RA - Assignment-2

1. [1] Cauchy but not monotone

Sequence $\frac{(-1)^n}{n}$

$a_1 = -1 \quad a_2 = \frac{1}{2} \quad a_3 = \frac{-1}{+3} \quad a_4 = \frac{1}{4}$

$a_2 > a_1$ but $a_3 < a_2$ and $a_2 > a_4$

hence this sequence is not a monotone.

taking 2 subsequences of this sequence

(1) $\longrightarrow$ when n is odd

(2) $\longrightarrow$ when n is even

$\frac{-1}{2n+1} \qquad \frac{1}{2n}$

both the subsequence converges to the same limit 0

and as it is convergent it is also cauchy

hence

1 Cauchy but not monotone.

[2] monotone but not cauchy

$a_n = n^2$

$a_1 = 1, \; a_2 = 4, \; a_3 = 9 \; \ldots \; a_3 > a_2 > a_1$

it is monotone, i.e monotonically increasing

and we also know it is not convergent

hence not Cauchy

Real Numbers $\longrightarrow$ complete metric space

1. [3] Bounded But not cauchy

Lets take the sequence

$a_n = (-1)^n$

taking 2 Subsequnces as n is odd (or) even

$a_{even} = 1$ $\quad a_{odd} = -1$

the subsequences are constant sequences and are not converging to the same limit

$\therefore$ hence it is not convergent i.e not cauchy

($\mathbb{R} \rightarrow$ complete metric space)

2. Given $f(x) = \dfrac{x^3}{1+x^2}$

WKT,

If f is continous on $\mathbb{R}$

$A \subseteq X$, Then f is uniformly continous on A if $\forall \varepsilon > 0, \exists \delta > 0$ s.t $x, y \in A$

when, $d(x,y) < \delta \Rightarrow \rho(f(x), f(y)) < \varepsilon$

$|f(x) - f(c)| < \varepsilon$

$$\left| \frac{x^3}{1+x^2} - \frac{c^3}{1+c^2} \right| < \varepsilon$$

$$|x-c| \cdot \left| \frac{x^2 + xc + c^2 + x^2c^2}{(1+x^2)(1+c^2)} \right|$$

$$|x-c| \cdot \left| 1 + \frac{xc - 1}{(1+x^2)(1+c^2)} \right|$$

always < 1 always $\leq \frac{1}{2}$

$$|x-c| \cdot \left| 1 - \left(\frac{1}{(1+x^2)(1+c^2)} \right) + \left(\frac{x}{1+x^2} \right) \left(\frac{c}{1+c^2} \right) \right|$$

$|x-c|\left(\frac{5}{4}\right) < \varepsilon$

$|x-c| < \frac{4\varepsilon}{5}$

considering $\delta = \frac{4\varepsilon}{5}$

Cleared continuity definition.

Now regarding uniform Continuity

def: $f: X \longrightarrow Y$

if f is continuous on $A \subseteq X$

$\forall\ \varepsilon > 0$ and $u \in A$ $\exists\ \delta > 0$ s.t $\forall\ x \in A$

and $d(x,y) < \delta$

$\rho(f(x), f(y)) < \varepsilon$

where δ is a function ε and u.

if we take $\delta = \frac{4\varepsilon}{5}$ it also already satisfied the uniform continuity on $\mathbb{R}$

Hence the given function uniformly Continous on $\mathbb{R}$.

3. given,

Sequences a_n and b_n are bounded

Sequence $c_n = a_n \cdot b_n$

To prove: $\lim(\sup c_n) = \lim\inf(a_n) \cdot \lim\inf(b_n)$

when $\lim\sup(a_n) < 0$ & $\lim\sup(b_n) < 0$

$\sup\{a_n\} < 0 \quad \forall n \ a_n < 0$ {By supremum Property}

$\sup\{b_n\} < 0 \quad \forall n \ b_n < 0$

$[\exists N_0 \in \mathbb{N}\ n \geq N_0]$

$c_n = a_n \cdot b_n$

$c_n = (-1)(-1)(a_n)(b_n)$

$c_n = (-a_n)(-b_n)$

consider a sequence a_n.

let $\inf(a_n) = -L$

By infimum property

$-L \leq a_n < -L+\epsilon$

$L > -a_n > (L-\epsilon)$

Let $\{-a_n\} = \{d_n\}$

$L-\epsilon < d_n < L$

By supremum Property

$\sup\{d_n\} = L$
$= -\inf\{a_n\}$

Similarly let $-b_n = e_n$

$\sup\{e_n\} = -\inf\{b_n\}$

$c_n = \{(-a_n)(-b_n)\}$

$c_n = d_n \cdot e_n$

$\sup\{c_n\} = \sup\{d_n\} \cdot \sup\{e_n\}$

$= -\inf\{a_n\} \cdot -\inf\{b_n\}$

$= \inf(a_n) \cdot \inf(b_n)$

as entire sequence satisfies

by Property

$\lim \sup(-a_n) = \liminf(a_n) \; \forall n \geq N_0$

$\lim \sup(-b_n) = \liminf(b_n) \forall n \geq N_0$

$\therefore \lim \sup(c_n) \leq \lim \inf(a_n) . \liminf(b_n)$

$\therefore$ Hence proved.

4. Given

$f: R \to R$ is a continuous function.

To prove, $f^{-1}(k)$ is closed where $k \in R$

~~*~~

considering a set, a singleton set containing k.

we know that any singleton set is a closed set as

the set of all limit points of S

$S \supset S'$ — $\emptyset$

Singleton set

then the complement of this singleton set $R - \{k\}$ is an open set.

we know that, if $f: X \to Y$ then

$\forall$ open set $U \subseteq Y$, $f^{-1}(U)$ is an openset in X iff f is continuous.

taking $U = R - \{k\} \subset R \leftarrow Y$

we can say that $f^{-1}(R - \{k\})$ is also open

then as before the complement of $f^{-1}(R - \{k\})$ would be closed

$\therefore$ Hence proved

$f^{-1}(k)$ is closed.

6) given,

$\{a_n\}$ is a sequence

$\{b_n\}$ is a non-decreasing convergent sequence

of the numbers

s.t,

$|a_{n+1} - a_n| \le b_{n+1} - b_n$

As $\{b_n\}$ is a convergent sequence

$\{b_n\}$ is a cauchy sequence

then,

$\forall \varepsilon > 0 \ \exists N_0 \in \mathbb{N}$ st $\forall$ $m, n \ge N_0$

$$|b_m - b_n| < \varepsilon_0$$

replacing $m \to n+p$ where $p = 1, 2, 3 \cdots$

$$|b_{n+p} - b_n| < \varepsilon_0$$

as $n+p > n$ and as $\{b_n\}$ is a non-decreasing convergent sequence we can,

$$b_{n+p} - b_n < \varepsilon_0 \quad \text{(No modulus)}$$

now,

$$b_{n+p} - b_n = (b_{n+p} - b_{n+p-1}) + \cdots (b_{n+2} - b_{n+1}) + (b_{n+1} - b_n)$$

$$\ge |a_{n+p} - a_{n+p-1}| + |a_{n+p-1} - a_{n+p-2}| + \cdots + |a_{n+2} - a_{n+1}| + |a_{n+1} - a_n|$$

$$\ge |a_{n+p} - a_n|$$

we can say this through Δgle inequality

$b_{n+p} - b_n \ge |a_{n+p} - a_n|$ and,

$b_{n+p} - b_n < \varepsilon_0$

$|a_{n+p} - a_n| < \varepsilon_0$

Replacing $n+p \to m$ where $p = 1, 2, 3 \cdots$

then,

$\forall \varepsilon_0 > 0 \ \exists \ N_0 \in \mathbb{N}$ s.t $\forall \ m, n \geq N_0$

$|a_m - a_n| < \varepsilon_0$

Hence proved that $\{a_n\}$ is a cauchy $\{\}$.

7) Given

f is a continuous mapping

$f : X \to Y$

$f(\overline{E}) \subset \overline{f(E)} \ \forall \ E \subset X$

The ~~closure~~ closure of set $A \to \bar{A}$

$\bar{A}$ = all the points of A + all limit points of A

Let $b \in f(\bar{E})$ as the function is continous

$\exists \ a \in \bar{E}$ s.t $f(a) = b$

now a is either a point of E or is a limit point of E.

in Either case let us take a convergent sequence in E that converges point of a_n is a as the function is continous.

$f(a_n)$ is also a convergent of $f(a)$ sequence converging.

and as $a_n \in E$, $f(a_n) \in f(E)$

if a is limit point of E

By def of limit point

$\forall \ \delta > 0 \qquad N'(a, \delta) \cap E \neq \emptyset$

as the function is continous $\forall \ \varepsilon > 0 \ \exists \delta > 0$

$\Rightarrow f(N'(a, \delta)) \subseteq N(f(a), \varepsilon)$

and

$$f(N'(a,\delta)) \cap f(E) \neq \emptyset$$

hence, $\forall \varepsilon > 0$

$$N'(f(a),\varepsilon) \cap f(E) \neq \emptyset$$

then $f(a)$ is a limit point of $f(E)$

$\overline{f(E)}$ = point of $f(E)$ + limit points of $f(E)$

as b is any arbitrary point, $b \in f(E)$

hence proved $f(\overline{E}) \subset \overline{f(E)}$ $\forall$ $E \subset X$

5)

i) Let A is nowhere dense in X

Then $int(\bar{A}) = \emptyset$

$\bar{A}$ is A + its limit points

Let Y be a set s.t $Y \subseteq A$

$\bar{Y} \subseteq \bar{A}$ (since limit points of Y are in limit points of A)

$int(\bar{Y}) \subseteq int(\bar{A})$

$int(\bar{A}) = \emptyset$

$\therefore$ $int(\bar{Y}) = \emptyset$

Hence proved.

2. Let $A_1, A_2 \ldots A_n$ be n nowhere dense subsets of X

$int(\bar{A_1}) = \emptyset$, $int(\bar{A_2}) = \emptyset$ .... $int(\bar{A_n}) = \emptyset$.

Let $B = A_1 \cup A_2 \cup A_3 \ldots A_n$

then $\bar{B} = \bar{A_1} \cup \bar{A_2} \cup \ldots \bar{A_n}$

$int \bar{B} = int(\bar{A_1} \cup \bar{A_2} \cup \ldots \bar{A_n})$

$= int(\bar{A_1}) \cup int(\bar{A_2}) \ldots \cup int(\bar{A_n})$

To prove

$int(C \cup D) = (int\ C) \cup (int\ D)$

Let $x \in int(C \cup D)$

$\exists$ open set $K \subseteq C$ / $K \subseteq D$

$x \in K$

$x \in \text{int}\, C \,/\, \text{int}\, D$

$x \in \text{int}\, C \cup \text{int}\, D$

$\text{int}\,(C \cup D) \subseteq (\text{int}\, C) \cup (\text{int}\, D)$

$x \in (\text{int}\, C) \cup (\text{int}\, D)$

$\exists$ open set k s.t $k \subseteq C / D$.

$k \subseteq C \not\subseteq D \Rightarrow k \subseteq C \cup D$

$x \in k \Rightarrow x \in \text{int}\,(C \cup D)$

$\Rightarrow \text{int}\, C \cup \text{int}\, D \subseteq \text{int}\,(C \cup D)$

$\therefore \text{int}\,(C \cup D) = \text{int}\, C \cup \text{int}\, D$

$\therefore \text{int}\, \bar{B} = \text{int}\,(\bar{A}_1) \cup \text{int}\,(\bar{A}_2) \ldots \text{int}\,(\bar{A}_n)$

$\emptyset \cup \emptyset$

Since A_1 is nowhere dense

$\text{int}\,(A_n) = \emptyset$

$\therefore \text{int}\, \bar{B} = \emptyset$

3. Given A is Nowhere dense.

$\text{int}\,(\bar{A}) = \emptyset$

$\bar{A} = A +$ limit points

$\bar{\bar{A}} = \overline{((\bar{A}))}$

$\bar{A} + \bar{A}$ limit points

As $\bar{A}$ consists of all its limitpoints

$\overline{(\bar{A})} = \bar{A}$

$\therefore \text{int}\,(\bar{\bar{A}}) = \text{int}\,(\bar{A}) = \emptyset$

$\therefore$ Hence proved.

4. X has no isolated points

For no where dense

If $A \subset X \Rightarrow int(\bar{A}) = \emptyset$

Let $F \{x_1, x_2, \dots x_n\}. F \subset X$

Since there are finite elements

$\bar{F} = F$

Since F has no isolated points

$x \in F \quad (F \cap N(x, \delta)) - \{x\} = \emptyset$

Let's assume

$int(\bar{F}) \neq \emptyset$

$\exists$ open set $V \subset X$ s.t

$V \subset \bar{F} = F$

Since $\bar{F} = F$

$\underbrace{V \subset F}_{\text{subset of } F} = \underbrace{F}_{\neq F}$ (not possible)

By contradiction $int(\bar{F}) = \emptyset$

F is no where dense.