Critical Review of
Pricing and Referral in Diffusion on Networks

Akshit Kumar
University of Michigan, Ann Arbor
akshitk@umich.edu

***Abstract*—When a new product or technology is introduced into the market, there is always skepticism about its adoption due to the uncertainty in the quality of the proposed technology. Two ways that customers can learn about the quality of the introduced technology is trying it first hand (at a risk of being disappointed with the technology) or letting their peers or friends try it first and then "free-ride" on the information they get from their peers and friends. To study this adoption of new technology of uncertain value, Leduc et al. in [1] propose a dynamic game setting where consumers are assumed to be connected by a network and a monopolist seller chooses a profit-maximizing pricing policy. In [1], the authors study two class of policies - (a) inter-temporal price discrimination and (b) referral incentives and show the optimality of these policies over the whole space of pricing policies for certain types of networks.**

***Index Terms*—Network Games, Social Learning, Word-of-Mouth, Network Diffusion, Referral Incentives**

I. Introduction

Adoption of a new technology is a difficult choice, primarily because of the uncertainty in the quality of the introduced product or technology. This uncertainty leads to informational "free-riding" : potential consumers wait for other consumers (their friends and peers) to experience the technology and learn about the quality of the technology vicariously. This way the consumers can avoid the risk of experimentation and take more informed decisions. But this severely degrades the technology adoption process and can lead to inefficiencies in diffusion processes. This means that only the consumers with relatively small number of friends (low degree) are enticed to try the product or technology early since they are not very likely to know about the quality of the product from their friends. Since only the people with small degree experience the product, even if they share their experience, only a small fraction of the population is informed about the technology which considerably delays the information diffusion process in the network. Given the risks of experimentation, it is more socially efficient to have consumers who have a large number of friends (high degree) to experience the product or technology early since their experience can be accessed by more number of people and this would require low number of experimenters to achieve a high level of awareness (information) about the technology in the society (network). This can be seen extensively in advertising, when firms pay "celebrities" to try and their product and endorse it since they they have a large following of people (high degree).

In [1], the authors study the problem in a two period network game setting in which a monopolist seller induces people to experiment with the product in the first period using two types of incentives which are price discounts and referral rewards. This idea of incentivizing using referral rewards can be seen in practice as well, as reported in [2], [3]. Such referral incentives have been employed by new companies like Uber and Airbnb and large companies like Amazon when they are rolling out new product like Amazon prime and Uber Eats. Even the idea of price discounts is not new as it can be seen in practice,for example when a new retail store opens up, they offer "early-bird" discounts to attract people to shop at those retail stores and then people can go and spread information about that retail store to their friends and this acts as a way of advertising their retail store to more people and driving in new customers. Intuitively then, it seems that referral incentives should encourage consumers/customers/agents with a large friend circle (high degree) to adopt early because they have a large number of friends and peers to refer to and hence they can earn large referral rewards. On the other hand, price discounts should attract people with "small degree" to adopt early as they are less likely to know about the product from their friends.

Referral rewards solve a two-fold problem, not only do they solve the "informational inefficiency" problem but they also increase the monopolist's profits. Also to induce a referral reward incentive, the monopolist does not require the full knowledge of the underlying social network which is not the case with the price discrimination policy, hence implementing a price discrimination policy is a harder task.

One interesting aspect this problem presents is of *tradeoff* from the perspective of the monopolist. Any profit maximizing monopolist ideally wishes to minimize the number of early adopter since the monopolist needs to make payments in the form of either referrals or price discounts but at the same point would want the information diffusion to be maximum. Hence the optimal pricing policy would depend on the underlying social network.

The optimal pricing policy should therefore depend on how heterogeneous or homogeneous the underlying social network is. If the social network is heterogeneous i.e a small number of nodes (i.e agents) have large degree then referral incentive can be more efficient than price discrimination policy whereas if the social network is homogeneous (close to d-regular graph), then referral incentives are not very efficient and [1] shows that for d-regular networks, inter-temporal price discrimination is optimal. More over referral incentives in this case turn out to be a more expensive policy for the monopolist because not only does this make the monopolist pay the agents but they only work when the product is of high quality.

The approach used in this article by the authors builds on the works of [4]–[6] in network diffusion especially the use of mean field approach used in studying network diffusion. [1] is also related to some previous work in pricing with network effects as considered in [7]–[9], differing from these paper in the aspect that [1] considers the dynamic learning about product quality rather than other forms of complementaries.

In this review, I start by presenting the dynamic network game setup in a finite setting, stating the basic assumptions, defining the payoffs and setting up notation in Section II. In Section III, I will introduce the mean-field approximation, compare the closeness of the approximation for a star network and compute the mean field equilibrium. In Section IV, dynamic pricing and information diffusion will be discussed and Section V will focus on profit maximization where we will compare the performance of the two class of pricing policies introduced. In this section, we will digress from looking into the effect of mean and variance of the degree distribution as they do not provide any intuition. Section VI will discuss some critiques on the assumptions, some extensions which the authors propose and some of my personal extensions. Section VII will conclude. For the purposes of brevity only a proof sketch will be provided for the stated theorems.

II. Dynamic Game with Finite number of Agents

Consider a set of agents $\{1, 2, \ldots, N\}$ connected through a network. Each agent i is aware of its own degree d_i but oblivious to the degree of other agents (even its neighbors). The game occurs in two periods which are indexed as $t \in \{0, 1\}$. Let $X_{i,t}$ denote the number of neighbours of i who adopt at time t. The quality of the product is assumed to be either *High* or *Low* and is denoted by a unknown state variable $\theta \in \{H, L\}$ and this variable is only known to monopolist and all the agents are unaware of θ. All the agents have a common and fixed prior about the quality of the product i.e p is the probability that $\theta = H$. Depending on θ and time t when the product is adopted, the value of the product is as

	$t=0$	$t=1$
$\theta = H$	$A_0^H > 0$	$A_1^H > 0$
$\theta = L$	$A_0^L < 0$	$A_1^L < 0$

Note that in this assumption, the authors of [1] assume that the product is universally *High* or universally *Low* and this is not a very practical assumption to make. One relaxation on this assumption could be that the valuation are drawn from a distribution but then it makes the problem more complex and hence for cleaner and more tractable analysis, this universal valuation assumption is made. Next we define the payoff to the players with normalized prices to zero in both periods. An agent who adopted in the first period $t = 0$, earns a referral reward of η for *each* of it's neighbors who adopt in the second period $t = 1$. The implications of this assumption are unrealistic and discussed in Section VI. The payoff to agent i are hence summarized in the table as

	$t=0$	$t=1$
$\theta = H$	$A_0^H + \eta X_{i,1}$	A_1^H
$\theta = L$	A_0^L	A_1^L

One interesting thing to note is that in the case of $\theta = L, t = 0$, the payoff is A_0^L instead of $A_0^L + \eta X_{i,1}$ because of a key assumption, [1] makes. This assumption is similar to the word of mouth process described in [10]

Assumption 1. *If any neighbour of agent i adopts at $t = 0$, then agent i learns the quality of the technology* perfectly *prior to choosing her action at $t = 1$.*

This assumption means that all the agents are truthful about their experience of the product and relay this information perfectly to all their neighbors. Since all the agents are assumed to be truthful about their experience, the agents can update their posterior probability perfectly.

Assumption 2. $pA_0^H + (1-p)A_0^L < 0 \iff \bar{A} := p(A_0^H + A_1^H) + (1-p)A_0^L < pA_1^H$.

To check the relevance of this assumption, let us assume the contrary, that is $pA_0^H + (1-p)A_0^L > 0 \implies$ expected value (in spite of the referral reward) is positive and hence the agents will prefer to adopt the technology early and this makes the adoption problem trivial.

Assumption 3. $pA_1^H + (1-p)A_1^L < 0$.

While the absence of this assumption does not make the problem trivial like the previous assumption, the implication of this assumption greatly simplifies the 2nd period ($t = 1$) adoption strategy by the agents. If none of agent i's neighbors have adopted the product at $t = 0$ and agent i has also not adopted the product at $t = 0$, then at $t = 1$, agent i will not adopt as it the posterior is the same as the prior the agent i has on the quality of the product and the expected value is negative, hence agent i is better off adopting the product at all. Hence the second stage strategy is greatly simplified as

Remark 1. *Suppose an agent i does not adopt at $t = 0$. Then if $X_{i,0} > 0$, agent i adopts at $t = 1$ only if $\theta = H$, otherwise does not adopt.*

The above remark simplifies the decision problem at $t = 1$ but the decision problem at $t = 0$ is intractable in the general setting and is a function of the network. To tackle this issue and make the problem more tractable, we make the mean-field approximation which is discussed in the next section.

III. Mean Field Approximation

Mean field model for network games has been extensively studied in [11]. The mean field assumptions that [1] makes are as follows

Mean Field Assumption 1. *Each agent assumes that the degree of her neighbors are drawn i.i.d according to the edge-*

prespective distribution $\tilde{f}(d) = \frac{f(d)d}{\sum_i f(i)i}$*, where* $f(d)$ *is the degree distribution.*

Mean Field Assumption 2. *Each agent* i *assumes that each of her neighbors adopts at* $t = 0$ *with probability* α *independently of others. This implies that* $X_{i,0} \sim$ *Binomial*(d_i, α) *where* d_i *is the degree of agent* i

Before I discuss the validity of the assumptions, optimal response of agents and mean field equilibrium, let us formalize our terminology.

Definition 1. *A mean field strategy* $\mu : \mathbb{N}_+ \to [0, 1]$ *specifies for every* $d > 0$*, the probability than an agent with degree* d *adopts at* $t = 0$.

Let $\mathcal{M}$ denote the set of all mean field strategies.

Definition 2. *A mean-field strategy* $\mu^\star$ *constitutes a mean-field equilibrium if* $\mu^\star \in \mathcal{BR}(\alpha(\mu^\star))$*, where* $\alpha(\mu) = \sum_{d\geq 1} \tilde{f}(d)\mu(d)$.

Definition 3. *A mean field strategy is a* double threshold strategy *if* $\exists d_L, d_U \in \mathbb{N} \cup \{\infty\}$ *such that*

$$\begin{aligned} d < d_L &\implies \mu(d) = 1; \\ d_L < d < d_U &\implies \mu(d) = 0; \\ d > d_U &\implies \mu(d) = 1 \end{aligned}$$

d_L, d_U *are referred to as the lower and upper thresholds respectively.*

Definition 4. *A dynamic pricing policy is a triplet* $(P_0, P_1, \eta) \in \mathbb{R} \times \mathbb{R}_+ \times \mathbb{R}_+$ *consisting of the prices for first period, second period and a referral fee.*

Let $\mathcal{D} = \{(P_0, P_1, 0)\}$ denote the class of inter-temporal price discrimination (two-price) policy and $\mathcal{R} = \{(P, P, \eta)\}$ denote the class of referral incentive policies.

While [1] establishes the validity of the mean-field approximation for a complete and star network under both classes of pricing policies - price discrimination and referral incentives, in this report I will only focus on the referral incentives for star network under the finite game setting and the mean-field approximation.

Let us consider a star graph as demonstrated in Fig. 1 with $n - 1$ nodes on the periphery and 1 central node which is denoted as n. In the finite game setting with the full information, the possible pure strategy Nash equilibriums are that the central node adopts early and the periphery nodes adopt later $\implies \mu_n = 1, \mu_{-n} = 0$ and vice versa. Under the mean-field assumption, we have a double threshold strategy where $\mu(d) = 1, \forall 1 < d < n \implies \mu_n = 1$, hence the mean field equilibrium will select the strategy where the central node adopts early and the periphery nodes use the information of the central node to make decision and hence eliminates the implausible equilibria we get in the case of the periphery nodes adopting early in the finite game with full information setting, demonstrating that atleast for star network under the referral incentive policy (which will be shown to be optimal later), mean field assumption is a fairly sound assumption to make.

Remark 1 simplified the optimal response/decision of the agent for $t = 1$, therefore we next characterize the optimal strategies of the players at $t = 0$. To talk about the response at $t = 0$, we first need to compute the expected payoff to agent i, in the case she adopts early and the case where she defers the decision to the next time period.

1) Agent adopts The expected payoff will be a function of the α and d. In the case $\theta = H$, the agent realizes the value of the product in both the periods $t = 0, 1$, hence with probability p, she enjoys a value of $A_0^H + A_1^H$ and if $\theta = L$, then she realizes a value of A_0^L only as she can discount using the product in the next period. Apart from that the expected number of people who might not adopt in the first period is $d_i - \mathbb{E}[X_{i,0}] = d_i - d_i\alpha = d_i(1-\alpha)$ which follows from the Mean Field Assumption 2. Also neighbors of i will only adopt at $t = 1$ in the case $\theta = H$ hence the expected number of neighbors of i who adopt at $t = 1$ is $p(1 - \alpha)d_i$ and for each of its neighbors, i accrues an referral reward of η and pays P_0 for adopting early, hence the expected payoff $\Pi^{\text{adopt}}(\alpha, d_i) = p(A_0^H + A_1^H) + (1 - p)A_0^L + \eta p(1 - \alpha)d_i - P_0$.
2) Agent defers Since the agent only adopts at $t = 1$ if any of its neighbor adopts and the technology is *High* and pays P_1 for it. $\Pi^{\text{defer}}(\alpha, d_i) = p(A_1^H - P_1)\mathbb{P}(X_{i,1} > 0) = p(A_1^H - P_1)\left(1 - (1 - \alpha)^{d_i}\right)$

The optimal response at $t = 0$ is simplified and we can make this concrete using the following definition

Definition 5. *Let* $\mathcal{BR}_d(\alpha)$ *denote the best response set for a degree* d *agent given* α*, then*

$$\begin{aligned} \Pi^{adopt}(\alpha, d) > \Pi^{defer}(\alpha, d) &\implies \mathcal{BR}_d(\alpha) = \{1\} \\ \Pi^{adopt}(\alpha, d) < \Pi^{defer}(\alpha, d) &\implies \mathcal{BR}_d(\alpha) = \{0\} \\ \Pi^{adopt}(\alpha, d) = \Pi^{defer}(\alpha, d) &\implies \mathcal{BR}_d(\alpha) = [0, 1] \end{aligned}$$

Let $\mathcal{BR}(\alpha) \in \mathcal{M}$ *denote the space of mean-field best responses given* α*,* $\mathcal{BR}(\alpha) = \Pi_{d\geq 1}\mathcal{BR}_d(\alpha)$

Having discussed the validity of the mean field equilibrium and the best response at $t = 0$, we next discuss the mean field equilibrium which has been defined before. Before calculating the mean field equilibrium, it is only natural to first establish the existence of an equilibrium.

Theorem 1. *There exists a mean field equilibrium to the technology adoption game. If* $\mu^\star$ *and* $\mu'^\star$ *are mean field equilibria then* $\alpha(\mu^\star) = \alpha(\mu'^\star)$

Proof. I give an outline of the proof. To show existense, [1] makes use of the Kakutani Fixed Point Theorem on the correspondence $\Phi(\alpha) = \alpha(\mathcal{BR}(\alpha))$. Φ trivially has a compact domain because of compactness of $[0, 1]$ and $\Phi(\alpha)$ is non-empty. Next they show that $\Phi(\alpha)$ has a closed graph which begins by showing that best response set has a closed graph using the bounded convergence theorem. To prove the uniqueness of $\alpha(\mu^\star)$, they show that $\forall d \geq 1, \Pi^{\text{adopt}}(\alpha, d) - \Pi^{\text{defer}}(\alpha, d)$

is strictly decreasing in α, then showing that $\alpha' > \alpha \implies \mathcal{BR}_d(\alpha') \preceq \mathcal{BR}_d(\alpha)$, next $\mu(d) \geq \mu'(d) \implies \alpha(\mu) \geq \alpha(\mu')$, using these one can come up with a contradiction which prohibits the existence of more than one $\alpha(\mu^\star)$. □

The next theorem characterizes what the mean-field equilibrium should look like and proves that any mean field equilibrium for the technology game must be a double threshold strategy. The theorem and a proof sketch are given next

Theorem 2. *If $\mu^\star$ is a mean-field equilibrium then*

1) *$\mu^\star$ is a double threshold strategy*
2) *the upper and lower thresholds are unique i.e. $\exists!(d_L^\star, d_U^\star)$ are valid thresholds for every mean field equilibrium.*

Proof. The proof sketch is as follows. Consider $\Delta\Pi(\alpha, d) = \Pi^{\text{adopt}}(\alpha, d) - \Pi^{\text{defer}}(\alpha, d)$, this function is convex in d because it is a sum of an affine function in d (i.e $\eta p d(1-\alpha)$) and convex function in d (i.e $(1-\alpha)^d$). If $\Delta\Pi(\alpha, x) < 0$ then the inverse image $(x, y) \cap \mathbb{N}_+, x \geq 1$ is also convex. The integers in $(x, y) \cap \mathbb{N}_+$ represent the degrees for which $\Delta\Pi(\alpha, d) < 0 \implies$ the consumers delay the adoption process and they have a strict best response $\mathcal{BR}_d(\alpha) = \{0\}, d \in (x, y) \cap \mathbb{N}_+$. Hence the degrees of agents for whom early adoption is a strict best response are located outside the interval $(x, y) \cap \mathbb{N}_+$. This is true for any pair of (μ, α) where $\mu \in \mathcal{BR}(\alpha), 0 < \alpha < 1 \implies$ this holds for the mean field equilibrium as well. The uniqueness of the upper and lower threshold follows from construction and the uniqueness of $\alpha(\mu^\star)$. $d_U^\star = \inf\{z : \mathcal{BR}_d(\alpha^\star) = \{1\}, \forall d > z\}, d_L^\star = \sup\{z : \mathcal{BR}_d(\alpha^\star) = \{1\}, \forall d < z\}$.
Intuition : Apart from the rigorous proof which follows from the convexity of $\Delta\Pi$, one way of seeing why it should be a double threshold policy is because there is an increasing term which is $\eta p d(1-\alpha)$ and a decreasing term in $p(1-\alpha)^d$, hence when we plot the function $\Delta\Pi$ with d, it should cross the axis atmost twice giving rise to a double threshold policy. □

There are two corollaries which follow from the above theorem which are fairly easy to derive.

Corollary 1. *If $\eta = 0$ and let $(\mu^\star, \alpha^\star)$ be a mean field equilibrium, then $\mu^\star(d) = 1, \forall d < d_L^\star$ and $\mu^\star(d) = 0, \forall d > d_L^\star$*

Before we discuss the proof of the corollary, the implication of this corollary is that if there is no referral reward then there is no incentive for the high degree nodes to adopt early and hence the only early adopters will be the people with low degree ($< d_L^\star$).

Proof. Setting $\eta = 0 \implies \Delta\Pi(\alpha, d) = p(A_0^H) + (1-p)A_0^L + p(1-\alpha)^d A_1^H$, which is a decreasing function in d and hence will cross axis only once at $d_L^\star$ and beyond that degree no agent will adopt as $\Delta\Pi(\alpha, d) < 0$. □

Corollary 2. *$\exists \hat{\eta} < \infty$ such that $\forall \eta \geq \hat{\eta}, (\mu^\star, \alpha^\star)$ is a mean field equilibrium such that $\mu^\star(d) = 1, \forall d \geq d_U^\star$ and $\mu^\star(d) = 0, \forall d < d_U^\star$*

This corollary implies that there exists a referral reward which will only induce all the high degree nodes to adopt early and none of the low degree nodes adopt early.

Proof. The proof of this corollary follows from showing that $\Delta\Pi(\alpha, d)$ is increasing in d for sufficiently large η which is done using induction. The base case is follows easily where we show that $\Delta\Pi(\alpha, 1) < \Delta\Pi(\alpha, 2)$. $\Delta\Pi(\alpha, 2) - \Delta\Pi(\alpha, 1) = p(\eta - A_1^H) - \alpha(p(\eta - A_1^H)) + p(1-\alpha)^2 A_1^H, \therefore \eta > A_1^H \implies \Delta\Pi(\alpha, 1) < \Delta\Pi(\alpha, 2)$. Therefore a sufficient condition is $\hat{\eta} = A_1^H$. □

Next I will breifly discuss information diffusion and informational efficiency without going into the details for the purposes of brevity.

IV. Information Diffusion

We start by defining a few terms to discuss the topic of information diffusion.

Definition 6. *Define $\beta : \mathcal{M} \to [0, 1]$ by $\beta(\mu) = \sum_{d \geq 1} f(d)\mu(d)$. A β-strategy $\mu \in \mathcal{M}$ is a mean field strategy for which $\beta(\mu) = \beta$. It is a strategy which leads a fraction $\beta \in [0, 1]$ of agents to adopt early.*

At the first look $\beta(\mu)$ and $\alpha(\mu)$ seem related in the fact that they only differ in the random variable but $\beta(\mu)$ is the expected fraction of early adopters (overall) while $\alpha(\mu)$ is the expected fraction of early adopters in the neighborhood of a given node. Hence such a definition now allows us to compare the informational access $\alpha(\mu)$ of a strategy and the informational efficiency of a strategy is defined next in terms of the these two quantities - $\beta(\mu)$ and $\alpha(\mu)$.

Definition 7. *The informational efficiency of a strategy $\mu \in \mathcal{M}$ is a mapping $\mathcal{E} : \mathcal{M} \to \mathbb{R}_+$, which normalizes the informational access by the mass of agents generating information signals. It is expressed as $\mathcal{E}(\mu) = \frac{\alpha(\mu)}{\beta(\mu)}$.*

Using the quantity defined above - the "informational efficiency", we can discuss the performance of the two strategies - (a) inter temporal price discrimination and (b) referral incentives. A dynamic pricing policy *without* referral incentives (in our case inter-temporal price discrimination) leads to an adoption strategy with minimal informational efficiency. In fact it allocates the mass $\beta(\mu^\star)$ of early adopters in a way that diffuses information in the worst possible manner - making the lowest degree agents adopt early. On the other hand, a dynamic pricing policy *with* referral incentives achieves maximum informational efficiency as it induces mass $\beta(\mu^\star)$ of early adopters to be the highest degree nodes so that information diffuses maximally. One more important thing to realize is that there is a cost associated with $\beta(\mu^\star)$ since these agents need to be incentivised (in terms of discounts or referrals) to adopt early.

In the next section we will study this tradeoff - where we want large information diffusion in the network but don't want to pay a lot to induce it. In the next section, we will formalize the profit maximization of the monopolist and discuss some results in regards to $d-$regular networks and networks with only two types of degrees and discuss the star network as a special case of these type of networks.

V. Profit Maximization

The dynamic game setup is in the realm of asymmetric information setting - monopolist knows the quality of the product but the consumers are oblivious to the quality. While in [1], they formulate the optimization problem as a function of θ, I will make slight changes to the assumptions that they make (as in [11]) in [1] and still show that most of the qualitative results remain the same. The reason why [1] discuss the optimization problem as a function of θ is to discuss the two types of equilibria - *pooling* and *separating* equilibria.

I assume that the monopolist markets the technology with the belief that $\mathbb{P}_{\text{monopolist}}(\theta = H) = 1$. Since the monopolist always prices the product irrespective of its actual quality, the agents can not infer anything about the quality of the product from (P_0, P_1, η). Hence the agents hold the prior belief that $\mathbb{P}(\theta = H) = p$. The profit is therefore a function mapping defined as $\pi : \mathbb{R}^3 \to \mathbb{R}$ given a pricing policy (P_0, P_1, η). Therefore

$$\pi(P_0, P_1, \eta) = \beta(\mu^\star)P_0 + \gamma^\star P_1 - \eta(1-\alpha^\star)\mathbb{E}[\mu^\star(d)d]$$

where $\beta(\mu^\star)$ is the fraction of early adopters in the mean field equilibrium, $\gamma^\star$ is the fraction of late adopters when $\theta = H$ and given as $\gamma^\star = \sum_{d\geq 1} f(d)(1-\mu^\star(d))\left(1-(1-\alpha(\mu^\star))^d\right)$ and $(1-\alpha^\star)\mathbb{E}[\mu^\star(d)d]$ is the number of referrals that need to be given out to the early adopters when $\theta = H$. We can simplify $\mathbb{E}[\mu^\star(d)d] = \sum_{d\geq 1} f(d)\mu^\star(d)d$. Note here that we are looking at the average profit which is the total profit divided by the number of consumers.

Notice here that profit is the income to the monopolist, so we are assuming the production cost of the technology to be 0. Under the two classes of pricing policies which are defined Section III, the profit maximization problem can be stated as

1) Inter-temporal price discrimination (Two Price Policy) : $\hat{\pi}_{\mathcal{D}} = \max_{(P_0,P_1,\eta)\in\mathcal{D}} \pi(P_0, P_1, \eta) = \max_{P_0,P_1} \pi(P_0, P_1, 0)$
2) Referral Incentives / Rewards : $\hat{\pi}_{\mathcal{R}} = \max_{(P_0,P_1,\eta)\in\mathcal{R}} \pi(P_0, P_1, \eta) = \max_{P,\eta} \pi(P, P, \eta)$

From this point forward, we are only interested in the performance of the $\mathcal{D}-$ and $\mathcal{R}-$ policies. Next we show that a particular two-price is optimal over the space of $\mathcal{D}-$ price policies.

Theorem 3. *(Optimal Two Price Policy). For any degree distribution $f(d)$, $(P_0, P_1, 0) = (\bar{A}, A_1^H, 0)$ is always an optimal two-price policy.*

Proof. I only present the high level overview of the proof. First, we can rule out prices which will lead to the product not being adopted. If $P_0 > \bar{A} \implies \Pi^{\text{adopt}}(\alpha, d) = \bar{A} - P_0 < 0 \implies \mathcal{BR}_d(\alpha) = \{0\}, \forall d, \alpha$. Hence no one will adopt in the first period which will lead to no one adopting in the second period either and hence the profit to the monopolist is 0. Hence we can focus on $P_0 \leq \bar{A}$. For the second period prices, if $P_1 > A_1^H \implies \Pi^{\text{defer}} < 0$ and the agent's best response in the case is to not adopt late either, and hence the monopolist will lose on the profits that it could make had the price been lower. Hence we should be interested in $P_1 \leq A_1^H$. Using these bounds and some properties of increasing sequences, the proof can be made rigorous but the underlying idea to get an upper bound on the profits in the class of two price policies is to use these upper bounds on P_0 and P_1. □

Next we will consider two kind of networks - (a) d-regular network and (b) networks with two degrees. It turns out that for d-regular networks, the two price policy is optimal over the entire space of pricing policy.

Theorem 4. *Suppose the network is d-regular i.e. $f(d) = 1$ for some d and $f(d) = 0$ otherwise. Then*

(i) $\hat{\pi}_{\mathcal{D}} = \hat{\pi}, \forall d$
(ii) $\hat{\pi}_{\mathcal{D}} > \hat{\pi}_{\mathcal{R}}$
(iii) $\lim_{d\to\infty} \hat{\pi}_{\mathcal{D}} = \lim_{d\to\infty} \hat{\pi}_{\mathcal{R}}$

Proof.

(i) $\pi_{\mathcal{D}} = \beta(\mu^\star)P_0 + \gamma^\star P_1$ and on a d-regular network $\beta(\mu^\star) = \mu^\star(d) = \alpha^\star, \gamma^\star = (1-\alpha^\star)\left(1-(1-\alpha^\star)^d\right)$. Using the argument used in the proof of Theorem 3, $P_0 \in [0, \bar{A}], P_1 \in [0, A_1^H]$, again using the property of increasing sequence, we can show that $\hat{\pi}_{\mathcal{D}} = \hat{\pi}$.

(ii) To show that $\hat{\pi}_{\mathcal{R}}$ is strictly dominated by $\hat{\pi}_{\mathcal{D}}$, $\pi(P, P, \eta) = (\beta(\mu^\star) + \gamma^\star)P - \eta d(1-\alpha^\star)\mu^\star(d)$. Fixing p, any $\alpha^\star$ can be achieved by setting $\eta = \frac{1}{p(1-\alpha^\star)d}(P - \bar{A} + p(1-(1-\alpha^\star)^d)(A_1^H - P))$, plugging this η back into the expression $\pi(P, P, \eta)$ followed by some algebraic manipulation gives us the required result.

(iii) As $d \to \infty$ the graph becomes complete and either class of policies can induce a vanishing fraction of nodes to adopt early and in the next stage the entire network is informed and hence both the class of policies will give the same profit tending to A_1^H.

□

Next, we will discuss the networks with 2 degrees. For this case, I give the theorem as presented in [1] without giving the proof for this theorem. Nonetheless, I will discuss some implications of this theorem in the context of a star network.

Theorem 5. *Suppose we have a two-degree network such that $f(d_u) = q$ and $f(d_l) = 1-q$ for some $d_u \geq d_l$. Then for any d_l*

(i)

$$\lim_{q\to 0}\lim_{d_u\to\infty} \hat{\pi}_{\mathcal{D}} < \lim_{q\to 0}\lim_{d_u\to\infty} \hat{\pi}_{\mathcal{R}} = \lim_{q\to 0}\lim_{d_u\to\infty} \hat{\pi}$$

(ii)

$$\lim_{q\to 1}\lim_{d_u\to\infty} \hat{\pi}_{\mathcal{D}} = \lim_{q\to 1}\lim_{d_u\to\infty} \hat{\pi}_{\mathcal{R}} = \lim_{q\to 1}\lim_{d_u\to\infty} \hat{\pi}$$

(iii) for any $q \in (0,1)$, $\lim_{d_u\to\infty} \hat{\pi}_{\mathcal{R}} < A_1^H$

To gain intuitive insight in the result, consider a star network as in Fig. 1. This time instead of imagining a fixed degree of the central node, let the degree of the central node tend to infinity. In that case, the referral incentive will *only* induce the central node to adopt early and achieve maximum information diffusion. The total incentivising cost can be shown to converge to 0 while the total revenue converge to A_1^H which is the total surplus of the second period. If we consider the inter-temporal price discrimination pricing policy on the other hand, this policy will lead to a non-trivial fraction of degree 1 nodes adopting early and thus a non-trivial fraction has to be incentivized.

Second part of the theorem can be viewed to be the same as part (iii) of Theorem 4 where the graph tends to a complete graph. The third part of the theorem implies that as long as a non-trivial fraction of consumers has to be incentivized, then the referral policy can't capture the total surplus of the second period as a non-trivial fraction will has to be incentivized.

In the next section, I will critique some assumptions the authors make and discuss some possible extensions to this line of work. The authors in their paper also consider some extensions to the model which I will briefly touch upon.

VI. Critique and Extensions

One of the assumptions that the authors make is that the product is either universally *High* or universally *Low* and that all the consumers realize the *same numerical value* on experiencing the product. While this assumption greatly simplifies the analysis, it is not a very reasonable to assume because no two individuals will have the same valuation for a product. Some possible relaxation to this restrictive model could be to discrete the valuation. In the real world, the valuation may belong to $\mathbb{R}$ but reducing it to just a binary valuation is too restrictive. Discretization to say more threshold of valuations can give a richer set of policies and will be closer to the real world scenario as well. Since the decisions of the agents in the second round does not depend on the valuations, this model extension only changes the calculation of the profits and since the profits are being calculated in the expected sense, my intuition is that extending this model should not be very difficult.

In the analysis of the paper we assumed that the monopolist prices the product as if it is of *High* quality while in [1] they consider the problem in the both settings - product being *High* and *Low* and show that the profit in the case the product is *Low* is the same as the profit of the first stage in the case the product is *High*. One of the reasons for this conclusion is the assumption that when the early adopters experience the product, they *perfectly* know the state of the product and they *truthfully* communicate the state of the product to their neighbors. Therefore, obvious extensions to this line of work is to relax these assumptions of *perfect experience* and *truthful communication*.

If we relax the assumption of *truthful communication* to allowing the consumers to lie if they wish to, then in the case $\theta = H$, there is no incentive for the consumer to lie but in the case $\theta = L$, the consumer will have propensity to lie so that in the second stage, some of its neighbors may adopt giving the liar some referral rewards. Hence this case makes for more interesting equilibria and its not even clear what the structure of the equilibria will be like. This case also makes the second stage decision a bit more complex. Imagine the case that if the product is of low quality, then the early adopter lies with probability l and tells the truth with probability $1 - l$, in this case a node which did not adopt in the first stage will receive signals from its neighbors and then in the case that all the neighbors say that the product is *High*, the node can only increase its posterior probability and not perfectly know the state of the product and in the case that even one of them says that the product is *Low*, the second stage node knows perfectly the state cause they know that all players are utility maximizing. This relaxation does make the analysis more complicated but can result in capturing richer equilibrium structures which are closer to the real world assumption.

Another possible extension could be to relax the assumption of *perfect experience*, the user may experience the product partially, the experience of the product could possibly be modeled as a binary symmetric channel. Then again this would complicate how the period decisions will be taken as in the second stage, the users will not have the *perfect* knowledge of the quality of the product and in this model we can further explore if the adopters are *truthful* or *probabilistic liars*.

Some of the extensions which the authors discuss are (a) un-informed monopolist and (b) nonlinear referral payments. In this case of the un-informed monopolist, the authors assume that even the monopolist doesn't know the state of the product and has the same prior as the consumers. In the case of non-linear payments, they restrict the maximum number of referral rewards which can be received. In the previous model, the authors focused on linear referral payments in which the expected referral scaled linearly as the degree of the consumer. This case is interesting because now this can lead to a three adoption threshold policy but the authors show that η can be chosen appropriately so that a three adoption policy doesn't emerge.

VII. Conclusion

In this critical review, I analyzed the work in [1] which dealt with the problem of a dynamic game for technology adoption. The authors considered two simple class of pricing policies - (a) inter-temporal price discrimination and (b) referral incentives. The insights obtained from this paper can be used by a monopolist is designing its pricing policies. Apart from discussing the results from an intuitive perspective, I also discussed some extensions to the problem which are closer to the real world assumption. As for the numerical simulation of the result, I reproduce one of the numerical simulations which the authors of [1] do as well. This is the comparison of the optimal profit for the two price policy and referral policy for a d-regular network with degree d. The parameters for this simulation are the ones used in [1]. We can see from the plot that it matches the results of Theorem 4.

VIII. APPENDIX

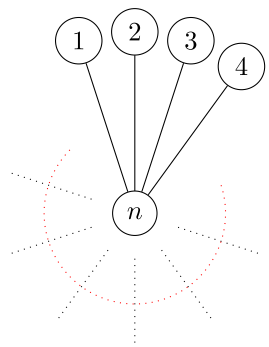

Fig. 1. Star Network

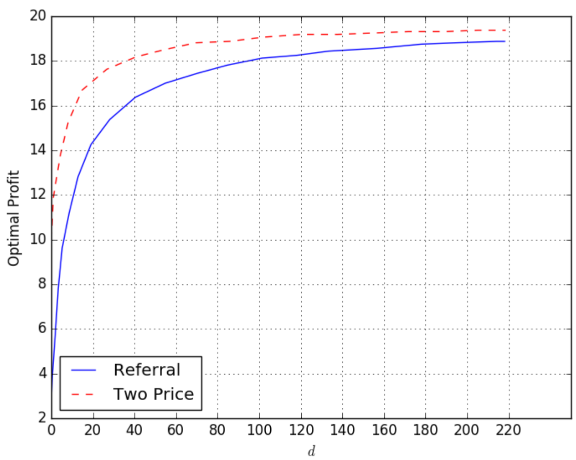

Fig. 2. Optimal profits of two-price and referral incentives for d regular networks. Parameters are $A_1^H = 20, A_0^H = 10, A_1^L = -20, A_0^L = -10, p = 0.4$

REFERENCES

[1] M. V. Leduc, M. O. Jackson, and R. Johari, "Pricing and referrals in diffusion on networks," *Games and Economic Behavior*, vol. 104, pp. 568–594, 2017.

[2] D. Hull, "Teslas musk takes page from paypal with referral incentive," *Bloomberg Business News*, August 2015.

[3] S. Patel, "5 brands that succeeded by incentivizing customer referrals," *Forbes*, January 2017.

[4] M. O. Jackson and L. Yariv, "Diffusion on social networks," in *Economie Publique*, Citeseer, 2005.

[5] M. O. Jackson and L. Yariv, "Diffusion of behavior and equilibrium properties in network games," *American Economic Review*, vol. 97, no. 2, pp. 92–98, 2007.

[6] A. Galeotti, S. Goyal, M. O. Jackson, F. Vega-Redondo, and L. Yariv, "Network games," *The review of economic studies*, vol. 77, no. 1, pp. 218–244, 2010.

[7] J. Farrell and G. Saloner, "Standardization, compatibility, and innovation," *the RAND Journal of Economics*, pp. 70–83, 1985.

[8] M. L. Katz and C. Shapiro, "Network externalities, competition, and compatibility," *The American economic review*, vol. 75, no. 3, pp. 424–440, 1985.

[9] Y.-J. Chen, Y. Zenou, and J. Zhou, "Competitive pricing strategies in social networks," 2015.

[10] A. Campbell, "Word-of-mouth communication and percolation in social networks," *American Economic Review*, vol. 103, no. 6, pp. 2466–98, 2013.

[11] M. V. Leduc, *Mean-field models in network game theory*. PhD thesis, Stanford University, 2014.