NESS - Midsem

(1) (2.5 marks) **Linearity :**

$$y_1 = \sum_{n\in Z} \alpha x_1(nT)\,\delta(t-nT) = \alpha \sum_{n\in Z} x_1(nT)\,\delta(t-nT)$$

$$y_2 = \sum_{n\in Z} \beta x_2(nT)\,\delta(t-nT) = \beta \sum_{n\in Z} x_2(nT)\,\delta(t-nT)$$

Now, we have $y(t)$ as the output when $x(t)$ is our input.

For $\alpha x_1(t) + \beta x_2(t)$ as the input, we have

$$y'(t) = \sum_{n\in Z} (\alpha x_1(nT) + \beta x_2(nT))\,\delta(t-nT)$$

$$= \sum_{n\in Z} \alpha x_1(nT)\,\delta(t-nT) + \sum_{n\in Z} \beta x_2(nT)\,\delta(t-nT)$$

$$= \alpha \sum_{n\in Z} x_1(nT)\,\delta(t-nT) + \beta \sum_{n\in Z} x_2(nT)\,\delta(t-nT)$$

$$= \alpha y_1(t) + \beta y_2(t).$$

$\therefore$ it is linear.

Time-Invariance : Consider input as $x(t-t_0)$.

$$y'(t) = \sum_{n\in Z} x(nT-t_0)\,\delta(t-nT)$$

$$y(t-t_0) = \sum_{n\in Z} x(nT)\,\delta(t-t_0-nT).$$

$y'(t) \neq y(t-t_0)$. $\therefore$ not time-invariant.

$$LT\{\delta(t-t_0)\} = \int_{-\infty}^{\infty} \delta(t-t_0)\, e^{-st}\, dt.$$

Sifting property: $\int_{-\infty}^{\infty} f(t)\,\delta(t-t_0)\,dt = f(t_0)$

$$\boxed{LT\{\delta(t-t_0)\} = e^{-st_0}}$$

Given : $x(t) = e^{-t}\,u(t)$.

$$y(t) = \sum_{n=-\infty}^{\infty} e^{-nT}\,\delta(t-nT)\,u(nT).$$

For $n<0$, $u(nT)=0$.

$$\therefore \quad y(t) = \sum_{n=0}^{\infty} e^{-nT}\,\delta(t-nT).$$

$$LT\{y(t)\} = \int_{-\infty}^{\infty} \left(\sum_{n=0}^{\infty} e^{-nT}\,\delta(t-nT)\right) e^{-st}\,dt$$

$$= \sum_{n=0}^{\infty} \int_{-\infty}^{\infty} e^{-nT}\,\delta(t-nT)\,e^{-st}\,dt.$$

$$= \sum_{n=0}^{\infty} e^{-nT} \underbrace{\int_{-\infty}^{\infty} e^{-st}\,\delta(t-nT)\,dt}_{LT\{\delta(t-nT)\} = e^{-snT}}.$$

$$= \sum_{n=0}^{\infty} e^{-n(s+1)T} \rightarrow \infty \text{ series geometric progression}$$

$$LT\{y(t)\} = \frac{1}{1-e^{-(s+1)T}}, \quad |e^{-(s+1)T}| < 1$$

$$\Rightarrow R(s+1) > 0 \Rightarrow R(s) > -1$$

$$\therefore \quad \boxed{LT\{y(t)\} = \frac{1}{1-e^{-T(s+1)}}, \quad R(s) > -1}$$

Ans-2

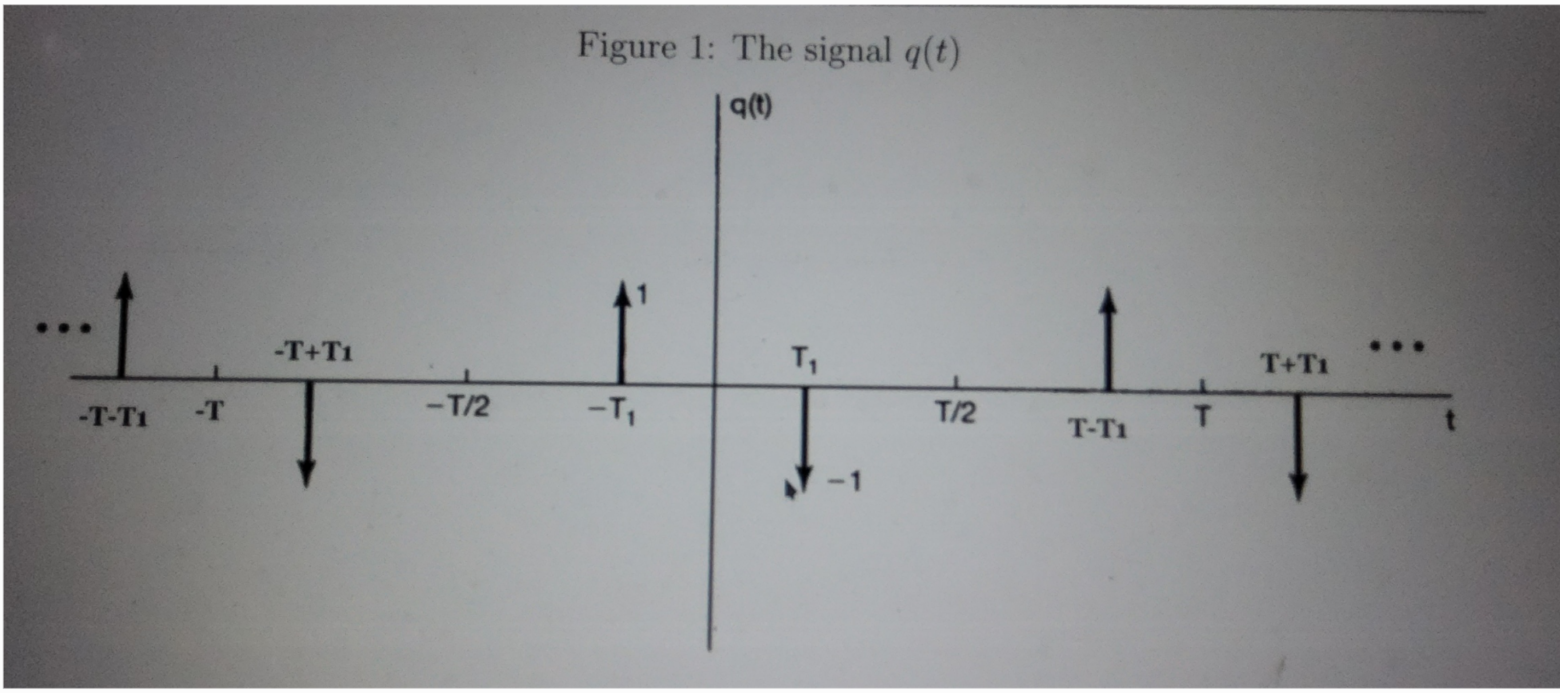


Figure 1: The signal $q(t)$

(a)

Fundamental period of a periodic signal is the smallest positive non-zero T such that signal satisfies :-

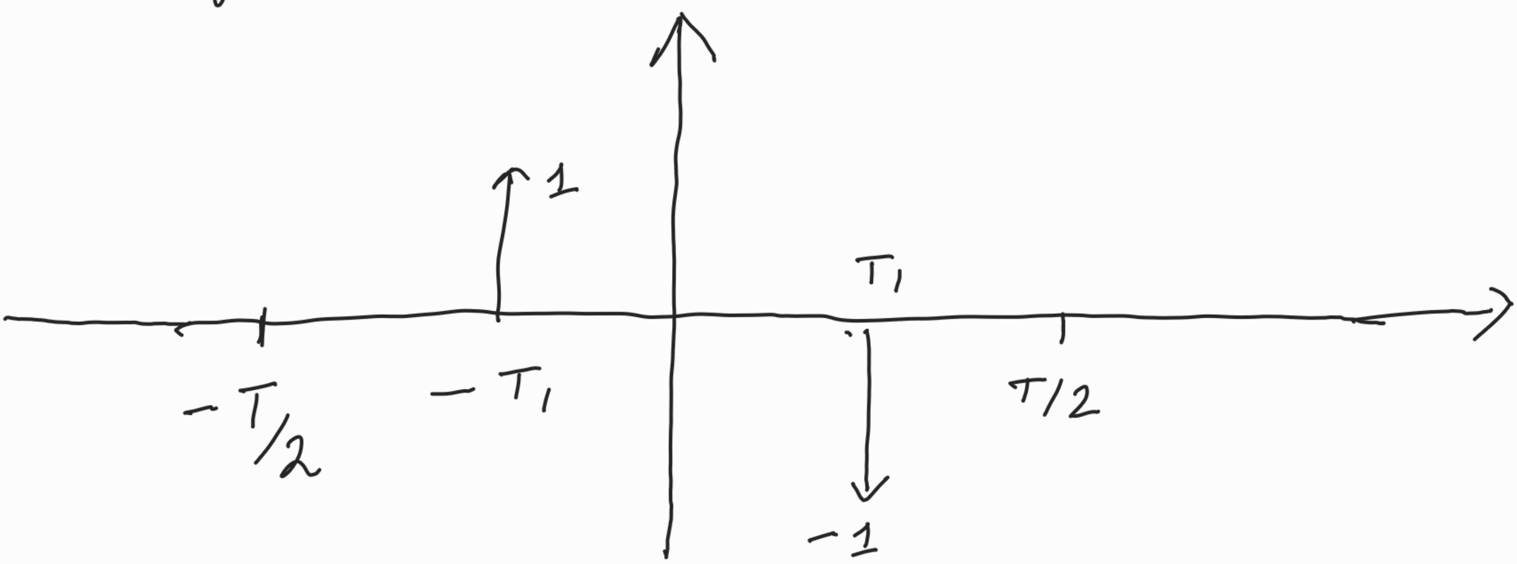
$$q(t+T) = q(t) \quad \forall\, t$$

→ In other words, signals repeats itself every T units of Time.

To prove T is fund. period, we need to show there's no smaller positive value that satisfied the periodicity condition.

→ The interval B/w 2 successive positive impulses is T.

→ The interval B/w 2 successive negative impacts is T.



$$\boxed{q(t) = \delta(t+T_1) - \delta(t-T_1), \quad -T/2 \le t < T/2}$$

↳ This is how we can represent $q(t)$ for a single period

For any time,

$$q(t+T) = q(t)$$

Because,

- If t corrosponds to a impulse, $t+T$ will corrospends to the same impulse in the next period.
- If t is between, $t+T$ will be at same relative position B/w impulses in the next period.

→ Any smaller value than T would not capture the full pattern of the signal, so we can conclude T is fundamental period of signal $q(t)$.

(b)

Fourier series for a periodic signal x(t) is written as -

$$x(t) = a_0 + \sum_{k=1}^{\infty} a_k \cos(k\omega_0 t) + b_k \sin(k\omega_0 t)$$

$$a_0 = \frac{1}{T}\int_{\langle T \rangle} q(t)\,dt$$

$$a_0 = \frac{1}{T}\int_{-T/2}^{T/2}\left[\delta(t+T_1) - \delta(t-T_1)\right]dt$$

$$a_0 = \frac{1}{T}\left[1 - 1\right]$$

$\boxed{a_0 = 0}$ → [DC component $a_0 = 0$ since area of Both impulse is 1]

$$a_K = \frac{2}{T} \int_{\langle T \rangle} q(t) \cos(K \omega_0 t)\, dt$$

$$= \frac{2}{T} \int_{-T/2}^{T/2} \left[\delta(t + T_1) - \delta(t - T_1) \right] \cos(K \omega_0 t)\, dt$$

$$\left[\because \int \delta(t - t_0) f(t)\, dt = f(t_0) \right]$$

$$= \frac{2}{T} \left[\cos(-K \omega T_1) - \cos(K \omega T_1) \right]$$

$$= \frac{2}{T} \left[\cos(K \omega T_1) - \cos(K \omega T_1) \right]$$

$$\boxed{a_K = 0 \quad \forall\ K}$$

$$b_k = \frac{2}{T} \int_{-T/2}^{T/2} q(t) \sin(k\omega t)\, dt$$

$$= \frac{2}{T} \int_{-T/2}^{T/2} \left[\delta(t+T_1) - \delta(t-T_1) \right] \sin(k\omega t)\, dt$$

$$= \frac{2}{T} \left[\sin(-k\omega T_1) - \sin(k\omega T_1) \right]$$

$$\boxed{b_k = \frac{-4}{T} \sin(k\omega T_1) \quad \forall\ k}$$

Exponential form

$$c_n = \frac{1}{T}\int_{\langle T\rangle} q(t)e^{-jn\omega t}\,dt$$

$$c_n = \frac{1}{T}\int_{-T/2}^{T/2}\left[\delta(t+T_1) - \delta(t-T_1)\right] e^{-jn\omega t}\,dt$$

$$\left[\because \delta(t-t_0)\,f(t) = f(t_0)\right]$$

$$c_n = \frac{1}{T}\left[\left(e^{jn\omega T_1} - e^{-jn\omega T_1}\right)\right]$$

$$\boxed{c_n = \frac{1}{T}\left[2j\sin(n\omega T_1)\right]}$$

(2)

(c) $x(t) \longrightarrow a_k$.

$\frac{d}{dt} x(t) \longrightarrow\ ?\ (a_k')$

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{jk\omega_0 t}.$$

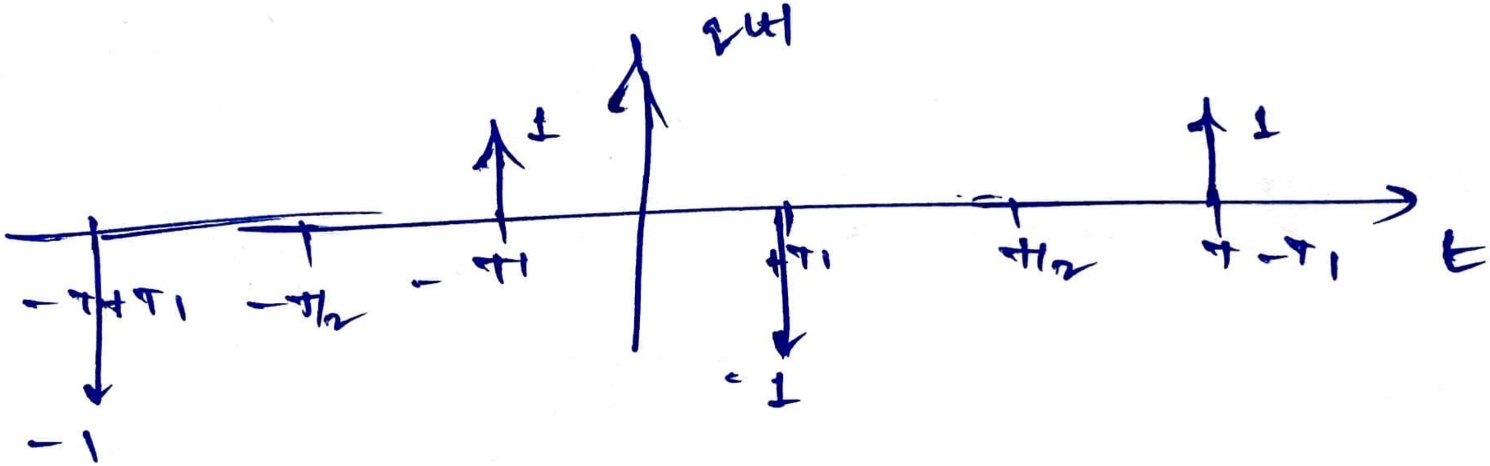
$$\Rightarrow \frac{d}{dt} x(t) = \sum_{k=-\infty}^{\infty} a_k \frac{d}{dt} \left(e^{jk\omega_0 t}\right)$$

$$= \sum a_k e^{jk\omega_0 t} [jk\omega_0]$$

$$= jk\omega_0 \left(\sum a_k e^{jk\omega_0 t}\right)$$

$$\boxed{a_k' = jk\omega_0 a_k.}$$

(d) $y(t) = \int_{\tau=-\infty}^{t} q(\tau)\, d\tau$

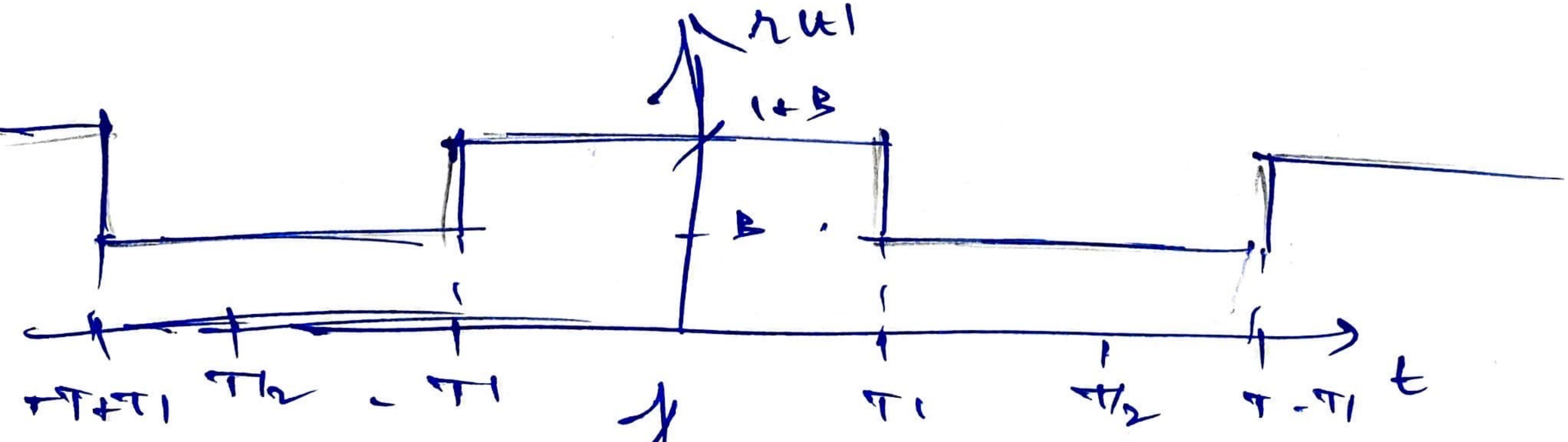


Consider the interval $-T/2 \longrightarrow T/2$.

To compute integral in this region, we need to know the value of summation of impulse from $-\infty$ to $-T/2$.

Assume that this summation value = B
where $B \in \mathbb{R}$.

So, $q(t)$ is train of impulses and $r(t)$ will be "B + square pulse" of amplitude 1



Amplitude | Height of pulse = 1.

(#) NOTE : For evaluation, any specific value of B is also accepted i.e it needn't be generalised.

$$r(t) = \int_{-\infty}^{t} q(s)\,ds$$

$$\Rightarrow \frac{d}{dt}\underbrace{(r(t))}_{A_k} = \underbrace{q(t)}_{B_k}$$

$$\Rightarrow B_k = A_k(jk\omega_0) \quad \text{[From part (c)]}$$

$$\Rightarrow \boxed{A_k = \frac{B_k}{jk\omega_0}}$$

So, F.S coefficients of $r(t)$ = $\dfrac{\text{F.S. coefficients of } q(t)}{jk\omega_0}$

For $r(t)$:

If $r(t)$ is assumed to be "B + square wave"

then, $\boxed{a_0 = B}$

But $\boxed{a_0 = \frac{0}{j(0)\omega_0} = \text{Undefined}}$ is also accepted.

From part (b)

$$\boxed{a_k = \frac{0}{jk\omega_0} = 0}$$

$$\boxed{b_k = \frac{-\frac{4}{T}\sin(k\omega_0 T_1)}{jk\omega_0}}$$

3(a). $x(t) = \sin(300\pi t) + \cos(500\pi t) + \cos(2000\pi t)$

$$= \frac{e^{j300\pi t} - e^{-j300\pi t}}{\cancel{4}2j} + \frac{e^{j500\pi t} + e^{-j500\pi t}}{2} + \frac{e^{j2000\pi t} + e^{-j2000\pi t}}{2}$$

We know that $e^{j\omega t}$ are eigenfunctions of LTI systems, meaning the output for an input of $e^{j\omega t}$ will be $H(j\omega)e^{j\omega t}$.

$$\Rightarrow \cancel{X(s)}\ H(j300\pi) = \frac{1}{j300\pi - 2} - \frac{1}{j300\pi + 3}$$

$$H(j500\pi) = \frac{1}{j500\pi - 2} - \frac{1}{j500\pi + 3}$$

$$H(j2000\pi) = 0.$$

Thus, we can say,

$$y(t) = \frac{1}{2j}\left[H(j300\pi)e^{j300\pi t} - H(-j300\pi)e^{-j300\pi t}\right]$$

$$+ \frac{1}{2}\left[H(j500\pi)e^{j500\pi t} + H(-j500\pi)e^{-j500\pi t}\right].$$

3) b)

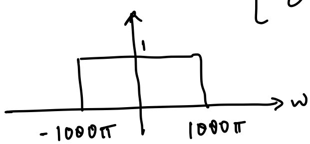
$$H(s) = \begin{cases} \frac{1}{s-2} - \frac{1}{s+3} & |w| \leq 1000\pi \\ 0 & \text{otherwise} \end{cases}$$

$$ROC(H(s)) \to \{ s \in \mathbb{C}, -3 < \sigma < 2 \}$$

Since, $s = \sigma + jw$

We can write, $H(s) = \left(\frac{1}{s-2} - \frac{1}{s+3} \right) \cdot rect\left(\frac{w}{2000\pi} \right)$

here, $rect\left(\frac{w}{1000\pi}\right) = \begin{cases} 1 & -1000\pi \leq w \leq 1000\pi \\ 0 & \text{otherwise} \end{cases}$ (or) $rect\left(\frac{w}{W}\right) = \begin{cases} 1 & -\frac{W}{2} \leq w \leq \frac{W}{2} \\ 0 & \text{otherwise} \end{cases}$



An idea than could be used here is to find the inverse Laplace transform for the values of σ in ROC.

$$\therefore h(t) = \mathcal{L}^{-1}\left(\left(\frac{1}{s-2} - \frac{1}{s+3} \right) \cdot rect\left(\frac{w}{2000\pi} \right) \right)$$

We know, $\mathcal{L}(x(t) * y(t)) = X(s) \cdot Y(s)$

$$\therefore h(t) = \underbrace{\mathcal{L}^{-1}\left(\left(\frac{1}{s-2} - \frac{1}{s+3} \right) \right)}_{\text{From what was taught in class the inverse can be found}} * \underbrace{\mathcal{L}^{-1}\left(rect\left(\frac{w}{1000\pi} \right) \right)}_{\text{The inverse of the rect function can be found in textbook.}}$$

$\mathcal{L}^{-1}\left(rect\left(\frac{w}{W}\right)\right) \to \frac{W}{2\pi} \, sinc\left(\frac{Wt}{2}\right)$

↳ Essentially the IFT since $\sigma = 0$