# Tutorial 3

1. Show the validity of the sequent using proof by contradiction.

$$a \to \neg a \vdash \neg a$$

2. Show the validity of the sequent using proof by contradiction.

$$a \to b, a \to \neg b \vdash \neg a$$

3. Show the validity of the sequent using proof by contradiction and without using Modus Tollens rule.

$$a \to (b \to c), a, \neg c \vdash \neg b$$

4. Show the validity of the sequent using proof by contradiction.

$$a \wedge \neg b \to c, a, \neg c \vdash b$$

5. Prove the following proposition:
Let a, b be integers. If ab is even, then at least one of a or b is even.

Sol: Suppose that a and b are both odd. Then there are integers k and l so that a = 2k + 1 and b = 2l + 1. Therefore, we have ab = (2k + 1)(2l + 1) = 4kl + 2k + 2l + 1 = 2(2kl + k + l) + 1, so ab is odd. Thus, by contrapositive, if ab is even, we must have at least one of a or b is even.

6. Suppose $x, y \in R$. If $y^3 + yx^2 \leq x^3 + xy^2$, then $y \leq x$.

Sol: Suppose it is not true that $y \leq x$, so $y > x$. Then $y - x > 0$. Multiply both sides of $y - x > 0$ by the positive value $x^2 + y^2$:

$$(y - x)(x^2 + y^2) > 0 \cdot (x^2 + y^2)$$

Expanding and simplifying:

$$(y - x)(x^2 + y^2) = yx^2 + y^3 - x^3 - xy^2$$

So:

$$yx^2 + y^3 - x^3 - xy^2 > 0$$

Rearrange the terms:

$$y^3 + yx^2 > x^3 + xy^2$$

Therefore:

$$y^3 + yx^2 > x^3 + xy^2$$

Thus, it is not true that $y^3 + yx^2 \leq x^3 + xy^2$.

7. Use a proof by contraposition to show that if $x + y \geq 2$, where $x$ and $y$ are real numbers, then $x \geq 1$ or $y \geq 1$.

Sol: To prove the statement by contraposition, we need to show the contrapositive of the original statement. The original statement is:

**If** $x + y \geq 2$, **then** $x \geq 1$ **or** $y \geq 1$.

The contrapositive of this statement is:

**If** $x < 1$ **and** $y < 1$, **then** $x + y < 2$.

**Proof by Contraposition:**

1. Assume $x < 1$ and $y < 1$.
2. Add the inequalities $x < 1$ and $y < 1$:

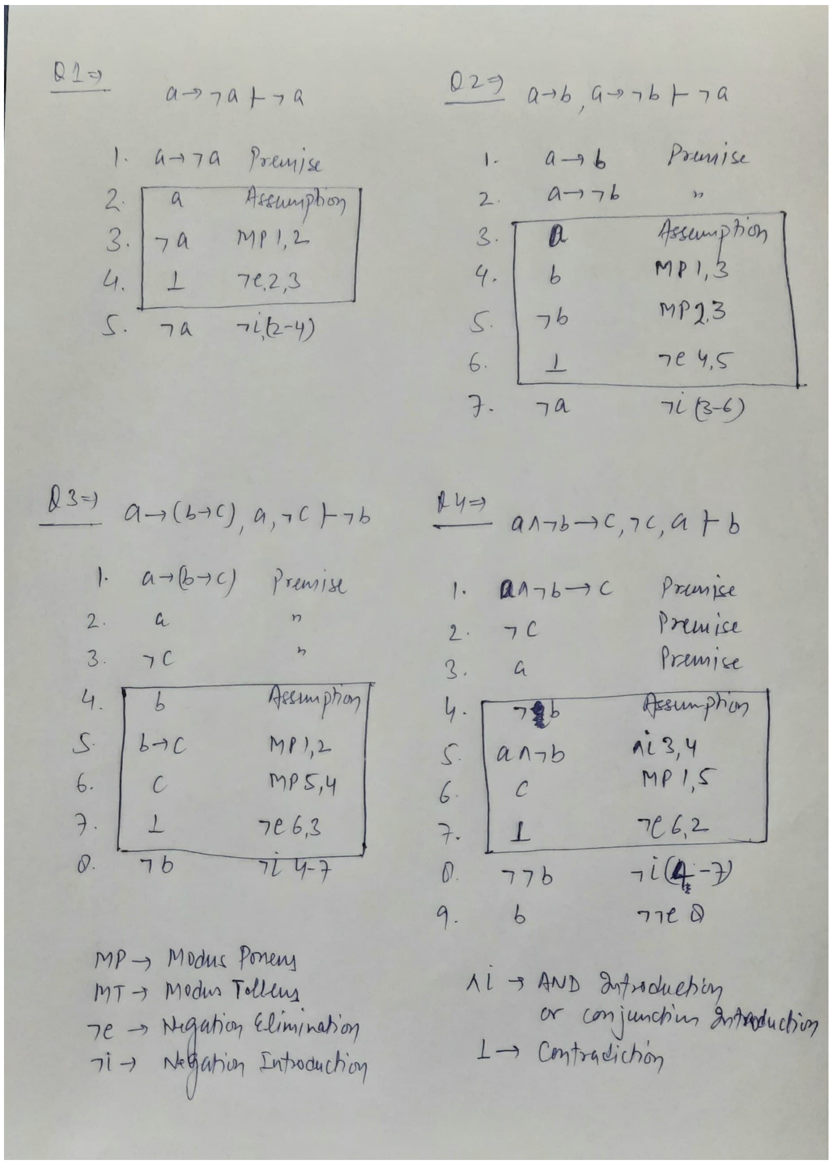
$$x + y < 1 + 1$$

3. Simplify the right-hand side:

$$x + y < 2$$

4. Conclude that $x + y < 2$ when $x < 1$ and $y < 1$.

Since we have shown that if $x < 1$ and $y < 1$, then $x + y < 2$, this proves the contrapositive of the original statement.



Q1 ⇒ $a \to \neg a \vdash \neg a$

| | | |
|---|---|---|
| 1. | $a \to \neg a$ | Premise |
| 2. | $a$ | Assumption |
| 3. | $\neg a$ | MP 1,2 |
| 4. | $\bot$ | $\neg e$ 2,3 |
| 5. | $\neg a$ | $\neg i$ (2-4) |

Q2 ⇒ $a \to b, a \to \neg b \vdash \neg a$

| | | |
|---|---|---|
| 1. | $a \to b$ | Premise |
| 2. | $a \to \neg b$ | " |
| 3. | $a$ | Assumption |
| 4. | $b$ | MP 1,3 |
| 5. | $\neg b$ | MP 2,3 |
| 6. | $\bot$ | $\neg e$ 4,5 |
| 7. | $\neg a$ | $\neg i$ (3-6) |

Q3 ⇒ $a \to (b \to c), a, \neg c \vdash \neg b$

| | | |
|---|---|---|
| 1. | $a \to (b \to c)$ | Premise |
| 2. | $a$ | " |
| 3. | $\neg c$ | " |
| 4. | $b$ | Assumption |
| 5. | $b \to c$ | MP 1,2 |
| 6. | $c$ | MP 5,4 |
| 7. | $\bot$ | $\neg e$ 6,3 |
| 8. | $\neg b$ | $\neg i$ 4-7 |

Q4 ⇒ $a \wedge \neg b \to c, \neg c, a \vdash b$

| | | |
|---|---|---|
| 1. | $a \wedge \neg b \to c$ | Premise |
| 2. | $\neg c$ | Premise |
| 3. | $a$ | Premise |
| 4. | $\neg b$ | Assumption |
| 5. | $a \wedge \neg b$ | $\wedge i$ 3,4 |
| 6. | $c$ | MP 1,5 |
| 7. | $\bot$ | $\neg e$ 6,2 |
| 8. | $\neg\neg b$ | $\neg i$ (4-7) |
| 9. | $b$ | $\neg\neg e$ 8 |

MP → Modus Ponens
MT → Modus Tollens
$\neg e$ → Negation Elimination
$\neg i$ → Negation Introduction

$\wedge i$ → AND Introduction or conjunction Introduction
$\bot$ → Contradiction

Figure 1: Q.(1-4) Solutions