Linear Algebra

Assignment-1 after midsem [redacted]

1. $x = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$ $\quad y = \begin{pmatrix} 3 \\ 4 \\ 2 \end{pmatrix}$

By Gram. Schmidt Orthogonalisation process

let $\{v_1, v_2\}$ be orthogonal vectors

$v_1 = x$

$v_2 = y - \text{proj}_x(y)$

$$v_2 = y - \frac{\langle x\, y \rangle}{\langle x\, x \rangle} \cdot x$$

$$v_2 = \begin{pmatrix} 3 \\ 4 \\ 2 \end{pmatrix} - \frac{1\cdot 3 + 1\cdot 4 + 0\cdot 2}{1+1+0} \cdot \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

$$v_2 = \begin{pmatrix} 3 \\ 4 \\ 2 \end{pmatrix} - \begin{pmatrix} \frac{7}{2} \\ \frac{7}{2} \\ 0 \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ 2 \end{pmatrix}$$

The orthogonal basis Subspace of R^3 spanned the vectors x, y are $\{v_1, v_2\} \Rightarrow \left\{ \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \begin{pmatrix} -\frac{1}{2} \\ +\frac{1}{2} \\ 0 \end{pmatrix} \right\}$

2.

(a) given, A and B are square matrix and AB is invertible

To prove Both A and B are invertible

wkt, AB is invertible

then, $\exists$ $(AB)^{-1}$ s.t $(AB)^{-1}(AB) = I$ — ①

$(AB)(AB)^{-1} = I$ — ②

let $(AB)^{-1} = C$

then, from ②

$$A\underbrace{BC}_{\text{let } B \cdot C \text{ be a matrix, } H} = I$$

$A \cdot H = I$

By this $\exists$ matrix H s.t $A \cdot H = I$

So, $\therefore$ A is invertible

If Right inverse exist, since A is square matrix Left inverse also exists

from ①

$CAB = I$

$$\underbrace{(CA)}_{\text{Let } C \cdot A \text{ be a matrix } H} \cdot B = I$$

By this $\exists$ matrix H s.t $H \cdot B = 0$

So, $\therefore$ B is invertible.

If Left inverse exist, since B is Square matrix Right inverse also exists

2(b) given matrix A is symmetric and invertible.

To show :- A^{-1} also symmetric

wkt, A is invertible

the $\exists$ a matrix P such that

$A \cdot P = I$

let $P = A^{-1}$

$A \cdot A^{-1} = I$

and A is symmetric then

$A^T = A$

$(AA^{-1})^T = I^T \longrightarrow I$ is also symmetric

$(A^{-1})^T \cdot A^T = I$ $\quad [(AB)^T = B^T \cdot A^T]$

$(A^{-1})^T \cdot A \cdot A^{-1} = I \cdot A^{-1}$ $\quad [A^T = A]$

$(A^{-1})^T I = A^{-1} \Rightarrow (A^{-1})^T = A^{-1}$

3) (a) For A should be invertible, A should be square matrix & $|A| \neq 0$

$$|A| = \begin{vmatrix} k & -k & 3 \\ 0 & k+1 & 1 \\ k & -8 & k-1 \end{vmatrix}$$

$$= k\begin{vmatrix} k+1 & 1 \\ -8 & k-1 \end{vmatrix} + k\begin{vmatrix} 0 & 1 \\ k & k-1 \end{vmatrix} \; 3\begin{vmatrix} 0 & k+1 \\ k & -8 \end{vmatrix}$$

$$= k(k^2-1+8) + k(-k) + 3(-(k^2+k))$$

$$= k^3+7k-k^2-3k^2-3k$$

$$|A| = k^3-4k^2+4k \neq 0$$

$$k(k^2-4k+4) \neq 0,$$

$$k(k-2)^2 \neq 0$$

$$k \neq 0 \;, \; k-2 \neq 0$$

$$k \neq 2$$

$k \in \mathbb{R} \setminus \{0,2\}$ for A to be invertible

(b) Similarly $|A| \neq 0$

$$|A| = \begin{vmatrix} k & k & 0 \\ k^2 & 2 & k \\ 0 & k & k \end{vmatrix}$$

$$= k\begin{vmatrix} 2 & k \\ k & k \end{vmatrix} - k\begin{vmatrix} k^2 & k \\ 0 & k \end{vmatrix} - 0(\;)$$

$$= k(2k-k^2) - k(k^3)$$

$$|A| = -k^3+2k^2-k^4 \neq 0$$

$$\Rightarrow -k^2(k^2+k-2) \neq 0$$

$$-k^2(k^2+2k-k-2) \neq 0$$

$$-k^2(k+2)(k-1) \neq 0$$

$$k \neq 0 \;, \; k \neq -2, \; k \neq 1$$

$k \in \mathbb{R} \setminus \{-2,0,1\}$ for A to be invertible

$K(K+2K+8)+0$

4) given, $\{u_1, u_2, u_3\}$, $u_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $u_2 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$, $u_3 = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$

(a) Let's check given vectors are orthogonal or not,

$\langle u_1, u_2 \rangle = 1\cdot1 + 1\cdot(-1) + 1\cdot0 = 0$

$\langle u_2, u_3 \rangle = 1\cdot1 - 1\cdot1 \; 0\cdot-2 = 0.$

$\langle u_3, v_1 \rangle = 1\cdot1 + 1\cdot1 - 2\cdot1 = 0$

* all the vectors are non-zero and orthogonal to each other. hence orthogonal basis.

Let each coloumn in matrix A be u_1, u_2, u_3.

* if set of vectors are orthogonal then set also called basis

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 0 & -2 \end{bmatrix} \Rightarrow \det A = 1(2) - 1(-2-1) + 1(1)$$

$$|A| = 6 \neq 0$$

The vectors are also linearly independent detA to the matrix is ~~row reduce~~ invertible implies vectors are linearly independent and also every pairwise vectors are orthogonal implies orthogonal basis

(b) co-ordinates of w w.r.t basis $\{u_1, u_2, u_3\}$

$$\Downarrow$$

$$w = \begin{bmatrix} 7 \\ 9 \\ 10 \end{bmatrix}$$

WKT,

$$w = c_1 u_1 + c_2 u_2 + c_3 u_3$$

$$c_1 = \frac{\langle w, u_1 \rangle}{\langle u_1, u_1 \rangle}, \quad c_2 = \frac{\langle w, u_2 \rangle}{\langle u_2 u_2 \rangle}, \quad c_3 = \frac{\langle w, u_3 \rangle}{\langle u_3 u_3 \rangle}$$

$$c_1 = \frac{7+9+10}{1+1+1} \quad c_2 = \frac{7\cdot 1 + 9(-1) + 10(0)}{1+1+0} \quad c_3 = \frac{7\cdot 1 + 9\cdot 1 + 10(-2)}{1+1+4}$$

$$c_1 = \frac{26}{3} \quad c_2 = \frac{-2}{2} \quad c_3 = \frac{-4}{6} = -\frac{2}{3}$$

$$c_1 = \frac{26}{3} \quad c_2 = -1 \quad c_3 = -\frac{2}{3}$$

Co-ordinates of w w.r.t $\{u_1, u_2, u_3\} = \left(\frac{26}{3}, -1, \frac{-2}{3}\right)$

5(a) For given matrix to be orthogonal

let $A = \begin{bmatrix} 1/3 & 1/2 & 1/5 \\ 1/3 & -1/2 & 1/5 \\ -1/3 & 0 & 2/5 \end{bmatrix}$

should satisfy $A \cdot A^T = I$

$$A \cdot A^T = \begin{bmatrix} 1/3 & 1/2 & 1/5 \\ 1/3 & -1/2 & 1/5 \\ -1/3 & 0 & 2/5 \end{bmatrix} \begin{bmatrix} 1/3 & 1/3 & -1/3 \\ 1/2 & -1/2 & 0 \\ 1/5 & 1/5 & 2/5 \end{bmatrix}$$

$$A \cdot A^T = \begin{bmatrix} \frac{1}{9}+\frac{1}{4}+\frac{1}{25} & \frac{1}{9}-\frac{1}{4}+\frac{1}{25} & \frac{-7}{225} \\ \frac{1}{9}-\frac{1}{4}+\frac{1}{25} & \frac{361}{900} & -7/225 \\ -\frac{1}{9}+\frac{2}{25} & \frac{-7}{225} & 61/225 \end{bmatrix}$$

$$A \cdot A^T = \begin{bmatrix} \frac{361}{900} & -89/900 & -7/225 \\ -89/900 & 361/900 & -7/225 \\ -7/225 & -7/225 & 61/225 \end{bmatrix} \neq I_{3\times3}$$

The given matrix is not orthogonal

(b) $\cancel{A_1 = \begin{pmatrix} a & b \\ b & a \end{pmatrix}}$

let $A_{2\times2} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ then for A to be an orthogonal matrix.

then,

$$A \cdot A^T = I$$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} a & c \\ b & d \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\Rightarrow \begin{bmatrix} a^2+b^2 & ac+bd \\ ac+bd & c^2+d^2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$ac+bd=0 \quad a^2+b^2=1 \quad c^2+d^2=1$

By solving, $c = \frac{-bd}{a}$

$$\Rightarrow \frac{b^2d^2}{a^2}+d^2=1$$

$(b^2 + a^2)d^2 = a^2$

to be matrix A as orthogonal $a^2+b^2=1$
because magnitude of coloumn should be
equal to $1 \Rightarrow (\sqrt{a^2+b^2})^2 = 1 \Rightarrow a^2+b^2=1$

$a^2 = d^2$ $\boxed{\begin{matrix} a=d \\ c=-b \end{matrix}}$ (or) $\boxed{\begin{matrix} a=-d. \\ c=b \end{matrix}}$

$c = -\frac{bd}{a}$

$c = \mp\frac{b\cancel{a}}{\cancel{a}}$

$c^2 = b^2$

Substituting values back into matrix.

$\begin{bmatrix} a & b \\ \mp b & -a \end{bmatrix}$ $\begin{bmatrix} a & b \\ -b & a \end{bmatrix}$

here
$a=-d$
$c=b$

let's consider
b as $-b$

$\begin{bmatrix} a & -b \\ b & a \end{bmatrix}$

hence, 2x2 orthogonal matrix A must have $\begin{pmatrix} a & -b \\ b & a \end{pmatrix}$ or $\begin{pmatrix} a & b \\ b & -a \end{pmatrix}$.