Assignment-5

Q3) Given V, W are finite dimensional vector spaces
$L(V, W)$ is set of all Linear Transformations
Let,

$\dim(V) = n$

$\dim(W) = m$

let $\dim(V) = n$, $\dim(W) = m$

basis of $V = (\bar{\alpha}_i)_{i=1}^{n}$, $W = (\beta_i)_{i=1}^{m}$

basis V spans V and W spans W

now by Linear transformation $T \in L(V, W)$
$T : V \to W$

on $(\bar{\alpha}_i)_{i=1}^{n}$ we get

$$T(\bar{\alpha}_i)_{i=1}^{n} \Rightarrow (T\bar{\alpha}_i)_{i=1}^{n} \in W$$

as basis of W spans W

$(T\bar{\alpha}_i)_{i=1}^{n}$ can be represented as the Linear Combination of $(\bar{\beta}_j)_{j=1}^{m}$

$$T\bar{\alpha}_i = \sum_{j=1}^{m} c_{ij} \bar{\beta}_j$$

there are n $\bar{\alpha}$ vectors that each are represented as combination of m vectors
then we can represent this transformation from V to W using $m \times n$ constants.
Let us represent each linear transformation from V to W using an $m \times n$ matrix consisting constants $\left((c_{ij})_{i=1}^{n}\right)_{j=1}^{m}$
Now By defining standard basis for $F^{m \times n}$

$\therefore \dim(L(V, m)) = n \cdot m$

$\dim(L(V, W)) = \dim(V) \times \dim(m)$ $\therefore$ Hence proved.

2) Given

$L(V,W)$ is the set of all Linear transformation from V to W then,

$\forall \bar{\alpha} \in V, \forall T \in L(V,W)$
$T\bar{\alpha} \in W$

$\forall \bar{\alpha} \in V$ the vector addition of elements $T, V \in L(V,W)$ be defined as

$$(T+V)\bar{\alpha} = T\bar{\alpha} + V\bar{\alpha}$$

and the scalar multiplication of an element $c \in F$ with $T \in L(V,W)$ be defined as

$$cT(\bar{\alpha}) = c(T\bar{\alpha})$$

now, $T, U \in L(V,W)$

$T : V \to W$ $\quad (T+V)\bar{\alpha} = T\bar{\alpha} + U\bar{\alpha}$

$U : V \to W$

appling the Linear transformation $T+U$ on $\bar{\alpha}$ we get the above eq^n, then

$T\bar{\alpha} \in W, U\bar{\alpha} \in W$ as W is a vector space

$T\bar{\alpha} + U\bar{\alpha} \in W$

$\alpha \in V$

$T, U \in L(V,W)$

then, $(T+V) \in L(V,W)$

$\forall \bar{\alpha} \in V, T, U \in L(V,W)$

$(T+U)\bar{\alpha} = T\bar{\alpha} + U\bar{\alpha}$ s.t

$(T \cdot 1V) \in L(V,W)$

now, take a Linear Transformation $T \in \mathcal{L}(V, W)$

$T: V \to W$

and $c \in F$

applying L.T

$cT(\bar{\alpha}) = c(T\bar{\alpha})$

$T\vec{\alpha} \in W$, $c(T\bar{\alpha}) \in W$

as W is a vector space

then, $\forall T \in \mathcal{L}(V, W)$

$\forall c \in F$

$cT \in \mathcal{L}(V, W)$

as both the vector addition and scalar multiplication is defined and closed over $\mathcal{L}(V, W)$ and matches the required condition. $\mathcal{L}(V, W)$ is a vector space

∴ Hence proved.

① To prove $T: V \to W$ T be a Linear Transformation

$$\text{rank}(T) + \text{nullity}(T) = \dim(V)$$

Sol:-

image of T in W

$$\text{rank}(T) = \dim(\text{im}(T))$$

$$\text{nullity}(T) = \dim(\ker(T))$$

Nullspace of T the no of unique vectors that are mapped to $\bar{0}$ by T from V to W

$\forall \alpha \in (\ker(T))$ $\quad T\bar{\alpha} = 0$

$\forall \bar{\alpha}, \bar{\beta} \in (\ker(T))$ s.t $\quad T \cdot \bar{\alpha} = 0, \; T\bar{\beta} = 0$

$$c\alpha + \beta \in (\text{kernel}(T))$$

since $T\bar{\alpha}=0$ $\quad T(c\bar{\alpha})=\bar{0}$

$T\bar{\beta}=0$ $\quad T(c\bar{\alpha}+\bar{\beta})=0$ $\quad \in \ker(T)$

This shows that $\ker(T)$ which is Nullity(T) is a subspace.

So, $\exists$ basis in $\ker(T)$ which spans $\ker(T)$

from the basis of V which has dim n $(k_1 \dots k_n)$

which has dimension $n(k_1 \dots k_n)$

let's take γ vectors which gives $T(k_1)=0$

so, Any other $Tx=0$

$$x=\sum_{i=1}^{\gamma} c_i k_i$$

So the dim.(of Nullity) of T is γ.

so, $n-\gamma$ independent vectors remaining

These map to W

& $\forall \alpha \in V \notin \ker(T)$ $\quad T(\alpha)\neq 0$

$$\sum_{i=\gamma+1}^{n} c_i k_i \neq 0$$

unless $\forall i\ c_i=0$.

so, $T\left(\sum_{i=\gamma+1}^{n} c_i k_i\right)\neq 0$ $\quad \sum_{i=\gamma+1}^{n} c\,T(k_i)\neq 0$

so, each $T(k_i)$ is $\longrightarrow$ form a vector space

linearly independent

which forms, basis of W

so, $\dim(W)=\dim(\operatorname{im}(T))=n-\gamma$

$\dim(\ker(T))=\gamma$

$\dim(V)=n$

$\Rightarrow n-\cancel{\gamma}+\cancel{\gamma}=\dim(\operatorname{im}(T))+\dim(\ker(T))$

$=\dim(V)$

$\therefore$ Hence proved.