1. Use set builder notation to give a description of each of these sets.
   a) {0, 3, 6, 9, 12} **b)** {-3, -2, -1, 0, 1, 2, 3} **c)** {*m*, *n*, *o*, *p*}

   Solution:

   a) $\{0, 3, 6, 9, 12\} = \{3n \mid n \in \mathbb{N} \wedge n \leq 4\}$

   b) $\{-3, -2, -1, 0, 1, 2, 3\} = \{x \in \mathbb{Z} \mid |x| \leq 3\}$

   c) $\{m, n, o, p\} = \{c \mid c \texttt{ is a letter } \wedge m \leq c \leq p\}$

2. Suppose that A, B, and C are sets such that A ⊆ B and B ⊆ C. Show that A ⊆ C.
   Solution:

   Suppose that x ∈ A. Because A ⊆ B, this implies that x ∈ B. Because B ⊆ C, we see that x ∈ C. Because x ∈ A implies that x ∈ C, it follows that A ⊆ C.

3. Can you conclude that A = B if A and B are two sets with the same power set?

   Solution:
   Yes, we can. First note the following:
   $\mathrm{P}(A) = \mathrm{P}(B)$ means that $\forall X, X \in \mathrm{P}(A) \leftrightarrow X \in \mathrm{P}(B)$, so (by the definition of powerset) also $\forall X, X \subseteq A \leftrightarrow X \subseteq B$.
   We know that $A \subseteq A$, therefore also $A \subseteq B$, similarly we know $B \subseteq B$ and so $B \subseteq A$.
   From the two: $A \subseteq B$ and $B \subseteq A$ we can conclude $A = B$.
4. Prove that P(A) ⊆ P(B) if and only if A ⊆ B.
   Solution:

   For the "if" part, given A ⊆ B, we want to show that that P(A) ⊆ P(B), i.e., if C ⊆ A then C ⊆ B. But this follows directly Q.3. For the "only if" part, given that P(A) ⊆ P(B), we want to show that A ⊆ B. Suppose a ∈ A. Then {a} ⊆ A, so {a} ∈ P(A). Since P(A) ⊆ P(B), it follows that {a} ∈ P(B), which means that {a} ⊆ B. But this implies a ∈ B, as desired.
5. Show that if A ⊆ C and B ⊆ D, then A × B ⊆ C × D.

   Solution: This is true. Pick $(x, y) \in A \times B$. Then $x \in A$ and $y \in B$. Since $A \subseteq C$ by assumption, $x \in C$, and similarly, $y \in D$. Thus $(x, y) \in C \times D$, so that $A \times B \subset C \times D$.
6. Prove the domination law and complementation law in set identities using set builder notation and logical equivalences.

   Solution:
   Complementation law
   A'' = {x | ¬(x ∈ A')}={x |¬(¬x ∈ A)} ={x | x ∈ A} = A
   where A'' and A' represents the double and single complement of A respectively.
   Idempotent Laws
   a) A ∪ U = {x | x ∈ A ∨ x ∈ U}={x | x ∈ A ∨ T} = {x | T} = U
   b) A ∩∅={x | x ∈ A ∧ x ∈ ∅} = {x | x ∈ A ∧ F}={x | F}=∅

7. What can you say about the sets A and B if we know that
   a) $A \cup B = A$? b) $A \cap B = A$?
   c) $A - B = A$? d) $A \cap B = B \cap A$?
   e) $A - B = B - A$?

   Solution:
   a) $B \subseteq A$ b) $A \subseteq B$ c) $A \cap B = \emptyset$ d) Nothing, because this is always true e) $A = B$
8. Show that if A is an infinite set, then whenever B is a set, $A \cup B$ is also an infinite set
   Solution:
   If $A \cup B$ were finite, then it would have n elements for some natural number n. But A already has more than n elements, because it is infinite, and $A \cup B$ has all the elements that A has, so $A \cup B$ has more than n elements. This contradiction shows that $A \cup B$ must be infinite
9. Determine whether f is a function from the set of all bit strings to the set of integers if
   a) *f (S)* is the position of a 0 bit in *S*.
   b) *f (S)* is the number of 1 bits in *S*.
   c) *f (S)* is the smallest integer *i* such that the *i*th bit of *S* is 1 and *f (S)* = 0 when *S* is the empty string, the
   string with no bits.

   Solution:
   a) Not a function b) A function c) Not a function
10. Give an explicit formula for a function from the set of integers to the set of positive integers that is
   a) one-to-one, but not onto.
   b) onto, but not one-to-one.
   c) one-to-one and onto.
   d) neither one-to-one nor onto.

   Solution:
   a) The function f (x) with $f(x) = 3x + 1$ when $x \geq 0$ and $f(x) = -3x + 2$ when $x < 0$
   b) $f(x) = |x| + 1$
   c) The function f (x) with $f(x) = 2x + 1$ when $x \geq 0$ and $f(x) = -2x$ when $x < 0$
   d) f (x) = x2 + 1
11. Show that the function $f(x) = |x|$ from the set of real numbers to the set of nonnegative real numbers is not invertible, but if the domain is restricted to the set of nonnegative real numbers, the resulting function is invertible.

   Solution:
   The function is not one-to-one, so it is not invertible. On the restricted domain, the function is the identity function on the nonnegative real numbers, $f(x) = x$, so it is its own inverse.

12. Suppose that g is a function from A to B and f is a function from B to C.
    a) Show that if both f and g are one-to-one functions, then f ◦ g is also one-to-one.
    b) Show that if both f and g are onto functions, then f ◦ g is also onto

    Solution:
    a) Let x and y be distinct elements of A. Because g is oneto-one, g(x) and g(y) are distinct elements of B. Because f is one-to-one, f (g(x)) = (f ◦ g)(x) and f (g(y)) = (f ◦ g)(y) are distinct elements of C. Hence, f ◦ g is one-to-one.
    b) Let y ∈ C. Because f is onto, y = f (b) for some b ∈ B. Now because g is onto, b = g(x) for some x ∈ A. Hence, y = f (b) = f (g(x)) = (f ◦g)(x). It follows that f ◦g is onto.
13. If *f* and *f* ◦ *g* are onto, does it follow that *g* is onto? Justify your answer.

    Solution:
    No. For example, suppose that A = {a}, B = {b, c}, and C = {d}. Let g(a) = b, f (b) = d, and f (c) = d. Then f and f ◦ g are onto, but g is not
14. Prove or disprove each of these statements about the floor and ceiling functions.
    a) $\lceil \lfloor x \rfloor \rceil = \lfloor x \rfloor$ for all real numbers x.
    b) $\lfloor 2x \rfloor = 2\lfloor x \rfloor$ whenever x is a real number.
    c) $\lceil x \rceil + \lceil y \rceil - \lceil x + y \rceil = 0$ or 1 whenever x and y are real numbers.
    d) $\lceil xy \rceil = \lceil x \rceil \lceil y \rceil$ for all real numbers x and y.
    e) $\left\lceil \frac{x}{2} \right\rceil = \left\lfloor \frac{x+1}{2} \right\rfloor$ for all real numbers x.

    a) True; because $\lfloor x \rfloor$ is already an integer, $\lceil \lfloor x \rfloor \rceil = \lfloor x \rfloor$. **b)** False; $x = \frac{1}{2}$ is a counterexample. **c)** True; if x or y is an integer, then by property 4b in Table 1, the difference is 0. If neither x nor y is an integer, then $x = n + \epsilon$ and $y = m + \delta$, where n and m are integers and ϵ and δ are positive real numbers less than 1. Then $m + n < x + y < m + n + 2$, so $\lceil x + y \rceil$ is either $m + n + 1$ or $m + n + 2$. Therefore, the given expression is either $(n + 1) + (m + 1) - (m + n + 1) = 1$ or $(n + 1) + (m + 1) - (m + n + 2) = 0$, as desired. **d)** False; $x = \frac{1}{4}$ and $y = 3$ is a counterexample. **e)** False; $x = \frac{1}{2}$ is a counterexample.