Week 5

Question: Which of the following is a measure of the amount of variance explained by a principal component in Principal Component Analysis (PCA)?

1. Covariance
2. Correlation
3. Mean absolute deviation
4. Eigenvalue

Correct Answer: 4. **Eigenvalue**

In Principal Component Analysis (PCA), the eigenvalue of a principal component is a measure of the amount of variance explained by that component. Each eigenvalue corresponds to a principal component and indicates how much of the data's total variance is captured by that component.

What is/are the limitations of PCA?

It can only identify linear relationships in the data.

It can be sensitive to outliers in the data.

Q. Which of the following is a property of eigenvalues of a symmetric matrix?

1. All eigenvalues of a symmetric matrix are real numbers.
2. Eigenvectors corresponding to distinct eigenvalues are orthogonal.
3. A symmetric matrix can be diagonalized by an orthogonal matrix.
4. The algebraic multiplicity equals the geometric multiplicity for eigenvalues.
5. Eigenvalues of a positive definite symmetric matrix are all positive.

If the eigenvalues of a matrix A are 3 and 4, we can find the eigenvalues of A^3 by raising each eigenvalue of A to the power of 3.

Step-by-Step Solution

1. **Identify the Eigenvalues of A:**
   Given eigenvalues of A are 3 and 4.
2. **Property of Eigenvalues in Powers of Matrices:**
   If λ is an eigenvalue of a matrix A, then λ^k is an eigenvalue of A^k.
   Therefore, the eigenvalues of A^3 are 3^3 and 4^3.
3. **Calculate 3^3 and 4^3:**
   - $3^3 = 27$
   - $4^3 = 64$
4. **Conclusion:** The eigenvalues of A^3 are **27 and 64**.

If we have a 12×12 matrix having entries from R, how many linearly independent eigenvectors corresponding to real eigenvalues are possible for this matrix?

Questions 6-9 are based on common data.

Consider the following data points x_1, x_2, x_3 to answer following questions: $x_1 = \begin{bmatrix} 2 \\ 2 \end{bmatrix}$, $x_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, $x_3 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

6) What is the mean of the given data points x_1, x_2, x_3?

Step 1: Sum the vectors

$$x_1 + x_2 + x_3 = \begin{bmatrix} 2 \\ 2 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 2+1+2 \\ 2+2+1 \end{bmatrix} = \begin{bmatrix} 5 \\ 5 \end{bmatrix}$$

Step 2: Calculate the mean

To find the mean, divide the sum by the number of vectors (which is 3):

$$\text{Mean} = \frac{1}{3}\begin{bmatrix} 5 \\ 5 \end{bmatrix} = \begin{bmatrix} \frac{5}{3} \\ \frac{5}{3} \end{bmatrix} \approx \begin{bmatrix} 1.67 \\ 1.67 \end{bmatrix}$$

Conclusion

The mean of the given data points x_1, x_2, and x_3 is:

$$\begin{bmatrix} 1.67 \\ 1.67 \end{bmatrix}$$

Step 1: Calculate the Mean Vector $\bar{x}$

The mean vector $\bar{x}$ is calculated as:

$$\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$$

7) The covariance matrix $C = \frac{1}{n}\sum_{i=1}^{n}(x - \bar{x})(x - \bar{x})^T$ is given by: ($\bar{x}$ is mean of the data points)

Step 2: Calculate $(x_i - \bar{x})$

Now, compute $(x_i - \bar{x})$ for each data point:

1. For x_1:

$$x_1 - \bar{x} = \begin{bmatrix} 2 \\ 2 \end{bmatrix} - \begin{bmatrix} 1.67 \\ 1.67 \end{bmatrix} = \begin{bmatrix} 0.33 \\ 0.33 \end{bmatrix}$$

2. For x_2:

$$x_2 - \bar{x} = \begin{bmatrix} 1 \\ 2 \end{bmatrix} - \begin{bmatrix} 1.67 \\ 1.67 \end{bmatrix} = \begin{bmatrix} -0.67 \\ 0.33 \end{bmatrix}$$

3. For x_3:

$$x_3 - \bar{x} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} - \begin{bmatrix} 1.67 \\ 1.67 \end{bmatrix} = \begin{bmatrix} 0.33 \\ -0.67 \end{bmatrix}$$

Step 3: Compute Outer Products

Now we compute the outer products:

1. For $x_1 - \bar{x}$:

$$(x_1 - \bar{x})(x_1 - \bar{x})^T = \begin{bmatrix} 0.33 \\ 0.33 \end{bmatrix} \begin{bmatrix} 0.33 & 0.33 \end{bmatrix} = \begin{bmatrix} 0.1089 & 0.1089 \\ 0.1089 & 0.1089 \end{bmatrix}$$

2. For $x_2 - \bar{x}$:

$$(x_2 - \bar{x})(x_2 - \bar{x})^T = \begin{bmatrix} -0.67 \\ 0.33 \end{bmatrix} \begin{bmatrix} -0.67 & 0.33 \end{bmatrix} = \begin{bmatrix} 0.4489 & -0.2211 \\ -0.2211 & 0.1089 \end{bmatrix}$$

3. For $x_3 - \bar{x}$:

$$(x_3 - \bar{x})(x_3 - \bar{x})^T = \begin{bmatrix} 0.33 \\ -0.67 \end{bmatrix} \begin{bmatrix} 0.33 & -0.67 \end{bmatrix} = \begin{bmatrix} 0.1089 & -0.2211 \\ -0.2211 & 0.4489 \end{bmatrix}$$

Step 4: Sum the Outer Products

Now sum the outer products:

$$\text{Sum} = \begin{bmatrix} 0.1089 & 0.1089 \\ 0.1089 & 0.1089 \end{bmatrix} + \begin{bmatrix} 0.4489 & -0.2211 \\ -0.2211 & 0.1089 \end{bmatrix} + \begin{bmatrix} 0.1089 & -0.2211 \\ -0.2211 & 0.4489 \end{bmatrix}$$

Calculating the sums:

$$\text{Sum} = \begin{bmatrix} 0.1089 + 0.4489 + 0.1089 & 0.1089 - 0.2211 - 0.2211 \\ 0.1089 - 0.2211 - 0.2211 & 0.1089 + 0.1089 + 0.4489 \end{bmatrix} = \begin{bmatrix} 0.6667 & -0.3333 \\ -0.3333 & 0.6667 \end{bmatrix}$$

Step 5: Divide by n

Finally, divide by $n = 3$:

$$C = \frac{1}{3} \begin{bmatrix} 0.6667 & -0.3333 \\ -0.3333 & 0.6667 \end{bmatrix} = \begin{bmatrix} 0.2222 & -0.1111 \\ -0.1111 & 0.2222 \end{bmatrix}$$

Conclusion

Thus, the covariance matrix C is approximately:

$$C \approx \begin{bmatrix} 0.22 & -0.11 \\ -0.11 & 0.22 \end{bmatrix}$$

This matches your accepted answer format.

The maximum eigenvalue of the covariance matrix C is:

To find the maximum eigenvalue of the covariance matrix C, we start with the covariance matrix:

$$C = \begin{bmatrix} 0.22 & -0.11 \\ -0.11 & 0.22 \end{bmatrix}$$

Step 1: Find the Eigenvalues

The eigenvalues can be found by solving the characteristic equation:

$$\det(C - \lambda I) = 0$$

where I is the identity matrix and λ is the eigenvalue.

The characteristic equation becomes:

$$\det \begin{bmatrix} 0.22 - \lambda & -0.11 \\ -0.11 & 0.22 - \lambda \end{bmatrix} = 0$$

Step 2: Calculate the Determinant

The determinant of a 2×2 matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is calculated as $ad - bc$.

Thus,

$$\det(C - \lambda I) = (0.22 - \lambda)(0.22 - \lambda) - (-0.11)(-0.11)$$

Expanding this:

$$= (0.22 - \lambda)^2 - 0.0121 = 0$$

Expanding the first term:

$$(0.22 - \lambda)^2 = 0.0484 - 0.44\lambda + \lambda^2$$

So the equation becomes:

$$\lambda^2 - 0.44\lambda + (0.0484 - 0.0121) = 0$$

which simplifies to:

$$\lambda^2 - 0.44\lambda + 0.0363 = 0$$

Step 3: Use the Quadratic Formula

To find the eigenvalues λ, we use the quadratic formula:

$$\lambda = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

In our case:

- $a = 1$
- $b = -0.44$
- $c = 0.0363$

Substituting these values in:

$$\lambda = \frac{0.44 \pm \sqrt{(-0.44)^2 - 4 \cdot 1 \cdot 0.0363}}{2 \cdot 1}$$

Calculating the discriminant:

$$(-0.44)^2 = 0.1936$$

$$4 \cdot 1 \cdot 0.0363 = 0.1452$$

$$\text{Discriminant} = 0.1936 - 0.1452 = 0.0484$$

Now we calculate λ:

$$\lambda = \frac{0.44 \pm \sqrt{0.0484}}{2}$$

$$\sqrt{0.0484} \approx 0.220$$

So:

$$\lambda_1 = \frac{0.44 + 0.220}{2} = \frac{0.66}{2} = 0.33$$

$$\lambda_2 = \frac{0.44 - 0.220}{2} = \frac{0.22}{2} = 0.11$$

Conclusion

The eigenvalues of the covariance matrix C are approximately 0.33 and 0.11. Therefore, the maximum eigenvalue is:

$$\text{Maximum Eigenvalue} \approx 0.33$$

The eigenvector corresponding to the maximum eigenvalue of the given matrix C is:

What is an Eigenvector?

An **eigenvector** is a non-zero vector $\mathbf{v}$ that, when multiplied by a matrix A, yields a scalar multiple of itself. This relationship is expressed as:

$$A\mathbf{v} = \lambda\mathbf{v}$$

where λ is the **eigenvalue** corresponding to the eigenvector $\mathbf{v}$.

Example with our Covariance Matrix

For the covariance matrix we discussed:

$$C = \begin{bmatrix} 0.2222 & -0.1111 \\ -0.1111 & 0.2222 \end{bmatrix}$$

We found that the maximum eigenvalue $\lambda_1 \approx 0.3333$ had an eigenvector in the direction of $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

Scalar Multiples of Eigenvectors

An eigenvector is not unique. Any non-zero scalar multiple of an eigenvector is also an eigenvector. For example:

1. **Basic Eigenvector**:

$$\mathbf{v} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

2. **Another Valid Eigenvector**:
   If we multiply by -1:

$$\mathbf{v} = -1 \cdot \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

3. **Different Scaling**:
   Let's take a different scale, say $k = 0.71$:

$$\mathbf{v} = \begin{bmatrix} -0.71 \\ 0.71 \end{bmatrix} \quad (\text{This is } -0.71 \cdot \begin{bmatrix} 1 \\ -1 \end{bmatrix})$$

Validity of Different Representations

- **Direction Matters**: The key point is that while the magnitude (length) of the eigenvector can change, its direction remains the same. Each of these vectors points in the same line (they are all along the line spanned by $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$).
- **Valid Solutions**: Thus, $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} -0.71 \\ 0.71 \end{bmatrix}$ are both valid representations of the same eigenvector since they lie along the same direction.

Summary

1. **Eigenvectors are not unique**: You can scale them by any non-zero constant.
2. **Multiple Valid Forms**: Thus, you can express the same eigenvector in various forms, such as $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$, $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$, or $\begin{bmatrix} -0.71 \\ 0.71 \end{bmatrix}$, and they will all represent valid eigenvectors corresponding to that eigenvalue.

What is the determinant of a 2×2 matrix that has eigenvalues of 4 and 5?

The determinant of a square matrix can be calculated using its eigenvalues. For a 2×2 matrix, the determinant is equal to the product of its eigenvalues.

Given the eigenvalues $\lambda_1 = 4$ and $\lambda_2 = 5$, the determinant $\det(A)$ is calculated as follows:

$$\det(A) = \lambda_1 \cdot \lambda_2 = 4 \cdot 5 = 20.$$

Thus, the determinant of the 2×2 matrix is **20**.

the determinant of a 3×3 matrix is equal to the product of its eigenvalues.

Suppose the eigenvalues of a 3×3 matrix A are λ_1, λ_2, and λ_3. The determinant $\det(A)$ is given by:

$$\det(A) = \lambda_1 \cdot \lambda_2 \cdot \lambda_3.$$

Example

If the eigenvalues of the matrix are:

- $\lambda_1 = 2$
- $\lambda_2 = 3$
- $\lambda_3 = 4$

Then the determinant would be:

$$\det(A) = 2 \cdot 3 \cdot 4 = 24.$$

Q5. If we have a 12×12 matrix having entries from R, how many linearly independent eigenvectors corresponding to real eigenvalues are possible for this matrix?

A. 10
B. 24
C. 12
D. 6

Explanation
The matrix is 12×12, which implies it has up to 12 eigenvalues (since the number of eigenvalues of an n×n matrix is at most n).
Real Eigenvalues and Linearly Independent Eigenvectors: A real matrix may have real or complex eigenvalues. If all the eigenvalues are real and distinct, the matrix will have 12 linearly independent eigenvectors. If some eigenvalues are complex or have multiplicity greater than 1, the number of independent eigenvectors will decrease, though it will still be equal to or less than the size of the matrix.
Scenarios for Real Eigenvalues:

- If **all** the eigenvalues are real and distinct, there will be exactly **12** linearly independent eigenvectors corresponding to these real eigenvalues.
- If **not all** eigenvalues are real, but say 6 or 10 of them are real, the number of linearly independent eigenvectors corresponding to real eigenvalues could be **6** or **10**, respectively.
- The matrix could also have complex eigenvalues (in conjugate pairs), which would reduce the number of real eigenvalues and, thus, the number of independent real eigenvectors.

Correct Options:

- **6**: It's possible that 6 of the eigenvalues are real, corresponding to 6 linearly independent real eigenvectors.
- **10**: It's also possible that 10 of the eigenvalues are real, corresponding to 10 linearly independent real eigenvectors.
- **12**: If all eigenvalues are real, there will be 12 linearly independent eigenvectors.

Q1. Given below are the possible numbers of linearly independent eigenvectors for a 5×5 matrix. Choose the incorrect option.

1. 1
2. 4
3. 5
4. 6

Correct Answer: 4. 6

Explanation:

For a 5×5 matrix, the maximum number of linearly independent eigenvectors is equal to its dimension, which is **5**. This limit is because a 5×5 matrix can have at most 5 linearly independent eigenvectors, corresponding to its maximum rank and number of eigenvalues.

- **Option 1 (1)** and **Option 2 (4)** are possible, as a matrix with fewer distinct eigenvalues can have fewer than 5 linearly independent eigenvectors.
- **Option 3 (5)** is also valid, representing the case where the matrix has 5 distinct eigenvalues and hence 5 linearly independent eigenvectors.
- **Option 4 (6)** is incorrect, as it exceeds the dimension of the matrix and is impossible for a 5×5 matrix.

Q2.

Which of the following is an eigenvalue of matrix A = $\begin{bmatrix} 0.3 & 0.8 \\ 0.7 & 0.2 \end{bmatrix}$

1. 1
2. 3
3. -1
4. 2

Solution :

To find the eigenvalues of matrix $A = \begin{bmatrix} 0.3 & 0.7 \\ 0.8 & 0.2 \end{bmatrix}$, we need to solve the characteristic equation:

$$\det(A - \lambda I) = 0$$

where λ is the eigenvalue and I is the identity matrix.

The characteristic equation becomes:

$$\det\left(\begin{bmatrix} 0.3 & 0.7 \\ 0.8 & 0.2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

$$\det\left(\begin{bmatrix} 0.3 - \lambda & 0.7 \\ 0.8 & 0.2 - \lambda \end{bmatrix}\right) = 0$$

$$(0.3 - \lambda)(0.2 - \lambda) - (0.7)(0.8) = 0$$

$$(0.06 - 0.5\lambda + \lambda^2) - 0.56 = 0$$

$$0.06 - 0.56 - 0.5\lambda + \lambda^2 = 0$$

$$\lambda^2 - 0.5\lambda - 0.5 = 0$$

Now, we can use the quadratic formula to solve for λ:

$$\lambda = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

where $a = 1, b = -0.5$, and $c = -0.5$.

$$\lambda = \frac{-(-0.5) \pm \sqrt{(-0.5)^2 - 4(1)(-0.5)}}{2(1)}$$

$$\lambda = \frac{0.5 \pm \sqrt{0.25 + 2}}{2}$$

$$\lambda = \frac{0.5 \pm \sqrt{2.25}}{2}$$

$$\lambda = \frac{0.5 \pm 1.5}{2}$$

So, the eigenvalues of matrix A are:

$$\lambda_1 = \frac{0.5 + 1.5}{2} = \frac{2}{2} = 1$$

$$\lambda_2 = \frac{0.5 - 1.5}{2} = \frac{-1}{2} = -0.5$$

Among the given options, 1 is an eigenvalue of matrix A. Therefore, the correct answer is $\text{Eigenvalue} = 1$.

To determine the eigenvalues of the matrix A:

$$A = \begin{bmatrix} 0.3 & 0.8 \\ 0.7 & 0.2 \end{bmatrix}$$

we can use the characteristic polynomial, which is derived from the determinant of $A - \lambda I$, where λ is the eigenvalue and I is the identity matrix.

Step 1: Set Up the Characteristic Equation

The characteristic polynomial is found by calculating:

$$\det(A - \lambda I) = 0$$

Calculating $A - \lambda I$:

$$A - \lambda I = \begin{bmatrix} 0.3 - \lambda & 0.8 \\ 0.7 & 0.2 - \lambda \end{bmatrix}$$

Step 2: Compute the Determinant

The determinant is calculated as:

$$\det(A - \lambda I) = (0.3 - \lambda)(0.2 - \lambda) - (0.8)(0.7)$$

Step 3: Expanding the Determinant

Expanding the determinant:

$$\det(A - \lambda I) = (0.3 - \lambda)(0.2 - \lambda) - 0.56$$

$$= 0.06 - 0.5\lambda + \lambda^2 - 0.56$$

$$= \lambda^2 - 0.5\lambda - 0.5$$

Step 4: Set the Determinant to Zero

Setting the characteristic polynomial to zero:

$$\lambda^2 - 0.5\lambda - 0.5 = 0$$

Step 5: Solve the Quadratic Equation

We can use the quadratic formula to find the eigenvalues:

$$\lambda = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

where $a = 1, b = -0.5, c = -0.5$.

Calculating the discriminant:

$$b^2 - 4ac = (-0.5)^2 - 4(1)(-0.5) = 0.25 + 2 = 2.25$$

Now substituting back into the quadratic formula:

$$\lambda = \frac{0.5 \pm \sqrt{2.25}}{2}$$
$$= \frac{0.5 \pm 1.5}{2}$$

This yields two possible solutions:

1. $\lambda = \frac{2}{2} = 1$
2. $\lambda = \frac{-1}{2} = -0.5$

Answer

Therefore, the eigenvalues of matrix A include:

1 (which is one of the options given).

Q4. What is the determinant of a 2x2 matrix with eigenvalues λ1 and λ2?

1. λ1 + λ2
2. λ1 - λ2
3. λ1∗λ2
4. λ1 / λ2

Solution :

- The determinant of a 2x2 matrix is the product of its eigenvalues.
- So, for a 2x2 matrix with eigenvalues λ1 and λ2, the determinant can be expressed as:
- det(A)=λ1×λ2

Here's the text from the image:

Q4. What is the determinant of a 2x2 matrix with eigenvalues λ1 and λ2?

1. λ1 + λ2

2. λ1 - λ2
3. λ1 * λ2
4. λ1 / λ2

Solution:

- The determinant of a 2x2 matrix is the product of its eigenvalues.
- So, for a 2x2 matrix with eigenvalues λ1 and λ2, the determinant can be expressed as:
- det(A) = λ1 * λ2

Therefore, the correct answer is option 3.

Questions 5-9 are based on common data.

Consider the following data points $x1, x2, x3$ to answer following questions: $x1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, x2 = \begin{bmatrix} 0 \\ 2 \end{bmatrix}, x3 = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$

5) What is the mean of the given data points $x1, x2, x3$?

- $\begin{bmatrix} 3 \\ 3 \end{bmatrix}$
- $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$
- $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$
- $\begin{bmatrix} 0.5 \\ 0.5 \end{bmatrix}$

To find the mean, you add up the corresponding elements of each data point and then divide by the total number of data points (which is 3 in this case).

For the first component:

$$\text{Mean}(x_1) = \frac{1+0+2}{3} = \frac{3}{3} = 1$$

For the second component:

$$\text{Mean}(x_2) = \frac{1+2+0}{3} = \frac{3}{3} = 1$$

So, the mean of the given data points x_1, x_2, x_3 is:

$$\text{Mean} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

6) The covariance matrix $C = \frac{1}{n}\sum_{i=1}^{n}(x-\overline{x})(x-\overline{x})^T$ is given by: ($\overline{x}$ is mean of the data points)

- ○ $\begin{bmatrix} 0.33 & -0.33 \\ -0.33 & 0.33 \end{bmatrix}$
- ○ $\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$
- ○ $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$
- ○ $\begin{bmatrix} 0.67 & -0.67 \\ -0.67 & 0.67 \end{bmatrix}$

Given the data points

$$x_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \xrightarrow{\text{Transpose}} x_1 = [1,1]^T$$

,

$$x_2 = [0,2]^T$$

, and

$$x_3 = [2,0]^T$$

, and the mean vector

$$\bar{x} = [1,1]^T$$

, we can compute the covariance matrix as follows:

First, compute the difference between each data point and the mean vector:

$$x_1 - \bar{x} = [1,1]^T - [1,1]^T = [0,0]^T$$
$$x_2 - \bar{x} = [0,2]^T - [1,1]^T = [-1,1]^T$$
$$x_3 - \bar{x} = [2,0]^T - [1,1]^T = [1,-1]^T$$

Then, compute the outer product of each difference and sum them:

$$\Sigma = \frac{1}{3}\left((x_1-\bar{x})(x_1-\bar{x})^T + (x_2-\bar{x})(x_2-\bar{x})^T + (x_3-\bar{x})(x_3-\bar{x})^T\right)$$
$$= \frac{4}{3}\left([0,0][0,0]^T + [-1,1][-1,1]^T + [1,-1][1,-1]^T\right)$$
$$= \frac{1}{3}\left(\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} + \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}\right)$$
$$= \frac{1}{3}\begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix} = \begin{bmatrix} 2/3 & -2/3 \\ -2/3 & 2/3 \end{bmatrix}$$

To calculate the maximum eigenvalue of the covariance matrix C, you can use the fact that the maximum eigenvalue is the largest root of the characteristic equation of C. The characteristic equation is given by:

$\det(C - \lambda I) = 0$

Where:

- λ is the eigenvalue.
- I is the identity matrix.

Given the covariance matrix C:

$C = \begin{pmatrix} \frac{2}{3} & -\frac{2}{3} \\ -\frac{2}{3} & \frac{2}{3} \end{pmatrix}$

Let's calculate the maximum eigenvalue:

1. First, we form $C - \lambda I$:

$C - \lambda I = \begin{pmatrix} \frac{2}{3} - \lambda & -\frac{2}{3} \\ -\frac{2}{3} & \frac{2}{3} - \lambda \end{pmatrix}$

1. Then, we calculate the determinant of $C - \lambda I$:

$\det(C - \lambda I) = \left(\frac{2}{3} - \lambda\right)^2 - \left(-\frac{2}{3} \times -\frac{2}{3}\right)$

$= \left(\frac{2}{3} - \lambda\right)^2 - \frac{4}{9}$

$= \frac{4}{9} - \frac{4}{3}\lambda + \lambda^2 - \frac{4}{9}$

$= \lambda^2 - \frac{4}{3}\lambda$

1. Set the determinant to zero and solve for λ to find the eigenvalues.

$\lambda^2 - \frac{4}{3}\lambda = 0$

$\lambda\left(\lambda - \frac{4}{3}\right) = 0$

From this, we get two eigenvalues:

$\lambda_1 = 0$
$\lambda_2 = \frac{4}{3}$

Since we're interested in the maximum eigenvalue, the maximum eigenvalue of C is $\lambda_2 = \frac{4}{3}$.

8) The eigenvector corresponding to the maximum eigenvalue of the given matrix C is:

- $\begin{bmatrix} 0.7 \\ 0.7 \end{bmatrix}$
- $\begin{bmatrix} -0.7 \\ 0.7 \end{bmatrix}$
- $\begin{bmatrix} -1 \\ 0 \end{bmatrix}$
- $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$

Given that the maximum eigenvalue of C is $\lambda_{max} = \frac{4}{3}$, we need to find the corresponding eigenvector by solving the equation:

$(C - \lambda_{max} I)v = 0$

Substituting the values:

$$\left(\underbrace{\begin{pmatrix} \frac{2}{3} & -\frac{2}{3} \\ -\frac{2}{3} & \frac{2}{3} \end{pmatrix}}_{C} - \underbrace{\frac{4}{3}}_{\lambda_{max}} \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}}_{I}\right) v = 0$$

$$\left(\begin{pmatrix} \frac{2}{3} & -\frac{2}{3} \\ -\frac{2}{3} & \frac{2}{3} \end{pmatrix} - \begin{pmatrix} \frac{4}{3} & 0 \\ 0 & \frac{4}{3} \end{pmatrix}\right) v = 0$$

$$\begin{pmatrix} \frac{2}{3} - \frac{4}{3} & -\frac{2}{3} \\ -\frac{2}{3} & \frac{2}{3} - \frac{4}{3} \end{pmatrix} v = 0$$

$$\begin{pmatrix} -\frac{2}{3} & -\frac{2}{3} \\ -\frac{2}{3} & -\frac{2}{3} \end{pmatrix} v = 0$$

take any equ

Now, let's solve the system of equations:

$$-\frac{2}{3}v_1 - \frac{2}{3}v_2 = 0$$
$$-\frac{2}{3}v_1 - \frac{2}{3}v_2 = 0$$

$V = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$

From the first equation, we can solve for v_1 in terms of v_2:

$v_1 = -v_2$

So, any vector of the form $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$ is an eigenvector corresponding to the maximum eigenvalue

You can also normalize this eigenvector by dividing it by its norm to obtain a unit eigenvector. In this case, the normalized eigenvector would be $\frac{1}{\sqrt{2}}\begin{pmatrix} -1 \\ 1 \end{pmatrix}$.

9) The data points $x1, x2, x3$ are projected on the eigenvector calculated above. After projection what will be the new coordinate of $x2$?(Hint: $\begin{bmatrix} -0.7 \\ 0.7 \end{bmatrix} = \frac{1}{\sqrt{2}}\begin{bmatrix} -1 \\ 1 \end{bmatrix}$)

- $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$
- $\begin{bmatrix} 0 \\ 2 \end{bmatrix}$
- $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$
- $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

To project a vector x onto another vector w, you can use the formula:

$$\text{proj}_w(x) = \frac{x^T w}{\|w\|^2} \cdot w$$

where x^T is the transpose of x, w is the vector onto which x is being projected, and $\|w\|^2$ is the squared norm of w.

Given $x_2 = \begin{pmatrix} 0 \\ 2 \end{pmatrix}$ and $w = \begin{pmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix}$, we can compute the projection using this formula.

First, we calculate $x^T w$:

$$x^T w = \begin{pmatrix} 0 & 2 \end{pmatrix} \begin{pmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} = 0 \cdot (-1/\sqrt{2}) + 2 \cdot (1/\sqrt{2}) = \frac{2}{\sqrt{2}} = \sqrt{2}$$

Next, we calculate $\|w\|^2$:

$$\|w\|^2 = \left\| \begin{pmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} \right\|^2 = (-1/\sqrt{2})^2 + (1/\sqrt{2})^2 = \frac{1}{2} + \frac{1}{2} = 1$$

Now, we can compute the projection:

$$\text{proj}_w(x_2) = \frac{\sqrt{2}}{1} \cdot \begin{pmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \end{pmatrix} = \begin{pmatrix} -1 \\ 1 \end{pmatrix}$$

So, the new coordinate of x_2 after projection onto the eigenvector w is $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$.

10) If the dominant eigenvalue of the matrix A is less than 1 and all the eigenvalues are distinct and non-zero, which of the following is true for the sequence S given by $|Ax|, |A^2x|, |A^3x|, \ldots$?($|v|$ represents the magnitude of vector v)

- ○ S converges to 1
- ○ S converges to 0
- ○ S converges to maximum eigenvalue of A
- ○ S grows bigger and never converges to any point.

If the dominant eigenvalue of matrix A is less than 1 and all the eigenvalues are distinct and non-zero, then for any vector x, the sequence $S = \{|Ax|, |A^2x|, |A^3x|, \ldots\}$ converges to the zero vector.

Here's why:

1. Since the dominant eigenvalue of A is less than 1, let's denote it as $\lambda_1 < 1$.
2. Let $\lambda_2, \lambda_3, \ldots, \lambda_n$ be the remaining distinct non-zero eigenvalues of A.
3. For each eigenvalue $\lambda_i, i = 2, 3, \ldots, n, |\lambda_i| > 1$ (since all eigenvalues are distinct and non-zero).
4. As n goes to infinity, $|\lambda_i|^n$ goes to infinity for all $i = 2, 3, \ldots, n$.
5. As a result, $|A^n x| = |\lambda_1|^n |x|$ tends to zero as n goes to infinity, since $|\lambda_1| < 1$.

Therefore, the sequence $S = \{|Ax|, |A^2x|, |A^3x|, \ldots\}$ converges to the zero vector. So, the correct statement is:

S converges to 0.