GRADING SCHEME (HALF 1)

Assignments : ~20%
Quiz1 : 10%
Midsem : 15-20%
Tutorial Quiz : remaining

Total = 50%

LINEAR ALGEBRA

SHIVANSH SANTOKI

1st Half Topics:

- Field
- System of Linear Equations
- Vector, Spaces
- Linear Transformation

FIELD:

→ A set F with two binary operations [add^n '+' & multiplication '.'] satisfying following rules:

1) Addition is Commutative

$$x+y=y+x \quad \forall\ x,y \in F$$

Note that in "binary operations", the closure property is implicit.

2) Addition is Associative

$$x+(y+z)=(x+y)+z \quad \forall\ x,y,z \in F$$

3) $\exists$ a unique element 0 (zero) $\in F$, s.t.

$$x+0=x \quad \forall\ x \in F$$

i.e. an additive identity.

4) For each $x \in F$, $\exists$ an additive inverse, (here denoted by '$-x$') s.t. $x+(-x)=0$.

5) Multiplication is Commutative,

$$x \cdot y = y \cdot x \quad \forall\ x,y \in F$$

6) Multiplication is Associative,

$$x \cdot (y \cdot z) = (x \cdot y) \cdot z \quad \forall\ x,y,z \in F$$

7) $\exists$ a unique non-zero element $1(\text{one}) \in F$ s.t. $x \cdot 1 = x \;\forall\; x \in F$.

8) For each $x \neq 0 \in F$ $\exists\; x^{-1}$ s.t. $x \cdot x^{-1} = 1$

9) Multiplication is distributive over addition.
$x \cdot (y+z) = x \cdot y + x \cdot z \quad \forall\; x, y, z \in F$

→ The set F satisfying all of the above rules is a Field $(F, +, \cdot)$.

Eg: $(\{0, 1\}, +, \cdot)$

+	0	1
0	0	1
1	1	0

·	0	1
0	0	0
1	0	1

→ Elements of field are called scalers

→ To Prove: $-x = (-1)x \Rightarrow x + (-1)x = 0$

Proof: $x + (-1)x$

$= (1) \cdot x + (-1) \cdot x$

$= x \cdot (1 + -1)$

$= x \cdot 0 = x \cdot 0 + 0$ [As $y + 0 = y \;\forall\; y \in F$]

$$= x\cdot 0 + 0$$
$$= x\cdot 0 + x + (-x)$$
$$= x\cdot(0+1) + (-x)$$
$$= x\cdot(1) + (-x)$$
$$\therefore \cancel{x} + (-1)x = \cancel{x} + (-x)$$
$$\therefore \boxed{(-1)x = -x}$$ Hence, Proved.

Assignment Que 1 : Deadline: 12 Jam Midnight

1] For a Field F $a, b, c \in F$

Prove: (1) $a\cdot b = b\cdot c \Rightarrow a = c$

(2) $a+b = b+c \Rightarrow a = c$

2] Prove that any subfield of $(\mathbb{C}, +, \cdot)$ must contain every rational number.

NOTE: Subfield: A set S is a subfield of a field $(F, +, \cdot)$, if $S \subset F$ & $(S, +, \cdot)$ is a field.

$\rightarrow$ The Binary operations of S & F must be same.

System of Linear Equations

n unknown scalars:
$(x_1, x_2, \ldots\ldots, x_n)$

$$(1.1)\quad \begin{cases} A_{11}x_1 + A_{12}x_2 + A_{13}x_3 + \cdots\cdots + A_{1n}x_n = y_1 \\ A_{21}x_1 + A_{22}x_2 + \cdots\cdots\cdots + A_{2n}x_n = y_2 \\ \vdots \\ A_{m1}x_1 + \cdots\cdots\cdots\cdots + A_{mn}x_n = y_m \end{cases}$$

→ Here all A_{ij} and x_j belong to the same field.

→ If all y_i $1 <= i <= m$ are zero, then it is a homogeneous system.

→ Multiply c_1 on both sides in eq - ①
c_2 on both sides in eq - ②
⋮
c_n " " " " eq - ③

After adding, we get:

$$(c_1A_{11} + c_2A_{21} + c_3A_{31} + \cdots\cdots + c_mA_{m1})x_1 +$$
$$(c_2A_{12} + c_2A_{22} + \cdots\cdots - + c_mA_{m2})x_2 + \cdots\cdots$$
$$= c_1y_1 + c_2y_2 + \cdots\cdots + c_my_m \rightarrow (A)$$

linear combination of a system of linear eq^n (1.1)

→ All solution of system 1.1 will satisfy the linear eq^n (A). [If all $c_i \neq 0$]

→ Converse is not true.

→ Now, if we create a new system of linear equations 1.2 by taking linear combinations of Linear equations in 1.1 then the solutions of the system 1.1 will satisfy the equations of system 1.2.

Equivalent System of Linear Equations:

→ Solutions of 1.2 may or may not sol^{n}s of 1.1.

→ If all of the solutions of 1.2 are solutions of 1.1 then both the systems are said to be equivalent which happens when equations in 1.1 are the linear combination of linear eq^{n}s in 1.2.

Matrix & Elementary Row Operation:

$$\begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ \vdots & & & \\ A_{m1} & A_{m2} & \cdots & A_{mn} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix} = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{pmatrix}$$

$\hookrightarrow m$

$$A \qquad X \qquad Y$$

$$AX = Y \Rightarrow 1.1$$

→ Matrices are defined over a field.
It is a function that maps pairs of integers (i, j) to scalers.

$$A_{ij} \in F \quad \forall \quad 1 \leq i \leq m \quad 1 \leq j \leq n$$

⇒ Elementary Row Operations:

1. Multiplication of a row by a non-zero scalar $(c \in F, c \neq 0)$

element

$$e(A_{ij}) = \begin{cases} c(A_{rj}) & \text{if } i = r \text{ for } c \neq 0 \\ A_{ij} & \text{otherwise} \end{cases}$$

2. Replacing a row (say 'r') by row r + (scalar times) any other row (say 's')

$$e(A_{ij}) = \begin{cases} A_{rj} + cA_{sj}, & \text{if } i = r \\ A_{ij}, & \text{otherwise} \end{cases}$$

③ Interchange any two row

$$e(A_{ij}) = \begin{cases} A_{rj} & \text{if } i = s \\ A_{sj} & \text{if } i = r \\ A_{ij} & \text{o.w} \end{cases}$$

※ **Theorem:** To each elementary row operation e, $\exists$ an elementary row operation e_1, such that,

$$e_1(e(A)) = e(e_1(A)) = A$$

[e_1 is of same type as e]

Row Equivalent: A, B $m \times n$ F

Def^n: A & B are two $m \times n$ matrices over F

A is row equivalent to B if A can be obtained by performing finite sequence of elementary row operations on B.

If A is row equivalent to B then B is row equivalent to A.

i.e. $A = e_n(e_{n-1} \cdots\cdots e_2(e_1(B))\cdots)$

$\therefore e_1^{-1} e_2^{-1} \cdots\cdots e_{n-1}^{-1} e_n^{-1}(A) = B$

※ Assignment - 2

Que 1: Row equivalence is an equivalence relation. (Define & Prove)

※ Theorem: If A & B are two row equivalent $m \times n$ matrices over field F, then the homogeneous equations $Ax=0$ and $Bx=0$ have exactly the same solutions.

Eg: $A = \begin{bmatrix} 2 & -1 & 3 & 2 \\ 1 & 4 & 0 & -1 \\ 2 & 6 & -1 & 5 \end{bmatrix} \xrightarrow{① = ① + -2 \times ②} \begin{bmatrix} 0 & -8 & 3 & 4 \\ 1 & 4 & 0 & -1 \\ 2 & 6 & -1 & 5 \end{bmatrix}$

Sol^n for $Ax=0$ & $A'x=0$ will be same.

$A' = \begin{bmatrix} 0 & 0 & 1 & -11/3 \\ 1 & 0 & 0 & 17/3 \\ 0 & 1 & 0 & -5/3 \end{bmatrix}$

$\therefore x_3 = 11/3\, x_4$

$x_1 = -17/3\, x_4$

$x_2 = 5/3\, x_4$

Sol^n: $(-17/3\, a, 5/3\, a, 11/3\, a, a)$

Def^n: An $m \times n$ matrix A over F is row-reduced if

① the 1st non-zero entry in each non-zero row is 1.

② each column of A which contains leading non-zero entry of a row has all other entries 0.

Eg:

$$\begin{bmatrix} 0 & 1 & 0 & x_1 \\ 1 & 0 & 0 & x_2 \\ 0 & 0 & 1 & x_3 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Assignment - 2:

Que-2: Every $m \times n$ matrix over a field F is row-equivalent to a row reduced matrix.

(Prove the above theorem)

※ Row-reduced echelon matrix:

Def^n: An $m \times n$ matrix R over field F is row-reduced echelon matrix if:

① R is row-reduced

② Every non zero rows occur before zero rows.

③ if rows $1, 2, \ldots, r$ are the non-zero rows & if the leading non-zero entry of row i $(i = 1, 2, \ldots, r)$ occurs in column K_i, then $K_1 < K_2 < \ldots < K_r$

i.e.

Either R is 0 matrix ($R_{ij} = 0 \ \forall \ 1 \le i \le m, 1 \le j \le n$) or $\exists \ r \in \mathbb{N}$ $1 \le r \le m$ & $K_1, K_2, \ldots, K_r \in \mathbb{N}$, with $1 \le K_i \le n$ &

ⓐ $R_{ij} = 0 \ \forall \ i > r$

$R_{ij} = 0 \ \forall \ j < K$

ⓑ $R_{iK} = \delta_{ij}$, $1 \le i \le r$, $1 \le j \le r$

$$\delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{o.w.} \end{cases}$$

ⓒ $k_1 < k_2 < k_3 < \dots < k_r$

Consider the following system:

$$A_{m\times n} X_{n\times 1} = O$$

$(x_1, x_2, \dots, x_n) \Rightarrow$ if $x_i = 0 \;\forall\, 1 \le i \le n$ [Trivial Solution]

$\to RX = 0$

Row reduced echelon $\leftarrow$ $\hookrightarrow m\times n$

m linear equations

n unknowns

Let r be the no. of non-zero rows in R.

$\therefore m - r \to$ Trivial Eq^{n}s.

$r \to$ Non-Trivial Eq^{n}s.

$i \in \{1, 2, \dots, r\}$

$\Rightarrow$ k_i for row i x_{k_i} will occur only in i^{th} eq^n.

$\therefore$ Let $(x_1, x_2, \dots, x_n) \equiv (x_{k_1}, x_{k_2}, \dots, U_1, \dots U_{n-r})$

Thus,

$$r\left\{\begin{aligned} & x_{k_1} + \sum_{j=0}^{n-r} C_{1j} U_j = 0 \\ & \vdots \\ & x_{k_r} + \sum_{j=0}^{n-r} C_{rj} U_j = 0 \end{aligned}\right.$$

$$m-r\left\{\begin{aligned} & 0 = 0 \\ & \vdots \\ & 0 = 0 \end{aligned}\right.$$

[After matrix multiplication]

Eg:

$$R = \begin{bmatrix} 0 & 1 & -3 & 0 & 1/2 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{array}{l} \} \text{ Non-Trivial} \\ \\ \} \text{ Trivial} \end{array}$$

$\therefore x_{k_1} = x_2$, $x_{k_2} = x_4$ | $x_1, x_3, x_5 \to U_1, U_2, U_3$

$$\begin{aligned} x_2 - 3x_3 + \tfrac{1}{2}x_5 &= 0 \\ x_4 + 2x_5 &= 0 \end{aligned} \quad \left\{ \begin{array}{l} C_{12} = -3 \quad C_{13} = 1/2 \\ C_{23} = 2 \end{array} \right.$$

Assignment - 2:

Que - 3: If A is $m \times n$ matrix, $m < n$. Then $AX = 0$ always has a non-trivial sol^n. (Prove)

Que - 4: A is a square matrix $n \times n$. $AX = 0$ has only trivial solutions iff A is row equivalent to $I_{n \times n}$ (Identity matrix) (Prove)

※ Homogeneous & Non-Homogeneous System:

Homogeneous $\downarrow$ $A_{m\times n} X_{n\times 1} = 0$

Non-Homogeneous $\downarrow$ $A_{m\times n} X_{n\times 1} = Y_{m\times 1}$

Consider $A_{m\times n} X_{n\times 1} = Y_{m\times 1}$

After row operations

$R_{m\times n} X_{n\times 1} = Z_{m\times 1}$

Let $A' = [A_{m\times n} \mid Y_{m\times 1}]$ — Augmented Form

Perform same set of elementary row operations in $Y_{m\times 1}$ too.

$R' = [R_{m\times n} \mid Z_{m\times 1}]$

If there are r non-zero rows in $R_{m\times n}$ then $Z_{r+1} = 0$

$\vdots$

$Z_m = 0$

Eg: $A = \begin{bmatrix} 1 & -2 & 1 \\ 2 & 1 & 1 \\ 0 & 5 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$

After certain no. of row operations

$$\begin{bmatrix} 1 & 0 & 3/5 \\ 0 & 1 & -1/5 \\ 0 & 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} \frac{1}{5}(y_1+2y_2) \\ y_2 - 2y_1 \\ y_3 - y_2 + 2y_1 \end{bmatrix}$$

$$x_1 + \tfrac{3}{5} x_3 = \tfrac{1}{5}(y_1 + 2y_2)$$

$$x_2 - \tfrac{1}{5} x_3 = y_2 - 2y_1$$

x_3 → unknown scalar

[We can assign any value]

MATRIX MULTIPLICATION:

$\Rightarrow C_{m\times p} = A_{m\times n} B_{n\times p}$

$$\boxed{C_{ij} = \sum_{r=1}^{n} A_{ir} B_{rj}}$$

$\Rightarrow$ If $B = \begin{bmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_n \end{bmatrix}$ then, $\boxed{\gamma_i = A_{i1}\beta_1 + A_{i2}\beta_2 + \cdots + A_{in}\beta_n}$

β ← represents whole row

i^{th} row in $C_{m\times p}$

$\Rightarrow$ let $B = [B_1 \; B_2 \; \ldots\ldots \; B_p]$ then $\boxed{AB = [AB_1 \; AB_2 \; \ldots\ldots \; AB_n]}$

where, $B_i = \begin{bmatrix} B_{1i} \\ B_{2i} \\ \vdots \\ B_{ni} \end{bmatrix}$

※ Theorem:

A, B, C are matrices over F.

AB and (AB)C are defined.

Then, BC is defined & $A(BC) = (AB)C$

Proof: $\Rightarrow$ $(AB)C$ exist

$\therefore$ let $C = n\times p$ matrix & $AB = m\times n$

$\therefore$ Let $A = m\times a$ $B = a\times n$

$\therefore$ BC is also defined

$BC = a\times p$

$\therefore$ $A(BC)$ is also defined.

Now, to show : $[A(BC)] = [(AB)C]$,

$$[A(BC)]_{ij} = \sum_r A_{ir}(BC)_{rj}$$

$$\text{And } (BC)_{rj} = \sum_k B_{rk} C_{kj}$$

$$\therefore [A(BC)]_{ij} = \sum_r A_{ir} \sum_k B_{rk} C_{kj}$$

$$= \sum_r \sum_k A_{ir} B_{rk} C_{kj}$$

$$= \sum_k \sum_r A_{ir} B_{rk} C_{kj}$$

$$= \sum_k \left(\sum_r A_{ir} B_{rk} \right) C_{kj}$$

$$= \sum_K (AB)_{ik} C_{kj} = [(AB)C]$$

※ Elementary Matrix :

Def^n : A square mat $E_{n\times n}$ is an elementary mat, if it can be obtained by doing a single elementary row operation on the Identity matrix $(\mathbb{1})_{n\times n}$

Eg: In $I_{3\times3}$ $R_3 \to cR_3$ (c)

then $E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & c \end{bmatrix}$

※ Assignment-2 :

Que-5: **Thm:** Let e be an elementary row operation & E be an elementary matrix $E = e(\mathbb{1})$,

Then, $\underset{m\times n}{e(A)} = \underset{m\times m}{EA}$

※ Corollary: Let A, B be $m\times m$ matrices over field F. Then B is row-equivalent to A iff $B = PA$, where P is a product of $m\times m$ elementary matrices.

Proof: $E_i = e_i(\mathbb{1})$

$$B = e_k e_{k-1} \cdots\cdots e_1(A)$$
$$= E_k E_{k-1} \cdots\cdots E_1(A)$$
$$\therefore \boxed{B = PA}$$

※ Invertible Matrix :

Def^n: Let $A_{n\times n}$ be a square matrix over the field F.

Let B be a left inverse of A $(BA = I)$;

Let C be a right inverse of A $(AC = I)$;

If both left and right inverses exist,

$BA = AC = I$,

then A is said to be invertible.

Lemma: If A is an invertible matrix, i.e.
$BA = I = AC$, then
$\boxed{B = C}$

Proof: $B = BI = B(AC) = (BA)C = IC = C$.

※ Theorem: Let A and B be $n \times n$ matrices over F.

(i) If A is invertible, so is A^{-1} and $(A^{-1})^{-1} = A$.

(ii) If both A and B are invertible, then so is AB, and $(AB)^{-1} = B^{-1}A^{-1}$

Proof: (ii) Let $(AB)X = I$ ---- ①

$(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1}$

$= AA^{-1} = I$ ..... ②

From ① and ②,

$(AB)X = (AB)(B^{-1}A^{-1})$

$\therefore X = B^{-1}A^{-1}$

$\therefore \boxed{(AB)^{-1} = B^{-1}A^{-1}}$

(i) $AA^{-1} = I$

let $XA^{-1} = I$

$\therefore X = (A^{-1})^{-1}$

& $XA^{-1} = AA^{-1}$

$\therefore X = A$

$\therefore \boxed{A = (A^{-1})^{-1}}$

* Assignment-2:

Que-6: Thm: $n \times n$ matrix A,

Prove that the following are equal.

(i) A is invertible

(ii) $AX = 0$ has only a trivial solⁿ

(iii) $AX = Y$ has a solⁿ X for each $n \times 1$ matrix Y.

NOTE:

1) A is invertible $\longleftrightarrow$ A is row-equivalent to $I_{n \times n}$

CHAPTER 2 :

VECTOR SPACE

→ A vector space (or linear space) consists of the following :

1. a field F of scalars ;
2. a set V of objects, called vectors.
3. a rule (or operation), called vector addition, which associates with each pair of vectors α, β in V, a vector $\alpha+\beta$ in V, called the sum of α and β, in such a way that

   (a) add^n is commutative : $\alpha+\beta = \beta+\alpha$;

   (b) add^n is associative : $\alpha+(\beta+\gamma) = (\alpha+\beta)+\gamma$;

   (c) $\exists$ a unique vector 0 in V, called the zero vector, such that $\alpha+0 = \alpha \quad \forall \alpha \in V$.

   (d) for each vector α in V, $\exists$ a unique vector $-\alpha$ in V such that $\alpha+(-\alpha) = 0$.

4. a rule (or operation), called scalar multiplica^n, which associates with each scalar c in F and vector α in V a vector $c\alpha$ in V, called the product of c and α, in such a way that

   (a) $1\alpha = \alpha \quad \forall \alpha \in V$

   (b) $(c_1c_2)\alpha = c_1(c_2\alpha)$

   (c) $c(\alpha+\beta) = c\alpha + c\beta$

   (d) $(c_1+c_2)\alpha = c_1\alpha + c_2\alpha$

Prove: $c\bar{0} = \bar{0}$

$c\bar{0} = c(\bar{0}+\bar{0}) = c\bar{0} + c\bar{0} \quad [\bar{0}+\bar{0} = \bar{0}]$

$[\alpha + \bar{0} = \alpha]$

$\therefore \boxed{c\bar{0} = \bar{0}}$

Prove: $-\bar{\alpha} = (-1)\bar{\alpha}$

Proof: $0\bar{\alpha} = \bar{0} \quad \Rightarrow \bar{\alpha} + 0\bar{\alpha} = (1+0)\bar{\alpha} = \bar{\alpha}$

$(1-1)\bar{\alpha} = \bar{0} \quad \therefore \boxed{0\bar{\alpha} = \bar{0}}$

$$\left.\begin{aligned} &\therefore \bar{\alpha} + (-1)\bar{\alpha} = \bar{0} \\ &\& \ \bar{\alpha} + (-\bar{\alpha}) = \bar{0} \end{aligned}\right\} \boxed{-\bar{\alpha} = (-1)\bar{\alpha}}$$

Prove: $c\bar{\alpha} = \bar{0}$

$\Rightarrow$ either $c = 0$ or $\bar{\alpha} = \bar{0}$

Eg: (1) The field C of complex numbers may be regarded as a vector space over the field R.

Add^n: $\alpha = (x_1 + iy_1)$

$\beta = (x_2 + iy_2)$

$\alpha + \beta = (x_1 + x_2) + i(y_1 + y_2)$

Multi^{n}: $c(\alpha)$

$= c(x_1 + iy_1)$

$= cx_1 + icy_1$

(2) The space of polynomial functions over a field F.

$f(x) = c_0 + c_1 x + \text{------} + c_n x^n$

$c_0, c_1, \ldots, c_n \in F$

Linear Combination of Vectors:

Def^n: A vector β in V is said to be a linear combination of the vectors $\alpha_1, \ldots, \alpha_n$ in V provided $\exists$ scalars $c_1, c_2, \ldots, c_n$ in F s.t.

$$\beta = c_1\alpha_1 + \cdots + c_n\alpha_n$$

$$\beta = \sum_{i=1}^{n} c_i\alpha_i$$

Other extensions of the associative property of vector add^n & distributive properties (c) & (d) of scalar multi^n apply to linear combinations:

$$\sum_1^n c_i\alpha_i + \sum_1^n d_i\alpha_i = \sum_1^n (c_i+d_i)\alpha_i$$

$$c\sum_1^n c_i\alpha_i = \sum_1^n (cc_i)\alpha_i$$

SUBSPACES:

Def^n: Let V be a vector space over the field F. A subspace of V is a subset W of V which is itself a vector space over F with the operations of vector addition and scalar multiplication on V.

→ Eg: R^n: $\bar{x} = (x_1, x_2, x_3, \ldots, x_n)$

$x_i \in R$

R^n over R

	subset	subspace	
$\{\bar{0}\} \subset V$			
$\{(1, x_2, \ldots, x_n)\}$	✓	✗	no $\bar{0}$ in subset
$\{(0, x_2, \ldots, x_n)\}$	✓	✓	

※ Theorem: A non-empty subset W of V is a subspace of V iff for each pair of vectors α, β in W and each scalar c in F the vector $c\alpha + \beta$ is again in W.

Proof: $\leftarrow$

Let non-empty $W \subset V$ & $c\bar{\alpha} + \bar{\beta} \in W$ $\forall$ $\bar{\alpha}, \bar{\beta} \in W$ and all scalars $c \in F$.

As $W \rightarrow$ non-empty

① $\therefore \exists\ \bar{\alpha} \in W$

let $c = -1$ & $\bar{\beta} = \bar{\alpha}$

$\therefore c(\bar{\alpha}) + \bar{\beta} = -1(\bar{\alpha}) + \bar{\alpha} = \bar{0} \in W$

② let $\bar{\beta} = \bar{0}$ & $\bar{\alpha} \in W$

then, $c\bar{\alpha} + \bar{0} \in W$

$\therefore \boxed{c\bar{\alpha} \in W}$

Assignment - 3 (Question - 1)

※ Assignment - 3:

Que 2: An $m \times n$ matrix A (over $\mathbb{C}$) is Hermitian if $A_{ij} = \overline{A_{ji}}$.

Any 2×2 Hermitian matrix is of the form

$$\begin{bmatrix} z & x+iy \\ x-iy & w \end{bmatrix} \quad w, x, y, z \in R$$

Show that,

the set of all Hermitian matrices is not a subspace of the space of all $m \times n$ matrices over $\mathbb{C}$.

(How, Why, What if the field is $\mathbb{R}$ instead of $\mathbb{C}$).

※ **Theorem:** Let V be a vector space over the field F. The intersection of any collection of subspace of V is a subspace of V.

Proof: Let $\{W_i\}$ be a collection of subspaces of V.

Let $W = \bigcap_i W_i$

$\therefore$ $W \to$ the set of all elements belonging to every W_α.

Since $\bar{0} \in W_i \ \forall i$

$\bar{0} \in \bigcap_i W_i$

$\therefore \bar{0} \in W$

$\therefore$ W is non-zero.

Let $\bar{\alpha}, \bar{\beta} \in W$ & $c \in F$

$\therefore \bar{\alpha}, \bar{\beta} \in W_i \ \forall i$

$\therefore$ As W_i is a subspace $\forall i$

$\therefore c\bar{\alpha} + \bar{\beta} \in W_i \ \forall i$

$\therefore \boxed{c\bar{\alpha} + \bar{\beta} \in W}$ By previous theorem,

W is a subspace.

Subspace Spanned:

Def^n: Let S be a set of vectors in a vector space V. The subspace spanned by S is defined to be the intersection W of all subspaces of V which contain S. When S is a finite set of vectors, $S = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$, we shall simply call W the subspace spanned by the vectors $\alpha_1, \alpha_2, \ldots, \alpha_n$.

※ Assignment - 3

Que-3: Prove the following theorem:

The subspace spanned by a non-empty subset S of a vector space is the set of all linear combinations of of vectors in S.

Def^n: If $S_1, S_2, \ldots, S_k$ are subsets of a vector space V, the set of all sums

$$\alpha_1 + \alpha_2 + \cdots + \alpha_k$$

of vectors α_i in S_i is called the sum of the subsets $S_1, S_2, \ldots, S_k$ and is denoted by $S_1 + S_2 + \cdots + S_k$

$$= \sum_{1}^{k} S_i$$

eg: $S_1 = \{\bar{x}, \bar{y}\}$

$S_3 = \{\bar{1}\}$

$S_2 = \{\bar{z}\}$

$$S_1 + S_2 + S_3 = \{\bar{x} + \bar{z} + \bar{1},\ \bar{y} + \bar{z} + \bar{1}\}$$

If $W_1, W_2, \ldots, W_k$ are subspaces of V, then the sum
$W = \sum_1^k W_i$ is also a subspace spanned by
$\bigcup_1^k W_i$.

Assignment-3:

Que-4: (a) Prove that $\bigcup_1^k W_i$ is not a subspace.

(b) Prove:
$\sum_1^k W_i$ is spanned by $\bigcup_1^k W_i$.

※ BASES: (singular: Basis)

Def^n: Let V be a vector space over F. A subset S of V is said to be linearly dependent if there exist distinct vectors $\alpha_1, \alpha_2, \ldots, \alpha_n$ in S and scalars $c_1, \ldots, c_n$ in F, not all of which are 0, such that

$$c_1\alpha_1 + \cdots + c_n\alpha_n = \bar{0}.$$

A set which is not linearly dependent is called linearly independent.

Consequences: [LD - Linearly Dependent LID = Linearly Independent]

1. Any set which contains a LD set is LD.
2. Any subset of a LID set is LID.
3. Any set which contains the 0 vector is LD.

Def^n of basis: $V \rightarrow$ Vector Space.

A basis for V is a linearly independent set of vectors in V which spans the space V.

Assignment-3:

Que-5: Find the number of bases of the solution space of the matrix $A_{m\times n}$.

i.e. $\{X : A_{m\times n} X_{n\times 1} = 0\}$

Prove it.

If B is finite set, then V is finite dimensional.

Theorem: Let V be a vector space which is spanned by a finite set of vectors $\bar{\beta}_1, \bar{\beta}_2, \ldots, \bar{\beta}_m$. Then any independent set of vectors in V is finite and contains no more than m elements.

Que-6 (Assignment-3)

Collorary ①: If V is a finite-dimensional vector space, then any two bases of V have the same (finite) number of elements.

Proof: V has a finite bases:

$$\{\beta_1, \beta_2, \beta_3, \ldots, \beta_m\}$$

By previous theorem, every basis of V is finite and contains no more than m elements.

Thus if $\{\alpha_1, \ldots, \alpha_n\}$ is a basis, $n \le m$.

And, by the same argument $m \le n$.

$$\left.\begin{array}{l} n \le m \\ m \le n \end{array}\right\} \boxed{m = n}$$

Thus, the dimension of a finite-dimensional vector space is the number of elements in a basis for V.

② Let V be a finite-dimensional vector space and let $n = \dim V$.

Then,

(a) any subset of V which contains more than n elements is linearly dependent.

(b) no subset of V which contains fewer than n vectors can span V.

Lemma: Let S be a linearly independent subset of a vector space V. Suppose β is a vector in V which is not in the subspace spanned by S. Then the set obtained by adjoining β to S is linearly independent.

Proof:

※ **Theorem:** If W is a subspace of a finite-dimensional vector space V, every linearly independent subset of W is finite and is part of a (finite) basis for W.

Proof: Suppose S_0 is a linearly independent subset of W. If S is a linearly independent subset of W containing S_0, then S is also linearly independent subset of V.

Since, $V \to$ finite dimensional

S contains no more than $\dim V$ elements.

Now,

we extend S_0 to a basis for W, as follows

If S_0 spans W, then $S_0 \to$ basis for W

If not,

then find a vector $\beta_1 \notin \text{span}(S_0)$ & $\beta_1 \in W$

let $S_1 = S_0 \cup \{\beta_1\}$

$\therefore S_1$ is independent (using previous lemma)

Now,

if $\text{span}(S_1) = W$, then $S_1 \to$ basis

else,

find a vector $\beta_2 \in W$, $\notin \text{span}(S_1)$,

let $S_2 = S_1 \cup \{\beta_2\}$

$\therefore S_2 \to$ independent

In this way we reach a set,

$$S_m = S_0 \cup \{\beta_1, \ldots, \beta_m\}$$

which is a basis for W.

※ Assignment - 4:

Que-1: Let A be an $n \times n$ matrix over a field F, and suppose the row vectors of A form a linearly independent set of vectors in F^n. Then A is invertible.

Que-2: If W_1 and W_2 are finite-dimensional subspaces of a vector space V, then $W_1 + W_2$ is finite-dimensional and

$$\dim W_1 + \dim W_2 = \dim(W_1 \cap W_2) + \dim(W_1 + W_2)$$

※ Coordinates:

→ The coordinates of a vector $\bar{\alpha}$ in V relative to the bases β will be the scalars which serve to express α as a linear combination of the vectors in the basis.

→ Natural Coordinates:

scalars which serve to express α as a linear combination of the vectors in the standard basis of F^n.

$$\alpha = (x_1, x_2, \ldots\ldots, x_n) = \sum_{1}^{n} x_i \varepsilon_i$$

→ Ordered Basis:

If V is a finite-dimentional vector space, an ordered basis for V is a finite sequence of vectors which is linearly independent and spans V.

If $\alpha_1, \ldots\ldots, \alpha_n$ is an ordered basis for V, then the set $\{\alpha_1, \alpha_2, \ldots\ldots, \alpha_n\}$ is a basis for V.

Hence,
the ordered basis is the set, together with the specified ordering.

Now, suppose V is a finite-dimensional vector space over the field F and that

$$\beta = \{\alpha_1, \ldots\ldots, \alpha_n\}$$

is an ordered basis for V.

Given α in V, there is a unique n-tuple $(x_1, \ldots, x_n)$ of scalars such that

$$\alpha = \sum_1^n x_i \alpha_i$$

To Prove: The n-tuple is Unique

Proof: Suppose $\exists$ two n-tuples:

$$\left.\begin{aligned} \alpha &= \sum_1^n x_i \alpha_i \\ \& \quad \alpha &= \sum_1^n z_i \alpha_i \end{aligned}\right\} \quad \sum_1^n x_i \alpha_i = \sum_1^n z_i \alpha_i$$

$$\boxed{\therefore \sum_1^n (x_i - z_i)\alpha_i = 0}$$

As $\{\alpha_1, \ldots, \alpha_n\}$ is a linearly independent set.

$$\boxed{x_i - z_i = 0 \ \forall i}$$

$$\boxed{\therefore x_i = z_i \ \forall i}$$

$\therefore$ n-tuple is UNIQUE

$[\bar{\alpha}]_B \Rightarrow$ coordinates of $\bar{\alpha}$ w.r.t. the basis B.

$\rightarrow$ $\bar{\alpha} + \bar{\delta} = \sum_1^n x_i \bar{\alpha_i} + \sum_1^n y_i \bar{\alpha_i} = \sum_1^n (x_i + y_i)\bar{\alpha_i}$

$\therefore$ coordinates of $\bar{\alpha} + \bar{\delta} = (x_1 + y_1, x_2 + y_2, \ldots, x_n + y_n)$.

Thus, $\boxed{[\bar{\alpha}]_B + [\bar{\beta}]_B = [\bar{\alpha} + \bar{\beta}]_B}$

Also,

$$\boxed{[c\bar{\alpha}]_B = c[\bar{\alpha}]_B}$$

$\rightarrow$ Suppose that V is n-dimensional and that

$$B = \{\alpha_1, \ldots, \alpha_n\} \text{ and } B' = \{\alpha_1', \ldots, \alpha_n'\}$$

are two ordered bases for V.

As α_j' is in V, we can write its coordinates w.r.t. B

$$\therefore \boxed{\alpha_j' = \sum_1^n P_{ij} \alpha_i} \quad 1 \le j \le n$$

$\therefore$ let $x_1', x_2', \ldots, x_n'$ be the coordinates of a given vector $\bar{\alpha}$ in the basis B'.

$$\therefore \alpha = \sum_{1}^{n} x_j' \alpha_j'$$

$$= \sum_{j=1}^{n} x_j' \left(\sum_{i=1}^{n} P_{ij} \alpha_i \right)$$

$$= \sum_{j=1}^{n} \sum_{i=1}^{n} (P_{ij} x_j') \alpha_i$$

$$\boxed{\bar{\alpha} = \sum_{i=1}^{n} \left(\sum_{j=1}^{n} P_{ij} x_j' \right) \alpha_i} \quad 1 \le i \le n$$

Let P be the $n \times n$ matrix whose i,j entry is the scalar P_{ij} and let X and X' be the coordinate matrices of the vector $\bar{\alpha}$ in the ordered bases B and B' respectively.

$$\therefore \boxed{X = P X'}$$

B and B' $\to$ linearly independent sets,

$\therefore X = 0$ if and only if $X' = 0$

$$\therefore \boxed{X' = P^{-1} X} \Rightarrow \begin{aligned} [\alpha]_B &= P[\alpha]_{B'} \\ [\alpha]_{B'} &= P^{-1}[\alpha]_B \end{aligned}$$

※ Theorem :

Let V be an n-dimensional vector space over the field F, and let B and B' be two ordered bases of V. Then there is a unique, necessarily invertible, $n \times n$ matrix P with entries in F such that,

(i) $[\alpha]_B = P[\alpha]_{B'}$

(ii) $[\alpha]_{B'} = P^{-1}[\alpha]_B$

Write example 19

Linear Transformations :

Definition :

Let V and W be vector spaces over the field F. A linear transformation from V into W is a function T from V into W such that,

$$T(c\alpha+\beta) = c(T\alpha) + T\beta$$

$$\forall \; \alpha, \beta \in V \quad \text{and} \quad c \in F$$

$$\rightarrow T(\bar{0}) = T(\bar{0}+\bar{0})$$

$$= T(\bar{0}) + T(\bar{0})$$

$$\boxed{T(\bar{0}) = \bar{0}_W}$$

※ Theorem :

Let V be a finite-dimensional vector space over the field F and let $\{\alpha_1, \ldots, \alpha_n\}$ be an ordered basis for V. Let W be a vector space over the same field F and let $\beta_1, \ldots, \beta_n$ be any vectors in W. Then there is precisely one linear transformation T from V into W such that,

$$T\alpha_j = \beta_j \quad , 1 \le j \le n$$

Proof :

Assignment - 5 :

Que 1: Prove the following theorem:

Let V and W be vector spaces over the field F and let T be a linear transformation from V into W. Suppose that V is finite-dimensional then,

$$\boxed{\text{rank}(V) + \text{nullity}(T) = \dim V}$$

→ **Nullspace:** $\{\bar{\alpha} \mid \bar{\alpha} \in V,\ T\bar{\alpha} = \bar{0}_W\}$

→ $\text{Image}(T) = \text{Range}(T)$

$$= \{T\bar{\alpha} \mid \forall\, \bar{\alpha} \in V\}$$

→ Proving that the Image(T) is a subspace.

let $\left.\begin{array}{l} T\bar{\alpha}_1 = \bar{\beta}_1 \\ T\bar{\alpha}_2 = \bar{\beta}_2 \end{array}\right\} \in \text{Image}(T)$

$\bar{\alpha}_1, \bar{\alpha}_2 \in V$

$\therefore c\bar{\alpha}_1 + \bar{\alpha}_2 \in V$

$\therefore T(c\bar{\alpha}_1 + \bar{\alpha}_2) = c(T\bar{\alpha}_1) + T\bar{\alpha}_2$

$= c\beta_1 + \beta_2 \in \text{Image}(T)$

※ The Algebra of Linear Transformation:

※ Theorem: V & W are vector space over F.

Let $T: V \to W$ & $U: V \to W$ be linear transformations. The $f^n (V+U): V \to W$ defined as $(T+U)\bar{\alpha} = T\bar{\alpha} + U\bar{\alpha}$ $(\forall \bar{\alpha} \in V)$ as a linear transformation

The f^n $(cT): V \to W$ defined as $(cT)\bar{\alpha} = c(T\bar{\alpha})$ $\forall \bar{\alpha} \in V$ is a linear transformation.

Then, the set $L(V, W)$ of all linear transformations $T: V \to W$ with add^n and scalar multiplication as defined earlier, forms a vector space over F.

(Assignment-5 Que-2)

※ Theorem: Let V and W be vector spaces,

$\dim(V) = n$, $\dim(W) = m$.

Then, $\dim(L(V, W)) = \dim(V) \times \dim(W) = \boxed{nm}$

(Assignment-5 Que-3) $\quad \delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{o.w.} \end{cases}$

※ Theorem: V, W, Z are vector spaces over F.

$T: V \to W$, $U: W \to Z$ are linear transformations.

Then, their composition $U \cdot T: V \to Z$ is also a linear transformation.

Proof: $(UT)(c\bar{\alpha}+\bar{\beta})$

$= U(T(c\bar{\alpha}+\bar{\beta}))$

$= U(cT\bar{\alpha}+T\bar{\beta})$

$= \boxed{cU\cdot T\bar{\alpha} + U\cdot T\bar{\beta}}$

Given: $T: V \to W$ s.t.

$T(c\bar{\alpha}+\bar{\beta}) = cT\bar{\alpha} + T\bar{\beta} \ \forall \bar{\alpha},\bar{\beta} \in V$

$U: W \to Z$ s.t.

$U(c\bar{\alpha}+\bar{\beta}) = cU\bar{\alpha} + U\bar{\beta} \ \forall \bar{\alpha},\bar{\beta} \in W$

NOTE: 1) If $T \in L(V,V)$, then T is a linear operator.

2) If $T \in L(V,V)$, then T^n i.e. $\underbrace{T\circ T\circ \cdots \circ T}_{n \text{ times}}$ is well defined.

3) $T^0 = 1$: Identity Transformation.

4) If $T\bar{\alpha} = U\bar{\alpha} \ \forall \bar{\alpha}$ then, T & U are same.

※ Lemma: V is a vector space over F. let $U, T_1, T_2 \in L(V,V)$ let $c \in F$.

1. $1U = U1 = U$
2. $U(T_1+T_2) = UT_1 + UT_2$
   $(T_1+T_2)U = T_1U + T_2U$
3. $c(UT_1) = (cU)T_1 = U(cT_1)$

※ Inverse of Linear Transformation:

→ $T \in L(V,V)$, $\exists\ T^{-1}$ s.t.

$$\boxed{T\circ T^{-1} = 1 = T^{-1}T}$$

→ f^n $T: V \to W$ is invertible if $\exists$ a $f^n: T^{-1}: W \to V$ s.t. $T^{-1}\circ T$ is identity f^n on V & $T\circ T^{-1}$ is identity function on W.

→ Further, T is invertible iff,

1. T is $1:1$, i.e., $T\bar{\alpha} = T\bar{\beta} \Rightarrow \bar{\alpha} = \bar{\beta}$
2. T is onto, i.e.

$$\text{Range}(T) = W.$$

※ **Theorem:** V & W are vector spaces over F. $T: V \to W$ be a linear transformation. If T is invertible then $\exists$ a (unique) inverse f^n $T^{-1}: W \to V$ which is a linear transforma^n.

Proof: Suppose $\beta_1, \beta_2 \in W$

To prove: $T^{-1}(c\bar{\beta}_1 + \bar{\beta}_2) = cT^{-1}\beta_1 + T^{-1}\beta_2$

→ Consider $\bar{\alpha}_1, \bar{\alpha}_2 \in V$, then $T\bar{\alpha}_1, T\bar{\alpha}_2 \in W$

$$(T\bar{\alpha}_1 = \beta_1, \; T\bar{\alpha}_2 = \beta_2)$$

$$\therefore c\bar{\beta}_1 + \bar{\beta}_2 = cT\bar{\alpha}_1 + T\alpha_2$$

$$= T(c\bar{\alpha}_1 + \bar{\alpha}_2)$$

$$\therefore T^{-1}(c\bar{\beta}_1 + \bar{\beta}_2) = T^{-1}T(c\bar{\alpha}_1 + \bar{\alpha}_2)$$

$$= c\bar{\alpha}_1 + \bar{\alpha}_2$$