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EE14B127

Programming Assignment 2
X Items or Less
Communication Networks (EE5150)

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Abstract

This assignment deals with finding the optimal value of items which should be decided upon to minimize the average customer waiting times in the queue. First, we start by mathematically formulating the problem and we find that the problem becomes an optimization problem wherein we need to find the optimal solution x^* such that average waiting time in the queues is minimized.

1 Problem Statement

Customers arrive at a grocery store's checkout counter according to a Poisson process with rate 1 per minute. Each customer carries a number of items that is uniformly distributed between 1 and 40. The store has two checkout counters, each capable of processing items at a rate of 15 per minute. To reduce the customer waiting time in queue, the store manager considers dedicating one of the two counters to customers with x items or less and dedicating the other counter to customers with more than x items. Write a small computer program to find the value of x that minimizes the average customer waiting time.

2 Formulation of the Problem

Let the assume that people with x items or less go to the Counter 1 and people with items equal to $x+1$ or above go to Counter 2. Since the number of items a customer carries is uniformly distributed between 1 and 40, the arrival rate for the first counter, $\lambda_1 = \frac{x}{40}$ and the arrival rate for the second counter is $\lambda_2 = \frac{40-x}{40}$. Let S_1 denote the customer service time on the Counter 1 and S_2 denote the customer service time at Counter 2.

Let us consider the Counter 1, we need to calculate the expected service time at Counter 1 i.e. we need to calculate $E[S_1]$. Now r is the service rate therefore the time to serve 1 item will be $\frac{1}{r}$, therefore to serve i items, the time taken will be $\frac{i}{r}$. Now the customer at the first counter is bringing x or less items, which are uniformly distributed hence we get

$$E[S_1] = \sum_{i=1}^{x} \frac{1}{x}\frac{i}{r} \tag{1}$$

$$= \frac{1}{xr}\sum_{i=1}^{x} i \tag{2}$$

$$= \frac{1}{xr}\frac{x(x+1)}{2} \qquad \left(\sum_{i=1}^{N} n = \frac{N(N+1)}{2}\right) \tag{3}$$

$$= \frac{x+1}{2r} = \frac{x+1}{30} \qquad \text{(Given } r = 15\text{)} \tag{4}$$

Now the load at the first counter $\rho_1 = \lambda_1 \times E[S_1]$. Since this is an infinite queue, for stability we need that $\rho_1 < 1$, which gives us the following constraints

$$\rho_1 = \lambda_1 \times E[S_1] = \frac{x(x+1)}{1200} < 1 \tag{5}$$

$$x(x+1) < 1200 \tag{6}$$

Now since x is an integer, we can solve $x(x+1) = 1200$ and take the floor function, we get that $x \leq 34$. This gives us an upper-bound on the value of x which is required for stability of the system.

Now by the *Pollaczek-Khinchin* formula for M/D/1 queue, we get that waiting time in the Counter 1 is given by $W_1 = \frac{\lambda_1 E[S_1^2]}{2(1-\rho_1)}$, therefore we need to calculate the second moment of S_1, which is given by

$$E[S_1^2] = \sum_{i=1}^{x} \frac{1}{x}\left(\frac{i}{r}\right)^2 \tag{7}$$

$$= \frac{1}{xr^2}\sum_{i=1}^{x} i^2 \tag{8}$$

$$= \frac{1}{xr^2}\frac{x(x+1)(2x+1)}{6} \qquad \left(\sum_{n=1}^{N} n^2 = \frac{n(n+1)(2n+1)}{6}\right) \tag{9}$$

$$= \frac{(x+1)(2x+1)}{6r^2} \tag{10}$$

Now we can do a similar analysis for the Counter 2 as well. We need to calculate $E[S_1]$ and $E[S_1^2]$ to get the values of ρ_2 and W_2. Using the similar arguments as presented above we get that

$$E[S_2] = \sum_{i=x+1}^{40} \frac{1}{40-x}\frac{i}{r} = \frac{x+41}{30} \tag{11}$$

$$E[S_2^2] = \sum_{i=x+1}^{40} \frac{1}{40-x}\left(\frac{i}{r}\right)^2 \tag{12}$$

Now again for stability, we need $\rho_2 < 1$, which means $\frac{(40-x)(x+1)}{1200} < 1$,using the fact that $\rho_2 = \lambda_2 \times E[S_2]$, from this we get that $x \geq 21$.

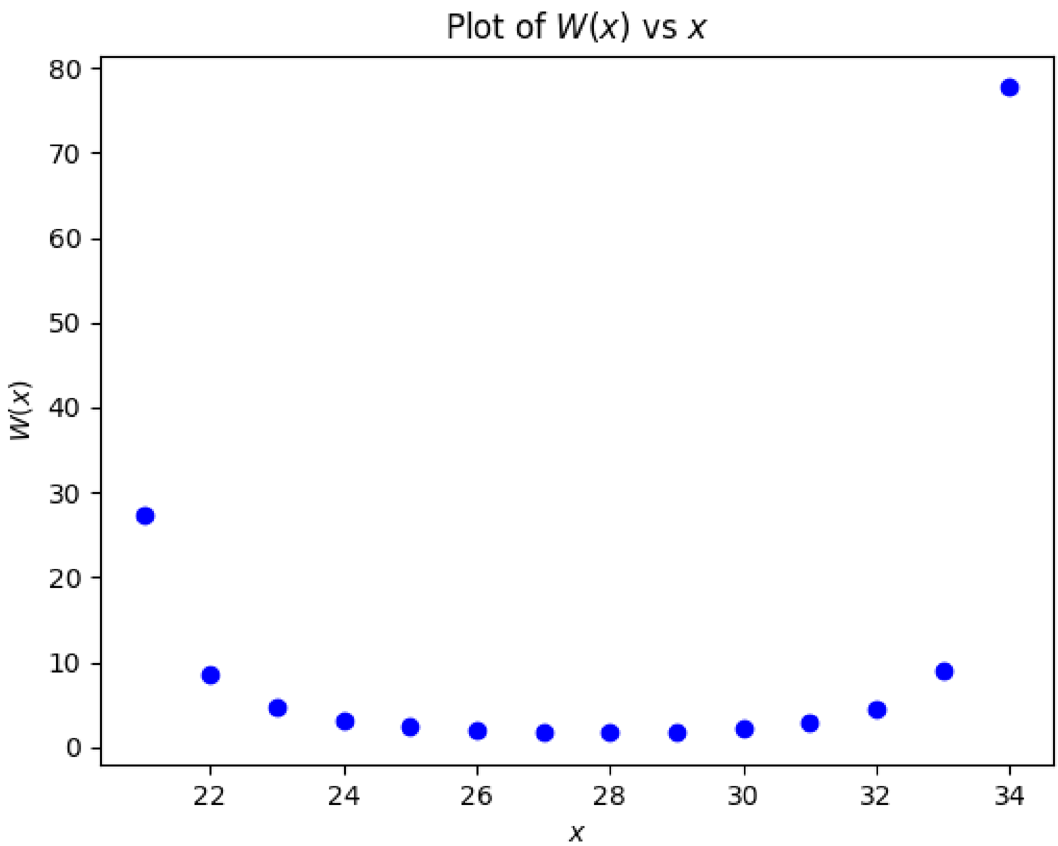
The average waiting time in the system W is given by $W(x) = \frac{x}{40}W_1 + \frac{40-x}{40}W_2$ i.e. with probability $\frac{x}{40}$ I will go to the first counter and face a waiting time of W_1 and with probability $\frac{40-x}{40}$, I will go to the second counter and face a waiting time of W_2.

So now our objective is to find the value of x which minimizes $W(x)$ which can be formally written down as

$$\begin{aligned} \underset{x}{\text{minimize}} \quad & W(x) \\ \text{subject to} \quad & 21 \leq x \leq 34 \end{aligned}$$

To find the optimal value of x, we can iterate over the feasible set and find the point at which $W(x)$ is minimized, which is done using a Python script.

3 Optimal Value of x



We can clearly see that the optimal value of x is achieved at $x = 28$. Hence for $x = 28$ is the value which should be chosen to minimize the average waiting time