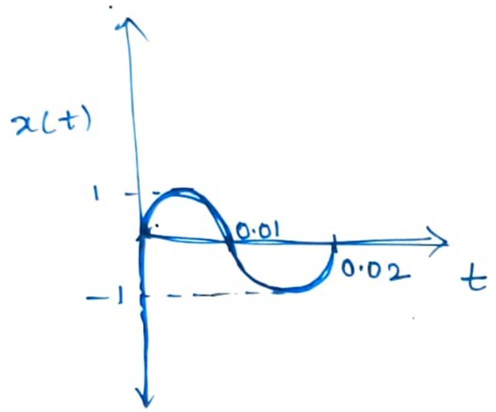
Assignment - 1 2024102014

1. Nyquist Rate and Quantization

Q.1) $x(t) = \sin(100\omega t) = \sin(100\pi\omega t)$

$$f = \frac{\omega}{2\pi} = \frac{100\pi}{2\pi} = 50Hz$$



Nyquist Rate

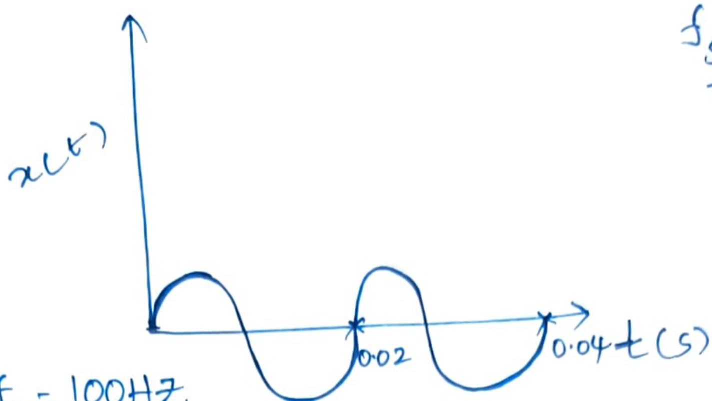
$$f_{sampling} \geq 2 f_{max}$$

$f_{max} = f_{\substack{Bandwidth \\ (used)}} = f_{\substack{max\ frequencey\ of\ x(t) \\ (term)}}$

$$f_s = f_{Sampling} \geq 2 \times 50 = 100Hz$$

(1) $f_s = 50\ Hz$

it's almost equal to signal frequencey only 1 sample taken per cycle so The sample point's all aligned to a constant value and such that reconstruction of signal is not possible. The reconstructed signal could appear as a DC signal or signal with higher time period.



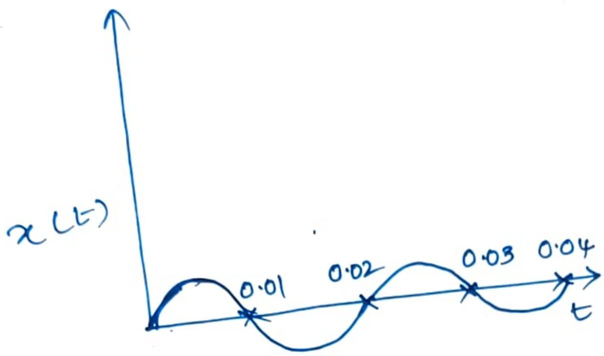
$f_s = 50HZ$

$T_s = \frac{1}{50}$ [redacted] s

$T_s = 0.02$ s

The samples can also be other values with T_s (0.02s) i.e (t = x, t = x + T_s) so.on
↓ for sample1 cycle
↓ for sample 2 cycle

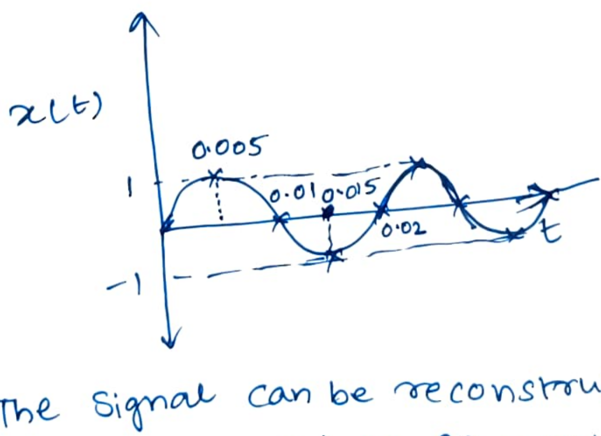
(ii) $f_s = 100HZ$



2 samples per cycle
exactly equals to nyquist rate, $f_s = 2f$
The samples can also be other values with time difference (0.01) = T_s, signal can be reconstructed but still some noise appears in signals since no.of levels are low.

(iii) $f_s = 200HZ$

4 samples per cycle



$f_s \geq 2f$

The signal can be reconstructed more accurately than compared to ($f_s = 100HZ$).

200HZ also samples more than nyquist rate hence reconstruction possible.

Q2)

$$Q_{noise} = \frac{\Delta^2}{12}$$

given,
Quantizer has (256 levels) 8bits

$$\Delta = \frac{\text{Signal Range}}{\text{No. of levels}} = \frac{x_+ - x_-}{256}$$

1. $x(t) = 2\cos(t)$

$x_+ = 2(1)$
$x_- = 2(-1)$

$$\frac{\text{Signal range}}{\text{levels}} = \frac{2-(-2)}{256} = \frac{1}{64}$$

$$Q_{noise} = \frac{\Delta^2}{12} = \frac{1}{(64)^2 \times 12}$$

2. $x(t) = -|\sin(t)|$

$max = x_+ = 0$
$min = x_- = -1$
Signal range $= 0-(-1)$

$$\frac{\text{Signal range}}{\text{Levels}} = \frac{1}{256}$$

$$Q_{noise} = \frac{1}{(256)^2 \times 12}$$

3. $x(t) = \begin{cases} t, & -T/2 < t < T/2 \\ -t+T, & T/2 < t < 3T/2 \end{cases}$

$x_+ = T/2$

$x_- = -T/2$

$x_+ - x_- = \frac{T}{2} - \left(-\frac{T}{2}\right) = T$

$\Delta = \frac{T}{256}$

$Q_{noise} = \left(\frac{T}{256}\right)^2 \times \frac{1}{12}$

from all above we can see that if more levels are used to quantize less amplitude then noise decreases.

2. Introduction to probability

3) given,

6th August 2022 is humid.

Prob; weather of tomorrow will be same as today is p.
for

No. of days from ~~7~~th August 2022 to 18th November 2022 is $25+30+31+18 = 104$

Probability that tomorrow's is same as today $= p$

Probability that tomorrow's weather changes $= 1-p$

$\boxed{E + \bar{E} = 1}$

given that there's only two types of weather's are there (humid & dry)

So, if 2 times the weather changes in following days then the weather in the ~~same~~ present day will also same as (3rd day)
(1st day)

So, for every even no. of changing [weather from present to next day] will be same ~~as~~ weather as present day.

So, in n days (n choose r) ^nC_r to be (1-P) such that (r is even) weather changes from present to next day.
and remaining days will atomatically same with probability P.

The current day is humid and if above conditions are followed the nth day will also be same.

i.e, all cases should be added.
(possibilites)

$$P_n = {}^nC_0 (1-P)^0 (P)^n + {}^nC_2 (1-P)^2 (P)^{n-2}$$

$$\ldots + {}^nC_n (P)^0 (1-P)^n + {}^nC_{n-1} (P)^1 (1-P)^{n-1}$$

(if n is even) (if n is odd)

assuming P_n as Partial Binomial expansion

$$2P_n = 2\left[{}^nC_0 (1-p)^0 (p)^n + {}^nC_2 (1-p)^2 (p)^{n-2}\right] +$$

$$\left[+{}^nC_1 (1-p)^1 (p)^n - {}^nC_1 (1-p)^1 (p)^n + - + - + - \ldots\right]$$

adding and subtracting missing terms

$$2P_n = [p+(1-p)]^n + [p-(1-p)]^n$$

$$P_n = \frac{1+(2p-1)^n}{2}$$

Substituting $n = 104$

$$P_{104} = \frac{1+(2p-1)^{104}}{2}$$

WKT, $0 \le p \le 1 \Rightarrow 0 \le 2p \le 2 \Rightarrow -1 \le 2p-1 \le 1$

$\Rightarrow (-1)^n \le (2p-1)^n \le (1)^n$

$-1 \le (2p-1)^n \le 1 \Rightarrow 0 \le 1+(2p-1)^n \le 2$

$\Rightarrow 0 \le \frac{1+(2p-1)^n}{2} \le 1$

hence probability is valid (also valid for $n=0,1$)

4) To find probability that two or more persons from group of n people have the same birthday.

$P = 1-$[Probability n people having distinict birthday]

$P = 1-p'$

$$P' = \frac{365}{365} \times \frac{364}{365} \times \cdots \times \frac{365-n+1}{365}$$

assigning 1 date to 1 person to be distinict

$$P' = \frac{(365)!}{(365)^n (365-n)!}$$

$$P' = \begin{cases} \dfrac{(365)!}{(365)^n (365-n)!} & \text{if } n \le 365 \\ 0 & \text{if } n > 365 \end{cases}$$

$$P = 1 - \frac{365!}{(365)^n (365-n)!}$$

$$P = 1 - P'$$

if $n = 23$ we get,

$$P \approx 0.5$$

Q5) four sided dice.

1,1	1,2	1,3	1,4
2,1	2,2	2,3	2,4
3,1	3,2	3,3	3,4
4,1	4,2	4,3	4,4

$X \rightarrow$ result on first die

$Y \rightarrow$ result on second die.

$\max(X,Y) = m = P(A)$

$\min(X,Y) = 2 = P(B)$

(1) $m = 1$

$\boxed{P(A) = 0}$ $\max(X,Y) = 1$ is not possible

hence probability $= 0$.

$P(A|B) = 0$ (also zero)

(ii) $m=2$

$$n(A) = \max(x,y)=2 = (1,2),(2,1),(2,2) \quad [n(S)]$$

$$\min(x,y)=2 = n(B) = (2,2),(3,2),(2,4),(2,3),(4,2)$$

$$n(A\cap B)=1$$

$$P(A|B) = \frac{P(A\cap B)}{P(B)} = \frac{\frac{n(A\cap B)}{n(S)}}{\frac{n(B)}{n(S)}} = \frac{\frac{1}{16}}{\frac{5}{16}} = \frac{1}{5}$$

(iii) $m=3$

$$n(A) = (1,3),(3,1),(2,3),(3,2),(3,3) = 5 = \max(x,y)=3$$

$$n(B) = (2,3)\{3,2\} \quad (2,4),(4,2),(2,2) = 5 \quad \min(x,y)=2$$

$$\frac{P(A\cap B)}{P(B)} = \frac{\frac{n(A\cap B)}{n(S)}}{\frac{n(B)}{n(S)}} = P(A|B) = \frac{\frac{2}{16}}{\frac{5}{16}} = \frac{2}{5}$$

Q6) given,
A man is known to tell the truth 3/4 times

1) Reports it to be an Ace of red suits in 1st pick
2) without replacement Reports it to be a hearts card which is known to be an Ace. (2nd pick)

Q6)

To find the probability he was telling the truth the first time

Event space

P (I) = Ace heartcard , Truth Ace others False

P (II) = Ace Diamond . Truth Ace others (Black suit) False

P (III) = Ace Diamond , Truth Ace heart . Truth

P (IV) = Not red Ace , False Ace heart , T

P (V) = Ace ~~Not~~ black suit , False Ace diamond (or) other black Ace , F

P (VI) = Not Ace (Any) , False , Ace Diamond (or) Black suit , F

$P_1 = \frac{1}{52} \times \frac{3}{4} \times \frac{3}{51} \times \frac{1}{4}$ $\quad P_2 = \frac{1}{52} \times \frac{3}{4} \times \frac{1}{51} \times \frac{3}{4}$

$P_3 = \frac{1}{52} \times \frac{3}{4} \times \frac{2}{51} \times \frac{1}{4}$ $\quad P_4 = \left(1 - \frac{2}{52}\right) \times \frac{1}{4} \times \frac{1}{51} \times \frac{3}{4}$

$P_5 = \frac{2}{52} \times \frac{1}{4} \times \frac{2}{51} \times \frac{1}{4}$ $\quad P_6 = \left(1 - \frac{4}{52}\right) \frac{1}{4} \times \frac{3}{51} \times \frac{1}{4}$

By Bayes theorem.

$$P(A/B) = \frac{P_1 + P_2 + P_3}{P_1 + P_2 + P_3 + P_4 + P_5 + P_6} = \frac{12}{161} \approx 0.0753$$

Q7) Given,

the pmf of a r.v X is given as

$$P(i) = c \cdot \frac{\lambda^i}{i!} \quad , i = 0,1,2$$

$$\lambda > 0$$

(a) $P(X=0) = P(0) = \frac{c\lambda^0}{0!} = c$

$P(X=0) = c$

(b) $P(X>2) = 1 - P(X \le 2) = 1 - (P(X=2) + P(X=1) + P(X=0))$

$$= 1 - c\left(\frac{\lambda^2}{2!} + \frac{\lambda^1}{1!} + \frac{\lambda^0}{0!}\right)$$

$$P(X>2) = 1 - c\left(\frac{\lambda^2}{2} + \lambda + 1\right)$$

now for the value of c,

we can add all mf values for $i=0,1,2$

then, $\sum_{n=0}^{\infty} P(X=n) = 1$

$$\sum_{n=0}^{\infty} \frac{c\lambda^n}{n!} \Rightarrow c \sum \frac{\lambda^n}{n!}$$

WKT, $\sum_{n=0}^{\infty} \frac{\lambda^n}{n!} = e^{\lambda}$

$c \cdot e^{\lambda} = 1$

$c = e^{-\lambda}$

(a) $P(X=0) = e^{-\lambda}$

(b) $P(X>2) = 1 - e^{-\lambda}\left(\frac{\lambda^2}{2} + \lambda + 1\right)$

Q8) Given,

X is a r.v with Parameter λ

$$P_X(k) = \frac{e^{-\lambda}\lambda^k}{k!}, \quad k = 0, 1, 2$$

$$P\{X = k\} = P_X(k) = \frac{e^{-\lambda}\lambda^k}{k!}$$

differentiating above eqn we get

$$\frac{d\,P_X(X=k)}{d\lambda} = \frac{-e^{-\lambda}\lambda^k}{k!} + \frac{e^{-\lambda}k\lambda^{k-1}}{k!} = 0$$

from here

$$\frac{e^{-\lambda}\lambda^k}{k!} = \frac{e^{-\lambda}k\lambda^{k-1}}{k!}$$

$$\boxed{\lambda = k}$$