MOUNTAINS OF THE MOON UNIVERSITY

$$\forall \tau \in \mathbb{R}\ \exists \delta : \delta = \mu\tau$$

MAT1201: REAL ANALYSIS

FOR SECODARY TEACHERS

ABOUT THE AUTHOR

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Published Research Work by the Author

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These lecture notes were prepared to serve and facilitate the teaching of Bachelor Science with Education at Mountains of the Moon University. The notes are also relevant to students offering Computer Science.

The Author welcomes the readers to the content of the course unit that introduces the mathematical logical thinking and proving for the existence of certain mathematical concepts. The contribution of different Authors whose work was used in compiling this hand book is appreciated. Students are encouraged to read these notes carefully and internalize the necessary concepts for use in the subsequent course units. (For any error please notify the Author at ndungoissa@mmu.ac.ug).

Wishing you all the best, enjoy Real Analysis.

PREFACE

This course unit introduces students to the concepts of mathematics that are the building blocks of mathematical reasoning and mathematical proofs. The course unit handles concepts such as logic, methods of proof, sets, functions, real number properties, sequences and series, limits and continuity and differentiation. Real analysis provides students with the basic concepts and approaches for internalising and formulation of mathematical arguments. The course unit is aimed at:

- Providing learners with the knowledge of building mathematical statements and constructing mathematical proofs.
- Giving learners an insight on the concepts of sets and the relevant set theories that are vital in the development of mathematical principles.
- Demonstrating to learners the concepts of sequences and series with much emphasis on the bound and convergence of sequences and series.
- Providing students with the knowledge of limits, continuity and differentiation of functions that will serve as an introduction to calculus.

By the end of the course unit, students should be able to:

- Construct truth tables to prove mathematical statements or propositions
- Use relevant methods of proof in constructing proofs of simple mathematical principles
- Operate sets, proof basic set principles and have ability to explain the set concepts such as closure of a set, boundary point, open set and neighborhood of a point.
- State and prove the axioms of real numbers and use the axioms in explaining mathematical principles and definitions.
- Construct proofs of theories involved in sequences such as convergent, boundedness, and Cauchy properties as well as showing understanding of the connection between bondedness and convergent.
- Obtain the limit of a function, construct relevant proofs for the existence of limits and perform algebra on limits.
- State and prove the rules of differentiations and show understanding of the application of the concept of differentiation and the connection between limits, continuity and differentiation.
  The course will be delivered through: (1) three hours of lecture per session (2) a combination of lectures, discussions and presentations. Students will be given lecture notes on each unit but students will be required to make use of the university E-library for personal reading when answering the assignments.

The course is will be assessed through: (1) Course Work Assessment (class exercises, assignments & tests (2) End of semester examination. The pass mark for this course unit is **50%.**

Table of Contents

ABOUT THE AUTHOR

PREFACE III

UNIT ONE: LOGIC AND METHODS OF PROOF 1

1.1 PROPOSITIONS 1
1.2 CONNECTIVES 1
EXERCISE 1.1 2
1.3 TRUTH TABLES AND TRUTH VALUES 2
EXERCISE 1.2 4
1.4 TAUTOLOGY, CONTRADICTIONS AND EQUIVALENCE 4
EXERCISE 1.3 4
1.5 OPEN SENTENCE AND QUANTIFIES 5
EXERCISE 1.4 5
1.6 NEGATION OF A QUANTIFIER 5
1.7 OVERGENERALIZATION AND COUNTEREXAMPLES 6
1.8 METHODS OF PROOF IN MATHEMATICS 6
EXERCISE 1.5 8
EXERCISE 1.6 9

UNIT TWO SETS AND FUNCTIONS 10

2.1 DEFINITIONS ABOUT SETS 10
2.2 INTERVAL AND INEQUALITIES 11
2.3 OPERATIONS ON SETS 11
EXERCISE 2.1 11
2.4 INDEXED FAMILIES OF SETS 13
EXERCISE 2.2 13
2.5 FUNCTIONS 13
EXERCISE 2.3 14
2.6 CARDINALITY: THE SIZE OF A SET 14
2.7 SETS OF REAL NUMBERS 14
2.8 ORDER ON SETS AND ORDERED SETS 14
2.9 BOUNDED SETS 15
EXERCISE 2.4 15
2.10 NEIGHBORHOODS 15
2.11 TYPES OF POINTS FOR SETS 16
EXERCISE 2.5 16
2.12 OPEN SETS AND TOPOLOGY 16
2.13 CLOSED SET 17
EXERCISE 2.6 17

UNIT THREE REAL NUMBERS AND THEIR PROPERTIES 18

3.1 REAL NUMBERS 18
3.2 AXIOMS OF REAL NUMBERS 18
3.2.1 THE FIELD AXIOMS 18
EXERCISE 3.1 20
3.2.2 THE ORDER AXIOM 20
3.2.3 THE COMPLETENESS AXIOM 21
3.3 THE ARCHIMEDEAN PROPERTY OF REAL NUMBERS 22
3.4 THE EUCLIDEAN SPACE 22

UNIT FOUR: SEQUENCES AND SERIES 24

4.1 SEQUENCES 24
EXERCISE 4.1 24
4.2 BOUNDED SEQUENCES 24
4.3 CONVERGENT AND DIVERGENT SEQUENCES 25
EXERCISE 4.2 27
EXERCISE 4.3 29
4.4 ALGEBRA OF LIMITS OF SEQUENCES 29
EXERCISE 4.4 30
4.5 MONOTONE SEQUENCE 30
4.6 SUBSEQUENCES 32
4.7 CAUCHY SEQUENCES 33
EXERCISE 4.5 35
4.8 INFINITE SERIES 36
4.9 GEOMETRIC SERIES 37
4.10 PROPERTIES OF INFINITE SERIES 38
4.11 CONVERGENT CRITERION FOR SERIES 38
4.11.1 LIMIT OF nth TERM TEST FOR DIVERGENCE 38
4.11.2 INTEGRAL TEST 38
4.11.3 P-SERIES TEST 40
EXERCISE 4.6 40
4.11.4 COMPARISON TEST 40
4.11.5 ALTERNATING TEST 42
4.11.6 ABSOLUTE CONVERGENCE 43
4.11.7 RATIO TEST 43
4.11.8 ROOT TEST 43

UNIT FIVE LIMITS AND CONTINUITY 44

5.1 LIMIT OF A FUNCTION 44
EXERCISE 5.1 46
5.2 ALGEBRA OF LIMITS 47
EXERCISE 5.2 48
5.3 CONTINUITY OF FUNCTIONS 49

EXERCISE 5.3 51
5.4 THE INTERMEDIATE VALUE THEORY 51
5.5 UNIFORM CONTINUITY 52
EXERCISE 5.4 52
5.6 DISCONTINUITY 52

UNIT SIX: DIFFERENTIATION 54

6.1 DERIVATIVE OF A FUNCTION 54
6.2 RULES OF DIFFERENTIATION 55
6.4 GENERALIZATION OF MEAN VALUE THEOREM (MVT) 56
6.5 L'HOSPITAL RULE 56
EXERCISE 6.1 56
REFERENCES 57

UNIT ONE

LOGIC AND METHODS OF PROOF

1.1 Propositions
A proposition (statement) is a sentence that is either true or false (but not both).

Examples of propositions are:

(1) John was born on 20th February 2019. (2) 3+6 = 11. (3)$\sqrt{2}$ is irrational.

Examples of non-propositions are:

(1) What is the date today? (2) How old are you? (3) This statement is true.

There are two types of statements/propositions:

1. Atomic/simple propositions: These cannot be divided into smaller propositions.
   We use capital letters to denote simple propositions.
   Examples include: (1) John's leg is broken (2) 5 is a prime number (3) $\sqrt{2}$ is irrational
2. Compound propositions: These can be broken down into smaller propositions. They are constructed by using connectives.
   Examples include: (1) $3 \leq 7$ (2) n^2 is odd whenever n is an odd integer

1.2 Connectives
Connectives are symbols used to construct compound statements/propositions from simple statements.

The most commonly used connectives are:

	Connective	English Meaning	Symbol
1	Conjunction	and/but/yet, although	$\wedge$
2	Disjunction	or	$\vee$
3	Implication	If … then	$\Rightarrow$
4	Biconditional	If and only iff	$\Leftrightarrow$
5	Negation	Not	$\neg$

Examples

1) $P\wedge Q$ means P and Q. 2) $P\Rightarrow Q$ means if P then Q. 3) $P\vee Q$ means Por Q

Example 1.1

Write the following compound statements into symbolic form

1) Jim is a lawyer but he is not a crook
   Solution
   Let P = Jim is a Lawyer and Q = Jim is a crook
   Then we have $P\wedge\neg Q$
2) Although our Professor is young but he is knowledgeable
   Let R = our Professor is young and S = our Professor is knowledgeable
   Then we have $R\wedge S$.

3) If you don't attend class, then you either read a book or you will fail the exam.
   Let P = you attend class Q = you read a book and R = you pass the exam
   Then we have $\neg P \Rightarrow (Q \vee \neg R)$.
   NB. Brackets are use for punctuations.
4) If Lucy has a credit in MAT1 or has a credit in MAT2 and MAT3, then she does MAT7.
   Let P = Lucy has a credit in MAT1, Q= Lucy has a credit in MAT2, R= Lucy has a credit in MAT3 and S = Lucy does MAT7
   Then we have $P \vee (Q \wedge R) \Rightarrow S$
5) The lights are on if and only if John or Mary is at home.
   Let R = the lights are on, S= John is at home and T = Mary is at home
   Then we have $R \Leftrightarrow (S \vee T)$

Exercise 1.1
Write the following compound propositions in symbolic form
1. If I go home and find lunch ready then I will not go to the restaurant
2. Either you pay your rent or I will kick you out of the apartments
3. Jolly will leave home and will not come back
4. If I go to Kampala, I will bring for you biscuits and bread.

1.3 Truth tables and truth values

A truth table is a convenient device to specify all possible truth values of a given atomic or compound propositions. We use truth tales to determine the truth or falsity of a compound statement based on the truth or falsity of its constituent atomic propositions.

We use T or 1 to indicate that a statement is true and F or 0 to indicate that the statement is false. So 1 and 0 are called truth values.

Truth tables for connectives

1. **Conjunction**

Let P and Q be two propositions, the proposition $P \wedge Q$ is called the conjunction of P and Q. $P \wedge Q$ is true if and only if both P and Q are true.

P	Q	$P \wedge Q$
T	T	T
T	F	F
F	T	F
F	F	F

2. **Disjunction**

Let P and Q be two propositions, the proposition $P \vee Q$ is called the disjunction of P and Q. The proposition $P \vee Q$ is true if and only if at least one of the atomic propositions is true. It is only false when both atomic propositions are false.

P	Q	$P \vee Q$
T	T	T
T	F	T
F	T	T

F	F	F

3. **Implication**

Let P and Q be two propositions, the proposition $P \Rightarrow Q$ is called the implication of P and Q. The proposition $P \Rightarrow Q$ simply means that P implies Q. P is called the hypothesis (condition or antecedent) and Q is called the conclusion (consequence). There are many ways of stating that P implies Q.

- If P then Q
- Q if P
- P is sufficient for Q
- Q is necessary for P
- P only if Q
- Q whenever P

$P \Rightarrow Q$ is only false when P is true and Q is false.

P	Q	$P \Rightarrow Q$
T	T	T
T	F	F
F	T	T
F	F	T

4. **Biconditional**

Let P and Q be two propositions, the proposition $P \Leftrightarrow Q$ is called the biconditional of P and Q. It simply means that $P \Rightarrow Q$ and $Q \Rightarrow P$. It is called biconditional because it represents two conditional statements. There are many ways of stating that P implies Q.

- P if and only if Q (P iff Q)
- P implies Q and Q implies P
- P is necessary and sufficient for Q
- Q is necessary and sufficient for P
- P is equivalent to Q

The proposition $P \Leftrightarrow Q$ is true when P & Q are both true and if P & Q are both false

P	Q	$P \Leftrightarrow Q$
T	T	T
T	F	F
F	T	F
F	F	T

5. **Negation**

Let P be a proposition, the proposition $\neg P$ meaning "not P" is used to denote the negation of P. If P is true then $\neg P$ is false and vice versa.

P	$\neg P$
T	F
F	T

Example 1.2

Let P and Q be propositions. Construct the truth table for the proposition $(P\wedge Q)\Rightarrow(P\vee Q)$

Solution

P	Q	$P\wedge Q$	$P\vee Q$	$(P\wedge Q) \Rightarrow P\vee Q$
T	T	T	T	T
T	F	F	T	T
F	T	F	T	T
F	F	F	F	T

Exercise 1.2
Let P, Q and R be propositions. Construct the truth tale for the proposition $\neg(P\wedge Q)\vee R$.

1.4 Tautology, contradictions, and Equivalence

Definition: A compound proposition is a tautology if it is always true irrespective of the truth values of its atomic propositions. If on the other hand a compound statement is always false regardless of its atomic propositions, we say that such proposition is a contradiction. Theorems in mathematics are always true and are examples of tautologies.

Example 1.3

The statement $P\vee\neg P$ is always true while the statement $P\wedge\neg P$ is always false

P	**$\neg$P**	**$P\vee\neg P$**	**$P\wedge\neg P$**
T	F	T	F
F	T	T	F

Exercise 1.3

1. Let P, Q and R be proposition. Use truth tales to prove that:

a) $\neg(P\wedge Q)\equiv P\vee\neg Q$
b) $\neg(P\vee Q)\equiv \neg P\wedge\neg Q$
c) $P\Rightarrow Q\equiv \neg P\vee Q$
d) $\neg(P\Rightarrow Q)\equiv P\wedge\neg Q$
e) $P\wedge(Q\vee R)\equiv(P\wedge Q)\vee(P\wedge R)$
f) $(P\vee Q)\vee R\equiv P\vee(Q\vee R)$

Hint: To prove for example a), show that $\neg(P\wedge Q)\Rightarrow P\vee\neg Q$ is a tautology (always true).

2. Use mathematical logic to discuss the validity of the following argument:
   If girls are beautiful, they are popular with boys. Ugly girls are unpopular with boys. Intellectual girls are ugly. Therefore, beautiful girls are not intellectual.

1.5 Open sentence and quantifies

Definition: An open sentence (also called a predicate) is a sentence that contains variable(s) and whose truth or falsity depend(s) on the value(s) assigned for the variable(s). We represent open sentences with capital letters followed by the variable(s) in the bracket. Eg $P(x)$, $Q(x, y)$

Definition: The collection of all allowed values of the variable(s) in an open sentence is called the universe if discourse.

Examples of open sentence are:

1) $x + 4 \leq 9$ 2) It has 4 legs 3) $x < y$

Universal quantifier-($\forall$): To say that $P(x)$ is true for all x in the universe of discourse we write $(\forall x)P(x)$. $\forall$ is called the universal quantifier.

$$\forall \text{ means } \begin{cases} \text{all} \\ \text{for all} \\ \text{for every} \\ \text{for each} \end{cases}$$

Existential quantifier ($\exists$): To say that there is (at least one) x in the universe of discourse for which $P(x)$ is true we write $(\exists x)P(x)$. $\exists$ is called the existential quantifier.

$$\exists \text{ means } \begin{cases} \text{there is} \\ \text{there exist} \\ \text{for some} \end{cases}$$

NB: Quantifying an open sentence makes it a proposition.

Exercise 1.4

1. Translate the following statements using word notation
   a) $(\exists y)(\forall x)P(x, y)$ b) $(\exists x)\neg P(x)$ c) $(\exists x)(\exists y)P(x, y)$ d) $(\forall x)(\exists y)P(x, y)$
2. Write the following statements using qualifiers

a) For each real number $x>0$, $x^2+x-6 = 0$
b) There is a real number $x>0$ such that $x^2+x-6 = 0$
c) The square of any real number is non-negative
d) For each integer x, there is an integer y such that $x+y = -1$
e) There is an integer x such that for each integer y, $x+y = -1$

1.6 Negation of a quantifier

To negate a statement that involves the quantifiers $\forall$ and $\exists$, change $\forall$ to $\exists$ and $\exists$ to $\forall$ then negate the open sentence.

Examples

1) $\neg[(\forall x)P(x)] \equiv (\exists x)\neg P(x)$
2) $\neg$[all birds can fly]$\equiv$ [there is at least one bird that does not fly]
3) $\neg[(\forall x)(\exists y)P(x, y)] \equiv (\exists x)(\forall y)\neg P(x, y)$

1.7 Overgeneralization and Counterexamples

Overgeneralization occurs when a pattern searcher discovers a pattern among finitely many cases and then claims that the pattern is general (when in fact it is not). To disprove a general statement such as $(\forall x)P(x)$, we must exhibit one x for which P(x) is false. i.e. $(\exists x)\neg P(x)$.

Examples

1) Statement $(\forall x \in \mathbb{R})(x < x^2)$ is false, x = ½ is a counterexample since $\frac{1}{2} \nless (\frac{1}{2})^2$.
2) For all real numbers x and y |x+y| = |x|+|y|. This statement is false, the counterexample is when x =1 and y = -1 since $|1+-1| \neq |1|+|-1|$ or $0 \neq 2$.
3) For all prime numbers p, 2p+1 is prime. This statement is false, the counterexample is p = 7.

1.8 Methods of proof in Mathematics

A proof is a process of establishing the truth of an assertion, it is a sequence of logical sound arguments which establishes the truth of a statement in question.

1) **Direct Method**

Suppose $P \Rightarrow Q$, in this method we assume that P is true and proceed through a sequence of logical steps to arrive at the conclusion that Q is also true.

Example 1.4

a) Show that if m is an even integer and n is an odd integer then m+n is an odd integer.

Assume that m is an even integer and n is an odd integer.

Then m = 2k and n = $2\ell+1$ for some integer k and ℓ.

Therefore $m + n = 2k + 2\ell + 1 = 2(k + \ell) + 1 = 2d + 1$ for some integer d = $k+\ell$. Since d is an integer then m+n is an odd integer.

b) Show that if n is an even integer, then n^2 is also an even integer.

Assume that n is an even integer, then n = 2k for some integer k.

Now $n^2 = (2k)^2 = 4k^2 = 2(2k)^2 = 2c^2$ for some integer c = 2k. Since c is an integer the n^2 is an even integer.

2) **Contrapositive Method**

Associated with the implication $P \Rightarrow Q$ is the logical equivalent statement $\neg Q \Rightarrow \neg P$, the contrapositive of the conditional statement $P \Rightarrow Q$. So, one way to prove the conditional statement $P \Rightarrow Q$ is to give a direct proof of the contrapositive statement $\neg Q \Rightarrow \neg P$. The first step is to write down the negation of the conclusion then follow it by a series of logical steps that this leads to the negation of the hypothesis of the original conditional statement.

Example 1.5

a) Show that if $n^2$is even integer then n is an even integer.

We will show that if n is odd then n^2 is odd. Assume that n is odd. Then $n = 2k + 1$ for some integer k.

Now $n^2 = (2k+1)^2 = 4k^2 + 4k + 1 = 2(2k^2 + 2k) + 1 = 2d + 1$ for some integer $d = 2k^2 + 2k$. Since d is an integer then n^2 is an odd integer. Thus, by contrapositive if n^2 is even then n is itself even.

b) Show that if 3n is odd integer then n is an odd integer.
By contrapositive, we show that if n is even integer then 3n is an even integer.
Now n = 2p and 3n =3*2p =2(3p) = 2c for some integer c = 3p. Since c is an integer then 3n is even and so by contrapositive, if 3n is odd then n is odd

3) **Contradiction Method**

Assume that P is true and Q is false i.e. P∧¬Q is true. Then show a series of logical steps that leads to a contradiction or impossibility or absurdity. This will mean that the statement P∧¬Q must have been fallacious and therefore, its negation ¬[P∧¬Q] must be true. Since ¬[P⇒Q]≡[P∧¬Q], it follows that [P⇒Q]≡ ¬[P∧¬Q] and hence P⇒Q must be true.

Definition: A real number r is said to be rational if there are integers n and m (m≠0) such that $r = \frac{n}{m}$ with greatest common divisor between [n, m] = 1. We denote the set of rations by $\mathbb{Q}$. A real number that is not rational is said to be irrational.

Example 1.6 (proofs by contradiction)

a) Show that √2 is irrational. That is there is no integer p and q such that $\sqrt{2}=\frac{p}{q}$ or there is no rational number p such that $p^2 = 2$.

By contradiction, we assume that √2 is rational.
So, we have $\sqrt{2}=\frac{p}{q}$. And so $2 = \frac{p^2}{q^2}$ thus $2q^2 = p^2$ ………………….. (*)
From (*), observe that p^2 is even and so p is even and can be written as p = 2c for some integer c.
So (*) becomes: $2q^2 = (2c)^2 = 4c^2$ and so $q^2 = 2c^2$…………………. (**)
Similarly, from (**) it is observed that q^2 is even and so q is itself even. Therefore, both p and q are even integers; implying that they both have a common factor of 2. This contradict the fact that the GCD [p, q] =1 and thus √2 is not rational but rather it is irrational.

b) Show that if 3n is an odd integer then n is an odd integer.
We will use contradiction. Assume that 3n is an odd integer and n is even integer.
Then 3n = 2k+1 and n = 2d for some integers k and d.
Therefore 3n =2k+1 = 2(3d) = 2c for some integer c. Since c is an integer then 3n is an even integer which indicates that 3n is both odd and even which is impossible. Thus, if 3n is odd n cannot be even but rather n is odd

c) If r is rational and x is irrational then prove that r+x and rx are irrational.
Let r+x be rational (by contradiction), then $r+x = \frac{p}{q}$ with $q \neq 0$, GCD [p, q] = 1.

Since r is rational, then $r = \frac{n}{m}$ and then $x = \frac{p}{q} - \frac{n}{m} = \frac{pm-qn}{qm}$ which is rational. This indicates that x is both rational and irrational which is absurd and so r+x is irrational.

Similarly, suppose rx is rational, then rx = $\frac{p}{q}$ for some integers p, q; q≠ 0 and GCD [p, q] =1. Since r is rational then r = $\frac{n}{m}$ and then rx =$\frac{pn}{qm}$ which is rational. This indicate that x is both rational and irrational; which is impossible and thus the assumption that rx is ration is false so rx is irrational

Exercise 1.5

Use the method of contradiction to prove the following:

a) If n is a positive integer which is not perfect square then $\sqrt{n}$ is irrational.

b) $\sqrt{12}$ is irrational c) $\sqrt{7}$ is irrational

4) **Mathematical induction Method**

It is a method that proves statements of the type $\forall\, n \in N$, A(n) holds. Here A(n) is statement depending on n. e.g. $1 + 2 + 3 + 4 + \cdots + n = \frac{1}{2}\, n(n+1)$.

Definition

Let $n_o \in N$ and let A(n) be statement depending on n∈N such that:

(1) A(n) is true.

(2) For every $k \geq n_o$ if A(k) is true then A(k+1) is also true. That is $\forall\, k \geq n_o$, A(k)⇒ $A(k+1)$.

Then A(n) is true for n≥ N.

Steps for mathematical induction

1- Initial stage# verify that $A(n_o)$ is true

2- Induction hypothesis# A(n) is true for some n say n = k

3- Induction step# deduce (from step1 and 2) that A(n) is true for n = k+1

4- Conclusion# by the PMI, A(n) is true for all integers n≥ N

Example 1.7

1) Prove that "7 is a divisor of $3^{2n} - 2^n$"

Step 1: For n =1 we have $3^2 - 2^1 = 9 - 2 = 7$ True

Step 2: $3^{2n} - 2^n$is divisible by 7 for some n = k

Step 3: set n = k+1, step 2 implies that ∃ j: $3^{2n} - 2^n$ =7j

$3^{2(k+1)} - 2^{(k+1)} = 9 * 3^{2k} - 2 * 2^k$

$=9 * (7j + 2^k) - 2 * 2^k$

$= 9\,(7j) + (9-2)*2^k$

$= 7(9\,j+2^k)$ which is divisible by 7

Step 4: It follows from the PMI that 7 is a divisor of $3^{2n} - 2^n$ for all n ≥ 1 ■

2) Prove that for all n∈ N, 1+4+9+ …+ $n^2 = \frac{1}{6}(2n^3 + 3n^2 + n)$

Solution

Check that the statement holds for n_o=1; $1=\frac{1}{6}(2 * 1^3 + 3 * 1^2 + 1) = 1$

A(n) holds for some n = k that is 1+4+9+ …+ $k^2 = \frac{1}{6}(2k^3 + 3k^2 + k)$

Check that the statement holds for n = k+1

That is 1+4+9+ …+ $k^2 + (k+1)^2 = \frac{1}{6}(2k^3 + 3k^2 + k) + (k+1)^2$

$= \frac{1}{6}(2(k+1)^3 + 3(k+1)^2 + (k+1))$

Which is the RHS of A(k+1). Therefore, the statement is true for n = k+1

It follows by PMI that A(n) holds for all n∈ N ■

Exercise 1.6

Prove the following statements by PMI

i) $1+2+3+\ldots+n = \frac{1}{2}(n^2+n)$ for all $n \geq 0$

ii) $1+3+5+\ldots. + (2n-1) = n^2$ for all $n \geq 1$

UNIT TWO

SETS AND FUNCTIONS

2.1 Definitions about sets

Definition: A set is a well-defined collection of objects that share a certain property or properties. The term well-defined means that the objects contained in the set are totally determined and so any given object is either in the set or not in the set.

We use capital letters to denote sets and small letters to denote the elements in the set. A set can be described by listing its members (roster method) or by defining a rule or a function that describes its elements. E.g. B = $\{x \in N: n = k^2$ for some $k \in N\}$.

Examples of sets include set of integers, set of rational numbers, set of counting numbers etc.

Definition: Let A and B be sets. We say that:

a) B is a subset of A (or is contained in A) denoted by $B \subseteq A$ if every element of B is an element of A. i.e. $(\forall x)(x \in B \to x \in A)$.
b) $A = B$ if $(A \subseteq B) \wedge (B \subseteq A)$ i.e $(\forall x)(x \in A \Leftrightarrow x \in B)$.
c) If B is a subset of A and $A \neq B$ then B is a proper subset of A written as $B \subset A$
d) We say that a set is empty if it contains no element e.g. $\{x \in \mathbb{R}: x^2+1 = 0\}$ is empty since $x^2+1=0$ has no solution in $\mathbb{R}$.

Proposition 2.1

1) If A is empty then $A \subseteq B$ for any set B
2) All empty sets are equal
3) For any set A, $A \subseteq A$, if $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$.

Theorem 2.1

Given any two sets A and B, if $A = B$ then $(A \subseteq B) \wedge (B \subseteq A)$

Proof

Let $x \in A$, since $A = B$ then $x \in B$ and so we have

$A = B \Leftrightarrow (\forall x)[(x \in A) \Leftrightarrow (x \in B)]$

$\Leftrightarrow (\forall x)[(x \in A) \Rightarrow (x \in B) \wedge (x \in B) \Rightarrow (x \in A)]$

$\Leftrightarrow (\forall x)[(x \in A) \Rightarrow (x \in B)] \wedge (\forall x)[(x \in B) \Rightarrow (x \in A)]]$

$\Leftrightarrow (A \subseteq B) \wedge B \subseteq A)$ ■

2.2 Interval and inequalities

Let $a, b \in \mathbb{R}$ then only and only one of the statements a<b, a>b and a = b is true.

Finite intervals

Open interval (a, b) or $\{x \in \mathbb{R} | a < x < b\}$

Closed interval [a, b] or $\{x \in \mathbb{R} | a \le x \le b\}$

Half-closed interval [a, b), (a, b] or $\{x \in \mathbb{R} | a \le x < b\}$, $\{x \in \mathbb{R} | a < x \le b\}$ respectively.

Infinite intervals

Open interval $(a, \infty), (\infty, a)$ or $\{x \in \mathbb{R} | x > a\}, \{x \in \mathbb{R} | x < a\}$ respectively

Closed interval $[a, \infty), (\infty, a]$ or $\{x \in \mathbb{R} | x \ge a\}$, $\{x \in \mathbb{R} | x \le a\}$ respectively

$-\infty$ and ∞ can not be boundaries on the subset of $\mathbb{R}$ since they are not members of $\mathbb{R}$.

2.3 Operations on sets

Definition: Let A be a set. The power of set A denoted by P(A) is the set whose elements are all the subsets of A. that is $P(A) = \{B: B \subseteq A\}$.

Example: Let A = {x, y, z}. P(A) = {ϕ, {x}, {y}, {z}, {x, y}, {x, z}, {y, x}, {x, y, z}}

Definition: Let A and B be subsets of a universal set U:

a) The union of A and B denoted by AUB is the set of all elements in U that are either in A or B or both. i.e. $AUB = \{x \in U: (x \in A) \vee (x \in B)\}$.
b) The intersection of A and B denoted by $A \cap B$ is the set containing all elements that are both A and B. i.e. $A \cap B = \{x \in U: (x \in A) \wedge (x \in B)\}$.
c) Sets A and B are said to be disjoint or mutually exclusive if $A \cap B = \phi$.
d) The complement of A relative to B denoted by B-A or B\A is a set of elements of B that are not in A. i.e. $B-A = \{x \in U: (x \in B) \wedge (x \notin A)\}$.
e) The complement of A denoted by A^c or A^I is the set of all elements in U that are not in A. i.e. $A^c = \{x \in U: x \notin A)\}$.
f) The symmetric difference between A and B denoted by $A \rhd B$ is a set given by $A \rhd B = \{(B-A) \vee (A-B)$.

Exercise 2.1

Let A and B be subsets of a universal set U = {all counting numbers}. Let A = {1,2,4,5,6, 12,13} and B = {3,5,6,7,8,9,10}. Write down the elements for each of the following sets.

a) $P = \{x \in U: (x \in A) \vee (x \in B)\}$
g) $Q=\{x \in U: (x \in B) \wedge (x \notin A)\}$
b) $R= \{x \in U: (x \in A) \wedge (x \in B)\}$
c) $S = \{x \in U: x \notin A)\}$

Proposition 2.2

Let A, B, and C be subsets of a universal set U.

a) i) $A \cup B = B \cup A$ and ii) $A \cap B = B \cap A$ (Commutative law)
b) i) $(A \cap B) \cap C = A \cap (B \cap C)$ and ii) $(A \cup B) \cup C = A \cup (B \cup C)$ (Associative law)
c) i) $A \cap A = A$ and ii) $A \cup A = A$ (Idempotent law)
d) i) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ and ii) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ (Distributive law)
e) i) $(A \cup B)^I = A^I \cap B^I$ and ii) $(A \cap B)^I = A^I \cup B^I$ (Demorgan's law)
f) $A-(B \cup C) = (A-B) \cap (A-C)$ (Distributive law)

We prove d) ii), e) i) and f). the rest can be proved in the same way by the reader.

d) ii) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$. Let $x \in A \cap (B \cup C)$

$x \in A \cap (B \cup C) \Leftrightarrow (x \in A) \wedge (x \in B \cup C)$

$\Leftrightarrow (x \in A) \wedge [(x \in B) \vee (x \in C)]$

$\Leftrightarrow [(x \in A) \wedge (x \in B)] \vee [(x \in A) \wedge (x \in C)]$

$\Leftrightarrow x \in (A \cap B) \cup (A \cap C)$. Thus $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

e) i) $(A \cup B)^I = A^I \cap B^I$

Let $x \in (A \cup B)^I$, then $x \notin (A \cup B)$

Therefore $\neg[x \in (A \cup B)]$

$x \in (A \cup B)^I \Leftrightarrow \neg[x \in (A \cup B)]$

$\Leftrightarrow \neg[(x \in A) \vee (x \in B)]$

$\Leftrightarrow (x \notin A^c) \wedge (x \notin B^c)$

$\Leftrightarrow x \in (A^c \wedge B^c)$. Therefore $(A \cup B)^I = A^I \cap B^I$

f) $A-(B \cup C) = (A-B) \cap (A-C)$

Let $x \in A-(B \cup C)$

$x \in A-(B \cup C) \Leftrightarrow (x \in A) \wedge [x \notin (B \cup C)]$

$\Leftrightarrow (x \in A) \wedge \neg[x \in (B \cup C)]$

$\Leftrightarrow (x \in A) \wedge \neg[(x \in B) \vee (x \in C)]$

$\Leftrightarrow (x \in A) \wedge [\neg(x \in B) \wedge \neg(x \in C)]$

$\Leftrightarrow [(x \in A) \wedge \neg(x \in B)] \wedge [(x \in A) \neg(x \in C)]$

$\Leftrightarrow (x \in A - B) \wedge (x \in A - C)$

$\Leftrightarrow x \in (A - B) \cap (A - C)$. Therefore $A-(B \cup C) = (A-B) \cap (A-C)$ ■

2.4 Indexed families of sets

Instead of using the alphabetical letters, due to existence of large collection of sets, we usually index sets using some convenient indexing sets.

Suppose I is a set and for each $i \in I$ there corresponds one and only one subset A_i of a universal set U. The collection $\{A_i : i \in I\}$ is called an indexing family of sets or indexed collection of sets.

Definition:
Let $\{A_i : i \in I\}$ be an indexed family of subsets of the universal set U.

a) The union of the family $\{A_i : i \in I\}$ denoted by $\bigcup_{i\in I} A_i$ is the set of all those elements of U which belong to at least one of the A_i. That is $\bigcup_{i\in I} A_i = \{x \in U : x \in A_i \text{ for some } i \in I\}$ or $\bigcup_{i\in I} A_i = \{x \in U : (\exists i \in I)(x \in A_i)$.

b) The intersection of the family $\{A_i : i \in I\}$ denoted by $\bigcap_{i\in I} A_i$ is the set of all those elements of U which belong to all the A_i. That is $\bigcap_{i\in I} A_i = \{x \in U : x \in A_i \text{ for each } i \in I\}$ or $\bigcap_{i\in I} A_i \{x \in U : (\forall i \in I)(x \in A_i)$.

Exercise 2.2
Let $\{A_i : i \in I\}$ be an indexed family of subsets of the universal set U and let B be a subset of U. Prove that:

a) $B \cap (\bigcup_{i\in I} A_i) = \bigcup_{i\in I}(B \cap A_i)$ b) $B - \bigcup_{i\in I} A_i = \bigcap_{i\in I}(B - A_i)$ c) $\bigcap_{i\in I} A_i = \bigcap_{i\in I} A_i^c$

2.5 Functions

Definition: Let X and Y be sets, a function f from X to Y denoted by $f: X \mapsto Y$ is a rule that assigns to each $x \in X$ a unique element $y \in Y$.

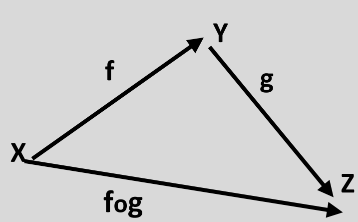
Definition: Let X and Y be sets, a function f from X to Y denoted by $f: X \mapsto Y$ is said to be:

a) Injective (one-to-one) if for each $y \in Y$ there is at most one $x \in X$ such that $f(x) = y$. Equivalently, f is injective if for all $x_1, x_2 \in X$, $f(x_1) = f(x_2)$ implies that $x_1 = x_2$. That is $(\forall x_1, x_2 \in X)\big(f(x_1) = f(x_2)\big) \to x_1 = x_2$

b) Surjective (onto) if for every $y \in Y$ there is an $x \in X$ such that $f(x) = y$. That is $(\forall y \in Y)(\exists x \in X)(f(x) = y)$.

c) Bijective: If it is both injective and surjective

Definition
Let X, Y and Z be sets, $f: X \mapsto Y$ and $g: Y \mapsto Z$ be functions. The composition of f and g denoted by fog is the function $fog: X \mapsto Z$ defined by $(fog)(x) = g[f(x)]$. As illustrated in the figure below.



Theorem 2.2

Let $f: X \mapsto Y$ and $g: Y \mapsto Z$ such that $\text{ran}(f) \subseteq \text{dom}(g)$, then:

a) If f and g are onto then so is the composition function gof.
b) If f and g are one-to-one then so is the composition function gof.
c) If gof is one-to-one then so is f.
d) If gof is onto then so is g.

Proof

a) Let z ∈ Z. Since g is onto there is a y ∈ Y such that g(y) = z since f is onto there is a x ∈X such that f(x) = y. Therefore $(gof)(x) = g[f(x)] = g(y) = z$. Hence gof is onto
b) Let x_1, x_2be in X such that $(gof)(x_1) = (gof)(x_2)$. Then
$g[f(x_1)] = g[f(x_2)]$
$f(x_1)] = f(x_2)$ since g is one-to-one
$x_1 = x_2$ since f is one -to-one
So gof is one-to -one
c) Let x_1, x_2be elements of X such that$f(x_1) = f(x_2)$ then $(gof)(x_1) = (g[f(x_1)] = g[f(x_2)]$ Since gof is one-to-one it follows that $x_1 = x_2$ thus f is one-to-one
d) Let z ∈ Z we must produce a y ∈ Ysuch that g(y) = z. Since is onto there is an x ∈X such that gof(x) = g[f(x)] = z. Let $y = f(x)$ $(\in Y)$ then g(y) = z which proves that g is onto. ■

Exercise 2.3
1) Let $f: X \mapsto Y$ be a bijection. Then $f^{-1}: X \mapsto Y$ is a bijection
2) Let $f: X \mapsto Y$ and $g: Y \mapsto Z$ be bijections then $(gof)^{-1} = f^{-1}og^{-1}$

2.6 Cardinality: The size of a set

Definition: Two sets A and B are said to have same cardinality, denoted by |A| = |B| if there is a one-to-one function from A onto B.
Sets with same cardinality are said to be equipotent or equinumerous.

2.7 Sets of real numbers

The sets of real numbers have unique description for example:

Set of counting numbers, integers, real numbers, rational numbers, whole numbers, even numbers etc.

2.8 Order on sets and ordered sets

Order of a set

Let S be a non-empty set. An order on a set S is a relation denoted by "<" with the following properties:

1) If x ∈ S and y ∈ S then one and only one of the statements x<y, x>y and x = y is true.
2) If x, y, z ∈ S and if x<y, y<z then x<z.

Ordered set

A set S is said to be ordered if an order is defined on S.

2.9 Bounded sets

Let S be an ordered set and $E \subseteq S$. If there exist a $b \in S$ such that $x \leq b \ \forall x \in E$ then we say that E is bounded above and b is known as an upper bound of E.

Let S be an ordered set and $E \subseteq S$. If there exist a $p \in S$ such that $x \geq p \ \forall x \in E$ then p is the lower bound of E.

Least upper bound (supremum)

Suppose S is an ordered set, $E \subseteq S$ and E is bounded above. Suppose there exist a $\alpha \in S$ such that:

i) α is an upper bound of E ii) if $g<\alpha$ then, g is not an upper bound of E. Then α is called a least upper bound of E or supremum of E written as $\sup E = \alpha$. i.e. α is the least member of the set of upper bounds of E.

Greatest lower bound (infimum)

Suppose S is an ordered set, $E \subseteq S$ and E is bounded below. Suppose there exist a $k \in S$ such that:

i) k is a lower bound of E ii) if $g>k$ then, g is not a lower bound of E. Then k is called a greatest lower bound of E or infimum of E written as $\inf E = k$. i.e. k is the greatest member of the set of lower bounds of E.

Note:

1) That the supremum and the infimum are not necessarily members of E.

E.g. Let E be a set of numbers of the form $\frac{1}{n}$ where n is a counting number. $\{E = 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5} \ldots\ldots\}$. then the $\sup E = 1$ which belongs to set E and the $\inf E = 0$ which does not belong to set E.

2) If the sup E exist and is not in S then set S does not contain the least upper bound property.

3) An ordered set which has the least upper bound property also has the greatest lower bound property.

Exercise 2.4

a) State the least upper bound property and greatest lower bound property of a set S.
b) Let E be a non-empty subset of an ordered set, suppose a is lower bound of E and b is an upper bound of E. Prove that $a \leq b$.

2.10 Neighborhoods

A neighborhood of a point $p \in \mathbb{R}$ is an interval of the form $(p-\delta, p+\delta)$ where $\delta > 0$ for any positive real number. Thus, the neighborhood consists of all points distance less than δ from p.

$(p-\delta, p+\delta) = \{x \in \mathbb{R}: |x-p| < \delta\}$. E.g. (1.2, 2.8) is a neighborhood of 2.

2.11 Types of points for sets

We shall consider sets that are subsets of ℝ or subset of extended ℝ. Note that ℝ does not include ∞ and $-\infty$ while extended ℝ includes ∞ and $-\infty$.

Definition (exterior point of S)
Consider $S \subset \mathbb{R}$, a point $p \in \mathbb{R}$ is said to an **interior point** if it has a neighborhood U laying entirely outside S. i.e. with $U \subset S$.
Definition (interior point of S)
Consider $S \subset \mathbb{R}$, a point $p \in \mathbb{R}$ is said to an **exterior point** if it has a neighborhood U laying entirely inside S. i.e. with $U \subset S^c$.
Definition (boundary point of S)
Consider $S \subset \mathbb{R}$, a point $p \in \mathbb{R}$ is said to **a boundary point of S** if it is neither interior or exterior point of S. Thus, p is a boundary point S if its neighborhood intersects both S and S^c.

Example 2.1

1) For the set $A = [-\infty, 4)U\{5, 9\}U[6, 7]$ decide which of the following are true or which is false.
   a) -6 is an interior point of A (**T**)
   b) 6 is an interior point of A (**F**)
   c) 9 is a boundary point of A (**T**)
   d) 5 is a boundary point of A (T)

Interior, exterior and boundary of a set

The set of all interior points of a set S is denoted by S^o and is called the interior of S, the set of all boundary points of S is denoted by ∂S is called the boundary of S while the set of all points of S is the exterior of S and is denoted by S^{ext}.

Thus, the extended ℝ line is split up into 3 parts: $\mathbb{R} = S^o \cup \partial S \cup S^{ext}$.

Example 2.2

For the set $A = [-\infty, 4) \cup \{5, 9\} \cup [6, 7)$

i) $A^o = [-\infty, 4) \cup (6, 7)$ ii) $\partial A = \{4,5,9,7,6\}$ iii) $A^c = [4, 5) \cup (5, 6) \cup [7, 9) \cup (9, \infty]$

iv) The interior of the complement of $A^c = (4, 4)U(5, 6)U(7, 9)U(9, \infty]$

Exercise 2.5

1) For the set $B = (-\infty, -5)U\{2,3,8\}U[4, 7]$ decide which of the following are true or which is false.
   a) -6 is an interior point of B b) -5 is an interior point of B c) 5 is a boundary point of B
   d) 7 is a boundary point of B e) 4 is an interior point of B
2) For the set $D = \{-4, 8\} \cup [1, 7) \cup [9, \infty]$ find D^o, ∂D, D^c and $(D^c)^o$

2.12 Open sets and Topology

We say that a set is **open** if it does not contain any of its boundary points. Eg $(2, 3)\cup(5, 9)$ is open. Note that every point of an open set is an interior point. Thus, if a set is open it means that $S^o = S$. This means that a union of open sets is open.
The collection of all open subsets of ℝ is called the **topology** of ℝ .

Theorem 2.3

If two sets A and B are open then $A \cap B$ is open.

Proof

Take a point $p \in (A \cap B)$, then p is in both A and B. Since $p \in A$ and A is open there is a neighborhood U of p which is a subset of A i.e. $U \subset A$.

Similarly, there is a neighborhood V of p which is a subset of B i.e. $V \subset B$.

But then $V \cap U$ is a neighborhood of p which is a subset of both A and B, so $V \cap U \subset A \cap B$. Thus, every point in $A \cap B$ is open. ■

2.13 Closed set

A set S is said to be closed if it contains all its boundary points. i.e. $\partial S \subset S$

e.g. B = [4, 8] U [9, ∞]

$\partial B = (4, \infty)$

$\partial B \subset B$ so, B is closed

A = [4, 5) is not closed.

N.B: If a set is open then its complement is closed and the converse is true.

Exercise 2.6

Prove the following theorems:

a) If a set is open then is complement is closed
b) The closure of a set is closed
c) The closure of a set is the smallest closed set containing S.
d) Every closed and bounded subset of ℝ is compact. Conversely every compact subset of ℝ is closed and bounded. (Heine Borel theory)

UNIT THREE

REAL NUMBERS AND THEIR PROPERTIES

3.1 Real numbers

Definition: Real numbers are numbers that appear on a number line. They form an open set written as (∞, -∞); ±∞ are not included in the set and so they are not real numbers.

Real numbers are divided into two types, rational numbers and irrational numbers.

Rational Numbers:

Any number that can be expressed as the quotient of two integers (fraction).
Any number with a decimal that repeats or terminates.
Subsets of Rational Numbers:
Integers: rational numbers that contain no fractions or decimals {…,-2, -1, 0, 1, 2, …}
Whole Numbers: all positive integers and the number 0 {0, 1, 2, 3, … }
Natural Numbers (counting numbers): all positive integers (not 0) {1, 2, 3, … }

Irrational Numbers:
Any number that cannot be expressed as a quotient of two integers (fraction).
Any number with a decimal that is non-repeating and non-terminal (doesn't repeat and doesn't end).
Examples of irrational numbers include: π, √2, √3 etc.

3.2 Axioms of Real numbers

3.2.1 The Field axioms

Definition: A field is a set $\mathbb{F}$ together with two binary operations $+: \mathbb{F}\times\mathbb{F} \Longrightarrow \mathbb{F}$ (called addition) and $\times : \mathbb{F}\times\mathbb{F} \Longrightarrow \mathbb{F}$ (called multiplication) such that for all x, y, z$\in \mathbb{F}$ the following are satisfied.

1. Closure law: x, y $\in \mathbb{F}$ then x+y$\in \mathbb{F}$ and $xy\in \mathbb{F}$.
2. Commutative law: $x+y=y+x$ and $xy=yx, \forall x,y \in \mathbb{F}$
3. Associative law: $x+(y+z)=(x+y)+z$ and $x(yz)=(xy)z, \forall x,y,z \in \mathbb{F}$
4. Existence of inverse:
   a) Additive inverse-For any $x\in\mathbb{F}, \exists -x \in \mathbb{F}: x+(-x)=(-x)+x=0$
   b) Multiplicative inverse-For any $x\in\mathbb{F}, \exists \frac{1}{x} \in \mathbb{F}: x\left(\frac{1}{x}\right)=\left(\frac{1}{x}\right)x=1$
5. Existence of identity:
   a) Additive identity-For any $x\in\mathbb{F}, \exists\, 0\in \mathbb{F}: x+0=0+x=x$
   b) Multiplicative identity- For any $x\in\mathbb{F}, \exists\, 1 \in \mathbb{F}: x.1=1.x=x$
6. Distributive law: For $\forall x,y,z \in \mathbb{F}, x(y+z)=(y+z)x=xy=xz$

Theorem 3.1

Let $x,y,z \in \mathbb{F}$ then,

a) If $x+y=x+z$, then $y=z$
b) If $x+y=x$, then $y=0$
c) If $x+y=0$, then $x=-y$
d) $-(-x)=x$

Proof

a) Suppose $x + y = x + z$

$$\begin{aligned} y &= 0 + y \\ &= (-x + x) + y \\ &= -x + (x + y) \\ &= -x + y + z \\ &= (-x + x) + z \\ &= 0 + z \\ &= z \end{aligned}$$

Thus $y = z$

b) Take $z = 0$ in a)

$$x + y = x + 0$$
$$\Rightarrow y = 0$$

c) Take $z = -x$ in a)

$$x + y = x + (-x)$$
$$y = -x$$

d) Let $-(-x) = A$

If $A + -x = 0,$

Then $A + -x + x = 0 + x$

$$A + 0 = x$$
$$A = x$$

Thus, $-(-x) = x$ ■

Theorem 3.2

Let $x, y\ and\ z \in \mathbb{F}$

a) If $x \neq 0$ and $xy = xz$ then, $y = z$
b) If $x \neq 0$ and $xy = x$ then, $y = 1$
c) If $x \neq 0$ and $xy = 1$ then, $y = \frac{1}{x}$
d) If $x \neq 0$ then, $\frac{1}{\frac{1}{x}} = x$

Proof

a) Suppose $xy = xz$

Since $y = 1.y = \left(\frac{1}{x}.x\right)y$

$$y = \frac{1}{x}(xy)$$

$$y = \frac{1}{x}(xz)$$

$$y = \left(\frac{1}{x}.x\right)z$$

$$y = 1.z$$

Thus, $y = z$

b) Suppose $xy = x$

From $xy = xz$ in a) put $z = 1$ and get $xy = x \implies y = 1$

c) Take $z = \frac{1}{x}$ in a)

$xy = \frac{1}{x} . x$

$\Rightarrow y = \frac{1}{x}$

d) Since $\frac{1}{x} . x = 1$ then c) gives $x = \frac{1}{\frac{1}{x}}$ ■

Exercise 3.1
Use the field axioms to prove the following propositions.

a) $(-1)(-1) = 1$
b) $(-).x = -x$
c) $0.x = 0$
d) If $x \neq 0, y \neq 0$ then, $xy \neq 0$

3.2.2 The order axiom

An ordered field is a field $\mathbb{F}$ on which an order relation < is defined such that:

i) (Trichotomy)-for every $x, y \in \mathbb{F}$ exactly one of the following holds: $x < y, x > y, x = y$.

ii) (Transitivity)- for all $x, y, z, x < y \wedge y < z \implies x < z$

iii) For all x, y, z in $\mathbb{F}, x < y \Rightarrow x + z < y + z$, furthermore if $z > 0$, then $xz < yz$.

Theorem 3.3

The following propositions holds for any ordered field.

i) If $x > 0$ then $-x < 0$ and vice versa
ii) If $x > 0$ and $y < z$ then $xy < xz$
iii) If $x < 0$ and $y < z$ then $xy > xz$
iv) If $x \neq 0$ then $x^2 > 0$ in particular, 1>0
v) If $0 < x < y$ then $0 < \frac{1}{y} < \frac{1}{x}$

Proof

i) If $x > 0$ then $0 = -x + x > -x + 0$. So $-x < 0$

If $x < 0$ then $0 = -x + x < -x + 0$. So $-x > 0$

ii) Since $Z > y$ we have $z - y > y - y = 0$

which means that $z - y > 0$. Also $x > 0$

Therefore $x(z - y) > 0$

$$xz - xy > 0$$

$$xz - xy + xy > 0 + xy$$

$$xz > xy \; of \; xy < xz$$

iii) Since $y < z \Rightarrow -y + y < -y + z$

$\Rightarrow z - y > 0$. Also $x > 0 \Rightarrow -x > 0$

Therefore $-x(z - y) > 0$

$$-xz + xy > 0$$

$$-xz - xz + xy > 0 + xz$$

$$xy > xz$$

iv) If $x > 0$ then $x.x > 0 \;\Rightarrow x^2 > 0$

If $x < 0$ then $-x > 0 \;\Rightarrow (-x).(-x) > 0 \;\Rightarrow x^2 > 0$

i.e. If $x > 0$, then $x^2 > 0$, Since $1^2 = 1$, then 1>0

v) If $y > 0$ and $v \leq 0$ then $y.v \leq 0$. But $\left(y.\frac{1}{y}\right) = 1 > 0 \Rightarrow \frac{1}{y} > 0$

Like wise $\frac{1}{x} > 0$, as $x > 0$. If we multiply both sides of the inequality $x < y$ by the positive quantity $\frac{1}{x}.\frac{1}{y}$ we obtain $\left(\frac{1}{x}\right)\left(\frac{1}{y}\right)x < \left(\frac{1}{x}\right)\left(\frac{1}{y}\right)y$. i.e. $\frac{1}{y} < \frac{1}{x}$ ■

3.2.3 The completeness axiom

Definition: An ordered field φ is said to be complete if every subset S of φ which is bounded above has the least upper bound.

Proposition: A nonempty subset S of an ordered field φ can have at most one least upper bound.

Proof

Suppose λ and v are both least upper bounds of S. Then by the definition of least upper bound, we have $\lambda \leq v \leq \lambda$, thus $\lambda = v$. ■

Theorem 3.4 (Characterization of supremum)

Let S be a nonempty subset of an ordered field φ and $M \in \varphi$. Then $M = supS$ if and only if:

i) M is an upper bound for S

ii) For any $\varepsilon \in \varphi$ with ε>0, there is an element $s \in S$ such that $M - \varepsilon < S$.

Proof

Assume that M is a supremum for S. i.e. $M = supS$. Then, by definition, M is an upper bound for S. If there is an $\varepsilon' \in \varphi$ with $\varepsilon' > 0$ for which $M - \varepsilon' \geq S$ for all $s \in S$, then $M - \epsilon'$ is an upper bound for S which is smaller than M, a contradiction.

For the converse, assume that i) and ii) hold. Since S is bounded above, it has a supremum, A (say). Since M is an upper bound for s, we must have that $A \leq M$. If $A < M$, then with $\epsilon = M - A$, there is an element $s \in S$ such that $M - (M - A) < S \leq A$. i.e. $A < A$ which is absurd.

Therefore $A = M$ and so M is the supremum of S. ■

Theorem 3.5

Let A and B be nonempty subsets of ℝ which are bounded above. Then the set $S = \{a + b : a \in A \; \& \; b \in B\}$ is bounded above and $supS = supA + supB$

Proof (Left to the reader)

3.3 The Archimedean Property of Real numbers

Theorem 3.6 (Archimedean Property)

The set ℕ of natural numbers is not bounded above.

Proof

Assume that ℕ is bounded above. By the completeness axiom, $sup\mathbb{N}$ exists. Let $m = sup\mathbb{N}$ then with $\epsilon = 1$, there is an element $k \in \mathbb{N}$ such that $m - 1 < k$. This implies that $m < k + 1 \leq m$ which is impossible. Thus ℕ is not bounded above ■

Corollary:

The Archimedean property implies the following

1) For every real number b there exist an integer m such that $m < b$.
2) Given any number x there exist an integer k such that $x - 1 \leq k < x$
3) If x and y are two positive real numbers there exist a natural number n such that $nx > y$
4) If $\epsilon > 0$ then there exists an $n \in \mathbb{N}$ such that $\frac{1}{n} < \epsilon$

Theorem 3.7 (Density of rational in reals)

If x and $y \in \mathbb{R}$ and $x < y$ then there exists a rational number r such that $x < r < y$. That is in between any two distinct numbers there is a rational number. Or ℚ is dense in ℝ.

Proof (lest to the reader).

3.4 The Euclidean space

Definition: Let $\underline{x}$ and $\underline{y}$ be vectors in $\mathbb{R}^k$. The inner product or scaler product of $\underline{x}$ and $\underline{y}$ is defined as $\underline{x} \,.\, \underline{y} = \sum_{i=1}^{k} x_i . y_i = (x_1y_1 + x_2y_2 + \cdots x_ky_k)$

And the norm of $\underline{x}$ is defined by $\parallel x \parallel = (x.x)^{\frac{1}{2}} = \sum_{i=1}^{k}(x_i^2)^{\frac{1}{2}}$

The vector $\mathbb{R}^k$ with the above inner product and norm is called Euclidean $k - space$.

Theorem 3.8

Let $\underline{x}, \underline{y} \in \mathbb{R}$ then:

i) $\| x^2 \| = \underline{x}.\underline{x}$

ii) $\| \underline{x}.\underline{y} \| \leq \| \underline{x} \| \underline{y} \|$ (Cauchy Schwarz's inequality)

Proof

i) Since $\| \underline{x} \| = (\underline{x}.\underline{x})^{\frac{1}{2}}$ therefore $\| \underline{x} \|^2 = \underline{x}.\underline{x}$

ii) For $l \in \mathbb{R}$ we have:

$0 \leq \| \underline{x} - ly \|^2 = (x - ly)(x - ly)$

$= \underline{x}\left(\underline{x} - l\underline{y}\right) + (-l\underline{y})(\underline{x} - l\underline{y}) = \underline{x}\underline{x} - l\underline{x}\underline{y} - l\underline{x}\,\underline{y} + l^2\underline{y} = \| \underline{x} \|^2 - 2l\left(\underline{x}.\underline{y}\right) + l^2 \| \underline{y} \|^2$

Now put $l = \frac{\underline{x}.\underline{y}}{\|\underline{y}\|^2}$ (certain real number) $\Rightarrow 0 \leq \| \underline{x} \|^2 - 2\frac{(\underline{x}.\underline{y})(\underline{x}.\underline{y})}{\|\underline{y}\|^2} + \frac{(\underline{x}.\underline{y})^2}{\|\underline{y}\|^4} \| \underline{y} \|^2$

$\Rightarrow 0 \leq \| \underline{x} \|^2 - \frac{\left(\underline{x}.\underline{y}\right)^2}{\| \underline{y} \|^2} = \| \underline{x} \|^2 \| \underline{y} \|^2 - \| \underline{x}.\underline{y} \|^2$

$\Rightarrow 0 \leq (\| \underline{x} \| \| \underline{y} \| + \| \underline{x}.\underline{y})(\| \underline{x} \| \| \underline{y} \| - \| \underline{x}.\underline{y})$ which holds if

$0 \leq \| \underline{x} \| \| \underline{y} \| - \| \underline{x}.\underline{y}$ i.e. $\| \underline{x}.\underline{y} \leq \| \underline{x} \| \| \underline{y} \|$ ■

Theorem 3.9

Suppose $x, y, z \in \mathbb{R}^n$ then:

a) $\| \underline{x} + \underline{y} \| \leq \| \underline{x} \| + \| \underline{y} \|$ (Triangle inequality)
b) $\| \underline{x} - \underline{z} \| \leq \| \underline{x} - \underline{y} \| + \| \underline{y} - \underline{z} \|$

Proof

a) Consider $\| \underline{x} + \underline{y} \|^2 = (\underline{x} + \underline{y})(\underline{x} + \underline{y}) = \underline{x}.\underline{x} + 2\underline{x}.\underline{y} + \underline{y}\,\underline{y}$

$\leq \| \underline{x} \|^2 + 2\underline{x}.\underline{y} \| + \| \underline{y} \|^2 = \left(\| \underline{x} \| + \| \underline{y} \| \right)^2$

Thus $\| \underline{x} + \underline{y} \|^2 = \left(\| \underline{x} \| + \| \underline{y} \| \right)^2 \quad \Rightarrow \| \underline{x} + \underline{y} \| \leq \| \underline{x} \| + \| \underline{y} \|$

b) We have $\| \underline{x} + \underline{z} \| = \| \underline{x} - \underline{y} + \underline{y} + \underline{z} \| = \| (\underline{x} - \underline{y}) + (\underline{y} + \underline{z}) \|$
The triangle inequality suggests that $\| (\underline{x} - \underline{y}) + (\underline{y} + \underline{z}) \| \leq \| \underline{x} - \underline{y} \| + \| \underline{y} + \underline{z} \|$
Thus $\| \underline{x} - \underline{z} \| \leq \| \underline{x} - \underline{y} \| + \| \underline{y} - \underline{z} \|$ ■

UNIT FOUR

SEQUENCES AND SERIES

4.1 Sequences

Definition: A sequence is a function whose domain is the set N of natural numbers. If f is such a sequence, let $f(x) = x_n$ denote the value of the sequence f at $n \in \mathbb{N}$. In this case we denote the sequence f by $(x)_{n=1}^{\infty}$ of simply (x_n).

An infinite sequence is an unending set of real numbers which are determined according to some rule. A sequence is normally defined by giving a formula for the n^{th} term.

Examples

1) $\left(\frac{n}{n+1}\right)$ is the sequence $\left(\frac{1}{2}, \frac{2}{3}, \frac{3}{4}. \ldots\right)$
2) (-1^n) is the sequence $(-1, 1, -1, 1, \ldots.)$
3) (2^n) is the sequence $(2, 4, 8, \ldots..)$

We can also use recursive formulas e.g. $x_{n+1} = \frac{x_n + x_{n+1}}{3}$ were $x_1 = 0$ and $x_2 = 1$, then the terms of the sequence (x_n) are $(0, 1, \frac{1}{3}, \frac{4}{7}, \frac{7}{27}, \ldots..)$.

Remark

(1) The order of the terms of the sequence is an important part of the definition of the sequence. For example, the sequence $(1, 5, 7, \ldots.)$ is not the same as the sequence $(1, 7, 5, \ldots.)$.
(2) There is a distinction between the terms of a sequence and the values of a sequence. A sequence has infinitely many terms while its values may or may not be finite.
(3) It is not necessary for the terms of a sequence to be different. For example, $(1, 2, 2, 2, 2, \ldots \ldots.)$ is a particularly good sequence.

Exercise 4.1
Write down the first five terms of the following sequences

a) $\left(\frac{n^2\_2n}{3n}\right)$ b) $\left(\frac{\cos n\pi}{n^2}\right)$ c) $\left(\sin\frac{n\pi}{2}\right)$

4.2 Bounded sequences

Definition: A sequence (x_n) is said to be:

(1) **Bounded above** if there is $k \in \mathbb{R}$ such that $x_n \leq k \ \forall n \in \mathbb{N}$.
(2) **Bounded below** if there is $k \in \mathbb{R}$ such that $x_n \geq k \ \forall n \in \mathbb{N}$.
(3) **Bounded** if it bounded below and bounded above; otherwise it is **unbounded**.

It is easy to see that a sequence (x_n) is bounded if and only if there is a positive real number Msuch that $|x_n| \leq M$ for all $n \in \mathbb{N}$.

Examples

a) The sequence $\left(\frac{1}{n}\right)$ is bounded since $0 < \frac{1}{n} \leq 1$ for all $n \in \mathbb{N}$.
b) The sequence (-1^n) is not bounded above and is not bounded below.
c) The sequence $\left(n + \frac{1}{n}\right)$ is bounded below by 2 but is not bounded above.

4.3 Convergent and divergent sequences

Convergence of a sequence is concerned with the behaviour of the sequence as n increases.

Definition: A sequence (x_n) is said to converge to real number ℓ if given $\epsilon > 0$, there exists a natural number N(which depends of ϵ) such that $|x_n - \ell| < \epsilon$ for all $n \geq \mathbb{N}$.
Or $(\forall \epsilon > 0)(\exists N \backslash in \mathbb{N}): (\forall n \geq N) \Rightarrow |x_n - \ell| < \epsilon$
If (x_n) converges to ℓ, then we say the ℓ is the limit of (x_n) as n increases without bound and we write $\lim_{n\to\infty} x_n = \ell$ or $x_n \to \ell$ as $n \to \infty$.

Note that if a sequence does not converge to a real number, it is said to diverge.

Definition: A sequence (x_n) is said to diverge to ∞ denoted by $x_n \to \infty$ as $n \to \infty$ if for any particular real number m there is an $N \in \mathbb{N}$ such that $x_n > m$ for all $n \geq N$.
Similarly, (x_n) diverges to $-\infty$ denoted by $x_n \to -\infty$ as $n \to \infty$ if for any particular real number k there is an $N \in \mathbb{N}$ such that $x_n < k$ for all $n \geq N$.

Example 4.1

1. Show that a sequence (x_n) converges to zero if and only if the sequence $(|x_n|)$ converges to zero.

Solution

Assume that the sequence (x_n) converges to zero. Then given $\epsilon > 0$, there exists a natural number N (which depends on ϵ) such that $|x_n - 0| < \epsilon \ \forall n \geq \mathbb{N}$.

Now for all $n \geq N$ we have $||x_n| - 0| = |x_n| < \epsilon$ That is the sequence $(|x_n|)$ converges to zero.

For the converse, assume that the sequences $(|x_n|)$ converges to zero. That is $\epsilon > 0$, there exists a natural number N (which depends on ϵ) such that:

$||x_n| - 0| = |x_n| < \epsilon \ \forall n \geq \mathbb{N}$. It follows that the sequence x_n converges to zero

2. Show that $\lim_{n\to\infty} \frac{1}{n} = 0$

Solution

Let $\epsilon > 0$ be given. We can find a $N \in \mathbb{N}$ such that $\left|\frac{1}{n} - 0\right| < \epsilon \ \forall \ n \geq N$. By Archimedean property, there is an $N \in \mathbb{N}$ such that $0 \leq \frac{1}{N} < \epsilon$.

Thus if $n \geq N$ then we have that $\left|\frac{1}{n} - 0\right| = \frac{1}{n} \leq \frac{1}{N} < \epsilon$.

That is $\lim_{n\to\infty} \frac{1}{n} = 0$.

3. Show that $\lim_{n\to\infty}\left(1-\frac{1}{2^n}\right)=1$

   Solution

   Let $\epsilon>0$ be given. We need to find an $N\in\mathbb{N}$ such that $\left|\left(1-\frac{1}{2^n}\right)-1\right|<\epsilon\ \forall\, n\geq N$

   Noting that: $\left|\left(1-\frac{1}{2^n}\right)-1\right|=\frac{1}{2^n}=\frac{1}{(1+1)^n}$ and

   $(1+1)^n=\sum_{k=0}^{n}\binom{n}{k}=\binom{n}{0}+\binom{n}{1}+\binom{n}{2}+\cdots+\binom{n}{n}\geq\binom{n}{0}+\binom{n}{1}=1+n$

   We have $\frac{1}{2^n}=\frac{1}{(1+1)^n}\leq\frac{1}{n+1}<\frac{1}{n}$.

   Now by Archimedean property, there is a $N\epsilon\mathbb{N}$ such that $0<\frac{1}{N}<\epsilon$. Therefore, for all $n\geq N$ we have: $\left|1-\frac{1}{2^n}\right|-1=\frac{1}{2^n}<\frac{1}{n}\leq\frac{1}{N}<\epsilon$.

   Thus $\lim_{n\to\infty}\left(1-\frac{1}{2^n}\right)=1$

4. Show that the sequence (-1^n) diverges.

   Solution

   Assume that the sequence converges to a number say ℓ. Then with $\epsilon=\frac{1}{2}$, there is an $N\in\mathbb{N}$ such that $|-1^n-\ell|<\frac{1}{2}\ \ \forall\, n\geq N$.

   In particular, $|(-1)^{n+1}-\ell|<\frac{1}{2}$

   Therefore, for all $n\geq N$,

   $2=|(-1)^n-(-1)^{n+1}|\leq|(-1)^n-\ell|+|\ell-(-1)^{n+1}|\leq\frac{1}{2}+\frac{1}{2}=1$, which is impossible.

   Thus (-1^n) diverges.

5. Show that the sequence $(1+(-1)^n)$ diverges.

   Solution

   Assume that the sequence converges to some real number ℓ. Then with $\epsilon=1$ there exists a number $N\in\mathbb{N}$ such that $|(1+(-1)^n-\ell|<1$ for all $n\geq N$.

   Now if $n>N$ is odd, then we have $|(1+(-1)^n-\ell|=|\ell|<1$. Hence $-1<\ell<1$.

   And if $n>N$ is even we have $|(1+(-1)^n-\ell|=|2+\ell|<1$. Hence $2<\ell<3$. A contradiction.

Exercise 4.2

1. Show that if $x \in \mathbb{R}$ and $|x| < 1$ then $\lim_{n\to\infty} x^n = 0$
2. Suppose that (x_n) is a sequence such that $x_n > 0$ for all $N \in \mathbb{N}$. Show that $x_n \to \infty$ as $n \to \infty$ if and only iff $\lim_{n\to\infty} \frac{1}{x_n} = 0$

Theorem 4.1

Let (s_n) and (t_n) be sequences of real numbers and let $s \in \mathbb{R}$ if for some positive real number k and some $N_1 \in \mathbb{N}$ we have $|s_n - s| \le k|t_n|$ for all $n \ge N_1$ and if $\lim_{n\to\infty} t_n = 0$ then $\lim_{n\to\infty} s_n = s$.

Proof

Let $\epsilon > 0$ be given. Since $t_n \to 0$ as $n \to \infty$, there exist an $N_n \in \mathbb{N}$ such that $|t_n| < \frac{\epsilon}{k}$ for all $n \ge N_2$.

Let $N = max\ (N_1, N_2)$, then for all $n \ge N$ we have $|s_n - s| \le k|t_n| < \frac{\epsilon}{k}.k = \epsilon$.

Thus $\lim_{n\to\infty} s_n = s$. ∎

Example 4.2

Show that $\lim_{n\to\infty} \sqrt[n]{n} = 1$

Solution

Since $\sqrt[n]{n} \ge 1$ for each $n \in \mathbb{N}$, there is a nonnegative real number a_n such that $\sqrt[n]{n} = 1 + a_n$. Thus, by binomial theorem we have:

$$n = (1 + a_n)^n = \sum_{k=1}^{n} \binom{n}{k} a_n^k = 1 + na_n + \frac{n(n-1)a_n^2}{2} + \cdots + a_n^n \ge 1 + \frac{n(n-1)a_n^2}{2}$$

Therefore, $n - 1 \ge \frac{n(n-1)a_n^2}{2}$ hence $a_n^2 \le \frac{2}{n}$ or $a_n = \sqrt{\frac{2}{n}}$ for all $n \ge 2$

Now since $\left|\sqrt[n]{n} - 1\right| = |a_n| = a_n \le \sqrt{\frac{2}{n}}$ and $\lim_{n\to\infty} \sqrt{\frac{2}{n}} = 0$ we have by theorem 4.1 that $\lim_{n\to\infty} \sqrt[n]{n} = 1$

Theorem 4.2 (Uniqueness of limits)

Let (s_n) be a sequence of real number. If $\lim_{n\to\infty} s_n = \ell_1$ and $\lim_{n\to\infty} s_n = \ell_2$, then $\ell_1 = \ell_2$.

Proof

Let $\epsilon > 0$ be given. Then there exist natural numbers N_1 and N_2 such that

$|s_n - \ell_1| < \frac{\epsilon}{2}$ for all $n \ge N_1$ and $|s_n - \ell_2| < \frac{\epsilon}{2}$ for all $n \ge N_2$.

Let $N = max\ (N_1,\ N_2)$, then for all $n \ge N$ we have:

$|\ell_1 - \ell_2| = |\ell_1 + s_n - s_n + \ell_2| \leq |s_n - \ell_1| + |(s_n - \ell_2)| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$

Thus, $|\ell_1 - \ell_2| = \epsilon$ and since $0 \leq |\ell_1 - \ell_2| < \epsilon$ holds for every $\epsilon > 0$, we have $\ell_1 - \ell_2 = 0$ and so $\ell_1 = \ell_2$. Thus, a sequence (s_n) converges to only and only one limit (the limit of a sequence (s_n) is unique) ■

Proposition 4.1

A sequence (x_n) converges to $\ell \in \mathbb{R}$ if and only if for each $\epsilon > 0$, the set $\{n: x_n \notin (\ell - \epsilon, \ell + \epsilon\}$ is finite.

Theorem 4.3

Every convergent sequence of real numbers is bounded.

Proof

Let (s_n) be a sequence of real numbers which converges to s, then with $\epsilon = 1$ there exists an $N \in \mathbb{N}$ such that $|s_n - s| < 1 \ \forall\, n \geq N$

By the triangle inequality, we have that: $|s_n| \leq |s_n - s| + |s| \leq 1 + |s|$ for all $n \geq N$

Let $M = max\, \{|s_1|, |s_2|, \dots |s_N|, |s| + 1\}$. Then $|s_n| \leq M$ for all $n \in N$. That is the sequence (s_n) is bounded. ■

The converse of Theorem 4.3 is not necessarily true. There are sequences which are bounded but do not converge. E.g. the sequences (-1^n) is bounded but not convergent.

Theorem 4.4 (Squeeze theory)

Suppose that (s_n), (t_n) and (u_n) are sequences such that $s_n \leq t_n \leq u_n$ for all $n \in N$. If $\lim\limits_{n\to\infty} s_n = \ell = \lim\limits_{n\to\infty} u_n$, then $\lim\limits_{n\to\infty} t_n = \ell$.

Proof.

Let $\epsilon > 0$ be given. Then there exist $N_1, N_2 \in \mathbb{N}$ such that:

$|s_n - \ell| < \epsilon$ for all $n \geq N_1$ and $|u_n - \ell| < \epsilon$ for all $n \geq N_2$

That is $\ell - \epsilon < s_n < \ell + \epsilon$ for all $n \geq N_1$ and $\ell - \epsilon < u_n < \ell + \epsilon$

Let $N = max\, \{N_1, N_2\}$. Then for all $n \geq N$, we have

$\ell - \epsilon < s_n \leq t_n \leq u_n < \ell + \epsilon$ and consequently $|t_n - \ell| < \epsilon$ for all $n > N$.

That is $\lim\limits_{n\to\infty} t_n = \ell$. ■

Example 4.3

1. Show that $\lim\limits_{n\to\infty} \frac{cos\frac{n\pi}{2}}{n^2} = 0$
   Solution
   Since $o \leq \left|\frac{cos\frac{n\pi}{2}}{n^2} - 0\right| = \left|\frac{cos\frac{n\pi}{2}}{n^2}\right| \leq 1.\frac{1}{n^2}$ and $\lim\limits_{n\to\infty} \frac{1}{n^2} = 0$ it follows that $\lim\limits_{n\to\infty} \frac{cos\frac{n\pi}{2}}{n^2} = 0$
2. Show that for every x, with $|x| < 1$ $\lim\limits_{n\to\infty} = nx^n = 0$.

Solution

Without loss of generality, assume that $x \neq 0$ and $n > 1$. Since $|x| < 1$, then there is a positive real number a such that $\frac{1}{|x|} = 1 + a$.

Then $\frac{1}{x^n} = (1+a)^n = (1+a_n)^n = \sum_{r=1}^{n} \binom{n}{r} a^r \geq \frac{n(n-1)a^2}{2}$ for some $a > 0$ then

$|x^n| \leq \frac{2}{n(n-1)a^2}$ this implies that $|nx^n| \leq \frac{2}{(n-1)a^2}$ and so,

$$\Rightarrow \frac{-2}{(n-1)a^2} \leq nx^n \leq \frac{2}{(n-1)a^2}$$

Since $\lim\limits_{n\to\infty} \frac{-2}{(n-1)a^2} = 0 = \lim\limits_{n\to\infty} \frac{2}{(n-1)a^2}$. We have by the squeeze theorem, that $\lim\limits_{n\to\infty} = nx^n = 0$

Exercise 4.3

1) Show that for any $x \in \mathbb{R}$, $\lim\limits_{n\to\infty} = \frac{x^n}{n!} = 0$
2) Show that $\lim\limits_{n\to\infty} = \frac{1}{n!} = 0$
3) Show that $\lim\limits_{n\to\infty} = \frac{n}{2^n} = 0$
4) Find the n^{th} term of the sequence $\left(\frac{2}{1}, \frac{4}{3}, \frac{8}{5}, \frac{16}{7}, \frac{32}{9}, \ldots\right)$

Theorem 4.5

Let S be a subset of $\mathbb{R}$ which is bounded above. Then there exists a sequence (s_n) in S such that $\lim\limits_{n\to\infty} s_n = supS$.

Proof

Let $c = supS$. By the characterization of supremum in Theory 3.4, for each $n \in N$ there exists $s_n \in S$such that $c - \frac{1}{n} < s_n \leq c$. Since $\lim\limits_{n\to\infty} \left(c - \frac{1}{n}\right) = c = \lim\limits_{n\to\infty} c$, we have by squeeze theorem that

$\lim\limits_{n\to\infty} s_n = c = supS$ ■

4.4 Algebra of Limits of sequences

Theorem 4.6

Let (s_n) and (t_n) be sequences of real numbers which converges to s and t respectively. Then

i) $\lim\limits_{n\to\infty} (s_n + t_n) = s + t$

ii) $\lim\limits_{n\to\infty} s_n t_n = st$

iii) $\lim\limits_{n\to\infty} \frac{s_n}{t_n} = \frac{s}{t}$ if $t_n \neq 0$ for all $n \in \mathbb{N}$ and $t \neq 0$

Proof

i) Let $\epsilon > 0$ be given. Then there exist N_1 and N_2 in $\mathbb{N}$ such that: $|s_n - s| < \frac{\epsilon}{2}$ for all $n \geq N_1$ and $|t_n - t| < \frac{\epsilon}{2}$ for all $n \geq N_2$.

Let $N = max\,\{N_1, N_2\}$. Then for all $n \geq N$, we have

$$|s_n + t_n - (s + t)| = |(s_n - s) + (t_n - t)|$$

$$\leq |s_n - s| + t_n - t| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

Hence, $\lim\limits_{n\to\infty}(s_n + t_n) = s + t$

ii) Let $\epsilon > 0$ be given. Now,

$$|s_n t_n - (st)| = |s_n t_n - st_n + st_n - st|$$

$$= |(s_n - s)t_n + (t_n - t)s|$$

$$\leq |s_n - s||t_n| + t_n - t||s| \,.$$

Since t_n is convergent, it is bounded. There exist a $k \in \mathbb{N}$ such that $t_n \leq k$ for all $n \in \mathbb{N}$.

Thus $|s_n t_n - st| \leq |s_n - s|k + |t_n - t|s$

Let $M = max\,\{k, |s|\}$. Then $|s_n t_n - st| \leq M(|s_n - s| + |t_n - t|)$

Since $s_n \to s$ and $t_n \to t$ and $n \to \infty$, Then there exist N_1 and N_2 in $\mathbb{N}$ such that: $|s_n - s| < \frac{\epsilon}{M}$ for all $n \geq N_1$ and $|t_n - t| < \frac{\epsilon}{M}$ for all $n \geq N_2$.

Let $N = max\,\{N_1, N_2\}$. Then for all $n \geq N$, we have:

$$|s_n t_n - st| \leq M(|s_n - s| + |t_n - t|)$$

$$< M|s_n - s| + M|t_n - t|$$

$$< M\left(\frac{\epsilon}{M}\right) + M\left(\frac{\epsilon}{M}\right) = \epsilon$$

Therefore, $\lim\limits_{n\to\infty} s_n t_n = st$

iii) (left to the reader) ■

Exercise 4.4
Show that if the sequence (s_n) converges to s, then the sequence s_n^2 converges to s^2.

4.5 Monotone Sequence

Definition: Let (s_n) be a sequence of real numbers we say that (s_n) is:

a) Increasing if for each $n \in \mathbb{N}$, $s_n \leq s_{n+1}$
b) Strictly increasing if for each $n \in \mathbb{N}$, $s_n < s_{n+1}$
c) Decreasing if for each $n \in \mathbb{N}$, $s_n \geq s_{n+1}$
d) Strictly decreasing if for each $n \in \mathbb{N}$, $s_n > s_{n+1}$
e) Monotone if (s_n) is increasing of decreasing
f) Strictly monotone if (s_n) is strictly increasing or decreasing

Remark

An increasing sequence (s_n) is bounded below by s_1 a decreasing sequence is bounded below by t_1 it therefore follows that an increasing sequence is bounded if and only if it is bounded above and a decreasing sequence is bounded if an only if it is bounded below.

Examples

1. The sequence $(1, 1, 2, 3, 5, \dots.)$ is increasing
2. The sequence $(3,\ 1, 0,\ 0, -3, \dots.)$ is decreasing
3. The sequence (n^2) is strictly increasing
4. The sequence $(-n)$ is strictly decreasing

Theorem 4.7

Let a sequence (s_n) be a bounded sequence.

i) If (s_n) is monotonically increasing then it converges to its supremum.
ii) If (s_n) is monotonically decreasing then it converges to its infimum.

Proof

Let $s_1 = sups_n$ and $s_2 = inf\, s_n$ and take $\epsilon > 0$

i) Since $sups_n = s_1$ there exists s_{n0} such that $s_1 - \epsilon < s_{n0}$.

Since s_n is increasing then $s_1 - \epsilon < s_{n0} < s_n < s_1 < s_1 + \epsilon$

$\Rightarrow s_1 - \epsilon < s_n < s_1 + \epsilon$ for all $n > n_0$

$\Rightarrow |s_n - s_1| < \epsilon$ for all $n > n_0$

$\Rightarrow \lim_{n\to\infty} s_n = s_1$

ii) Since $inf\, s_n = s_2$ there exists s_{n1} such that $s_{n1} < s_2 + \epsilon$.

Since s_n is decreasing then $s_2 - \epsilon < s_n < s_{n1} < s_2 < s_2 + \epsilon$

$\Rightarrow s_2 - \epsilon < s_n < s_1 + \epsilon$ for all $n > n_1$

$\Rightarrow |s_n - s_2| < \epsilon$ for all $n > n_1$. $\Rightarrow \lim_{n\to\infty} s_n = s_2$ ■

Theorem 4.8

A monotone sequence converges if and only if it is bounded.

Proof

We already proved in Theorem 4.3 that every convergent sequence is bounded. To prove the converse let (s_n) be a bounded increasing sequence and let $S = \{s_n | n \in \mathbb{N}\}$. Since S is bounded above it has a supremum, $supS = s$ say. We claim that $\lim_{n\to\infty} S = s$. Let $\epsilon > 0$ be given, by the characterization of supremum, there exist $s_N \in S$ such that $s - \epsilon < s_N \le s_n < s + \epsilon$ for all $n \ge N$. Thus, $|s_n - s| < \epsilon$ for all $n \ge N$

The proof for the case when the sequence (s_n) is decreasing is similar. ■

Example 4.4

Show that $\left(\frac{n+1}{n}\right)$ is a convergent sequence

Solution

We show that $\left(\frac{n+1}{n}\right)$ is monotone and bounded. Its convergence will ben follow from theorem 4.8.

Monotonicity: Let $s_n = \left(\frac{n+1}{n}\right)$ then, $\frac{s_{n+1}}{s_n} = \frac{n+2}{n+1} \times \frac{n}{n+1} = \frac{n^2+2n}{(n+1)^2} < \frac{n^2+2n+1}{(n+1)^2} = 1$

This $s_n > s_{n+1}$ for all $n \in \mathbb{N}$ so the sequence $\left(\frac{n+1}{n}\right)$ is monotone decreasing.

Another proof for monotone: Consider $f(x) = \frac{x+1}{x}$, $f'(x) = \frac{-1}{x^2} < 0$ for all $x \in [1, \infty)$. Thus, f is decreasing on $[1, \infty)$. Therefore $f(n) > f(n+1)$ i.e. $\frac{n+1}{n} > \frac{n+2}{n+1}$ for all $n \in \mathbb{N}$.

Boundedness: $\frac{n+1}{n}$ is bounded below by 1. So $\frac{n+1}{n}$ is a convergent sequence by Theorem 4.8.

4.6 Subsequences

If the terms of the sequence (s_n) are contained in other sequences (t_n) then (s_n) is a subsequence of (t_n).

Definition:
Let (s_n) be a sequence of real numbers and let $(n_k)_k \in \mathbb{N}$ be a sequence of natural numbers such that $n_1 < n_2 < \cdots$. Then the sequence (s_{nk}) is called a subsequence of (s_n). That is a subsequence (s_{nk}) of sequence (s_n) is strictly increasing function $\phi: k \mapsto s_{nk}$.

Example:

Let (s_n) be the sequence $(1, 2, \frac{1}{2}, 3, \frac{1}{3}, \dots)$ then $(1, \frac{1}{2}, \frac{1}{3}, \dots..)$ and $(1, 2, 3, \dots)$ are subsequences of (s_n).

Theorem 4.9

Let (s_n) be a sequence which converges to s. Then any subsequences of (s_n) converges to s.

Proof

Let (s_{nk}) be a subsequence of (s_n) and let $\epsilon > 0$ be given. Then there is an $N \in \mathbb{N}$ such that $|s_n - s| < \epsilon$ for all $n \geq N$. Thus when $k \geq N$ we have that $n_k \geq K \geq N$ and so $|s_{nk} - s| < \epsilon$ for all $K \geq N$. Thus, $\lim_{n\to\infty} s_{nk} = s$. ■

Theorem 4.10 (Bolzano Weierstrass theorem for sequences)

Every bounded infinite sequence $(s_n)$of real numbers has a convergent subsequence.

Proof (is left to the reader)

4.7 Cauchy Sequences

Definition: A sequences (s_n) is said to be Cauchy if given any $\epsilon > 0$, there exists a $N \in \mathbb{N}$ such that
$|s_n - s_m < \epsilon|$ for all $n, m \geq N$.
Or $(\forall \epsilon > 0)(\in N \in \mathbb{N})(\forall\, n, m \geq N)\colon (n \geq M)\Lambda(m \geq N) \Rightarrow (|s_n - s_m| < \epsilon$.
Or (s_n) is a Cauchy sequence if $\lim_{n\to\infty} |s_n - s_m| = 0$.

Example 4.5

Show that the sequence $(s_n) = \frac{n+1}{n}$ is a Cauchy sequences.

Solution

For all $n, m \in \mathbb{N}$, $|s_n - s_m| = \left|\frac{n+1}{n} - \frac{m+1}{m}\right| = \left|\frac{m-n}{mn}\right| < \frac{m+n}{mn}$.

Therefore if $m \geq n$ then, $|s_n - s_m| < \frac{m+n}{mn} < \frac{2m}{mn} = \frac{2}{n}$

Let $\epsilon > 0$ be given. Then there is an $N \in \mathbb{N}$ such that $\frac{1}{N} < \frac{\epsilon}{2}$. Thus, for all $n \geq N$ we have

$|s_n - s_m| = \left|\frac{n+1}{n} - \frac{m+1}{m}\right| = \left|\frac{m-n}{mn}\right| < \frac{2}{n} \leq \frac{2}{N} < \epsilon$. Hence, (s_n) is Cauchy. ∎

Theorem 4.11

Every convergent sequence (s_n) is a Cauchy sequence.

Proof

Assume that (s_n) converges to s. Then given $\epsilon > 0$ there exists an $N \in \mathbb{N}$ such that $|s_n - s| < \frac{\epsilon}{2}$ for all $n \geq N$. Now for all $n, m \geq N$ we have that:

$$|s_n - s_m| = |(s_n - s + s - s_m)| \leq |s_n - s| + |s - s_m|$$
$$= |s_n - s| + |s_m - s| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

Thus (s_n) is a Cauchy sequence. ∎

Theorem 4.12

Every Cauchy sequence (s_n) is bounded.

Proof

Let $\epsilon = 1$, then there exists $N \in \mathbb{N}$ such that $|s_n - s_m| < 1$ for all $n, m \geq N$. Choose a $k \geq N$ and observe that $|s_n| = |s_n - s_k + s_k| \leq |s_n - s_k| + |s_k| < 1 + s_k$

Let $M = max\,\{|s_1|, |s_2|, \ldots . |s_N|, |s_k| + 1\}$.

Then $(s_n) < M$ for all $n \geq N$ and therefore, (s_n) is bounded. ∎

Theorem 4.13

Every Cauchy sequence (s_n) of real numbers converges.

Proof

Let (s_n) be Cauchy, by Theorem 4.12, (s_n) is bounded and therefore by Bolzano Weierstrass theorem, (s_n) has a subsequence (s_{nk}) that converges to some real number ℓ.

We claim that (s_n) converges to ℓ. Let $\epsilon > 0$ be given. Then there exist $N_1, N_2 \in \mathbb{N}$ such that:

$|s_n - s_m| < \frac{\epsilon}{2}$ for all $n.m \geq N_1$ and

$|s_{nk} - \ell| < \frac{\epsilon}{2}$ for all $n, m \geq N_2$

Let $M = max\{N_1, N_2\}$. Then for all $n \geq N$ we have:

$$|s_n - l| \leq |s_n - s_{nk}| + |s_{nk} - \ell| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon$$

Therefore $\lim_{n\to\infty} s_n = \ell$ and so (s_n) is convergent.

Theorem 4.11 and theorem 4.13 combined together gives the Cauchy's convergent criterion for sequences: **"A sequence (s_n) of real numbers converges if and only if it is Cauchy."**

Example 4.6

1) Use Cauchy criterion to show that the sequence $\left(\frac{-1^n}{n}\right)$ converges.

Solution

We need to show that the sequence $\left(\frac{-1^n}{n}\right)$ in Cauchy. To that end $\epsilon > 0$ and $s_n = \left(\frac{-1^n}{n}\right)$. then, for all $n, m \in \mathbb{N}$ with $m \geq n$ $|s_n - s_m| = \left|\frac{(-1)^n}{n} - \frac{(-1)^m}{m}\right| \leq \frac{1}{n} + \frac{1}{m} \leq \frac{1}{n} + \frac{1}{n} = \frac{2}{n}$. Now there is an $N \in \mathbb{N}$ such that $\frac{2}{N} < \epsilon$. Thus, for all $n \geq N$, we have $|s_n - s_m| = \left|\frac{(-1)^n}{n} - \frac{(-1)^m}{m}\right| \leq \frac{2}{N} < \epsilon$.

Thus $\left(\frac{-1^n}{n}\right)$ is a Cauchy sequence and so it converges.

2) Show that the sequence $s_n = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$ diverges.

Solution

It suffices to show that (s_n) is not a Cauchy sequence. Now for $n, m \in \mathbb{N}$ with $n > m$, we have:

$$|s_n - s_m| = \left|\left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}\right) - \left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{m}\right)\right| = \left|\frac{1}{m+1} + \frac{1}{m+2} + \cdots + \frac{1}{n}\right|$$

$$= \frac{1}{m+1} + \frac{1}{m+2} + \cdots + \frac{1}{n}$$

$$> > \underbrace{\frac{1}{n} + \frac{1}{n} + \cdots .. + \frac{1}{n}}_{n-m\ times} = \frac{n-m}{n}$$

In particular if we take $n = 2m$ we get

$$|s_n - s_m| = \left|\left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}\right) - \left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{m}\right)\right| > \frac{n-m}{n} = \frac{1}{2}$$

$|s_n - s_m| > \frac{1}{2}$ thus, (s_n) is not a Cauchy sequence and so it diverges.

Exercise 4.5
Show that every subsequence of a bounded sequence is bounded.

4.8 Infinite series

When the individual terms of a sequences s_n are summed a series of real numbers is obtained. If (a_n) is an infinite sequence then:$\sum_{n=1}^{\infty} a_n = a_1 + a_2 + \cdots + a_n + \cdots$ is an infinite series. Then numbers $a_1, a_2, a_3, \ldots$ are terms of the series.

Definition: For the infinite series $\sum_{n=1}^{\infty} a_n$, the n^{th} partial sum is given by:

$S_n = a_1 + a_2 + a_3 + \cdots + a_n$.

If the sequence of partial sum (S_n) converges to S then, the series $\sum_{n=1}^{\infty} a_n$ converges. The limit S is called the sum of the series.

$S = a_1 + a_2 + a_3 + \cdots + a_n + \cdots$ so $S = \sum_{n=1}^{\infty} a_n$.

If (S_n) diverges then $\sum_{n=1}^{\infty} a_n$ diverges.

Example 4.7

a) The series $\sum_{n=1}^{\infty} \frac{1}{2n} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots$ has the following partial sums

$S_1 = \frac{1}{2}, S_1 = \frac{1}{2} + \frac{1}{4} = \frac{3}{4}, \ldots., S_n = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{2^n} = \frac{2^n - 1}{2^n}$

Because $\lim\limits_{n\to\infty} \frac{2^n - 1}{2^n} = 1$. It follows that the series converge and its sum is 1.

b) The n^{th} partial sum of the series $\sum_{n=1}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1}\right) = \left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \cdots$ is given by

$S_n = 1 - \frac{1}{n+1}$ because the $\lim\limits_{n\to\infty} \left(\frac{1}{n} - \frac{1}{n+1}\right) = 1$. Thus, the series converge and its sum is 1.

c) The series $\sum_{n=1}^{\infty} 1 = 1, 1, 1, 1, 1, \ldots$ diverges because $S_n = n$ and the sequence of partial sums diverges.

NB. The series in Example 4.7b) is a **telescoping series** of the form $(b_1 - b_2) + (b_2 - b_3) + (b_3 \_ b_4) + \cdots . (b_{n-1} - b_n) + (b_n - b_{n+1})$.

Because the sum of a telescoping series is given by $S_n = b_1 - b_{n+1}$, it follows that a telescoping series will converge if and only if b_n approaches a finite number as $n \to \infty$. Moreover, if the series converges its sum is $S_n = b_1 - \lim\limits_{n\to\infty} b_{n+1}$.

Writing a series in telescoping form

Find the sum of the series $\sum_{n=1}^{\infty} \frac{2}{4n^2 - 1}$.

Solution

Using partial fractions, we write $a_n = \frac{2}{4n^2 - 1} = \frac{2}{(2n-1)(2n+1)} = \frac{2}{2n-1} + \frac{2}{2n+1}$.

The partial sums is $S_n = \left(\frac{1}{1} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{5}\right) + \cdots + \left(\frac{1}{2n-1} - \frac{1}{2n+1}\right) = 1 - \frac{1}{2n+1}$ so the series converge and its sum is $\lim\limits_{n\to\infty} S_n = \lim\limits_{n\to\infty} \left(1 - \frac{1}{2n-1}\right) = 1 - \lim\limits_{n\to\infty} \frac{1}{2n-1} = 1 - 0 = 1$.

4.9 Geometric series

A geometric series is given by $\sum_{n=0}^{\infty} ar^n = a + ar + ar^2 + \gg .. > ar^n + \cdots$. Provided $a \neq 0$, r is called the ratio.

Theorem 4.14

A geometric series with ratio r diverges if $|r| \geq 1$ and if $0 < |r| < 1$ the series converges to the sum $\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}$

Proof

It is easy to see that the series diverge if $r = \pm 1$. If $0 < |r| < 1$ then

$S_n = a + ar + ar^2 + \ldots + ar^{n-2} + ar^{n-1}$. Let us multiply this equation by r to yield:

$rS_n = ar + ar^2 + ar^3 + \ldots + ar^{n-1} + ar^n$

Subtracting rS_n from S_n we get:

$(r-1)S_n = a - ar^n$

$S_n = \frac{a(1-r^n)}{1-r}$, with $r \neq 1$

When $|r| \geq 1$, it follows that $r^n \to \infty$ as $n \to \infty$ that is $\lim_{n\to\infty} \frac{a(1-r^n)}{1-r} = \infty$. Thus, the series diverge

When $|r| < 1$, it follows that $r^n \to 0$ as $n \to \infty$ that is $\lim_{n\to\infty} \frac{a(1-r^n)}{1-r} = \frac{a}{1-r}$. Thus, the series converge and the sum is $\frac{a}{1-r}$.

Example 4.8

a) The geometric series $\sum_{n=0}^{\infty} \frac{3}{2^n} = 3\left(\frac{1}{2}\right)^n$
$= 3(1) + 3\left(\frac{1}{2}\right) + 3(\left(\frac{1}{2}\right) + \cdots$ has $r = \frac{1}{2}$ and $a = 3$. Because $0 < |r| < 1$ then the series converge and its sum is $S_n = \frac{a}{1-r} = \frac{3}{1-\frac{1}{2}} = 6$.

b) The geometric series $\sum_{n=0}^{\infty} \left(\frac{3}{2}\right)^n = 1 + \frac{3}{2} + \frac{9}{4} + \cdots$ has $r = \frac{3}{2}$. Since $|r| \geq 1$ then the series diverge.

c) Use a geometric series to write $0.\overline{08}$ as a ratio of two integers.

Solution

For the repeated decimal $0.\overline{08}$, we write $0.08080808080808 \ldots \ldots \ldots . = \frac{8}{10^2} + \frac{8}{10^4} + \frac{8}{10^6} + \cdots$

$$= \sum_{n=0}^{\infty} \left(\frac{8}{10^2}\right)\left(\frac{1}{10^2}\right)^n$$

For this series we have $a = \frac{8}{10^2}$ and $r = \frac{1}{10^2}$ so the series converge and the sum is given by

$$S_n = 0.\overline{08} = \frac{a}{1-r} = \frac{\frac{8}{10^2}}{\frac{1}{10^2}} = \frac{8}{99}$$

4.10 Properties of infinite series
Theorem 4.15

Let $\sum a_n$ and $\sum b_n$ be convergent series and let A, B and c be real numbers. If $\sum a_n = A$ and $\sum b_n = B$, the following series converged to the indicated sums.

1) $\sum_{n=1}^{\infty} ca_n = cA$ 2) $\sum_{n=1}^{\infty}(a_n \pm b_a) = A \pm B$

4.11 Convergent criterion for series
4.11.1 Limit of n^{th} term test for divergence
We first provide a proposition whose contrapositive gives the desired test criterion for divergence.

Proposition 4.1 (limit of n^{th} term for a convergent series)

If $\sum_{n=1}^{\infty} a_n$ converges then $\lim_{n\to\infty} a_n = 0$

Proof

Assume that $\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} S_n = \ell$ then because $S_n = S_{n-1} + a_n$ and $\lim_{n\to\infty} S_n = \lim_{n\to\infty} S_{n-1} = \ell$ it follows that $\ell = \lim_{n\to\infty} S_n = \lim_{n\to\infty}(S_{n-1} + a_n) = \lim_{n\to\infty}(S_{n-1}) + \lim_{n\to\infty}(a_n) = \ell + \lim_{n\to\infty} a_n$

That is $\ell = \ell + \lim_{n\to\infty} a_n$ and so $\lim_{n\to\infty} a_n = 0$ ■

Proposition 4.2 (limit of n^{th} term test for divergence)

If $\lim_{n\to\infty} a_n \neq 0$ then $\sum_{n=0}^{\infty} a_n$ diverges. (Contrapositive of proposition 4.1)

Example 4.9

1) For the series $\sum_{n=0}^{\infty} 2^n$ we have:

$\lim_{n\to\infty} 2^n = \infty$ so, the limit of the n^{th} term is not zero so the series diverge.

2) For the series $\sum_{n=1}^{\infty} \frac{n!}{2n!+1}$ we have $\lim_{n\to\infty} \frac{n!}{2n!+1} = \frac{1}{2} \neq 0$ so, the series diverge.

4.11.2 Integral test
Proposition 4.3

If f is positive, continuous and decreasing for $x \geq 1$ and $a_n = f(n)$ then:

$\sum_{n=1}^{\infty} a_n$ and $\int_1^{\infty} f(x)\, dx$ either both diverge or converge.

Proof

Begin by partitioning the interval $[1, n]$ into $n-1$ unit interval as illustrated on Figure 4.1 a and 4.1b. The total area of the inscribed rectangles and the circumscribed rectangles are as follows:

$\sum_{i=2}^{n} f(i) = f(2) + f(3) + \cdots f(n)$ inscribed rectangles

$\sum_{i=1}^{n=1} f(i) = f(1) + f(2) + \cdots f(n-1)$ circumscribed rectangles

The exact area under the graph if from $x = 1$ to $x = n$ lies between the inscribed area and the circumscribed area.

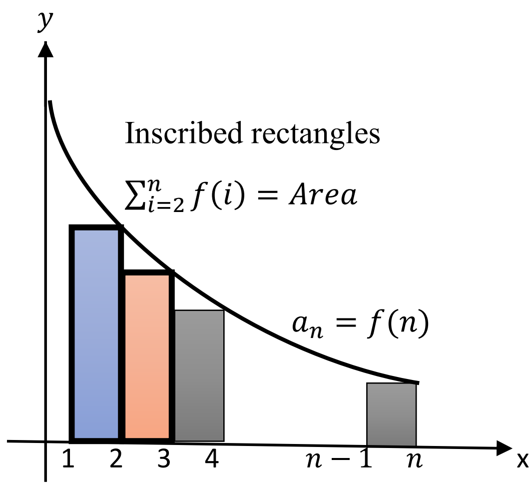


Figure 4.1a

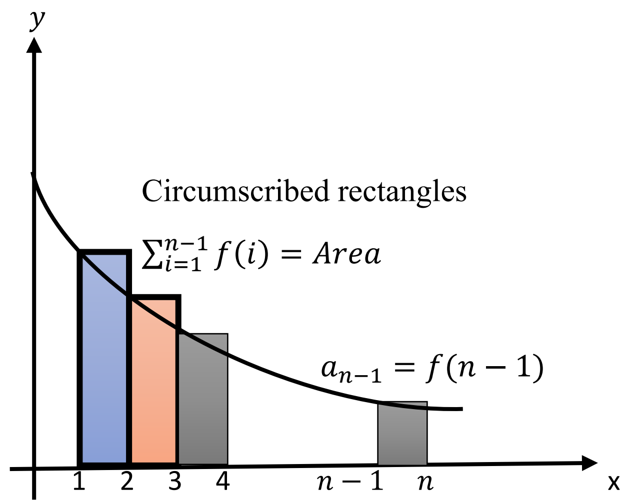


Figure 4.1b

$\sum_{i=2}^{n} f(i) \le \int_1^n f(x)\,dx \le \sum_{i=1}^{n=1} f(i)$ (1)

Using the n^{th} partial sum, $S_n = f(1) + f(2) + \cdots f(n)$ we write (1) as

$S_n - f(1) \le \int_1^n f(x)\,dx \le S_{n-1}$, assuming that $\int_1^\infty f(x)\,dx$ to ℓ it follows that for $n \ge 1$

$S_n - f(1) \le \ell \Rightarrow s_n = \ell + f(1)$

Consequently s_n is bounded and monotonic and by Theorem 4.7, it converges. So $\sum a_n$ converges.

For the other direction proof;

Assume that the improper integral $\int_1^\infty f(x)\,dx$ diverges, then $\int_1^n f(x)\,dx$ approaches infinity as $n \to \infty$ and the inequality $S_{n-1} \ge \int_1^n f(x)\,dx$ implies that (S_n) diverges and so $\sum a_n$ diverges. ■

Example 4.10

Apply the integral test to the series $\sum_{n=1}^{\infty} \frac{n}{n^2+1}$

Solution

$f(x) = \frac{x}{x^2+1}$ is positive and continuous for $x \ge 1$. We find $f'(x) = \frac{-x^2+1}{(x^2+1)^2} < 0$ for $x > 1$ and so f is decreasing. f satisfies the conditions for the integral test.

So $\int_1^\infty \frac{x}{x^2+1}\,dx = \frac{1}{2}\int_1^\infty \frac{2x}{x^2+1}\,dx = \frac{1}{2}\lim_{b\to\infty}\int_1^b \frac{2x}{x^2+1}\,dx = \frac{1}{2}\lim_{b\to\infty}[ln(b^2+1) + ln\,2] = \infty$ so the series diverge.

4.11.3 P-series test

Proposition 4.4

A series of the form $\sum_{n=1}^{\infty}\frac{1}{n^p}=\frac{1}{1^p}+\frac{1}{2^p}+\frac{1}{3^p}+\cdots$

1) Converge if $p>1$ and 2) Diverges if $p\leq 1$

Example 4.11

Discuss the convergence and divergence of

a) Harmonic series and b) p-series with $p=2$.

Solution

a) By the p-series test, it follows that for the harmonic series $\sum_{n=1}^{\infty}\frac{1}{n}=\frac{1}{1}+\frac{1}{2}+\frac{1}{3}+\cdots, p=1$ diverges.
b) It follows from the p-series test that the series $\sum_{n=1}^{\infty}\frac{1}{n^2}=\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\cdots, p=2>1$ so the series converges.

Exercise 4.6

1) Use the integral test to determine the divergence and convergence of the following series

a) $\sum_{n=1}^{\infty}\frac{1}{n+3}$ b) $\sum_{n=1}^{\infty}\frac{2}{3n+5}$ c) $\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+\frac{1}{9}+\cdots$ d) $\sum_{n=1}^{\infty}\frac{1}{n^2+1}$

2) Explain why the integral test does not apply to the following series

a) $\sum_{n=1}^{\infty}\frac{-1^n}{n}$ b) $\sum_{n=1}^{\infty}2+\frac{\sin n}{n}$ c) $\sum_{n=1}^{\infty}\frac{(\sin n)^2}{n}$

3) use the p-series test to determine the convergence of divergence of the following series.

a) $\sum_{n=1}^{\infty}\frac{1}{\sqrt[5]{n}}$ b) $1+\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+\cdots$

4.11.4 Comparison test

Direct comparison

Proposition 4.5

Let $0<a_n\leq b_n$ for all n

1. If $\sum_{n=1}^{\infty}b_n$ converges then $\sum_{n=1}^{\infty}a_n$ converges. 2. If $\sum_{n=1}^{\infty}a_n$ diverges then $\sum_{n=1}^{\infty}b_n$diverges

Proof

To prove the first property, let $L=\sum_{n=1}^{\infty}b_n$ and let $s_n=a_1+a_2+\cdots a_n$. Because $0<a_n\leq b_n$ the sequences $s_1,s_2,s_3,\ldots.$ is nondecreasing and bounded above by L, so it must converge. Because $\lim_{n\to\infty}s_n=\sum_{n=1}^{\infty}a_n$ it follows that $\sum a_n$ converges.

The second property is logically equivalent to the first. ■

Example 4.7

1) Determine the convergent and divergent of the series $\sum_{n=1}^{\infty}\frac{1}{1+3^n}$

Solution

The series $\sum_{n=1}^{\infty}\frac{1}{1+3^n}$ resembles $\sum_{n=1}^{\infty}\frac{1}{3^n}$ converging geometric series. Term by term comparison yields: $a_n = \frac{1}{2+3^n} < \frac{1}{3^n} = b_n$ so, since $\sum_{n=1}^{\infty}\frac{1}{3^n}$ converges then $\sum_{n=1}^{\infty}\frac{1}{1+3^n}$ also converges.

2) Determine the convergent and divergent of the series $\sum_{n=1}^{\infty}\frac{1}{2+\sqrt{n}}$.

Solution

The series $\sum_{n=1}^{\infty}\frac{1}{2+\sqrt{n}}$ resembles $\sum_{n=1}^{\infty}\frac{1}{\sqrt{n}}$ divergent p-series. And $\frac{1}{2+\sqrt{n}} \leq \frac{1}{\sqrt{n}}$ $n \geq 2$ which does not meet the requirement for divergence. We also compare with $\sum_{n=1}^{\infty}\frac{1}{n}$ divergent harmonic series.

$a_n = \frac{1}{n} \leq \frac{1}{2+\sqrt{n}} = b_n$ $n \geq 4$ and by the direct comparison test, the given series converge.

Limit comparison

Proposition 4.6

Suppose that $a_n > 0$, $b_n > 0$, and $\lim_{n\to\infty}\frac{a_n}{b_n} = \ell$ where ℓ is finite and positive. Then the two series $\sum a_n$ and $\sum b_n$ either both diverge of converge.

Proof

Because $a_n > 0$, $b_n > 0$ and $\lim_{n\to\infty}\frac{a_n}{b_n} = \ell$ there exists $N > 0$ such that $0 < \frac{a_n}{b_n} = \ell + 1$ for $n \geq N$. This implies that $0 < a_n < (\ell + 1)b_n$. So, by the direct comparison test, the convergence of $\sum b_n$ implies the convergence of $\sum a_n$. Similarly, the fact that $\lim_{n\to\infty}\frac{b_n}{a_n} = 1/\ell$ can be used to show that the convergence of $\sum a_n$ implies the convergence of $\sum b_n$. ■

NB: Some examples of p-series to use in comparison tests for given series are in the table below

Given series	**p-series**	**Conclusion**
$\sum \frac{\mathbf{1}}{\mathbf{3n^2 - 4n + 5}}$	$\sum \frac{1}{n^2}$	Both series converge
$\sum \frac{\mathbf{1}}{\sqrt{\mathbf{3n - 2}}}$	$\sum \frac{1}{\sqrt{n}}$	Both series diverge
$\sum \frac{\mathbf{n^2 - 10}}{\mathbf{4n^{5+n^3}}}$	$\sum \frac{1}{n^3}$	Both series converge

The table above suggests that when choosing a series for comparison one disregards all but the highest powers of n in both the numerators and the denominator.

Example 4.8

1) Show that the general harmonic series $\sum_{n=1}^{\infty}\frac{1}{an+b}$ diverge.

Solution

By comparing with $\sum_{n=1}^{\infty}\frac{1}{n}$ divergent harmonic series, we have $\lim\limits_{n\to\infty}\frac{1}{an+b}\div\frac{1}{n}=\frac{1}{a}$ because this limit is finite and positive then, the given series diverges.

2) Determine the convergence or divergence of $\sum\frac{\sqrt{n}}{n^2+1}$

Solution

Compare the series with $\sum\frac{\sqrt{n}}{n^2}=\sum\frac{1}{n^{\frac{3}{2}}}$ convergent p-series

Because $\lim\limits_{n\to\infty}\frac{\sqrt{n}}{n^2}\div\frac{1}{n^{\frac{3}{2}}}=\lim\limits_{n\to\infty}\frac{n^2}{n^2+1}=1$ then by limit comparison test the given series converge.

3) Determine the convergence or divergence of $\sum\frac{n2^n}{4n^3+1}$.

Solution

Compare with $\sum\frac{2^n}{n^2}$ divergent series. Now that the series diverge, by n^{th} term test from the limit

$\lim\limits_{n\to\infty}\frac{n2^n}{4n^3+1}\div\frac{2^n}{n^2}=\lim\limits_{n\to\infty}\frac{1}{4+\left(\frac{1}{n^3}\right)}=\frac{1}{4}$ so, the given series diverge.

4.11.5 Alternating test

Proposition 4.7

Let $a_n>0$ the alternating series $\sum(-1)^n a_n$ and $\sum(-1)^{n+1}a_n$ converges if the following conditions are satisfied.

1) $\lim\limits_{n\to\infty}a_n=0$. 2) $a_{n+1}\le a_n$ for all n.

Proof

Consider the alternating series $\sum(-1)^{n+1}a_n$. For this series, the partial sum (where $2n$)

$s_{2n}=(a_1-a_2)+(a_3-a_4)+(a_5-a_6)+\cdots(a_{2n-1}-a_{2n})$ has all nonnegative terms and therefore the sequence (s_{2n}) is a nondecreasing sequence, we can also write:

$s_{2n}=a_1-(a_2-a_3)-(a_4-a_5)-\cdots-(a_{2n-2}-a_{2n})-a_{2n}$ which implies that $s_{2n}\le a_1$ for every integer n. So, (s_{2n}) is bounded, nondecreasing and converges to some value L. Because $s_{2n-1}-a_{2n}=s_{2n}$ and $a_{2n}\to 0$ we have $\lim\limits_{n\to\infty}s_{2n-1}=\lim\limits_{n\to\infty}s_{2n}=\lim\limits_{n\to\infty}a_{2n}=L+\lim\limits_{n\to\infty}a_{2n}=L$

Because both s_{2n} and s_{2n-1} converges to the same limit L, it follows that (s_n) also converges to L. Consequently, the given alternating series converge. ■

4.11.6 Absolute convergence

Proposition 4.8

If a series $\sum |a_n|$ converges then the series $\sum a_n$ converge.

Because $0 \leq a_n + |a_n| \leq 2|a_n|$ for all n, the series $\sum_{n=1}^{\infty}(a_n + |a_n|)$ converges by comparison with the convergent series $\sum 2|a_n|$. Furthermore because $a_n = (a_n + |a_n|) + |a_n|$ we write e$\sum a_n = \sum(a_n + |a_n| - \sum |a_n|$ where both series on the right converge.

So, it follows that $\sum a_n$ converges. ■

The converse of proposition 4.8 is not true. For example, the alternating harmonic series $\sum \frac{(-1)^{n+1}}{n}$ converge by alternating series test. Yet the harmonic series diverge. This type of convergence is called **conditional.**

Note: (1) $\sum a_n$ is absolutely convergent $\sum |a_n|$ converges

(2) $\sum a_n$ conditionally converges if $\sum a_n$ converges but $\sum |a_n|$ diverges

4.11.7 Ratio test

Proposition 4.9

Let $\sum a_n$ be a series with nonzero terms:

1) $\sum a_n$ converges if $\lim\limits_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| < 1$
2) $\sum a_n$ diverges if $\lim\limits_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| > 1$
3) The ratio test is inconsistent if $\lim\limits_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = 1$

4.11.8 Root test

Proposition 4. 10

Let $\sum a_n$ be a series,

1) $\sum a_n$ converges absolutely if $\sqrt[n]{|a_n|} < 1$
2) $\sum a_n$ diverges absolutely if $\sqrt[n]{|a_n|} > 1$
3) The root test is inconsistent if $\sqrt[n]{|a_n|} = 1$

UNIT FIVE

LIMITS AND CONTINUITY

5.1 Limit of a function

Definition:
Let f be a function defined on an open interval containing c (expcept possibly at c) and let L be a real number. L is called a limit of f at c if given $\epsilon > 0$, there exists a $\delta > 0$ (depending on c and ϵ) such that: $|f - L| < \epsilon$ for all $x \in the\ domain\ of\ definition\ of\ f$ satisfying $0 < |x - a|\delta$. We write $\lim_{x\to c} f = L$.
Or

1) Suppose f is defined for all real numbers $x > K$ where $K \in \mathbb{R}$, then $\ell \in \mathbb{R}$ is the limit of f as x tends to ∞ if, given $\epsilon > 0$ there exist a real number K such that $|f - \ell| < \epsilon$ whenever $x > K$ and $\lim_{x\to c} f = \ell$.
2) Suppose f is defined for all real numbers $x < K$ where $K \in \mathbb{R}$, then $\ell \in \mathbb{R}$ is the limit of f as x tends to $-\infty$ denoted by and $\lim_{x\to c} f = \ell$ if, given $\epsilon > 0$ there exist a real number K such that $|f - \ell| < \epsilon$ whenever $x < K$.

Example 5.1

1) Show that $\lim_{x\to 2} x^2 = 4$.

Let $\epsilon > 0$ be given. We need to produce a δ>0 whenever $0 < |x - 2| < \delta$

Now we have $|x^2 - 4| = |(x + 2)(x - 2)| = |x - 2||x + 2|$

Consider all x which satisfy the inequality $|x - 2| < 1$. Then, for all such x we have $1 < x < 3$

Thus $|2 + 2 \leq |x| + 2 < 3 + 2 = 5$. Choose $\delta = \max\{1, \frac{\epsilon}{5}\}$. Then, whenever $0 < |x - 2| < \delta$ we have that $|x^2 - 4| < \epsilon$.

2) Show that $\lim_{x\to 3}(x^2 + 2x) = 15$.
Let $\epsilon > 0$ be given. We need to find a $\delta > 0$ such that: $|(x^2 - 2x) - 15| < \epsilon$ for all x satisfying $0 < |x - 3| < \delta$.
Note that $|(x^2 + 2x) - 15| = |(x + 5)(x - 3)| = |x + 5||x - 3|$ since we are interested in the value of x near 3, we may consider those values of x satisfying the inequality $|x - 3| < 1$ i.e. $2 < x < 4$. For all these values we have that $|x + 5| < 9$. Therefore, if $|x - 3| < 1$ we have that $|(x^2 + 2x) - 15| < 9|x - 3|$. Choose $\delta = \max\{1, \frac{\epsilon}{9}\}$, then, working backwards, we have that: $|(x^2 + 2x) - 15| < \epsilon$ for all x satisfying $0 < |x - 3| < \delta$
3) Show that $\lim_{x\to -1} \frac{2x+3}{x+2} = 1$.
Solution
Let $\epsilon > 0$ be given. We need to find a $\delta > 0$ such that: $\left|\frac{2x+3}{x+2} - 1\right| < \epsilon$ for all x satisfying $0 < |x + 1| < \delta$. Consider values of x which satisfy $|x + 1| < \frac{1}{2}$ i.e. $-\frac{3}{2} < x < -\frac{1}{2}$. Recognizing that $|x + 2| = |x - (-2)|$ as the distance from x to -2 we have

$|x+2| = |x-(-2)| > \left|-\frac{3}{2}-(-2)\right| = \frac{1}{2}$. Thus, $\left|\frac{2x+3}{x+2}-1\right| = \left|\frac{x+1}{x+2}\right| = \frac{|x+1|}{|x+2|} < 2|x+1|$

Choose $\delta = \max\left\{\frac{1}{2}, \frac{\epsilon}{2}\right\}$. Then whoever $o < |x+1| < \delta$ we have that $\left|\frac{2x+3}{x+2}-1\right| < \epsilon$.

4) Show that $\lim_{x\to 0} f(x)$ where $f(x) = \frac{|x|}{x} = \begin{cases} 1 \; if \; x > 0 \\ -1 \; if \; x < x \end{cases}$ does not exists.

Solution

Assume that the limit exists and $\lim_{x\to 0} f(x) = L$. Then with $\epsilon = 1$ there is a $\delta > 0$ such that $|f(x) - L| < 1$ for all x satisfying $0 < x < \delta$. Taking $x = \frac{-\delta}{2}$, we have that $|x| = \frac{\delta}{2} < \delta$ and so, $1 > |f(x) - L| = |-1-L| = |L+1| \Rightarrow |L+1| < 1$

$\Rightarrow -2 < L < 0$ …………………….. (*)

On the other hand, if $x = \frac{\delta}{2}$, we have $|x| = \frac{\delta}{2} < \delta$ so $-1 > |f(x) - L| = |1-L| = |L-1|$

$\Rightarrow |L-1| < 1$

$\Rightarrow 0 < x < 2$ ……………………… (**)

Therefore, there is no real number that simultaneously satisfy equations (*) and (**). So, $\lim_{x\to 0} f(x)$ does not exit.

5) Show that $\lim_{x\to 0} xsin\frac{1}{x} = 0$

Solution

Let $\epsilon > 0$ be given. We need to find a $\delta > 0$, such that: $\left|xsin\frac{1}{x}\right| < \epsilon$ for all x satisfying $0 < |x| < \delta$. We have $\left|xsin\frac{1}{x}\right| < |x| < \epsilon$. Which proves that that $\lim_{x\to 0} xsin\frac{1}{x} = 0$

6) Consider the function $f: \mathbb{R} \to \{0,1\}$ given by $f(x) = \begin{cases} 1 \; if \; x \in \mathbb{Q} \\ 0 \; if \; x \in \mathbb{R}\backslash\mathbb{Q} \end{cases}$

Show that if $a \in \mathbb{R}$, then that $\lim_{x\to a} f(x)$ does not exist.

Solution

Assume that there is an $L \in \mathbb{R}$ such that that $\lim_{x\to a} f(x) = L$. Then with $\epsilon = \frac{1}{4}$, there exist a $\delta > 0$ such that $|f(x) - L| < \epsilon$ for all x satisfying $0 < x - a < \delta$.

If $x \in \mathbb{Q}$ then, we have $|1 - L| < \frac{1}{4}$ whenever, $0 < x - a < \delta$.

If $x \in \mathbb{R}\backslash\mathbb{Q}$ then, we have $|L| < \frac{1}{4}$ whenever, $0 < x - a < \delta$.

Since the set $\{x \in \mathbb{R}: o < |x-a| < \delta\}$ contains both rational and irrationals, we have that $1 = |1-0| = |1-L+L| \leq |1-L| + |L| < \frac{1}{4} + \frac{1}{4} = \frac{1}{2}$.

$\Rightarrow 1 < \frac{1}{2}$, which is absurd. And so $\lim_{x\to a} f(x)$ does not exist.

7) Given $\lim_{x\to 3}(2x-5) = 1$, find δ such that $|2x-5) - 1| < 0.01$ whenever $0 < x - 3 < \delta$

Solution

Given $\epsilon = 0.01$. To find δ notice that $|(2x-5)-1| = |2x-6| = 2|x-3| < 0.01$

Choose $\delta = \frac{1}{2}(0.01) = 0.005$, then $0 < |x-3| < 0.005$ implies that

$|(2x-5)-1| = 2|x_3| < 2(0.005) = 0.001 = \epsilon$. So $\delta = 0.005$.

Exercise 5.1

Use $\epsilon - \delta$ definition of limits to prove that:

a) $\lim_{x \to 2}(3x - 2) = 4$

b) $\lim_{x \to 4}(x + 4) = 8$

c) $\lim_{x \to -3}(x^2 + 3x) = 0$

Theorem 5.1

Let f be defined in some open interval I containing $a \in \mathbb{R}$ except possibly at a. The $\lim_{x \to a} f(x) = L$ if and only if for every sequence $a_n \subset I\{a\}$ such that $\lim_{n \to \infty} a_n = L$ we have $\lim_{n \to \infty} f(a_n) = L$.

Prove

Assume that $\lim_{x \to a} f(x) = L$ and $a_n \subset I\{a\}$ be a sequence such that $\lim_{n \to \infty} a_n = L$. Then given $\epsilon > 0$, there exists a $\delta > 0$ and a $N \in \mathbb{N}$ such that $|f(x) - L| < \epsilon$ for all $x \in I$ satisfying $o < x < \delta$ and $|a_n - a| < \delta$ for all $n \geq N$. Now, $0 < |a_n - a| < \delta$ since $a_n \neq a$ for all $n \geq N$. Thus $|f(a_n) - L| < \epsilon$ for all $n \geq N$. Therefore, $\lim_{n \to \infty} f(a_n) \to L$.

For the converse, assume that for every $a_n \subset I\{a\}$ such that $\lim_{n \to \infty} a_n = L$, we have $\lim_{n \to \infty} f(a_n) = L$.

Claim that $\lim_{x \to a} f(x) = L$. If the claim were false, then there would exist an $\epsilon_o > 0$ such that for every $\delta > 0$ with $0 < |x - a| < \delta$, we have $|f(x) - L| \geq \epsilon_o$. Let $n \in N$ and take $\delta = \frac{1}{n}$, then we can find $a_n \in I\{a\}$ such that $0 < |a_n - a| < \frac{1}{n}$ and $|f(a_n) - L| \geq \epsilon_o$.

Clearly a_n is a sequence in $I \backslash \{a\}$ with the property that $\lim_{n \to \infty} a_n = a$ and $|f(a_n) - L| \geq \epsilon_o$ for all $n \in \mathbb{N}$. Thus that $\lim_{n \to \infty} f(a_n) \neq L$ this is a contradiction. ∎

Theorem 5.2 (uniqueness of limits)

Let f be a function which is defined on some open interval I containing a, except possibly at a. If $\lim_{x \to a} f(x) = L_1$ and $\lim_{x \to a} f(x) = L_2$ then $L_1 = L_2$

Proof

If $L_1 \neq L_2$, let $\epsilon = \frac{|L_1 - L_2|}{3}$. Then there is a $\delta_1 > 0$ and $\delta_2 > 0$ such that

$|f(x) - L_1| < \frac{\epsilon}{2}$ whenever $x \in I$ and $0 < |x - a| < \delta_1$ and $|f(x) - L_2| < \frac{\epsilon}{2}$ whenever $x \in I$ and $0 < |x - a| < \delta_2$. Let $\delta = \max\{\delta_1, \delta_2\}$. Then whenever, $0 < |x - a| < \delta$ we have

$$0 < |L_1 - L_2| \leq |f(x) - L_1| + |f(x) - L_2| < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \frac{|L_1 - L_2|}{6} + \frac{|L_1 - L_2|}{6} = \frac{|L_1 - L_2|}{3}$$

$\Rightarrow |L_1 - L_2| < \frac{|L_1 - L_2|}{3}$, which is impossible and so $L_1 = L_2$. ∎

5.2 Algebra of limits

Theorem 5.3

Let $L_1, L_2, a \in \mathbb{R}$. Suppose that f and g are real valued functions defined on some open interval I containing a, except possibly at a itself, and that $\lim_{x\to a} f(x) = L_1$ and $\lim_{x\to a} g(x) = L_2$ then,

a) $\lim_{x\to a}[f(x) \pm g(x) = L_1 \pm L_2$
b) $\lim_{x\to a} f(x)g(x) = L_1L_2$
c) $\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{L_1}{L_2}$ provided $g(x) \neq 0\ \forall x \in I$ and $L_2 \neq 0$

Proof

a) Let $\epsilon >= 0$ be given. Then there exists $\delta_1 > 0$ and $\delta_2 > 0$ such that:
$|f(x) - L_1| < \frac{\epsilon}{2}$ whenever, $x \in I$ and $0 < |x - a| < \delta_1$ and
$|g(x) - L_2| < \frac{\epsilon}{2}$ whenever, $x \in I$ and $0 < |x - a| < \delta_2$
Let $\delta = \max\{\delta_1, \delta_2\}$. Then, whenever $x \in I$ and $0 < |x - a| < \delta$ we have:
$$|[f(x) + g(x)] - [L_1 + L_2]| = |[f(x) - L_1 + [g(x) - L_2] < |f(x) - L_1| + |g(x) - L_2|$$
$$< \frac{\epsilon}{2} + \frac{\epsilon}{e} = \epsilon$$
Thus $\lim_{x\to a}[f(x) + g(x) = L_1 + L_2$
A similar argument shows that $\lim_{x\to a}[f(x) - g(x) = L_1 - L_2$

b) With $\epsilon = 1$, there exists a $\delta_1 > 0$ such that $|f(x) - L_1| < 1$ whenever $x \in I$ and $0 < |x - a| < \delta_1$.
This implies that $|f(x)| < |f(x) - L_1| + |L_1| < 1 < 1 + L_1$ whenever $x \in I$ $x \in I$ and $0 < |x - a| < \delta_1$.
Now $x \in I$ with $0 < |x - a| < \delta_1$ we have
$$\begin{aligned}|f(x)g(x) - L_1L_2| &= |f(x)g(x) - f(x)L_2 + f(x)L_2 - L_1L_2|\\ &\leq |f(x)||g(x) - L_2| + |L_2||f(x) - L_1|\\ &\leq (1 + L_1)|g(x) - L_2|L_2||f(x) - L_1|\end{aligned}$$
Given $\epsilon > 0$ there exists $\delta_2 > 0$ and $\delta_3 > 0$ such that
$|f(x) - L_1| < \frac{\epsilon}{2(1+L_2)}$ whenever $x \in I$ with $0 < |x - a| < \delta_2$ and
$|g(x) - L_2| < \frac{\epsilon}{2(1+L_2)}$ whenever $x \in I$ with $0 < |x - a| < \delta_3$
Let $\delta = \max\{\delta_1, \delta_2, \delta_3\}$. Then, whenever $x \in I$ with $0 < |x - a| < \delta$ we have
$$\begin{aligned}|f(x)g(x) - L_1L_2| &\leq (1 + L_1)\left[\frac{\epsilon}{2(1 + L_2)}\right] + L_2\left[\frac{\epsilon}{2(1 + L_2)}\right]\\ &< (1 + L_1)\left[\frac{\epsilon}{2(1+L_2)}\right] + (L_2 + 1)\left[\frac{\epsilon}{2(1+L_2)}\right]\\ &< \frac{\varepsilon}{2} + \frac{\epsilon}{2} = \varepsilon\end{aligned}$$
Thus $\lim_{x\to a} f(x)g(x) = L_1L_2$

c) It is enough to show that $\lim_{x\to a}\frac{1}{g}=\frac{1}{L_2}$ provided $g(x)\neq 0$ for all $x\in I$ and $L_2\neq 0$. Since $L_2\neq 0$, $\epsilon=\frac{|L_2|}{2}$>0. Therefore, there exists a $\delta_1>0$ such that $|g(x)-L_2|<\frac{|L_2|}{2}$ whenever $x\in I$ and $0<|x-a|<\delta_1$.

Now for all $x\in I$ satisfying $0<|x-a|<\delta_1$ we have

$$|L_2|\leq|L_2-g(x)|+|g(x)|<\frac{|L_2|}{2}+|g(x)|$$

$\Rightarrow |L_2|-\frac{|L_2|}{2}<|g(x)|$ that is $\frac{|L_2|}{2}<|g(x)|$ for all $x\in I$ satisfying $0<|x-a|<\delta_1$. It now follows that for all $x\in I$ satisfying $0<|x-a|<\delta_1$

$$\left|\frac{1}{g(x)}-\frac{1}{L_2}\right|<\left|\frac{L_2-g(x)}{L_2 g(x)}\right|<\frac{2|L_2-g(x)|}{|L_2|L\_2|}=\frac{2|L_2-g(x)|}{|L_2^2|}$$

Given $\epsilon>0$, there exist $\delta_2>0$ such that $|g(x)-L_2|<\frac{\epsilon L_2^2}{2}$ whenever $x\in I$ and $0<|x-a|<\delta_1$.

Let max $\{\delta_1,\delta_2,\delta_3\}$. Then, whenever $x\in I$ with $0<|x-a|<\delta$ we have

$$\left|\frac{1}{g(x)}-\frac{1}{L_2}\right|\leq\frac{2|L_2-g(x)|}{|L_2^2|}<\frac{\varepsilon L_2^2}{2}.\frac{2}{L_2^2}=\epsilon$$

Thus, $\lim_{x\to a}\frac{1}{g}=\frac{1}{L_2}$ and from part b) of theorem 5.3, $\lim_{x\to a}\frac{f(x)}{g(x)}=\lim_{x\to a}\frac{1}{g(x)}.f(x)=\frac{1}{L2}.L_1=\frac{L_1}{L_2}$ ∎

Theorem 5.4

Let $L_1,L_2,a\in\mathbb{R}$. Suppose that f and g are real valued functions defined on some open interval I containing a, except possibly at a itself and that $f(x)\leq g(x)$. If $\lim_{x\to a}f(x)=L_1$ and $\lim_{x\to a}g(x)=L_2$ then $L_1\leq L_2$.

Prove (Left to the reader)

Theorem 5.5

Let f,g and h be real valued function which are defined on some open interval I containing a, except possibly at a and that $f(x)\leq g(x)\leq h(x)$ for all $x\in I$ if $\lim_{x\to a}f(x)=L$, $\lim_{x\to a}h(x)=L$, then $\lim_{x\to a}g(x)=L$.

Exercise 5.2

1. Prove Theorem 5.4
2. Prove Theorem 5.5
3. Find the limits of

a) $\lim_{x\to -3}\frac{x^2+x-6}{x+3}$

b) $\lim_{x\to 0}\frac{\sqrt{x+1}-1}{x}$

5.3 Continuity of functions

Definition:

Continuity at appoint: A function $\boldsymbol{f}$ **is continuous at** $\boldsymbol{c}$ if the following conditions are met.

i) $f(c)$ is defined

ii) $\lim_{x\to c} f(x)$ exists

iii) $\lim_{x\to c} f(x) = f(c)$

Continuity on an open interval: a function $\boldsymbol{f}$ **is said to continuou**s at an open interval (a, b) if its continuous at each point in the interval. A function that is continuous on the entire real line $(-\infty, +\infty)$ is everywhere continuous.

Or

1) Let $D \subseteq \mathbb{R}$ and $f: D \to \mathbb{R}$. Then function $\boldsymbol{f}$ **is said to be continuous at** $\boldsymbol{a} \in \boldsymbol{D}$ if given $\epsilon > 0$ there exist a $\delta > 0$ such that $|f(x) - f(a)| < \epsilon$ whenever $x \in D$ and $|x - a| < \delta$.
2) Let $D \subseteq \mathbb{R}$ and $f: D \to \mathbb{R}$. Then function $\boldsymbol{f}$ **is said to be continuous at** $\boldsymbol{a} \in \boldsymbol{D}$ if for each $\epsilon - neighbourhood\ \ N(f(a), \epsilon)$ of $f(a)$ there is $\delta - neighbourhood\ N(a, \delta)$ of a such that $f(x) \in N(f(a), \epsilon)$ whenever $x \in N(a, \delta) \cap D$.

Example 5.2

1. Show that $f(x) = x^2$ is continuous on $\mathbb{R}$

Solution

Let $\epsilon > 0$ be given and $a \in \mathbb{R}$. We need to produce a $\delta > 0$ such that:

$|f(x) - f(a)| < \epsilon$ whenever $|x - a| < \delta$.

Now $|f(x) - f(a)| = |x^2 - a^2| = |(x + a)(x - a)|$ since we need the behavior of f near a, we may restrict our attention to those real numbers x that satisfy the inequality $|x - a| < 1$.

$i.e. a - 1 < x < a + 1$ therefore, for all those real numbers we have

$|x + a| \leq |x| + |a| \leq |a + 1| + |a| \leq 1 + 2|a|$. Now take $\delta = \min\left\{1, \frac{\epsilon}{1+2|a|}\right\}$. Then, $|x - a| < \delta$

Thus, f is continuous at a. Since a was arbitrary chosen from $\mathbb{R}$, it follows that f is continuous on $\mathbb{R}$.

2. Show that the function $f(x) = \begin{cases} x\sin\frac{1}{x} & if\ x \neq 0 \\ 0 & if\ x = 0 \end{cases}$ is continuous at 0.

Solution

Let $\epsilon > 0$ be given. We need to produce a $\delta > 0$ such that:

$f(x) - f(0) < \epsilon$ whenever $|x - 0| < \delta$. Now $|f(x) - f(0)| = \left|x\sin\frac{1}{x} - 0\right| = |x||\sin\frac{1}{x}| \leq |x|$

Choose $0 < \delta \leq \epsilon$. Then, $|x - 0| < \delta$ implies that $|f(x) - f(0)| = \left|x\sin\frac{1}{x} - 0\right| \leq |x| < \delta \leq \epsilon$

Thus, f is continuous at 0.

3. Show that the function $f: \mathbb{R} \to \{-1,1\}$ given by $f(x) = \begin{cases} 1 & if\ x \in \mathbb{Q} \\ -1 & if\ x \in \mathbb{R}\backslash\mathbb{Q} \end{cases}$ is discontinuous at every real number.
   Solution

Assume that f is continuous at some number $a \in \mathbb{R}$. Then, given $\epsilon = 1$ there exists a $\delta > 0$ such that $|f(x) - f(a) < \epsilon$ whenever $|x - a| < \delta$. Since rational numbers and irrational numbers are dense in $\mathbb{R}$, the interval $|x - a| < \delta$ contains both rational and irrationals. If $x \in \mathbb{Q}$ and $|x - a| < \delta,$ then $|1 - f(a)| < 1$ whenever $0 < f(a) < 2$,

On the other hand, if $x \in \mathbb{R}\backslash\mathbb{Q}$ and $|x - a| < \delta$ then $|-1 - f(a)| < 1$ whenever $-2 < f(a) < 0$. But there is no real number which simultaneous satisfy the inequalities $0 < f(a) < 2$ and $-2 < f(a) < 0$.

Therefore f is discontinuous at every $a \in \mathbb{R}$.

4. Show that $f(x) = \frac{1}{x}$ is continuous at 1.

Solution

Let $\epsilon > 0$ be given. We need to find a $\delta > 0$ such that $|f(x) - f(1)| < \epsilon$ whenever $|x - 1| < \delta$. Since we are interested in the values of x for which $|x - 1| < \frac{1}{2}$. These x values satisfy the inequality $\frac{1}{2} < x < \frac{3}{2}$. Now for all x which satisfy $|x - 1| < \frac{1}{2}$ we have

$|f(x) - f(1)| = \left|\frac{1}{x} - 1\right| = \left|\frac{x-1}{x}\right| < 2|x - 1|$. Chose $\delta = \min\left\{\frac{1}{2}, \frac{\epsilon}{2}\right\}$. Then, whenever $|x - 1| < \delta$ we have that $\left|\frac{1}{x} - 1\right| < \epsilon$

Theorem 5.6

Let $D \subset \mathbb{R}$ and $f: D \to \mathbb{R}$. Then f is continuous at $a \in D$ if and only if for every sequence $(a_n) \subset D$ such that $\lim_{n\to\infty} a_n = a$, we have that $\lim_{n\to\infty} f(a_n) = f(a)$.

Proof.

Suppose that f is continuous at $a \in D$ and that (a_n) a sequence in D such that the $\lim_{n\to\infty} a_n = a$. Given $\epsilon > 0$, there exist a $\delta > 0$ and an $N \in \mathbb{N}$ such that $|f(x) - f(a)| < \epsilon$ whenever $|x - a| < \delta$ and $|x - a_n| < \delta$ for all $n \geq N$.

Therefore $|f(x_n) - f(a_n) < \epsilon$ for all $n \geq N$. That is $\lim_{n\to\infty} f(a_n) = f(a)$.

For the converse, assume that for every sequence $(a_n) \subset D$ such that $\lim_{n\to\infty} a_n = a$ we have $\lim_{n\to\infty} f(a_n) = f(a)$ and that f is not continuous at a. Then, there exist an $\epsilon_o > 0$ such that for every $\delta > 0$ with $0 < |x - a| < \delta$ we have $|f(x) - f(a)| \geq \epsilon_o$. For $n \in \mathbb{N}$, let $\delta = \frac{1}{n}$ and then we can find $a \in D$ such that $0 < |a_n - a| > \frac{1}{n}$ and $|f(a_n) - f(a)| \geq \epsilon_o$.

Clearly, (a_n) is a sequence in D with the property that $\lim_{n\to\infty} a_n = a$ and $|f(a_n) - f(a)| \geq \epsilon_o$ for all $n \in \mathbb{N}$. That is $\lim_{n\to\infty} f(a_n) \neq f(a)$ a contradiction. ■

Example 5.3

Find the limit of the sequences $\left\{\ell_n\left(\frac{n+1}{n}\right)\right\}$, if it exists

Solution

Since $\lim_{n\to\infty}\frac{n+1}{n}=1$ and the function $f(x)=\ell_n x$ is continuous on $(0,\infty)$, if follows from Theorem 5.6 that $\lim_{n\to\infty}\ell_n\left(\frac{n+1}{n}\right)=\ell_n.1=0$

That is the sequence $\left\{\ell_n\left(\frac{n+1}{n}\right)\right\}$ converges to zero. ■

Exercise 5.3

Show that the function $f(x)=\begin{cases} x & if\ x\in\mathbb{Q} \\ -x & if\ x\ in\mathbb{R}\backslash\mathbb{Q}\end{cases}$ is continuous at $x=0$

Theorem 5.7

Let f and g be functions with a common domain $D\in\mathbb{R}$ and let $a\in D$. If f and g are continuous at a then so are the functions (i) $f\pm g$ (ii) cf for each $c\in\mathbb{R}$ (iii) $|f|$.

Theorem 5.8

Let f be a function which is continuous at $a\in\mathbb{R}$. Suppose that g is a function which is continuous at the point $f(a)$. Then the composition function gof is also continuous at a.

5.4 The intermediate value theory

Theorem 5.9 (Intermediate Value Theorem)

If f is a continuous function on a closed interval $[a,b]$ and $f(a)\neq f(b)$ then for each number k between $f(a)$ and $f(b)$ there is a point $c\in[a,b]$ such that $f(c)=k$.

Proof

For definiteness, assume that $f(a)<f(b)$.

Let $S=\{x\in[a,b]: f\leq k\}$. Then $S\neq\phi$ since $a\in S$. Thus $c=supS$ exists as a real number in $[a,b]$, by Theorem 4.5 there exists a sequence (x_n) in S such that $\lim_{n\to\infty}x_n=c$. Since $a\leq c\leq b$ for each $n\in\mathbb{N}$ we have that $a\leq c\leq b$, and so f is continuous at c. This implies that $\lim_{n\to\infty}f(x_n)=f(c)$.

As $f(x_n)\leq k$ for each $n\in\mathbb{N}$, we deduce that $f(c)\leq k$, and so, $c\in S$. It now remains to show that since $c\in S$, and $c=supS$, $c+\frac{1}{n}\notin S$ for each $n\in\mathbb{N}$. Also, since $k<f(b)$ we have that $c<b$. Therefore, there exists an $n\in\mathbb{N}$ such that $0<\frac{1}{N}<b-c$. Hence for each $n\geq N$ we have that $\frac{1}{n}<b-c$ i.e. $c+\frac{1}{n}<b$. this implies that for all $n\geq N$, $c+\frac{1}{n}\in[a,b]$ and $c+\frac{1}{n}\notin S$. Thus $f\left(c+\frac{1}{n}\right)>k$ for all $n\geq N$. By the continuity of f we obtain that $f(c)\geq k$ hence $f(c)=k$. ■

Theorem 5.9 (Fixed point theorem)

If f is continuous on a closed interval $[a, b]$ and $f(x) \in [a, b]$ for each $x \in [a, b]$, then f has a fixed point. i.e. there is a point $c \in [a, b]$ such that $f(c) = k$.

5.5 Uniform continuity

Definition:
Let $D \subseteq \mathbb{R}$ and $f: D \to \mathbb{R}$. The function f is said to be uniformly continuous on D if given any $\epsilon > 0$ there exist a $\delta > 0$ such that $|f(x) - f(y)| < \epsilon$ whenever $x, y \in D$ and $|x - y| < \delta$.
The most important point to note here is that δ does not depend on any particular point of the domain D- the same δ works for all paints of D.

Example 5.4

1) Show that the function$f(x) = x$ is uniformly continuous on $\mathbb{R}$.

Solution

Let $\epsilon > 0$ be given. We must produce $\delta > 0$ such that $|f(x) - f(y)| < \epsilon$ whenever $x, y \in \mathbb{R}$ and $|x - y| < \delta$. Since $|f(x) - f(y)| = |x - y|$ we may choose $0 < \delta \leq \epsilon$. Then for all $x, y \in \mathbb{R}$ with $|x - y| < \delta$ we have $|f(x) - f(y)| = |x - y| < \delta \leq \epsilon$. Thus f is uniformly continuous on $\mathbb{R}$.

2) Show that $f(x) = x^2$ is not uniformly continuous on $\mathbb{R}$.

Solution

Let $\epsilon > 0$ be given. We must show that for every $\delta > 0$ there exist $x, y \in \mathbb{R}$ such that $|x - y| < \delta$ and $|f(x) - f(y)| = |x^2 - y^2| \geq \epsilon$. Choose $x, y \in \mathbb{R}$ with $x - y = \frac{\delta}{2}$ and $x + y = \frac{2\epsilon}{\delta}$.

Then $|x - y| < \delta$ and $|x^2 - y^2| = |x - y||x + y| \geq \frac{2\epsilon}{\delta} . \frac{\delta}{2} = \epsilon$. Thus f is not continuous on $\mathbb{R}$.

3) Show that the function $f(x) = x^2$ is continuous on $[-1, 1]$.

Solution

Let $\epsilon > 0$ be given. Then for all $x, y \in [-1, 1]$ we have $|f(x) - f(y)| = |x^2 - y^2| = |x + y||x - y| \leq 2|x - y|$. Choose $\delta = \frac{\epsilon}{2}$. Then for all $x, y \in [-1, 1]$ with $|x - y| < \delta$ we have

$$|f(x) - f(y)| < |x^2 - y^2| = |x + y||x - y| \leq 2|x - y| < 2.\frac{\epsilon}{2} = \epsilon.$$

Therefore f is uniformly continuous on $[-1, 1]$.

Exercise 5.4
Show that $f(x) = \frac{1}{x}$ is not uniformly continuous on $[0, 1]$.

Theorem 5.10

If $f: D \to \mathbb{R}$ is uniformly continuous on D, then it is continuous on D.

5.6 Discontinuity

If x is a point in the domain of definition of the function f at which f is not continuous we say that $f$is discontinuous at x or f has a discontinuity at x.

If the function is defined on an interval, the discontinuity is divided into two:

(1) Let f be defined on (a, b). If f is discontinuous at point x and if $f(x^+)$ and $f(x^-)$ exist the f is said to have a discontinuity of the 1^{st} kind or simple discontinuity.

(2) Otherwise the discontinuity is said to be of the 2^{nd} kind.

UNIT SIX

DIFFERENTIATION

6.1 Derivative of a function

Definition:

Let f be defined and real valued on $[a,b]$. For any point $c \in [a,b]$, form a quotient $\frac{f(x)-f(c)}{x-c}$ and define $f'(x) = \lim_{x\to c} \frac{f(x)-f(c)}{x-c}$ provided the limits exists.

f' is called derivative of f.

Observation

1. If f' is defined at point x then f is differentiable at x.
2. $f'(c)$ exists if and only iff for a real number $\epsilon > 0$ $\exists$ a real number $\delta > 0$ such that $\left|\frac{f(x)-f(c)}{x-c}\right| < \epsilon$ whenever $|x-c| < \delta$.
3. $If\ x - c = h$ then, we have: $f'(c) = \lim_{h\to 0} \frac{f(c+h)-f(c)}{h}$.
4. f is differentiable at c if and only if c is removable discontinuity of the function $\phi(x) = \frac{f(x)-f(c)}{x-c}$.

Example 6.1

1. A function $f: \mathbb{R} \to \mathbb{R}$ defined by $f(x) = \begin{cases} x^2 \sin\frac{1}{x}; & x \neq 0 \\ 0 & x = 0 \end{cases}$

This function is differentiable at $x = 0$ because:

$$\lim_{x\to 0} \frac{f(x)-f(0)}{x-0} = \lim_{x\to 0} \frac{x^2 \sin\frac{1}{x} - 0}{x-0} = \lim_{x\to 0} \frac{x^2 \sin\frac{1}{x}}{x} = \lim_{x\to 0} x sin\frac{1}{x} = 0$$

2. Let $f(x) = x^n$, $n \geq 0$ (n is an integer), $x \in \mathbb{R}$. Then:

$$\lim_{x\to c} \frac{f(x)-f(c)}{x-c} = \lim_{x\to c} \frac{x^n - c^n}{x-c} = \lim_{x\to c} \frac{(x-c)(x^{n-1} + cx^{n-2} + \cdots + c^{n-2}x + c^{n-1})}{x-c}$$
$$= \lim_{x\to c} (x-c)(x^{n-1} + cx^{n-2} + \cdots + c^{n-2}x + c^{n-1})$$
$$= nc^{n-1}$$

Implying that f is differentiable everywhere and $f'(x) = nx^{n-1}$.

Theorem 6.1

Let f be defined on $[a,b]$, if f is differentiable at a point $x \in [a,b]$, then f is continuous at x.

Proof

We want that $\lim_{t\to x} \frac{f(t)-f(x)}{t-x} = f'(x)$ where $t \neq x$ and $a < t < b$.

Now $\lim_{t \to x}(f(t) - f(x)) = \lim_{t \to x} \frac{f(t)-f(x)}{t-x} . \lim_{t \to x}(t - x) = f'(x). 0 = 0$

Implying that $\lim_{t \to x} f(t) = f(x)$ which shows that f is continuous at x. ■

Note that the converse of Theorem 6.1 does not hold.

Example 6.2

Let f be defined by $f(x) = \begin{cases} x^2 & if \ \ x > 1 \\ x^3 & if \ x \leq 1 \end{cases}$

Then $D^+f(1) = \lim_{\substack{x \to 1+h \\ h \to 0}} \frac{f(x)-f(1)}{x-1} = \lim_{h \to 0} \frac{f(1+h)-f(1)}{1+h-1} = \lim_{h \to 0} \frac{(1+h)^2-(1)}{h} = \lim_{h \to 0} \frac{h^2+2h+1-1}{h} = 2$

And $D^-f(1) = \lim_{\substack{x \to 1-h \\ h \to 0}} \frac{f(x)-f(1)}{x-1} = \lim_{h \to 0} \frac{f(1-h)-f(1)}{1-h-1} = \lim_{h \to 0} \frac{(1-h)^3-(1)}{-h} = \lim_{h \to 0} \frac{1-3h+3h^2-h^3-1}{-h} = 3$

Since $D^+f(x) \neq D^-f(x)$ then $f'(1)$ does not exist.

6.2 Rules of differentiation

Theorem 6.2 (Rules of differentiation)

Suppose f and g are differentiable on $[a, b]$ and are differentiable at a point x on [a, b], then $f + g$, fg and $\frac{f}{g}$ are differentiable at x and

i) $(f + g)'(x) = f'(x) + g'(x)$ ii) $(fg)'(x) = g(x)f'(x) + f(x)g'(x)$

iii) $\left(\frac{f}{g}\right)'(x) = \frac{g(x)f'(x)-f(x)g'(x)}{[g(x)]^2}$

6.3 Local Maximum

Let f be a real valued function defined on a metric space X, we say that f has a local maximum at point $p \in X$ if there exists $\delta > 0$ such that $f(q) \leq f(p)$ $\forall q, p \in X$ with $d(p, q) < \delta$.

Local minimum is defined otherwise.

Theorem 6.3

Let f be defined on $[a, b]$, if f has a local maximum at a point $x \in [a, b]$ and if $f(t)$ exists then $f'(x) = 0$. (the analogy for local minimum is of course also true)

Proof

Choose δ such that $a < x - \delta < x < x + \delta < b$.

Now if $x - \delta < t < x$ then, $\frac{f(t)-f(x)}{t-x} \geq 0$. Taking limits at $t \to x$ we get

$f'(x) \geq 0$ ---(1)

If $x < t < x + \delta$ then, $\frac{f(t)-f(x)}{t-x} \le 0$. Again, taking limits as $t \to x$ we get

$f'(x) \le 0$ ---(2)

Combining (1) and (2) we get that $f'(x) = 0$. ∎

6.4 Generalization of Mean Value theorem (MVT)
Theorem 6.4

If f and g are continuous real valued functions on closed interval $[a, b]$, then there is a point $x \in (a, b)$ at which $[f(b) - f(a)]g'(x)] = [g(b) - g(a)]f'(x)$.

Theorem 6.5 (Lagrange's MTV)

Let f be (i) Continuous at $[a, b]$

(ii) Differentiable on $[a, b]$

Then $\exists$ a point $x \in (a, b)$ such that $\frac{f(b)-f(a)}{b-a} = f'(x)$

6.5 L'hospital rule
Theorem 6.6

Suppose $f'(x)$ and $g'(x)$ exist $g'(x) \neq 0$ and $f(x) = g(x) = 0$. Then $\lim_{t \to x} \frac{f(t)}{g(t)} = \frac{f'(x)}{g'(x)}$

Proof

$$\lim_{t \to x} \frac{f(t)}{g(t)} = \lim_{t \to x} \frac{f(t) - 0}{g(t) - 0} = \lim_{t \to x} \frac{f(t) - f(x)}{g(t) - g(x)}$$

$$= \lim_{t \to x} \frac{f(t)-f(x)}{t-x} . \frac{t-x}{g(t)-g(x)} = \lim_{t \to x} \frac{f(t)-f(x)}{t-x} . \lim_{t \to x} \frac{\frac{1}{g(t)-g(x)}}{t-x}$$

$$= f'(x) . \frac{1}{g\prime(x)} = \frac{f'(x)}{g'(x)}$$ ∎

Exercise 6.1

1. Prove the Lagrange's Mean value theorem
2. Let f be defined for all real numbers x and suppose that $|f(x) - f(y)| \le |x - y|^2 \; \forall \, x, y \in \mathbb{R}$. Prove that f is constant.
3. If $f'(x) > 0$ in (a, b) then prove that f is strictly increasing in (a, b) and let g be its inverse function, prove that the function g is differentiable and that $g'\big(f(x)\big) = \frac{1}{f(x)}, a < x < b.$
4. Suppose f is defined and differentiable for every $x > 0$ and $f(x) \to 0$ as $x \to +\infty$, put $g(x) = f(x + 1) - f(x)$. Prove that $g(x) \to o$ as $x \to +\infty$.
5. If $f(x) = |x^3|$, then compute $f'(x), f"(x), f'''(x)$ and show that $f'''(0)$ does not exist.
6. Suppose f is defined in the neighborhood of a point x and $f'(x)$ exist. Use Lagrange's mean value theorem to show that $\lim_{h \to 0} \frac{f(x+h)+f(x-h)-2f(x)}{h^2} = f"(x)$

End

Congratulation upon finishing this course unit. As you prepare to take your examination, I wish you the very best.

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