$\lambda(\lambda^2-6\lambda+8-8)=0.$$

$$\lambda(\lambda^2-6\lambda)=0$$

$$\lambda^2(\lambda-6)=0.$$

$$\lambda=0,0,6$$

$\underline{\lambda=6}$

$$\begin{pmatrix} -4 & 2 & 2 \\ 2 & -4 & 2 \\ 2 & 2 & -4 \end{pmatrix}=0.$$

$$\begin{bmatrix} -4 & 2 & 2 \\ -2 & -2 & 4 \\ -2 & 4 & -2 \end{bmatrix}=0 \quad R_3 \to R_3-R_2$$

$$\begin{bmatrix} -4 & 2 & 2 \\ 0 & -6 & 6 \\ 0 & 6 & -6 \end{bmatrix}=0. \quad R_3 \to R_2+R_3$$

$$-6y+6z=0$$

$$-4x+2y+2z=0 \quad \boxed{y=z}$$

$$-4x+4y=0$$

$$\boxed{x=y}$$

$$u_3=\begin{bmatrix}1\\1\\1\end{bmatrix}$$

after normallizing $= v_3=\frac{1}{\sqrt{3}}\begin{bmatrix}1\\1\\1\end{bmatrix}$

$\underline{\lambda=0}$

$$x+y+z=0.$$

$$u_1 = v_1 = \begin{bmatrix}1\\-1\\0\end{bmatrix} \quad u_2=\begin{bmatrix}1\\0\\-1\end{bmatrix}$$

Orthogonalizing V_2 by gram Schdmit proces

$$v_2 = u_2 - \frac{\langle v_1, u_2 \rangle}{\langle v_1, v_1 \rangle} v_1$$

$$= \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} - \frac{1}{2} \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

$$v_2 = \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} \cdot 2$$

normalizing $v_2 = \frac{1}{\sqrt{6}} \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}$

$$v_1 = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \quad v_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

$$v_3 = \frac{1}{\sqrt{6}} \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix}.$$

(b) eigen(values & vectors) of AA^T

$$|AA^T - \lambda I| = 0$$

$$\begin{pmatrix} 3-\lambda & 3 \\ 3 & 3-\lambda \end{pmatrix} = (3-\lambda)^2 - 9 = 0.$$

$$\lambda = 0, 6$$

$\underline{\underline{\lambda = 6}}$

$$\begin{pmatrix} -3 & 3 \\ 3 & -3 \end{pmatrix} u = 0 \qquad -3x + 3y = 0$$

$$x = y$$

$u_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ normalizing $u_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$\underline{\underline{\lambda = 0}}$ $\quad \begin{pmatrix} 3 & 3 \\ 3 & 3 \end{pmatrix} \quad x + y = 0 \quad x = -y$

$u_2 = \begin{bmatrix} -1 \\ +1 \end{bmatrix}$

normalizing

$u_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 1 \end{bmatrix}$

$U = [u_1, u_2]$

$V = [v_1 \; v_2 \; v_3]$.

$\Sigma = \begin{bmatrix} \sqrt{6} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

SVD: $A = U \Sigma V^T$

$= \begin{pmatrix} \frac{1}{\sqrt{2}} & -1/\sqrt{2} \\ 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix} \begin{pmatrix} \sqrt{6} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \\ -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{\sqrt{6}} & -\frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}} \end{pmatrix}$

The outer product form of A is

$A = \sum \sigma_i u_i v_i^T$ $\quad \boxed{i \in \mathbb{Z},\ 0 \leq i < \text{rank}(A)}$

$\sigma_1 = 6$

$\sigma_{2,3} = 0$

$A = \sigma_1 u_1 v_1^T + \sigma_2 u_2 v_2^T + \sigma_3 u_3 v_3^T$

$\sigma_1 = \sqrt{\lambda_1} = \sqrt{6}$.

$\sigma_2 = \sqrt{\lambda_2} = \sqrt{0} = 0$.

$\sigma_3 = \sqrt{\lambda_3} = 0$.

$A = \sqrt{6} \left[\sqrt{2} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right] \left[\sqrt{3} [1 \; 1 \; 1] \right]$

$A = \sqrt{6} \begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \end{bmatrix}$

$A = \frac{\sqrt{6}}{\sqrt{2} \times \sqrt{3}} \begin{bmatrix} 1 \\ 1 \end{bmatrix} [1 \; 1 \; 1] + 0 + 0 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} [1 \; 1 \; 1]$

Q3) $f(x_1, x_2) = x_1^2 + 8x_1x_2 + x_2^2$ — (1)

A is a symmetric matrix associated with Quadratic form

$$Ax = \begin{bmatrix} x_1^2 & \frac{x_1x_2}{2} \\ \frac{x_2x_1}{2} & x_2^2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$A = \begin{bmatrix} 1 & 8/2 \\ 8/2 & 1 \end{bmatrix} = \begin{pmatrix} 1 & 4 \\ 4 & 1 \end{pmatrix}$$

$|A - \lambda I| = 0$ to find eigen values.

$$|A - \lambda I| = \begin{vmatrix} 1-\lambda & 4 \\ 4 & 1-\lambda \end{vmatrix} = 0$$

$$(1-\lambda)^2 - 8\cancel{x2} = 0.$$

$$\lambda^2 + 1 - 2\lambda - 8\cancel{x2} = 0.$$

$$\lambda^2 - 2\lambda - 15 = 0.$$

$$\lambda = \frac{2 \pm \sqrt{4+60}}{2} = 5, -3$$

$\underline{\underline{\lambda = 5}}$

$$\begin{pmatrix} -4 & 4 \\ 4 & -4 \end{pmatrix} = 0.$$

$-x_1 + \cancel{y}^{x_2} = 0$ $\quad x_1 = \cancel{y}^{x_2}$.

$E_{v_1} = \{t \cdot \begin{bmatrix} 1 \\ 1 \end{bmatrix}\}$ $\quad v_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

normalize

$$v_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$\lambda = -3$

$$\begin{bmatrix} 4 & 4 \\ 4 & 4 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

$4x_1 = +4(\cancel{-y}) x_2$

$x = -\cancel{y}(x_2)$ $\quad V_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

normalize

$E_{V_2} = \left\{ t \cdot \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right\}$ $\quad V_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

V_1 & V_2 are already orthogonal.

$$Q = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{bmatrix} \quad QQ^T = I$$

$A = QDQ^T$ $\quad Q^T = Q^{-1}$ (orthogonal)

$Q^T A Q = Q^T Q D Q^T Q$

$Q^T A Q = D Q^T Q$

$Q^T A Q = D = \begin{pmatrix} 5 & 0 \\ 0 & -3 \end{pmatrix}$

$$\boxed{f(y) = 5y_1^2 - 3y_2^2}$$

Let $y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$

$x = Qy$

$x = \begin{pmatrix} \frac{1}{\sqrt{2}} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix}\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$

$x = \begin{pmatrix} \frac{y_1 + y_2}{\sqrt{2}} \\ \frac{y_1 - y_2}{\sqrt{2}} \end{pmatrix}$

Put back in og. eqn. 1 as x_1 & x_2

4. given

$$A = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix} \qquad A^T = \begin{bmatrix} 1 & 2 & 2 \\ 2 & 1 & 2 \\ 2 & 2 & 1 \end{bmatrix}$$

here $A = A^T$ (i.e A is a symmetric matrix)

by spectral theorem the given matrix can orthogonally diagonalizable.

$$A - \lambda I = \begin{bmatrix} 1-\lambda & 2 & 2 \\ 2 & 1-\lambda & 2 \\ 2 & 2 & 1-\lambda \end{bmatrix}$$

$|A - \lambda I| = 0.$ let $\frac{1-\lambda}{2} = x.$

$$A - \lambda I = \begin{bmatrix} x & 1 & 1 \\ 1 & x & 1 \\ 1 & 1 & x \end{bmatrix}$$

$$x(x^2+1) - 1(x-1) + 1(1-x)$$

$$x(x^2-1) - 1(x-1) - 1(x-1)$$

$$x(x^2-1) - 2(x-1)$$

$$x^3 - x - 2x + 2 = 0.$$

$$x^3 - 3x + 2 = 0.$$

$$\boxed{x = 1}$$

$$1 - 3 + 2 = 0.$$

$$\begin{array}{r|rrrr} x=1 & 1 & 0 & -3 & 2 \\ & 0 & 1 & 1 & -2 \\ \hline & 1 & 1 & -2 & \boxed{0} \end{array}$$

$$(x^2 + x - 2)(x - 1)$$

$$(x^2 + 2x - x - 2)(x-1) \Rightarrow (x+2)(x-1)^2$$

$x = \frac{1-\lambda}{2}$

$x = 1 = \frac{1-\lambda}{2}$ $\quad$ $x = 1 = \frac{1-\lambda}{2}$ $\quad$ $x = -2 = \frac{1-\lambda}{2}$

$1-\lambda = 2$ $\quad$ $1-\lambda = 2$ $\quad$ $1-\lambda = -4$

$\lambda_1 = -1$ $\quad$ $\lambda_2 = -1$ $\quad$ $\lambda_3 = 5.$

$\lambda = -1$ (algebraic multiplicity 2)
$\lambda = 5$ (algebraic multiplicity 1)

$\underline{\lambda_1 = -1}$

$$\begin{bmatrix} 1-(-1) & 2 & 2 \\ 2 & 2 & 2 \\ 2 & 2 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0.$$

$x + y + z = 0$

$$V_1 = u_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \qquad u_2 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

v_1 is not orthogonal to v_2

$$V_2 = u_2 - \frac{\langle V_1, u_2 \rangle}{\langle V_1, V_1 \rangle} V_1 = \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} - \frac{1}{2} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$$

$$= \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix} \times \cancel{\frac{1}{2}}$$

(scaling) negelected as constant because it's basis

$\underline{\lambda = 5}$

$$u_3 = \begin{bmatrix} -4 & 2 & 2 \\ 2 & -4 & 2 \\ 2 & 2 & -4 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

$R_2 \to R_2 + R_1$

$R_3 \to R_3 + R_1$

$$\begin{bmatrix} -4 & 2 & 2 \\ -2 & -2 & 4 \\ -2 & 4 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0 \qquad R_3 \to R_3 - R_2$$

$$\begin{bmatrix} -4 & 2 & 2 \\ 0 & -6 & 6 \\ 0 & 6 & -6 \end{bmatrix} \qquad R_3 \to R_2 + R_3$$

$$\begin{bmatrix} -4 & 2 & 2 \\ 0 & -6 & 6 \\ 0 & 0 & 0 \end{bmatrix}$$

$$-6y + 6z = 0$$

$$\boxed{y = z}$$

$$-4x + 2y + 2z = 0$$

$$-4x + 4y = 0$$

$$\boxed{x = y}$$

$$v_3 = u_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

v_1 and v_2 are orthogonal to u_3 (v_3)

Normallizing v_1, v_2, v_3

$$v_3 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

$$v_2 = \frac{1}{\sqrt{6}} \begin{pmatrix} 1 \\ 1 \\ -2 \end{pmatrix}$$

$$v_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$$

$\boxed{\text{Q is an orthogonal matrix}}$

$$A = QDQ^T$$

$$A = \underbrace{\begin{bmatrix} 1/\sqrt{6} & 1/\sqrt{2} & 1/\sqrt{3} \\ 1/\sqrt{6} & -1/\sqrt{2} & 1/\sqrt{3} \\ -2/\sqrt{6} & 0 & 1/\sqrt{3} \end{bmatrix}}_{Q} \underbrace{\begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 5 \end{bmatrix}}_{D} \underbrace{\begin{bmatrix} 1/\sqrt{6} & 1/\sqrt{6} & -2/\sqrt{6} \\ 1/\sqrt{2} & -1/\sqrt{2} & 0 \\ 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \end{bmatrix}}_{Q^T}$$