Maculaeen & Taylor Series formula

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} \cdot (x-a)^n$$

$a =$ real or complex number

$f^{(n)}(a) = n^{th}$ derivative of f evaluated at the Point a.

By using the above expansion formula

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots\cdots$$

$$\frac{\sin x}{x} = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \cdots\cdots$$

Let,

$$f(x) = \frac{\sin x}{x} = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} + \cdots\cdots$$

WKT,

Every polynomial can be represented as

$f(x) = (a + r_1)(b + r_2) \cdots\cdots$ degree of the Polynomial times.

as per roots of $\sin x / x$. (general)

$$f(x) = (\lambda)(x-\pi)(x+\pi)(x-2\pi)(x+2\pi)\ldots \quad \text{—①}$$

$$\frac{\sin x}{x} = 1 - \frac{x^2}{3!} + \frac{x^4}{5!} - \frac{x^6}{7!} \ldots \quad \text{—②}$$

The constant term from Expansion of ①

is $\lambda(-\pi)(\pi)(2\pi)(-2\pi)\ldots$

& from ② is '1'

$$\Rightarrow \lambda(-\pi)(\pi)(2\pi)(-2\pi)\ldots = 1$$

$$\lambda = \frac{1}{(-\pi)(\pi)(2\pi)(-2\pi)\ldots}$$

$$f(x) = \lambda \cdot \pi\left(1 - \frac{x}{\pi}\right)\left(1 + \frac{x}{\pi}\right)(-\pi)(-2\pi)\left(1 - \frac{x}{2\pi}\right)\ldots$$

$$f(x) = \lambda\left((\pi)(-\pi)(2\pi)(-2\pi)\ldots\right)\left(1 - \frac{x}{\pi}\right)\left(1 + \frac{x}{\pi}\right)\ldots$$

$$f(x) = \cancel{\lambda} \cdot \frac{1}{\cancel{\lambda}} \cdot \left(1 - \frac{x}{\pi}\right)\left(1 + \frac{x}{\pi}\right)\left(1 - \frac{x}{2\pi}\right)\ldots$$

$$f(x) = \left(1 - \frac{x^2}{\pi^2}\right)\left(1 - \frac{x^2}{4\pi^2}\right)\left(1 - \frac{x^2}{9\pi^2}\right)\ldots$$

$f(x) =$ co-eff(x^2) + remaining terms.

$$g(x) = -\frac{1}{\pi^2} - \frac{1}{4\pi^2} - \frac{1}{9\pi^2} - \ldots = \text{co-eff}(x^2)$$

$$g(x) = \left(\sum_{n=01}^{\infty} \frac{1}{n^2}\right)\left(-\frac{1}{\pi^2}\right) = -\frac{1}{3!} \quad \left[\text{from eq ②}\right]$$

$$\sum_{n=1}^{\infty} \frac{1}{n^2} = \left(-\frac{1}{6}\right)\left(-\pi^2\right) = \frac{\pi^2}{6}$$

$$g(x) = \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{22 \times 22}{49 \times 6} = \frac{484}{49 \times 6} = 1.646$$

$$\frac{29}{18} < 1.646 < \frac{31}{18}$$

$$1.6\overline{11} < 1.646 < 1.7\overline{22}$$

7. $$\sum_{n=1}^{\infty} \frac{n}{n^4+n^2+1}$$

$$\sum_{n=1}^{\infty} \frac{n^2+n+1-(n^2-n+1)}{2(n^4+n^2+1)}$$

$$\frac{1}{2}\sum_{n=1}^{\infty} \frac{n^2+n+1-(n^2-n+1)}{(n^2+n+1)(n^2-n+1)}$$

$$\frac{1}{2}\sum_{n=1}^{\infty} \frac{1}{n^2-n+1} - \frac{1}{n^2+n+1}$$

$$\underset{n\to\infty}{Lt}\ \frac{1}{2}\left[\frac{1}{1} - \cancel{\frac{1}{3}} + \cancel{\frac{1}{3}} - \cancel{\frac{1}{7}} + \cancel{\frac{1}{7}} \cdots \cancel{\cdot} \cancel{\cdot} \cancel{\cdot} - \frac{1}{n^2+n+1}\right]$$

$$\underset{n\to\infty}{Lt}\ \frac{1}{2}\left[1 - \frac{1}{n^2+n+1}\right]$$

which is equals to $\frac{1}{2} - \underset{n\to\infty}{Lt} \frac{1}{n^2+n+1}$

$$y = \frac{1}{2} - \underset{n\to\infty}{Lt} \left(\underbrace{\frac{1}{n^2}}_{l_1} \underbrace{\frac{1}{1+\frac{1}{n}+\frac{1}{n^2}}}_{l_2} \right)$$

$\frac{1}{n} \to 0$

$\frac{1}{n^2} \to 0$

$$\frac{1}{2} - \underset{n\to\infty}{Lt} (0)\left(\frac{1}{1+0+0}\right)$$

Let,

$$\underset{n\to\infty}{Lt} l_1 = \underset{n\to\infty}{Lt} \frac{1}{n^2}$$

$$\underset{n\to\infty}{Lt} l_2 = \underset{n\to\infty}{Lt} \frac{1}{\underbrace{1}_{l_3}+\underbrace{\frac{1}{n}}_{l_4}+\underbrace{\frac{1}{n^2}}_{l_5}}$$

$$\underset{n\to\infty}{Lt} l_3 = \underset{n\to\infty}{Lt} 1$$

$$\underset{n\to\infty}{Lt} l_4 = \underset{n\to\infty}{Lt} \frac{1}{n}$$

$$\underset{n\to\infty}{Lt} l_5 = \underset{n\to\infty}{Lt} \frac{1}{n^2}$$

By the properties of limits, wkT,

$$\underset{n\to c}{Lt} l_1 + l_2 = \underset{n\to c}{Lt} l_1 + \underset{n\to c}{Lt} l_2$$

$$\underset{n\to c}{Lt} l_1 . l_2 = \underset{n\to c}{Lt} l_1 . \underset{n\to c}{Lt} l_2$$

$$y = \frac{1}{2} - (0)\left(\frac{1}{1+0+0}\right)$$

$$y = \frac{1}{2}$$

1. Sequence $\{a_n\}$

$a_n - a_{n-2} \to 0$ when,

$n \to \infty$

$b_n = \dfrac{a_n - a_{n-1}}{n}$ — (1)

now, $b_{n-1} = \dfrac{a_{n-1} - a_{n-2}}{n-1}$

$b_{n+1} = \dfrac{a_{n+1} - a_n}{n+1}$

$b_{n-1} > \dfrac{a_{n-1} - a_{n-2}}{n}$ — (2)

$b_{n+1} < \dfrac{a_{n+1} - a_n}{n}$ — (3)

(1)+(2)

$b_n + b_{n-1} > \dfrac{a_n - a_{n-2}}{n}$

(1)+(3)

$b_n + b_{n+1} < \dfrac{a_{n+1} - a_{n-1}}{n}$

if $b_{n+1} + b_n < \dfrac{a_{n+1} - a_{n-1}}{n}$

then,

$b_n + b_{n-1} < \dfrac{a_n - a_{n-2}}{n-1}$

then,

$\dfrac{a_n - a_{n-2}}{n} < b_n + b_{n-1} < \dfrac{a_n - a_{n-2}}{n-1}$

As $n \to \infty$, both

$\dfrac{a_n - a_{n-2}}{n}, \dfrac{a_n - a_{n-2}}{n-1}$

converges to 0 as $a_n - a_{n-2} \to 0$ when $n \to \infty$ and $\frac{1}{n}, \frac{1}{n-1} \to 0$

WKT, By properties of limits

$$\lim_{x \to c} l_1 . l_2 = \lim_{x \to c} l_1 . \lim_{x \to c} l_2$$

Hence, $b_n + b_{n-1} \longrightarrow 0$ when $n \to \infty$

Let $c_{n-1} = b_n + b_{n-1}$

d_{n-1} converges to some x

$\forall\ \varepsilon > 0\ \exists\ N_0 \in \mathbb{N}$ s.t $\forall (n-1) \geq N_0$

$$|b_n + b_{n-1} - x| < \varepsilon \quad \text{— (8)}$$

By Law of Trichotomy

1) $b_n > b_{n-1}$ 2) $b_n < b_{n-1}$

3) $b_n = b_{n+1}$

(1) & (3) $b_n \geq b_{n-1}$

$|b_n + b_{n-1} - x| < \frac{\varepsilon}{2}$

$|2b_{n-1} - x| \leq |b_n + b_{n-1} - x| < \frac{\varepsilon}{2}$

$|b_n - b_{n-1}| = |b_n + b_{n-1} - x + x - 2b_{n-1}|$

$\leq |b_n + b_{n-1} - x| + |x - 2b_{n-1}|$

$|b_n - b_{n-1}| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2}$

$|b_n - b_{n-1}| < \varepsilon$

(2) $b_n < b_{n-1}$

$b_n < b_{n-1}$

$|b_n + b_{n-1} - x| < \frac{\varepsilon}{2}$

$|2b_n - x| < |b_n + b_{n-1} - x| < \frac{\varepsilon}{2}$

$|b_{n-1} - b_n| = |b_{n-1} + b_n - x + x - 2b_n|$

$\leq |b_{n-1} + b_n - x| + |x - 2b_n|$

$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2}$

$|b_n - b_{n-1}| < \varepsilon$

Hence $\forall\ \varepsilon > 0\ \exists\ N_0 \in \mathbb{N}$ s.t $(n-1) \geq N_0$

$$|b_n - b_{n-1}| < \varepsilon$$

taking $m, n \in \mathbb{N}$

$$|b_m - b_n| \leq |b_m - b_{m-1}| + |b_{m-1} - b_{m-2}| \cdots |b_n - b_{n+1}|$$

$$|b_n - b_{n-1}| < \frac{\varepsilon}{m+n-1} \quad \forall (n-1) \geq N_0$$

$$|b_m - b_n| \leq \frac{\varepsilon}{m+n-1} + \frac{\varepsilon}{m+n-1} + \cdots \frac{\varepsilon}{m+n-1}$$

$$|b_m - b_n| \leq \left(\frac{\varepsilon}{\cancel{m+n-1}}\right) \cancel{m+n-1}$$

let $m = n + p$

then, $\forall\ \varepsilon > 0\ \exists\ N_0 \in \mathbb{N}$ s.t $n \geq N_0$;

p starts with 1

$$|b_{n+p} - b_n| < \varepsilon$$

hence b_n is a cauchy sequence

then a cauchy can be convergent

Let b_n converge to M

then $\forall\ \varepsilon > 0\ \exists\ N_0 \in \mathbb{N}$ s.t $\forall n \geq N_0$

$$|b_n - y| < \varepsilon/2 \quad \text{--- (4)}$$

then

$$|b_{n+1} - y| < \varepsilon/2 \quad \text{--- (6)}$$

from (4) & (6)

$$|b_n + b_{n+1} - y - y| < |b_n - y| + |b_{n+1} - y|$$

$|b_n + b_{n+1} - 2y| < \varepsilon$ by Agle inequality

WKT, $|b_n + b_{n+1} - x| < \varepsilon$ [from (8)]

$x = 2y \qquad y = \frac{x}{2}$

we know that $x = 0$

hence $y = 0$

$\therefore$ b_n is a convergent sequence which converges to 0.

4. fact $(1+n^{-1})^n \longrightarrow e$ as $n \to \infty$

Since $x \to \infty$, we assume $x > 1$

Let $[x] = p$. Then $p \le x < p+1$

and $1 + \frac{1}{p+1} < 1 + \frac{1}{x} \le 1 + \frac{1}{p}$

It Follows that

$$\left(1+\frac{1}{p+1}\right)^p < \left(1+\frac{1}{x}\right)^x < \left(1+\frac{1}{p}\right)^{p+1}$$

Taking limit as $x \to \infty$ and noting that as $x \to \infty$, $p \to \infty$

We have $\lim_{x\to\infty} \left(1+\frac{1}{x}\right)^x \ge \lim_{p\to\infty} \left(1+\frac{1}{p+1}\right)^p$

$$= \lim_{p\to\infty} \frac{1}{\left(1+\frac{1}{p+1}\right)^{p+1}} \left(1+\frac{1}{p+1}\right) = e$$

Also $\lim_{x\to\infty} \left(1+\frac{1}{x}\right)^x \le \lim_{p\to\infty} \left(1+\frac{1}{p}\right)^{p+1}$

$$= \lim_{p\to\infty} \left(1+\frac{1}{p}\right)^p \left(1+\frac{1}{p}\right) = e$$

It follows that $\lim_{x\to\infty} \left(1+\frac{1}{x}\right)^x = e$.

2. $$\lim_{n\to\infty} \sum_{k=1}^{n} \left(\sqrt{1+\frac{k}{n^2}}-1\right)$$

multiply and divide conjugate

$$\underset{n\to\infty}{Lt} \sum_{k=1}^{\infty} \frac{\left(\sqrt{1+\frac{k}{n^2}}-1\right)\left(\sqrt{1+\frac{k}{n^2}}+1\right)}{\left(\sqrt{1+\frac{k}{n^2}}+1\right)}$$

$$\underset{n\to\infty}{Lt} \sum_{k=1}^{\infty} \cancel{1}+\frac{k}{n^2}\cancel{-1}\left(\frac{1}{\sqrt{1+\frac{k}{n^2}}+1}\right)$$

$$\underset{n\to\infty}{Lt} \sum_{k=1}^{\infty} \frac{k}{n^2}\left(\frac{1}{\sqrt{1+\frac{k}{n^2}}+1}\right)$$

Expanding

$$= \underset{n\to\infty}{Lt}\left(\frac{1}{n^2}\left(\frac{1}{\sqrt{1+\frac{1}{n^2}}+1}\right)+\frac{2}{n^2}\left(\frac{1}{\sqrt{1+\frac{2}{n^2}}+1}\right) \dots + \frac{n}{n^2}\left(\frac{1}{\sqrt{1+\frac{n}{n^2}}+1}\right)\right.$$

when $n\to\infty, \frac{1}{n}\to 0$ and by limit properties $\lim_{x\to c} f(x)\cdot g(x) = \lim_{x\to c} f(x) \cdot \lim_{x\to c} g(x)$

and $\underset{x\to c}{lt} \frac{f(x)}{g(x)} = \underset{x\to c}{lt} f(x) \Big/ \underset{x\to c}{Lt}\, g(x)$

$$= \underset{n\to\infty}{Lt}\left(\frac{1}{n^2}\left(\frac{1}{\sqrt{1+0}+1}\right)+\frac{2}{n^2}\left(\frac{1}{\sqrt{1+0}+1}\right) \dots + \frac{n}{n^2}\left(\frac{1}{\sqrt{1+0}+1}\right)\right)$$

$\left[\text{When } n\to\infty \quad \frac{1}{n^2}, \frac{2}{n^2}, \frac{3}{n^2} \cdots \frac{n}{n^2} \to 0.\right]$

$$= \lim_{n\to\infty} \left(\frac{1}{n^2}\left(\frac{1}{2}\right) + \frac{2}{n^2}\left(\frac{1}{2}\right) + \cdots \frac{n}{n^2}\left(\frac{1}{2}\right) \right)$$

$$= \frac{1}{2} \lim_{n\to\infty} \left(\frac{1}{n^2} + \frac{2}{n^2} + \cdots + \frac{n}{n^2} \right)$$

$$= \frac{1}{2} \lim_{n\to\infty} \sum_{k=1}^{n} \left(\frac{k}{n^2} \right). \quad \left[\because \lim_{n\to c} f\cdot g = \left(\lim_{n\to c} f\right)\left(\lim_{n\to c} g\right)\right]$$

$$= \frac{1}{2} \lim_{n\to\infty} \left(\frac{1}{n^2} \right) \left[\sum_{k=1}^{n} k \right] \quad \left[\because \sum_{i=1}^{n} i = \frac{n(n+1)}{2} \right]$$

$$= \frac{1}{2} \lim_{n\to\infty} \left(\frac{1}{n^2} \right) \left(\frac{n(n+1)}{2} \right)$$

$$= \frac{1}{2\cdot 2} \lim_{n\to\infty} \left[\frac{n^2+n}{n^2} \right]$$

$$= \frac{1}{4} \lim_{n\to\infty} \left[1 + \frac{1}{n} \right] \quad \lim_{x\to c}(f+g) = \lim_{x\to c} f + \lim_{x\to c} g$$

$$= \frac{1}{4} \left[\lim_{n\to\infty} 1 + \lim_{n\to\infty} \left(\frac{1}{n} \right) \right]$$

$$= \frac{1}{4}(1) \quad \text{RHS}$$

———o———

Hence proved

5) $\lim_{x \to 0} \frac{(1+x)^a - 1}{x} = a.$

from Binomial Expansion, for real number n,

$$(1+x)^n = \sum_{k=0}^{\infty} \binom{n}{k} x^k \quad \left[\binom{n}{k} = \frac{n(n-1)\cdots(n-k+1)}{k!} \right]$$

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{2!}x^2 + \cdots$$

Substituting above expansion in our problem with $n = a$;

$$LHS = \lim_{x \to 0} \frac{(1+x)^a - 1}{x} = \lim_{x \to 0} \frac{\left(1 + ax + \frac{a(a-1)}{2!}x^2 + \cdots\right) - 1}{x}$$

$$= \lim_{x \to 0} \frac{\cancel{x}\left(a + \frac{a(a-1)}{2!}x + \cdots\right)}{\cancel{x}}$$

WKT,

$$\lim_{x \to c} f(x) + g(x) = \lim_{x \to c} f(x) + \lim_{x \to c} g(x)$$

$$= \lim_{x \to 0} a + \lim_{x \to 0} \frac{a(a-1)}{2!}x + \lim_{x \to 0} \frac{a(a-1)(a-2)}{3!}x^2$$

$$\lim_{x \to c} g(x) f(x) = \lim_{x \to c} g(x) \cdot \lim_{x \to c} f(x)$$

$$= a + \frac{a(a-1)}{2}\left(\lim_{x \to 0} x\right) + \frac{a(a-1)(a-3)}{6}\left(\lim_{x \to 0} x^2\right)$$

$$= a + 0 + 0 + 0 \cdots = a$$

$$\lim_{x \to 0} \frac{(1+x)^a - 1}{x} = a = R.H.S$$

Hence proved

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8 (a) $\sum_{n=1}^{\infty} u_n$, $u_n = \left(\frac{a^n}{(n!)^{1/n}}\right)$

Now again by Cauchy's root test

$\sum u_n$ (i) converges if $\lim_{n \to \infty} \sqrt[n]{u_n} = l < 1$

(ii) diverges if $\lim_{n \to \infty} \sqrt[n]{u_n}$ $l > 1$

$$\sqrt[n]{u_n} = \frac{a}{(n!)^{1/n^2}}$$

$$\frac{a}{(n \cdot n \cdot n \cdots)^{1/n^2}} < \frac{a}{(1 \cdot 2 \cdot 3 \cdot 4 \cdots n)^{1/n^2}} < \frac{a}{(1 \cdot 1 \cdot 1 \cdot 1)^{1/n^2}}$$

so now we have

$$\frac{a}{n^{1/n}} < \frac{a}{[1\cdot 2\cdot 3 \cdots a]^{1/n^2}} < a$$

Now $\underset{n\to\infty}{Lt} (n)^{1/n} = 1$

$$\underset{n\to\infty}{Lt} \left(\frac{a}{n^{1/n}}\right) < \underset{n\to\infty}{Lt} \sqrt[n]{u_n} < \underset{n\to\infty}{Lt} a$$

$$a(1) < \underset{n\to\infty}{Lt} \sqrt[n]{u_n} < a.$$

So we have $\underset{n\to\infty}{Lt} \sqrt[n]{u_n} = a$ by

Sandwhich theorem

So if $a<1$ the Series will be converging

& if $a>1$ the Series will be diverging

8(b) $\sum_{n=1}^{\infty} a^n\left(1+1/n\right)^n$

in the above series the value of n^{th} term

$$(a_n) = a^n(1+1/n)^n$$

applying root test,

$\sum u_n$ (i) converges if $\lim_{n\to\infty} \sqrt[n]{u_n} = l < 1$

(ii) diverges if $\lim_{n\to\infty} \sqrt[n]{u_n}$ $l > 1$

$$L = \lim_{n\to\infty} \sqrt[n]{|a_n|} = \lim_{n\to\infty} \sqrt[n]{a^n(1+1/n)^n}$$

$$= \lim_{n\to\infty} \sqrt[n]{a^n} \cdot \lim_{n\to\infty} \sqrt[n]{\left(1+\frac{1}{n}\right)^n}$$

$$= \lim_{n\to\infty} |a| \cdot \lim_{n\to\infty} \left(1+1/n\right)$$

$$l = |a| \cdot 1$$

as we know $a > 0$. given

hence, series is convergent if $a < 1$ $(\because a > 0)$

$\therefore$ convergent if $0 < a < 1$

$\Rightarrow$ series is divergent if $a > 1$

for the case $a = 1$; we cannot conclude nature of the series, for $a = 1$

by applying test for divergence,

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \left(1+1/n\right)^n = e$$

$\sum a_n = \sum (1)^n \left(1+\frac{1}{n}\right)^n = \sum \left(1+\frac{1}{n}\right)^n$

WKT,

$$\left(1+\frac{1}{n}\right)^n > 1$$

$\forall n \in N$

$$\frac{1}{n} > 0$$

$$1+\frac{1}{n} > 0$$

$$1+\frac{1}{n} > 1$$

$$\left(1+\frac{1}{n}\right)^n > (1)^n$$

hence we can say that

$\sum a_n = \sum \left(1+\frac{1}{n}\right)^n$ is divergent

Since $\lim_{n\to\infty} a_n \neq 0$

we can conclude series diverges for $a=1$

3. Let us, $\{a_n\}$ not a convergent sequence

then according to the definition of convergent series

$$|a_n - l| \geq \varepsilon_0$$

given, $a_n \geq 1$ $\therefore a_n + \frac{1}{a_n}$ is convergent

if $a_n \geq 1$ then

$$0 < \frac{1}{a_n} \leq 1$$

$$\left|a_n + \frac{1}{a_n} - L\right| > \varepsilon_0 - |L - l|$$

according to our assumption:

$\forall \varepsilon_0 > 0 \ \exists N_0 \in \mathbb{N}$ s.t $\forall n \geq N_0$

$$|a_n - l| \geq \varepsilon_0$$

let us take an $\varepsilon_0 > |L - l|$

such that $\varepsilon_0 - |L - l| > 0$

let. $\varepsilon_0 - |L - l| = \varepsilon$

then $\left|a_n + \frac{1}{a_n} - L\right| > \varepsilon$

But by def convergent series.

$\forall \varepsilon_0 > 0 \ \exists N_0 \in N$ s.t $\forall n \geq N_0$

let $a_n + \frac{1}{a_n}$ converge to L

$$|a_n - l| = \left|a_n + \frac{1}{a_n} - L + L - l - \frac{1}{a_n}\right|$$

$$\leq \left|a_n + \frac{1}{a_n} - L\right| + \left|L - l - \frac{1}{a_n}\right|$$

as $0 < \frac{1}{a_n} \leq 1$

$$< \left|a_n + \frac{1}{a_n} - L\right| + |L - l|$$

$$\left|a_n + \frac{1}{a_n} - L\right| + |L - l| > \varepsilon_0$$

$\forall \varepsilon_0 > 0 \ \exists N_0 \in \mathbb{N}$ s.t $\forall n \geq N_0$

$$\left|a_n + \frac{1}{a_n} - L\right| < \varepsilon_0$$

but

taking $\varepsilon_0 < \varepsilon$

there will not exist an $N_0 \in \mathbb{N}$

s.t $\forall n \geq N_0$

the defition does not satisfy

But we know that $a_n + \frac{1}{a_n}$ is convergent we have reached a contradiction

$\therefore a_n$ is convergent