PART - 2

Linear Algebra

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5%	→ TUT-QUIZ
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MATRICES:

→ Equality of 2 Matrices:

A & B are said to be equal if they have the same order and their corresponding elements are equal.

i.e. $A = [a_{ij}]_{m \times n}$ $\quad a_{ij} = b_{ij} \ \forall \ i, j$

$B = [b_{ij}]_{m \times n}$

TYPES OF MATRICES:

1) Square Matrix $\quad m = n$ (Rows = Column)
2) Zero Matrix $\quad a_{ij} = 0 \ \forall \ i, j$
3) Diagonal Matrix $\quad a_{ij} = 0 \ \forall \ i \neq j$
   - Scalar Matrix $\quad a_{ij} = a \ ; \ a \neq 0$
     (All diagonal elements are same)
   - Identity Matrix $\quad a_{ij} = \begin{cases} 0, & i \neq j \\ 1, & i = j \end{cases}$
4) Upper Triangular Matrix: $\boxed{a_{ij} = 0 \text{ if } i > j}$
5) Lower Triangular Matrix: $\boxed{a_{ij} = 0 \text{ if } i < j}$

MATRIX OPERATION:

1) Addition of Matrices
2) Multiplication of Matrices
3) Transpose of Matrix

1) For addition $\longrightarrow$ same order $m \times n$

$C = A + B$

$C_{ij} = [a_{ij} + b_{ij}] = [a_{ij}] + [b_{ij}]$

Properties:

$(A+B) = (B+A)$

$(A+B)+C = A+(B+C)$

$k(lA) = (kl)A$

$(k+l)A = kA + lA$

2) Matrix Multiplication:

$A = [a_{ij}]_{m \times n}$ → n(column) of A

$B = [b_{ij}]_{n \times r}$ → n(Row) of B

$C = AB$

$= [c_{ij}]_{m \times r}$

$$\boxed{c_{ij} = \sum_{k=1}^{n} a_{ik} b_{kj}}$$

Properties:

(i) $A(BC) = (AB)C$

(ii) $A(B+C) = AB + AC$

(iii) $k(AB) = (kA)B = A(kB)$

(iv) $I_n A = A I_n = A$

where A is $n \times n$ matrix

$I_n = n \times n$ Identity matrix

(Multiplicative Identity)

Q. If A, B are square matrices of same size, $(A+B)^2 = A^2 + 2AB + B^2$?

No, $(A+B)^2 = A^2 + AB + BA + B^2$

Proof of $A(B+C)$:

$[A(B+C)]_{ij} = a_i \cdot (b_j + c_j)$

$= a_i \cdot b_j + a_i \cdot c_j = (AB)_{ij} + (AC)_{ij} = (AB+AC)_{ij}$

$\therefore \boxed{A(B+C) = AB + AC}$

Inverse of a Matrix :

If A is a $n \times n$ matrix, an inverse of A is a $n \times n$ matrix A', such that $AA' = A'A = I$

Eg: $A = \begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}$

$$A' = \begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix}$$

→ Zero matrix will not have an inverse.

Result: If $A_{n \times n}$ is an invertible matrix, then the inverse is unique.

Proof: Let A' and A'' be two inverses of A

then,

$$AA' = A'A = I \longrightarrow (1)$$
$$AA'' = A''A = I \longrightarrow (2)$$

(Associative)

$$A' = A'I = A'(AA'') = (A'A)A''$$

(Identity) From (2)

$$= IA'' = A''$$

From (1)

$$\therefore \boxed{A' = A''}$$

NOTE: If $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then $A^{-1} = \dfrac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

if $ad \neq bc$.

Properties:

(1) If A is an invertible matrix, A^{-1} is also invertible, then. $(A^{-1})^{-1} = A$.

(2) If A is an invertible matrix, and c is a non-zero scalar, then

$$(cA)^{-1} = \frac{1}{c} A^{-1}$$

(3) If A and B are invertible matrices of same size, then AB is also invertible, and

$$(AB)^{-1} = B^{-1}A^{-1}$$

(4) If A is invertible, then A^n is invertible for all non-negative integer n.

$$(A^n)^{-1} = (A^{-1})^n$$

Proof: (1) let X be an inverse of A^{-1}.

$$\left.\begin{aligned} \therefore\ XA^{-1} &= A^{-1}X = I \\ \text{and, } AA^{-1} &= A^{-1}A = I \end{aligned}\right\} \quad \boxed{(A^{-1})^{-1} = A}$$

(2) Let X be an inverse of cA.

$$\therefore X(cA) = (cA)X = I \quad \text{....(iii)}$$

$$\text{and, } \left(\tfrac{1}{c}A^{-1}\right)(cA) = \left(\tfrac{1}{c} \times c A^{-1}\right)(A) = A^{-1}A = I$$

$\hookrightarrow$ (i)

$$(cA)\left(\frac{1}{c}A^{-1}\right) = A\left(c \times \frac{1}{c}A^{-1}\right) = AA^{-1} = I \longrightarrow (iii)$$

From (i), (ii) and (iii),

$$\boxed{X = \frac{1}{c}A^{-1}}$$

(3) Similarly,

let $x \rightarrow$ Inverse of AB

$$\therefore X(AB) = (AB)X = I \text{ ----- } (i)$$

$$\rightarrow (AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1}$$

$$= A(IA^{-1}) = AA^{-1} = I \text{ ---- } (ii)$$

$$\rightarrow (B^{-1}A^{-1})(AB) = B^{-1}(A^{-1}A)B$$

$$= B^{-1}IB = B^{-1}B = I \text{ ---- } (iii)$$

From (i), (ii) and (iii),

$$\boxed{X = B^{-1}A^{-1}}$$

(4) Let $X \rightarrow$ Inverse of A^n,

To prove: $XA^n = A^nX = I \Rightarrow X = (A^{-1})^n$

$\rightarrow$ Base case: $n = 1$

$$AA^{-1} = A^{-1}A = I \quad (\text{True})$$

Suppose it is true for $n = k$,

$$\therefore A^k(A^{-1})^k = (A^{-1})^k A^k = I$$

i) $\therefore A \cdot A^k \cdot (A^{-1})^k \cdot (A^{-1}) = A \cdot I \cdot A^{-1} = A \cdot A^{-1} = I$

$$\therefore \boxed{A^{k+1} \cdot (A^{-1})^{k+1} = I}$$

ii) Similarly, $(A^{-1})^{k+1} \cdot A^{k+1} = I$

$\therefore$ It is true for $k+1$.

Hence, inductively we proved that

$$A^n \cdot (A^{-1})^n = I \quad \forall \ n \geq 0$$

Elementary Matrices:

→ An elementary matrix is a matrix that can be obtained by performing elementary row operation on identity matrix.

Eg:

$$E_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$ is equivalent to $3R_2$ on I_4

$$E_2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$ is equivalent to $R_1 \leftrightarrow R_3$ on I_4

$$E_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & -2 & 0 & 1 \end{pmatrix}$$ is equivalent to $R_4 \to R_4 - 2R_2$ on I_4

※ Inverse of an Elementary Matrix :

$\Rightarrow E_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \quad E_1^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$

Why?

$I_3 \xrightarrow{R_{23}} E_1 \xrightarrow{R_{23}} I_3$

$\Rightarrow E_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$E_2^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/4 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$I_3 \xrightarrow{R_2 \to 4R_2} E_2 \xrightarrow{R_2 \to 1/4 R_2} I_3$

$\Rightarrow E_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{pmatrix}$

$E_3^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix}$

$I_3 \xrightarrow{R_3 \to R_3 - 2R_1} E_3 \xrightarrow{R_3 \to R_3 + 2R_1} I_3$

How to find inverse?

$\to$ We know that $E = e(I)$

Let X = Inverse of E

$\therefore XE = EX = I$

$\therefore Xe(I) = e(I)X = I$

Let $X = e^{-1}(I)$

$\therefore Xe(I) = e^{-1}e(I \cdot I)$

$= \boxed{I}$

$\therefore X = e^{-1}(I)$

Hence, if $E = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$e: R_2 \to R_2 + R_1$

$\therefore e^{-1}: R_2 \to R_2 - R_1$

$\boxed{E^{-1} = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}}$

Theorem:

Let E be an elementary matrix obtained by performing elementary row operation on $I_{n\times n}$.
If the same elementary row operation is done on $n\times n$ matrix A, the resultant is same as EA.

Fundamental Theorem of Invertible Matrices:

(a) A is an invertible matrix

(b) $AX=B$ has a unique solution for any $B\in R$.

(c) $AX=0$ has only trivial solution

(d) The row reduced echolen form of A is I_n.

(e) A is a product of elementary matrices

$$(a)\longleftrightarrow(b)\longleftrightarrow(c)\longleftrightarrow(d)\longleftrightarrow(e)$$

Proof: To Prove: $(a)\rightarrow(b)\rightarrow(c)\rightarrow(d)\rightarrow(e)$

$(a)\rightarrow(b)$

Given: $AA^{-1}=A^{-1}A=I$

Proving that $AX=B$ has a solution:

Suppose $X=A^{-1}B$ is a solution.

$$\therefore A(A^{-1}B)=(AA^{-1})B = \boxed{B}$$

Now, proving that the solution is unique:

Let y be a solution of $AX=B$

$\therefore Ay=B$

$\therefore A^{-1}(Ay)=A^{-1}B$

$\therefore (A^{-1}A)y=A^{-1}B$

$\therefore Iy=\boxed{y=A^{-1}B}$ Hence, unique.

(b) $\to$ (c)

Given: $AX=B$ has a unique sol^n.

let $B=0$

$\therefore AX=0$ also has a unique sol^n.

A homogeneous system of linear equations always has $X=0$ as a solution.

As, $AX=0$ has only one unique sol^n,

$AX=0$ only has trivial sol^n $(X=0)$.

(c) $\to$ (d)

Let R be a row-reduced echelon form of A.

As $AX=0$ has only $X=0$ as it's solution,

$RX=0$ has only trivial sol^n.

Suppose R has r non zero rows.

$\therefore \exists$ $n-r$ free scalars

$\therefore$ there will be infinite solutions, if $n-r>0$

$\therefore n-r \leq 0$ As $r \leq n$

$n-r=0$

$\therefore \boxed{r=n} \Rightarrow \boxed{R=I_n}$

(d) $\to$ (e)

Given: $R=I_n$

Suppose R is obtained from A after performing n elementary row operations

$\therefore R = e_n e_{n-1} \cdots\cdots e_1(A)$

As, $E_i = e_i(I)$

$R = E_n E_{n-1} \cdots\cdots E_1(A) = I$

As elementary matrices are invertible,

$A = E_1^{-1} E_2^{-1} \cdots\cdots E_n^{-1}(I)$

As product of invertible matrices is invertible,

$\boxed{A = E_1^{-1} E_2^{-1} \cdots E_n^{-1}}$ is a product of invertible matrices.

(e) $\rightarrow$ (a)

Suppose $A = E_1 E_2 \cdots E_n$ (Product of n invertible matrices)

As E_i is invertible $\forall\ 1 \le i \le n$

$\exists\ E_i^{-1}$ s.t. $E_i^{-1}E_i = E_iE_i^{-1} = I$

$\therefore A(E_n^{-1}E_{n-1}^{-1}\cdots E_1^{-1}) = I$

let $\boxed{X = E_n^{-1}E_{n-1}^{-1}\cdots E_1^{-1}}$

$\exists\ X$ s.t. $\boxed{AX = I}$

Similarly,

$(E_n^{-1}E_{n-1}^{-1}\cdots E_1^{-1})A = I$

$\therefore \exists\ X$ s.t.

$\boxed{XA = I}$

$\therefore AX = XA = I$

$\therefore A$ is invertible.

$\therefore I = E_3 E_2 E_1 A$

$\therefore \boxed{E_1^{-1}E_2^{-1}E_3^{-1} = A}$

$(e_1)^{-1} = R_1 \rightarrow R_1 + R_2$

$(e_2)^{-1} = R_2 \rightarrow R_2 + R_1$

$(e_3)^{-1} = R_2 \rightarrow 3R_2$

$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$e_3 \uparrow R_2 \rightarrow \frac{1}{3}R_2$

Alt. method $\begin{pmatrix} 2 & 3 \\ 1 & 3 \end{pmatrix} \xrightarrow[e_1]{R_1 \rightarrow R_1 - R_2} \begin{pmatrix} 1 & 0 \\ 1 & 3 \end{pmatrix} \xrightarrow[e_2]{R_2 \rightarrow R_2 - R_1} \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}$

Q: Express $A = \begin{pmatrix} 2 & 3 \\ 1 & 3 \end{pmatrix}$ as product of elementary matrices.

$A = \begin{pmatrix} 2 & 3 \\ 1 & 3 \end{pmatrix}$ $\begin{pmatrix} 1 & 0 \\ 1 & 3 \end{pmatrix} \xrightarrow[(e_3)]{R_1 \rightarrow R_1 + R_2} \begin{pmatrix} 2 & 3 \\ 1 & 3 \end{pmatrix}$ $\Rightarrow E_1 = \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}$

$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ $\uparrow R_2 \rightarrow R_2 + R_1$ (e_2)

$E_2 = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$

$I \xrightarrow[(e_1)]{R_2 \rightarrow 3R_2} \begin{pmatrix} 1 & 0 \\ 0 & 3 \end{pmatrix}$

$E_3 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$

$\boxed{\therefore \begin{pmatrix} 2 & 3 \\ 1 & 3 \end{pmatrix} = E_3 E_2 E_1}$

* Transpose of a Matrix :

→ The transpose of a $m \times n$ matrix $A = [a_{ij}]_{m \times n}$ is a $n \times m$ matrix $A^T = [a_{ji}]_{n \times m}$ with $\boxed{a_{ji}' = a_{ij}}$

for $1 \le i \le m$, $1 \le j \le n$

Eg: $A = \begin{pmatrix} 1 & 6 \\ 8 & 2 \\ 4 & 7 \end{pmatrix}_{3 \times 2}$ $A^T = \begin{pmatrix} 1 & 8 & 4 \\ 6 & 2 & 7 \end{pmatrix}_{2 \times 3}$

→ The transpose of a matrix is exchanging the rows & columns of the matrix.

rows of A = column of A^T

columns of A = rows of A^T

⇒ If you have 2 column vectors

$$U = \begin{bmatrix} U_1 \\ U_2 \\ \vdots \\ U_n \end{bmatrix}_{n \times 1} \qquad V = \begin{bmatrix} V_1 \\ V_2 \\ \vdots \\ V_n \end{bmatrix}_{n \times 1}$$

$$U \cdot V = [U_1 \,----\, U_n] \begin{bmatrix} V_1 \\ \vdots \\ V_n \end{bmatrix} = U^T V = U_1V_1 + U_2V_2 + ---- + U_nV_n$$

$$\therefore \boxed{\vec{U} \cdot \vec{V} = \vec{U}^T \vec{V}}$$

Properties :

① $(A^T)^T = A$

② $(AB)^T = B^T A^T$

③ $(A+B)^T = A^T + B^T$

④ $(KA)^T = KA^T$

Proof of $(AB)^T = B^T A^T$:

known : 1) $\text{row}_i(A) = \text{column}_i(A^T)$

2) $a_{ji}^T = a_{ij}$

$$\therefore [(AB)^T]_{ij} = (AB)_{ji}$$
$$= \text{row}_j(A) \cdot \text{column}_i(B)$$
$$= \text{column}_j(A^T) \cdot \text{row}_i(B^T)$$
$$= \text{row}_i(B^T) \cdot \text{column}_j(A^T)$$
$$= (B^T A^T)_{ij}$$

※ DETERMINANT :

$\rightarrow$ $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}_{2\times 2}$

$$\det A = a_{11}a_{22} - a_{12}a_{21}$$

Determinant of a 3x3 matrix :

$\rightarrow$ $A = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$

$$\det A = +a_{11}\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12}\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13}\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

$$\therefore \det A = a_{11}\det A_{11} - a_{12}\det A_{12} + a_{13}\det A_{13}$$

$$= \boxed{\sum_{j=1}^{3} (-1)^{1+j} a_{1j} \det A_{1j}}$$

where, $\det A_{ij}$ is called the (i,j)- minor of A

Determinant of a nxn matrix :

$$\det A = |A| = a_{11}A_{11} - a_{12}A_{12} + \cdots + (-1)^{1+n}a_{1n}A_{1n}$$

$$= \boxed{\sum_{j=1}^{n} (-1)^{1+j} a_{1j} A_{1j}} \quad (A_{ij} = \text{minor of } a_{ij})$$

Cofactor :

Combine a minor with it's plus or minus sign. To this end, we define the (i,j)-cofactor of A to be:

$$\boxed{C_{ij} = (-1)^{i+j} A_{ij}}$$

$$\therefore \boxed{\det A = \sum_{j=1}^{n} a_{ij} C_{ij}}$$

※ The Laplace Expansion Theorem:

The determinant of an $n \times n$ matrix $A = [a_{ij}]$, where $n \geq 2$, can be computed as

$$\det A = a_{i1}C_{i1} + a_{i2}C_{i2} + \cdots + a_{in}C_{in}$$
$$= \sum_{j=1}^{n} a_{ij}C_{ij}$$

(which is the cofactor expansion along the i^{th} row) and also as

$$\det A = a_{1j}C_{1j} + a_{2j}C_{2j} + \cdots + a_{nj}C_{nj}$$
$$= \sum_{i=1}^{n} a_{ij}C_{ij}$$

(the cofactor expansion along the j^{th} column).

Theorem 4.2: The determinant of a triangular matrix is the product of the entries on its main diagonal.

Specifically, if $A = [a_{ij}]$ is an $n \times n$ triangular matrix, then

$$\boxed{\det A = a_{11}a_{22}\cdots a_{nn}}$$

Properties of Determinant:

1. If A has a zero row/column, then $\det A = 0$.
2. If B is obtained by interchanging two rows/column of A, then $\det B = -\det A$
3. If A has two identical rows/columns, then $\det A = 0$.
4. If B is obtained by multiplying a row/column of A by k, then $\det B = k \det A$

(5) If A, B and C are identical except that the i^{th} - row/column of C is the sum of the i^{th} row/column of A and B, then

$$\det C = \det A + \det B$$

(6) If B is obtained by adding a multiple of one row/column of A to another row/column, then

$$\det B = \det A$$

Prove all of the above properties:

(1) Suppose i^{th} row (column) is zero in $A_{n\times n}$

Row: $\therefore \det A = \sum_{j=1}^{n} a_{ij} C_{ij}$ (As $a_{ij} = 0 \ \forall j$)

$$\boxed{\det A = 0}$$

Column: $\det A = \sum_{j=1}^{n} a_{ji} C_{ji}$ (As $a_{ji} = 0 \ \forall j$)

$$\boxed{\det A = 0}$$

(3) $A \rightarrow$ has two identical rows, swap them to obtain the matrix B.

clearly, $B = A$

$\therefore \det B = \det A$

Also, by (2), $\det B = -\det A$ $\Big\}$ $\det A = -\det A$

$$\boxed{\therefore \det A = 0}$$

(4) Suppose $b_{ij} = k a_{ij}$ for $j = 1, \dots, n$

$\therefore$ expanding them along i^{th} row,

$$\det B = \sum_{j=1}^{n} b_{ij} C_{ij} = \sum_{j=1}^{n} k a_{ij} C_{ij} = k \sum_{j=1}^{n} a_{ij} C_{ij}$$

$$= \boxed{k \det A}$$

(5) $c_{ij} = a_{ij} + b_{ij} \ \forall j$

else, $c_{rj} = a_{rj} = b_{rj} \ \forall r \neq i$ (the cofactors C_{ij} of the elements in the expanding along i^{th} row, i^{th} rows of A, B and C are identical.)

$$\det C = \sum_{j=1}^{n} c_{ij} C_{ij}$$

$$= \sum_{j=1}^{n} (a_{ij} + b_{ij}) C_{ij} = \sum_{j=1}^{n} a_{ij} C_{ij} + \sum_{j=1}^{n} b_{ij} C_{ij}$$

$$= \boxed{\det A + \det B}$$

(6)

$R_i \to R_i + kR_r \Rightarrow B$

$$\det B = \sum_{j=1}^{n} b_{ij} C_{ij}$$

$$= \sum_{j=1}^{n} (a_{ij} + k a_{rj}) C_{ij}$$

$$= \sum_{j=1}^{n} a_{ij} C_{ij} + \boxed{k \sum_{j=1}^{n} a_{rj} C_{ij}}$$

$$= \det A + k(0) \qquad \hookrightarrow \det C$$

$$= \det A$$

where i^{th} row is replaced with r^{th} row.

So, now r^{th} and i^{th} row are identical

$\therefore \det C = 0$

(2)

Determinants of Elementary Matrices:

Theorem 4.4:

Let E be an $n \times n$ elementary matrix.

1. If E results from interchanging two rows of I_n, then
$$\det E = -1.$$

2. If E results from multiplying one row of I_n by k, then $\det E = k$.

3. If E results from adding a multiple of one row of I_n to another row, then $\det E = 1$.

Can be directly derived from the part 2, 4 & 6 of the previous theorem.

Lemma 4.5: Let B be an $n \times n$ matrix and let E be an $n \times n$ elementary matrix. Then
$$\det(EB) = (\det E)(\det B)$$

Theorem 4.6: A square matrix A is invertible if and only if $\det A \neq 0$.

※ CRAMER'S RULE:

→ Let A be an invertible $n \times n$ matrix, and let $b \in \mathbb{R}^n$. Then the unique sol^n x of the system of equation $Ax = b$ is given by

$$\boxed{x_i = \frac{\det(A_i(b))}{\det(A)}}$$ for $i = 1, 2, \dots, n$

where $A_i(b)$ is obtained by replacing i^{th} column of A with b.

Eg: $\left.\begin{aligned} x_1 + 2x_2 &= 2 \\ -x_1 + 4x_2 &= 1 \end{aligned}\right\}$ let $A = \begin{bmatrix} 1 & 2 \\ -1 & 4 \end{bmatrix}$, $b = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$\therefore \det A = \begin{vmatrix} 1 & 2 \\ -1 & 4 \end{vmatrix} = 4 + 2 = \boxed{6}$

$$\left.\begin{aligned} \therefore x_1 &= \frac{\det(A_1(b))}{\det(A)} = \frac{\begin{vmatrix} 2 & 2 \\ 1 & 4 \end{vmatrix}}{6} = \frac{6}{6} = \boxed{1} \\ x_2 &= \frac{\det(A_2(b))}{\det(A)} = \frac{\begin{vmatrix} 1 & 2 \\ -1 & 1 \end{vmatrix}}{6} = \frac{3}{6} = \boxed{1/2} \end{aligned}\right\} \boxed{X = \begin{bmatrix} 1 \\ 1/2 \end{bmatrix}}$$

→ $$\boxed{A^{-1} = \frac{1}{\det(A)} \operatorname{adj}(A)}$$

Eg: let $A = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 2 & 4 \\ 1 & 3 & -3 \end{bmatrix}$ ⇒ $|A| = -18 - 2(-10) - 1(4)$
$= -22 + 20 = \boxed{-2}$

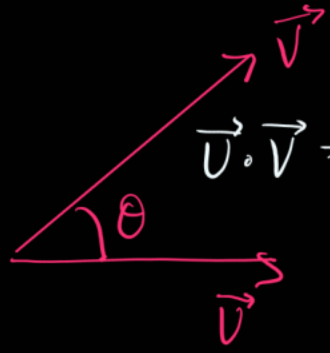
$\begin{matrix} + & - & + \\ - & + & - \\ + & - & + \end{matrix}$

$$\operatorname{adj}(A) = \begin{bmatrix} -18 & 10 & 4 \\ 3 & -2 & -1 \\ 10 & -6 & -2 \end{bmatrix}^T = \begin{bmatrix} -18 & 3 & 10 \\ 10 & -2 & -6 \\ 4 & -1 & -2 \end{bmatrix}$$

$$\therefore A^{-1} = \frac{1}{\det(A)} \operatorname{adj}(A) = -\frac{1}{2} \begin{pmatrix} -18 & 3 & 10 \\ 10 & -2 & -6 \\ 4 & -1 & -2 \end{pmatrix} = \boxed{\begin{pmatrix} 9 & -3/2 & -5 \\ -5 & 1 & 3 \\ -2 & 1/2 & 1 \end{pmatrix}}$$

※ INNER PRODUCT SPACE:

Definition:



$$\vec{U}\cdot\vec{V} = |\vec{U}||\vec{V}|\cos\theta$$

An inner product on a vector space V is an operation that aligns to every pair of vectors U and V in V a real number $\langle U, V\rangle$ such that the following properties hold $\forall$ vectors U, V & $W \in V$ and all scalars c:

1. $\langle U, V\rangle = \langle V, U\rangle$
2. $\langle U, V+W\rangle = \langle U, V\rangle + \langle U, W\rangle$
3. $\langle cU, V\rangle = c\langle U, V\rangle$
4. $\langle U, U\rangle \geq 0$ and $\langle U, U\rangle = 0$ iff $U = 0$

A vector space with a inner product is called an inner product space.

If V and U are two vectors is an innerproduct space V:

1. length of V is $\|V\| = \sqrt{\langle V, V\rangle}$
2. The distance b/w U & V is $d(V, U) = \|U - V\|$.
3. U and V are orthogonal if $\langle U, V\rangle = 0$

Eg: 1) R^n is an inner product space $(u, v) = u \cdot v$

But dot product is not the only inner product that can be defined on R^n.

2)

NOTE: $\{u_1, u_2, \ldots, u_k\}$ where every pair is orthogonal then, $u_i \cdot u_j = 0$ if $u_i \neq u_j$.

→ If $\{u_1, u_2, \ldots, u_n\}$ is an orthogonal set of non-zero values in R^n then they are linearly independent.

If $c_1, c_2, \ldots, c_k$ are scalars such that,

$$c_1 v_1 + c_2 v_2 + \cdots + c_k v_k = 0$$

$$\left(\sum_{i}^{k} c_i v_i\right) \cdot v_1 = 0 \cdot v_1$$
$$= 0$$

we know that $v_i v_j = 0$ if $i \neq j$

$\therefore \; c_i (v_i \cdot v_i) = 0 \;\; \forall i$

$\therefore \; \boxed{c_i = 0 \;\; \forall i}$

Eg: $(R^2, +) \to (R, +, \cdot)$

$\langle u, v \rangle = 2u_1v_1 + 3u_2v_2$

Is this an inner product?

Solution:

(i) $\langle u, v \rangle$

$= 2u_1v_1 + 3u_2v_2$

$= 2v_1u_1 + 3v_2u_2$

$= \boxed{\langle v, u \rangle}$

(ii) $\langle u, v+w \rangle$

$= 2u_1(v_1+w_1) + 3u_2(v_2+w_2)$

$= 2u_1v_1 + 2u_1w_1 + 3u_2v_2 + 3u_2w_2$

$= (2u_1v_1 + 3u_2v_2) + (2u_1w_1 + 3u_2w_2)$

$= \boxed{\langle u, v \rangle + \langle u, w \rangle}$

(iii) $\langle cu, v \rangle$

$= 2(cu_1)v_1 + 3(cu_2)v_2$

$= c[2u_1v_1 + 3u_2v_2]$

$= \boxed{c\langle u, v \rangle}$

(iv) $\langle u, u \rangle$

$= \underset{\geq 0}{2u_1^2} + \underset{\geq 0}{3u_2^2}$

$\boxed{\therefore \langle u, u \rangle \geq 0}$

If $u = 0 \Rightarrow u_1 = u_2 = 0$

$\therefore \boxed{\langle u, u \rangle = 0}$

Complex Dot Product:

If $u = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix}$ and $v = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix}$ are vectors in $\mathbb{C}^n$, then the complex dot product of u and v is defined by

$$\boxed{u \cdot v = \bar{u}_1 v_1 + \cdots\cdots + \bar{u}_n v_n}$$

Prove that it is a inner product:

→ The norm (length) of a complex vector v is defined as in the real case:

$$\|v\| = \sqrt{v \cdot v}$$

(General form: $\|v\| = \sqrt{\langle v, v \rangle}$)

Distance b/w two complex vectors:

$$d(u, v) = \|u - v\|$$

Eg: let $u = \begin{pmatrix} i \\ 1 \end{pmatrix}$ and $v = \begin{pmatrix} 2-3i \\ 1+5i \end{pmatrix}$

(a) $u \cdot v$

$$u \cdot v = \sum_{1}^{2} \bar{u}_i v_i = (-i)(2-3i) + (1)(1+5i)$$
$$= -2i + 3(-1) + 1 + 5i$$
$$= \boxed{3i - 2}$$

(b) $\|u\|$

$$\|u\| = \sqrt{u \cdot u}$$
$$= \sqrt{1+1} = \boxed{\sqrt{2}}$$

(c) $||v||$

$$||v|| = \sqrt{4+9+25+1} = \boxed{\sqrt{39}}$$

(d) $d(u,v)$

$$d(u,v) = ||u-v||$$

$$= \left|\left|\begin{pmatrix} 4i-2 \\ -5i \end{pmatrix}\right|\right| = \sqrt{16+4+25}$$

$$= \boxed{\sqrt{45}}$$

NOTE: Let $v = \begin{pmatrix} v_1 \\ \vdots \\ \vdots \\ v_n \end{pmatrix}$ then, $||v|| = \sqrt{|v_1^2| + |v_2^2| + \cdots + |v_n^2|}$

※ Properties:

(a) $u \cdot v = \overline{v \cdot u}$

(b) $u \cdot (v+w) = u \cdot v + u \cdot w$

(c) $(cu) \cdot v = \bar{c}(u \cdot v)$ and $u \cdot (cv) = c(u \cdot v)$

(d) $u \cdot u \geq 0$ and $u \cdot u = 0$ iff $u = 0$.

Proofs:

(a) $\langle u, v \rangle = \sum_i \bar{u}_i v_i$

As $\bar{\bar{v}} = v$

$$= \sum_i \bar{u}_i \bar{\bar{v}}_i$$

As $\overline{pq} = \bar{p}\bar{q}$

$$= \sum_i \overline{u_i \bar{v}_i}$$

As $\overline{p+q} = \bar{p}+\bar{q}$

$$= \overline{\sum_i u_i \bar{v}_i}$$

$$= \overline{\sum_i \bar{v}_i u_i}$$

$$= \boxed{\overline{\langle v, u \rangle}}$$

(b) $\langle u, v+w \rangle$

$$= \sum_i \bar{u}_i (v_i + w_i)$$

$$= \sum_i \bar{u}_i v_i + \sum_i \bar{u}_i w_i$$

$$= \boxed{\langle u, v \rangle + \langle u, w \rangle}$$

(c) $\langle cu, v \rangle = \sum_i \overline{cu_i}\, v_i$

$$= \sum_i \bar{c}\, \bar{u}_i v_i$$

$$= \bar{c} \sum_i \bar{u}_i v_i$$

$$= \boxed{\bar{c} \langle u, v \rangle}$$

(d) $\langle u, u \rangle = \sum_i \bar{u}_i u_i$

$$= \sum_i |u_i|^2 \geq 0$$

If $u = 0$

then $\sum_i |u_i|^2 = 0$

Conjugate Transpose:

If A is a complex matrix, then the conjugate transpose of A is the matrix A^* defined by

$$\boxed{A^* = \bar{A}^T}$$

$\bar{A}$ refers to the matrix whose entries are the complex conjugates of the corresponding entries of A.

If $A = [a_{ij}]$

then, $\bar{A} = [\overline{a_{ij}}]$

Properties:

(a) $\bar{\bar{A}} = A$

(b) $\overline{A+B} = \bar{A} + \bar{B}$

(c) $\overline{cA} = \bar{c}\bar{A}$

(d) $\overline{AB} = \bar{A}\bar{B}$

(e) $(\bar{A})^T = (\overline{A^T})$

Proofs:

(a) let $A = [a_{ij}]$

$\bar{A} = [\overline{a_{ij}}]$

$\bar{\bar{A}} = [\overline{\overline{a_{ij}}}] = [a_{ij}]$

$\boxed{\therefore \bar{\bar{A}} = A}$

(b) $\overline{A+B} = [\overline{a_{ij}+b_{ij}}]$

$= [\overline{a_{ij}} + \overline{b_{ij}}]$

$= [\overline{a_{ij}}] + [\overline{b_{ij}}]$

$\therefore \overline{A+B} = \boxed{\bar{A} + \bar{B}}$

(d) $\overline{AB}$

$(AB)_{ij} = \sum_k a_{ik} b_{kj}$

$(\overline{AB})_{ij} = \sum_k \overline{a_{ik} b_{kj}}$

As $\forall\, p, q \in \mathbb{C}$ $(\overline{p+q} = \bar{p} + \bar{q})$ & $\overline{pq} = \bar{p}\bar{q}$

$= \sum_k \overline{a_{ik} b_{kj}}$

$= \sum_k (\overline{a_{ik}})(\overline{b_{kj}})$

$= (\bar{A}\bar{B})_{ij}$

$\boxed{\therefore \overline{AB} = \bar{A}\bar{B}}$

More Properties:

(a) $(A^*)^* = A$

(b) $(A+B)^* = A^* + B^*$

(c) $(cA)^* = \bar{c}A^*$

(d) $(AB)^* = B^* A^*$

Proofs:

(d) $(AB)^* = (\overline{AB})^T = (\bar{A}\bar{B})^T$

$= \bar{B}^T \bar{A}^T$

$= \boxed{B^* A^*}$

※ If $u = \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix}$ and $v = \begin{pmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{pmatrix}$

if $w_1, \dots, w_n$ are positive scalars and are vectors in R^n, then

$$\langle u, v \rangle = w_1 u_1 v_1 + \dots + w_n u_n v_n$$

This defines an inner product on R^n, called a weighted dot product.

Prove that this is a inner product:

(i) $\langle u, v \rangle = \langle v, u \rangle$

$$\Rightarrow \langle u, v \rangle = \sum_i w_i u_i v_i$$

$$= \sum_i w_i v_i u_i$$

$$= \boxed{\langle v, u \rangle}$$

(ii) $\langle u, v + w \rangle$

$$= \sum_i w_i u_i (v_i + w_i)$$

$$= \sum_i w_i u_i v_i + \sum_i w_i u_i w_i$$

$$= \boxed{\langle u, v \rangle + \langle u, w \rangle}$$

(iii) $\langle cu, v \rangle = c \langle u, v \rangle$

$$\langle cu, v \rangle = \sum_i c w_i u_i v_i$$

$$= c \sum_i w_i u_i v_i$$

$$= \boxed{c \langle u, v \rangle}$$

(iv) $\langle u, u \rangle \geq 0$

$$\langle u, u \rangle = \sum_i w_i u_i u_i$$

$$= \sum_i w_i (u_i)^2$$

$(u_i)^2 \geq 0$ & $w_i \geq 0$

$$\therefore \sum_i w_i u_i^2 \geq 0$$

$$\boxed{\therefore \langle u, u \rangle \geq 0}$$

※ Orthogonal Projections: (Pg 562)

An orthogonal set of vectors in an inner product space V is a set $\{v_1, \ldots, v_k\}$ of vectors from V such that $\langle v_i, v_j \rangle = 0$ whenever, $v_i \neq v_j$.

A set of vectors is an orthonormal set of vectors if it is an orthogonal set of unit vectors.

Theorem: (Pg: 396)

Let $\{v_1, \ldots, v_k\}$ be an orthogonal basis for a subspace W of R^n and w be any vector of W. Then there are unique scalars $c_1, c_2, \ldots, c_k$ such that,

$$w = \sum_{i=1}^{k} c_i v_i$$

where $\boxed{c_i = \frac{w \cdot v_i}{v_i \cdot v_i}}$ for $i = 1, 2, \ldots, k$

If inner product is defined in the question, then

$$\boxed{c_i = \frac{\langle w, v_i \rangle}{\langle v_i, v_i \rangle}}$$

Proof:

Since, $\{v_1, \ldots, v_k\}$ is a basis of W,

$\exists$ scalars $c_1, \ldots, c_k$ such that $w = \sum_{i=1}^{k} c_i v_i$

$$w \cdot v_i = (c_1 v_1 + \cdots + c_k v_k) v_i$$
$$= c_1 (\cancel{v_1 \cdot v_i})^{0} + \cdots + c_i (v_i \cdot v_i) + \cdots c_i (\cancel{v_k \cdot v_i})^{0}$$
$$= c_i (v_i \cdot v_i)$$

$$\therefore \boxed{c_i = \frac{w \cdot v_i}{v_i \cdot v_i}}$$

Orthogonal Projection:

→ We can define the orthogonal projection $proj_W(v)$ of a vector v onto a subspace W of an inner product space. If $\{u_1, \ldots, u_k\}$ is an orthogonal basis for W, then

$$\boxed{proj_W(v) = \frac{\langle u_1, v\rangle}{\langle u_1, u_1\rangle} u_1 + \cdots + \frac{\langle u_k, v\rangle}{\langle u_k, u_k\rangle} u_k}$$

Then the component of v orthogonal to W is the vector:

$$\boxed{perp_W(v) = v - proj_W(v)}$$

Que: $\langle f, g\rangle = \int_a^b f(x)g(x)\,dx$

Prove that this is a inner product.

(1) $\langle v, u\rangle = \langle u, v\rangle$

$\therefore \int_a^b f(x)g(x) = \int_a^b g(x)f(x)\,dx$ True

(2) $\langle u, v+w\rangle = \langle u, v\rangle + \langle u, w\rangle$

(3) $\langle cu, v\rangle = c\langle u, v\rangle$ $\int_a^b (cf(x))g(x)\,dx = c\int_a^b f(x)g(x)\,dx$

(4) $\langle u, u\rangle \geq 0$ $\langle u, u\rangle = 0$ iff $u = 0$ $\int_a^b \underbrace{f(x)^2}_{+ve}\,dx$

Q: Find an orthogonal basis for the subspace W of R^3 given by

$$W = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} : x - y + 2z = 0 \right\}$$

Sol^n:

$\rightarrow x - y + 2z = 0$

$x = y - 2z$

$\therefore W = \left\{ \begin{pmatrix} y-2z \\ y \\ z \end{pmatrix} \right\}$ $y, z \in R$

Let $w_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$

$w_2 = \begin{pmatrix} y-2z \\ y \\ z \end{pmatrix}$

As W is orthogonal,

$w_1 \cdot w_2 = 0$

$\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} y-2z \\ y \\ z \end{pmatrix} = 0$

$\therefore y - 2z + y = 0$

$\boxed{y = z}$

$\therefore w_2 = \begin{pmatrix} -y \\ y \\ y \end{pmatrix}$

Let $y = 1$

$\boxed{\therefore w_2 = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}}$

$\left(w_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \right)$

$\boxed{W_B = \{w_1, w_2\}}$

$\hookrightarrow$ Orthogonal Basis of Subspace W.

※ The Gram-Schmidt Process:

→ We want to find an orthogonal basis for a subspace W of R^n.

→ The idea is to begin with a arbitrary basis $\{x_1, \ldots, x_k\}$ and to "orthogonalize" it one vector at a time.

→ For example:

Let $W = \text{span}(x_1, x_2)$, where

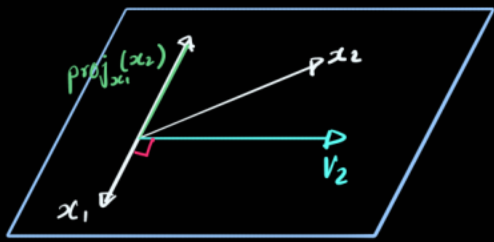
$$x_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \text{ and } x_2 = \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}$$

Construct an orthogonal basis for W.

Sol^n: Starting with x_1,

we get a second vector that is orthogonal to it by taking the component of x_2 orthogonal to x_1.

Constructing v_2 orthogonal to x_1



$x_1 \perp V_2$

Algebraically, we set $v_1 = x_1$, so

$$v_2 = \text{perp}_{x_1}(x_2) = x_2 - \text{proj}_{x_1}(x_2)$$

$$= x_2 - \left(\frac{x_1 \cdot x_2}{x_1 \cdot x_1}\right) x_1$$

$$= \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix} - \left(\frac{-2}{2}\right)\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$$

$$\boxed{v_2 = \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}}$$

Hence, $\{v_1, v_2\}$ is an orthogonal set of vectors in W.

So, it is linearly independent set.

$\rightarrow$ $\{v_1, v_2\}$ is a basis for W, since $\dim W = 2$.

NOTE:

If we had taken $v_1 = x_2$, we would have obtained a different set of orthogonal vectors in W.

※ Theorem: The Gram-Schmidt Process:

Let $\{x_1, \ldots, x_k\}$ be a basis for a subspace W of R^n and define the following:

$$v_1 = x_1 \qquad W_1 = \text{span}(x_1)$$

$$v_2 = x_2 - \text{proj}_{v_1}(x_2) \qquad W_2 = \text{span}(x_1, x_2)$$

$$v_3 = x_3 - \text{proj}_{v_1}(x_3) - \text{proj}_{v_2}(x_3) \qquad W_3 = \text{span}(x_1, x_2, x_3)$$

$$\vdots$$

$$v_k = x_k - \text{proj}_{v_1}(x_k) - \cdots - \text{proj}_{v_{k-1}}(x_k) \qquad W_k = \text{span}(x_1, \ldots, x_k)$$

Then, for each $i \in \{1, \ldots, k\}$, $\{v_1, \ldots, v_i\}$ is an orthogonal basis for W_i. In particular $\{v_1, \ldots, v_k\}$ is an orthogonal basis for W.

☆

Proof: Proof by Induction:

→ Let $v_1 = x_1$

Clearly, $\{v_1\}$ is an orthogonal basis for $W_1 = \text{span}(x_1)$.

→ Suppose,

for some $i < k$,

$\{v_1, \dots, v_i\}$ is an orthogonal basis for W_i.

Then,

$$v_{i+1} = x_{i+1} - \text{proj}_{v_1}(x_{i+1}) \dots - \text{proj}_{v_i}(x_{i+1})$$

$$= x_{i+1} - \left(\frac{v_1 \cdot x_1}{v_1 \cdot v_1}\right) v_1 - \dots - \left(\frac{v_i \cdot x_{i+1}}{v_i \cdot v_i}\right) v_i$$

By induction hypothesis,

$\{v_1, \dots, v_i\}$ is an orthogonal basis for $\text{span}(x_1, \dots, x_i) = W_i$

Hence, $\boxed{v_{i+1} = x_{i+1} - \text{proj}_{W_i}(x_{i+1}) = \text{perp}_{W_i}(x_{i+1})}$

→ ∴ By orthogonal Decomposition Theorem, v_{i+1} is orthogonal to W_i. — ①

By definition, $v_1, v_2, \dots, v_i$ are linear combinations of $x_1, \dots, x_i$ and hence, are in W_{i+1}. → ②

Also,

$v_{i+1} \neq 0$, since otherwise $x_{i+1} = \text{proj}_{W_i}(x_{i+1})$ implies x_{i+1} is in W_i.

But, since $W_i = \text{span}(x_1, \dots, x_i)$, $[\{x_1, \dots, x_i\}]$

this is impossible.

→ Hence, from ① and ②,

we conclude that, $\{v_1, \dots, v_{i+1}\}$ is a set of linearly independent vectors in W_{i+1}.

Since, $\dim W_{i+1} = i+1$,

$\{v_1, \dots, v_{i+1}\}$ is a basis for W_{i+1}.

Hence, Proved.

Eg: Apply the Gram-Schmidt Process to construct an orthogonal basis for the subspace $W = \text{span}(x_1, x_2, x_3)$ of $\mathbb{R}^4$, where $x_1 = \begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \end{pmatrix}$, $x_2 = \begin{pmatrix} 2 \\ 1 \\ 0 \\ 1 \end{pmatrix}$, $x_3 = \begin{pmatrix} 2 \\ 2 \\ 1 \\ 2 \end{pmatrix}$

Sol^n: Let $v_1 = x_1$

Now, we compute component of x_2 orthogonal to $W = \text{span}(v_1) = v_1$

$$v_2 = \text{perp}_{W_1}(x_2)$$

$$= x_2 - \text{proj}_{W_1}(x_2)$$

$$v_2 = x_2 - \left(\frac{v_1 \cdot x_2}{v_1 \cdot v_1}\right) v_1 \quad \left[\begin{array}{l} x_1 \cdot x_2 = 2 - \cancel{1} + \cancel{1} = 2 \\ x_1 \cdot x_1 = 1+1+1+1 = 4 \end{array}\right]$$

$$= x_2 - \left(\frac{2}{4}\right) v_1$$

$$= x_2 - \frac{1}{2}(v_1) = \begin{pmatrix} 2 \\ 1 \\ 0 \\ 1 \end{pmatrix} - \frac{1}{2}\begin{pmatrix} 1 \\ -1 \\ -1 \\ 1 \end{pmatrix}$$

$$\boxed{v_2 = \begin{pmatrix} 3/2 \\ 3/2 \\ 1/2 \\ 1/2 \end{pmatrix}}$$

For easy calculation, $\Rightarrow$ "scale" v_2

$\therefore$ let $v_2 = 2\begin{pmatrix} 3/2 \\ 3/2 \\ 1/2 \\ 1/2 \end{pmatrix}$

$$\boxed{v_2 = \begin{pmatrix} 3 \\ 3 \\ 1 \\ 1 \end{pmatrix}}$$

Let $W_2 = \text{span}(v_1, v_2)$

$$\therefore v_3 = \text{perp}_{W_2}(x_3)$$

$$= x_3 - \text{proj}_{W_2}(x_3)$$

$$= x_3 - \left(\frac{v_1 \cdot x_3}{v_1 \cdot v_1}\right) v_1 - \left(\frac{v_2 \cdot x_3}{v_2 \cdot v_2}\right) v_2$$

$v_1 \cdot x_3 = 1$

$v_1 \cdot v_1 = 4$

$v_2 \cdot x_3 = 15$

$v_2 \cdot v_2 = 20$

$$\therefore V_3 = x_3 - \left(\frac{1}{4}\right)\begin{pmatrix}1\\-1\\-1\\1\end{pmatrix} - \left(\frac{15}{20}\right)\begin{pmatrix}3\\3\\1\\1\end{pmatrix}$$

$$= \begin{pmatrix}2\\2\\1\\2\end{pmatrix} - \frac{1}{4}\begin{pmatrix}1\\-1\\-1\\1\end{pmatrix} - \frac{3}{4}\begin{pmatrix}3\\3\\1\\1\end{pmatrix}$$

$$= \begin{pmatrix}2-1/4-9/4\\2+1/4-9/4\\1+1/4-3/4\\2-1/4-3/4\end{pmatrix} = \begin{pmatrix}-1/2\\0\\1/2\\1\end{pmatrix}$$

"Scale" $V_3 \Rightarrow$ Let $\boxed{V_3 = \begin{pmatrix}-1\\0\\1\\2\end{pmatrix}}$

$$\{V_1, V_2, V_3\} = \left\{\begin{pmatrix}1\\-1\\-1\\1\end{pmatrix}, \begin{pmatrix}3\\3\\1\\1\end{pmatrix}, \begin{pmatrix}-1\\0\\1\\2\end{pmatrix}\right\}$$

Q: Construct an orthogonal basis for P_2 with respect to the inner product

$$\langle f, g\rangle = \int_{-1}^{1} f(x)g(x)\,dx$$

by applying the Gram-Schmidt Process to the basis $\{1, x, x^2\}$.

Sol^n: Let $x_1 = 1$

$x_2 = x$ & $x_3 = x^2$. Also, let $v_1 = x_1$

$$\therefore \langle v_1, v_1\rangle = \int_{-1}^{1} dx = \boxed{2}$$

$$\rightarrow \langle v_1, x_2\rangle = \int_{-1}^{1} (1)(x)\,dx$$

$$= \left.\frac{x^2}{2}\right)_{-1}^{1} = \frac{1}{2} - \frac{1}{2} = \boxed{0}$$

$$\rightarrow v_2 = x_2 - \left(\frac{\langle v_1, x_2\rangle}{\langle v_1, v_1\rangle}\right) v_1$$

$$= x_2 - 0 = x_2 = x$$

$$\boxed{\therefore V_2 = x}$$

$$\rightarrow V_3 = x_3 - \left(\frac{\langle v_1, x_3\rangle}{\langle v_1, v_1\rangle}\right) v_1 - \left(\frac{\langle v_2, x_3\rangle}{\langle v_2, v_2\rangle}\right) v_2$$

where,

$$\langle v_2, v_2\rangle = \int_{-1}^{1} x^2\,dx = \left.\frac{x^3}{3}\right]_{-1}^{1} = \boxed{2/3}$$

$$\langle v_1, x_3\rangle = \int_{-1}^{1} (1)(x^2)\,dx$$

$$= \left.\frac{x^3}{3}\right)_{-1}^{1} = \boxed{2/3}$$

$$\langle v_2, x_3\rangle = \int_{-1}^{1} (x)(x^2)\,dx = \boxed{0}$$

$$\therefore V_3 = x^2 - \left(\frac{2/3}{2}\right)(1) - \left(\frac{0}{2/3}\right)(x)$$

$$\boxed{V_3 = x^2 - 1/3}$$

$\therefore$ Orthogonal Basis $= \{1, x, x^2 - 1/3\}$.

※ The Cauchy-Schwarz Inequality:

→ Let u and v be vectors in an inner product space V. Then

$$|\langle u,v\rangle| \le \|u\|\,\|v\|$$

with equality holding if and only if u and v are scalar multiples of each other.

Proof: → If $u=0$, then the inequality is actually an equality, since

$$\boxed{|\langle 0,v\rangle| = 0 = \|0\|\|v\|}$$

→ If $u \neq 0$,
then let W be the subspace of V spanned by u. Since $\text{proj}_W(v) = \left(\frac{\langle u,v\rangle}{\langle u,u\rangle}\right)u$ and $\text{perp}_W(v) = v - \text{proj}_W(v)$ are othogonal,
we can apply pythagoras theorem,

$$\|v\|^2 = \|\text{proj}_W(v) + (v - \text{proj}_W(v))\|^2$$
$$= \|\text{proj}_W(v) + \text{perp}_W(v)\|^2$$
$$= \|\text{proj}_W(v)\|^2 + \|\text{perp}_W(v)\|^2$$

It follows that

$$\|v\|^2 \ge \|\text{proj}_W(v)\|^2$$

Now, $\|\text{proj}_W(v)\|^2 = \left\langle \left(\frac{\langle u,v\rangle}{\langle u,u\rangle}\right)u, \left(\frac{\langle u,v\rangle}{\langle u,u\rangle}\right)u \right\rangle = \left(\frac{\langle u,v\rangle}{\langle u,u\rangle}\right)^2 \langle u,u\rangle$

$$= \frac{\langle u,v\rangle^2}{\langle u,u\rangle}$$

$$= \boxed{\frac{\langle u,v\rangle^2}{\|u\|^2}}$$

So, we have,

$$\frac{\langle u,v\rangle^2}{\|u\|^2} \le \|v\|^2 \Rightarrow \langle u,v\rangle^2 \le \|u\|^2\|v\|^2$$

$$\boxed{\therefore\ |\langle u,v\rangle| \le \|u\|\,\|v\|}$$

Equality: iff $\|v\|^2 = \|\text{proj}_W(v)\|^2$, which is true only when $v = \text{proj}_W(v)$.
∴ If this is so, then v is a scalar multiple of u.
Conversely, if $v = cu$, ⇒ $\text{perp}_W(v) = v - \text{proj}_W(v) = cu - \left(\frac{\langle u,cu\rangle}{\langle u,u\rangle}\right)u = cu - cu$
∴ Equality holds. $= \boxed{0}$

※ Triangular Inequality Theorem:

$$\|u+v\| \leq \|u\| + \|v\|$$

Proof: LHS $= \|u+v\| = \sqrt{\langle u+v, u+v \rangle}$

$$\|u+v\|^2 = \langle u+v, u+v \rangle$$
$$= \langle u+v, u \rangle + \langle u+v, v \rangle$$
$$= \underbrace{\langle u, u \rangle}_{= \|u\|^2} + \underbrace{\langle u, v \rangle}_{< \|u\| \|v\|} + \underbrace{\langle v, u \rangle}_{< \|v\| \|u\|} + \underbrace{\langle v, v \rangle}_{= \|v\|^2}$$

$$\leq \|u\|^2 + 2\|u\|\|v\| + \|v\|^2$$

$$\|u+v\|^2 \leq (\|u\| + \|v\|)^2$$

$$\therefore \boxed{\|u+v\| \leq \|u\| + \|v\|}$$

※ The Orthogonal Decomposition Theorem:

Let W be a subspace of R^n and let v be a vector on R^n. Then there are unique vectors w in W and $w_\perp$ in $W^\perp$ such that

$$\boxed{v = w + w^\perp}$$

Proof:

To Prove: 1) Such a decomposition exist.

2) It is unique.

Part 1: Let $\{u_1, \dots, u_k\}$ be an orthogonal basis for W.

Let $w = \text{proj}_W(v)$ and,

let $w^\perp = \text{perp}_W(v)$

Then, $w + w^\perp = \text{proj}_W(v) + \text{perp}_W(v)$

$= \boxed{v}$

→ Clearly,

$w = \text{proj}_W(v)$ is in W, since it is a linear combination of basis vectors $u_1, \dots, u_k$.

→ To show that $w^\perp \in W^\perp$,

it is enough to show that $w^\perp$ is orthogonal to each of the basis vector u_i.

$$\begin{aligned} \to u_i \cdot w^\perp &= u_i \cdot \text{perp}_W(v) \\ &= u_i \cdot (v - \text{proj}_W(v)) \\ &= u_i \cdot \left(v - \left(\frac{u_1 \cdot v}{u_1 \cdot u_1}\right)u_1 - \left(\frac{u_2 \cdot v}{u_2 \cdot u_2}\right)u_2 - \dots - \left(\frac{u_k \cdot v}{u_k \cdot u_k}\right)u_k\right) \\ &= u_i \cdot v - \left(\frac{u_1 \cdot v}{u_1 \cdot u_1}\right)\cancelto{0}{(u_1 \cdot u_i)} - \dots - \left(\frac{u_i \cdot v}{u_i \cdot u_i}\right)(u_i \cdot u_i) \dots - \left(\frac{u_k \cdot v}{u_k \cdot u_k}\right)\cancelto{0}{(u_k \cdot u_i)} \end{aligned}$$

Since, $u_i \cdot u_j = 0 \;\forall\; u_i \neq u_j$,

$$\therefore u_i \cdot w^\perp = u_i \cdot v - 0 - \left(\frac{u_i \cdot v}{\cancel{u_i \cdot u_i}}\right)\cancel{(u_i \cdot u_i)}$$

$$= u_i \cdot v - u_i \cdot v -$$

$\boxed{\therefore u_i \cdot w^\perp = 0} \;\forall\; i$

$\therefore$ $w^\perp$ is in $W^\perp$

2) Uniqueness:

Suppose, we have another de-composition $v = w_1 + w_1^\perp$, where $w_1 \in W$ and $w_1^\perp \in W^\perp$. Then $w + w^\perp = w_1 + w_1^\perp$

$$\therefore w - w_1 = w_1^\perp - w^\perp$$

Since, $w - w_1 \in W$ and $w_1^\perp - w^\perp \in W^\perp$,

we know that this common vector $\in W \cap W^\perp = \{0\}$

$\therefore w - w_1 = 0$ and $w_1^\perp - w^\perp = 0$

$\therefore \boxed{w = w_1}$ and $\boxed{w^\perp = w_1^\perp}$

* Orthogonal Complement:

Let W be a subspace of R^n. We say that a vector v in R^n is orthogonal to W if v is orthogonal to every vector in W. The set of all vectors that are orthogonal to W is called the orthogonal complement of W, denoted by $W^\perp$.

That is,

$$\boxed{W^\perp = \{v \text{ in } R^n : v \cdot w = 0 \ \forall\ w \in W\}}$$

---

* Theorem: Let W be a subspace of R^n.

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a) $W^\perp$ is a subspace of R^n.

b) $(W^\perp)^\perp = W$

c) $W \cap W^\perp = \{0\}$

d) If $W = \text{span}(w_1, \ldots\ldots, w_k)$, then

1

Proof:

a) $\rightarrow$ $W^\perp$ is non-empty

$0 \cdot w = 0 \quad \forall\ w \in W$

$\boxed{0 \in W^\perp}$ — (i)

Let $u, v \in W^\perp$

$u \cdot w = v \cdot w = 0 \quad \forall\ w \in W$

$(u+v) \cdot w = u \cdot w + v \cdot w = 0 + 0 = 0 \quad \forall\ w \in W$

$\boxed{\therefore u+v \in W^\perp}$ — (ii)

$(cu) \cdot w = c(u \cdot w) = c(0) = 0 \quad \forall\ w \in W$

$\boxed{\therefore cu \in W^\perp}$ — (iii)

From (i), (ii) & (iii) $W^\perp$ is a subspace.

b) let $w \in W$ and $x \in W^{\perp}$,

then $w \cdot x = 0$

this implies that, $w \in (W^{\perp})^{\perp}$

$$\boxed{\therefore W \subseteq (W^{\perp})^{\perp}} \longrightarrow ①$$

Let $v \in (W^{\perp})^{\perp}$

By, orthogonal discomposition theorem,

$v = w + w^{\perp}$ for unique vector w in W & $w^{\perp}$ in $W^{\perp}$

$v \cdot w^{\perp} = 0$

$\therefore (w + w^{\perp}) \cdot w^{\perp} = w \cdot w^{\perp} + w^{\perp} \cdot w^{\perp}$

$= 0 + w^{\perp} \cdot w^{\perp} = 0$

$\therefore \boxed{w^{\perp} = 0}$

$\therefore v = w + w^{\perp} = w + 0$

$\boxed{v = w}$

$\therefore \boxed{v \in W}$

$$\boxed{\therefore (W^{\perp})^{\perp} \subseteq W} \longrightarrow ②$$

From ① and ② $\boxed{(W^{\perp})^{\perp} = W}$

※ Theorem:

The columns of an $m \times n$ matrix Q forms an orthonormal set iff $Q^TQ = I_n$.

Proof:

To show $Q^TQ = I_n$,

$$(Q^TQ)_{ij} = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{if } i \neq j \end{cases}$$

Let q_i be the i^{th} column of Q (or i^{th} row of Q^T).

$$(Q^TQ)_{ij} = (i^{th} \text{ row of } Q^T) \cdot (j^{th} \text{ column of } Q)$$

$$\boxed{(Q^TQ)_{ij} = q_i \cdot q_j} \longrightarrow (1)$$

Now, the columns Q form an orthogonal set iff

$$q_i \cdot q_j = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$$

which by equation (1) holds iff,

$$\boxed{(Q^TQ)_{ij} = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j \end{cases}}$$

※ Orthogonal Matrices:

An $n \times n$ matrix Q whose columns form an orthonormal set, is called orthogonal matrix.

Orthonormal:

A set of orthogonal unit vectors.

Que:

$$\begin{array}{c} \begin{array}{ccc} A & B & C \end{array} \\ Q = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{3} & x \\ 0 & 1/\sqrt{3} & y \\ -1/\sqrt{2} & 1/\sqrt{3} & z \end{pmatrix} \end{array}$$

For what values of x, y, z, Q is orthogonal matrix.

Sol^n: $\|C\| = 1$

$$\therefore \boxed{x^2 + y^2 + z^2 = 1} \longrightarrow 1$$

$$\langle A, C \rangle = 0$$

$$x/\sqrt{2} - z/\sqrt{2} = 0 \Rightarrow \boxed{x = z} \longrightarrow 2$$

$$\langle B, C \rangle = 0$$

$$x + y + z = 0 \Rightarrow \boxed{y = -2x} \longrightarrow 3$$

From 1, 2 & 3,

$$x^2 + 4x^2 + x^2 = 6x^2 = 1$$

$$\boxed{x = \pm 1/\sqrt{6}}$$

$$\boxed{\therefore (x, y, z) = \left(1/\sqrt{6}, -2/\sqrt{6}, 1/\sqrt{6}\right) \text{ or } \left(-1/\sqrt{6}, 2/\sqrt{6}, -1/\sqrt{6}\right)}$$

※ **Theorem:** A square matrix Q is orthogonal iff $Q^{-1} = Q^T$.

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Proof: 2

※ QR factorization :

→ Let A be a $m \times n$ matrix with linearly independent columns.

Then A can be factorized as $A = QR$ where Q is an $m \times n$ matrix with orthogonal column and R is invertible uppertriangular matrix.

→ Let $a_1, a_2, \ldots, a_n$ be the linearly independent columns of A and let $q_1, q_2, \ldots, q_n$ be the orthogonal vectors obtained by applying Gram-Schmidt process to $\{a_1, a_2, \ldots, a_n\}$ with normalization.

$$W_i = \text{span}\{a_1, a_2, \ldots, a_i\}$$
$$= \text{span}\{q_1, q_2, \ldots, q_i\}$$

∴ there are scalars $r_{1i}, r_{2i}, \ldots, r_{ii}$ such that

$$\left.\begin{aligned} a_1 &= r_{11} q_1 \\ a_2 &= r_{12} q_1 + r_{22} q_2 \\ a_3 &= r_{13} q_1 + r_{23} q_2 + r_{33} q_3 \\ &\vdots \\ a_n &= r_{1n} q_1 + r_{2n} q_2 + \cdots + r_{nn} q_n \end{aligned}\right\} \boxed{a_i = r_{1i} q_1 + r_{2i} q_2 + \cdots + r_{ii} q_i}$$

which can be written in matrix form as

$$A = [a_1\ a_2 \cdots a_n] = [q_1\ q_2 \cdots q_n] \begin{pmatrix} r_{11} & r_{12} & \cdots & r_{1n} \\ 0 & r_{22} & \cdots & r_{2n} \\ \vdots & 0 & \ddots & \vdots \\ 0 & 0 & \cdots & r_{nn} \end{pmatrix} = QR$$

∴ $A = QR$

As Q is an orthogonal set,

$Q^{-1} = Q^T$

$\boxed{\therefore Q^T A = R}$

Eg: Find QR factorization of

$$A = \begin{pmatrix} 1 & 2 & 2 \\ -1 & 1 & 2 \\ -1 & 0 & 1 \\ 1 & 1 & 2 \end{pmatrix}$$

Sol^n: The orthonormal basis for col(A) produced by Gram-Schmidt Process is:

$$q_1 = \begin{pmatrix} 1/2 \\ -1/2 \\ -1/2 \\ 1/2 \end{pmatrix}, \quad q_2 = \begin{pmatrix} 3\sqrt{5}/10 \\ 3\sqrt{5}/10 \\ \sqrt{5}/10 \\ \sqrt{5}/10 \end{pmatrix}, \quad q_3 = \begin{pmatrix} -\sqrt{6}/6 \\ 0 \\ \sqrt{6}/6 \\ \sqrt{6}/3 \end{pmatrix}$$

$$\therefore Q = \begin{pmatrix} 1/2 & 3\sqrt{5}/10 & -\sqrt{6}/6 \\ -1/2 & 3\sqrt{5}/10 & 0 \\ -1/2 & \sqrt{5}/10 & \sqrt{6}/6 \\ 1/2 & \sqrt{5}/10 & \sqrt{6}/3 \end{pmatrix}$$

$$A = QR$$

$$Q^T A = Q^T Q R = R$$

$$\therefore Q^T = \begin{pmatrix} 1/2 & -1/2 & -1/2 & 1/2 \\ 3\sqrt{5}/10 & 3\sqrt{5}/10 & \sqrt{5}/10 & \sqrt{5}/10 \\ -\sqrt{6}/6 & 0 & \sqrt{6}/6 & \sqrt{6}/3 \end{pmatrix}$$

$$\therefore R = Q^T A = \begin{pmatrix} 1/2 & -1/2 & -1/2 & 1/2 \\ 3\sqrt{5}/10 & 3\sqrt{5}/10 & \sqrt{5}/10 & \sqrt{5}/10 \\ -\sqrt{6}/6 & 0 & \sqrt{6}/6 & \sqrt{6}/3 \end{pmatrix} \begin{pmatrix} 1 & 2 & 2 \\ -1 & 1 & 2 \\ -1 & 0 & 1 \\ 1 & 1 & 2 \end{pmatrix} = \boxed{\begin{pmatrix} 2 & 1 & 1/2 \\ 0 & \sqrt{5} & 3\sqrt{5}/2 \\ 0 & 0 & \sqrt{6}/2 \end{pmatrix}}$$

※ Eigenvalues & Eigenvectors:

Definition: Let A be an $n \times n$ matrix. A scalar λ is called an eigenvalue of A if there is a nonzero vector x such that $Ax = \lambda x$. Such a vector x is called an eigen-vector of A corresponding to λ.

Eg: Show that $x = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is an eigenvector of $A = \begin{pmatrix} 3 & 1 \\ 1 & 3 \end{pmatrix}$ and find the corresponding eigenvalue.

Sol^n: $Ax = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \lambda \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$$\therefore \begin{bmatrix} 4 \\ 4 \end{bmatrix} = \lambda \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\therefore \boxed{\lambda = 4}$$

Eg: Show that 5 is an eigenvalue of $A = \begin{pmatrix} 1 & 2 \\ 4 & 3 \end{pmatrix}$ and determine all eigenvectors corresponding to this eigenvalue.

$\rightarrow Ax = \lambda x$

$(A - \lambda I)x = 0$

for $x \neq 0$,

$\det(A - \lambda I) = 0$

$$\therefore \begin{vmatrix} 1-\lambda & 2 \\ 4 & 3-\lambda \end{vmatrix} = 0$$

$\therefore (\lambda - 1)(\lambda - 3) - 8 = 0$

$\therefore \lambda^2 - 4\lambda + 3 - 8 = 0$

$\lambda^2 - 4\lambda - 5 = 0$

$(\lambda - 5)(\lambda + 1) = 0$

$$\boxed{\lambda = -1, 5}$$

for $\lambda = 5$

Let $x = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

$(A - \lambda I)x = 0$

$$\therefore \begin{pmatrix} -4 & 2 \\ 4 & -2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$4x_1 = 2x_2$

$\boxed{x_2 = 2x_1}$ $\Rightarrow \left\{ t\begin{pmatrix} 1 \\ 2 \end{pmatrix} \,\middle|\, t \in \mathbb{R} \right\}$

Eigenspace:

Let A be an $n \times n$ matrix and let λ be an eigenvalue of A. The collection of all eigenvectors corresponding to λ, together with the zero vector is called the eigenspace of λ and is denoted by E_λ.

※ Eigenvalues and Eigenvectors of n×n Matrices:

→ The eigenvalues of a square matrix A are precisely the solution λ of the equation $\det(A - \lambda I) = 0$.

$$Ax = \lambda x$$
$$Ax - \lambda x = 0$$
$$\boxed{\therefore (A - \lambda)x = 0}$$

→ When we expand $\det(A - \lambda I)$, we get a polynomial in λ, called the characteristic polynomial of A.

→ The equation $\det(A - \lambda I) = 0$ is called the characteristic equation.

Steps to find the eigenvalues and eigenvectors of a matrix:

Step 1: Compute the characteristic polynomial $\det(A - \lambda I)$ of A.

Step 2: Find the eigenvalues of A by solving the characteristic equation $\det(A - \lambda I) = 0$ for λ.

Step 3: For each eigenvalue λ, find the null space of the matrix $A - \lambda I$. This is the eigenspace E_λ, the nonzero vectors of which are the eigenvectors of A corresponding to λ.

Step 4: Find a basis for each eigenspan.

Eg: Find the eigenvalues and the corresponding eigenspaces of

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & -5 & 4 \end{pmatrix}$$

Sol^n: let $\lambda \to$ eigenvalue

characteristic equation: $\det(A-\lambda I)=0$

$$A-\lambda I = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & -5 & 4 \end{pmatrix} - \lambda \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 1 \\ 2 & -5 & 4-\lambda \end{pmatrix}$$

$\to \det(A-\lambda I)=0$

$\therefore -\lambda((\lambda-4)(\lambda)+5)-1(-2)$

$\therefore -\lambda(\lambda^2-4\lambda+5)+2$

$\therefore -\lambda^3+4\lambda^2-5\lambda+2=0$

$\therefore \boxed{\lambda^3-4\lambda^2+5\lambda-2=0}$

$\therefore \lambda^3-\lambda^2-3\lambda^2+3\lambda+2\lambda-2=0$

$\therefore (\lambda-1)(\lambda^2-3\lambda+2)=0$

$\therefore (\lambda-1)(\lambda-1)(\lambda-2)=0$

$\therefore \boxed{\lambda=1,2}$

For $\lambda=2$,

$$\begin{pmatrix} -2 & 1 & 0 \\ 0 & -2 & 1 \\ 2 & -5 & 2 \end{pmatrix} \xrightarrow{R_3 \to R_3+R_1} \begin{pmatrix} -2 & 1 & 0 \\ 0 & -2 & 1 \\ 0 & -4 & 2 \end{pmatrix} \xrightarrow{R_1 \to -\frac{1}{2}R_1} \begin{pmatrix} 1 & -1/2 & 0 \\ 0 & -2 & 1 \\ 0 & -4 & 2 \end{pmatrix}$$

$$\begin{pmatrix} 1 & -1/2 & 0 \\ 0 & -2 & 1 \\ 0 & -4 & 2 \end{pmatrix} \xrightarrow[\text{then, } R_3 \to R_3-2R_2]{R_1 \to R_1-1/4 R_2} \begin{pmatrix} 1 & 0 & -1/4 \\ 0 & -2 & 1 \\ 0 & 0 & 0 \end{pmatrix} \xrightarrow{R_2 \to (-1/2)R_2} \boxed{\begin{pmatrix} 1 & 0 & -1/4 \\ 0 & 1 & -1/2 \\ 0 & 0 & 0 \end{pmatrix}}$$

$\therefore x_1 - 1/4 x_3 = 0$

$\boxed{4x_1 = x_3}$

$x_2 - 1/2 x_3 = 0$

$\therefore \boxed{2x_2 = x_3}$

$\therefore \boxed{x = t\begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix}}$

For $\lambda=1$,

$(A-\lambda I)x=0$

let $x=\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$

$$A-\lambda I = \begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 2 & -5 & 3 \end{pmatrix} \xrightarrow{R_1 \to -R_1} \begin{pmatrix} 1 & -1 & 0 \\ 0 & -1 & 1 \\ 2 & -5 & 3 \end{pmatrix}$$

$$\xrightarrow{R_3 \to R_3-2R_1} \begin{pmatrix} 1 & -1 & 0 \\ 0 & -1 & 1 \\ 0 & -3 & 3 \end{pmatrix} \xrightarrow{R_2 \to (-1)R_2} \begin{pmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ 0 & -3 & 3 \end{pmatrix} \xrightarrow[\text{then, } R_3 \to R_3+3R_2]{R_1 \to R_1+R_2} \boxed{\begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}}$$

$$\therefore \begin{pmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

$\therefore x_1 - x_3 = 0$

$x_2 - x_3 = 0$

$\boxed{x_1 = x_2 = x_3}$

$\boxed{\therefore x = t\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}}$

NOTE:

→ **Algebraic Multiplicity:**

It is multiplicity of an eigenvalue as a root of the characteristic equation.

→ **Geometric Multiplicity** of an eigenvalue λ is defined as the dimension of its corresponding eigenspace.

Theorem:

→ A square matrix A is invertible iff 0 is not an eigenvalue of A.

Proof: We know that, a square matrix is invertible iff $\det A \neq 0$.

(→) Square matrix is invertible.

$\therefore \det A \neq 0$

$\therefore \det(A - 0I) = \boxed{\det(A) \neq 0}$

$\therefore \lambda$ is not an eigen value.

(←) Suppose zero is not an eigenvalue,

$\therefore \det(A - 0I) \neq 0$

$\therefore \det(A) \neq 0$

$\therefore A$ is invertible.

Theorem:

Let A be a square matrix with eigenvalue λ and the corresponding eigen-vector x.

a) For any positive integer n, λ^n is an eigenvalue of A^n with corresponding eigenvector x.

b) If A is invertible, then $1/\lambda$ is an eigenvalue of A^{-1} with corresponding eigenvector x.

c) If A is invertible, then for any integer n, λ^n is an eigenvalue of A^n with corresponding eigenvector x.

Proof:

Eg: Compute $\begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix}^{10} \begin{bmatrix} 5 \\ 1 \end{bmatrix}$

Sol^n: Let $A = \begin{pmatrix} 0 & 1 \\ 2 & 1 \end{pmatrix}$ and $x = \begin{pmatrix} 5 \\ 1 \end{pmatrix}$

We need to find $A^{10}x$.

$\rightarrow$ The eigenvalues of A are:

$\det(A - \lambda I) = 0$

$\therefore \begin{vmatrix} -\lambda & 1 \\ 2 & 1-\lambda \end{vmatrix} = 0$

$(\lambda - 1)\lambda - 2 = 0$

$\therefore \lambda^2 - \lambda - 2 = 0$

$\boxed{\lambda = -1, 2}$

Computing eigen vectors:

For $\lambda = -1$, $AV_1 = -V_1$

and for $\lambda = 2$, $AV_2 = 2V_2$

For $\lambda = -1$,

$$A - \lambda I = \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix} \to \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \quad \boxed{x_1 + x_2 = 0}$$

$A + I$

$$\therefore \boxed{x_1 = x_2 = 0} \qquad \boxed{\therefore v_1 = t_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix}}$$

For $\lambda = 2$,

$$A - \lambda I = \begin{pmatrix} 0 & 1 \\ 2 & 1 \end{pmatrix} - 2 \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} -2 & 1 \\ 2 & -1 \end{pmatrix} \to \begin{pmatrix} -2 & 1 \\ 0 & 0 \end{pmatrix}$$

$$\therefore -2x_1 + x_2 = 0$$

$$\boxed{x_2 = 2x_1} \Rightarrow \boxed{v_2 = t_2 \begin{pmatrix} 1 \\ 2 \end{pmatrix}}$$

→ Using the previous theorem(c),

if eigenvalue of A is λ with corresponding eigenvector being x, for the same eigenvector, eigenvalue of A^{10} will λ^{10} (if A is invertible).

→ Now, let $c_1, c_2 \in \mathbb{R}$ s.t.,

$$x = c_1 v_1 + c_2 v_2$$

$$\therefore \begin{pmatrix} 5 \\ 1 \end{pmatrix} = c_1 \begin{pmatrix} 1 \\ -1 \end{pmatrix} + c_2 \begin{pmatrix} 1 \\ 2 \end{pmatrix}$$

$$\therefore \cancel{c_1} + c_2 = 5$$

$$\cancel{c_1} + 2c_2 = 1$$

$$3c_2 = 6$$

$$\boxed{c_2 = 2} \Rightarrow \boxed{c_1 = 3}$$

$$\therefore x = 3v_1 + 2v_2$$

$$\therefore A^{10}x = 3A^{10}v_1 + 2A^{10}v_2$$

Using that theorem,

$$A^{10}v_1 = (\lambda_1)^{10} v_1$$

$$= (-1)^{10} v_1 = \boxed{v_1}$$

$$A^{10}v_2 = (\lambda_2)^{10} v_2$$

$$= (2)^{10} v_2$$

$$= \boxed{1024 v_2}$$

$$\therefore A^{10}x = 3v_1 + 2(1024)v_2$$

$$= 3\begin{pmatrix} 1 \\ -1 \end{pmatrix} + 2048 \begin{pmatrix} 1 \\ 2 \end{pmatrix} = \boxed{\begin{pmatrix} 2051 \\ 4093 \end{pmatrix}}$$

※ Theorem : Let A be an $n \times n$ matrix & let $\lambda_1, \lambda_2, \ldots, \lambda_m$ be distinct eigen vectors of A with corresponding eigen vectors $v_1, \ldots, v_m$, then $v_1, \ldots, v_m$ are linearly independent.

Proof: Let $v_1, \ldots, v_m$ are linearly dependent.

Let v_{k+1} be the first vector which is linearly dependent on $\{v_1, \ldots, v_k\}$.

$\therefore \exists\ c_1, \ldots, c_k$ (scalars) such that,

$$\boxed{v_{k+1} = \sum_{1}^{k} c_i v_i} \longrightarrow (1)$$

Multiplying eqn (1) by A on both side

$$A v_{k+1} = A(c_1 v_1 + \cdots + c_k v_k)$$

$$\therefore \lambda v_{k+1} = c_1(Av_1) + c_2(Av_2) + \cdots + c_k(Av_k)$$

$$\therefore \lambda v_{k+1} = c_1 \lambda_1 v_1 + c_2 \lambda_2 v_2 + \cdots + c_k \lambda_k v_k \longrightarrow (2)$$

Multiplying eq^n ① by λ_{k+1} on both sides,

$$\lambda_{k+1} V_{k+1} = c_1 \lambda_{k+1} V_1 + \cdots + c_k \lambda_{k+1} V_k \longrightarrow ③$$

eq^{n}③ − eq^{n}②,

$$\therefore 0 = c_1 V_1 (\lambda_{k+1} - \lambda_1) + c_2 V_2 (\lambda_{k+1} - \lambda_2) + \cdots + c_k V_k (\lambda_{k+1} - \lambda_k)$$

$$\therefore \sum_{i=1}^{k} c_i V_i (\lambda_{k+1} - \lambda_i) = 0$$

As $\{V_1, \ldots, V_k\}$ are linearly independent

$$\therefore c_i(\lambda_{k+1} - \lambda_i) = 0 \quad \forall\, 1 \le i \le k$$

$$\therefore \boxed{c_i = 0} \quad \forall\, 1 \le i \le k$$

$$\therefore V_{k+1} = \sum_{i=1}^{k} c_i V_i = \boxed{0}$$

It is impossible since eigenvectors cannot be zero.

$\therefore$ Contradiction.

Hence, $\{v_1, \ldots, v_n\}$ are linearly Independent.

Similar Matrices

Definition Let A and B be $n \times n$ matrices. We say that A ***is similar to*** B if there is an invertible $n \times n$ matrix P such that $P^{-1}AP = B$. If A is similar to B, we write $A \sim B$.

$$\boxed{P^{-1}AP = B \Rightarrow A \sim B}$$

→ If A is similar to B, $P^{-1}AP = B$

$$\therefore PP^{-1}APP^{-1} = PBP^{-1}$$

$$\therefore \boxed{A = PBP^{-1}}$$

$$\therefore \boxed{AP = PB}$$

Theorem 4.21 Let A, B, and C be $n \times n$ matrices.

a. $A \sim A$
b. If $A \sim B$, then $B \sim A$.
c. If $A \sim B$ and $B \sim C$, then $A \sim C$.

Proof: (a) $A = IAI \quad [I^{-1} = I]$

$\therefore A = I^{-1}AI$

$\therefore \boxed{A \sim A}$

(b) $A \sim B$

$\exists$ Invertible matrix P such that,

$P^{-1}AP = B$

$\therefore PP^{-1}APP^{-1} = PBP^{-1}$

$\therefore A = PBP^{-1}$

Let $P_1 = P^{-1}$

$P_1^{-1} = (P^{-1})^{-1} = P$

$\therefore \boxed{A = P_1^{-1}BP_1}$

$\therefore \boxed{B \sim A}$

(c) Given $A \sim B$
$B \sim C$

$\therefore P_1^{-1}AP_1 = B$

and, $P_2^{-1}BP_2 = C$

$\therefore P_2^{-1}P_1^{-1}AP_1P_2 = P_2^{-1}BP_2$

Let $P_3 = P_1P_2$

$\therefore P_3^{-1} = P_2^{-1}P_1^{-1}$

$\therefore P_3^{-1}AP_3 = P_2^{-1}BP_2 = C$

$\therefore \boxed{P_3^{-1}AP_3 = C}$

$\therefore \boxed{A \sim C}$

Q: $A = \begin{pmatrix} 1 & 2 \\ 0 & -1 \end{pmatrix} \quad B = \begin{pmatrix} 1 & 0 \\ -2 & -1 \end{pmatrix}$

Find the matrix P such that $A \sim B$

Sol^n: let $P = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$

$AP = PB$

$\therefore \begin{pmatrix} 1 & 2 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} 1 & 0 \\ -2 & -1 \end{pmatrix}$

$$\therefore \begin{pmatrix} a+2c & b+2d \\ -c & -d \end{pmatrix} = \begin{pmatrix} a-2b & -b \\ c-2d & -d \end{pmatrix}$$

$\therefore \cancel{a}+2c = \cancel{a}-2b$

$\boxed{c=-b} \rightarrow (1)$

$b+2d = -b$

$2d = -2b$

$\therefore \boxed{d=-b} \rightarrow (2)$

$-c = c-2d$

$2c = 2d$

$\boxed{d=c} \rightarrow (3)$

$$\therefore \boxed{d=c=-b}$$

$$\therefore P = \begin{pmatrix} a & b \\ -b & -b \end{pmatrix}$$

Let $a=1$, $b=-1$ $\Rightarrow \boxed{P = \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}}$

Theorem 4.22 Let A and B be $n \times n$ matrices with $A \sim B$. Then

a. $\det A = \det B$
b. A is invertible if and only if B is invertible.
c. A and B have the same rank.
d. A and B have the same characteristic polynomial.
e. A and B have the same eigenvalues.
f. $A^m \sim B^m$ for all integers $m \geq 0$.
g. If A is invertible, then $A^m \sim B^m$ for all integers m.

Proof: (a) $\det A = \det B$

$\Rightarrow P^{-1}AP = B$

$\det(P^{-1}) \det(A) \det(P) = \det(B)$

$\therefore \dfrac{1}{\cancel{\det(P)}} \det(A) \cancel{\det(P)} = \det(B)$

$\therefore \boxed{\det(A) = \det B}$

(d) $\det(B-\lambda I) = \det(P^{-1}AP - \lambda I)$

As $I \sim I$

$P^{-1}IP = I$

$$= \det(P^{-1}AP - \lambda(P^{-1}IP))$$
$$= \det(P^{-1}AP - P^{-1}(\lambda I)P)$$
$$= \det(P^{-1}(A-\lambda I)P)$$
$$= \cancel{\det(P^{-1})} \det(A-\lambda I) \cancel{\det(P)}$$

$$\boxed{\det(B-\lambda I) = \det(A-\lambda I)}$$

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* Diagonalisation of Matrices

Definition An $n \times n$ matrix A is ***diagonalizable*** if there is a diagonal matrix D such that A is similar to D—that is, if there is an invertible $n \times n$ matrix P such that $P^{-1}AP = D$.

An $n\times n$ matrix A is diagonalizable, if $\exists$ a diagonal matrix D, such that $\boxed{A \sim D}$.

$\therefore \exists P$ s.t. $\boxed{P^{-1}AP = D}$

Theorem 4.23 Let A be an $n \times n$ matrix. Then A is diagonalizable if and only if A has n linearly independent eigenvectors.

More precisely, there exist an invertible matrix P and a diagonal matrix D such that $P^{-1}AP = D$ if and only if the columns of P are n linearly independent eigenvectors of A and the diagonal entries of D are the eigenvalues of A corresponding to the eigenvectors in P in the same order.

Proof: $\Rightarrow$ Suppose $A \sim D$,

$\exists P$ s.t. $\therefore P^{-1}AP = D \Rightarrow \boxed{AP = PD}$

Let the columns of P be $p_1, \ldots, p_n$

and the diagonal entries of D be $\lambda_1, \ldots, \lambda_n$.

$$A[P_1, \ldots, P_n] = P \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & \lambda_n \end{bmatrix}$$

$$\therefore [AP_1, \ldots, AP_n] = [P_1, \ldots, P_n] \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & & \vdots \\ \vdots & & \ddots & 0 \\ 0 & \cdots & 0 & \lambda_n \end{bmatrix}$$

$$\therefore [AP_1, \ldots, AP_n] = [\lambda_1 P_1, \ldots, \lambda_n P_n]$$

$$\therefore \boxed{AP_i = \lambda_i P_i}$$

This proves that the column vectors of P are eigenvectors of A, whose corresponding eigen values are $\lambda_1, \ldots, \lambda_n$.

Since P is invertible,
it's column vectors are linearly independent.
[By Fundamental Theorem of Invertible Matrices].

Hence, A has n linearly independent eigen vectors.

Fundamental Theorem of Invertible Matrices:

$A \to n \times n$ matrix.
The following are equivalent:

(a) A is an invertible matrix

(b) $AX = B$ has a unique solution for any $B \in \mathbb{R}$.

(c) $AX = 0$ has only trivial solution

(d) The row reduced echolen form of A is I_n.

(e) A is a product of elementary matrices

(f) rank of $A = n$

(g) nullity of $A = 0$

(h) The column vectors of A are linearly independent.

Q. If possible, find a matrix D that diagonalises

$$A = \begin{bmatrix} -1 & 0 & 1 \\ 3 & 0 & -3 \\ 1 & 0 & -1 \end{bmatrix}$$

Sol^n: Eigen values:

$$\det(A - \lambda I) = 0$$

$$\therefore \begin{vmatrix} -1-\lambda & 0 & 1 \\ 3 & -\lambda & -3 \\ 1 & 0 & -1-\lambda \end{vmatrix} = 0$$

$$\therefore -1\big((1+\lambda)(\lambda(1+\lambda))\big) + 1(0+\lambda) = 0$$

$$\therefore \lambda = (\lambda+1)^2(\lambda)$$

$$\therefore \lambda\left[(\lambda+1)^2 - 1\right] = 0$$

$$\therefore \lambda\left[\lambda^2 + 2\lambda\right] = 0$$

$$\therefore \lambda^2(\lambda+2) = 0$$

$$\boxed{\therefore \lambda = 0, -2}$$

For $\lambda = 0$,

$$\begin{pmatrix} -1 & 0 & 1 \\ 3 & 0 & -3 \\ 1 & 0 & -1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\boxed{x_1 = x_3}$$

$$\to \left\{ t\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + s\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \right\}$$

$t, s \in \mathbb{R}$

For $\lambda = -2$,

$$\begin{pmatrix} 1 & 0 & 1 \\ 3 & 2 & -3 \\ 1 & 0 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

$$\therefore \boxed{x_1 = -x_3}$$

$$\to 3x_1 + 2x_2 - 3x_3 = 0$$

$$\therefore 6x_3 = 2x_2 \to \boxed{x_2 = 3x_3}$$

$x_1 = t$

$x_3 = -t \to x_2 = -3t$

$$\therefore E_{-2} = \left\{ t\begin{pmatrix} 1 \\ -3 \\ -1 \end{pmatrix} \right\}.$$

$\therefore$ eigen vectors $\Rightarrow \left\{ \underset{w_1}{\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}} \underset{w_2}{\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}} \underset{w_3}{\begin{pmatrix} 1 \\ -3 \\ -1 \end{pmatrix}} \right\}$

$\therefore W_1, W_2$ and W_3 are linearly independent.

Hence, A is diagonalizable

$\Rightarrow \therefore P = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -3 \\ 1 & 0 & -1 \end{pmatrix}$

$$P^{-1} = \frac{Adj(P)}{det(P)} = \frac{-1}{2} \begin{pmatrix} -1 & 3 & -1 \\ 0 & -2 & 0 \\ -1 & 3 & 1 \end{pmatrix}^T$$

$$= \boxed{\begin{pmatrix} 1/2 & 0 & 1/2 \\ -3/2 & 1 & -3/2 \\ 1/2 & 0 & -1/2 \end{pmatrix}}$$

$\Rightarrow \boxed{D = P^{-1} A P}$

Q. If possible find a matrix P that diagonalizes

(a) $A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 2 & -5 & 4 \end{pmatrix}$

$$\det(A-\lambda I) = \begin{vmatrix} -\lambda & 1 & 0 \\ 0 & -\lambda & 1 \\ 2 & -5 & 4-\lambda \end{vmatrix} = 0$$

$\therefore \ -\lambda(\lambda(\lambda-4)+5) - 1(0-2) = 0$

$\therefore \ -\lambda(\lambda^2-4\lambda+5) + 2 = 0$

$\therefore \ -\lambda^3+4\lambda^2-5\lambda+2 = 0$

$\therefore \ \lambda^3-4\lambda^2+5\lambda-2$

$\lambda^3-\lambda^2-3\lambda^2+3\lambda+2\lambda-2 = 0$

$(\lambda^2-3\lambda+2)(\lambda-1) = 0$

$(\lambda-1)(\lambda-2)(\lambda-1) = 0$

$\boxed{(\lambda-1)^2(\lambda-2) = 0} \Rightarrow \boxed{\lambda = 1, 2}$

$\rightarrow$ For $\lambda = 1$,

$$A-\lambda I = \begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 2 & -5 & 3 \end{pmatrix}$$

Let v be the linearly independent vectors

$$v = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

$$\Rightarrow (A - \lambda I)v = 0$$

$$\therefore \begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 2 & -5 & 3 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

$\therefore \boxed{x_1 = x_2}$ $\boxed{x_2 = x_3}$ | $x_1 - 5x_2 + 3x_3 = 0$ ✓

$\therefore$ Basis $= \left\{ t\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \mid t \in R \right\}$

$\lambda = 2$, $(A - \lambda I)v = 0$

$$\begin{pmatrix} -2 & 1 & 0 \\ 0 & -2 & 1 \\ 2 & -5 & 2 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

$$\therefore \left.\begin{matrix} \boxed{x_2 = 2x_1} \\ \boxed{x_3 = 2x_2} \end{matrix}\right\} \boxed{x_3 = 4x_1}$$

$\therefore$ Basis $= \left\{ t\begin{pmatrix} 1 \\ 2 \\ 4 \end{pmatrix} \mid t \in R \right\}$

$\therefore$ Only 2 linearly independent eigen vectors.

$\therefore$ Not diagonalizable.

(b) $A = \begin{pmatrix} -1 & 0 & 1 \\ 3 & 0 & -3 \\ 1 & 0 & -1 \end{pmatrix}$

$\det(A - \lambda I) = 0$

$\therefore \begin{vmatrix} -1-\lambda & 0 & 1 \\ 3 & -\lambda & -3 \\ 1 & 0 & -1-\lambda \end{vmatrix} = 0$

$\therefore -(1+\lambda)\left[\lambda(1+\lambda) - 0\right] + 1(0+\lambda) = 0$

$\therefore \lambda\left[-(1+\lambda)^2 + 1\right] = 0$

$\therefore \lambda\left[\lambda^2 + 2\lambda\right] = 0 \Rightarrow \lambda^2(\lambda+2) = 0$

$\lambda = 0, -2$

For $\lambda = 0$,

$A - \lambda I = A = \begin{pmatrix} -1 & 0 & 1 \\ 3 & 0 & -3 \\ 1 & 0 & -1 \end{pmatrix}$

$\therefore \begin{pmatrix} -1 & 0 & 1 \\ 3 & 0 & -3 \\ 1 & 0 & -1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$

$x_1 = x_3$ $\therefore$ Basis $= \left\{ t\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, s\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \middle| \, t, s \in \mathbb{R} \right\}$ $\boxed{\dim = 2}$

For $\lambda = -2$,

$A - \lambda I = A + I = \begin{pmatrix} 1 & 0 & 1 \\ 3 & 2 & -3 \\ 1 & 0 & 1 \end{pmatrix}$

$\therefore \begin{pmatrix} 1 & 0 & 1 \\ 3 & 2 & -3 \\ 1 & 0 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$

$\therefore \boxed{x_1 = -x_3}$

$3x_1 + 2x_2 - 3(-x_1) = 0$

$6x_1 + 2x_2 = 0$

$\boxed{x_2 = -3x_1}$

Basis $= \left\{ t\begin{pmatrix} 1 \\ -3 \\ -1 \end{pmatrix} \middle| \, t \in \mathbb{R} \right\}$ $\boxed{\dim = 1}$

Hence, there are 3 linearly independent eigen vectors.

$\therefore P = [P_1 \; P_2 \; P_3]$ Column vectors are
↳ Linearly independent eigen vectors

$$\therefore P = \begin{pmatrix} 0 & 1 & -1 \\ 1 & 0 & 3 \\ 0 & 1 & 1 \end{pmatrix} \Rightarrow \det P = -1(1-0) - 1(1) = \boxed{-2}$$

$$\therefore P^{-1} = \frac{Adj(P)}{|P|}$$

$$= \frac{1}{-2}\begin{pmatrix} -3 & -1 & 1 \\ -2 & 0 & 0 \\ 3 & -1 & -1 \end{pmatrix}^T$$

$$= -\frac{1}{2}\begin{pmatrix} -3 & -2 & 3 \\ -1 & 0 & -1 \\ 1 & 0 & -1 \end{pmatrix} = \boxed{\begin{pmatrix} 3/2 & 1 & -3/2 \\ 1/2 & 0 & 1/2 \\ -1/2 & 0 & 1/2 \end{pmatrix}}$$

$$\boxed{D = P^{-1}AP}$$

Steps: Given $A\ (n\times n)$

(1) Find all eigen values of A.

(2) Then find all corresponding eigen vectors.

(3) If there are n linearly independent eigen vectors of A, then compute P.

$$P = [\, P_1 \; \ldots \; P_n \,]$$

(4) Compute $P^{-1}AP = D$ to verify.

Q. Find the values of $\begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix}^{10}$ using diagonalization.

Sol^n:

$$A = \begin{pmatrix} 0 & 1 \\ 2 & 1 \end{pmatrix}$$

$$\det(A - \lambda I) = 0$$

$$\therefore \begin{vmatrix} -\lambda & 1 \\ 2 & 1-\lambda \end{vmatrix} = 0$$

$$\therefore \lambda(\lambda - 1) - 2 = 0$$

$$\therefore \lambda^2 - \lambda - 2 = 0$$

$$\begin{matrix} -\lambda & 1 \\ 2 & 1-\lambda \end{matrix}$$

$$\therefore (\lambda - 2)(\lambda + 1) = 0$$

$$\boxed{\lambda = -1, 2}$$

$\Rightarrow \lambda = -1$

$$(A - \lambda I)v = 0 \Rightarrow \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\boxed{x_1 = -x_2}$$

$$\therefore \text{Basis} = \left\{ \begin{pmatrix} 1 \\ -1 \end{pmatrix} \right\}$$

$\lambda = 2$

$$(A - \lambda I)v = 0$$

$$\therefore \begin{pmatrix} -2 & 1 \\ 2 & -1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\left.\begin{matrix} \therefore -2x_1 + x_2 = 0 \\ \boxed{x_2 = 2x_1} \end{matrix}\right\} \text{Basis} = \left\{ \begin{pmatrix} 1 \\ 2 \end{pmatrix} \right\}$$

$$\boxed{\therefore P = \begin{pmatrix} 1 & 1 \\ -1 & 2 \end{pmatrix}} \longrightarrow P^{-1} = \frac{1}{2+1}\begin{pmatrix} 2 & -1 \\ 1 & 1 \end{pmatrix}$$

$$= \boxed{\begin{pmatrix} 2/3 & -1/3 \\ 1/3 & 1/3 \end{pmatrix}}$$

We know that,

$$P^{-1}AP = D \quad \left(\text{where } D = \begin{pmatrix} -1 & 0 \\ 0 & 2 \end{pmatrix} \right)$$

$$D^{10} = (P^{-1}AP)^{10} = \underbrace{(P^{-1}AP)}_{1}\underbrace{(P^{-1}AP)}_{2} \cdots \underbrace{(P^{-1}AP)}_{10}$$

$$\therefore \quad P^{-1}A^{10}P = D^{10}$$

$$\therefore \boxed{A^{10} = PD^{10}P^{-1}}$$

$$A^{10} = \begin{pmatrix} 1 & 1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} -1 & 0 \\ 0 & 2 \end{pmatrix}^{10} \begin{pmatrix} 2/3 & -1/3 \\ 1/3 & 1/3 \end{pmatrix}$$

$$A^{10} = \frac{1}{3} \begin{pmatrix} 1 & 1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & (2)^{10} \end{pmatrix} \begin{pmatrix} 2 & -1 \\ 1 & 1 \end{pmatrix}$$

$$= \frac{1}{3} \begin{pmatrix} 1 & 2^{10} \\ -1 & (2)^{11} \end{pmatrix} \begin{pmatrix} 2 & -1 \\ 1 & 1 \end{pmatrix}$$

$$= \frac{1}{3} \begin{pmatrix} 2+2^{10} & 2^{10}-1 \\ 2^{11}-2 & 2^{11}+1 \end{pmatrix}$$

$$= \frac{1}{3} \begin{pmatrix} 1026 & 1023 \\ 2046 & 2049 \end{pmatrix}$$

$$= \boxed{\begin{pmatrix} 342 & 341 \\ 682 & 683 \end{pmatrix}}$$

⁂ Orthogonal Diagonilazation of Symmetric Matrices :

Definition A square matrix A is ***orthogonally diagonalizable*** if there exists an orthogonal matrix Q and a diagonal matrix D such that $Q^T AQ = D$.

Q: If possible, diagonalize the matrix

$$A = \begin{pmatrix} 1 & 2 \\ 2 & -2 \end{pmatrix}$$

Solⁿ: $\det(A - \lambda I) = 0$

$$\therefore \begin{vmatrix} 1-\lambda & 2 \\ 2 & -2-\lambda \end{vmatrix} = 0$$

$$\therefore -(1-\lambda)(2+\lambda) - 4 = 0$$

$$(\lambda - 1)(\lambda + 2) - 4 = 0$$

$$\lambda^2 + \lambda - 2 - 4 = 0$$

$$\lambda^2 + \lambda - 6 = 0$$

$$\boxed{(\lambda + 3)(\lambda - 2) = 0}$$

$\lambda = 2$,

$$(A - \lambda I)v = 0$$

$$\begin{pmatrix} -1 & 2 \\ 2 & -4 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0$$

$$\boxed{x_1 = 2x_2}$$

$$\therefore \text{Basis} = \left\{ \begin{pmatrix} 2 \\ 1 \end{pmatrix} \right\}$$

$\lambda = -3$,

$$(A - \lambda I)v = 0$$

$$\therefore \begin{pmatrix} 4 & 2 \\ 2 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = 0 \Rightarrow \boxed{x_2 = -2x_1}$$

$$\therefore \text{Basis} = \left\{ \begin{pmatrix} 1 \\ -2 \end{pmatrix} \right\}$$

$\therefore P = \begin{pmatrix} 2 & 1 \\ 1 & -2 \end{pmatrix}$

$\therefore Q = \frac{1}{\sqrt{4+1}} \begin{pmatrix} 2 & 1 \\ 1 & -2 \end{pmatrix}$

$= \boxed{\frac{1}{\sqrt{5}} \begin{pmatrix} 2 & 1 \\ 1 & -2 \end{pmatrix}}$

$\det Q = \frac{1}{\sqrt{5}}(-5)$

$= \boxed{-\sqrt{5}}$

$\boxed{Q^T = \frac{1}{\sqrt{5}} \begin{pmatrix} 2 & 1 \\ 1 & -2 \end{pmatrix}}$

$\Rightarrow D = Q^T A Q$

$= \frac{1}{\sqrt{5}} \begin{pmatrix} 2 & 1 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ 2 & -2 \end{pmatrix} \frac{1}{\sqrt{5}} \begin{pmatrix} 2 & 1 \\ 1 & -2 \end{pmatrix}$

$= \frac{1}{5} \begin{pmatrix} 2 & 1 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} 4 & -3 \\ 2 & 6 \end{pmatrix}$

$= \frac{1}{5} \begin{pmatrix} 10 & 0 \\ 0 & -15 \end{pmatrix}$

$D = \boxed{\begin{pmatrix} 2 & 0 \\ 0 & -3 \end{pmatrix}}$

Theorem 5.17 If A is orthogonally diagonalizable, then A is symmetric.

Proof: If A is orthogonally diagonalizable, then there exists an orthogonal matrix Q and a diagonal matrix D such that

$$Q^T A Q = D$$

Since, $Q^{-1} = Q^T$

we have $Q^T Q = I = Q Q^T$

$$\therefore Q Q^T A Q Q^T = Q D Q^T$$

$$\therefore \boxed{A = Q D Q^T}$$

$$A^T = (Q^T)^T D^T Q^T$$

$$= Q D^T Q^T$$

Since, D is a diagonal matrix, $D = D^T$

$$\boxed{\therefore A^T = Q D Q^T = A}$$

Hence, A is symmetric.

Theorem 5.18 If A is a real symmetric matrix, then the eigenvalues of A are real.

Recall that the complex conjugate of a complex number $z = a + bi$ is the number

Proof: Suppose that λ is an eigenvalue of A with corresponding eigenvector v. Then $Av = \lambda v$,

and taking complex conjugates, we have $\overline{Av} = \overline{\lambda v}$

But then, as $A = \bar{A}$ (A is real)

$$A\bar{v} = \bar{A}\bar{v} = \overline{Av} = \overline{\lambda v} = \bar{\lambda}\bar{v}$$

$$\begin{bmatrix} A = \bar{A} \\ A = A^T \end{bmatrix}$$

Using the fact that A is symmetric,

$$(A\bar{v})^T = (\bar{\lambda}\bar{v})^T$$

$$\therefore \bar{v}^T A^T = \bar{v}^T A = \bar{\lambda}(\bar{v})^T$$

$$\therefore \lambda(\bar{v}^T v) = (\bar{v}^T)(\lambda v) = \bar{v}^T(Av) = (\bar{v}^T A)v$$
$$= (\bar{\lambda}\bar{v}^T)v$$
$$= \bar{\lambda}(\bar{v}^T v)$$

$$\therefore \lambda(\bar{v}^T v) = \bar{\lambda}(\bar{v}^T v)$$
$$\therefore \lambda(\bar{v}^T v) - \bar{\lambda}(\bar{v}^T v) = 0$$
$$\therefore (\lambda - \bar{\lambda})(\bar{v}^T v) = 0$$
$$\therefore \boxed{\lambda = \bar{\lambda}} \Rightarrow \lambda \text{ is Real.}$$

Theorem 5.19 If A is a symmetric matrix, then any two eigenvectors corresponding to distinct eigenvalues of A are orthogonal.

Proof: let $v_1, v_2 \to$ eigenvectors corresponding to distinct eigenvalues $(\lambda_1 \neq \lambda_2)$

$$\therefore Av_1 = \lambda_1 v_1 \text{ and } Av_2 = \lambda_2 v_2$$

$\to$ Using $A^T = A$ and $x \cdot y = x^T y$,

$$\lambda_1(v_1 \cdot v_2) = (\lambda_1 v_1) \cdot v_2 = (Av_1) \cdot v_2$$
$$= (Av_1)^T v_2$$
$$= v_1^T A^T v_2$$
$$= v_1^T (A v_2)$$
$$= v_1^T (\lambda_2 v_2)$$
$$= \lambda_2 (v_1^T v_2)$$
$$= \lambda_2 (v_1 \cdot v_2)$$

$$\lambda_1(v_1 \cdot v_2) = \lambda_2(v_1 \cdot v_2)$$
$$\therefore (\lambda_1 - \lambda_2)(v_1 \cdot v_2) = 0$$

As $\lambda_1 \neq \lambda_2$

$$\boxed{v_1 \cdot v_2 = 0}$$

Q: Verify the result of Theorem 5.19 for

$$A = \begin{pmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{pmatrix}$$

$\underline{\underline{\text{Sol}^n}}$: $\det(A - \lambda I) = 0$

$$\therefore \begin{vmatrix} 2-\lambda & 1 & 1 \\ 1 & 2-\lambda & 1 \\ 1 & 1 & 2-\lambda \end{vmatrix} = 0$$

$$\therefore (2-\lambda)\left[(2-\lambda)^2 - 1\right] - 1\left[2-\lambda - 1\right] + 1\left[1 - (2-\lambda)\right] = 0$$

$$\therefore (2-\lambda)\left[4 + \lambda^2 - 4\lambda - 1\right] - 2(1-\lambda) = 0$$

$$\therefore (2-\lambda)(\lambda^2 - 4\lambda + 3) + 2(\lambda - 1) = 0$$

$$\therefore (2-\lambda)(\lambda-3)(\lambda-1) + 2(\lambda-1) = 0$$

$$\therefore (\lambda-1)\left[(\lambda-3)(2-\lambda) + 2\right] = 0$$

$$\therefore (\lambda-1)\left[\lambda^2 - 5\lambda + 4\right] = 0$$

$$\therefore \boxed{(\lambda-1)^2(\lambda-4) = 0}$$

For $\lambda = 1$,

$$\begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0 \qquad \therefore \text{Basis} = \left\{ \underset{V_1}{\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}} \underset{V_2}{\begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}} \right\}$$

$$\therefore \boxed{x_1 + x_2 + x_3 = 0}$$

For $\lambda = 4$

$$\begin{pmatrix} -2 & 1 & 1 \\ 1 & -2 & 1 \\ 1 & 1 & -2 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

$$\therefore \left.\begin{aligned} x_2+x_3 &= 2x_1 \\ x_1+x_3 &= 2x_2 \\ x_1+x_2 &= 2x_3 \end{aligned}\right\} \boxed{x_1 = x_2 = x_3}$$

$$\text{Basis} = \left\{ \underbrace{\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}}_{V_3} \right\}$$

Verifying,

$$V_2 \cdot V_3 = 1-1 = 0$$

$$V_1 \cdot V_3 = 1-1 = 0$$

Hence, verified.

Theorem 5.20 **The Spectral Theorem**

Let A be an $n \times n$ real matrix. Then A is symmetric if and only if it is orthogonally diagonalizable.

$$Q = [q_1\ q_2\ \cdots\cdots\ q_n]$$

$$Q^T = \begin{bmatrix} q_1^T \\ q_2^T \\ \vdots \\ q_n^T \end{bmatrix} \longrightarrow A = [q_1 \ldots q_n] \begin{pmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ 0 & \cdots & & \lambda_n \end{pmatrix} \begin{bmatrix} q_1^T \\ \vdots \\ q_n^T \end{bmatrix}$$

$$\therefore \boxed{A = \lambda_1 q_1 q_1^T + \lambda_2 q_2 q_2^T + \cdots\cdots + \lambda_n q_n q_n^T}$$

↳ spectral decomposition form of a symmetric matrix.

Eg: $A = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$ Orthogonally diagonalize the matrix.

Sol^n: $\det(A - \lambda I) = 0$

$\hookrightarrow \lambda = 4, 1, 1$

$$E_4 = \left\{ t\begin{pmatrix}1\\1\\1\end{pmatrix} \mid t \in R \right\}$$

$$E_1 = \left\{ p\begin{pmatrix}-1\\0\\1\end{pmatrix} + q\begin{pmatrix}-1\\1\\0\end{pmatrix}, p, q \in R \right\}$$

$\rightarrow v_1 = \begin{pmatrix}1\\1\\1\end{pmatrix}, v_2 = \begin{pmatrix}-1\\0\\1\end{pmatrix}, v_2 = \begin{pmatrix}-1\\1\\0\end{pmatrix}$

$\rightarrow$ Gram - Schmidt Process:

v_2 and v_3 are not orthogonal

$\therefore$ let $v_1' = v_1$

$v_2' = v_2$

$(\because v_1 \cdot v_3 = 0)$ & $v_3' = v_3 - \left(\frac{v_2 \cdot v_3}{v_2 \cdot v_2}\right) v_2$

$$= \begin{pmatrix}-1\\1\\0\end{pmatrix} - \left(\frac{1}{2}\right)\begin{pmatrix}-1\\0\\1\end{pmatrix}$$

$$= \begin{pmatrix}-1\\1\\0\end{pmatrix} - \begin{pmatrix}-1/2\\0\\1/2\end{pmatrix}$$

$$= \boxed{\begin{pmatrix}-1/2\\1\\-1/2\end{pmatrix}}$$

$\therefore \{v_1, v_2, v_3'\}$ orthogonal but not orthonormal

$\therefore \{q_1, q_2, q_3\}$

$$= \left\{ \begin{pmatrix}1/\sqrt{3}\\1/\sqrt{3}\\1/\sqrt{3}\end{pmatrix}, \begin{pmatrix}-1/\sqrt{2}\\0\\1/\sqrt{2}\end{pmatrix}, \begin{pmatrix}-1/\sqrt{6}\\2/\sqrt{6}\\1/\sqrt{6}\end{pmatrix} \right\}$$

$$\rightarrow q_1 q_1^T = \begin{pmatrix}1/3 & 1/3 & 1/3\\1/3 & 1/3 & 1/3\\1/3 & 1/3 & 1/3\end{pmatrix}$$

$$\rightarrow q_2 q_2^T = \begin{pmatrix}1/2 & 0 & -1/2\\0 & 0 & 0\\-1/2 & 0 & 1/2\end{pmatrix}$$

$$\rightarrow q_3 q_3^T = \begin{pmatrix}1/6 & -1/3 & -1/6\\-1/3 & 2/3 & 1/3\\-1/6 & -1/3 & 1/6\end{pmatrix}$$

$$\boxed{A = 4 q_1 q_1^T + q_2 q_2^T + q_3 q_3^T}$$

Q. Find a 2x2 matrix with eigen-values $\lambda_1 = 3, \lambda_2 = -2$ and corresponding eigenvectors

$$v_1 = \begin{pmatrix} 3 \\ 4 \end{pmatrix} \quad v_2 = \begin{pmatrix} -4 \\ 3 \end{pmatrix}$$

Sol^n: As $v_1 \cdot v_2 = 0$,

$\exists\ Q \in \mathbb{R}^{2\times 2}$ such that,

$$Q^T A Q = \begin{pmatrix} 3 & 0 \\ 0 & -2 \end{pmatrix}$$

Let $Q = (q_1\ q_2)$

$$\therefore A = \lambda_1 q_1 q_1^T + \lambda_2 q_2 q_2^T$$

$$q_1 = \begin{pmatrix} 3/5 \\ 4/5 \end{pmatrix} \quad q_2 = \begin{pmatrix} -4/5 \\ 3/5 \end{pmatrix}$$

$$\therefore A = 3 \begin{pmatrix} 9/25 & 12/25 \\ 12/25 & 16/25 \end{pmatrix} + (-2) \begin{pmatrix} 16/25 & -12/25 \\ -12/25 & 9/25 \end{pmatrix}$$

$$\boxed{A = \begin{pmatrix} -1/5 & 12/5 \\ 12/5 & 6/5 \end{pmatrix}}$$

* Quadratic Form :

→ An expression of the form

$ax^2 + by^2 + cxy$

is called a quadratic form of x, y.

$ax^2 + by^2 + cz^2 + dxy + exz + fyz$

↳ of x, y, z

$$= \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} a & c/2 \\ c/2 & b \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$= \begin{bmatrix} x & y & z \end{bmatrix} \begin{bmatrix} a & d/2 & e/2 \\ d/2 & b & f/2 \\ e/2 & f/2 & c \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

Definition: A quadratic form in n variable is a function $f: \mathbb{R}^n \to \mathbb{R}$ of the form

$$f(x) = x^T A x$$

where A is a symmetric $n \times n$ matrix and x is in $\mathbb{R}^n$. We refer to A as the matrix associated with f.

Q: What is the quadratic form with associated matrix $A = \begin{bmatrix} 2 & -3 \\ -3 & 5 \end{bmatrix}$

Sol^n:
$$\begin{bmatrix} a & c/2 \\ c/2 & b \end{bmatrix}$$

$$\left.\begin{array}{l} \therefore c = -6 \\ a = 2 \\ b = 5 \end{array}\right\} \boxed{2x^2 + 5y^2 - 6xy}$$

Q. Find the matrix associated with the quadratic form.

$$f(x_1, x_2, x_3) = 2x_1^2 - x_2^2 + 5x_3^2 + 6x_1x_2 - 3x_1x_3$$

$$\left.\begin{array}{l} \therefore a = 2 \\ b = -1 \\ c = 5 \\ d = 6 \\ f = -3 \\ e = 0 \end{array}\right\} \begin{bmatrix} 2 & 3 & -3/2 \\ 3 & -1 & 0 \\ -3/2 & 0 & 5 \end{bmatrix}$$

※ Principal Axis Theorem:

Let A be a $n \times n$ symmetric matrix associated with a quadratic form $x^T A x$.

If Q is orthogonal matrix such that $Q^T A Q = D$, (D is a diagonal matrix), then $x = QY$

transforms the quadratic form to $y^T D y$.

Proof: Pg 436 (let $f(x) =$ - - - . . .

Q: Find a change in variable that transforms the quadratic form $f(x_1, x_2) = 5x_1^2 + 4x_1x_2 + 2x_2^2$ into one with no cross-product terms.

Sol^n: Corresponding matrix,

$$A = \begin{pmatrix} 5 & 2 \\ 2 & 2 \end{pmatrix}$$

$\det(A - \lambda I) = 0$

$\Rightarrow \boxed{\lambda_1 = 6} \quad \boxed{\lambda_2 = 1}$

$\Rightarrow$ The corresponding eigenvectors are:

$$\begin{pmatrix} 2/\sqrt{5} \\ 1/\sqrt{5} \end{pmatrix} \text{ and } \begin{pmatrix} 1/\sqrt{5} \\ -2/\sqrt{5} \end{pmatrix}$$

$\lambda = 6 \qquad \lambda = 1$

Also, $D = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix} = \boxed{\begin{pmatrix} 6 & 0 \\ 0 & 1 \end{pmatrix}}$

If we set $Q = \begin{pmatrix} 2/\sqrt{5} & 1/\sqrt{5} \\ 1/\sqrt{5} & -2/\sqrt{5} \end{pmatrix}$ then $D = Q^T A Q$ will be be a diagonal matrix.

Since we don't want any cross-pdt. terms, the change of variable $x = Qy$ will create the Diagonal Matrix D such that,

$f(y) = y^T D y$.

Let $y = [y_1 \; y_2]$

$$= [y_1 \; y_2] \begin{bmatrix} 6 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$$

$$= \boxed{6y_1^2 + y_2^2}$$

※ A quadratic form $f(x) = x^T A x$ is classified as one of the following: symmetric A is also called:

A) Positive definite if $f(x) > 0 \ \forall \ x \neq 0$

B) positive semidefinite if $f(x) \geq 0 \ \forall \ x$

C) negative definite if $f(x) < 0 \ \forall \ x \neq 0$

D) negative semidefinite if $f(x) \leq 0 \ \forall \ x$.

E) indefinite if $f(x)$ takes on both positive and negative values.

Theorem 5.22 Let A be an $n \times n$ symmetric matrix. The quadratic form $f(\mathbf{x}) = \mathbf{x}^T A\mathbf{x}$ is

a. positive definite if and only if all of the eigenvalues of A are positive.
b. positive semidefinite if and only if all of the eigenvalues of A are nonnegative.
c. negative definite if and only if all of the eigenvalues of A are negative.
d. negative semidefinite if and only if all of the eigenvalues of A are nonpositive.
e. indefinite if and only if A has both positive and negative eigenvalues.

Eg: Classify $f(x,y,z) = 3x^2 + 3y^2 + 3z^2 - 2xy - 2xz - 2yz$

as +ve definite
-ve definite
indefinite
none of these

Solⁿ:

$$A = \begin{pmatrix} 3 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 3 \end{pmatrix}$$

$\rightarrow \det(A - \lambda I) = 0$

$$\therefore \begin{vmatrix} 3-\lambda & -1 & -1 \\ -1 & 3-\lambda & -1 \\ -1 & -1 & 3-\lambda \end{vmatrix} = 0$$

$\therefore (3-\lambda)^3 - (3-\lambda) + 1\left(\cancel{(\lambda-3)} + 1\right) - 1\left(1 + \cancel{(\lambda-3)}\right)$

$(3-\lambda)(\lambda+2)(\lambda+4) = 0$

$\boxed{\lambda = 3, -2, -4}$

$\therefore$ Indefinite

※ Singular Values of a Matrix.

Definition If A is an $m \times n$ matrix, the ***singular values*** of A are the square roots of the eigenvalues of A^TA and are denoted by $\sigma_1, \ldots, \sigma_n$. It is conventional to arrange the singular values so that $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_n$.

For any $m \times n$ matrix A. A^TA is a $n \times n$ sym. matrix.

By spectral theorem, we can orthogonally diagonalize A^TA.

All eigenvalues of A^TA are real and non-negative.

(let λ be the eigen value of A^TA with corresponding unit eigen vector v)

$$
\begin{aligned}
0 \leq \|Av\|^2 &= (Av)\cdot(Av) \\
&= (Av)^T(Av) \\
&= v^TA^TAv \qquad (\lambda \to \text{eigenvalue of } A^TA \quad \therefore A^TAv = \lambda v) \\
&= v^T(\lambda v) \\
&= \lambda v^Tv \\
&= \lambda(v \cdot v) = \lambda \|v\|^2 = \boxed{\lambda}
\end{aligned}
$$

$\therefore \boxed{\lambda \geq 0}$

The Singular Value Decomposition:

$A \to m \times n$ matrix

$\therefore A^T A \to n \times n$ matrix

let $\lambda_1, \ldots, \lambda_k \to$ eigenvalues (of $A^T A$)

$\therefore$ singular values: $\sqrt{\lambda_1}, \ldots, \sqrt{\lambda_k}$

$= \sigma_1, \ldots, \sigma_k$

$\therefore \lambda_i = \|Av_i\|^2$

$\therefore \sigma_i = \sqrt{\|Av_i\|^2}$

$$\boxed{\sigma_i = \|Av_i\|}$$

i.e. the singular values of A are the lengths of the vectors $Av_1, \ldots, Av_n$.

Now, we want to show that $m \times n$ matrix A can be factored as $A = U \Sigma V^T$ where U is an $m \times m$ orthogonal matrix, V is an $n \times n$ orthogonal matrix and Σ is an $m \times n$ diagonal matrix.

$\to$ If the non-zero singular values of A are

$\sigma_1 \geq \sigma_2 \geq \ldots \geq \sigma_r > 0$

and $\sigma_{r+1} = \sigma_{r+2} = \ldots = \sigma_n = 0$

then Σ is of the form:

$$\Sigma = \begin{array}{c} \begin{array}{cc} \overbrace{\ }^{r} & \overbrace{\ }^{n-r} \end{array} \\ \left[\begin{array}{c|c} D & 0 \\ \hline 0 & 0 \end{array}\right] \begin{array}{l} \}r \\ \}m-r \end{array} \end{array}$$

where,

$$D = \begin{pmatrix} \sigma_1 & \cdots & \cdots & 0 \\ 0 & \sigma_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & \cdots & \sigma_r \end{pmatrix}$$

Eg: (of Σ matrix): for $r=2$,

$$\Sigma = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 3 & 0 \end{pmatrix}, \quad \Sigma = \begin{pmatrix} 2 & 0 \\ 0 & 2 \\ 0 & 0 \end{pmatrix}, \quad \Sigma = \begin{pmatrix} 8 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

$$\Sigma = \begin{pmatrix} 5 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Steps:

1. To construct the orthogonal matrix V, we first find an orthonormal basis $\{v_1, \ldots, v_n\}$ for $\mathbb{R}^n$ consisting of eigenvectors of the $n \times n$ symmetric matrix A^TA.

   Then $V = [v_1 \ldots v_n]$

   is an orthogonal $n \times n$ matrix.

2) For the orthogonal matrix U, we first note that $\{Av_1, ----, Av_n\}$ is an orthogonal set of vectors in R^n.

For $i \neq j$,

$$
\begin{aligned}
(Av_i)\cdot(Av_j) &= (Av_i)^T(Av_j) \\
&= v_i^T \underbrace{A^T A v_j}_{\hookrightarrow \lambda_j v_j} \\
&= v_i^T(\lambda_j v_j) \\
&= \lambda_j(v_i^T v_j) \\
&= \lambda_j(v_i \cdot v_j) \\
&= \boxed{0}
\end{aligned}
$$

(since the eigenvectors v_i are orthogonal).

$\rightarrow \sigma_i = ||Av_i||$ (let first r of these be nonzero)

$\therefore$ Let $\boxed{u_i = \frac{1}{\sigma_i} Av_i}$ $(i = \{1, ----, r\})$

This guarantees that $\{u_1, -----, u_r\}$ is an orthonormal set in R^m, but if $r < m$, it will not be a basis for R^m.

In this case, we extend the set $\{u_1, ---, u_r\}$ to an orthonormal basis $\{u_1, ----, u_m\}$ for R^m.

Then we set,

$$\boxed{U = [u_1 \ u_2 \ --- \ u_m]}$$

Proof: (of the above method)

$V \to n\times n$ orthogonal

$\therefore V^T = V^{-1}$

$\therefore \to A = U\Sigma V^T$

$\therefore AV = U\Sigma$

We know that,

$$Av_i = \sigma_i u_i \quad \text{for } i = 1, \dots, r$$

and $\|Av_i\| = \sigma_i$

$\therefore Av_i = 0$ for $i > r$

$$\therefore AV = A[v_1 \; \dots \; v_n]$$

$$= [Av_1 \; \dots \; Av_n]$$

$$= [Av_1 \; \dots \; Av_r \; 0 \; \dots \; 0]$$

$$= [\sigma_1 u_1 \; \dots \; \sigma_r u_r \; 0 \; \dots \; 0]$$

$$= [u_1 \; \dots \; u_m] \left(\begin{array}{ccc|c} \sigma_1 & \cdots & 0 & \\ \vdots & & \vdots & 0 \\ 0 & \cdots & \sigma_r & \\ \hline & 0 & & 0 \end{array}\right)$$

$\boxed{AV = U\Sigma}$ as required.

This was the proof of The Singular Value Decomposition Theorem:

Theorem 7.13 **The Singular Value Decomposition**

Let A be an $m \times n$ matrix with singular values $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r > 0$ and $\sigma_{r+1} = \sigma_{r+2} = \cdots = \sigma_n = 0$. Then there exist an $m \times m$ orthogonal matrix U, an $n \times n$ orthogonal matrix V, and an $m \times n$ matrix Σ of the form shown in Equation (1) such that

$$A = U\Sigma V^T$$

Q Find a singular value decomposition for the following matrices.

(a) $A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}_{2\times 3}$

$$A^TA = \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\det(A^TA - \lambda I) = 0$$

$$\therefore \begin{vmatrix} 1-\lambda & 1 & 0 \\ 1 & 1-\lambda & 0 \\ 0 & 0 & 1-\lambda \end{vmatrix} = 0$$

$$\therefore (1-\lambda)^3 - 1((1-\lambda)) = 0$$

$$\therefore (1-\lambda)((1-\lambda)^2 - 1) = 0$$

$$\therefore (\lambda-1)(\lambda^2 - 2\lambda + 1 - 1) = 0$$

$$\therefore \lambda(\lambda-1)(\lambda-2) = 0$$

$$\boxed{\lambda = 0, 1, 2}$$

$\lambda_1 \geq \lambda_2 \geq \lambda_3$

$\therefore \lambda_1 = 2, \lambda_2 = 1, \lambda_3 = 0$

for $\lambda_3 = 0$,

let $v_3 = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$

$$A^T A v_3 = \lambda_3 v_3$$

$$\therefore \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\left.\begin{aligned} \therefore x_1 &= -x_2 \\ x_3 &= 0 \end{aligned}\right\} \boxed{v_3 = \begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \\ 0 \end{pmatrix}}$$

$$\Sigma = \begin{pmatrix} \sigma_1 & 0 & 0 \\ 0 & \sigma_2 & 0 \end{pmatrix}$$

$$= \boxed{\begin{pmatrix} \sqrt{2} & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}}$$

Similarly,

for $\lambda_2 = 1$,

let $v_2 = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$

$$A^T A v_2 = \lambda_2 v_2$$

$$\therefore x_1 + x_2 = x_1 \qquad x_1 + x_2 = x_2$$

$$x_2 = 0 \qquad x_1 = 0$$

$$\therefore \boxed{v_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}}$$

Similarly for

for $\lambda_1 = 2$, $(A^T A - \lambda_1 I) v_1 = 0$

$$\begin{pmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\left.\begin{aligned} \therefore x_1 &= x_2 \\ x_3 &= 0 \end{aligned}\right\} \boxed{v_1 = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{pmatrix}}$$

$$\therefore V = [v_1 \; v_2 \; v_3] = \boxed{\begin{pmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 0 & 1 & 0 \end{pmatrix}}$$

For U,

$$u_1 = \frac{1}{\sigma_1} A v_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{pmatrix}$$

$$= \boxed{\begin{pmatrix} 1 \\ 0 \end{pmatrix}}$$

$$u_2 = \frac{1}{\sigma_2} A v_2 = \frac{1}{1} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

$$= \boxed{\begin{pmatrix} 0 \\ 1 \end{pmatrix}}$$

These vectors already form an orthonormal basis of R^2.

$\therefore$ we have $U = [u_1 \; u_2]$

$$= \boxed{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}$$

$$\therefore A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} \sqrt{2} & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \\ 0 & 0 & 1 \\ -1/\sqrt{2} & 1/\sqrt{2} & 0 \end{pmatrix}$$

$$= \boxed{U \Sigma V^T}$$

(b) $A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}_{3\times 2}$

$\rightarrow A^T A = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}$

$\therefore A^T A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$

$\rightarrow \det(A^T A - \lambda I) = 0$

$\therefore \begin{vmatrix} 2-\lambda & 1 \\ 1 & 2-\lambda \end{vmatrix} = 0$

$\therefore (\lambda - 2)^2 - 1 = 0$

$\therefore (\lambda - 2 - 1)(\lambda - 2 + 1) = 0$

$\therefore (\lambda - 3)(\lambda - 1) = 0$

$\boxed{\lambda = 1, 3}$

$\therefore \lambda_1 = 3$ and $\lambda_2 = 1$

$\boxed{\sigma_1 = \sqrt{3}}$ and $\boxed{\sigma_2 = 1}$

(As $\sigma_1 \geq \sigma_2 > 0$)

$\rightarrow$ Now,

$$\boxed{\Sigma_{3\times 2} = \begin{pmatrix} \sqrt{3} & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}}$$

Finding $V = [v_1 \ v_2]$ where $\{v_1, v_2\}$ are forms a orthonormal basis of $\mathbb{R}^2$. ($n = 2$, in this que)

$\therefore$ for $\lambda = 3$,

$(A^T A - \lambda_1 I) v_1 = 0$

$\therefore \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$\therefore x_1 = x_2 \rightarrow \boxed{v_1 = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}}$

$\boxed{E_3 = \left\{ t\begin{pmatrix} 1 \\ 1 \end{pmatrix} \middle| t \in \mathbb{R} \right\}}$

For $\lambda = 1$,

$$(A^TA - \lambda_2 I) V_2 = 0$$

$$\therefore \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\therefore \boxed{x_1 = -x_2}$$

$$\Rightarrow \boxed{E_1 = \left\{ t \begin{pmatrix} 1 \\ -1 \end{pmatrix} \middle| \, t \in R \right\}}$$

$$\therefore \boxed{V_2 = \begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix}}$$

$$\therefore V = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix} \Rightarrow \boxed{V^T = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix}}$$

$\Rightarrow$ Finding U:

$\rightarrow$ let $U = [U_1 \; U_2 \; U_3]$ $\quad U_i = \frac{1}{\sigma_i} A V_i$

$$U_1 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}$$

$$= \frac{1}{\sqrt{3}} \begin{pmatrix} \sqrt{2} \\ 1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} = \boxed{\begin{pmatrix} 2/\sqrt{6} \\ 1/\sqrt{6} \\ 1/\sqrt{6} \end{pmatrix}}$$

$$U_2 = \frac{1}{1} \begin{pmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix}$$

$$\boxed{U_2 = \begin{pmatrix} 0 \\ 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix}}$$

Extending $[v_1\ v_2]$ to $[v_1\ v_2\ v_3]$

let $v_3 = e_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$

→ Using Gram-Schmidt Process to orthogonalize it.

$\therefore\ v_3' = \text{prep}_{\text{span}(v_1, v_2)}^{e_3}$

$$= e_3 - \left(\frac{v_1 \cdot e_3}{v_1 \cdot v_1}\right) v_1 - \left(\frac{v_2 \cdot e_3}{v_2 \cdot v_2}\right) v_2$$

$$= \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} - \left(\frac{1/\sqrt{6}}{1}\right)\begin{pmatrix} 2/\sqrt{6} \\ 1/\sqrt{6} \\ 1/\sqrt{6} \end{pmatrix} - \left(\frac{-1/\sqrt{2}}{1}\right)\begin{pmatrix} 0 \\ 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix}$$

$$= \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} - \begin{pmatrix} 1/3 - 0 \\ 1/6 + 1/2 \\ 1/6 - 1/2 \end{pmatrix}$$

$$v_3' = \begin{pmatrix} -1/3 \\ -2/3 \\ 1/3 \end{pmatrix}$$

$$\boxed{v_3 = \begin{pmatrix} 1/\sqrt{6} \\ 2/\sqrt{6} \\ -1/\sqrt{6} \end{pmatrix}}$$

$$\Rightarrow U = [v_1\ v_2\ v_3] = \boxed{\begin{pmatrix} 2/\sqrt{6} & 0 & 1/\sqrt{6} \\ 1/\sqrt{6} & 1/\sqrt{2} & 2/\sqrt{6} \\ 1/\sqrt{6} & -1/\sqrt{2} & -1/\sqrt{6} \end{pmatrix}}$$

Hence,

$$\begin{pmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}_{3\times 2} = \begin{pmatrix} 2/\sqrt{6} & 0 & 1/\sqrt{6} \\ 1/\sqrt{6} & 1/\sqrt{2} & 2/\sqrt{6} \\ 1/\sqrt{6} & -1/\sqrt{2} & -1/\sqrt{6} \end{pmatrix} \begin{pmatrix} \sqrt{3} & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix}$$

NOTE: 1) $\{U_1, U_2, \ldots, U_r\}$ are called left singular vectors.
2) $\{V_1, V_2, \ldots, V_r\}$ are called right singular vectors.

Theorem 7.14 **The Outer Product Form of the SVD**

Let A be an $m \times n$ matrix with singular values $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r > 0$ and $\sigma_{r+1} = \sigma_{r+2} = \cdots = \sigma_n = 0$. Let $\mathbf{u}_1, \ldots, \mathbf{u}_r$ be left singular vectors and let $\mathbf{v}_1, \ldots, \mathbf{v}_r$ be right singular vectors of A corresponding to these singular values. Then

$$A = \sigma_1 \mathbf{u}_1 \mathbf{v}_1^T + \cdots + \sigma_r \mathbf{u}_r \mathbf{v}_r^T$$

$$A = U \Sigma V^T = [U_1 \cdots\cdots U_m] \left[\begin{array}{ccc|c} \sigma_1 & \cdots & 0 & \\ \vdots & \ddots & \vdots & 0 \\ 0 & \cdots & \sigma_r & \\ \hline & 0 & & 0 \end{array}\right] \begin{bmatrix} V_1^T \\ \vdots \\ V_n^T \end{bmatrix}$$

$$= [U_1 \cdots U_r \cdots U_m] \left[\begin{array}{ccc|c} \sigma_1 & \cdots & 0 & \\ \vdots & \ddots & \vdots & 0 \\ 0 & \cdots & \sigma_r & \\ \hline & 0 & & 0 \end{array}\right] \begin{bmatrix} V_1^T \\ \vdots \\ V_r^T \\ \vdots \\ V_n^T \end{bmatrix}$$

$$= [U_1 \cdots U_r] \begin{bmatrix} \sigma_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \sigma_r \end{bmatrix} \begin{bmatrix} V_1^T \\ \vdots \\ V_r^T \end{bmatrix} + \cancel{[U_{r+1} \cdots U_m] [0] \begin{bmatrix} V_{r+1}^T \\ \vdots \\ V_m^T \end{bmatrix}} \rightarrow 0$$

$$= [U_1 \cdots U_r] \begin{bmatrix} \sigma_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \sigma_r \end{bmatrix} \begin{bmatrix} V_1^T \\ \vdots \\ V_r^T \end{bmatrix}$$

$$= \boxed{\sigma_1 U_2 V_1^T + \sigma_2 U_2 V_2^T + \cdots + \sigma_r U_r V_r^T}$$

Q. Find Outer Product of $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Sol^n:

$$\rightarrow A^T A = \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$\rightarrow$ Eigenvalues: $\det(A^TA - \lambda I) = 0$

$$\therefore \begin{vmatrix} 1-\lambda & 1 & 0 \\ 1 & 1-\lambda & 0 \\ 0 & 0 & 1-\lambda \end{vmatrix} = 0$$

$$\therefore (1-\lambda)^3 - 1(1-\lambda) = 0$$

$$\therefore (1-\lambda)[\cancel{1} + \lambda^2 - 2\lambda \cancel{-1}] = 0$$

$$\therefore \lambda(\lambda-1)(\lambda-2) = 0 \rightarrow \begin{cases} \lambda_1 = 2 \rightarrow \sigma_1 = \sqrt{2} \\ \lambda_2 = 1 \rightarrow \sigma_2 = 1 \\ \lambda_3 = 0 \end{cases}$$

$\rightarrow$ Eigenvectors:

For $\lambda = 2$,

$$\begin{pmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\left. \begin{array}{l} x_3 = 0 \\ x_1 = x_2 \end{array} \right\} \boxed{E_2 = \left\{ t \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \middle| \, t \in R \right\}}$$

For $\lambda = 1$,

$$\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$\therefore x_1 = x_2 = 0$

let $x_3 = t$

$$\therefore \boxed{E_1 = \left\{ t \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \middle| \, t \in R \right\}}$$

$\rightarrow$ For $\lambda = 0$,

$$\begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{array}{l} x_1 = -x_2 \\ x_3 = 0 \end{array} \Rightarrow \boxed{E_0 = \left\{ t \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \middle| \, t \in R \right\}}$$

$\Rightarrow \therefore v_1 = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{pmatrix}, \quad v_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$ and $v_3 = \begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \\ 0 \end{pmatrix}$

$\rightarrow U_1 = \frac{1}{\sigma_1} A v_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{pmatrix}$

$= \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1/2 \\ 1/2 \\ 0 \end{pmatrix} = \boxed{\begin{pmatrix} 1 \\ 0 \end{pmatrix}}$

$\rightarrow U_2 = \frac{1}{\sigma_2} A v_2 = \frac{1}{1} \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \boxed{\begin{pmatrix} 0 \\ 1 \end{pmatrix}}$

$\Rightarrow$ Outer Product form:

$$A = \sigma_1 U_1 V_1^T + \sigma_2 U_2 V_2^T + \cdots + \sigma_r U_r V_r^T$$

Here, $r = 2$

$$\boxed{\therefore A = \sqrt{2} \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} & 0 \end{pmatrix} + 1 \begin{pmatrix} 0 \\ 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \end{pmatrix}}$$

Ans

$$= \begin{pmatrix} 1 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Q. Find Outer Product form of $A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}$

$\Rightarrow$ $A^T A = \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$

$\det(A^TA - \lambda I) = 0$

$\therefore \lambda_1 = 3 \quad \lambda_2 = 1$

$\sigma_1 = \sqrt{3} \quad \sigma_2 = 1$

$\Rightarrow$ For $\lambda_1 = 3$,

Let X_1 be the eigen-vector

$X_1 = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

$\therefore (A^TA - \lambda I) X_1 = 0$

$\therefore \begin{pmatrix} -1 & 1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$\therefore \boxed{x_1 = x_2}$

$\boxed{E_3 = \left\{ t \begin{pmatrix} 1 \\ 1 \end{pmatrix} \mid t \in R \right\}}$

$\Rightarrow$ For $\lambda = 1$,

$(A^TA - \lambda I) X_2 = 0$

$\therefore \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$\therefore \boxed{x_1 = -x_2}$

$\therefore \boxed{E_1 = \left\{ t \begin{pmatrix} 1 \\ -1 \end{pmatrix} \mid t \in R \right\}}$

Let $V_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ 1 \end{pmatrix}$ and $V_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$U_1 = \frac{1}{\sigma_1} A V_1 \qquad U_2 = \frac{1}{\sigma_2} A V_2$

$= \frac{1}{\sqrt{2}\sqrt{3}} \begin{pmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ 1 \end{pmatrix} \qquad = \frac{1}{1 \times \sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix}$

$= \boxed{\begin{pmatrix} 2/\sqrt{6} \\ 1/\sqrt{6} \\ 1/\sqrt{6} \end{pmatrix}} \qquad = \boxed{\begin{pmatrix} 0 \\ 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix}}$

$\therefore$ Outer Product form

$= \sigma_1 U_1 V_1^T + \sigma_2 U_2 V_2^T$

$= \sqrt{3} \begin{pmatrix} 2/\sqrt{6} \\ 1/\sqrt{6} \\ 1/\sqrt{6} \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix} + 1 \begin{pmatrix} 0 \\ 1/\sqrt{2} \\ -1/\sqrt{2} \end{pmatrix} \begin{pmatrix} 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix}$

$= \boxed{\sqrt{3} \begin{pmatrix} 1/\sqrt{3} & 1/\sqrt{3} \\ 1/2\sqrt{3} & 1/2\sqrt{3} \\ 1/2\sqrt{3} & 1/2\sqrt{3} \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 1/2 & -1/2 \\ -1/2 & 1/2 \end{pmatrix}}$ Ans

※ Pseudo Inverse :

Definition Let $A = U\Sigma V^T$ be an SVD for an $m \times n$ matrix A, where $\Sigma = \begin{bmatrix} D & O \\ O & O \end{bmatrix}$ and D is an $r \times r$ diagonal matrix containing the nonzero singular values $\sigma_1 \geq \sigma_2 \geq \cdots \geq \sigma_r > 0$ of A. The ***pseudoinverse*** (or ***Moore-Penrose inverse***) of A is the $n \times m$ matrix A^+ defined by

$$A^+ = V\Sigma^+U^T$$

where Σ^+ is the $n \times m$ matrix

$$\Sigma^+ = \begin{bmatrix} D^{-1} & O \\ O & O \end{bmatrix}$$

Q: $A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

Find the pseudo inverse of A.

Sol^n: $A^+ = U\Sigma^+V^T$

$$A^TA = \begin{pmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$\rightarrow \det(A^TA - \lambda I) = 0$

$\therefore (1-\lambda)^3 - 1((1-\lambda) - 0) = 0$

$\therefore (1-\lambda)[(1-\lambda)^2 - 1] = 0 \Rightarrow (\lambda-1)(\lambda^2 - 2\lambda + \cancel{1} - \cancel{1}) = 0$

$\therefore \boxed{\lambda(\lambda-1)(\lambda-2) = 0}$

$\therefore \lambda_1 = 2,\ \lambda_2 = 1$ and $\lambda_3 = 0$

$\therefore \sigma_1 = \sqrt{2},\ \sigma_2 = 1$

$$\therefore \boxed{D = \begin{pmatrix} \sqrt{2} & 0 \\ 0 & 1 \end{pmatrix}} \Rightarrow D^{-1} = \begin{pmatrix} 1/\sqrt{2} & 0 \\ 0 & 1 \end{pmatrix} \Rightarrow \Sigma^{+} = \begin{pmatrix} D^{-1} & 0 \\ 0 & 0 \end{pmatrix}_{n \times m} = \boxed{\begin{pmatrix} 1/\sqrt{2} & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}}$$

→ for $\lambda_1 = 2$,

$$(A^T A - \lambda I)x = 0$$

$$\therefore \begin{pmatrix} -1 & 1 & 0 \\ 1 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

$$\therefore \left.\begin{matrix} x_1 = x_2 \\ x_3 = 0 \end{matrix}\right\} \boxed{E_2 = \left\{ t\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix} \middle| t \in \mathbb{R} \right\}} \rightarrow v_1 = \begin{pmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{pmatrix}$$

→ for $\lambda = 1$,

$$\boxed{E_1 = \left\{ t\begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \middle| t \in \mathbb{R} \right\}} \rightarrow v_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

→ for $\lambda = 0$,

$$\boxed{E_0 = \left\{ t\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix} \middle| t \in \mathbb{R} \right\}} \rightarrow v_3 = \begin{pmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \\ 0 \end{pmatrix}$$

Thus, $\boxed{V = \begin{pmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 0 & 1 & 0 \end{pmatrix}}$

$$\left.\begin{matrix} u_1 = \frac{1}{\sigma_1} A v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \\ u_2 = \frac{1}{\sigma_2} A v_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \end{matrix}\right\} U = [u_1\ u_2] = \boxed{\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}}$$

$$A^{+} = V \Sigma^{+} U^{T}$$

$$A^{+} = \begin{pmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 1/\sqrt{2} & 0 & -1/\sqrt{2} \\ 0 & 1 & 0 \end{pmatrix}\begin{pmatrix} 1/\sqrt{2} & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \times \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= \boxed{\begin{pmatrix} 1/2 & 0 \\ 1/2 & 0 \\ 0 & 1 \end{pmatrix}}$$

Ans.

Q. $A = \begin{pmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}$ Find Pseudo Inverse.

Sol^n: As solved earlier,

$$\Sigma = \begin{pmatrix} \sqrt{3} & 0 \\ 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \Sigma^{+} = \begin{pmatrix} 1/\sqrt{3} & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

$$U = \begin{pmatrix} 2/\sqrt{6} & 0 & -1/\sqrt{3} \\ 1/\sqrt{6} & 1/\sqrt{2} & 1/\sqrt{3} \\ 1/\sqrt{6} & -1/\sqrt{2} & 1/\sqrt{3} \end{pmatrix} \qquad V = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix}$$

$$A^{+} = V\Sigma^{+}U^{T} = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{2} \\ 1/\sqrt{2} & -1/\sqrt{2} \end{pmatrix} \begin{pmatrix} 1/\sqrt{3} & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} U^{T}$$

$$= \begin{pmatrix} 1/\sqrt{6} & 1/\sqrt{2} & 0 \\ 1/\sqrt{6} & -1/\sqrt{2} & 0 \end{pmatrix} \begin{pmatrix} 2/\sqrt{6} & 1/\sqrt{6} & 1/\sqrt{6} \\ 0 & 1/\sqrt{2} & -1/\sqrt{2} \\ -1/\sqrt{3} & -1/\sqrt{2} & 1/\sqrt{3} \end{pmatrix}$$

$$= \boxed{\begin{pmatrix} 1/3 & 2/3 & -1/3 \\ 1/3 & -1/3 & 2/3 \end{pmatrix}}$$

※ Hermition Matrix :

→ A square matrix $A_{n\times n}$ is said to be Hermitian iff $\boxed{A^* = A}$

Q. $A = \begin{pmatrix} 1 & 2+i \\ 2-i & i \end{pmatrix}$ Hermitiam ?

→ $A^* = \bar{A}^T$

$= \begin{pmatrix} 1 & 2-i \\ 2+i & -i \end{pmatrix}^T = \boxed{\begin{pmatrix} 1 & 2+i \\ 2-i & -i \end{pmatrix} \neq A}$ ✗

(ii) $A = \begin{pmatrix} 3 & 1+i \\ 1-i & 4 \end{pmatrix}$

$A^* = \begin{pmatrix} 3 & 1-i \\ i+i & 4 \end{pmatrix}^T = \boxed{\begin{pmatrix} 3 & 1+i \\ 1-i & 4 \end{pmatrix}}$ ✓

※ Unitary Matrix :

→ A square matrix is unitary if $U^* = U^{-1}$.

$$U^*U = UU^* = I \ ?$$

→ A square complex matrix A is called unitarily diagonizable if there exists a unitary matrix U and a diagonal matrix D s.t.

$$\boxed{D = U^*AU}$$

Step 1 : Compute the eigen-values of the matrix A.

Step 2 : Find the basis for each eigenspace.

Step 3 : Ensure that each eigen space contain orthonormal eigen vectors. [Use Gram-Schmidt Process if needed]

Step 4 : Find the matrix U whose columns are orthogonal eigenvectors.

Q: Find the unitary matrix U and diagonal matrix D such that $U^*AU = D$.

(a) $A = \begin{pmatrix} 2 & i \\ -i & 2 \end{pmatrix}$

$\det(A - \lambda I) = 0$

$\therefore \begin{vmatrix} 2-\lambda & i \\ -i & 2-\lambda \end{vmatrix} = 0$

$\therefore (2-\lambda)^2 + i^2 = 0$

$\lambda^2 - 4\lambda + 4 - 1 = 0$

$\therefore (\lambda-3)(\lambda-1) = 0$

$\underline{\underline{\lambda_1 = 3}} \quad \underline{\underline{\lambda_2 = 1}}$

For $\lambda_1 = 3$

$\begin{pmatrix} -1 & i \\ -i & -1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$\therefore \boxed{x_1 = ix_2}$ & $\boxed{x_2 = -ix_1}$

$\therefore E_1 = \left\{ t\begin{pmatrix} i \\ 1 \end{pmatrix} \middle| \, t \in \mathbb{C} \right\}$

For $\lambda_2 = 1$

$\begin{pmatrix} 1 & i \\ -i & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$

$\therefore x_1 + ix_2 = 0$

$$\Rightarrow \quad E_2 = \left\{ t\begin{pmatrix} -i \\ 1 \end{pmatrix} \,\middle|\, t \in \mathbb{C} \right\}$$

$$\Rightarrow \quad \langle v_1, v_2 \rangle = \overline{v_1} \cdot v_2$$

$$= \begin{pmatrix} -i \\ 1 \end{pmatrix} \cdot \begin{pmatrix} -i \\ 1 \end{pmatrix} = (i^2) + 1$$

$$= (-1) + 1 = \boxed{0} \quad \checkmark$$

$$\Rightarrow \quad U = \frac{1}{\sqrt{2}} \begin{pmatrix} i & -i \\ 1 & 1 \end{pmatrix}$$

$$D = U^* A U = \frac{1}{\sqrt{2}} \begin{pmatrix} -i & i \\ 1 & 1 \end{pmatrix}^T \begin{pmatrix} 2 & i \\ -i & 2 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix} \times \frac{1}{\sqrt{2}}$$

$$= \frac{1}{2} \begin{pmatrix} -i & 1 \\ i & 1 \end{pmatrix} \begin{pmatrix} 2 & i \\ -i & 2 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}$$

$$= \frac{1}{2} \begin{pmatrix} -3i & 3 \\ i & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}$$

$$D = \frac{1}{2} \begin{pmatrix} 6 & 0 \\ 0 & 2 \end{pmatrix} = \boxed{\begin{pmatrix} 3 & 0 \\ 0 & 1 \end{pmatrix}}$$

Ans

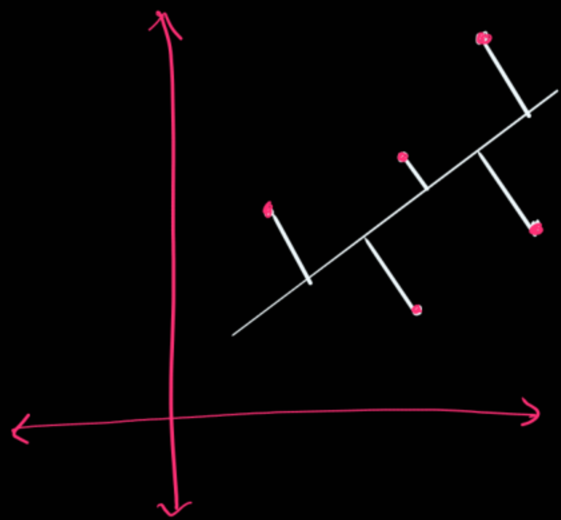
※ Principal Component Analysis :

→ Suppose you have a dataset of n points in a d-dimensional space.

$x_1, x_2, \ldots, x_n \in R^d$

→ Aim of PCA is to find a linear dimensional space that will approximate data.

$$\boxed{d_i = \frac{|ax_i + by_i + c|}{\sqrt{a^2+b^2}}}$$



$$f(a,b,c) = \sum_{i=1}^{n} d_i^2$$

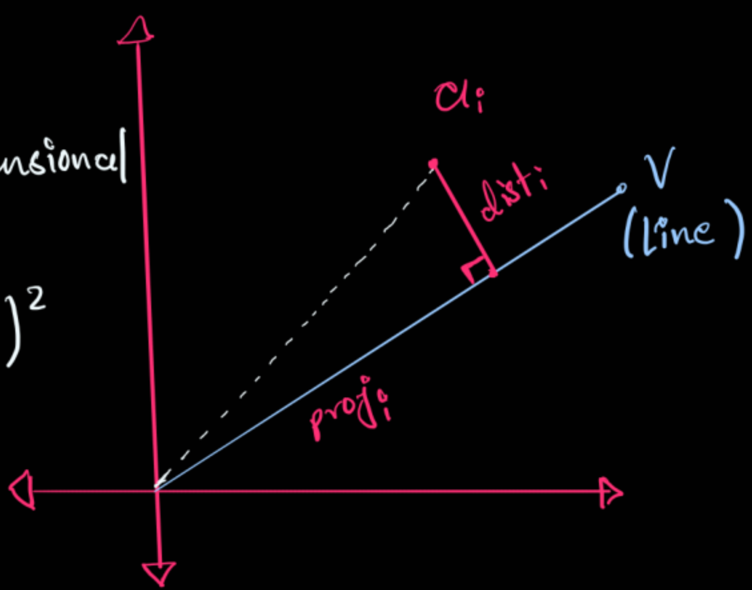
We want to find the line (a,b,c) such that the total sum of squared distanceed $f(a,b,c)$ is minimised.

※ Bestfit Subspace :

Let a_i be a point in d-dimensional space.

$(dist_i)^2 = a_{i_1}^2 + a_{i_2}^2 + \cdots + a_{i_d}^2 - (proj_i)^2$

$[\because$ Pythagoras$]$



Minimisation of $\sum_{i=1}^{n} (\text{dist}_i)^2 \Rightarrow$ maximisation of $(\text{proj}_i)^2$

Proj_i = length of projection of i^{th} row of A onto V.

$\Rightarrow$ Let A be $n \times d$ matrix where n rows are d-dimensional points of A. Singular vector V of A is a unit vector along its best fit line through the origin for the points of A.

Q. $A = \begin{pmatrix} \boxed{3/5 \quad 4/5} \\ \boxed{6 \quad 8} \\ \boxed{3 \quad 4} \end{pmatrix} \begin{array}{l} \rightarrow a_1 \\ \rightarrow a_2 \\ \rightarrow a_3 \end{array}$

Find the outer product representation of A.

Sol^n:

$$A^TA = \begin{pmatrix} 3/5 & 6 & 3 \\ 4/5 & 8 & 4 \end{pmatrix} \begin{pmatrix} 3/5 & 4/5 \\ 6 & 8 \\ 3 & 4 \end{pmatrix} = \boxed{\begin{pmatrix} 45.36 & 60.48 \\ 60.48 & 80.64 \end{pmatrix}}$$

Eigenvalues:

[Best fit Subspace = Span(V_1)]

$\det(A^TA - \lambda I) = 0$

$\therefore (45.36 - \lambda)(80.64 - \lambda) - (60.48)^2 = 0$

$[(60.48)^2 = 45.36 \times 80.64]$

$\therefore \boxed{\lambda = 0, 126}$ $\quad (126 = 80.64 + 45.36)$

$\hookrightarrow \sigma = \sqrt{126} = \boxed{11.225}$

(Skip this type of questions, calc. is not allowed)

∴ Final ans:

$$A = \sigma_1 U_1 V_1^T + \cdots + \sigma_r U_r V_r^T$$

$$\therefore \boxed{A = (11.225)\begin{pmatrix} 0.089 \\ 0.89 \\ 0.445 \end{pmatrix}\begin{pmatrix} 3/5 & 4/5 \end{pmatrix}}$$

$\rightarrow$ $proj_1 = |a_1 \cdot v| = (3/5 \quad 4/5) \cdot (3/5 \quad 4/5)$

$$= 9/25 + 16/25 = \boxed{1}$$

$\rightarrow$ $proj_2 = |a_2 \cdot v_1| = (6 \quad 8) \cdot (3/5 \quad 4/5) = \boxed{10}$

$\rightarrow$ $proj_3 = |a_3 \cdot v_1| = (3 \quad 4) \cdot (3/5 \quad 4/5) = \boxed{5}$

$$|Av_1|^2 = \left|\begin{pmatrix} 1 \\ 10 \\ 5 \end{pmatrix}\right|^2 = 1^2 + 10^2 + 5^2$$

$$= 126$$

$$\therefore \boxed{|Av_1| = 11.225}$$

Bestfit subspace $= \text{span}\left\{\begin{pmatrix} 3/5 \\ 4/5 \end{pmatrix}\right\}$

Steps:

(1) First singular vector (also called as right singular vector) v_1 of A.

$$\boxed{v_1 = \arg\max_{|v|=1} |Av|}$$

$\rightarrow$ meaning

$v_1 = v$ for which $|Av|$ is maximum

(2) First singular value $\delta_1(A)$ is defined as:

$$\delta_1(A) = |Av_1|$$

③ Finally,

Find a unit vector $V_2 \perp^r$ to V_1 that maximizes $|AV|^2$

$$\boxed{V_2 = \underset{\substack{V \perp V_1 \\ |V|=1}}{\arg\max} |AV|}$$

④ Repeat that for V_r perpendicular to $V_{r-1}, V_{r-2}, \dots, V_1$

until

$$\boxed{\underset{\substack{V \perp V_1, V_2, \dots, V_r \\ |V|=1}}{\arg\max} |AV| = 0}$$