Analysis of Coded Slotted Aloha from a Power Control perspective

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1 Introduction

Random access techniques are widely used in communication networks because of their simple distributed implementation and reasonable throughput guarantees. A particular example is the slotted ALOHA protocol that dates back to [], but is still operational for initial access acquisition in cellular, satellite and ad hoc networks []. The basic advantage of random access technique is that it completely avoids the need of network knowledge at each node, e.g. the total number of nodes etc, and works without any coordination overhead, that can grow exponentially with the number of nodes in the network.

The flip side of this uncoordinated communication paradigm is that packets sent simultaneously by different nodes collide, making them undecodable at the receiver node, prompting retransmissions. Thus, an important performance metric with random access is the *throughput* that counts the average number of packets successfully received per time slot. For the basic vanilla version of the slotted ALOHA protocol the throughput approaches $1/e$ as the number of nodes in the network grow to infinity. Thus, the overall slot occupancy is a constant even with no coordination among nodes.

There is a large body of work [] that has addressed the question of improving the throughput with random access, where primary recourse is to employ advanced signal processing at the receiver node. For example, performing successive interference cancellation (SIC) at the receiver, or exploiting the natural random delay seen by each nodes' transmission [].

The basic idea here is that each node transmits its packet in multiple slots (repetition), where packets received in slots without collision are decoded successfully, which are then used to peel off the interference they cause in other slots, increasing the throughput fundamentally []. Exploiting the random delay decreases the need for repetition.

This idea of using repetition code at the transmit side and employing SIC was formalized in [], which called it the irregular repetition slotted ALOHA (IRSA), where the packet repetition rate is chosen systematically and shown to achieve a throughput close to 1/2. A natural extension to the IRSA was to include non-trivial forward error correction codes in contrast to just using the repetition code, which is termed as coded slotted ALOHA (CSA). Using a judicious choice of codes, throughput larger than IRSA is achievable as expected, with also some analytical tractability via the iterated decoding framework typically used for LDPC codes.

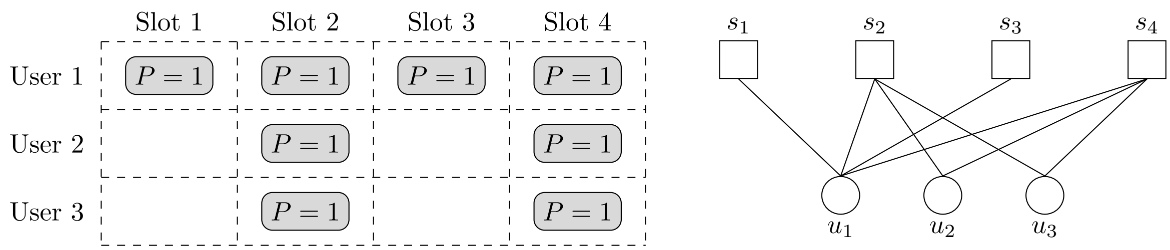
Even though prior work has exploited the SIC ability intelligently, one feature that it has neglected is that if the colliding packets have different power levels, some of them can still be decodable. Essentially, if two signals with power P_i, P_j overlap, then the i^{th} (j^{th}) packet is decodable if the signal to interference ratio $SIR_i = \frac{P_i}{P_j} > \beta$ ($SIR_j = \frac{P_j}{P_i} > \beta$), where $\beta > 1$ is typically fixed threshold. Thus, there is a case for making sure that the power profile of colliding packets in a slot is as different as possible, allowing simultaneous decoding of more than 1 packet, which can potentially allow a throughput of more than 1. However, since the network in distributed and each node has to make an autonomous decision, the power transmit decision is non-trivial.

In this paper, we propose a random access protocol, where nodes choose which slots to use for packet transmission similar to the slotted ALOHA, or IRSA, or CSA protocol, but employ different (random) power levels (following a distribution) for transmitting packets in slots chosen for transmission.

Suppose a node can choose between two different power levels $\{H, L\}, H > L$ to transmit, and since the decoding threshold $\beta > 1$, we want to design this protocol in such a way that for each slot, the number of packets with H power levels is at most 1, since such a slot can potentially carry more than 1 successfully decodable packet. We call this phenomena power multiplexing.

In general, the protocol can choose between n different power levels for transmission in any slot with random distribution D, to exploit as much of this power multiplexing as much as possible, together with the random slot occupancy distribution (S) as proposed in CSA, and the problem is jointly optimize the throughout over D and S.

1.1 How Multi-Level Power Transmission Leads to Fundamental Gain in Throughput

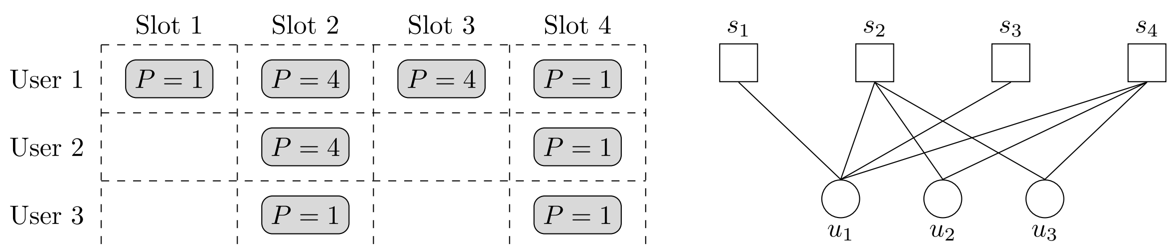


(a) (b)

Figure 1: (a) denotes a MAC frame with 3 users and 4 slots where each user employs a repetition code independently and uses the same power $P = 1$, (b) denotes the bipartite graph representation of the same.

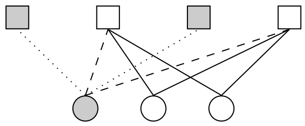
(a) Iteration 1 (b) Iteration 2

Figure 2: Graphical representation of the decoding process

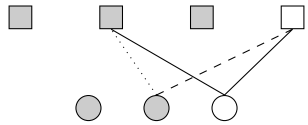


(a) (b)

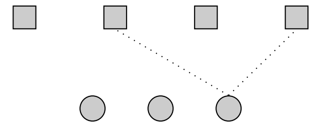
Figure 3: (a) denotes a MAC frame with 3 users and 4 slots where each user employs a repetition code independently and also employ power control by transmitting packets at different power levels chosen independently for the set $\mathcal{P} = \{1, 4\}$, (b) denotes the bipartite graph representation of the same.



(a) Iteration 1



(b) Iteration 2



(c) Iteration 3

Figure 4: Graphical representation of the decoding process

Before presenting our main results, we illustrate how introducing multiple power levels fundamentally changes the throughput for any of the random access scheme such as slotted ALOHA, IRSA or CSA. Consider the toy examples presented in Figure 9 and Figure 11. Figure 9 represents the case of IRSA where one user transmits in all the four slots and rest of the two users transmit in two slots, with identical power $P = 1$. When we perform SIC decoding process (this is equivalent to the peeling decoder), in the first iteration, we are able to decode the packet corresponding to user 1. However, in the second iteration, the decoding process gets stuck due to the collision of packets of user 2 and 3 and therefore the decoding process terminates. Hence we are only able to successfully decode one user over four slots resulting in a throughput of $T = 0.25$. Meanwhile, in Figure 11, employing two power levels $P = 1, 4$ randomly in conjunction with IRSA, in the second iteration, the packet corresponding to user 2 can also be decoded because the SIR is above the threshold β. Therefore, the peeling decoder which got stuck in the second iteration in case 1, kickstarts again enabled by the decoding of packet of user 2. In the second case, we are able to decode the packets of all the three users giving us a throughput of $T = 0.75$, thrice of what we had in the first case with no power control, by incurring a higher average transmit power.

What is important to note is that the increase in throughput by introducing random power levels is not additive or linear, but bootstraps the ability of the SIC process to unravel many more colliding packets. In Fig. **??**, we present actual simulation results, where SA represents slotted ALOHA, IRSA and CSA with just two power levels as $IRSA - PC$ and $CSA - PC$. It is clearly evident that there is a fundamental increase in the throughput that the proposed strategy provides over and above the slotted ALOHA (denoted as SA), IRSA and CSA, even with the use of just two power levels.

The remarkable feature is that introducing just two power levels that are used randomly in conjunction with random access protocol, throughput of more than 1 is achievable, i.e., on average more than one packet can be successfully decoded in each slot. Moreover, the proposed strategy achieves a throughput double that of the best known scheme (CSA) even with just two power levels. Of course the power consumption with the proposed strategy is higher than CSA, however, since the system is interference limited, naively increasing the transmitted power would not yield any benefit.

The main idea behind the improved performance of IRSA over ALOHA was to introduce repetition in transmission thereby increasing the average power transmission and exploit SIC capability. Thus, higher power consumption allowed higher throughput. CSA improved the power efficiency over IRSA by using more efficient codes compared to just repetition code in IRSA. To further exploit the SIC, as discussed before, we need to create a power profile at the receiver where some packets are at much higher power level than the rest. The proposed strategy accomplishes this by using multiple power levels with sufficient gaps between the levels, thus requiring larger power consumption, but allowing fundamental improvement in achievable throughputs.

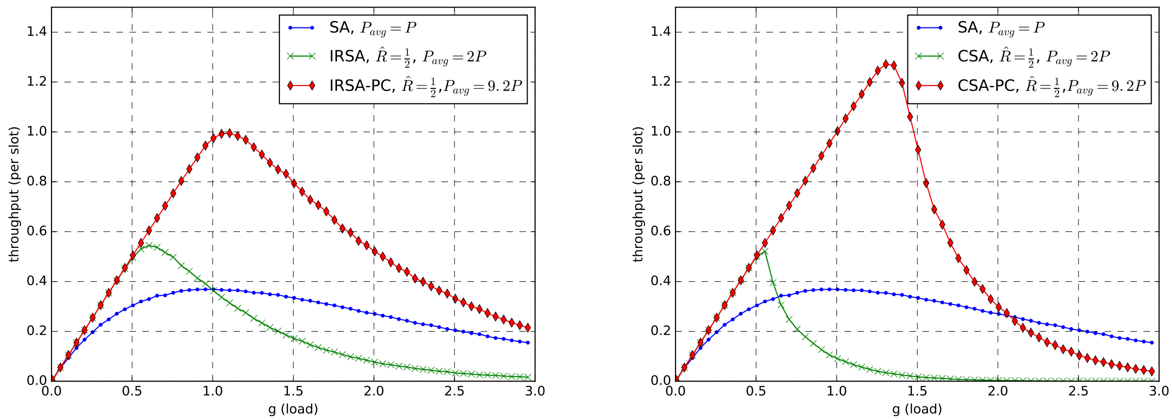


Figure 5: P is the minimum power required to ensure successful decoding at the base station and P_{avg} refers to ther average energy consumption per frame.

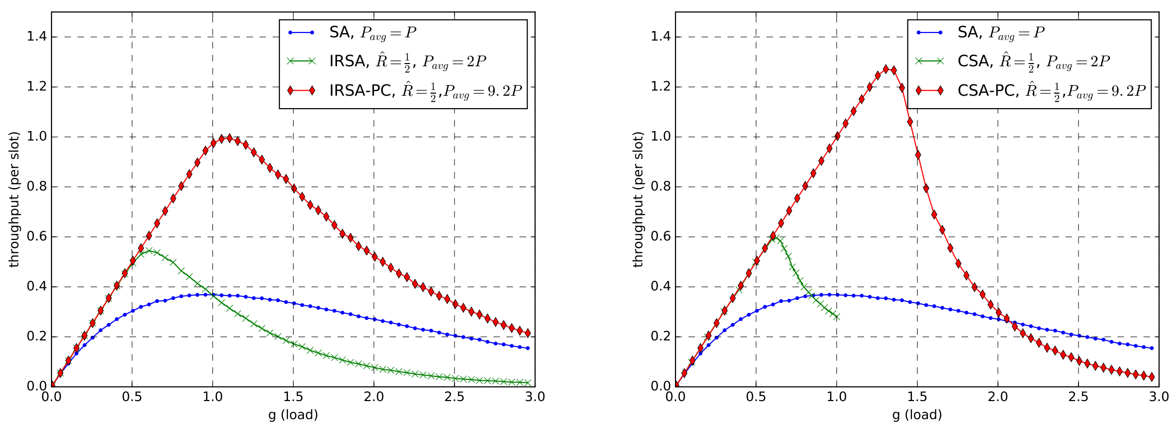


Figure 6: Note to Rahul : Plot with CSA data taken directly from the Liva's paper and plotted only uptil 1

1.2 Our results

In this paper, we propose a multi-level random (with distribution D) power transmit strategy which is used in conjunction with the random access protocol (with distribution S). With the transmit strategy, in the slot chosen by S, a packet is transmitted with power level P chosen according to D. The optimization problem we solve is to maximize the throughput as a function of D and S for ALOHA, IRSA and CSA.

For the case of ALOHA, we give an exact characterization of the throughput of the proposed strategy for two or more power levels, which can be directly optimized over D. With IRSA an exact expression for throughput is not possible, however, using the expressions that are derived for iterative decoding techniques, we derive an expression for the probability of successful decoding of any packet asymptotically. The derived expressions significantly reduce the complexity of solving the optimization problem of maximizing throughput over D

and S. Capacity upper bounds, that characterize what throughputs are not possible are also derived, that are shown to match the achievable throughputs via simulation. Similar analytical results are possible for CSA, but are omitted for brevity, since they do not bring any new ideas with it. Extensive simulation results are presented and throughput comparison with ALOHA, IRSA and CSA are presented that show that there is a fundamental improvement in throughput with the proposed strategy at the cost of higher average power consumption.

The main takeaways from the simulation results are as follows. i) Most of the throughput gain can be extracted via employing just two different power levels, and there is only mild incremental gain with more than two power levels. Thus, the complexity of practical implementation can be kept low by choosing only two power levels. ii) The proposed protocol doubles the throughput gain possible with the best CSA protocol and that too with only two power levels. iii) For two power levels talk about the optimizing distribution D.

1.3 Other Related Work

Random access protocols have been considered in variety of different contexts and not just maximizing the throughput. For example, an important metric is the stability of the system, that defines the largest packet arrival rates at nodes, such that the sum of the expected queue lengths across different nodes remains bounded. Extensive work has been reported in studying the stability properties of slotted ALOHA protocol [**?**, **?**, **?**, **?**]. Another interesting direction has been to understand the information theoretic limits of random access protocols as initiated in [1], where the nodes may not even be slot synchronized, which was later extended in [2–4].

For the throughput problem, extensions of the CSA protocol have been proposed in [5] for the erasure channel, with finite blocklength [6–8], with multiple parallel links for each slot that do not interfere with each other [9]. For the basics of iterated decoding of slotted ALOHA we refer the reader to [10].

During the course of writing the paper, we found that just the raw idea of using multiple power levels chosen randomly with random access protocols can be found in a two-page short note [11], however, without any analysis or simulation results.

2 System Model

We consider a massive time slotted random access model. M time slots are grouped together to form a frame. For each frame, there are N users who wish to communicate one packet of information each. We will assume that the users are slot-synchronized, and that each packet fits exactly in the duration of the slot. We are concerned with studying large systems, and hence assume that N and M are large. The users are uncoordinated, and hence have to choose a random slot for transmission without the knowledge of the other users. In case of collision of message packets, the packets can be decoded using physical layer algorithms as per the channel models described below. Once the packet is decoded, it can be *subtracted* from its slot, thus possibly enabling the recovery of other packets in the slot. This process will be called *intra-slot cancellation.* The *load* g is the average number of users per slot, and is given by $g = N/M$.

In the Irregular-Repetition-Slotted-Aloha(IRSA) model introduced by Liva in [12], each user first samples a repetition-number l from a probability distribution $\{\Lambda_l\}$(where l ranges from 1 to the maximum repetition possible value called deg). The user then creates l replicas of his packet and transmits each of them uniformly at random into l unique slot in a frame. Define R as the average repetition rate, $R \triangleq \sum_l l\Lambda_l$. Each replica also stores the location of the other packet-replicas. Therefore, if one of the replicas were to be decoded in one of the slots, then the other replicas can be *subtracted* from their respective slots. This process is called *inter-slot cancellation.* The same packet repetition procedure is followed in our model. In addition to the packet repetition, in our model, the user also chooses the power with which the packet replica is to be transmitted. Let $\mathcal{P}$ denote the set of powers which the user can select, and $\{\delta_k\}$ denote the probability distribution over the power levels $\mathcal{P}$. The transmitted power is sampled according to the distribution defined as $Pr(P = P_k) = \delta_k, \forall P_k \in \mathcal{P}$ (without loss of generality, assume that $P_k > P_{k+1}$). More precisely, first the user samples the number of repetitions for his packet from $\{\Lambda_l\}$, and then he chooses a power for each of his replicas *i.i.d.,* from $\{\delta_k\}$. We call our model IRSA-PC(stands for IRSA with Power Control). The average power per replica is $\sum_k P_k\delta_k$, and the average power per user is $P_{avg} = R\sum_k P_k\delta_k$. All the users use the common repetition distribution $\{\Lambda_l\}$, set of powers $\mathcal{P}$ and power distribution $\{\delta_k\}$.

IRSA, with an associated parameter $\{\Lambda_l\}$, represents a communication-scheme on the system. Similarly, other communication-strategies are possible such as Coded-Slotted-Aloha [13]. The *throughput* $T(g, \mathcal{P}, \delta, \Lambda)$ of the system(described by $\mathcal{P}$ and δ) and an associated communication-scheme(described by Λ) is the average number of user-packets decoded per slot for a specific load g. A throughput T is said to be achievable in a system, if there exists a communication-scheme such that a throughput T can be obtained on the system. When the throughput is maximized over all possible loads g, we call it the *capacity* T^*.

Definition 1 (Capacity)**.** *For a given set of powers $\mathcal{P}$ with probability distribution δ and repetition distribution Λ, the capacity $T^\star(\mathcal{P}, \delta, \Lambda)$ is defined as*

$$T^\star(\mathcal{P}, \delta, \Lambda) := \sup_{g \geq 0} T(g, \mathcal{P}, \delta, \Lambda) \tag{2.1}$$

We consider two channel models in this paper. The channels are considered to be without feedback and are interference limited, and hence noise can be ignored. The first channel is an ideal channel, where the received power P_{rec} is equal to the transmitted power P i.e., $P_{\text{rec}} = P$. This channel model enables a theoretical analysis of our IRSA-PC algorithm. The other channel model considered is the path-loss channel model which is parameterized by $d_{\min}$ and path-loss exponent α. Here, a base-station is the receiver, and the users are assumed to be distributed according to some spacial distribution around the base-station in a circle of radius r. If a user transmits a packet with a power P from a distance of d, then the received power is given by:

$$P_{rec} = \begin{cases} P & d \le d_{\min}, \\ P\left(\frac{d}{d_{\min}}\right)^{-\alpha} & d > d_{\min}. \end{cases}$$

At the receiver, a packet can be decoded if its *signal to interference ratio(SIR)* is sufficiently high. Let the set of users who have transmitted their packets in slot m be denoted by $\mathcal{R}_m^0$. Then, the $SIR^{(i,m)}$ of a user $i \in \mathcal{R}_m^0$ is given by:

$$SIR^{(i,m)} = \frac{P_{rec}^{(i)}}{\sum_{n \in \mathcal{R}_j^0 \setminus i} P_{rec}^{(n)}}$$

The packet i can be decoded in slot m if $SIR^{(i,m)} \ge \beta$, where β is the *capture threshold*. This phenomenon is called *capture effect* in random-access literature, and the packet is said to have being *captured*. In narrow-band communication systems(such is prevalent in large scale IoT cellular networks[reference?]), β is greater than 1. Throughout this paper, we assume that $\beta > 1$, which implies that at most one packet can be captured in a slot at a particular time. The decoding process follows an iterative procedure: at a particular iteration, packets are captured in slots wherever possible, after which these packets are subtracted from the all their slots using intra-slot and inter-slot cancellation. This process is repeated until no more packets can be captured. Such a process is called *successive interference cancellation(SIC)* in literature. Observe that if the set of powers were a singleton set, $\{P\}$, and the channel were ideal, then our model becomes equivalent to the collision channel model considered in [12], where a packet can be decoded in a slot(has $SIR \ge \beta$, in our notation), if and only if it is the only packet present in the slot.

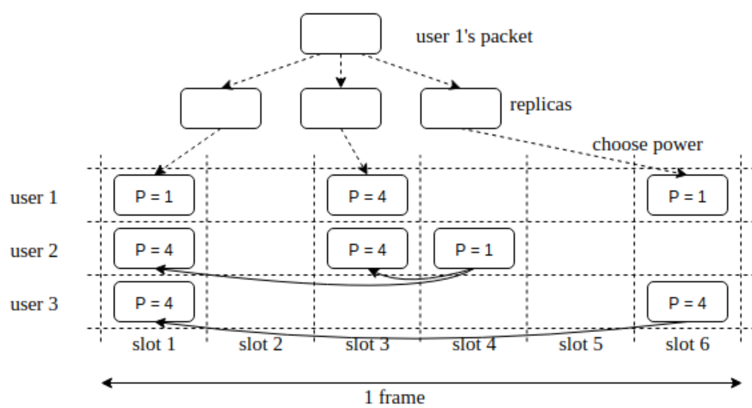


Figure 7: Illustration of the IRSA-PC model with $N = 3$ and $M = 6$. The set of powers is $\mathcal{P} = \{1, 4\}$, and the powers chosen by the users are indicated in the figure. The value of β is 2. In slots 1 and 3, none of the users have $SIR \ge \beta$ in the first iteration. User 2's packet is captured in slot 4 as it has no collisions. User 3's packet is captured in slot 6 as its $SIR \ge \beta$. In the next iteration, users 2 and 3's packets are subtracted from slot 1 by inter-slot cancellation which enables the capture of user 1's packet. Also, in slot 3, user 2's packet is subtracted using intra-slot cancellation which enables the capture of user 1's packet in slot 3 as well.

The primary goal of our work is to maximize the capacity of the random access system. The system enforces several constraints such as average and maximum power constraints(denoted by $P_{\text{avg,max}}$ and $P_{\max}$), and maximum average repetition rate ($R_{\max}$) and maximum possible repetition ($deg_{\max}$) constraints. With a slight abuse of notation, our objective is summarized by the following optimization problem:

$$
\begin{aligned}
&\underset{\Lambda_i,\delta_j}{\text{maximize}} \qquad T^*(\Lambda_i,\delta_i) \\
&\text{subject to} \\
&\qquad P_{\text{avg}} \leq P_{\text{avg,max}} \\
&\qquad P_1 \leq P_{\max} \\
&\qquad R \leq R_{\max} \\
&\qquad deg \leq deg_{\max}
\end{aligned} \tag{2.2}
$$

The channel model has been implicitly assumed in the optimization problem. In the following sections, we will analyze our model under various settings of the parameters $P_{\text{avg,max}}$, $P_{\max}$, $R_{\max}$, and $deg_{\max}$.

2.1 Graphical Model Description

One frame of the IRSA-PC model can be represented by a bipartite graph $\mathcal{G} = \{U, S, E\}$, where U is the set of N nodes representing the users, S is the set of M nodes representing the slots and E is the set of edges. An edge exists between a user node $u \in U$ and a slot node $s \in S$ if the user u has transmitted a packet-replica to the slot s in the given frame. If the user u transmits a packet-replica to slot s with power P, then we will say that the edge connecting u and s has power P. This construction is very similar to the Tanner graph representation of LDPC codes, with the user nodes acting like bit nodes, and the slot nodes acting like check nodes. Figure 8 shows the construction for the example in figure 7. In fact, the SIC algorithm can be interpreted as a peeling decoder of LDPC codes over erasure channel: in a slot, if a particular user u's packet is captured, then the node u and all the edges connected to it are deleted from $\mathcal{G}$. This process is repeated until either all edges are deleted, or no more can be deleted. This interpretation enables the use of the powerful density evolution tool to analyze the performance of the IRSA-PC model. In fact, [12] uses density evolution to analyze IRSA under the simple collision channel model. However, formulating density evolution for a general power control and channel model is difficult, primarily because it is difficult to compute the distribution of SIR and probability of capture of packet [14, 15] . In [16], the authors show that SIR follows the Pareto distribution for the Rayleigh-fading channel model with no power-control distribution, thus enabling a theoretical analysis. In [17], the authors studied IRSA under the path-loss channel model, however they only perform Monte-Carlo simulations for density evolution. A primary contribution of our work is to use approximations to the SIR and capture model in order to formulate density evolution equations. Our approximations are semi-rigorously justified and are numerically verified. In order to describe the density evolution equations in later sections, we need a modicum of notation, and we will do so next.

For our study, it suffices to summarize the bipartite-graph by its user(and slot) node degree distribution, which denotes the fraction of user(and slot) nodes with a particular degree. According to our IRSA-PC model, the probability that a user creates l replicas of his packet(which means that the corresponding user-node has a degree l in the bipartite graph) is equal to Λ_l. In the large system limit, the user-node degree distribution of the graph will be nearly identical to this probability distribution. Therefore, we use $\{\Lambda_l\}$ to denote both the probability distribution of the repetition, as well as the user-node degree distribution of the bipartite-graph. Similarly, let $\{\psi_l\}$ represent the slot-node degree distribution, where ψ_l represents the probability that a slot node has a degree l. These distributions can be succinctly represented in polynomial form as: $\Lambda(x) = \sum_l \Lambda_l x^l$ and $\psi(x) = \sum_l \psi_l x^l$. It is useful to also define edge-perspective degree distributions. Let λ_l be the probability that an edge is connected to a user node of degree l. Similarly, let ρ_l be the probability that an edge is connected to a slot node of degree l. Their corresponding polynomial representations are: $\lambda(x) = \sum_l \lambda_l x^{l-1}$ and $\rho(x) = \sum_l \rho_l x^{l-1}$. The edge and node perspective distributions can be computed from each other with the following simple relations: $\lambda(x) = \Lambda'(x)/\Lambda'(1)$, $\rho(x) = \psi'(x)/\psi'(1)$, $\Lambda(x) = \int_0^x \lambda(x)/\int_0^1 \lambda(x)$, and $\psi(x) = \int_0^x \rho(x)/\int_0^1 \rho(x)$. Since the slots are chosen uniformly at random, the slot node-perspective and edge-perspective degree distributions are determined by the load g and repetition rate R as: $\psi(x) = \rho(x) = e^{-gR(1-x)}$ (for the derivations, refer [12]).

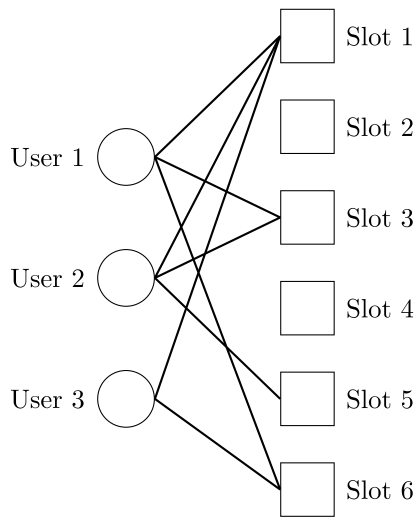


Figure 8: Graph construction for the example in figure 7.

2.2 How Power Control Leads to Gains in Throughput

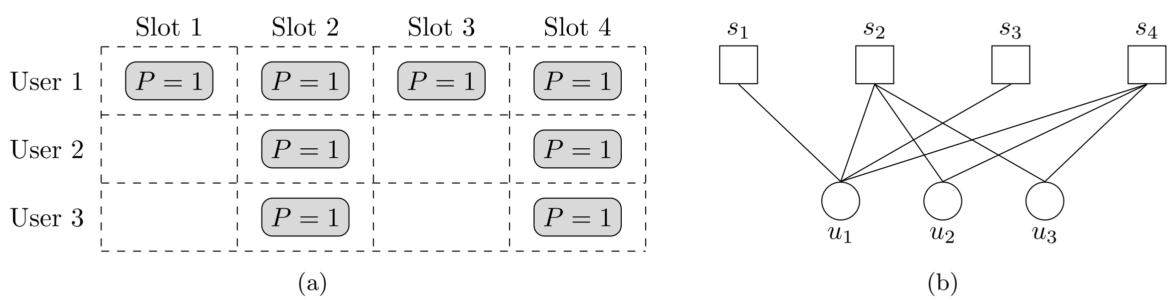


Figure 9: (a) denotes a MAC frame with 3 users and 4 slots where each user employs a repetition code independently and uses the same power $P = 1$, (b) denotes the bipartite graph representation of the same.

Figure 10: Graphical representation of the decoding process

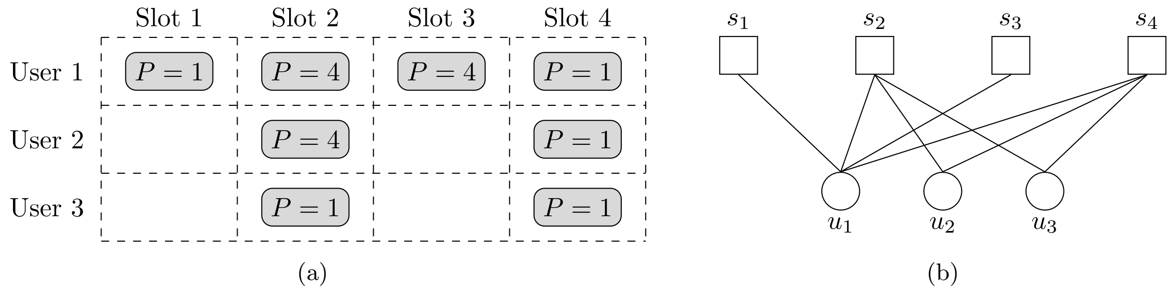


Figure 11: (a) denotes a MAC frame with 3 users and 4 slots where each user employs a repetition code independently and also employ power control by transmitting packets at different power levels chosen independently for the set $\mathcal{P} = \{1, 4\}$, (b) denotes the bipartite graph representation of the same.

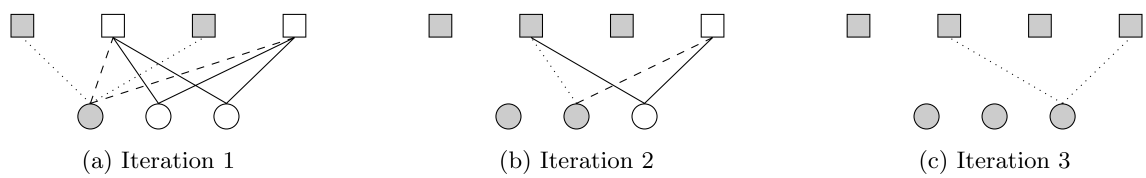


Figure 12: Graphical representation of the decoding process

Before diving into the analysis of how introducing power control on top of IRSA leads to gains in achievable throughput, it is important to get an intuition for how the power control aids the decoding process which in turn leads to increase in throughput. Note that this increase in throughput is not a linear increase but a result of improved sucessive interference cancellation process which is aided by the capture effect. Consider the toy examples presented in Figure 9 and Figure 11. Figure 9 represents the case of IRSA where one user transmits in all the four slots and rest of the two users transmit in two slots. When we perform SIC decoding process (this is equivalent to the peeling decoder), in the first iteration, we are able to decode the packet corresponding to user 1. However, in the second iteration, the decoding process gets stuck due to the collision of packets of user 2 and 3 and therefore the decoding process terminates. Hence we are only able to successfully decode one user over four slots resulting in a throughput of $T = 0.25$. Meanwhile, in Figure 11,in the case of introducing power control in addition to IRSA, in the second iteration, we are able to decode the packet corresponding to user 2 using the *capture effect*. Therefore, the peeling decoder which got stuck in the second iteration in case 1, kickstarts again enabled by the decoding of packet of user 2 using capture effect. In the second case, we are able to decode the packets of all the three users giving us a throughput of $T = 0.75$, thrice of what we had in the first case with no power control. This combined effect of SIC and capture effect results in the throughput gains which we will discuss in detail.

3 Slotted Aloha

Before discussing the IRSA-PC model and analyzing the performance of that model, we first describe the *vanilla Slotted Aloha (SA)* scheme over the collision channel. The vanilla SA scheme has the same setting as described in the System Model with N users trying to transmit a message in M slots but without any repetition of packets. In this setting $\Lambda(x) = x, \mathcal{P} = \{P\}, \{\delta_k\} = \{\delta_1 = 1\}$. The vanilla SA scheme is a very simple scheme in which each user independent of other users chooses a uniformly random slot, to transmit their packet. Each user only transmits a single copy of his/her message. Under the collision channel assumption, if in a particular slot only a single user has transmitted then the packet gets decoded. However, if more than one users transmit in a particular slot then none of the packets corresponding to these users can be decoded. Hence, throughput is essentially the probability of exactly one transmission happening in a particular slot. In the asymptotic case of $N \to \infty, M \to \infty$, throughput for the vanilla SA scheme can written as a function of the load $g, T_{\mathrm{SA}} = ge^{-g}$. The capacity is achieved for $g = 1, T^{\star}_{\mathrm{SA}} = \frac{1}{e}$. Note that in the vanilla Slotted Aloha scheme, since each user transmits only one packet in one frame, one cannot leverage SIC by performing inter-slot decoding. Additionally since all the users transmit the packet at the same power level and due to the ideal channel assumption in the analysis of vanilla Slotted Aloha scheme, even capture effect cannot be leveraged in order to get higher throughput.

3.1 Power Control in Slotted Aloha

IRSA makes use of the diversity in time via the repetition of packets across slots and performs SIC, leading to increased throughput. Instead of leverging the diversity in time, we first analyze the performance of Slotted Aloha by introducing diversity in the power of transmitted packets by leveraging the capture effect.

3.1.1 Two level Power Control in Slotted Aloha

We consider a dual power control slotted aloha (SA-DPC) scheme which is an augmentation of the vanilla SA scheme described above. In this scheme a user apart from choosing the slot in which the packet is to be transmitted, the user also chooses a power level with which the packet is transmitted. All users make this choice independent of each other and the power level chosen by the user can be considered as a random variable which takes values in $\mathcal{P} = \{P_1, P_2\}$ and is distributed as $\{\delta_1 = \delta, \delta_2 = 1 - \delta\}$.

Define $k \triangleq P_1/\beta P_2$, which denotes the maximum number of users that use the lower power level in a slot such that a high power message can be captured in that slot. Notice that in the case of two-level power control, because of the fact that $\beta > 1$, capture of a message is possible only if it has one higher power P_1, and there are a maximum of k other messages in the slot all with low power P_2. The following lemma gives a bound on the probability of the event that there are k interfering packets in a slot.

Lemma 1. *In the time slotted model presented above, the probability that k users choose to transmit in one particular slot is $\mathcal{O}\left(\frac{1}{poly(k)}\right)$.*

Proof. The slots chosen by each of the users are independent and are uniformly at random. Let $\Pr_j(k)$ denote the probability that there are k packet transmissions in slot j:

$$
\begin{aligned}
\Pr_j(k) &\overset{(a)}{=} \binom{N}{k}\left(\frac{1}{M}\right)^k\left(1-\frac{1}{M}\right)^{gM-k} \\
&\overset{(b)}{\approx} \frac{g^k e^{-g}}{k!} \\
&\overset{(c)}{\leq} \frac{g^m}{k(k-1)\dots(k-m+1)}\frac{g^{k-m}}{(k-m)!}e^{-g} \\
&\overset{(d)}{\leq} \frac{K}{k^m} \\
&\overset{(e)}{=} \mathcal{O}\left(\frac{1}{k^m}\right)
\end{aligned}
$$

where (a) follows from the fact that there are $\binom{N}{k}$ ways of having k packets transmitted and the probability of each such event is $\left(\frac{1}{M}\right)^k\left(1-\frac{1}{M}\right)^{gM-k}$ (b) follows from the Stirlings' approximation for large N, (c) follows from taking m to be finite, (d) is true for a suitably large enough K and follows from the fact that $\frac{g^{k-m}}{(k-m)!}\leq e^g$, (e) follows from the definition of $\mathcal{O}(.)$. Since this is true for any finite m, we get that $\Pr_j(k)=\mathcal{O}\left(\frac{1}{\text{poly}(k)}\right)$. □

Remark 1. *Lemma 1 points out that as k increases, $Pr_j(k)$ decreases atleast as fast as $\frac{1}{k^p}$ for some finite p. Notice that in the SA-DPC, we have defined that $P_1 = k\beta P_2$, hence taking $k\to\infty \implies P_1\to\infty$, which is an unrealistic assumption to make. Hence we choose k to be some integer such that P_1 is not unrealisably large and at the same time the probability of k packets occuring together in one slot is low (order of 10^{-2}). After some experimentations, we found $k=5$ to be a suitable value. So while all our results would follow by taking $k\to\infty$, for the purposes of constructing the set $\mathcal{P}$ of powers (details in upcoming sections) and for experiments, we would take $k=5$ and get a very close approximation to the case of $k\to\infty$.*

We will now describe a recursive method to compute the throughput in the SA-DPC scheme and extend this recursive technique to compute the throughput in the extension of SA-DPC scheme to n-power level control Slotted Aloha scheme.

Lemma 2. *Consider a packet with higher power P_1 which can be decoded if there are at most k lower power packets of power P_2. If the probability of choosing power P_1 is δ, and that of choosing P_2 is $1-\delta$, then for sufficiently large k, the probability of packet of power P_1 being decoded in a slot is $g\delta e^{-g\delta}$, where g is the load.*

Proof. From the capture effect, it follows that a packet of higher power P_1 can be decoded if there are k interferering lower power P_2 packets. Let A denote the event that packet of higher power P_1 gets decoded when there are atmost k interferring packets. Let B_i denote the event that one of the transmitted packets is of power P_1 and there are exactly i transmitted packets of power P_2. It follows that $A=\cup_{i=0}^{k}B_i$ and $B_i\cap B_j=\phi,\forall i,j$. It follows that

$$\begin{aligned}
\mathbb{P}(A) &= \mathbb{P}\left(\cup_{i=0}^{k}B_i\right)\\
&\overset{(a)}{=} \sum_{i=0}^{k}\mathbb{P}(B_i)\\
&\overset{(b)}{=} \sum_{i=0}^{k}\binom{N}{i+1}\left(\frac{1}{M}\right)^{i+1}\left(1-\frac{1}{M}\right)^{N-i-1}\binom{i+1}{1}\delta(1-\delta)^i\\
&\overset{(c)}{\approx} \sum_{i=0}^{\infty}\binom{N}{i+1}\left(\frac{1}{M}\right)^{i+1}\left(1-\frac{1}{M}\right)^{N-i-1}\binom{i+1}{1}\delta(1-\delta)^i\\
&\overset{(d)}{\approx} \sum_{i=0}^{\infty}\frac{g^i}{i!}e^{-g}\delta(1-\delta)^i\\
&\overset{(e)}{=} g\delta e^{-g\delta}
\end{aligned}$$

where (a) follows from fact that $B_i\cap B_j=\phi$, (b) follows from the $\mathbb{P}(B_i)=\binom{N}{i+1}\left(\frac{1}{M}\right)^{i+1}\left(1-\frac{1}{M}\right)^{N-i-1}\binom{i+1}{1}\delta(1-\delta)^i$, (c) follows from Remark 1, (d) follows from the Stirlings' approximation for the term $\binom{N}{i+1}$ and $g=N/M$, (e) follows from the Taylor Series expansion of e^x. □

Theorem 1. *In the case when $N\to\infty, M\to\infty$, the throughput of the SA-DPC scheme is $T_{SA\text{-}DPC}= g\delta e^{-g\delta}+(1+g\delta)g(1-\delta)e^{-g}$, where $g=\frac{N}{M}$.*

Proof. Let $\bar{N}_1$ denote the average number of power level P_1 packets decoded per slot and $\bar{N}_2$ denote the average number of power level P_2 packets decoded. Then $T = \bar{N}_1 + \bar{N}_2$. From Lemma 2 it follows that $\bar{N}_1 = g\delta e^{-g\delta}$. For calculating the value of $\bar{N}_2$, there are two cases where a lower power level P_2 packet is decoded : (a) the lower power level packet is the only packet in a given slot, (b) there are two packets in a given slot - one of higher power level P_1 and one of lower power level P_2. In case (b), the P_1 power level packet can be decoded using the capture effect and using SIC, the higher power level packet can be "subtracted" to decode the lower power level packet. Let $\bar{N}_{2a}, \bar{N}_{2b}$ denote the average number of packets decoded in case (a) and case (b) respectively.

$$
\begin{aligned}
\bar{N}_2 &= \bar{N}_{2a} + \bar{N}_{2b} \\
&\overset{(a)}{=} \binom{N}{1}\left(\frac{1}{M}\right)\left(1-\frac{1}{M}\right)^{gM-1}(1-\delta) + 2\binom{N}{2}\left(\frac{1}{M}\right)^2\left(1-\frac{1}{M}\right)^{gM-2}\delta\,(1-\delta) \\
&\overset{(b)}{=} g(1-\delta)e^{-g}\,(1+g\delta)
\end{aligned}
$$

where (a) follows from the fact that $\bar{N}_{2a}$ is the probability that only one low power packet transmission happens in a given slot and $\bar{N}_{2b}$ is the probability that exactly two packets get transimitted and they are of different power levels, (b) follows from taking $N \to \infty, M \to \infty$. Combining the results from Lemma 2 and calculation of $\bar{N}_2$, $T_{\text{SA-DPC}} = g\delta e^{-g\delta} + (1+g\delta)g(1-\delta)e^{-g}$. ☐

Setting $\delta = 0$ or $\delta = 1$, we get that for vanilla Slotted Aloha, $T_{\text{SA}} = ge^{-g}$ which is maximised for $g = 1$ giving $T^{\star}_{\text{SA}} = 0.37$. For a given value of the load, we can optimize δ to obtain the maximum throughput. The maximum value of throughput possible is 0.658, which is achieved for a load of $g = 1.75$ and $\delta = 0.4$. Let $\Delta T := T_{\text{SA-DPC}} - T_{\text{SA}}$, then for any non-trivial δ, we have that $\Delta T = g^2\delta(1-\delta)e^{-g}$. Hence $\forall \delta \in (0,1), \Delta T > 0$, which implies that for all loads, choosing a dual power scheme results in throughput gain over vanilla Slotted Aloha which can be seen in Figure 13.

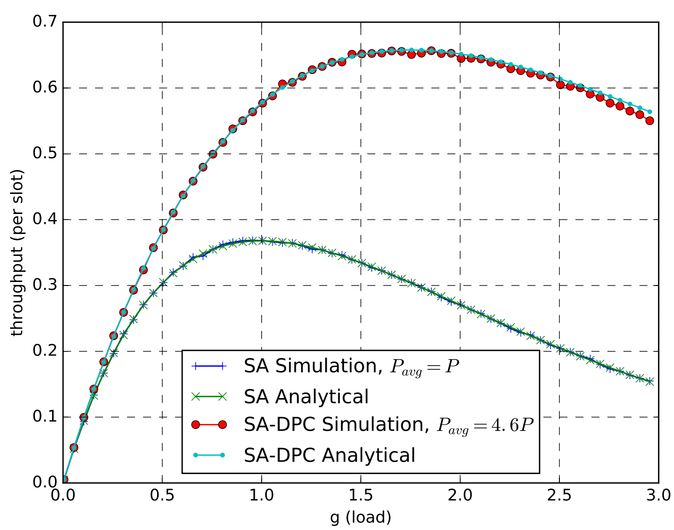


Figure 13: Plot comparing the performance of vanilla Slotted Aloha scheme and Slotted Aloha scheme with Dual Power Control. SA Simulation and SA-DPC Simulation curves correspond to the throughput curves obtained using simulation and the SA Analytical and SA-DPC Analytical correspond to the plot of closed form expression we have for T_{SA} and $T_{\text{SA-DPC}}$. Note that the simulation curves are for a finite number of slots $M = 1000$ whereas the closed form expressions T_{SA} and $T_{\text{SA-DPC}}$ correspond to $M \to \infty$.

3.1.2 n Level Power Control in Slotted ALOHA

In the n-level power control scheme for slotted aloha (SA-nPC), we generalise the SA-DPC scheme from two power levels to n-power levels. In this generalization, users can now choose to transmit their packet with power $P_i \in \mathcal{P}$ where $|\mathcal{P}|= n$. Assume that $P_i \geq k\beta P_{i+1}, \forall i \in [n-1]$, where $[m] \triangleq \{1, 2, \dots, m\}$

Theorem 2. *In the case $N \to \infty, M \to \infty$, throughput of the SA-nPC scheme is given as $T_{SA\text{-}nPC} = \sum_{i=1}^{n} \left(\prod_{j=1}^{i-1} (1 + g\delta_j) e^{-g\delta_j} \right) g\delta_i e^{-g\delta_i}$, where $g = \frac{N}{M}$.*

Proof. Let $\bar{N}_i, \forall i \in [n-1]$ denote the average number of packet of power level P_i decoded. Then $T_{\text{nPC-SA}} = \sum_{i=1}^{n} \bar{N}_i$. $\bar{N}_1 = g\delta_1 e^{-g\delta_1}$ follows from Lemma 2, $\bar{N}_2 = (1 + g\delta_1) g\delta_2 e^{-g(\delta_1+\delta_2)}$ follows from taking two cases - (a) there are at most k lower power level packets, (b) there are one packet of power level P_1 and at most k lower power level packets. The calculation of $\bar{N}_2$ is along the same lines as in the proof of Theorem 1. For computing $\bar{N}_3$, there are 4 cases:

(a) 0 P_1 power level packets, 0 P_2 power level packets and atmost k packets of power level $P_i, i > 3$.

(b) 0 P_1 power level packets, 1 P_2 power level packets and atmost k packets of power level $P_i, i > 3$.

(c) 1 P_1 power level packets, 0 P_2 power level packets and atmost k packets of power level $P_i, i > 3$.

(d) 1 P_1 power level packets, 1 P_2 power level packets and atmost k packets of power level $P_i, i > 3$.

Summing across the 4 cases gives us that $\bar{N}_3 = (1 + g\delta_1)(1 + g\delta_2) g\delta_3 e^{-g(\delta_1+\delta_2+\delta_3)}$. Consider the case of $\bar{N}_i$, there are 2^{i-1} cases to consider which are similar to the cases considered for computing $\bar{N}_3$. In those 2^{i-1} we consider all combinations of 0 and 1 packets of power level $P_j, j < i$ and atmost k packets of power level $P_j, j > i$. This count is given as $\bar{N}_i = (1 + g\delta_1)(1 + g\delta_2) \dots (1 + g\delta_{i-1}) g\delta_i e^{-g(\delta_1+\delta_2+\dots+\delta_i)} = \left(\prod_{j=1}^{i-1} (1 + g\delta_j) e^{-g\delta_j} \right) g\delta_i e^{-g\delta_i}$. Summing all these $\bar{N}_i$ gives us the required result. □

4 IRSA-PC

In this section, we will formulate a theoretical analysis of the IRSA-PC model under some assumptions on the set of powers $\mathcal{P}$. The analysis is motivated by the analysis of IRSA in [12], and hence it is briefly recapitulated below.

Recall that IRSA considers a collision channel model, where a packet can be decoded in a slot only if it has no collisions with any other packets. Let q denote the probability that an edge connected to a user node is unknown(which means that it hasn't been decoded until that time, and is not decoded at that iteration) at a given iteration, given that the packets corresponding to the other edges connected to the user node have been decoded(also called as the edges having been revealed or known) with probability $(1 - p)$ by the previous iteration. An edge connected to a user node of degree l remains unknown only if all the other edges connected to the user node remain unknown at that particular time, giving rise to $q = p^{l-1}$(in terms of the example in Figure 9, this means that at some iteration of the SIC algorithm at the user node u_1, the edge $s_1 - u_1$ is unkown only if all of $s_2 - u_1$, $s_3 - u_1$ and $s_4 - u_1$ are unknown, giving $q = p^3$ at that edge). Note to Rahul: This illustration using the example in Fig 9 is not accurate because Density Evolution works for large graphs only, and more importantly it makes the assumption that the graph is acyclic. The graph in the example is neither large nor acyclic. Should we say something about this? Similarly, let p denote the probability that an edge connected to a slot node is unknown at a particular iteration, given that the other edges connected to the slot node have been revealed with probability $(1-q)$ by the previous iteration. Under the collision channel model, an edge connected to a slot of degree l is revealed at a particular iteration only if all the other edges connected to the slot node have been revealed by that iteration, giving $(1-p) = (1-q)^{l-1}$(in terms of the example in Figure 9, this means that at some iteration of the SIC algorithm at the slot node s_2, the edge $u_1 - s_2$ is known only if both edges $u_2 - s_2$ and $u_3 - s_2$ have been revealed by the previous iteration, giving $1-p = (1-q)^2$ for that edge). Let q_i and p_i denote the probability that an edge connected to a user node and a slot node remain unknown at iteration i during the iterative SIC decoding process. Then, averaging the previously stated formulas over the respective edge-perspective distributions, we get the following density evolution equation:

$$\begin{aligned} q_1 &= 1, \\ 1 - p_i &= \sum_{l=1}^{N} \rho_l (1-q_i)^{l-1} = \rho(1-q_i), \quad i \geq 1, \\ q_i &= \sum_{l=1}^{deg} \lambda_l p_{i-1}^{l-1} = \lambda(p_{i-1}), \quad i \geq 2. \end{aligned} \tag{4.1}$$

The density evolution equations above generate a sequence of probabilities as $q_1 \to p_1 \to q_2 \to \cdots p_\infty$. At the end, a user node's packet is decoded if any of the edges connected to the user have been revealed. For a given load g, the relation between the throughput and packet loss probability is given as $T = g\,(1 - P_L(g))$ and the asympototic (as $M \to \infty$) packet loss probability is given as $P_L = \Lambda(p_\infty)$ [12].

Definition 2 (Asymptotic Throughput)**.** *Let $p_\infty := \liminf_{i\to\infty} p_i$. For a given load g, set of power $\mathcal{P}$ with probability distribution δ and repetition distribution Λ, the asymptotic throughput $T(g, \mathcal{P}, \delta, \Lambda)$ is defined as*

$$T(g, \mathcal{P}, \delta, \Lambda) := g\,(1 - \Lambda(p_\infty)). \tag{4.2}$$

For the sake of notational convenience we will drop the parameters $g, \mathcal{P}, \delta, \Lambda$ and refer to the asymptotic throughput as just T. Also note that p_i is a function of δ, Λ and the exact expression for the case of two power levels is presented in Lemma 3. Under our general system model(with power-control and the channel models considered), the update at the user node still remains the same, $q = \lambda(p)$. However, at a degree l slot node, a packet may be decoded even if it has collisions, if its $SIR \geq \beta$. Let $w_{l,t}$ denote the probability that an edge connected to a slot node of degree l is decoded at a particular iteration, given that t of the other edges connected to that slot node are still unknown at that iteration. Then, the update equation at the slot nodes can be written as:

$$1 - p = w_{l,0}(1-q)^{l-1} + \sum_{t=1}^{l-1} w_{l,t} \binom{l-1}{t} q^t (1-q)^{l-1-t}. \tag{4.3}$$

Since we ignore noise in our model, an edge connected to a slot with no collisions is revealed with probability 1: $w_{l,0} = 1$. In order to build up to the analysis of a more general IRSA-PC model, we will start by considering an IRSA-PC model with just two power levels.

4.1 IRSA-PC with Two Power Levels

The set of powers is $\mathcal{P} = \{P_1, P_2\}$, and the corresponding probability distribution is $\{\delta, (1-\delta)\}$. Let the power levels be such that, $P_1 \geq k\beta P_2$, where k is some positive number and β is the capture threshold. As a consequence of the nature of the packet capture model, it suffices to restrict ourselves to integer values of k(i.e., consider $k = \lfloor P_1/\beta P_2 \rfloor$.) From the capture model, this means that a packet-replica with power P_1 in a slot can be decoded when there are at most k collisions with packets of lower power P_2.

Lemma 3 (Density evolution recursion for IRSA-PC)**.** *Consider the IRSA-PC scheme with two power levels as described above. Let R be the average repetition rate as defined in the System model section. Let λ_l be the probability that an edge is connected to a user node of degree l and $\lambda(x) = \sum_l \lambda_l x^{l-1}$ be the corresponding polynomial representation. Let p_i denote the probability that an edge connected to a slot node of degree l is unknown at iteration i, then as $M \to \infty$ for a constant load g, we have*

$$p_i \approx 1 - (1-\delta)e^{-g\delta\lambda(p_{i-1})} - \delta e^{g\lambda(p_{i-1})\delta R} - \delta(1-\delta)g\lambda(p_{i-1})Re^{-g\lambda(p_{i-1})R} \tag{4.4}$$

$$q_{i+1} = \lambda(p_i) \tag{4.5}$$

Proof. Consider an edge connected to a slot node with degree l and t other edges connected to the slot node still being unknown at the particular iteration. If this chosen edge has higher power P_1(i.e., corresponds to a packet-replica transmitted with power P_1), then it can decoded at this iteration as long as the other t edges are of lower power P_2 and $0 \le t \le k$. Similarly, if the chosen edge is of lower power P_2, then it can be decoded if $t = 0$, or if $t = 1$ and the other edge is of higher power P_1(first the higher power edge gets decoded, thus leaving the lower power level packet with no collisions). Thus, the term $w_{l,t}$ in the density evolution equation in (4.3) for the two power level case can be written as:

$$w_{l,t} = \begin{cases} 1 & t = 0, \\ 2\delta(1-\delta) & t = 1, \\ \delta(1-\delta)^t & 2 \le t \le \min\{k, l-1\}, \\ 0 & t > \min\{k, l-1\}. \end{cases} \tag{4.6}$$

Averaging over the edge degree distributions, the DE update equation on the slot node side (4.3) for iteration i can be specifically written as:

$$p_i = 1 - \rho_1 - \sum_{l=2}^{N} \rho_l \left((1-q_i)^{l-1} + \sum_{t=1}^{\min\{k,l-1\}} \delta(1-\delta)^t \binom{l-1}{t} q_i^t (1-q_i)^{l-1-t} + (l-1)\,\delta(1-\delta)q_i(1-q_i)^{l-2} \right). \tag{4.7}$$

In order to simplify the expression, and also for the ease of extension to more general power control, we now apply some approximations to the above formula. Firstly, as a consequence of Lemma 1, $\min\{k, l-1\}$ can be approximated as $(l-1)$ for a large enough k. Hence,

$$p_i \overset{(a)}{\approx} 1 - \rho_1 - \sum_{l=2}^{N} \rho_l \left((1-q_i)^{l-1} + \sum_{t=1}^{l-1} \delta(1-\delta)^t \binom{l-1}{t} q_i^t (1-q_i)^{l-1-t} + (l-1)\,\delta(1-\delta)q_i(1-q_i)^{l-2} \right), \tag{4.8a}$$

$$\overset{(b)}{=} 1 - \rho_1 - \sum_{l=2}^{N} \rho_l \left((1-q_i)^{l-1} + \delta\left((1-\delta q_i)^{l-1} - (1-q_i)^{l-1} \right) + (l-1)\,\delta(1-\delta)q_i(1-q_i)^{l-2} \right), \tag{4.8b}$$

$$\overset{(c)}{=} 1 - (1-\delta)\,e^{-gq_iR} - \delta e^{gq_i\delta R} - \delta(1-\delta)gq_iRe^{-gq_iR}, \tag{4.8c}$$

$$\overset{(d)}{=} 1 - (1-\delta)e^{-g\delta\lambda(p_{i-1})} - \delta e^{g\lambda(p_{i-1})\delta R} - \delta(1-\delta)g\lambda(p_{i-1})Re^{-g\lambda(p_{i-1})R}. \tag{4.8d}$$

In the above derivation, (a) is obtained by using Lemma 1 to approximate $\min\{k, l-1\}$ as $(l-1)$, (b) is obtained by using the binomial formula, (c) is obtained by substituting the expression for the slot edge-perspective degree distribution, $\rho(x) = e^{-gR(1-x)}$ as $N \to \infty$ because $M \to \infty$ for a constant load g, (d) follows from equation (4.1) that $q_i = \lambda(p_{i-1})$. $\square$

For a given $\Lambda, \delta, g, \mathcal{P}$, we are interested in computing the asymptotic throughput $T(g, \Lambda, \delta, \mathcal{P})$ as defined in (4.2) for which we are required to compute $\Lambda(p_\infty)$. By iterating (4.4) and (4.5) for a sufficiently large number of times, we can approximately get the value of p_∞ and use that to compute $T(g, \Lambda, \delta, \mathcal{P})$. In practice we have found that 500 iterations of (4.4) and (4.5) are sufficient to obtain the value of p_∞. Now for a given $\lambda, \delta, \mathcal{P}$, using a binary search like approach we can also compute the capacity approximately as defined in (2.1) by discretizing and searching over finitely many g. The finer the discretization the better our approximation of optimal g for a given $\Lambda, \delta, \mathcal{P}$.

For ease of notation, we will make the following definitions:

$$f_q(p) = \lambda(p) \tag{4.9}$$

$$f_p(q) = 1 - (1-\delta)\, e^{-gqR} - \delta e^{-gq\delta R} - \delta(1-\delta) gqR e^{-gqR} \tag{4.10}$$

The Figure 14 shows the performance improvement that is possible by using power control with IRSA. The DE plots in the figure were obtained performing DE for the given set of parameters, and then plotting the throughput as $T = g(1 - \Lambda(p_\infty))$.

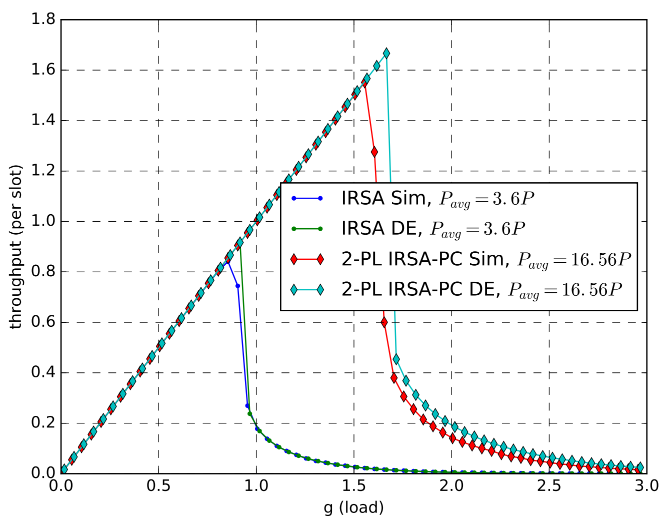


Figure 14: Performance improvement with 2-level Power Control with $\delta_1 = 0.4, \delta_2 = 0.6$ and $\Lambda(x) = 0.5x^2 + 0.28x^3 + 0.22x^8$. *DE* lines are those obtained by DE analysis of the respective IRSA and IRSA-DPC models, and *IRSA* lines are those corresponding simulation results of the respective models.

The implications of Figure 14 are discussed in detail in subsection 6.1. As $M \to \infty$, for all $g \le T^\star(\mathcal{P}, \delta, \Lambda)$, the throughput is $T = g$ which is evident from the linear increase in throughput T with respect to g of the DE plots in Figure 14. This means that the all the packets have been decoded, and hence there is no loss in packets. We call this region(parameterized by g) where $T(g, \mathcal{P}, \delta, \Lambda) = g$ as the *lossless region*. Also notice that $\forall g > T^\star(\mathcal{P}, \delta, \Lambda)$, we have $T < g$, which means that a fraction of the packets weren't decoded. We will call this the *lossy region*.

Remark 2. *Notice that in Figure 14, the capacity, as defined in* (2.1)*, is achieved for a load g such that $T^*(\mathcal{P}, \delta, \Lambda) = g$, i.e., it is achieved in the lossless region. After that point, there is a sudden drop in the throughput(one could call this a phase transition point). We have noticed this phenomenon in all our experiments, although a rigorous proof for this remains elusive. However, we will use this fact in the further sections in order to derive some theoretical results.*

As a consequence of Remark 2, we can redefine capacity(initially defined in (2.1)) as written below.

Definition 3 (Capacity)**.** *For a given set of powers $\mathcal{P}$ with probability distribution δ and repetition distribution Λ, the capacity $T^\star(\mathcal{P}, \delta, \Lambda)$ is defined as*

$$T^\star(\mathcal{P}, \delta, \Lambda) := \sup\{g \geq 0 \mid \liminf_{i\to\infty} p_i = 0\} \tag{4.11}$$

where p_∞ and g are related using the density evolution equations which we will characterize subsequently.

4.1.1 Design of IRSA-PC scheme

Under the IRSA-PC model with 2 power levels, the design task is find a repetition distribution $\Lambda(x)$ and the power-choice distribution $\{\delta_i\}$. Given these parameters and a certain load g, the degree distribution at the slot nodes becomes fixed: $\rho(x) = \sum_l \rho_l x^{l-1} = e^{-gR(1-x)}$, where recall that ρ_l is the probability that a randomly chosen edge is incident on a slot node of degree l. The coefficients ρ_l can be found by a Taylor series expansion of $\rho(x)$ around $x = 0$ resulting in:

$$\rho_l = \frac{1}{(l-1)!}(gR)^{l-1}e^{-gR}$$

So far in our analysis, we have been concerned with computing the capacity of a given system i.e given $\Lambda, \delta, \mathcal{P}$, we are interested in computing the capacity $T^\star(\mathcal{P}, \Lambda, \delta)$ which can been done by discretizing the load values g and using Lemma 3 to do a binary search and compute approximation to the capacity. Note that all our analytical results are for $|\mathcal{P}|= 2$ and extensions to higher levels are discussed subsequently but they lack neat closed form expressions. Next we are interested in looking at a more general optimization problem where instead of being given a particular Λ, δ, we optimize throughput over Λ and δ as well. In this more general optimization problem, the only two constraints we set are fixing the set of power $\mathcal{P}$ and maximum possible repetitions $d_{\max}$. The insights we draw from the analysis uptil this point along with the equations (4.9) and (4.10) derived in Lemma 3 would aid us in setting the right constraints for this optimization problem for the case of $|\mathcal{P}|= 2$

As has been illustrated in Remark 2, the capacity for a given $\Lambda, \delta, \mathcal{P}$ is attained at the maximum load such that the system still remains in the lossless region i.e $\Lambda(p_\infty) = 0$ which is equivalent to $p_\infty = 0$. Since we know that capacity is attained in the lossless region, instead of setting a non-convex constraint like $T^\star = g(1 - \Lambda(p_\infty))$, we can set a much simpler linear constraint of $T^\star = g$. Since all user packets are decoded in the lossless region, the probability of a user-edge being unknown $q_i \to 0$ as the iteration number $i \to \infty$. Note that since $q_i = \lambda(p_{i-1})$, $q_i \to 0$ as $i \to \infty$ is equivalent to $p_i \to 0$ as $i \to \infty$. A necessary and sufficient condition for q_i in the DE (for example, in (4.1)) to converge to 0 is that $q > f_q(f_p(q))$ for every $q \in (0, 1]$(this is clearly a sufficient condition as it leads to a decreasing sequence q_i. It also turns out to be a necessary condition [18]). This immediately leads to a tractable form of our optimization problem in (2.2):

$$
\begin{aligned}
\underset{g,\lambda_i,\delta_j}{\text{maximize}} \quad & T^\star \\
\text{subject to} \quad & \\
& T^\star = g, \\
& \lambda_i \geq 0,\ i = 2, \ldots, d_{\max}, \\
& \sum \lambda_i = 1, \\
& \delta_j \geq 0,\ j = 1, 2 \\
& \sum \delta_i = 1, \\
& q > f_q(f_p(q)),\ \forall q \in (0, 1]
\end{aligned}
\tag{4.12}
$$

The density evolution constraint $q > f_q(f_p(q)), \forall q \in (0,1]$ in the above optimization program consists of an uncountable number of constraints. In practical implementation, the density evolution constraint is only applied on a large finite number of points in $(0,1]$. The above optimization problem is in general a non-convex optimization problem, thus requiring non-convex solvers. In our work, Differential Evolution [19] is used to algorithm to solve the optimization problem.

The design of an IRSA-PC scheme using the optimization program (4.12) requires us to choose the maximum repetition degree $d_{\max}$ and the size of the set of powers available $|\mathcal{P}|$. In practical situations, these can be decided by system constraints such as power constraints. Moreover, since the optimization is done using a numerical solver, it carries no guarantees with respect to the optimal capacity possible. Therefore, we supplement our results with a number of throughput upper bounds next.

4.1.2 Upper bounds on the throughput for two power level IRSA-PC

The first upper bound will depend on the average repetition rate R of the repetition scheme. That is, if the average repetition rate $R > 1$ is fixed, what is the maximum throughput achievable?

Theorem 3 (Upper Bound 1)**.** *Consider a 2 power-level IRSA-PC scheme with average repetition rate R, and power control parameter δ. If a throughput of T is achievable, then the following relation must be satisfied:*

$$
\frac{(\delta^2 - 2)}{RT} + e^{-RT}\left(RT(1-\delta^2) + \delta(1-\delta)\right) + RTe^{-RT\delta} + \frac{1}{R} \leq 0 \tag{4.13}
$$

Proof. The proof uses the DE equations: $q = f_q(p)$ and $p = f_p(q)$. The DE can be visualized on an EXIT chart as show in Figure 15. EXIT charts is a popular model used to represent recursive computations in the iterative error correcting codes literature [20, 21].

Define the area between $f_q(p)$ as defined in (4.10) and the x-axis as A_q, and the area between $f_p(q)$ as defined in (4.9) and the y-axis as A_p:

$$
A_q = \int_0^1 f_q(p)dp, \qquad A_p = \int_0^1 f_p(q)dq
$$

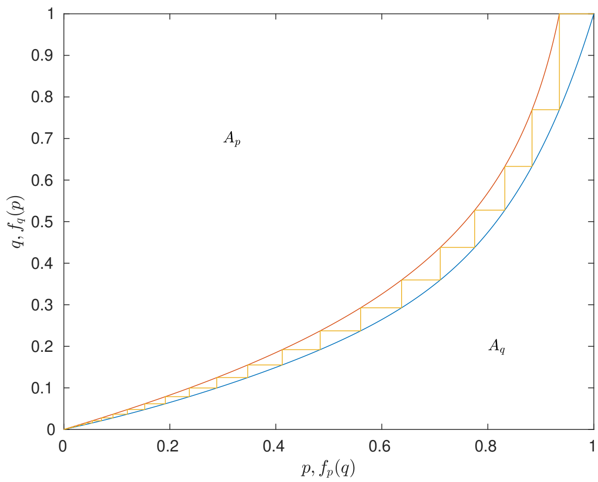


Figure 15: Exit Chart depiction of DE. $f_q(p)$ is plotted on the y-axis as a function of p in the x-axis. $f_p(q)$ is plotted on the x-axis as a function of q in the y-axis. The DE recursion starts at the point (1,1), and then proceeds by projections onto the $f_q(p)$ curve and $f_p(q)$ curve sequentially. In the lossless region, the DE converges to (0,0) on the chart. In the lossy region, the DE does not converge to (0,0).

To upper bound the throughput $T(g, \mathcal{P}, \delta, \Lambda)$ for a particular load g, it is sufficient to upper bound the capacity $T^*(\mathcal{P}, \delta, \Lambda)$. As postulated in Remark 2, the capacity is achieved when the system is in the lossless region. In terms of the DE equations, the system is in the lossless region implies that $p, q \to 0$.A necessary and sufficient condition for $p, q \to 0$ is that the two curves f_p, f_q in the EXIT chart in Figure 15 are non-intersecting [13]. Clearly, a necessary condition for the two curves to be non-intersecting is that the area covered by the two curves is not greater than 1: $A_p + A_q \leq 1$. The respective areas can be computed as:

$$A_q = \int_0^1 \lambda(p)dp = \frac{1}{\Lambda'(1)} = \frac{1}{R} \tag{4.14}$$

$$A_p = \int_0^1 f_p(q)dq = \frac{\delta^2 - 2}{RT} + e^{-RT}\left(\frac{1-\delta^2}{RT} + \delta(1-\delta)\right) + \frac{e^{-RT\delta}}{RT} + 1 \tag{4.15}$$

Substituting this in $A_p + A_q \leq 1$ gives us the result. □

Corollary 1 (Rate Independent Upper Bound)**.** *A two power-level IRSA-PC scheme with power control parameter δ has a universal upper bound on throughput as $T \leq (2 - \delta^2)$.*

Proof. The second and third terms in the LHS of (4.13) are both non-negative terms. Therefore, a necessary condition for (4.13) to be satisfied is: $\frac{(\delta^2-2)}{RT} + \frac{1}{R} \leq 0$, which gives us our result for all $R > 0$. □

Theorem 4 (Upper Bound 2)**.** *Consider a IRSA-PC scheme with the power distribution $\{\delta, 1-\delta\}$ and the fraction of messages being repeated twice, Λ_2, being fixed. If a throughput T is achievable, then it satisfies:*

$$T < \min\left(2 - \delta^2, \frac{1}{2\Lambda_2\left(1 + 2\delta^2 - 2\delta\right)}\right). \tag{4.16}$$

Proof. The Corollary 1 includes communication schemes with arbitrary rates, which clearly includes schemes with a particular fixed Λ_2 parameter. Therefore, $(2 - \delta^2)$ is an upper bound on schemes with the fixed parameter Λ_2. The other term in the upper bound expression is obtained by upper bounding the capacity as derived next.

Combining (4.9) and (4.10), the DE recursion is written as $q_{i+1} = f_q(f_p(q_i))$. As mentioned in Remark 2, the capacity is equal to the maximum load such that the system is still in the lossless region. Therefore, the DE recursion at the capacity achieving load g has to be such that $q > f_q(f_p(q))$ for every $q \in (0, 1]$. In the region where $q \to 0$, by taking the partial derivative of $f_q(f_p(q))$ with respect to q ,this condition becomes equivalent to $\left.\frac{\partial f_q(f_p(q))}{\partial q}\right|_{q=0} < 1$. This expression can be evaluated to obtain the result.

$$
\begin{aligned}
\left.\frac{\partial f_p(q)}{\partial q}\right|_{q=0} &= RT(1-\delta) + RT\delta^2 - RT\delta(1-\delta) \\
\left.\frac{\partial f_q(p)}{\partial p}\right|_{p=0} &= \lambda_2 \\
\left.\frac{\partial f_q(f_p(q))}{\partial q}\right|_{q=0} &= \left.\frac{\partial f_p(q)}{\partial q}\right|_{q=0} \left.\frac{\partial f_q(p)}{\partial p}\right|_{p=0} \\
&= \lambda_2 \left(RT(1-\delta) + RT\delta^2 - RT\delta(1-\delta)\right) \\
&\overset{(a)}{=} 2\Lambda_2 T \left(1 + 2\delta^2 - 2\delta\right)
\end{aligned}
$$

, where (a) is obtained by using the relation $R\lambda_2 = 2\Lambda_2$ and rearranging the terms. Now, using the inequality, $\left.\frac{\partial f_q(f_p(q))}{\partial q}\right|_{q=0} < 1$, completes the proof. □

Corollary 2 (Upper Bound 3)**.** *Consider a 2 power-level CSA system with coding rate R, and power control parameter δ. Let a throughput of g be achievable. Let $l(p)$ be a line which is tangent to the curve $f_p^{-1}(p)$ and also passes through the point $(1, 1)$. Let the point of contact of $l(q)$ and $f_p^{-1}(p)$ be (p_c, q_c). Define A_{min} as the area between the line $l(p)$ and the curve $f_p^{-1}(p)$ from the range (p_c, q_c) to $(1, 1)$. Then, the following equation must be satisfied:*

$$A_p + A_q + A_{min} \leq 1 \tag{4.17}$$

Proof. As mentioned in the proof of Theorem 3, a necessary and sufficient condition for the density evolution of p, q to converge to 0 is that the two curves $f_q(p)$ and $f_p^{-1}(q)$ do not intersect. This implies that the curve $f_q(p)$ lies below the curve $f_p^{-1}(q)$. Therefore, $f_q(p_c) \leq q_c$. Also, observe that, $f_q(p) = \lambda(x)$, is a convex function(since, $\lambda(x)$ is a polynomial with non-negative coefficients). Hence, $f_q(p)$ lies below the line $l(x)$(which is above the line connecting two points of $f_q(p)$). We have shown that the curve $f_q(p)$ cannot enter the area between the line $l(x)$ and $f_p^{-1}(x)$, which gives us our result. □

Since it is not easy to represent $A_{\min}$ in 2 analytically, we evaluate Upper Bound 3 numerically.

4.1.3 Upper Bound Comparisons

This section contains some comparisons of achieved throughputs to the upper bounds. The distributions Λ_1 and Λ_2 have been optimized according to (2.2) for the respective parameters.

$\Lambda(x)$	δ	g^*	**UB 1**	**UB 2**	**UB 3**	**Rate Ind. UB**
$\Lambda^{\text{liva}}(x) = 0.5x^2 + 0.28x^3 + 0.22x^8$	1	0.938	0.9695	1	0.962	1
$\Lambda^{\text{liva}}(x) = 0.5x^2 + 0.28x^3 + 0.22x^8$	0.4	1.67	1.756	1.84	1.738	1.84
$\Lambda^{1}(x) = 0.56x^2 + 0.21x^3 + 0.23x^8$	0.4	1.67	1.756	1.717	1.734	1.84
$\Lambda^{2}(x) = 0.6x^2 + 0.2x^3 + 0.2x^8$	0.6	1.517	1.589	1.581	1.579	1.64

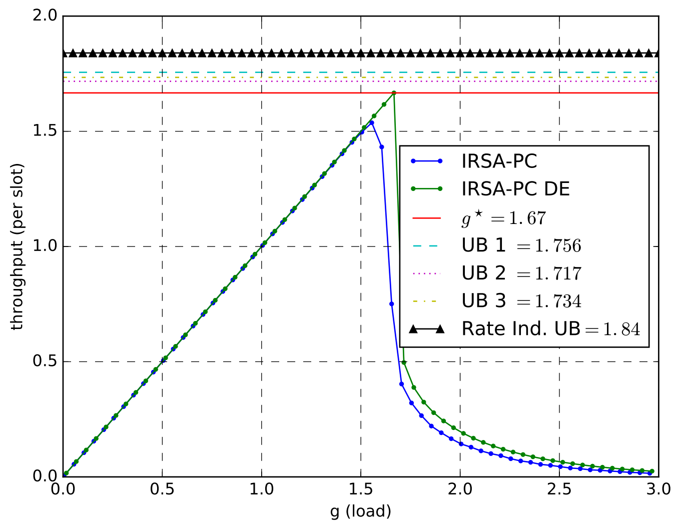


Figure 16: Comparing the various upper bounds with the actual throughput for the repetition distribution $\Lambda^1(x) = 0.56x^2 + 0.21x^3 + 0.23x^8$ with the 2 power levels distributed as $\delta = \{0.6, 0.4\}$

Note that $T^\star(\mathcal{P}, \delta, \Lambda) \leq \min\{\text{UB 2}, \text{UB 3}\}$. It is evident from the proof of UB 3 that UB 3 is a refinement of UB 1 and hence is a tighter upper bound albeit computationally more involved than computing UB 1. The Rate Independent UB follows as a corollary of UB 1 and is computationally trivial in comparison to UB 1. The reason for considering UB 2 is that for same Λ distributions UB 2 provides a tighter upper bound than UB 1 as illustrated in the case of $\Lambda^2(x)$ distribution. It is not clear how the UB 2 and UB 3 compare with each other for different Λ distributions and hence to upper bound the capacity $T^\star(\mathcal{P}, \delta, \Lambda)$, we consider the minimum of the two.

4.2 IRSA-PC with Three Power Levels

Lemma 4. *Consider the IRSA-PC scheme with three power levels $\mathcal{P} = \{P_1, P_2, P_3\}$ with the corresponding probability distribution $\{\delta_1, \delta_2, \delta_3\}$. The power levels are selected such that $P_1 \geq k_1\beta P_2, P_2 \geq k_2\beta P_3$ where k_1, k_2 are positive integers. Let q_i and p_i denote the probability that an edge connected to a user node and a* *slot node remain unknown respectively at iteration i, then as $M \to \infty$, we have that*

$$
\begin{aligned}
p_i = 1 - \rho_1 - \rho_2 & \left((1-q_i) + \sum_{j=1}^{3} \delta_j \left(1-\delta_j\right) q_i \right) \\
& - \sum_{l=3}^{N} \rho_l \left((1-q_i)^{l-1} + \sum_{t=1}^{l-1} \left(\delta_1 (1-\delta_1)^t + \delta_2 \left(\delta_3^t + \binom{t}{1} \delta_1 \delta_3^{t-1} \right) \right) \binom{l-1}{t} q_i^t (1-q_i)^{l-t-1} \right) \\
& - \sum_{l=3}^{N} \rho_l \left(\delta_3 \left(1-\delta_3\right) \binom{l-1}{1} \left(1-q_i\right)^{l-2} q_i + 2\delta_1\delta_2\delta_3 \binom{l-1}{2} \left(1-q_i\right)^{l-3} q_i^2 \right)
\end{aligned}
\tag{4.18}
$$

The proof of Lemma 4 follows on the same lines as Lemma 3. A brief sketch is provided as follows. Nodes with degrees 1 and 2 are considered separately. If the degree is 1, it means the slot has only message, therefore the message gets decoded. If the degree is 2, $\delta_1(1-\delta_1)q$ corresponds to the probability of P_1 message being decoded, $\delta_2(1-\delta_2)q$ corresponds to the probability of P_2 being decoded and $\delta_3(1-\delta_3)q$ corresponds to the probability of P_3 being decoded. In the higher degree terms in the first summation, $\delta_1(1-\delta_1)^t$ corresponds to the probability of P_1 being decoded, and $\delta_2 \left(\delta_3^t + \binom{t}{1}\delta_1\delta_3^{t-1}\right)$ corresponds to the probability of P_2 being decoded. The last summation corresponds to the probability of P_3 being decoded in the higher degree nodes.

Note that the expression for p_i in Lemma 4 is rather cumbersome and does not simplify into a closed form expression as in the case of Lemma 3. As the value of n increases, calculating a closed form expression for p_i become more and more complicated. Hence in the next section, we discuss a recursive algorithm for calculating the coefficients $w_{l,t}$ required in the expression for p_i.

4.3 General n Power Level with IRSA

In this section we will set up density evolution equations for a general n-level power control model for IRSA. Let the n power levels be $P_1, P_2, \ldots, P_n$(arranged in descending order) with probabilities of being chosen $\delta_1, \delta_2, \ldots, \delta_n$ respectively. As in the previous sections, we assume that there is a sufficient multiplicative gap between successive power levels: $P_i > k\beta P_{i-1}$. We need to basically compute $w_{l,t}$ in equation (**??**). We will compute it as follows:

$$
w_{l,t} = \sum_{i=1}^{n} \delta_i \sum_{j=0}^{t} \binom{t}{j} j!\, Pr_{i,j}^{\text{higher}} Pr_{i,t-j}^{\text{lower}}
\tag{4.19}
$$

Recall that $w_{l,t}$ is the probability that a a packet connected to a degree l slot with t unresolved packets gets resolved at this particular time. This packet could be of any power, which is represented by $\sum \delta_i$ in the above formula. Then it could have anything from 0 to t higher-power interferers(represented by j in the above formula) and $t-j$ lower power interferers respectively. The term $Pr_{i,j}^{\text{higher}}$ represents the probability that a node with Power i has j higher power interferers, such that it still gets decoded at this particular step. Similarly, $Pr_{i,t-j}^{\text{lower}}$ represents the probability that a packet with power P_i has $t-j$ lower power interferers such that the chosen packet still gets decoded. The $\binom{t}{j}$ term represents the choice of j locations for higher power interferers, and $j!$ represents the permutations in locations of the higher power interferers possible. We will now show how to compute $Pr_{i,t-j}^{\text{lower}}$ and $Pr_{i,j}^{\text{higher}}$.

Computing $Pr^{\textbf{lower}}_{i,t-j}$ Computation of $Pr^{\text{lower}}_{i,t-j}$ is simple because of our multiplicative gap assumption: $P_i > k\beta P_{i-1}$. With this assumption, the chosen packet can get "captured" for any number of low power interferers $t-j$. This term is therefore simply calculated as:

$$Pr^{\text{lower}}_{i,t-j} = (\delta_{i+1} + \ldots + \delta_n)^{t-j} \tag{4.20}$$

Computing $Pr^{\textbf{higher}}_{i,j}$ Computing this quantity is a bit more involved. Before actually giving the method of computation, let's first look at how the chosen packet gets decoded in the presence of j higher power interferers. If a packet with power P_i has j higher power interferers, then all these j higher power interferers need to be first captured before the capture of the chosen packet is possible. All the j higher power interferers can be captured in a single iteration step if and only if they all have distinct power levels. This can be seen easily by a contradiction example: if there is there are two interferers with the same power level, then neither of them have $SIR \geq \beta$(since $\beta > 1$). Therefore, neither of them can be captured. We use this fact in computing $Pr^{\text{higher}}_{i,j}$.

Firstly, it is easy to see that if $i \leq j$, it is not possible for the packet to have j unique higher power interferers. Thus, $Pr^{\text{higher}}_{i,j} = 0, \qquad \text{if } i \leq j$. We will do the computation using an iterative algorithm. It is easy to compute the following two quantities, which denote probabilities of 0 and 1 higher power interferers respectively:

$$\begin{aligned} Pr^{\text{higher}}_{i,0} &= 1 \\ Pr^{\text{higher}}_{i,1} &= \delta_1 + \ldots \delta_{i-1} \end{aligned} \tag{4.21}$$

Now, $Pr^{\text{higher}}_{i,j}$ for a general j can be computed using the quantities $Pr^{\text{higher}}_{i-1,j-1}$ and $Pr^{\text{higher}}_{i-1,j}$. Notice that higher power interferers for a packet with power P_i are all the higher power interferers for a packet with power P_{i-1} and the packet with power P_{i-1} itself. Therefore, $Pr^{\text{higher}}_{i,j}$ can be computed as:

$$Pr^{\text{higher}}_{i,j} = Pr^{\text{higher}}_{i-1,j} + \delta_{i-1} Pr^{\text{higher}}_{i-1,j-1} \tag{4.22}$$

With the initial conditions above and this iterative process, we can compute $Pr^{\text{higher}}_{i,j}$ for all i, j. We have thus laid down a procedure to compute $w_{l,t}$ for a general n-level power control model for CSA.

Remark 3 (Upper Bounds)**.** *We can derive upper bounds for the general n-level power-control scheme as well using the same techniques as used in Theorems 3 and 4. Since we haven't derived explicit closed-form expressions here, we will evaluate the bounds numerically.*

5 Approximating Path-Loss Model

We consider the path loss model as described in the System Model section. To the best of your knowledge, [17] is the only paper in which the authors attempt to study the IRSA scheme with the path loss model assumption but are unable to get any closed form or explicit expression for the density evolution equations required for asymptotic analysis. Instead the authors of [17] use Monte-Carlo simulations to infer the coefficients in the density evolution equations. This makes the job of optimizing the degree-distribution $\{\Lambda_l\}$ for the users a

Result: Returns the $w_{l,t}$
$w_{l,t} = 0$;
for i = *1 to n* **do**
  temp = 0 ;
  for j = *1 to t* **do**
    $Pr^{\text{lower}}_{i,t-j} = (\delta_{i+1} + \ldots + \delta_n)^{t-j}$;
    if *j=0* **then**
      $Pr^{\text{higher}}_{i,0} = 1$;
    else if *j=1* **then**
      $Pr^{\text{higher}}_{i,1} = \delta_1 + \ldots + \delta_{i-1}$;
    else
      $Pr^{\text{higher}}_{i,j} = Pr^{\text{higher}}_{i-1,j} + \delta_{i-1} Pr^{\text{higher}}_{i-1,j-1}$;
    end
    temp = temp + $\binom{t}{j} j!\, Pr^{\text{higher}}_{i,j} Pr^{\text{lower}}_{i,j}$;
  end
  $w_{l,t} = w_{l,t} + \delta_i * \text{temp}$
end

Algorithm 1: Algorithm to compute $w_{l,t}$ for n-level power control

difficult task due to the lack of an explicit closed form constraints in the optimization problem as stated in (4.12). We provide a framework to "approximate" the path loss model to n-power level IRSA model to make it amenable for theoretical analysis. Before going into the details of this approximation framework, it is instructive to understand in an intuitive manner as to why it is difficult to analyse the IRSA scheme under the path loss model. The heart of the analysis of the IRSA scheme is the capture effect which essentially says that packet of user i in slot m can be decoded as long as $SIR^{(i,m)} = \frac{P^{(i)}_{rec}}{\sum_{n \in \mathcal{R}^0_j \setminus i} P^{(n)}_{rec}} \geq \beta$. Assuming that the radial distance of all the users in more than $d_{\min}$, the expression for $SIR^{(i,m)}$ under the path loss model assumption is $SIR^{(i,m)} = \frac{P\left(\frac{r_i}{d_{\min}}\right)^{-\alpha}}{\sum_{n \in \mathcal{R}^0_j \setminus i} P\left(\frac{r_n}{d_{\min}}\right)^{-\alpha}}$ which can be simplified to $SIR^{(i,m)} = \frac{r_i^{-\alpha}}{\sum_{n \in \mathcal{R}^0_j \setminus i} r_n^{-\alpha}}$ where r_j is the radial distance of user j from the basestation. Assuming the users are distributed randomly around the basestation, r_ns can be considered as random variables, hence the denominator can be considered as a random sum of random variables $\sum_{n \in \mathcal{R}^0_j \setminus i} r_n^{-\alpha}$. Unlike [16], which showed that SIR follows the Pareto distribution under the Rayleigh fading channel model, we were unable to characterize SIR in the path loss model with any previously known distribution whose properties we could leverage for theoretical analysis of the capture effect. Hence, the need to approximate the path loss model with a n-power level IRSA-PC model. To do this, we show that the path loss model and n-power level IRSA model are "approximately"" equivalent. In addition to approximating the path loss model, this framework also acts as a heuristic to optimise the $\{\Lambda_l\}$ distribution for the path loss model. In the IRSA-PC model, we externally introduced the n-power levels to create diversity in the power of the received packets (under the ideal channel assumption). In the path loss model, this job is automatically done by "nature". Let us assume that $P_{\min}$ is the minimum power of a packet which can be received and decoded by the basestation. Having this minimum power assumption implies that there is a maximum distance constraint for the users, beyond which the packet of the user cannot be decoded by the basestation. Let $d_{\max}$ be this distance. Then we have that $d_{\max} = \left(\frac{P}{P_{\min}}\right)^{1/\alpha} d_{\min}$, where P is the common power with which all the users transmit their packets as mentioned in the System model. Let the power of the received packet corresponding to user i be P^i_{rec}, then it follows from our assumption that $P^i_{\text{rec}} \in [P_{\min}, P], \forall i \in \{1, 2, \ldots, N\}$.

5.1 Discretization For Approximation

To approximately show the equivalence of the n-power level IRSA-PC model and the path loss model, the first step is to discretize the set of possible received powers due to path loss which is $[P_{\min}, P]$. We can discretize this continous set into n power levels where $n = \left\lfloor \frac{\log P/P_{\min}}{\log k\beta} \right\rfloor + 1$ such that $P_i = k\beta P_{i+1}, P_1 = P, P_n = P_{\min}$. This discretized set of powers is the same set $\mathcal{P}$ (by construction) which is considered in the IRSA-PC model. In the IRSA-PC model apart from the power level, we also considered the distribution $\{\delta_k\}$ from which these power levels are sampled independently by the users. In the path loss model since all the users transmit the packets with a common power P, the variation in the power of the received packets arises due to their respective distances from the base station. The spatial distribution of the users around the basestation determines the $\{\delta_k\}$ distribution. Note that in the path loss model, there is one-to-one map between P^i_{rec} and r_i where r_i is the radial distance of the user from the basestation. Therefore just as we can discretize $[P_{\min}, P]$ into n-levels, discretization of $[0, d_{\max}]$ is also straightforward. For each $P_i \in \mathcal{P}, d_i = \left(\frac{P}{P_i}\right)^{1/\alpha} d_{\min}$. Define $d_0 := -d_1, d_{n+1} := d_n = d_{\max}$. Let $N_i = |\{j \in U : \frac{d_i + d_{i-1}}{2} \leq r_j \leq \frac{d_i + d_{i+1}}{2}\}|$ denote the number of users whose radial distance from the basestation lies between $\frac{d_i + d_{i-1}}{2}$ and $\frac{d_i + d_{i+1}}{2}$. This gives us the $\{\delta_k\}$ distribution as $\delta_i = \frac{N_i}{N}$ where N_i is number which depends on how the users are spatially distributied around the base-station.

5.2 Case Study

The best way to understand this framework for approximating the path loss model is with the help of an example. For this example let us assume that $P_{\min} = 0.01P$. Also make the additional assumption that $\beta = 2$ and the assumption of $k = 5$ follows from Remark 1. Then it follows from the argument above that $d_{\max} \approx 4.64 d_{\min}$ and we can discretize this path loss model in $n = \left\lfloor \frac{\log 100}{\log 10} \right\rfloor + 1 = 3$ levels where $P_1 = P, P_2 = 0.1P, P_3 = 0.01P = P_{\min}$.

Let L be the random variable denoting the position of a user relative to the base station. In terms of the polar coordinates this random variable L can be characterized by two random variables R_L and Θ_L which denote the radial distance of the user from the base station and the angle from a referenced axes respectively.

Hence we can define $L := (R_L, \Theta_L)$. In our example let $R_L \sim \text{Unif}[0, d_{\max}], \Theta_L \sim \text{Unif}[0, 2\pi]$ and assume that R is independent of Θ_L. Since R_L is independent of Θ_L, it implies that the joint PDF of the position random variable L can be written as $f_L(r, \theta) = \frac{1}{2\pi d_{\max}}$. The reason for choosing such a simple assumption on the spacial distribution of users is that the calculation of $\{\delta_k\}$ is greatly simplified. Without loss of generality, we can assume $d_{\min} = 1$, then we have that $d_1 = d_{\min} = 1, d_2 = \left(\frac{P}{0.1P}\right)^{\frac{1}{3}} \times d_{\min} = 2.15, d_3 = d_{\max} = 4.64$. Note that while calculating $\{\delta_k\}$, for this example, since the radial distance of the user is independent of the angle the user makes from a referenced axis. Also since the radial distance of the user is uniformly distributed between 0 and $d_{\max}$, it follows that $\delta_i = \frac{\frac{d_i + d_{i+1}}{2} - \frac{d_i + d_{i-1}}{2}}{d_{\max}} = \frac{d_{i+1} - d_{i-1}}{2d_{\max}}$. Since $d_0 = -d_1, d_4 = d_3$, we have that $\delta_1 = \frac{d_1 + d_2}{2d_{\max}} \approx 0.34, \delta_2 = \frac{d_3 - d_1}{2d_{\max}} \approx 0.39, \delta_3 = \frac{d_3 - d_2}{2d_{\max}} \approx 0.27$.

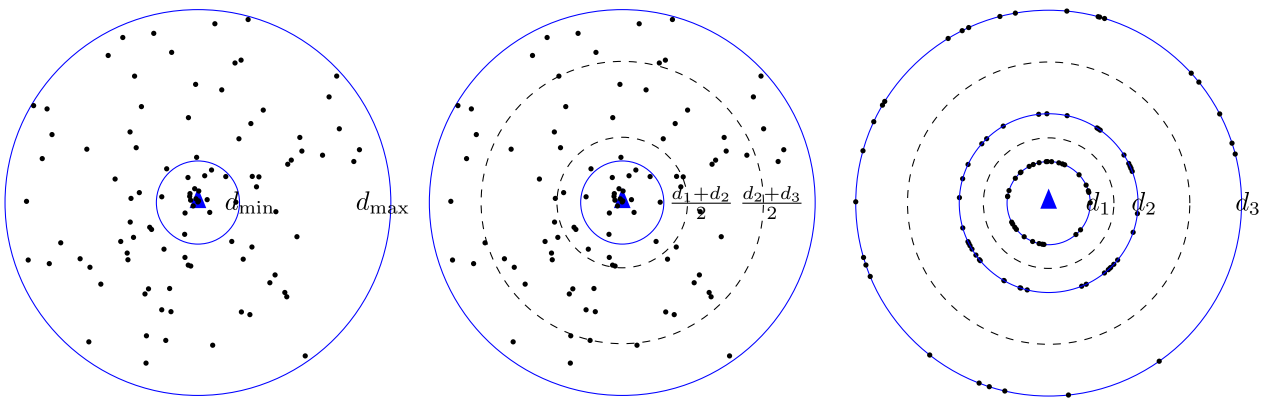


Figure 17: Sub-figure (a) describes a random distribution of the users around the basestation where the radius and angle with respect to a referenced axis are sample uniformly and independently from Unif$[0, d_{\max}]$ and Unif$[0, 2\pi]$ respectively. In sub-figure (b), the circular region around the base-station is divided into zones. Users whose radial distance from the base-station is between 0 and $\frac{d_1+d_2}{2} \approx 1.57$ belong to zone 1. Users whose radial distance is between $\frac{d_1+d_2}{2}$ and $\frac{d_2+d_3}{2}$ belong to zone 2 and similarly, for the users whose radial distance is between $\frac{d_2+d_3}{2}$ and d_3 belong to zone 3. In approximating the path-loss model with the n-power level IRSA model, we approximate that for all the users in zone 1, the received power at the base-station is P which is equivalent to approximating the radial distance of all the users in zone 1 with $r_i = d_1$ for all users i in zone 1. Similarly we can approximate the radial distance of all the users in zone 2 with d_2 and hence their received power of all the users in zone 2 can be approximated to be $0.1P$ and the same hold approximation is made for zone 3 users where their received power is approximated to be $0.01P$. This approximation is shown in sub-figure (c).

5.3 Closeness of approximation

While we don't give any theoretical guarantee as to how close the approximation of the path loss model is to the actual path loss model , we show via simulation results that the throughput achieved in both the cases are numerically quite close. For the case-study example considered above, we approximate the path loss model by discretizing and approximating the path loss model by a 3-power level IRSA model.

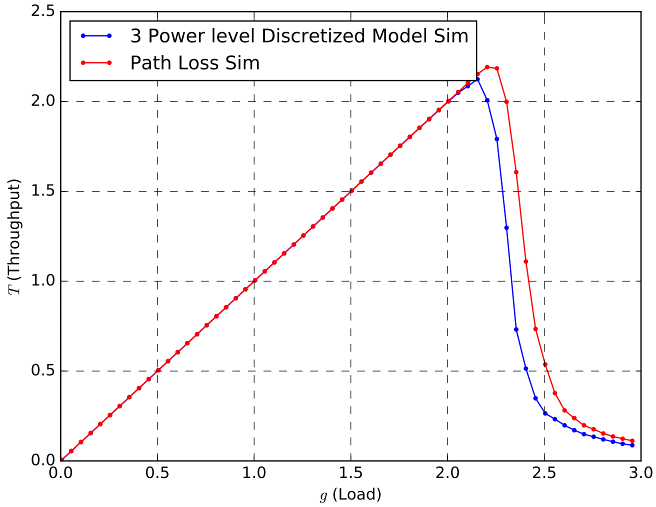


Figure 18: For this simulation, we fix the number of slots $M = 1000$. Each users independently decided how many times to repeat her message according to the distribution $\Lambda^{\text{liva}}(x) = 0.5x^2 + 0.28x^3 + 0.22x^8$. The radial distance of all the users is sampled independently from a uniform distribution over $[0, d_{\max}]$. Other parameters for the simulation (which are mentioned in the system model) are $\beta = 2, \alpha = 3, k = 5$.

As is evident from the simulation results above, the approximate path loss model lower bounds the performance of the actual path loss model. While we don't have a rigourous proof for this observation, the intuition for this claim is as follows. There is a lot of diversity in the received powers at the base station in the path loss model due to the $\left(\frac{d}{d_{\min}}\right)^{-\alpha}$ factor in the path loss model and this diversity helps in performing the decoding due to the capture effect. But this diversity in the received power is not mathematically tractable which makes calculation of density evolution equations really hard. By approximating the received powers by the discretized powers in the n-power level IRSA, we give up on the diversity for the sake of tractibility. This reduced diversity in the received powers, under-estimates the decoding due to the capture effect and hence gives a lower bound on the achieveable throughput in the path loss case.

5.4 Optimizing the Λ-distribution for path loss model

Lack of a closed form expression for the density evolution equation in the case of the path loss model, makes the optimization of the Λ-distribution difficult. In order to circumvent this problem, we make use of the approximate path loss model, for which we have a closed form expression for the density evolution equation due to the equivalence between the approximate path loss model and the n-power level IRSA model.

One important conjecture we make for the optimization of the Λ-distribution is $\text{OPT}_{\text{path loss}} \geq \text{OPT}_{\text{n-level}}$. So whichever distribution achieves the optimal throughput for the n-level power control, the same distribution should be the optimal distribution or be very close to the optimal distribution for the path loss model.

We don't have a theoretical proof for the above mentioned conjecture but provide an empirical result to demonstrate the conjecture.

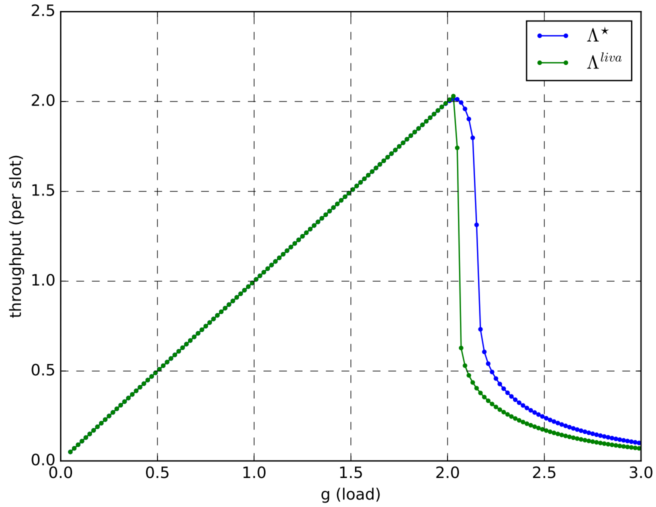


Figure 19: Plot comparing the asymptotic throughput (using density evolution equations) performance of the optimal distribution $\Lambda^{\star}$ and the $\Lambda^{\text{liva}}(x) = 0.5x^2 + 0.28x^3 + 0.22x^8$ under the 3-power level IRSA-PC scheme

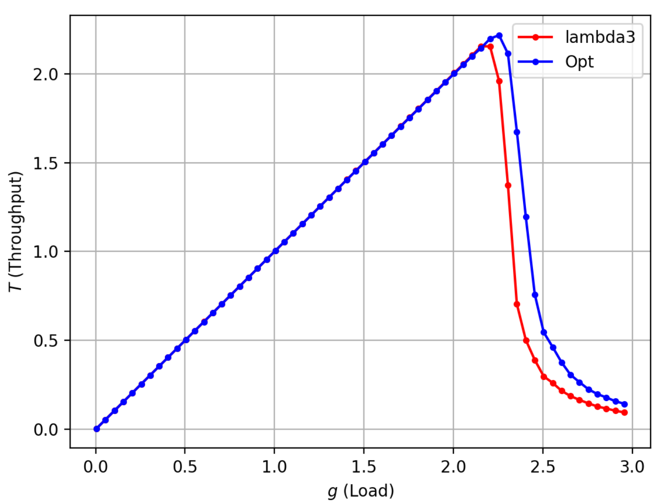


Figure 20: Plot comparing the simulated throughput performance of the optimal distribution $\Lambda^{\star}$ and the $\Lambda^{\text{liva}}(x) = 0.5x^2 + 0.28x^3 + 0.22x^8$ under the path loss model.

We optimize the $\{\Lambda\}$ distribution for $n = 3$ with $\{\delta\} = [0.34, 0.39, 0.27]$ and compare the performance of the optimal distribution in the n-level power control case with the $\Lambda^{\text{liva}}(x)$ distribution (Figure 19). We notice that the same optimal distribution for the n-level power control performs better than the Λ^{liva} distribution for the path loss model as well (Figure 20). The key take-away is the conjecture that the optimal distribution $\{\Lambda^{\star}\}$ for the n-power level IRSA scheme (which can obtained by solving a well-posed optimization problem) is also the optimal distribution for the corresponding path-loss model (which is difficult to obtain due to the lack of a closed form expression for the constraints in the optimization problem).

6 Simulation Results

In this section we provide some empirical and simulation results. For the simulations we assume that we can perfectly perform the successive interference cancellation at the MAC layer. For decoding in the physical layer we assume that a signal can be decoded as long as the SIR of the signal is greater than some factor β, which for the purposes of the simulation we assume to be $\beta = 2$. Along with the simulations we also plot the density evolution curve (suffixed by DE in the legends) to show the performance of the schemes in the asymptotic setting ($M \to \infty$). For all the non-asymptotic cases where we are looking at empirical average throughput performances, unless otherwise stated we assume the number of slots to be $M = 1000$.

6.1 Comparison of the vanilla IRSA with 2-power level IRSA scheme

In the vanilla IRSA, all the users transmit their packets with the common power $\mathcal{P} = \{P\}$. Let us say the average power use per user under the vanilla IRSA is $\hat{P}$. In the 2-level power scheme the set of powers are $\mathcal{P} = \{P, k\beta P\} = \{P, 10P\}$ which is distributed as a Bernoulli random variable where the probability of choosing the power P is 0.6. Hence the average power use per user under the 2-power level IRSA scheme is $4.6\hat{P}$. It is evident from the plots that asymptotic throughput nearly doubles (increases from 0.92 in the IRSA case to 1.67 in the IRSA-PC case) but it comes at a cost of increased (nearly 5 times) average power use per user. The important thing to note here is that vanilla IRSA scheme exploited the fact that throughput can be increased by adding redundancy across time by repeating packets across slots in a given frame. Now by introducing another dimension - diversity in power caused due to different power level of the transmitted packets, we can leverage the capture effect and decode more number of packets in a slot thereby achieving higher throughput. In conclusion, in the 2-level power IRSA scheme, we leverage both the redundancy in time and the diveristy in power to achieve the gains in throughput which are evident from the plot below.

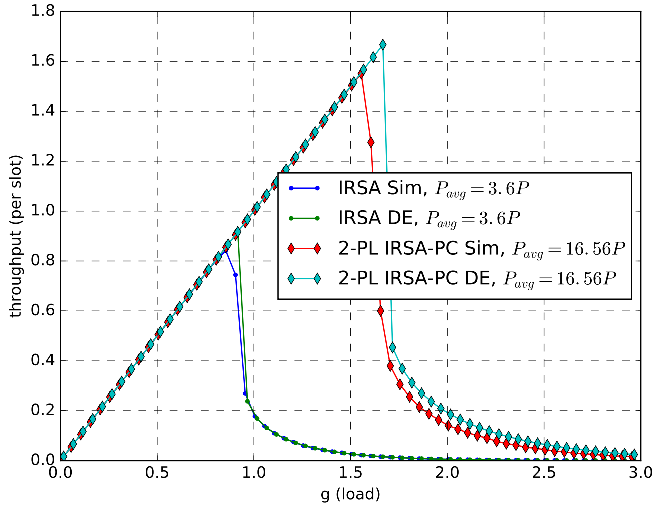


Figure 21: Comparison of vanilla IRSA scheme with 2-power level IRSA scheme. In both the scheme, the $\Lambda(x) = 0.5x^2 + 0.28x^3 + 0.22x^8$ distribution is the same.

6.2 Comparison of SA-DPC with 2-power level IRSA scheme

In the previous section, we demonstrated the gains in throughput one can achieve by introducing diversity in the power of the transmitted packets and then achieving additional decoding possible due to the capture effect. In this section, we highlight the gains in throughput one can achieve by introducing redundancy across time. In this comparison, we compare the 2-power level IRSA scheme with a variant of the vanilla Slotted Aloha scheme which is the dual power control Slotted Aloha (SA-DPC). For the comparison, each users chooses the power from set $\mathcal{P} = \{P, 10P\}$ with the distribution $\{\delta\} = [0.6, 0.4]$. For the SA-DPC case, users repeat their packets only once in a given frame and hence $\Lambda_{\text{SA-DPC}}(x) = x$, while in the 2-power level IRSA-PC scheme, users repeat their packets according to the distribution $\Lambda(x) = 0.5x^2 + 0.28x^3 + 0.22x^8$. In comparison to the vanilla Slotted Aloha scheme where the optimal throughput is 0.37, the optimal throughput in the SA-DPC scheme increases to 0.658. Hence, by introducing diversity due to different power levels, one can roughly double the achievable throughput. Now in addition to increasing the diversity due to the different power levels, leveraging the redundancy across time due to repetition gives optimal asymptotic throughput to be roughly 1.66 which is a 2.5× increase in throughput.

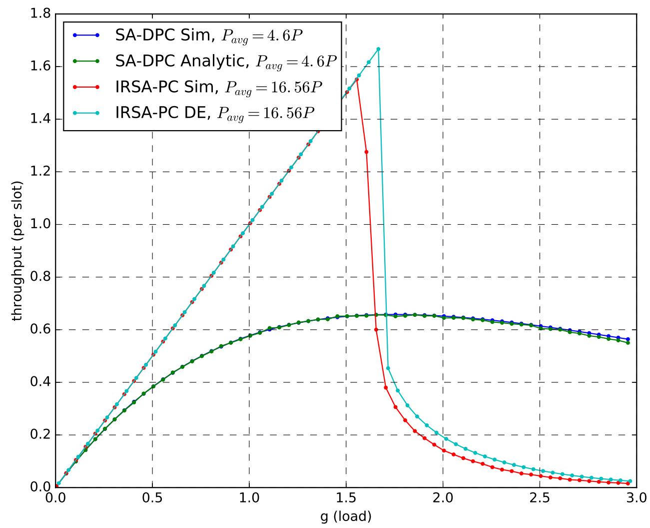


Figure 22: Comparison of SA-DPC with 2-power level IRSA scheme

6.3 Comparison of throughput with increasing levels of Power control IRSA

Intuitively it would seem that increasing the diversity in power by increasing the number of power levels should increase the optimal throughput. We consider n-power level IRSA-PC scheme for $n = 1, 2, 3$. Note that 1-power level IRSA-PC scheme is essentially the vanilla IRSA scheme. For the 2-power level IRSA-PC scheme, the set of powers are $\mathcal{P} = \{P, 10P\}$ which is distributed as $\{\delta\} = [0.6, 0.4]$. For the 3-power level IRSA-PC scheme, the set of powers are $\mathcal{P} = \{P, 10P, 100P\}$ which is distributed as $\{\delta\} = [0.34, 0.39, 0.27]$. Let the average power use per user in the vanilla IRSA be 1, then the average power use per user in the 2-power level IRSA-PC and 3-power level IRSA-PC scheme are $4.6, 31.24$ respectively. It is evident from the plot that increasing n increases the throughput but at the expense of increased power expenditure as well.

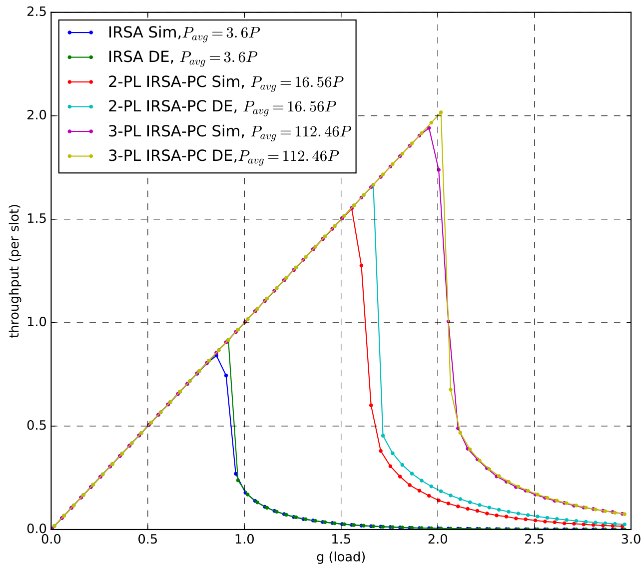


Figure 23: Comparison of n- level Power Control, $n = 1, 2, 3$

6.4 Comparison of different schemes in terms of packet loss probability

Throughput the paper, we considered the following schemes in detail - (a) vanilla Slotted Aloha (SA), (b) dual power control Slotted Aloha (SA-DPC), (c) IRSA , (d) 2 power level IRSA-PC and (e) 3 power level IRSA-PC scheme. The asymptotic performance of these five schemes is presented in terms of packet loss probability which corresponds to the probability of packets not being decoded in a particular frame. Let us denote that packet loss probability corresponding to load g by $\Pr_l(g)$. From [12], it follows that relationship between throughput T, packet loss probability $\Pr_l(g)$ and load g is given as $T = g(1 - \Pr_l(g))$ and $\Pr_l(g) = \Lambda(p_\infty)$. For SA and SA-DPC, it follows that $\Lambda(x) = x$ while for IRSA, 2 power level IRSA-PC and 3 power level IRSA-PC schemes, we use the Liva distribution $\Lambda^{\text{liva}}(x) = 0.5x^2 + 0.28x^3 + 0.22x^8$. For SA-DPC and 2 power level IRSA-PC scheme, the set of powers are given as $\mathcal{P} = \{P, 10P\}$ which are distributed as $\{\delta\} = [0.6, 0.4]$. For 3 power level IRSA-PC scheme, the set of powers are given as $\mathcal{P} = \{P, 10P, 100P\}$ which is distributed as $\{\delta\} = [0.34, 0.39, 0.27]$. From the plot, we can see that for the cases of IRSA, IRSA-PC with 2 power level and 3 power levels, the packet loss probability is very low (practically zero) right up the threshold throughput $g^\star$ and increases to 1 after the load increases beyond the threshold throughput. This is equivalent to the Figure 23 where for all the 3 schemes mentioned above, the throughput increases linearly with the load till a threshold throughput and then starts decreasing as the load increases which equivalent to increasing packet loss probability. Another thing to note here that as the number of power levels increase for the IRSA-PC scheme,the packet loss probability decreases. For the case of vanilla Slotted Aloha and Dual Power Control Slotted Aloha scheme, we can see that the packet loss probability curve in the dual power control SA scheme is lower than the packet loss probability curve in the vanilla SA scheme as we are leverging the capture effect to decode more packets in the dual power control SA scheme resulting in lower packet loss probability.

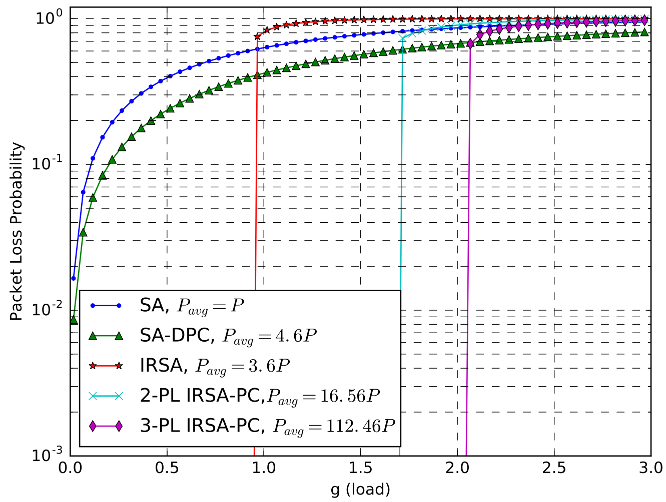


Figure 24: Asymptotic performance of SA, SA-DPC, IRSA, 2 power level IRSA-PC and 3 power level IRSA-PC schemes with respect to packet loss probability vs the load g

6.5 Comparison of throughput with varying number of slots M

In all our analysis, we talked about the asymptotic throughput which corresponds to setting $M \to \infty$. The density evolution equations give us the asymptotic throughput results which we are able to compute either as a closed form expression or through the Algorithm 1. While simulating though, we cannot set M to be arbitrarily large and hence vary the value of M from 100 to 5000. It follows from the plot that as M increases, the throughput curve from the simulations approaches the asymptotic throughput curve which we compute using the density evolution equations.

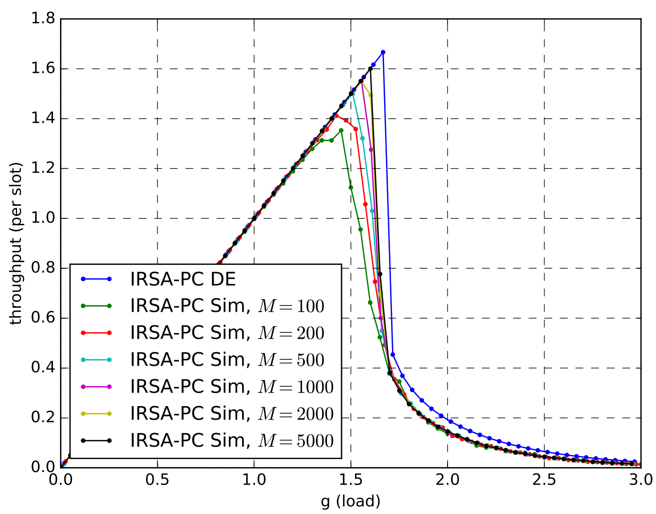


Figure 25: Comparison of throughput with varying number of slots M

6.6 Comparison of vanilla CSA scheme with 2 power level CSA scheme

Since theoretically analysing power control in the Coded Slotted Aloha scheme is intractable, we present some empirical results using simulations comparing the the throughput obtained using the CSA scheme and the 2-power level CSA scheme for different rates. We define the rate $\hat{R} := \frac{k}{\sum_{h=1}^{\theta} \Lambda_h n_h}$ where θ is the total number of codes considered, Λ_h is the probability of choosing component code $\mathscr{C}_h$ whose length is n_h. Each component code has a common dimension k. We consider rates $\hat{R} = \frac{1}{3}, \frac{2}{5}, \frac{1}{2}, \frac{3}{5}$. For the component codes and the respective probability distribution Λ for $k = 2$, refer to Table II of [13]. In the vanilla CSA, all the users transmit their packets with the common power $\mathcal{P} = \{P\}$. Let us say the average power use per user under the vanilla IRSA is $\hat{P}$. In the 2-level power scheme the set of powers are $\mathcal{P} = \{P, k\beta P\} = \{P, 10P\}$ which is distributed as a Bernoulli random variable where the probability of choosing the power P is 0.6. Hence the average power use per user under the 2-power level CSA scheme is $4.6\hat{P}$. One of the things evident from the simulations is that lower the value of rate $\hat{R}$, higher the throughput achievable in both cases - vanilla CSA scheme and power control CSA scheme. Another thing worth noting is the fact that introducing power control more than doubles the capacity. The increase in capacity in the case of CSA and CSA-PC for different rates are tabulated below.

$\hat{R}$	CSA Capacity	CSA-PC Capacity
$\frac{1}{3}$	0.654	1.602
$\frac{2}{5}$	0.645	1.583
$\frac{1}{2}$	0.521	1.271
$\frac{3}{5}$	0.426	1.026

Note that in the IRSA, on introducing 2 power levels we had a gain of $\sim 1.5\times$, whereas in CSA, introducing 2 power levels, gives a gain of more than $2\times$. Lack of a closed form expression for the density evolution equations for the CSA-PC scheme makes optimizing the code choice and their distribution a very difficult problem.

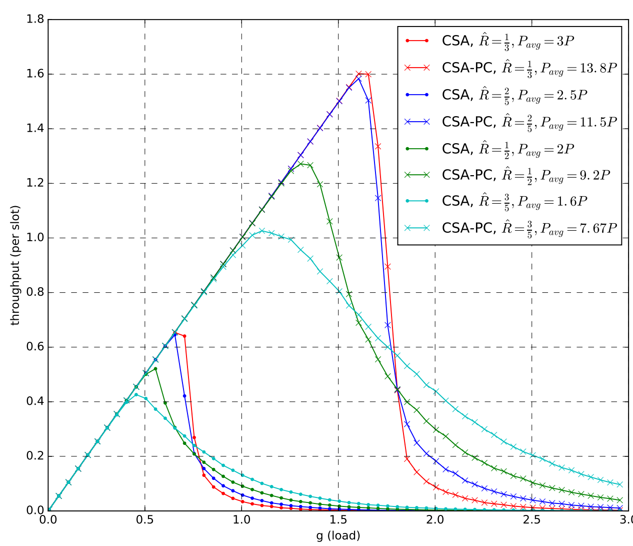


Figure 26: Comparison of vanilla CSA scheme with 2-power level CSA scheme for rates $\hat{R} = \frac{1}{3}, \frac{2}{5}, \frac{1}{2}, \frac{3}{5}$.

7 Conclusion

In this work, we built upon previous works such as [16, 17] to analyze Coded Slotted Aloha(first introduced in [12]) over realistic channel models. To this end, we first introduced a discrete power control model where the user is allowed to choose between a set of discrete power levels. We formulated the density evolution equations for this model. And, we also set-up an optimization problem which enables us to optimize over the power-control scheme and the packet repetition scheme to obtain the maximum throughput. We showed through simulation that an increase in maximum throughput from 0.81 to 1.35 is possible with the use of just two power levels. We then used this model to approximate the realistic path-loss channel model, and optimize the packet repetition scheme for it. We show through simulation that this is a reasonable approximation, and that the approximation helps in finding better packet-repetition distributions $\Lambda()$ for the path-loss model. For future work, we could attempt to approximate other realistic channel models with the discrete power-control model and analyze the results. Another plausible future work is to solve the general optimization problem formulated in (2.2).

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