Performance Limits of Tunable Servers with Finite Buffer Capacity and a Packet-Drop Probability Constraint

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VSRP Talk, 2017

Motivation: Immigration Queue

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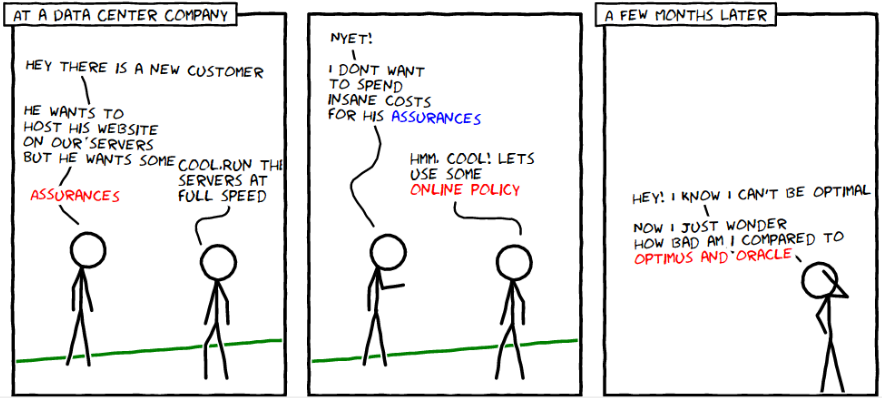
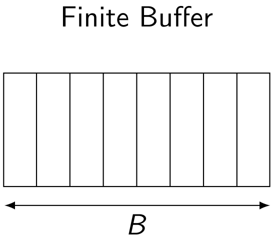


System Model



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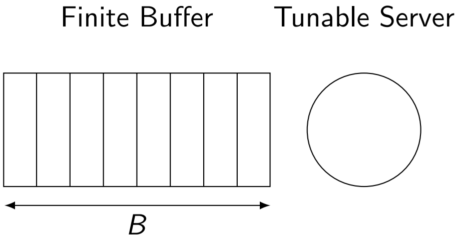

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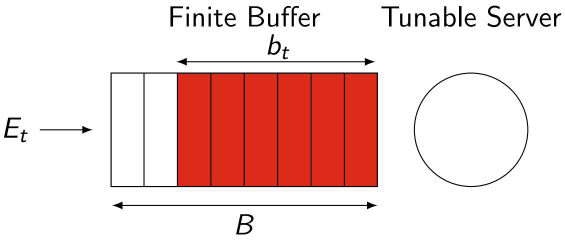

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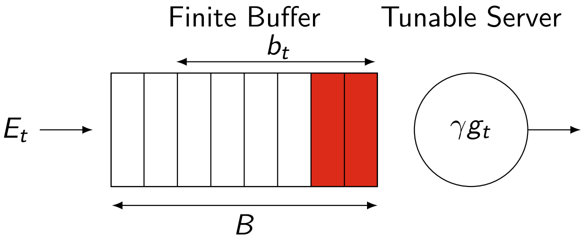

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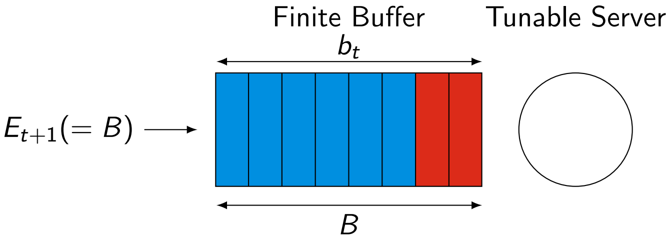

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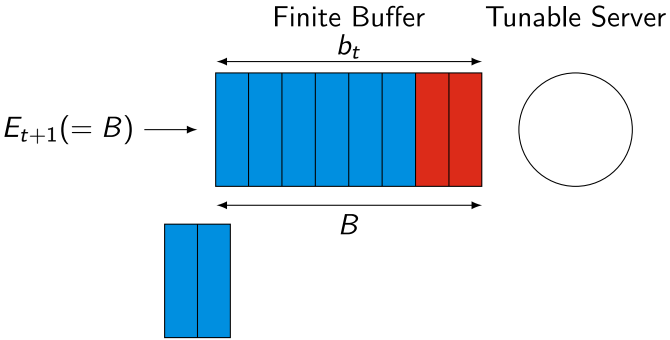

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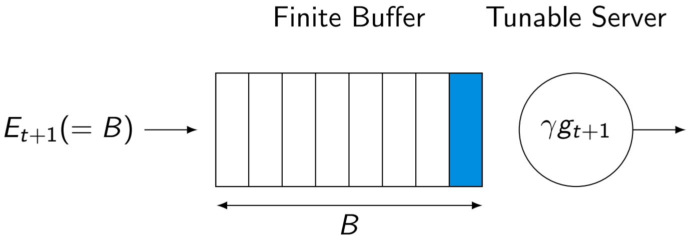

Problem Formulation

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So our objective is to

$$\underset{s}{\text{minimize}} \qquad \lim_{n\to\infty} \mathbb{E}\left[\frac{1}{n}\sum_{t=1}^{n} c_t\right]$$

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$$\begin{aligned} \underset{s}{\text{minimize}} \quad & \lim_{n \to \infty} \mathbb{E}\left[\frac{1}{n} \sum_{t=1}^{n} c_t\right] \\ \text{subject to} \quad & P_{dropping} \leq \alpha \end{aligned} \tag{1}$$

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$$\lim_{n\to\infty} \mathbb{E}\left[\frac{\sum_{t=1}^{n} g_t}{n} + \cancelto{0}{\frac{B}{n}}\right] \geq (1-\alpha)\,\mathbb{E}\left[\frac{\sum_{t=1}^{n} E_t}{n}\right]$$

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Now, the performance of an arbitrary policy is given by:

$$\begin{aligned}
\mathcal{J} &= \mathbb{E}\left[\lim_{n\to\infty} \frac{1}{n}\sum_{t=1}^{n} f(g_t)\right] \\
&\geq f\left(\mathbb{E}\left[\lim_{n\to\infty} \frac{1}{n}\sum_{t=1}^{n} g_t\right]\right) && \text{Using Jensen's Inequality} \\
&= f((1-\alpha)\mu) = \mathcal{J}*
\end{aligned}$$

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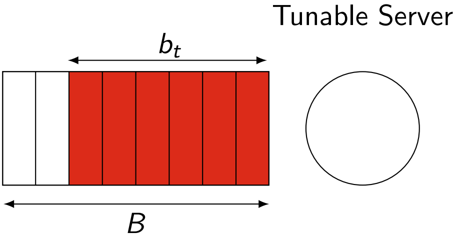
$$E_t = \begin{cases} B, & \text{w.p } p \implies \text{Queue gets full} \\ 0, & \text{w.p } 1-p \end{cases} \tag{2}$$

- Bernoulli distribution gives the worst performance (*Shaviv et al.,2016*)

λ-Fraction Policy

- If the number of packets in the buffer are b_t at time t, then



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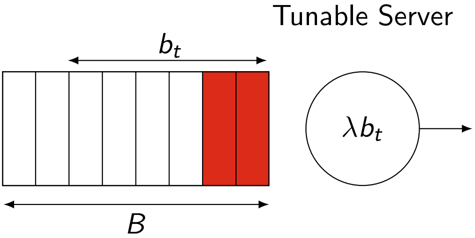

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- The Probability of Dropping can be calculated as:

$$P = \frac{\mathbb{E}_N \left[B(1-\lambda)^N \right]}{B}$$

- Setting $P = \alpha$ in the above equation we get:

$$\lambda = \frac{p(1-\alpha)}{p(1-\alpha)+\alpha}$$

Performance of the λ-Fraction Policy

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- The performance J_λ is given by:

$$J_\lambda = \frac{\mathbb{E}\left[\sum_{t=1}^{N} g_t^2\right]}{\mathbb{E}\, N} = \frac{B^2 p^2 (1-\alpha)^2}{1-(1-p)(1-\alpha)^2}$$

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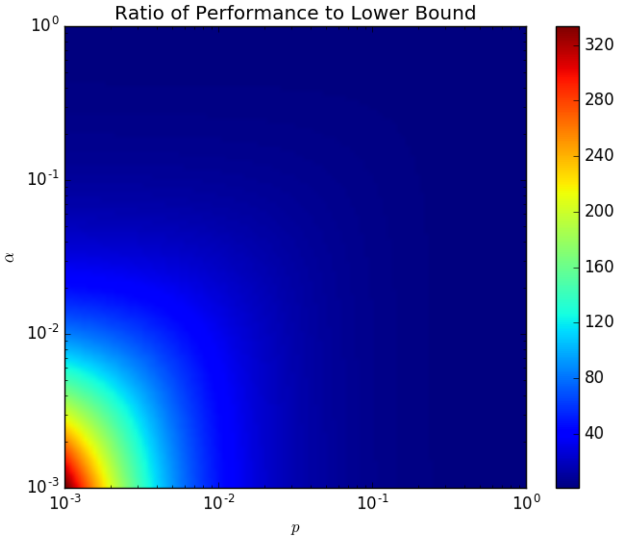
- Recollect that the lower bound $\mathcal{J}^*$ is given by:

$$\mathcal{J}^* = B^2 p^2 (1-\alpha)^2 \tag{3}$$

Performance of the λ-Fraction Policy



Small α,Small p Regime

Proposition

For $\alpha << 1, p << 1$, there exists no online policy that has a cost which is lower than $\Theta\left(\frac{B^2 p^2}{p+\alpha}\right)$

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$$
\begin{aligned}
\mathcal{J}_\lambda &= \frac{B^2 p^2 (1-\alpha)^2}{1-(1-p)(1-\alpha)^2} \\
&\approx \frac{B^2 p^2}{p+2\alpha} \qquad [\alpha << 1, p << 1]
\end{aligned}
$$

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Simple Case: $\alpha << p$

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$$P(N = 1)$$

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Simple Case: $\alpha << p$

$$P(N = 1)[B - g_1] \leq B\alpha$$
$$g_1 \geq B\left(1 - \frac{\alpha}{p}\right)$$

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Simple Case: $\alpha << p$

$$\begin{aligned} P(N=1)[B-g_1] &\leq B\alpha \\ g_1 &\geq B\left(1-\frac{\alpha}{p}\right) \\ &\approx B \end{aligned}$$

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$$\mathcal{J} = \frac{\mathbb{E}\left[\sum_{t=1}^{N} g_t^2\right]}{\mathbb{E}\, N}$$

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Simple Case: $\alpha << p$

$$g_1 \approx B$$

$$\begin{aligned} \mathcal{J} &= \frac{\mathbb{E}\left[\sum_{t=1}^{N} g_t^2\right]}{\mathbb{E}\, N} \\ &\geq \frac{g_1^2}{\mathbb{E}\, N} \end{aligned}$$

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\mathcal{J} &= \frac{\mathbb{E}\left[\sum_{t=1}^{N} g_t^2\right]}{\mathbb{E}\,N} \\
&\geq \frac{g_1^2}{\mathbb{E}\,N} \\
&\approx B^2 p
\end{aligned}$$

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Simple Case: $\alpha << p$

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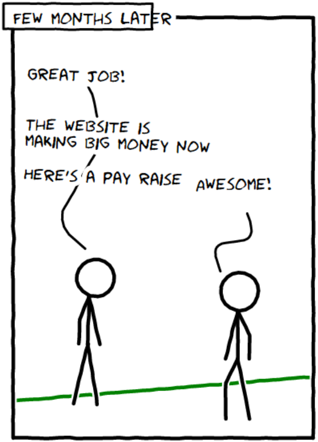
$$\boxed{\therefore \mathcal{J} \geq \Theta\left(\frac{B^2 p^2}{p+\alpha}\right)} \qquad [\text{For } \alpha << p]$$

Summary

- Tune server speed with convex cost function s.t., $P_{\text{drop}} \leq \alpha$
- Approximate online-policy for Bernoulli packet arrivals.
- Lower bounds for any online-policy;our policy is near optimal.
- Future Work:
  - Extend the policy to a general arrival distribution
  - Tighten the lower bound for small α for a general distribution

Thank You!


FEW MONTHS LATER
GREAT JOB!
THE WEBSITE IS MAKING BIG MONEY NOW
HERE'S A PAY RAISE
AWESOME!

Reference: I

James R. Perkins, R. Srikant
The Role of Queue Length Information In Congestion Control and Resource Pricing.
Proceedings of the 38th Conference on Decision and Control,1999.

Adam Wierman, Lachlan L.H. Andrew, Ao Tang
Power-aware speed scaling in processor sharing systems: Optimality and robustness
Performance Evaluation, 2012.

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Simple Case: $\alpha << p$

$$P(N = 1)$$

$$\begin{aligned} g_1 &\geq B\left(1 - \frac{\alpha}{p}\right) \\ &\geq B/2 \end{aligned}$$

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$$\begin{aligned} P(N=1)[B-g_1] &\leq B\alpha \\ g_1 &\geq B\left(1-\frac{\alpha}{p}\right) \\ &\geq B/2 \end{aligned}$$

Small α,Small p Regime

Case: $p \geq 2\alpha$

$$g_1 \geq B/2$$

$$\begin{aligned}
\mathcal{J} &= \frac{\mathbb{E}\left[\sum_{t=1}^{N} g_t^2\right]}{\mathbb{E}\, N} \\
&\geq \frac{g_1^2}{\mathbb{E}\, N} \\
&= \frac{B^2 p}{4} \qquad [\text{For } \alpha << 1, p \geq 2\alpha] \\
\therefore \mathcal{J} &\geq \Theta\left(\frac{B^2 p^2}{p+\alpha}\right)
\end{aligned}$$

- Recall: $\mathcal{J}_\lambda = \frac{B^2 p^2 (1-\alpha)^2}{1-(1-p)(1-\alpha)^2} \approx B^2 p \quad$ [For $\alpha << p$]

Small α,Small p Regime

$p \leq 2\alpha$

$$P[N \leq k]\left(B - \sum_{t=1}^{k} g_t\right) \leq B\alpha$$

$$\sum_{t=1}^{k} g_t \geq B\left(1 - \frac{\alpha}{pk}\right)$$

$$\sum_{t=1}^{k} g_t \geq B/2 \qquad \text{choosing } k = \left\lceil \frac{2\alpha}{p} \right\rceil$$

$$\begin{aligned}
\mathcal{J} &= \frac{\mathbb{E}\left[\sum_{t=1}^{N} g_t^2\right]}{\mathbb{E}\,N} \\
&\geq p \cdot P(N \geq k)[g_1^2 + g_2^2 + \cdots + g_k^2] \\
&\geq p(1 - pk) \cdot \frac{B^2}{4k} \\
&= \Theta\left(\frac{B^2 p^2}{\alpha}\right)
\end{aligned}$$

$$\therefore \mathcal{J} \geq \Theta\left(\frac{B^2 p^2}{\alpha + p}\right)$$