Linear Algebra Assignment-2 2024102014

1Q. Row-equivalence is an equivalence relation

A matrix said to be in equivalence

if, $A \equiv A$ Reflexive

$A \equiv B \Rightarrow B \equiv A$ symmetric

$A \equiv B, B \equiv C, A \equiv C$ Transitive

(i) Reflexive

$A \equiv A$

apply operation e on A and e be '1'

$e(A) = 1(A) = I(A) = A$

$\therefore$ Row-equivalence is an Reflexive relation

(ii) Symmetric

$A \equiv B \Leftrightarrow B \equiv A$

let sequence of (elementary) row operation matrices

$e_1, e_2, e_3 \dots e_k$ such that

$$B = e_k(e_{k-1}(\dots(e_2(e_1(A)))))$$

Since every operation e_i is invertible

apply $= e_k', e_{k-1}' \dots e_2', e_1'$ on B

$$A = e_k'(e_{k-1}'(\dots(e_3(e_2(e_1(B))))))$$

Hence, $A \Leftrightarrow B$

3. Transitivity ($A \equiv B$, $B \equiv C$, $C \equiv A$)

If $A \sim B$ and $B \sim C$, then there exist row operation such that $e_1 \dots e_k$ $k \in \mathbb{N}$

$$B = e_k(e_{k-1}(\cdots(e_2(e_1(A))))) \quad -(1)$$

and similarly $F_1, F_2 \dots F_m$ $m \in \mathbb{N}$

such that

$$C = F_m(\cdots(F_2(F_1(B))))$$

From (1)

$$C = \underbrace{F_m(F_{m-1}(\cdots(F_2(F_1(e_k(\cdots(e_2(e_1(A)))))))))}_{G}$$

G (Let set of elementary operation be G)

$C = GA$

$\Rightarrow A = G^{-1}(C)$

$\therefore C \Longleftrightarrow A$

Hence $A \equiv B$, $B \equiv C$, $C \equiv A$

So,

Row equivalence satisfies reflexivity, symmetry and transitivity, $\therefore$ it's an equivalence relation

2(Q).

② Q

$$A_{m\times n} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}_{m\times n}$$

find
The first non-zero element in coloumn 1 and swap rows

$$R_j \longleftrightarrow R_1$$

R_j contains non-zero element in coloumn in 1

$$R_1 \longrightarrow \frac{1}{a_{11}} R_1$$

$$A = \begin{bmatrix} 1 & \frac{a_{12}}{a_{11}} & \cdots & a_{1n}/a_{11} \\ a_{21} & a_{21} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}_{m\times n}$$

For $\forall\, i>1$, eliminating a_{i1} using $R_i \to R_i - a_{i1}R_1$. For row i,

$$R_i \longrightarrow \left[a_{i1} - a_{i1}\cdot 1,\ a_{i2} - a_{i1}\cdot\frac{a_{12}}{a_{11}} \cdots\cdots a_{in} - a_{i1}\cdot\frac{a_{1n}}{a_{11}} \right]$$

then,

$$A = \begin{bmatrix} 1 & \frac{a_{12}}{a_{11}} & \cdots & \frac{a_{1n}}{a_{11}} \\ 0 & a'_{22} & \cdots & a'_{2n} \\ \vdots & \vdots & & \vdots \\ 0 & a'_{m2} & & a'_{mn} \end{bmatrix}$$

Now, consider the submatrix A'

by rows $(2-m)$

Columns $(2-n)$

* if top left element zero the find the first non zero element and move to top
* if all elements are zero in 1st column in A' then move to another submatrix
* by Repeating above steps similarly,

we get
$$\begin{bmatrix} 1 & a_{12}/a_{11} & \cdots & \frac{a_{1n}}{a_{11}} \\ 0 & 1 & \cdots & a''_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a'''_{mn} \end{bmatrix}_{m\times n}$$

* Arrange zero rows at bottom
(If there are rows with all zeroes move them to bottom)
* Eliminate entries below pivot by (m,n).
* pivot $\longrightarrow$ each leading non zero entry

* then, now eliminate all entries above each pivot by $R_k \to R_k - (\text{coefficient}) R_i$, where R_i contains the pivot.

* Arrang zero rows if any at bottom.

By doing this unti submatrix row $(m-1 \text{ to } m)$, Column $(n-1 \text{ to } n)$

we can obtain row reduced matrix $R(A_{m\times n})$

by $A_{m\times n} = R(A_{m\times n})$.

Thus, every $m\times n$ matrix is row-equivalent to a row-reduced form.

3) $AX=0$ $A_{m\times n}$ $m<n$

$$A=\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}_{m\times n}, \quad X=\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

let $\{B\}$ be some (row) operation which makes A as row-reduced echelon

$$R_{(A)}=\{B\}(A)=e_k(e_{k-1}(\cdots e_2(e_1(A))))$$

$$R_{(A)}=A=\begin{bmatrix} \underbrace{\begin{matrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{matrix}}_{m} & \underbrace{\begin{matrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & & \vdots \\ b_{m1} & b_{m2} & \cdots & b_{mn} \end{matrix}}_{m-n} \end{bmatrix}_{m\times n} \quad (m<n)$$

here m / <m variables are, can be only fully constrained variable remaining n-m variables are "free variables" i-e they can take arbitary values

We can assign arbitary values (infinite), Leading to non-zero solutions for X

The eqn, from $A\cdot X=0$.

$$x_1+\sum^{n-m} b_{mj}x_{m+j}=0.$$

$$x_2+\sum_{1}^{n-m} b_{mj}\cdot x_{m+j}=0.$$

$$\vdots$$

$$x_m+\sum_{1}^{n-m} x_{m+j}\cdot b_{mj}=0.$$

depending on the values of x_1 ... to ... x_m

$\exists$ considerable values of x_{m+1} to x_n.

to ~~get~~ Obtain the ~~values~~ equations equals to RHS.

and can be vice versa based on values of any set of values $(x_{m+1} - x_n) \rightarrow \exists$ a solution.

hence there exist non-trivial solutions

$\therefore$ hence proved.

$A_{m\times n}$ matrix such that $m<n$, then $AX=0$ always has non-trivial solⁿ.

(4)

(1) A is row equivalent to I, then $AX=0$ has only trivial solution

~~Lent~~ Let matrix $A_{n\times n}$

By defⁿ $\exists$ a sequence of elementary row operations that A into I.

$$A \rightarrow e_k(e_{k-1}(\cdots(e_3(e_2(e_1(A)))))) = A' = I$$

by 3rd proof $EA = e(A) = e(I).A$ (multiple times we apply)

e be set of operations $e_k(e_{k-1}(e_{k-2}\cdots(e_2(e_1))))$

$e(I) = P$ Suchthat $PA = I$

apply on $AX = 0$

$$P(AX) = P(0)$$

$$(PA)X = IX = 0$$

This simplifies to $X=0$.

This shows that there's only soln to $AX=0$ is the trivial solution $X=0$

If A is row equivalent to I, the Homogenous system $AX=0$ has only trivial soln.

(ii) If ~~A is row~~ $AX=0$ has only trivial soln, then

* A is row equivalent to I.

$X = \{0, 0, \cdots 0\}$

* Assume $AX=0$ has only trivial soln

* By applying elementary row operations to bring echelon (row-reduced) form (nxn) only

* Since $AX=0$ has only the trivial sol^n^.

* every row of A is linearly independent

* The row reduced echelon form of A must

∴ have (leading 1s) in all n rows.

* unless any two or more rows are identical or zero rows present in A.

This can ensure.

* A can be transformed into identity matrix I.

* from (ii) & (i)

∴ A is row equivalent to I $\Leftrightarrow$ $AX=0$ has trivial soln

5 given $A_{m\times n}$

$E = e(\mathbb{1}_{m\times m})$ $\quad m,n \in \mathbb{N}$

$R_i = [a_{i1}, a_{i2}, a_{i3}, \dots a_{in}]$

$i,j \in (0,m) \in \mathbb{N}$

(case-1) let e be $R_i \rightarrow R_i + cR_j$

To prove: $e(A) = e(\mathbb{1}_{m\times m}) \cdot A = E\,A$

$$e(\mathbb{1}_{m\times m}) = e\Big(\underbrace{\begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & & 0 & 1 \end{bmatrix}}_{m}\Big)_{m\times m} \Big\} m$$

$$e(\mathbb{1}_{m\times m}) = \underbrace{\begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 \cdots 1 \cdots 0 & \cdots & 0 & 0 \\ 0 & 0 & 0\, C \cdots 1 & \cdots & 0 & 0 \\ \vdots & & & & & \vdots \\ 0 & 0 & 0 & 0 \cdots & 0 & 1 \end{bmatrix}}_{m}{}_{m\times m} = E$$

$$e(\mathbb{1}_{m\times m})\cdot A_{m\times n} = \begin{array}{c} \\ \\ (i) \\ (i+1) \\ \\ \end{array}\begin{bmatrix} 1 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 \cdots & 1 \cdots & 0 \\ 0 & 0 \cdots & C\,1 \cdots & 0 \\ 0 & 0 & \cdots & 0\;1 \end{bmatrix}_{m\times m} \cdot \underbrace{\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{k1} & a_{k2} & \cdots & a_{kn} \\ a_{k+1,1} & a_{k+1,2} & \cdots & a_{k+1,n} \\ \vdots & & & \vdots \\ & & & a_{mn} \\ a_{n1} & a_{n2} & & \end{bmatrix}}_{n}{}_{m\times n}$$

$$E\cdot A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & a_{kn} \\ a_{k1} & a_{k2} & \cdots & \\ a_{k+1,1}+c\cdot a_{k1} & a_{k+1,2}+c\cdot a_{k2} & \cdots & a_{k+1,n}+c\cdot a_{kn} \\ \vdots & & & \vdots \\ a_{m,1} & a_{m,2} & \cdots & a_{m,n} \end{bmatrix}_{m\times n} \quad -(1)$$

$$e(A) = e\left(\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{k1} & a_{k2} & & a_{kn} \\ a_{k+i,1} & a_{k+i,2} & & a_{k+i,n} \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}_{m\times n}\right)$$

Let i, j be $k+i, k$ Respectively, $i \in [-k, m-k] \in \mathbb{N}$

$$e(A) = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{k1} & a_{k2} & \cdots & a_{kn} \\ a_{k+i,1}+c\cdot a_{k1}, & a_{k+i,2}+c a_{k2} & \cdots & a_{k+i,n}+c\cdot a_{kn} \\ \vdots & & & \vdots \\ a_{m1}, & a_{m2} & \cdots & a_{mn} \end{bmatrix}_{m\times n} \longrightarrow (2)$$

From (1) & (2)

$$e(A) = E\cdot A$$

$$e(A) = e(\mathbb{I}_{m\times m})(A_{m\times n})$$

$\therefore$ Hence proved.

(case - II)

Let e be $R_i \longrightarrow cR_i$

$$I_{m\times m} \Rightarrow (\mathbb{I}_{m\times m}) = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & & \ddots & \\ 0 & 0 & \cdots & 1 \end{bmatrix}_{m\times m} \Longrightarrow e(I)_{m\times m} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & \vdots \\ 0 & 0 & c & 0 \\ 0 & 0 & \cdots & 1 \end{bmatrix}_{m\times m} = E$$

$$A_{m\times n} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1}, & a_{m2} & \cdots & a_{mn} \end{bmatrix}_{m\times n} \Rightarrow e(A_{m\times n}) = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & & \vdots \\ ca_{i1} & ca_{i2} & \cdots & a_{in} \\ a_{m1}, & a_{m2} & \cdots & a_{mn} \end{bmatrix}_{m\times n} \longrightarrow (1)$$

$$e(I)\cdot A = \underset{m\times m}{E}\cdot\underset{m\times n}{A} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & & \vdots \\ 0 & 0 & c & \\ 0 & \cdots & & 1 \end{bmatrix}_{m\times m} \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1}, & \cdots & a_{mn} \end{bmatrix}_{m\times n}$$

$$E\cdot A = \begin{bmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \\ ca_{i1} & \cdots & ca_{in} \\ a_{m1} & \cdots & a_{mn} \end{bmatrix}_{m\times n} \longrightarrow (2)$$

By ① & ②

$e(I) \cdot A = e(A)$

$E \cdot A = e(A)$

∴ Hence proved.

Case-(III) Interchanging rows

let e be $\begin{bmatrix} R_i \longrightarrow R_j \\ R_j \longrightarrow R_i \end{bmatrix}$

$$I_{m\times m} \xrightarrow{e(I)} \begin{bmatrix} 1 & 0 & \cdots & & & 0 \\ 0 & 1 & \cdots & & & 0 \\ \vdots & & & & & \\ 0 & 0 & \cdots & 0 & 1 & 0 \\ 0 & 0 & \cdots & 1 & 0 & 0 \\ \vdots & & & & & \\ 0 & 0 & \cdots & & & 1 \end{bmatrix}_{m\times m}$$

$$A \xrightarrow{e(A)} \begin{bmatrix} a_{11} & \cdots & a_{n1} \\ \vdots & & \\ a_{(i-1)1} & \cdots & a_{n(i-1)} \\ a_{j1} & \cdots & a_{jn} \\ \vdots & & \\ a_{(j-1)1} & \cdots & a_{(j-1)n} \\ a_{(i)1} & \cdots & a_{(i)n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix}_{m\times n}$$

exchanged

By multiplying $e(I)$ with A we get the same as $e(A)$

$$e(I)(A) = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & \cdots & 1 & 0 & 0 \\ 0 & \cdots & 1 & 0 & 0 \\ 0 & \cdots & & & 1 \end{bmatrix}_{m\times m} \begin{bmatrix} a_{11} & \cdots & a_{n1} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{bmatrix}_{m\times n} = \begin{bmatrix} a_{11} & \cdots & a_{n1} \\ \vdots & & \vdots \\ a_{j-11} & & a_{j-1n} \\ a_{j1} & & a_{jn} \\ a_{j-11} & & a_{j-1n} \\ a_{i1} & & a_{in} \\ \vdots & & \vdots \\ a_{m1} & & a_{mn} \end{bmatrix}_{m\times n}$$

∴ satisfied for row interchange property.

By above 3-cases we ~~have~~ come to know that

For, $e(I) = E$

$$e(A) = E \cdot A = e(I)A$$

$\therefore$ This saties for all 3 types of row operations

$\therefore$ Hence proved.

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Let, A is invertible (statement-1 be true).

$AA^{-1} = A^{-1}A = I$

(b) Let, $AX = 0$ (Pre-multiply A^{-1} on Both sides)

$A^{-1}AX = A^{-1}0$

$IX = A^{-1}0$

$IX = 0$

$IX = 0$ has simplifed to $X = 0$

$[x_1, x_2, x_3 \ldots x_n] = \{0\}$

$Ix = 0$ has only trivial soln

$\therefore$ Hence (b) statement proved

(c) from (b) $AX = 0$ only has trivial solns

$\{0, 0 \ldots, 0\}$ is the only Solution for $AX = 0$

$\Rightarrow$ "$AX = 0$ has only a trivial soln

A is row equivalent to I

$e_1(e_2 \cdots (e_k(A))) = I$

then for system $AX = Y$

By applying row operations
$e_1(e_2(\cdots(e_k)))$ on both sides

By theorem

$$e(A) = e(I)\cdot A$$

$$e_1(e_2 \cdots e_k(A)))\cdot X = e_1(e_2 \cdots e_k(I)\cdot A\cdot X$$

$$IX = e_1(e_2 \cdots e_k(I)\cdot Y$$

we get atleast one solution for any Y as each row of I gives each unknown.

(a) from (c)
$AX = Y$ has a solⁿ for every

$Y_{n\times 1}$ matrix

$Y = O_{n\times 1}$ Let, $Y = 0$

we get

$AX = 0$ has atleast a soln (unique soln)

for X,

we know that it also should have a trivial soln

then,

By theorem,

$A_{n\times n}$ is row equivalent to $\mathbb{I}_{n\times n}$
then $\exists$ set of elementary row operations
$e_k(e_{k-1}(\cdots e_2(e_1)))$ such that

$$e_k(\cdots(e_2(e_1(A))) = \mathbb{I}_{n\times n}$$

By Applying multiple times as $e(A) = EA$

$$\underbrace{\underset{n\times n}{E_k} \cdot \underset{n\times n}{E_{k-1}} \cdots \underset{n\times n}{E_2} \underset{n\times n}{E_1}}_{A^{-1}} A_{n\times n} = A_{n\times n} \cdot \underbrace{\underset{n\times n}{E_k} \cdot E_{k-1} \cdots E_{2\,n\times n} E_1}_{A^{-1}} = I_{n\times n}$$

$$A^{-1} \cdot A = AA^{-1} = I_{n\times n}$$

then A is Invertible

$\therefore$ Hence proved.