Q1) To show: if $\alpha, \beta \in W$, $c \in F$ then $c\alpha + \beta \in W$

[redacted] given, W is a subspace of V $\quad W \subset V$

then, W itself also a vector space

By def of subspace

it satisfies

eq — (1) Vector addition $\text{on } V \supset W$ (closure under addition)

eq — (2) Scalar multiplication on $V \supset W$ (closure under scalar multiplication)

from (2)

if $\alpha \in W$ $\quad c\alpha$ also $\in W$

then,

$c\alpha \in W$, $\beta \in W$

from (1) $\quad c\alpha + \beta \in W$

Thus,

if α, β in W and $c \in F$ then $c\alpha + \beta \in W$ holds.

To show :- W is a subspace of V

given, $\forall$ $\alpha, \beta \in W$ and all $c \in F$, $c\alpha + \beta \in W$

W is a non-empty set $\underset{\text{subset to}}{\subseteq}$ V [redacted] over F

To prove :- W is a subspace we need

(i) W itself should be a vector space (i.e zero vector $0 \in W$)

(ii) (closure under addition) over F of $W \overset{\text{subset}}{\subseteq} F$

(iii) (closure under scalar multiplication) [redacted] over F of $W \underset{\text{subset}}{\subseteq} F$

So, W is non-empty, pick any $\alpha \in W$, $c = 0 \in F$ and $\alpha = \beta \in W$ we get,

$$0\alpha + \alpha = 0 + \alpha = \alpha \in W$$

$0 \in F$ and any vector $\bar{\alpha} \in V$.

$0 \cdot \bar{\alpha} = (0+0) \cdot \bar{\alpha} = 0\bar{\alpha} + 0 \cdot \bar{\alpha}$ (by distributivity of scalar multiplication over scalar addition)

~~$\bar{\alpha} + 0$~~.

By a rule of scalar multiplication

$1\bar{\alpha} = \bar{\alpha}$ — ⑥ (by distributivity of scalar multiplication over scalar addition)

$(1+0)\bar{\alpha} = 1 \cdot \bar{\alpha} + 0 \cdot \bar{\alpha}$

from ⑥ $\bar{\alpha} = 1 \cdot \bar{\alpha} + 0 \cdot \bar{\alpha}$ — ⑤

but from uniqueness of vector $\bar{0}$ in rules of vector addition
$\bar{\alpha} + \bar{0} = \bar{\alpha}$ — ⑦

from ~~7~~ & ⑤ uniqueness

$$\boxed{0 \cdot \bar{\alpha} = \bar{0}} \quad — ③$$

~~Let $\alpha \in W$~~

Let $\alpha \in V$

WKT

$1 \cdot \alpha = \alpha$ (by scalar multiplication identity axiom)

$(1+(-1))\alpha = 1 \cdot \alpha + (-1)\alpha$ (distributivity of scalar multiplication over scalar addition)

$1+(-1)=0$ as in F (axiom additive inverse)

$0 \cdot \alpha = \alpha + (-1 \cdot \alpha)$

from (3) $0 = \alpha + (-1 \cdot \alpha)$ — (4)

By additive inverse axiom $\alpha + (-\alpha) = 0$.

but in (4) we have $\alpha + (-1 \cdot \alpha) = 0$.

$\therefore$ by uniqueness of the additive inverse.

$$\boxed{-1\cdot\alpha = -\alpha.} \quad -(8)$$

(3), (8) proved $\quad -1\cdot\alpha = -\alpha \;-(8) \qquad 0\cdot\bar{\alpha} = \bar{0} \;-(3).$

as proved before $\quad 0\alpha+\alpha = 0+\alpha = \alpha \in W$

Now choose

$c = \;$, $\beta = \alpha$,

Let c be additive inverse of 1 over F

$c\alpha + \beta = (-1)\alpha + \alpha = -\alpha+\alpha = 0.$

$= \alpha(c+1). = \alpha(0) = 0$

Thus, $\bar{0} \in W$ $\quad$ zero (Field axiom ($x+$ additive inverse of $x = 0$))

So, W contains the zero vector.

Let $\alpha \in W$ and $c \in F$,

by above ~~proof~~ proof's $\beta = 0$ (choose) $\in W$

$c\alpha+\beta = c\alpha + 0 = c\alpha.$

by given statement,

$c\alpha+\beta \in W$, it follows $c\alpha \in W$.

thus, W is closed under scalar multiplication

Let $\alpha, \beta \in W$, choose $c = 1$.

$c\alpha = 1\cdot\alpha = \alpha$ (by rule of multiplicative scalar property)

$c\alpha+\beta = 1\alpha+\beta = \alpha+\beta.$

Thus, W is closed under vector addition, W is subspace of V

bcz since W is non-empty, contains zero vector, and is closed under scalar multiplication and vector addition, W satisfies subspace axioms

Q2) Let, a matrix $A_{2\times2}$ over $\mathbb{C}^{2\times2}$ be $\begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix}$

$m_{ij} \in \mathbb{C}$

(1) Form of Hermitian matrix
To M be as Hermitian matrix

$$\forall\, m_{ij} = \overline{m}_{ji} \quad \text{i.e} \quad A_{ij} = \overline{A}_{ji} = A^{*}_{ij}$$

then,

$$A_{ij} = \begin{bmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{bmatrix} = \overline{A}_{ji} = \begin{bmatrix} \overline{m}_{11} & \overline{m}_{21} \\ \overline{m}_{12} & \overline{m}_{22} \end{bmatrix}$$

S.t $m_{11} = \overline{m}_{11}$ Hence, m_{11}, m_{22} be real
$m_{22} = \overline{m}_{22}$

and Let's denote $m_{21} = \overline{m}_{12}$ be $x - yi$

$m_{12} = \overline{m}_{21}$ be $x + yi$.

Hence, $A_{2\times2}$ matrix be in form of

$$\begin{bmatrix} a & x+iy \\ x-iy & b \end{bmatrix} \qquad x, y, a, b \in \mathbb{R}$$

(ii) Set of all $n\times n$ Hermitian matrices $(\mathcal{H})$ over $\mathbb{C}^{n\times n}$ is not a vector subspace of $\mathbb{C}^{n\times n}$. $n\in\mathbb{N}$

Conditions: for Subspace. (Subspace itself a vectorspace

(i) Contains $\bar{0}$ (zero vector)
(ii) Closed under addition
(iii) Closed under scalar multiplication $\}$ on V

Zero vector (Zero Matrix)$_{n\times n}$ is Hermitian matrix since, $Z_{n\times n}$ s.t $Z_{ij}=\bar{Z}_{ji}$

Now let $\vec{B}=\vec{0}$ and $\vec{A}\in\mathcal{H}$ s.t

$$A\in\mathcal{H} \Rightarrow A_{ij}=\begin{cases} x_i+iy_j & i\neq j \\ a_i & i=j\end{cases}$$

let $c\in\mathbb{C}$ Then,

$$(c\cdot A)^* = \bar{c}A^*$$

only when $c=\bar{c}$, meaning c is real

Hence, set of Hermitian matrices is not closed over under scalar multiplication of vectors.

and also For example

let c be any $A_{ij}\in A_{n\times n}$ s.t

$c=A_{i_0j_0}$ where $i_0\neq j_0$

then $c\vec{\alpha}$ or $c\vec{n}=B$ st,

$$B_{ij} = \begin{cases} ca_i & i=j \\ c(x_i + iy_i) & i \neq j, i \neq i_0, j \neq j_0, i_0 \\ (A_{ji})(\overline{A_{ij}}) & i = j_0, j = i_0 \\ (A_{ij})^2 & i = i_0, j = j_0 \end{cases} \quad A_{ij} \neq \overline{A_{ji}}$$

now B ~~does~~ doesnot belongs to set of Hermitian matrices. $B \notin H$

Hence, the set of Hermitian matrices $\mathcal{H}$ is not a subspace of vector space $\mathbb{C}_{n \times n}$

Hence proved.

(iii) If we replace $\mathbb{C}$ with $\mathbb{R}$, then all matrices are real

$\forall x \in \mathbb{R} \quad x = \overline{x}$

Then all symmetric matrices are Hermitian matrices.

Since, symmetric matrices properties satisfy all 3 properties of vector subspace

(i) A zero matrix is symmetric

(ii) Sum of any 2 symmetric matrices is also symmetric

(iii) multiplying Real scalar with symmetric matrix also includes symmetry.

(3). To prove: The subspace spanned by a non-empty subset S of a vector space V is the set of all linear combinations of vectors in S.

Let W be a subspace and S spans W.

Since W is subspace

$\therefore$ $\forall \alpha, \beta \in W$ $\quad c\alpha + \beta \in W$

W be the set of all vectors produced by linear combination of all vectors present in non-zero subset S, $S \subset V$, (vector space) V

let $\vec{\alpha}, \vec{\beta} \in W$, for any $\vec{\alpha} \in W$

$$\vec{\alpha} = \sum_{i=1}^{n} c_i \vec{a_i} \quad \text{then, } \vec{\beta} = \sum_{i=1}^{m} ll_i \vec{b_i}$$

$$(a_i)_{i=1}^{n} \subset S, (b_i)_{i=1}^{m} \subset S, (c_i)_{i=1}^{n} \subset F, (ll_i)_{i=1}^{n} \subset F$$

now,

$$c'\vec{\alpha} + \vec{\beta} = \sum_{i=1}^{n} c' c_i \vec{a_i} + \sum_{i=1}^{m} ll_i \vec{b_i}$$

$c'\vec{\alpha} + \vec{\beta}$ will also the linear combination of vectors and present in S.

then $c\vec{\alpha} + \vec{\beta} \in W$

as the choice of $\vec{\alpha}$ and $\vec{\beta}$ is arbitary

$\forall \vec{\alpha}, \vec{\beta} \in W$, $c \in F$ $\quad c\vec{\alpha} + \vec{\beta} \in W$

then W is a subspace of V and This subspace is spanned by S.

$\therefore$ Hence proved.

4) To prove:- $(W_i)_{i=1}^{k}$ are subspaces of V (vector space), then $\sum_{i=1}^{k} W_i$ is a subspace and spanned by the vector set formed by $\bigcup_{i=1}^{n} W_i$.

Let, $S = W_1 + W_2 + \cdots + W_k$

$S_i = \sum_{\forall i} \alpha_i \quad \alpha_i \in W$

$\forall \alpha_i$ s.t $\alpha_i \in W_i$, $c\alpha_i$ exists

so $\sum_{\forall i} c\alpha_i$ exists so, cS_i exists

$k_i = \sum \beta_i \quad \beta_i \in W_i$

Since $\forall_i \quad c\alpha_i + \beta_i \in W_i$

$\therefore \quad cS_i + k_i \in S$

Union $(M) = W_1 \cup W_2 \cup \cdots \cup W_k$

If $\alpha_i \in W_i$

$\alpha_i \in M$

$\therefore \quad \forall_i (\alpha_i \in M)$

$\therefore \quad \sum \alpha_i = S_i$

s.t Any S_i in S can be written as

$\sum \alpha_i \quad \alpha_i \in W_i$

Since $\alpha_i \; \forall_i \in M$

so $\sum_i c\alpha_i$ is written as

Linear combination of $\forall_i \; \alpha_i \in M$ same for β_i

So, Any element in S_i in S

follows $\sum_i \alpha_i$ $\alpha_i \in W_i$

$\forall i$ $\alpha_i \in M$

$\therefore$ Any element can be written as linear combination of elements in M

$\therefore$ M spans S.

Hence proved

5) $A \in F^{m \times n}$, $S_A = \{X \in F^n \mid AX = \vec{0}\}$, S_A is the Solution space of $AX = \vec{0}$, Find the number of linearly independent $x \in S_A$

$$AX = 0$$

let, A is in row-reduced echelon form with r Zero rows $\quad x = \{x_1, x_2 \dots x_{n-r}, u_1, \dots u_r\}$

$$x_k = \sum_{i=1}^{r} c_{ki} u_i \quad \text{where } 1 \le k \le n-r$$

~~if m<~~

Now, $S_A = \{x : AX = 0\}$

$$X = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n-r} \\ u_1 \\ \vdots \\ u_r \end{bmatrix} = \begin{bmatrix} \sum c_{1i} u_i \\ \sum c_{2i} u_i \\ \vdots \\ \sum c_{n-r\,i} u_i \\ u_1 \\ \vdots \\ u_r \end{bmatrix}$$

set of solutions can be represented as the sum of independent variables:

$$\begin{bmatrix} c_{11} u_1 \\ c_{21} u_1 \\ c_{31} u_1 \\ c_{n-r\,1} u_1 \\ u_1 \\ \vdots \end{bmatrix} + \begin{bmatrix} c_{12} u_2 \\ c_{22} u_2 \\ \vdots \\ c_{n-r\,2} u_2 \\ u_2 \\ \vdots \end{bmatrix} \Rightarrow \underbrace{\begin{bmatrix} c_{11} \\ c_{21} \\ c_{31} \\ \vdots \\ c_{n-r\,1} \\ 1 \\ \vdots \end{bmatrix} u_1 + \begin{bmatrix} c_{12} \\ c_{22} \\ c_{32} \\ \vdots \\ c_{n-r} \\ 1 \\ \vdots \end{bmatrix} u_2}$$

These matrices can be represented as vectors that are Independent.

then, the soln space can be produced by the linear combination of these vectors

$$u_1 \bar{V}_1 + u_2 \bar{V}_2 + \dots u_r \bar{V}_r \Rightarrow S_A$$

hence, there are exactly r independent vectors.

To prove:-

6) Given, V is spanned by $(\bar{\beta}_i)_{i=1}^{n}$ then prove that any independent set of vectors in V is finite and contains no more than n elements

→ every vector in V can be expressed as linear combination of $\{\vec{\beta}_1, \vec{\beta}_2 \cdots \vec{\beta}_n\}$

let $S \subset V = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_m\}$ be an independent set of vectors..

$\Downarrow$

s.t $c_1\vec{v}_1 + c_2\vec{v}_2 \cdots c_m\vec{v}_m = \bar{0}$ — (2)

where $c_1, c_2 \cdots c_m$ ~~are not all which are~~ ~~zero~~ must be zero

$\vec{v}_1, \vec{v}_2, \cdots \vec{v}_m \in S$

S is linearly independent then all of its subsets are also the same..

let $S_1 \subset S$,

and $(\vec{\beta}_i)_{i=1}^{n}$ spans $V \supset S$ then

$\vec{v}_i = c_{i1}\vec{\beta}_1 + c_{i2}\vec{\beta}_2 \cdots c_{in}\vec{\beta}_n$ — (1)

Let's consider $\overset{m=}{n+1}$ vectors in set S. $(m>n)$

from ①

$$\vec{v_1} = c_{11}\vec{\beta_1} + c_{12}\vec{\beta_2} + \cdots + c_{1n}\vec{\beta_n}$$

$$\vdots \qquad\qquad \vdots$$

$$\vec{v}_{n+1} = c_{(n+1)1}\vec{\beta_1} + c_{(n+1)2}\vec{\beta_2} + \cdots + c_{(n+1)n}\vec{\beta_n}$$

from ② $\boxed{c \to a}$ replace in ②

$$a_1(c_{11}\vec{\beta_1} + c_{12}\vec{\beta_2} + \cdots + c_{1n}\vec{\beta_n}) + a_2(\cdots) + \cdots + a_{n+1}(c_{(n+1)1}\vec{\beta_1} + \cdots c_{(n+1)n}\vec{\beta_n}) = \vec{0}$$

after re arranging

$$(a_1c_{11} + a_2c_{21} + \cdots)\vec{\beta_1} + (a_1c_{12} + \cdots a_{n+1}c_{(n+1)2})\vec{\beta_2}$$

$$+ \cdots + (a_1c_{1n} + a_2c_{2n} + \cdots + a_{n+1}c_{(n+1)n})\vec{\beta_n} = \vec{0}$$

since $\{\vec{\beta_1}, \cdots \vec{\beta_n}\}$ is a basis for V_g it is linearly independent

so, coefficients of each $\vec{\beta_i}$ must be zero.

we get n linear eqn's with $n+1$ variables, so system has infinitely many soln which is false.

$\exists$ a nontrivial soln where not all a_i are zero

so, m cannot be $>n$ so,

since S is an independent set and $\dim \leq n$;

S is finite