Discrete Mathematical Structures (UCS405)
Tutorial Sheet-02

1. Let $P(x)$, $Q(x)$, $R(x)$, and $S(x)$ be the statements "x is a duck," "x is one of my poultry," "x is an officer," and "x is willing to waltz," respectively. Express each of these statements using quantifiers; logical connectives; and $P(x)$, $Q(x)$, $R(x)$, and $S(x)$.
   a. No ducks are willing to waltz.
   b. No officers ever decline to waltz.
   c. All my poultry are ducks.
   d. My poultry are not officers.
   e. Does (d) follow from (a), (b), and (c)? If not, is there a correct conclusion?

Solution:

a) $\forall x(P(x) \to \neg S(x))$ **b)** $\forall x(R(x) \to S(x))$ **c)** $\forall x(Q(x) \to P(x))$ **d)** $\forall x(Q(x) \to \neg R(x))$

e) Yes. If x is one of my poultry, then he is a duck (by part (c)), hence not willing to waltz (part (a)). Since officers are always willing to waltz (part (b)), x is not an officer.

2. Let $P(x)$, $Q(x)$, and $R(x)$ be the statements "x is a professor," "x is ignorant," and "x is vain," respectively. Express each of these statements using quantifiers; logical connectives; and $P(x)$, $Q(x)$, and $R(x)$, where the domain consists of all people.
   a. No professors are ignorant.
   b. All ignorant people are vain.
   c. No professors are vain.
   d. Does (c) follow from (a) and (b)? Explain.

Solution:

(a) $\forall x(P(x) \to \neg Q(x))$

(b) $\forall x(Q(x) \to R(x))$

(c) $\forall x(P(x) \to \neg R(x))$

(d) The conclusion does not follow. There may be vain professors because the premises do not rule out the possibility that there are other vain people besides ignorant ones.

3. Use rules of inference to show that the hypotheses "If it does not rain or if it is not foggy, then the sailing race will be held and the lifesaving demonstration will go on," "If the sailing race is held, then the trophy will be awarded," and "The trophy was not awarded" imply the conclusion "It rained."

Solution:

Let R = "It rains"

F = "It is foggy"

S = "The sailing race will be held"

D = "Life-saving demonstrations will go on"

T = "The trophy will be awarded"

We can now proceed to prove the claim:

	Step	Reason
1	$\neg T$	Premise
2	$S \to T$	Premise
3	$\neg S$	Modus Tollens
4	$\neg S \lor \neg D$	Addition
5	$\neg (S \land D)$	DeMorgan's Law
6	$(\neg R \lor \neg F) \to (S \land D)$	Premise
7	$\neg(\neg R \lor \neg F)$	Modus Tollens
8	$R \land F$	DeMorgan's Law
9	R	Simplification

4. Consider
   Premises: If Claghorn has wide support, then he'll be asked to run for the senate. If Claghorn yells "Eureka" in Iowa, he will not be asked to run for the senate. Claghorn yells "Eureka" in Iowa.
   Conclusion: Claghorn does not have wide support.

   Determine whether the conclusion follows logically from the premises. Explain by representing the statements symbolically and using rules of inference.

Solution:

Let P: Claghorn has wide support.
Q: Claghorn will be asked to run for the senate.
R: Claghorn yells "Eureka" in Iowa.

Premises:

$P \to Q$
$R \to \neg Q$
R

Conclusion: $\neg P$

	Steps	Reasons
1.	$P \to Q$	Premise
2.	$\neg Q \to \neg P$	~~Premise~~ Contrapositive
3.	$R \to \neg Q$	Premise
4.	R	Premise
5.	$\neg Q$	Modus Ponens 3,4
6.	$\neg P$	Modus Ponens 2,5

Therefore, the conclusion follows logically from the premises.

5. Consider the following open propositions over the universe $U = \{-4, -2, 0, 1, 3, 5, 6, 8, 10\}$
   $P(x): x \geq 4$
   $Q(x): x^2 = 25$
   $R(x): s$ is a multiple of 2

   Find the truth values of
   a. $P(x) \wedge R(x)$
   b. $[\sim Q(x)] \wedge P(x)$

Solution:
The truth values for each element of U under the given conditions of propositions is given in the following table:

x	P(x)	Q(x)	R(x)	P(x) ∧ R(x)	[~ Q(x)] ∧ P(x)
-4	0	0	1	0	0
-2	0	0	1	0	0
0	0	0	1	0	0
1	0	0	0	0	0
3	0	0	0	0	0
5	1	1	0	0	0
6	1	0	1	1	1
8	1	0	1	1	1
10	1	0	1	1	1

6. Express each of these sentences into logical expression using predicates, quantifiers, and logical connectives.
   a. No one is perfect.
   b. Not everyone is perfect.
   c. All your friends are perfect.
   d. At least one of your friends is perfect
   e. Everyone is your friend and is perfect.
   f. Not everybody is your friend or someone is not perfect.
   g. At least one mail message, among the nonempty set of messages, can be saved if there is a disk with more than 10 kilobytes of free space.
   h. Whenever there is an active alert, all queued messages are transmitted.
   i. The diagnostic monitor tracks the status of all systems except the main console.
   j. Each participant on the conference call whom the host

Solution:

a. This means that everyone has the property of being not perfect: $\forall \boldsymbol{x} \neg \boldsymbol{P}(\boldsymbol{x})$.
Alternatively, we can write this as $\neg \exists \boldsymbol{x}\ \boldsymbol{P}(\boldsymbol{x})$., which says that there does not exist a perfect person.
b. This is just the negation of "Everyone is perfect": $\neg \forall \boldsymbol{x}\ \boldsymbol{P}(\boldsymbol{x})$.
c. If someone is your friend, then that person is perfect: $\forall \boldsymbol{x}\,(\boldsymbol{F}(\boldsymbol{x}) \rightarrow \boldsymbol{P}(\boldsymbol{x}))$. Note the use of conditional statements with universal quantifiers.
d. We do not have to rule out your having more than one perfect friend. Thus we have simply $\exists \boldsymbol{x}\,(\boldsymbol{F}(\boldsymbol{x}) \wedge \boldsymbol{P}(\boldsymbol{x}))$. Note the use of conjunction with existential quantifiers.
e. The expression is $\forall \boldsymbol{x}\,(\boldsymbol{F}(\boldsymbol{x}) \wedge (\boldsymbol{P}(\boldsymbol{x}))$. Note that here we did use a conjunction with the universal quantifier, but the sentence is not natural (who could claim this?). We could also have split this up into two quantified statements and written $\forall \boldsymbol{x}\,(\boldsymbol{F}(\boldsymbol{x})) \wedge \forall \boldsymbol{x}\,(\boldsymbol{P}(\boldsymbol{x}))$.
f. This is a disjunction. The expression is $(\neg \forall \boldsymbol{x}\ \boldsymbol{F}(\boldsymbol{x})) \vee (\exists \boldsymbol{x} \neg \boldsymbol{P}(\boldsymbol{x}))$.
g. $(\exists \boldsymbol{x}\,\boldsymbol{F}(\boldsymbol{x}, \mathbf{10})) \rightarrow \exists \boldsymbol{x}\,\boldsymbol{S}(\boldsymbol{x}))$, where $F(x,y)$ is "Disk x has more than y kilobytes of free space," and S(x) is "Mail message x can be saved"
h. $(\exists \boldsymbol{x}\,\boldsymbol{A}(\boldsymbol{x})) \rightarrow \forall \boldsymbol{x}(\boldsymbol{Q}(\boldsymbol{x}) \rightarrow \boldsymbol{T}(\boldsymbol{x}))$, where A(x) is "Alert x is active," Q(x) is "Message x is queued," and T(x) is "Message x is transmitted"
i. $\forall \boldsymbol{x}\,((\boldsymbol{x} \neq \boldsymbol{main\ console}) \rightarrow \boldsymbol{T}(\boldsymbol{x}))$, where T(x) is "The diagnostic monitor tracks the status of system x"
j. $\forall \boldsymbol{x}\left(\neg \boldsymbol{L}(\boldsymbol{x}) \rightarrow \boldsymbol{B}(\boldsymbol{x})\right)$, where L(x) is "The host of the conference call put participant x on a special list" and B(x) is "Participant x was billed"

7. Let $D_x = N$ and $D_y = N^0$. Define $P(x,y)$ as "x divides y".
Find the truth values along with proper reasons for the following quantified predicates:
   i. $\forall x\, P(x, 0)$
   ii. $\forall x\, P(x, x)$
   iii. $\forall y\, \exists x\, P(x, y)$
   iv. $\exists y\, \forall x\, P(x, y)$
   v. $\forall x\, \forall y\, [(P(x,y) \wedge P(y,x)) \rightarrow (x = y)]$

vi. $\forall x\, \forall y\, \forall x\, [(P(x,y) \wedge P(y,x)) \rightarrow P(x,z)]$

Solution:

Solution : Let $D_x = \mathbf{N}$ and $D_y = \mathbf{N}^0$, and define two variable predicate P as

$$P(x,y) = x \text{ divides } y.$$

1. $\forall x P(x,0) = T$.
2. $\forall x P(x,x) = T$.
3. $\forall y \exists x P(x,y) = T$.
   Given any number y, there exists x, say $x = 1$, such that x divides y.
4. $\exists y \forall x P(x,y) = T$. Such a y is 0.
5. $\forall x \forall y[(P(x,y) \wedge P(y,x)) \rightarrow (x = y)] = T$.
   Given any x and y, suppose that $(P(x,y) \wedge P(y,x))$ is true.
   $$P(x,y) \Rightarrow y = ax, a \in \mathbf{N}^0;$$
   $$P(y,x) \Rightarrow x = by, b \in \mathbf{N}.$$
   Thus, $x = by = b(ax) = abx$, and hence $ab = 1$. Because both a and b are nonnegative integers, we know that $a = b = 1$. Therefore, $x = y$.
6. $\forall x \forall y \forall z[(P(x,y) \wedge P(y,z)) \rightarrow P(x,z)] = T$.
   Given any x, y, and z, suppose that $(P(x,y) \wedge P(y,z))$ is true.
   $$P(x,y) \Rightarrow y = ax, a \in \mathbf{N}^0;$$
   $$P(y,z) \Rightarrow z = by, b \in \mathbf{N}^0.$$
   Thus, $z = by = b(ax) = abx$. Since $ab \in \mathbf{N}^0$, therefore, $P(x,z)$ is true.

8. Identify the error or errors in this argument that supposedly shows that if $\forall x\,(P(x) \vee Q(x))$ is true then $\forall x\, P(x) \vee \forall x\, Q(x)$ is true.

a.	∀x(P(x) ∨ Q(x))	Premise
b.	P(c) ∨ Q(c)	Universal instantiation from (1)
c.	P(c)	Simplification from (2)
d.	∀xP(x)	Universal generalization from (3)
e.	Q(c)	Simplification from (2)
f.	∀xQ(x)	Universal generalization from (5)
g.	∀x(P(x) ∨ ∀xQ(x))	Conjunction from (4) and (6)

Solution:

Steps 3 and 5 are incorrect; simplification applies to conjunctions, not disjunctions.

9. Let k be a positive integer. Show that $1^k + 2^k + 3^k + \cdots + n^k$ is $O(n^{k+1})$.

   Solution:

   $\mathbf{1^k + 2^k + 3^k + \cdots + n^k \leq n^k + n^k + n^k + \cdots n^k = n.n^k = n^{k+1}}$.

10. Prove that if n is an integer and 3n + 2 is even, then n is even using
    a. proof by contraposition.
    b. proof by contradiction.

Solution:

(a) We must prove the contrapositive:

If n is odd, then 3n + 2 is odd. Assume that n is odd. Then we can write n = 2k + 1 for some integer k . Then 3n + 2 = 3(2k + 1) + 2 = 6k + 5 = 2(3k + 2) + 1. Thus 3n + 2 is two times some integer plus 1, so it is odd.

(b) Suppose that 3n + 2 is even and that n is odd. Since 3n + 2 is even, so is 3n. If we add subtract an odd number from an even number, we get an odd number, so 3n − n = 2n is odd. But this is obviously not true. Therefore our supposition was wrong, and the proof by contradiction is complete.

11. Proof by contrapositive that for every real number $x \epsilon [0, \pi/2]$, we have $\sin x + \cos x \geq 1$.

Solution:

Suppose for the sake of contradiction that this is not true.

Then there exists an $x \epsilon [0, \pi/2]$ for which $\sin x + \cos x < 1$.

Since x ∈ [0, π/2], neither sin x nor cos x is negative, so 0 ≤ sin x + cos x < 1.

Thus $0^2 \leq (sin\ x + cos\ x)^2 \leq 1^2$, which gives $0^2 \leq sin^2 x + 2\ sin\ x\ cos\ x + cos^2 x < 1^2$

As $sin^2 x + cos^2 x = 1$, this becomes 0 ≤ 1+2 sin x cos x < 1, so 1+2 sin x cos x < 1.

Subtracting 1 from both sides gives 2 sin x cos x < 0.

But this contradicts the fact that neither sin x nor cos x is negative.

12. What is wrong with the following proof? Explain your answer with a valid explanation.

Prove that the statement $\sqrt{2} + \sqrt{6} < \sqrt{15}$ is true.

"Proof":

Step 1. $\sqrt{2} + \sqrt{6} < \sqrt{15}$

Step 2. $(\sqrt{2} + \sqrt{6})^2 < 15$

Step 3. $8 + 2\sqrt{12} < 15$

Step 4. $2\sqrt{12} < 7$

Step 5. 48 < 49

Solution:

It may seem that the above argument is correct as we have reached a true statement (48 < 49), but this is not the case. It is important to remember that statement X ⇒ true statement does NOT mean that statement X is necessarily true! We assumed that √2+ √6 < √15 is true, whereas this is what we need to prove. Therefore, our implications are going in the wrong direction. Valid proof would be of the form true statement ⇒ statement X showing that X is true.

13. Derive the formula for the following expression: $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{2^n}$

Also prove the validity of your formula using mathematical induction.

Solution:

$$\sum_{j=1}^{n} \frac{1}{2^j} = (2^n - 1)/2^n$$

Basis Step: P (1) is true because $\frac{1}{2} = (2^1 - 1)/2^1$

Inductive Step: Assume that $\sum_{j=1}^{k} \frac{1}{2^j} = (2^k - 1)/2^k$

Then

$$\sum_{j=1}^{k+1} \frac{1}{2^j} = (\sum_{j=1}^{k} \frac{1}{2^j}) + \frac{1}{2^{k+1}} = \frac{2^k - 1}{2^k} + \frac{1}{2^{k+1}} = \frac{2^{k+1} - 2 + 1}{2^{k+1}} = \frac{2^{k+1} - 1}{2^{k+1}}$$