Mid sem. Model answer.

1. Prove that every convergent sequence of real numbers is bounded.

Proof: Let $\{S_n\}$ be a sequence of real numbers which converges to S.

Then $|S_n - S| < \varepsilon \quad \forall\, n \geq N \in \mathbb{N}$.

Consider $\varepsilon = 1$.

Then we have

$$|S_n - S| < 1 \quad \forall\, n \geq N_1 \in \mathbb{N}.$$

Therefore

$$\begin{aligned} |S_n| &= |S_n - S + S| \\ &\leq |S_n - S| + |S| \\ &\leq 1 + |S| \quad \forall\, n \geq N_1. \end{aligned}$$

Let $M = \max\{|S_1|, |S_2| \dots |S_N|, 1+|S|\}$.

Then $|S_n| \leq M \quad \forall\, n \in \mathbb{N}$.

Therefore the sequence $\{S_n\}$ is bounded.

2. State and prove squeeze theorem.

Ans:

Statement: If $\{s_n\}$, $\{t_n\}$ and $\{u_n\}$ are sequences such that $s_n \leq t_n \leq u_n$ $\forall\, n \in \mathbb{N}$. Then if $\lim_{n\to\infty} s_n = \lim_{n\to\infty} u_n = l$, then $\lim_{n\to\infty} t_n = l$.

Proof: Let $\varepsilon > 0$ is given.

Then $|s_n - l| < \varepsilon \quad \forall\, n \geq N_1 \in \mathbb{N}$.

and $|u_n - l| < \varepsilon \quad \forall\, n \geq N_2 \in \mathbb{N}$.

So we have $l - \varepsilon < s_n < l + \varepsilon \quad \forall\, n \geq N_1$

$l - \varepsilon < u_n < l + \varepsilon \quad \forall\, n \geq N_2$

Consider $N = \max\{N_1, N_2\}$.

Then $l - \varepsilon < s_n \leq t_n \leq u_n < l + \varepsilon$.

$\forall\, n \geq N$.

$\Rightarrow \quad l - \varepsilon < t_n < l + \varepsilon \quad \forall\, n \geq N$

$\Rightarrow \quad |t_n - l| < \varepsilon \quad \forall\, n \geq N$.

Therefore $\lim_{n\to\infty} t_n = l$ $\square$.

3. Define Cauchy sequence and prove that all convergent sequences of real numbers are Cauchy sequences.

Ans. ~~Eo~~

Cauchy sequence:

A sequence is said to be Cauchy if for a given arbitrary $\varepsilon > 0$, $\exists\, N \in \mathbb{N}$ s.t. $|S_n - S_m| < \varepsilon \ \forall\ (n, m) \geq N$.

To prove the given statement in the question, we need to prove all convergent Cauchy sequences (of $\mathbb{R}$) are ~~Cauchy~~ and also all convergent sequences are Cauchy.

i) Every convergent sequence is Cauchy.

Consider a convergent sequence

$\lim\limits_{n\to\infty} S_n = l$.

Then $|S_n - l| < \varepsilon/2 \quad \forall\ n \geq N_1 \in \mathbb{N}$.

and $|S_m - l| < \varepsilon/2 \quad \forall\ m \geq N_2 \in \mathbb{N}$.

Consider $N = \max\{N_1, N_2\}$.

Therefore

$$|S_n - S_m| = |S_n - l + l - S_m|$$
$$\leq |S_n - l| + |S_m - l|.$$
$$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon \quad \forall\, n \geq N.$$

which is Cauchy $\square$.

Every Cauchy sequence is convergent.

Let $\{S_n\}$ be a Cauchy sequence.
Therefore Cauchy sequence $\{S_n\}$ is bounded (Do not need to prove this).
Then by BW theorem, we know that $\{S_n\}$ has a subsequence $\{S_{n_k}\}$ which converges to some $l \in \mathbb{R}$.

Therefore.

$|S_n - S_m| < \varepsilon/2 \quad \forall\, (n, m) \geq N_1 \in \mathbb{N}.$

$|S_{n_k} - l| < \varepsilon/2 \quad \forall\, n_k \geq N_2 \in \mathbb{N}.$

Let $M = \max\{N_1, N_2\}$

Therefore

$$|S_n - l| = |S_n - S_{n_k} + S_{n_k} - l|$$
$$\leq |S_n - S_{n_k}| + |S_{n_k} - l|$$
$$< \varepsilon/2 + \varepsilon/2 = \varepsilon \quad \square.$$

4. If $\lim_{x \to c} f(x) = l_1$ and $\lim_{x \to c} g(x) = l_2$.

P.T. a) $\lim_{x \to c} \left(f(x) + g(x) \right) = l_1 + l_2$

b) $\lim_{x \to c} \left(f(x)\, g(x) \right) = l_1 l_2$.

Ans: a) $\cancel{|f(x) - l_1| < \varepsilon/2 \quad \forall\, n \geqslant N_1 \in \mathbb{N}}$.

$\cancel{|g(x) - l_2| < \varepsilon/2 \quad \forall\, n \geqslant N_2 \in \mathbb{N}}$.

~~Consider~~ $\cancel{N = \max}$

a) $|f(x) - l_1| < \varepsilon/2 \quad \forall\, x \in N'(c, \delta_1) \cap D$.

$|g(x) - l_2| < \varepsilon/2 \quad \forall\, x \in N'(c, \delta_2) \cap D$.

Consider $\delta = \min(\delta_1, \delta_2)$.

$$
\begin{aligned}
&|f(x) + g(x) - l_1 - l_2| \\
&\leqslant |f(x) - l_1| + |g(x) - l_2| \\
&< \varepsilon/2 + \varepsilon/2 = \varepsilon \quad \forall\, x \in N'(c, \delta) \cap D.
\end{aligned}
$$

$\square$.

b)
$$
\begin{aligned}
&|f(x)\, g(x) - l_1 l_2| \\
&= |f(x)\, g(x) - l_1 g(x) + l_1 g(x) - l_1 l_2| \\
&\leqslant |g(x)|\, |f(x) - l_1| + |l_1|\, |g(x) - l_2|.
\end{aligned}
$$

Now. $|g(x)| \leq M \in \mathbb{R}$.

(Since $g(x)$ is bounded).

and consider $K = \max(M, |l_1|)$.

So, Now, $|f(x) - l_1| < \varepsilon/_{2K}$ $\forall$ $x \in N'(c, \delta_1) \cap D$

$|g(x) - l_2| < \varepsilon/_{2K}$ $\forall x \in N'(c, \delta_2) \cap D$.

$\delta = \min(\delta_1, \delta_2)$.

Therefore.

$|f(x)g(x) - l_1 l_2| < K.(|f(x) - l_1| + |g(x) - l_2|)$.

$= \varepsilon$.

$\forall x \in N'(c, \delta) \cap D$.

$\square$.

5. Consider $\{x_n\}$ be a sequence convergi[ng] to l. S.t. $y_n = \frac{x_1 + x_2 + \cdots x_n}{n}$ also converges to l.

Ans. Problem 3.15 of the Handbook.