LDPC Codes for Compressive Sensing

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Outline

Agenda

- Review Compressed Sensing along with the linear programming relaxation, **CS-LPD**

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- Review Channel Coding along with the linear programming relaxation, **CC-LPD**
- Interpretation of the bridge connection between **CC-LPD** and **CS-LPD**
- Illustrate the results with example of parity check matrix of (7,4) Hamming Code

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Motivation

- Consider the classical linear algebra problem of solving $y = Ax$ for some *measurement matrix* $A \in \mathbb{R}^{mxn}$, $x \in \mathbb{R}^n$ and $y \in \mathbb{R}^m$. So given y, can we recover x?

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- In case $m < n$ and A is full rank, the system is under-determined system and can have no solutions or infinite solutions.
- The under-determined system seems like an impossible problem to solve. Can we do better?

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Example

Consider the sinusoidal function, say $x(t) = sin(\omega t)$, in time domain the signal takes continuous values but in the frequency domain, it can be represented as a “spiky” signal by taking the Fourier transform

Definition

A signal x is said to be s-sparse if at most s coordinates of x are non-zero.

Does Sparsity Help?

The central idea to the field of compressive sensing theory is the following:

Theorem

If every $2s$ distinct columns in A are linearly independent (assuming $m \geq 2s$), then any s-sparse signal can be reconstructed uniquely from Ax.

Proof.

Suppose, there exist two s-sparse signals $x, x^{'} \in \mathbb{R}^n$ such that $Ax = Ax^{'}$,then $A(x - x^{'}) = 0$. Now $x - x^{'}$ is $2s$-sparse,so there is a linear dependence between $2s$ columns of A, hence a contradiction. $\square$

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$$\begin{aligned} &\min_x \|x\|_0 \\ &\text{s.t. } A.x = y \end{aligned} \tag{1}$$

- ℓ_0 minimization is an NP-hard problem and is computationally intractable.

Basis Pursuit

- We can relax the ℓ_0-minimization problem by ℓ_1-minimization problem.
- This is a convex optimization problem and can be solved in polynomial time by linear programming methods.
- Intuition for relaxing it to an ℓ_1 minimization problem is given below :

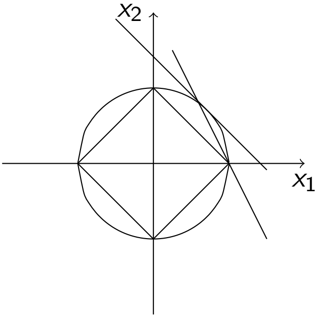

Figure: ℓ_1 minimization vs ℓ_2 minimization

Formalizing the Compressive Sensing Problem

- Let us consider a measurement matrix H_{CS} and a system of equations given by $y = H_{CS}x$. The compressive sensing problem boils down to the following problem : Given y, recover x assuming x is s-sparse.
- The optimal solution to the problem can be achieved by solving the following minimization problem, to be referred as **CS-OPT**.

$$\begin{aligned} &\min_x \|x\|_0 \\ &\text{s.t. } H_{CS}.x = y \end{aligned} \tag{CS-OPT}$$

- Since **CS-OPT** is an NP-hard problem,we relax it and refer to it as **CS-LPD**

$$\begin{aligned} &\min_x \|x\|_1 \\ &\text{s.t. } H_{CS}.x = y \end{aligned} \tag{CS-LPD}$$

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- To characterize the condition, we need the *nullspace characterization* of "good" measurement matrices.
- A measurement matrix H_{CS} is said to be s-good if ℓ_1-minimization recovers correctly all s-sparse vectors x
- A necessary and sufficient condition for H_{CS} to be s-good is the following *nullspace property* :

$$\|v_I\|_1 < \frac{1}{2}\|v\|_1$$

for all $v \in \mathbb{R}^n : Av = 0, v \neq 0, I \subset \{1, ..., n\}, |I| \leq s$

So when is **CS-OPT** and **CS-LPD** equivalent?

Theorem

Let H_{CS} be a measurement matrix. Also, assume that $y = H_{CS}x$ and that x has at most k non zero elements i.e. $\|x\|_0 \leq k$. Then the estimate $\hat{e}$ given by **CS-LPD** *will be equal to the estimate $\hat{e}$ given by* **CS-OPT** *if H_{CS} satisfies the nullspace property.*

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- In conclusion, the estimate of **CS-OPT** and **CS-LPD** are equal as long as the H_{CS} is s-good.

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- Areas of Application of Compressive Sensing include

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  - Communications
  - Image and Video Processing
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Outline

Block codes

H_{CC}: Parity check matrix of dimension r x m
A vector $c \in \{0,1\}^n$ *lies in the codespace if and only if*

$$H_{CC}.c = 0$$

The codespace($\mathcal{C}$)=$\{x| < h_i, x >= 0\ \forall i \in \{1, \ldots, m\}\}$

The MLD formulation

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- Seen as a cost of assigning a bit value of 1.
- Let f_i denote the LP variables over which we minimise cost. There is one variable corresponding to each bit.

Equivalent MLD formulations

- $$min_f \sum_{i=1}^{n} \gamma_i f_i \qquad \text{(CC-MLD)}$$
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Equivalent MLD formulations

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- Since the optimal values of a convex optimisation problem occurs at the extreme points, we can say that the 2 problems are equivalent.
- The convex hull has exponential description complexity.

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Relaxing CC-MLD

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- $S_j = \{x \mid H_{CC,j}.x = 0 (mod\ 2)\}$
- $$\begin{aligned} conv(C_{CC}) &= conv(S_1 \cap S_2 \cap \ldots S_m) \\ conv(C_{CC}) &\subseteq \ conv(S_1) \cap conv(S_2) \cap \ldots conv(S_m) \end{aligned} \tag{2}$$

CC-LPD

- $$\mathcal{P}(H_{CC}) = \cap_{i=1}^{m} conv(S_i)$$

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- $$min_f \sum_{i=1}^{n} \gamma_i f_i \quad \text{s.t. } f \in \mathcal{P}(H_{CC}) \tag{CC-LPD}$$
- A relaxation to the original problem.

Illustration of H_{CC}

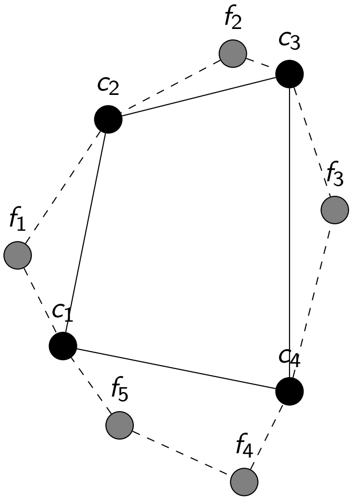

Figure: The darkened lines represent the convex hull of the codespace, and the dotted lines represent the polytope $\mathcal{P}(H_{CC})$. The gray nodes represent fractional

Outline

Fractional distance

- $$d_{frac} = min \sum_{i=1}^{N} |y_i - f_i| \qquad (3)$$
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- It can then be shown that CC-LPD can correct upto $[\frac{d_{frac}-1}{2}]$ errors.

Outline

The fundamental cone

- *Definition:* The fundamental cone $\mathcal{K}(H_{CC})$ is defined as the set of vectors $\nu = (\nu_1, \nu_2, \ldots, \nu_n) \in \mathbb{R}^n$such that $\forall$ $1 \leq i \leq n$ and $1 \leq j \leq m$, we have the following:

$$\nu_i \geq 0$$

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- $$H = \begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{bmatrix}$$

- $\nu \in \mathcal{K}$ if: $\nu_1 = \nu_2 = \nu_3,\ \nu_5 = \nu_6 = \nu_7$
  $2\nu_1 \geq \nu_4,\ 2\nu_5 \geq \nu_4$

Condition for correct decoding

- *Lemma:* Let $\mathcal{S}$ denote the set of indices flipped over during transmission over a BSC channel. If the parity check matrix H_{CC} is such that for all vectors(ω) in the fundamental cone of the parity matrix, the following holds:

$$||\omega_{\mathcal{S}}||_1 < ||\omega_{\overline{\mathcal{S}}}||_1 \tag{4}$$

. If the above condition holds then CC-LPD will decode to the same point as CC-MLD.

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  . If the above condition holds then CC-LPD will decode to the same point as CC-MLD.
- This is the same condition as that from the Nullspace property of the CS-LPD
- We dig a little deeper to form the connection between CC-LPD and CS-LPD

Forming the bridge

- Let H be a zero one measurement matrix. Then

$$\nu \in \textit{Nullsp}(H) \implies |\nu| \in \mathcal{K}(H)$$

Let $\omega = |\nu|$. It can be shown that ω is in the fundamental cone of H. Thus, corresponding to every ν, we have an $\omega = |\nu|$ such that ν lies in the fundamental cone of H.

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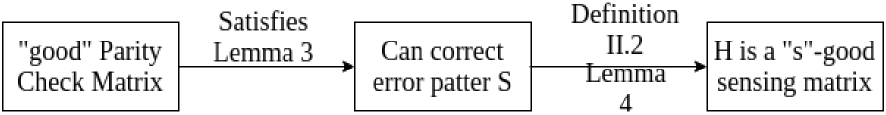
- Note that the result is one way, i.e for every point in the Nullspace of H we can find a point in the conic hull of $\mathcal{K}$
- If H is a good LDPC code then any vector in the nullspace of H will satisfy the null space property provided it satisfies property(4)
- Since H is a good LDPC code, it will satisfy property (4) for a significant value of $|\mathcal{S}|$ and hence the H will act as a $|\mathcal{S}|$ good sensing matrix.

Intuition using block diagram



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- Hence, there can be no point in the nullspace of H which violates the nullspace property for s sparse solutions.
- Hence, we can say that such a matrix is s-good.
- The notion of a good matrix depends on the channel being used.For a BSC channel, the performance results translate directly,i.e an s error correcting code can act as s-good matrix. For other channels, the performance translations are slightly more complicated.

Outline

Constructing the parity check matrix

- Theoretical results state that for a binary symmetric channel (BSC), if a parity-check matrix can correct any k bit-flipping errors under **CC-LPD**,then the same matrix taken as a measurement matrix over reals can be used to recover all k-sparse error signals under **CS-LPD**.
- Consider the parity check matrix of $(7,4)$ Hamming code:

$$H = \begin{bmatrix} 1 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 \end{bmatrix}$$

- The parity check matrix H can correct upto 1-bit flipping error under **CC-LPD**

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- For H, any $2(2s)$ columns are linearly independent and hence can be used to recover all $1(s)$-sparse vectors
- Use the l1-Magic Solver in MATLAB to compute the ℓ_1-minimization estimate to **CS-LPD**
- Consider the following 1-sparse vector x

$$x = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0.9244 \\ 0 \\ 0 \end{bmatrix}$$

Recovery of s-sparse signals

- Consider $x \mapsto Ax$, let $y = Hx$

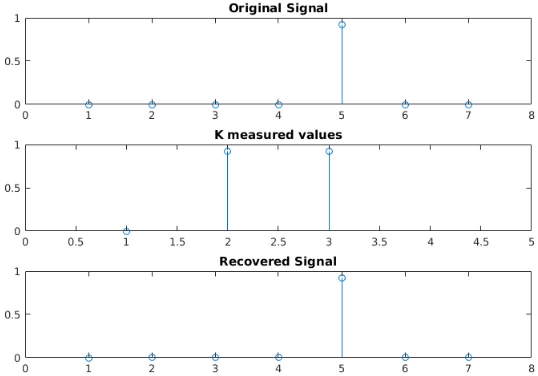
$$y = \begin{bmatrix} 0 \\ 0.9244 \\ 0.9244 \end{bmatrix}$$

- Pass y and H into the ℓ_1 minimization solver and let the estimate be $\hat{x}$

$$\hat{x} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0.9244 \\ 0 \\ 0 \end{bmatrix}$$

Verification of theoretical results

- We can see that $x = \hat{x}$.
- We are able to perfectly recover the 1-sparse signal using ℓ_1-minimization solver
- This verifies the theoretical results



Case of Approximate Recovery of 2-sparse vector

- $x = \begin{bmatrix} 0 \\ 0 \\ 0.6073 \\ 0 \\ 0 \\ 0 \\ 0.0708 \end{bmatrix}$: Original signal
- $y = \begin{bmatrix} 0.0708 \\ 0 \\ 0.6780 \end{bmatrix}$: The measurement vector
- $\hat{x} = \begin{bmatrix} 0 \\ 0 \\ 0.6071 \\ 0 \\ 0 \\ 0 \\ 0.0707 \end{bmatrix}$: Estimate of the original signal

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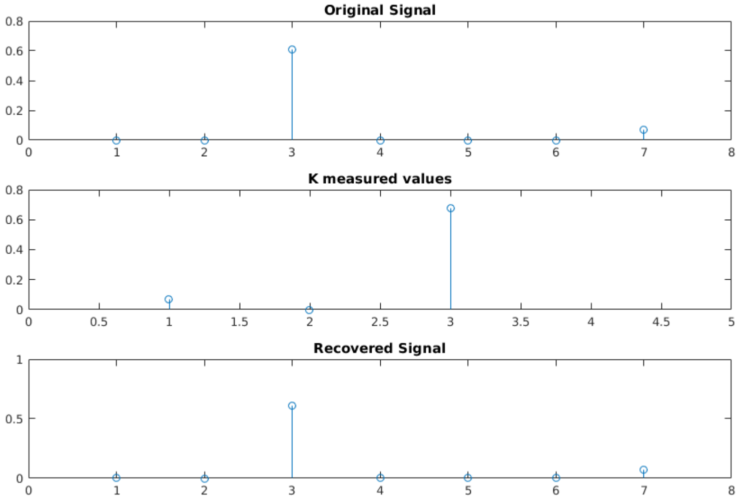
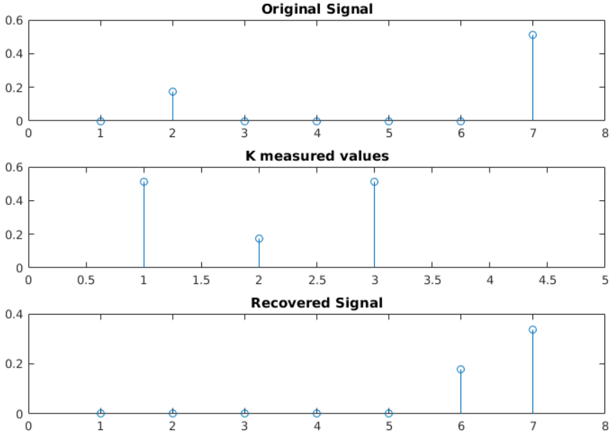


Case of Incorrect Recovery of 2-sparse vector

- $x = \begin{bmatrix} 0 \\ 0.1767 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0.5137 \end{bmatrix}$: Original Signal
- $y = \begin{bmatrix} 0.5137 \\ 0.1767 \\ 0.5137 \end{bmatrix}$: Measurement Vector
- $\hat{x} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0.1766 \\ 0.3370 \end{bmatrix}$: Estimate of the original signal

Case of Incorrect Recovery of 2-sparse vector



Summary

- Established a bridge connection between **CS-LPD** and **CC-LPD**.
- Presented a deterministic polynomial time algorithm to form a zero-one sensing matrix
- Showed that a good LDPC 0-1 parity check matrix will make a good sensing matrix for BSC
- Illustrated the results with a small example
- Outlook
  - Didn't show performance guarantees translation for other channels like AWGN and BEC
  - Didn't cover the concept of M-cover of a graph and equivalent definition of fractional weights