Bolzano and Accumulation

Bolzano weistrass theorem

every bounded sequence has a convergent subsequence

Every sequences has a monotonic subsequence → there exits a bounded monotonic subsequence

any bounded monotonic subsequence is convergent

Accumulation point vs Limit of a sequence

Limit of a sequence

lim Xn=X

for every epsilon >0 ; there exists a N such that for every n>N |Xn-X|<epsilon

Accumulation Point

If x is an accumulation point and if for every epsilon >0 for every N in natural numbers

there exists at least one element n>N such that

|Xn-X|<epsilon

consider the sequence

$$(-1)^n * n/n + 1$$

It changes to positive and negative values after each term so it doesn't converge to a single point so limit doesn't exits

but it contains accumulation points {-1,+1}

Theorem

If x is an Accumulation point of set S, then there exists infinitely many elements of S that are within Epsilon distance of x.

Proof:

for Contradiction , assume that there are finitely many elements of S that are within epsilon distance of x

let the elements are s1 s2 s3 ... only k elements

|x-s1|<epsilon ,|x-s2|<epsilon ,|x-s3|<epsilon ...

let epsilon′ = min(|x-s1|,|x-s2|,....)

|x-s1|≥epsilon′ ,|x-s2|≥epsilon′ ,|x-s3|≥epsilon′ ...

Take z such that |x-z|<epsilon′<epsilon

→ z doesn’t belong to {s1,s2,s3....}

→if z is a member of S it should belong to {s1,s2,s3...} because |z-x|<epsilon but it contradicts previous statement so z is not a member in S so there exists no element in the N(x , Epsilon′)

so No element it the sets such that |Z-x|<epsilon′

according to the def of accumulation point there should exists a point such that |A-x|<epsilon for all epsilon >0

so No element it the sets such that |Z-x|<epsilon′ so x is not an accumulation point

so its a contradiction so our assumption is wrong

so

If x is an Accumulation point of set S, then there exists infinitely many elements of S that are within Epsilon distance of x.

QED