Performance Limits of Tunable Servers with Finite Buffer Capacity and a Packet-Drop Probability Constraint

Parikshit S Hegde, Akshit Kumar, Rahul Vaze

1 Problem Statement

Consider a tunable server with a finite buffer whose power consumption(cost-function) is a convex function of its service rate. If the buffer is full at the time of a packet arrival, then the packet gets dropped. We are interested in the probability of such an event, and we denote it as P_{Drop}. We consider the problem of tuning the speed of the server to minimize its expected cost, with a specified hard constraint on the Packet Drop probability.

Notation:

B : The capacity of the buffer
E_t: The number of packet arrivals at time t
b_t: The number of packets in the buffer at time t
g_t: The number of packets served at time t
$f_c(\cdot)$: The server cost function
P_{Drop}: Probability of Packet Drop
α: The specified constraint on P_{Drop}
$\mathcal{J}$: The Expected Cost when operating with a policy g_t on an arrival process E_t

Our Problem-Statement can be stated as follows:

$$\begin{aligned} \underset{g_t}{\text{minimize}} \quad & \mathcal{J} = \lim_{n\to\infty} \mathbb{E}\left[\frac{1}{n}\sum_{t=1}^{n} f_c(g_t)\right] \\ \text{subject to} \quad & P_{\text{Drop}} \le \alpha \end{aligned} \tag{1.1}$$

2 A General Lower Bound

Here, we will derive a lower-bound on the expected-cost of any policy on arrival process E_t with a mean $\mu = \mathbb{E}[E_t]$. We will call this lower bound $\mathcal{J}^*$.

For the server to have dropped less than α fraction of the packet arrivals, the sum of total number of packets serviced and the number of packets left in the buffer at the end of the experiment need to be greater than $(1-\alpha)$ fraction of the number of packet arrivals. In the best case scenario, at the end of the experiment the buffer will be filled to its capacity. Therefore, this gives us the following lower bound on the number of packets that need to be serviced:

$$\lim_{n\to\infty} \mathbb{E}\left[\sum_{t=1}^{n} g_t + B\right] \geq (1-\alpha)\,\mathbb{E}\left[\sum_{t=1}^{n} E_t\right] \tag{2.1}$$

$$\lim_{n\to\infty} \mathbb{E}\left[\frac{\sum_{t=1}^{n} g_t}{n} + \cancelto{0}{\frac{B}{n}}\right] \geq (1-\alpha)\,\mathbb{E}\left[\frac{\sum_{t=1}^{n} E_t}{n}\right] \tag{2.2}$$

$$= (1-\alpha)\mu \tag{2.3}$$

Since the cost function of the server is a convex function, we can use the Jensen's Inequality to obtain a lower bound on the Expected Cost:

$$\begin{aligned}
\mathcal{J} &= \mathbb{E}\left[\lim_{n\to\infty} \frac{1}{n}\sum_{t=1}^{n} f_c(g_t)\right] && (2.4)\\
&\geq f\left(\mathbb{E}\left[\lim_{n\to\infty} \frac{1}{n}\sum_{t=1}^{n} g_t\right]\right) \qquad \text{(Using Jensen's Inequality)} && (2.5)\\
&\geq f_c((1-\alpha)\mu) \qquad \text{[From Equation 2.3]} && (2.6)\\
&\triangleq \mathcal{J}^* && (2.7)
\end{aligned}$$

It turns out that for some probability distributions, and for small values of α, this lower-bound is very loose. We will attempt to tighten this lower bound in the future sections as and when the need arises.

3 (0,B) Bernoulli Arrival:

We will now consider a very special arrival distribution:

$$E_t = \begin{cases} B, & \text{w.p } p \implies \text{Queue gets full} \\ 0, & \text{w.p } 1-p \end{cases} \tag{3.1}$$

We can see that it is a Bernoulli Distribution characterized by the parameter p. We consider this distribution primarily for two reasons: it is easy to analyze, and as we will show later, the policy designed for this distribution will be a feasible policy and have lesser cost for any other distribution with the same mean.

3.1 λ-Fraction Policy:

We will now go ahead and design a policy for the $(0, B)$ Bernoulli Arrival Distribution. Our policy says that we need to service a fixed fraction of the packets in the buffer at each time instant, where the fraction is determined by the arrival distribution and the Packet Drop Probability constraint.

$$g_t = \lambda b_t \qquad (\ \lambda \text{ is a fixed fraction}) \tag{3.2}$$

Given a fixed fraction λ, we will evaluate the Packet Drop Probability:

When an arrival happens, since the size of the arrival is B, the number of packets that are present in the buffer just prior to the arrival (which is $b_{t-1} - g_{t-1}$) will get dropped.

Let N be the random variable over the number of time-steps between two arrivals. Since the arrival distribution is iid Bernoulli, N is a geometric random variable with the same parameter p.

It can be seen that for the λ fraction policy, the number of packets left after t time-steps since last arrival will be: $B(1-\lambda)^t$. Therefore, the packet drop probability can be derived as follows:

$$\mathbb{E}[b_{N-1} - g_{N-1} | N = n] = B(1-\lambda)^n \tag{3.3}$$

$$\begin{aligned}\mathbb{E}[b_{N-1} - g_{N-1}] &= \mathbb{E}_N[\mathbb{E}[b_{n-1} - g_{n-1}|N]] && (3.4)\\ &= \Sigma_{n=1}^{\infty} B(1-\lambda)^n p(1-p)^{n-1} && (3.5)\\ &= \frac{Bp(1-\lambda)}{\lambda + p - \lambda p} && (3.6)\\ & && (3.7)\end{aligned}$$

Using the Renewal Reward Theorem, the Packet Drop Probability can be written as:

$$\begin{aligned}P_{\text{Drop}} &= \frac{\mathbb{E}[b_{N-1} - g_{N-1}]}{B} && (3.8)\\ P_{\text{Drop}} &= \frac{p(1-\lambda)}{\lambda + p - \lambda p} && (3.9)\\ & && (3.10)\end{aligned}$$

Setting $P_{\text{Drop}} = \alpha$, we get

$$\boxed{\lambda = \frac{p(1-\alpha)}{p + \alpha(1-p)}} \tag{3.11}$$

3.1.1 Performance of the λ-Fraction Policy

The expected cost can be written as follows by using the Renewal Reward Theorem:

$$\mathcal{J} = \frac{\mathbb{E}\left[\sum_{t=1}^{N} f_c(g_t)\right]}{\mathbb{E}\left[N\right]} \tag{3.12}$$

From here on, we assume that our convex cost function is simply a square function: $f_c(x) = x^2$. This is a reasonable assumption to make because the cost function of microprocessors has been observed to be close to a square-function in practice. Using this assumption, we can now exactly evaluate the performance of our $\lambda-$Fraction Policy.

$$\begin{aligned}\mathcal{J} &= \frac{\mathbb{E}_N\left[B^2\lambda^2 + B^2\lambda^2(1-\lambda)^2 + \cdots B^2\lambda^2(1-\lambda)^{2(N-1)}\right]}{\mathbb{E}\, N} && (3.13)\\ &= \frac{pB^2\lambda^2\, \mathbb{E}_N[1-(1-\lambda)^{2N}]}{1-(1-\lambda)^2} && (3.14)\end{aligned}$$

Evaluating the expectation oven N we get:

$$\mathcal{J} = \frac{pB^2\lambda^2}{1-(1-p)(1-\lambda)^2} \tag{3.15}$$

Substituting the optimal value for λ from Equation 3.11, we get:

$$\mathcal{J} = \frac{B^2p^2(1-\alpha)^2}{1-(1-p)(1-\alpha)^2} \tag{3.16}$$

$$= \frac{\mu^2(1-\alpha)^2}{1-(1-p)(1-\alpha)^2} \tag{3.17}$$

$$= \frac{\mathcal{J}^*}{1-(1-p)(1-\alpha)^2} \tag{3.18}$$

We can see that there is a multiplicative gap of $\frac{1}{1-(1-p)(1-\alpha)^2}$ as compared to the lower-bound. This gap grows unbounded as α and p become small. This could mean one of two things: either our policy performs really bad under those conditions, or the lower bound becomes very weak under those conditions. Fortunately for us, it turns out that the latter is true. We will now tighten the lower bound in the following section.

3.2 Tight Lower Bound for small α and p

Before going ahead and tightening the bound, we can first observe that the cost of the λ-Fraction policy can be written as below for small α and p using a Binomial Approximation:

$$\mathcal{J} \approx \frac{B^2p^2(1-\alpha)^2}{2\alpha+p} \tag{3.19}$$

We now make the following proposition:

Proposition 1. *For $\alpha << 1, p << 1$, there exists no online policy that has a cost which is lower than $\Theta\left(\frac{B^2p^2}{p+\alpha}\right)$ for the $(0, B)$ Bernoulli Arrival Process with parameter p.*

Looking back at Equation 3.19, we can see that the above Proposition essentially tells us that there exists no online-policy that can do better than the $\lambda-$Fraction Policy, in terms of the order. For the proof, we will split it into two cases.

Proof:

Case 1: $\alpha, p << 1$ **and** $\alpha < \frac{p}{2}$

$$\mathbb{E}\left[\left(B - \sum_{t=1}^{N} g_t\right)^+\right] \leq B\alpha \tag{3.20}$$

$$P[N=1](B-g_1)^+ + P[N=2](B-g_1-g_2)^+ + \cdots \leq B\alpha \tag{3.21}$$

$$P(N=1)[B-g_1] \leq B\alpha \tag{3.22}$$

$$g_1 \geq B\left(1-\frac{\alpha}{p}\right) \tag{3.23}$$

$$\geq \frac{B}{2} \qquad [\because \alpha < \frac{p}{2}] \tag{3.24}$$

$$\mathcal{J} = \frac{\mathbb{E}\left[\sum_{t=1}^{N} g_t^2\right]}{\mathbb{E}\,N} \tag{3.25}$$

$$= \frac{g_1^2 + P(N \geq 2)g_2^2 + P(N \geq 3)g_3^2 + \cdots}{\mathbb{E}\,N} \tag{3.26}$$

$$\geq \frac{g_1^2}{\mathbb{E}\,N} \tag{3.27}$$

$$\geq \frac{B^2 p}{4} \tag{3.28}$$

$$\boxed{\therefore \mathcal{J} \geq \Theta\left(\frac{B^2p^2}{p+\alpha}\right)} \tag{3.29}$$

- Equation 3.20 comes from the formulation of the Packet Drop Probability using the Renewal Reward Theorem.
- Equation 3.22 is obtained by just considering the first term in the LHS of Equation 3.21
- Equation 3.25 comes from the Renewal Reward Theorem.
- Equation 3.27 is obtained by just considering the first term in numerator of Equation 3.26
- Equation 3.28 is obtained by substituting the value of g_1 from Equation 3.24 and $\mathbb{E}[N] = \frac{1}{p}$

Case 2: $\alpha, p << 1$ **and** $\alpha > \frac{p}{2}$

$$\mathbb{E}\left[\left(B - \sum_{t=1}^{N} g_t\right)^+\right] \leq B\alpha \tag{3.30}$$

$$P[N=1](B-g_1)^+ + P[N=2](B-g_1-g_2)^+ + \cdots \leq B\alpha \tag{3.31}$$

$$P[N \leq k]\left(B - \sum_{t=1}^{k} g_t\right) \leq B\alpha \tag{3.32}$$

$$\sum_{t=1}^{k} g_t \geq B\left(1 - \frac{\alpha}{pk}\right) \tag{3.33}$$

$$\sum_{t=1}^{k} g_t \geq B/2 \qquad \text{choosing } k = \left\lceil \frac{2\alpha}{p} \right\rceil \tag{3.34}$$

$$\mathcal{J} = \frac{\mathbb{E}\left[\sum_{t=1}^{N} g_t^2\right]}{\mathbb{E}\,N} \tag{3.35}$$

$$\geq p \cdot P(N \geq k)[g_1^2 + g_2^2 + \cdots + g_k^2] \tag{3.36}$$

$$\geq p(1-pk) \cdot \frac{B^2}{4k} \tag{3.37}$$

$$= \Theta\left(\frac{B^2p^2}{\alpha}\right) \tag{3.38}$$

$$\therefore \mathcal{J} \geq \Theta\left(\frac{B^2p^2}{\alpha+p}\right) \tag{3.39}$$

- Equation 3.30 comes from the formulation of Packet Drop Probability using Renewal Reward Theorem

- In Equation 3.32 we take the first k terms from LHS of Equation 3.31, and also additionally say that the minimum number of packets that can be dropped for every $N \leq k$ is $\left(B - \sum_{t=1}^{k} g_t\right)$
- In Equation 3.33 we make the approximation that $P(N \leq k) = 1 - (1-p)^k \approx pk$. From the choice of k in Equation 3.34, we can see that this is a reasonable approximation.
- In Equation 3.36, we take the terms corresponding to $N \geq k$, and within each of them we only consider the summation $g_1^2 + g_2^2 + \cdots + g_k^2$.
- Given the inequality in Equation 3.34, the minimum of the RHS of Equation 3.36 can be achieved by setting $g_1 = g_2 = \cdots = g_k = \frac{B}{2k}$.
- Substituting the chosen value of k we get the Equation 3.38.

From Equation 3.29 and 3.39 , we can see that $\mathcal{J} \geq \Theta\left(\frac{B^2p^2}{\alpha+p}\right)$, $\forall \alpha, p << 1$. $\square$

4 Bernoulli is the Worst Distribution

Observe that our policy just works with the mean of the arrival distribution. In this section we will show that: 1) if the policy is feasible on a $(0, B)$ Bernoulli Arrival Distribution with a certain mean, then it is feasible on an arbitrary Arrival Distribution with the same mean(here, feasibility is with respect to maintaining the Packet Drop Probability Constraint), 2) the Expected Cost of the policy on an arbitrary Arrival Distribution is upper bounded by the Expected Cost of the policy on the $(0, B)$ Bernoulli Arrival Distribution with the same mean. The claims 1 and 2 are independent, and hence we will prove them independently in the following two sections.

4.1 Feasibility:

We will prove the feasibility for a very special arrival distribution: $(0, x)$ Bernoulli Arrival distribution, where $0 < x \leq B$. We also make an additional assumption that $\frac{B}{x}$ is an integer.

Consider the following arrival distribution:

$$E_t = \begin{cases} x & \text{with probability } q \\ 0 & \text{with probability } 1-q \end{cases} \tag{4.1}$$

The λ-Fraction policy is oblivious to the nature of the distribution and only works with mean of the distribution $\mu = qx$. It goes ahead as though the arrival distribution is $\{0, B\}$ bernoulli with the parameter $p = \frac{\mu}{B}$.

To prove that the $\lambda-$Fraction policy is feasible, consider the following construction: Assume that $\frac{B}{x}$ is an integer. Partition the buffer into $\frac{B}{x}$ parts each of size x. For the purpose of the proof assume that $\frac{B}{x}$ is 10.

When the arrival happens at the $(10k+i)$th step, push those packets into the i^{th} part of the buffer. If that part of the buffer gets full, drop the remaining packets. One can observe that the number of packets dropped in this construction will be more than the number dropped in the original 'intact-buffer' case. Therefore, if our policy is feasible under this construction, it will be feasible for the 'intact-buffer' case as well.

Now consider one of the parts of the buffer. We will look at the fraction of packets that get dropped in this part alone. By symmetry, the fraction packets dropped in all the other parts will be the same. For this part alone, the arrival is happening at every 10^{th} step with a probability of q. Therefore, if we were to hypothetically compress the 10 steps to 1 step, then as far as this part is concerned the arrival process is the same as the original {0,full-buffer} bernoulli arrival. Thus, a $\lambda-$Fraction policy with $\lambda' = \frac{q(1-\alpha)}{q(1-\alpha)+\alpha}$ will be a feasible policy.

Now, our policy working with $\lambda = \frac{p(1-\alpha)}{p(1-\alpha)+\alpha}$ at every time step. Therefore, at the end of 10 time-steps the fraction of packets that will be left over is $(1-\lambda)^{10}$. If this fraction is smaller compared to the fraction $(1-\lambda')$, then our policy is feasible. This can indeed be verified to be true.

4.2 Bernoulli Upper Bounds the Cost

Note: This proof is exactly the same as in the Ayfer paper, with the concave function replaced by a convex function.

Later in the proof we will make use of the following Lemma:

Lemma 1. *Let $f(z)$ be a convex function in the interval $[0, B]$, and let Z be a RV confined to the same interval, i.e., $0 \le Z \le B$. Let $\hat{Z} \in \{0, B\}$ be a Bernoulli RV with $Pr(\hat{Z} = B) = \mathbb{E}\, Z/B$. Then*

$$\mathbb{E}[f(Z)] \le \mathbb{E}[f(\hat{Z})] \tag{4.2}$$

Proof: By the convexity of f, for any $z \in [0, B]$,

$$f(z) \le \frac{z}{B} f(B) + \frac{B-z}{B} f(0) \tag{4.3}$$

Setting $z = Z$ and taking expectation yields

$$\mathbb{E}[f(Z)] \le \frac{\mathbb{E}\, Z}{B} f(B) + \left(1 - \frac{\mathbb{E}\, Z}{B}\right) f(0) \tag{4.4}$$

$$= \mathbb{E}[f(\hat{Z})] \tag{4.5}$$

□

We now need to prove that:

$$\mathcal{J}(g, E) \le \mathcal{J}(g, \hat{E}) \tag{4.6}$$

Where E is an arbitrary arrival distribution and $\hat{E}$ is the corresponding $(0, B)$ Bernoulli Distribution. g indicates the $\lambda-$Fraction Policy

Consider the following quantity:

$$\mathcal{J}_n(g, E, x) \triangleq \frac{1}{n} \sum_{t=1}^{n} \mathbb{E}\left[(\lambda b_t)^2 | b_1 = x\right] \tag{4.7}$$

We now make the following Proposition:

Proposition 2. *For any $x \in [0, B]$ and any integer $n \ge 1$:*

$$\mathcal{J}_n(g, E, x) \le \mathcal{J}_n(g, \hat{E}, x) \tag{4.8}$$

Proof: We will prove this by induction.
Base Case: Since in both cases $b_1 = x$, the number of packets serviced in both cases will be λx

$$\mathcal{J}_1(g, E, x) = \mathcal{J}_1(g, \hat{E}, x) = (\lambda x)^2 \tag{4.9}$$

Induction Hypothesis: Assume $\mathcal{J}_{n-1}(g, E, x) \leq \mathcal{J}_{n-1}(g, \hat{E}, x)$ and that $\mathcal{J}_{n-1}(g, \hat{E}, x)$ is monotonic non-decreasing and convex in x.

First, observe the following relationship:

$$n\,\mathcal{J}_n(g, E, x) = (\lambda x)^2 + (n-1)\,\mathbb{E}\left[\mathcal{J}_{n-1}(g, E, b_2)\right] \tag{4.10}$$

where the expectation is over the RV $b_2 = \min\{(1-\lambda)x + E_2, B\}$. From the Induction Hypothesis we have:

$$n\,\mathcal{J}(g, E, x) \leq (\lambda x)^2 + (n-1)\,\mathbb{E}\left[\mathcal{J}_{n-1}(g, \hat{E}, b_2)\right] \tag{4.11}$$

Let us now look more closely at $J_{n-1}(g, \hat{E}, b_2)$:

$$\mathcal{J}_{n-1}(g, \hat{E}, b_2) = \mathcal{J}_{n-1}(g, \hat{E}, \min\{(1-\lambda)x + E_2, B\}) \tag{4.12}$$

$$= \min\{\mathcal{J}_{n-1}(g, \hat{E}, (1-\lambda)x + E_2), \mathcal{J}_{n-1}(g, \hat{E}, B)\} \tag{4.13}$$

The second equality above comes from the Induction Hypothesis that $\mathcal{J}_{n-1}(g, E, \cdot)$ is non-decreasing. Also, by Induction Hypothesis, $f_1(z) \triangleq \mathcal{J}_{n-1}(g, \hat{E}, (1-\lambda)x + z)$ is convex. Also, $\mathcal{J}_{n-1}(g, E, B)$ is just a constant. Therefore, $f_2(z) \triangleq \mathcal{J}_{n-1}(g, E_1, \min\{(1-\lambda)x + z, B\})$ is a minimum of a convex function and a constant, hence it is also convex. Therefore, we can now invoke Lemma 1 on f_2:

$$\mathbb{E}\left[\mathcal{J}_{n-1}(g, \hat{E}, b_2)\right] = \mathbb{E}[f_2(E_2)] \tag{4.14}$$

$$\leq \mathbb{E}[f_2(\hat{E}_2)] \tag{4.15}$$

$$= \mathbb{E}\left[\mathcal{J}_{n-1}(g, \hat{E}, \hat{b_2})\right] \tag{4.16}$$

where, $b_2 \triangleq \min\{(1-\lambda)x + \hat{E}_2, B\}$. Substituting this in Equation 4.11:

$$n\,\mathcal{J}_n(g, E, x) \leq (\lambda x)^2 + (n-1)\,\mathbb{E}[\mathcal{J}_{n-1}(g, E, \hat{b_2})] \tag{4.17}$$

$$= n\,\mathcal{J}_n(g, \hat{E}, x) \tag{4.18}$$

Proved. □
We can now simply apply the limit $n \to \infty$ and set $x = 0$ to obtain Equation 4.6

5 Bounds For Altered Arrival Distribution

Proposition 3. *Let the arrival distribution be such that* $P\left(E_t \geq \frac{B}{2} + \Theta(B)\right) \geq \Theta\left(\frac{\mu}{B}\right)$. *Then, for* $\alpha << 1$ *and* $\frac{\mu}{B} << 1$, *there exists no online algorithm whose performance is better than* $\Theta\left(\frac{\mu^2}{\frac{\mu}{B}+\alpha}\right)$.

Proof. Let us denote $P(E_t \geq y) = q$, where $y = \frac{B}{2} + \Theta(B)$ and $q \geq \Theta\left(\frac{\mu}{B}\right)$

Case 1: $\alpha < \frac{q}{2}$

$$\begin{aligned}
P_{drop} &= \frac{\mathbb{E}[drop]}{\mathbb{E}[arrival]} && (5.1)\\
&= \frac{\mathbb{E}[drop]}{\mu} && (5.2)\\
&\geq \frac{q\,\mathbb{E}[\text{drop from arrival } E_t \geq y]}{\mu} && (5.3)\\
&= \frac{\mathbb{E}[\text{drop from arrival } E_t \geq y]}{\Theta(B)} \qquad \left(q \geq \Theta(\frac{\mu}{B})\right) && (5.4)\\
& && (5.5)
\end{aligned}$$

Now, we have a necessary condition that

$$\mathbb{E}[\text{drop from arrival } E_t \geq y] \leq \alpha\Theta(B) \tag{5.6}$$

Now we will only consider those drops that happen due to another packet $E_{t+N} \geq y$, where N is a Random Variable over the interval length. So, in the first time step itself, we have

$$\begin{aligned}
P(N=1)\{\Theta(B) - g_1\} &\leq \alpha\Theta(B) \qquad (\Theta(B) - g_1 \text{ packets will be dropped if } E_t \text{ occurs at } N=1) && (5.7)\\
q\{\Theta(B) - g_1\} &\leq \alpha\Theta(B) \qquad (P(N=1) = q) && (5.8)\\
g_1 &\geq \Theta(B)\left(1 - \frac{\alpha}{q}\right) && (5.9)
\end{aligned}$$

We can set $g_1 = \Theta(B)$ and calculate the performance

$$\begin{aligned}
\mathcal{J} &\geq q g_1^2 && (5.10)\\
\therefore \mathcal{J} &\geq \Theta(B\mu) && (5.11)\\
\mathcal{J} &= \Theta\left(\frac{\mu^2}{\frac{\mu}{B} + \alpha}\right) && (5.12)
\end{aligned}$$

For α comparable to q, a similar extension to that of the $(0, B)$ Bernoulli case holds. For some integer $k \geq 1$

$$\begin{aligned}
P(N \leq k)\{\Theta(B) - g_1 - g_2 - \cdots\cdots - g_k\} &\leq \alpha\Theta(B) && (5.13)\\
qk\{\Theta(B) - g_1 - g_2 - \cdots\cdots - g_k\} &\leq \alpha\Theta(B) && (5.14)\\
\sum_{i=1}^{k} g_i &\geq \Theta(B)\left(1 - \frac{\alpha}{qk}\right) && (5.15)\\
& && (5.16)
\end{aligned}$$

Now we can set $k = \lfloor \frac{2\alpha}{q} \rfloor$ and we get $\sum_{i=1}^{k} g_i \geq \Theta(B)$ Again we can calculate the performance in this case

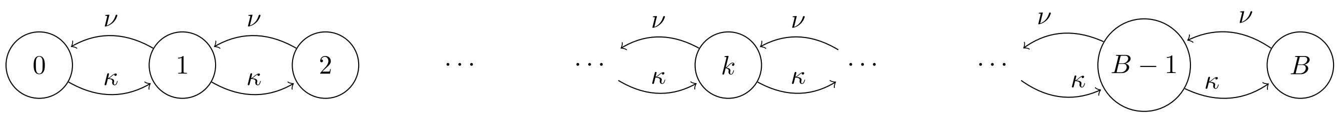
$$\begin{aligned}
\mathcal{J} &\geq qP(N \geq k)\sum_{i=1}^{k} g_i^2 && (5.17)\\
&= q\Theta(1)k\left(\frac{\Theta(B)}{k}\right)^2 && (5.18)\\
&= \frac{q^2\Theta(B^2)}{2\alpha} && (5.19)\\
\therefore \mathcal{J} &\geq \Theta\left(\frac{\mu^2}{\frac{\mu}{B} + \alpha}\right) && (5.20)
\end{aligned}$$

□

6 Considering a CTMC Model

6.1 CTMC Model

Consider the following CTMC model where arrival is a Poisson process with rate κ and the service is also a Poisson process with rate ν. Let us define $\rho = \frac{\kappa}{\nu}$. Let us define π to be a probability vector, where π_i denotes the probability of being in state i.



On writing the balance equation for the CTMC model, we get that

$$\kappa\pi_0 = \nu\pi_1 \implies \pi_1 = \frac{\kappa}{\nu}\pi_0 \implies \pi_1 = \rho\pi_0 \tag{6.1}$$

$$\kappa\pi_1 = \nu\pi_2 \implies \pi_2 = \frac{\kappa}{\nu}\pi_1 \implies \pi_2 = \rho^2\pi_0 \tag{6.2}$$

$$\kappa\pi_{k-1} = \nu\pi_k \implies \pi_k = \frac{\kappa}{\nu}\pi_{k-1} \implies \pi_k = \rho^k\pi_0 \tag{6.3}$$

Now, since π is a probability vector, $\sum_i \pi_i = 1$, therefore we get the following geometric summation

$$\pi_0 + \pi_1 + \pi_2 + \cdots\cdots + \pi_B = 1 \tag{6.4}$$

$$\pi_0\left(1 + \rho + \rho^2 + \cdots\cdots + \rho^B\right) = 1 \tag{6.5}$$

$$\implies \pi_0 = \frac{1}{\sum_{i=0}^{B}\rho^i} \tag{6.6}$$

$$\therefore \pi_k = \frac{\rho^k}{\sum_{i=0}^{B}\rho^i} \tag{6.7}$$

6.2 Calculating the Drop Probability

Using PASTA(Poisson Arrivals See Time Averages), we can argue that the probability of drop is the same as the probability of being in state B

$$\therefore P_{drop} = \pi_B = \frac{\rho^B}{\sum_{i=0}^{B}\rho^i}$$

Now, depending on the value of ρ, two cases arrives
Case 1: $\rho \geq 1$, in this case the arrival rate is at least as large as the service rate.
Since $\rho \geq 1$, $\rho^m \geq \rho^n$ for $m > n$.
$\therefore \rho^B \geq \rho^{B-1} \geq \rho^{B-2} \geq \cdots\cdots \geq \rho^2 \geq \rho \geq 1$
Hence we have the following equation,

$$\sum_{i=0}^{B}\rho^i = 1 + \rho + \rho^2 + \cdots\cdots + \rho^B \leq \rho^B + \rho^B + \rho^B + \cdots\cdots \rho^B = B\rho^B \tag{6.8}$$

Using (6.8), we get a necessary condition that $P_{drop} \geq \frac{\rho^B}{B\rho^B} = \frac{1}{B}$ and we need $P_{drop} \leq \alpha$, therefore we get a necessary condition that $\alpha \geq \frac{1}{B}$

Case 2: $\rho < 1$, in this case the arrival rate is strictly less than the service rate Since $\rho < 1$, $\rho^m < \rho^n$ for $m > n$

$\therefore \rho^B < \rho^{B-1} < \rho^{B-2} < \cdots\cdots < \rho^2 < \rho < 1$
So again we get a similar equation as above,

$$\sum_{i=0}^{B} \rho^i = 1 + \rho + \rho^2 + \cdots\cdots + \rho^B \leq 1 + 1 + 1 + \cdots\cdots 1 = B \tag{6.9}$$

Using (6.9), we get a necessary condition that $P_{drop} \geq \frac{\rho^B}{B}$ and we need $P_{drop} \leq \alpha$, therefore we get a necessary condition that $\alpha \geq \frac{\rho^B}{B}$. This implies that

$$\nu \geq \frac{\kappa}{(\alpha B)^{\frac{1}{B}}} \tag{6.10}$$

7 Constant Policy

In the previous section we had looked at a CTMC formulation of the problem in which case the packet arrival process is Poisson with rate κ and service of the packet is also Poisson with rate ν. In this case,we consider the Bernoulli packet arrivals with the following distribution

$$E_t = \begin{cases} 1, & \text{w.p } q \\ 0, & \text{w.p } 1-q \end{cases} \tag{7.1}$$

We define our policy to be

$$g_t = \min\{b_t, g\} \tag{7.2}$$

where g is some constant service we provide at each time step.

7.1 Calculating the drop probability

$$P_{drop} = \sum_{T=1}^{\infty} P[\text{drop happens at time T}] \tag{7.3}$$

A necessary condition for the drop to happen at time T is

$$\sum_{t=1}^{T} E_t - g_t \geq B \tag{7.4}$$

$$\implies \sum_{t=1}^{T} E_t \geq B + Tg \tag{7.5}$$

Let $N^{(T)}$ be the number of arrivals required for the drop to happen at time T,therefore we have

$$N^{(T)} = B + T.g \tag{7.6}$$

$$\geq T.g \tag{7.7}$$

So, we can write the drop probability as a binomial distribution

$$P_{drop} = \sum_{T=1}^{\infty} \binom{T}{N^{(T)}} q^{N^{(T)}} (1-q)^{T-N^{(T)}} \tag{7.8}$$

Using Stirling's approximation, we get that

$$\binom{n}{k} \leq \left(\frac{en}{k}\right)^k \tag{7.9}$$

Also using the fact that

$$q < 1 \implies q^m < 1 \qquad \forall m > 0 \tag{7.10}$$

Therefore we upper bound the P_{drop}

$$\begin{aligned}
P_{drop} &\leq \sum_{T=1}^{\infty} \left(\frac{eT}{N^{(T)}}\right)^{N^{(T)}} q^{N^{(T)}} && (7.11)\\
&\leq \sum_{T=1}^{\infty} \left(\frac{eT}{Tg}\right)^{N^{(T)}} q^{N^{(T)}} \qquad \text{(Using 7.7)} && (7.12)\\
&= \sum_{T=1}^{\infty} \left(\frac{eq}{g}\right)^{N^{(T)}} && (7.13)\\
&= \sum_{T=1}^{\infty} \left(\frac{eq}{g}\right)^{B+Tg} \qquad \text{(Using 7.6)} && (7.14)\\
&= \left(\frac{eq}{g}\right)^{B} \sum_{T=1}^{\infty} \left(\frac{eq}{g}\right)^{Tg} && (7.15)\\
&= \left(\frac{eq}{g}\right)^{B} \frac{\left(\frac{eq}{g}\right)^{g}}{1-\left(\frac{eq}{g}\right)^{g}} \qquad \text{(Summation of infinite series)} && (7.16)\\
&\leq \left(\frac{eq}{g}\right)^{B} M \qquad \text{(For small } \alpha\text{, we can take a constant } M\text{)} && (7.17)
\end{aligned}$$

Now, using the sufficient condition that,

$$\alpha \geq \left(\frac{eq}{g}\right)^{B} M \tag{7.18}$$

Using this sufficient condition, we get the following condition on g

$$g \geq \frac{eq}{\left(\frac{\alpha}{M}\right)^{\frac{1}{B}}} \tag{7.19}$$

Setting $g = \frac{eq}{\left(\frac{\alpha}{M}\right)^{\frac{1}{B}}}$

7.2 Calculation of Cost

$$\begin{aligned}
Cost &\leq g^2 \qquad (g \geq \min\{b_t, g\}) && (7.20)\\
&= \left(\frac{eq}{\left(\frac{\alpha}{M}\right)^{\frac{1}{B}}}\right)^2 && (7.21)
\end{aligned}$$