Probability and Statistics (IT302)

31st August 2020 Monday 9:45 AM-10:15 AM

Class 11

Mean of a Random Variable

Now consider the following. If two coins are tossed 16 times and X is the number of heads that occur per toss, then the values of X are 0, 1, and 2. Suppose that the experiment yields no heads, one head, and two heads a total of 4, 7, and 5 times, respectively. The average number of heads per toss of the two coins is then

$$\frac{(0)(4) + (1)(7) + (2)(5)}{16} = 1.06.$$

This is an average value of the data and yet it is not a possible outcome of $\{0, 1, 2\}$. Hence, an average is not necessarily a possible outcome for the experiment. For instance, a salesman's average monthly income is not likely to be equal to any of his monthly paychecks.

Let us now restructure our computation for the average number of heads so as to have the following equivalent form:

$$(0)\left(\frac{4}{16}\right) + (1)\left(\frac{7}{16}\right) + (2)\left(\frac{5}{16}\right) = 1.06.$$

Source : Probability & Statistics for Engineers & Scientists, by Ronald E. Walpole, Raymond H. Myers, Sharon L. Myers, Keying Ye, 9th Edition, Prentice Hall.

Mean of a Random Variable Contd.

- The numbers 4/16, 7/16, and 5/16 are the fractions of the total tosses resulting in 0, 1, and 2 heads, respectively. These fractions are also the relative frequencies for the different values of *X* in our experiment.

- In fact, then, we can calculate the mean, or average, of a set of data by knowing the distinct values that occur and their relative frequencies, without any knowledge of the total number of observations in our set of data.

- Therefore, if 4/16, or 1/4, of the tosses result in no heads, 7/16 of the tosses result in one head, and 5/16 of the tosses result in two heads, the mean number of heads per toss would be 1.06 no matter whether the total number of tosses were 16, 1000, or even 10,000.

Mean of a Random Variable Contd.

This method of relative frequencies is used to calculate the average number of heads per toss of two coins that we might expect in the long run. We shall refer to this average value as the **mean of the random variable** X or the **mean of the probability distribution of** X and write it as μx or simply as μ when it is clear to which random variable we refer.

It is also common among statisticians to refer to this mean as the **mathematical expectation, or the expected value of the random variable *X*, and denote it as $E(X)$**.

Source : Probability & Statistics for Engineers & Scientists, by Ronald E. Walpole, Raymond H. Myers, Sharon L. Myers, Keying Ye, 9th Edition, Prentice Hall.

Mean of a Random Variable Contd.

Assuming that 1 fair coin was tossed twice, we find that the sample space for our experiment is $S = \{HH, HT, TH, TT\}$. Since the 4 sample points are all equally likely, it follows that

$P(X = 0) = P(TT) = 1/4$, $P(X = 1) = P(TH) + P(HT) = 1/2$, and $P(X = 2) = P(HH) = 1/4$,

where a typical element, say TH, indicates that the first toss resulted in a tail followed by a head on the second toss. Now, these probabilities are just the relative frequencies for the given events in the long run. Therefore, $\boldsymbol{\mu = E(X) = (0)(1/4) + (1)(1/2) + (2)(1/4) = 1}$**.**

This result means that a person who tosses 2 coins over and over again will, on the average, get 1 head per toss.

Source : Probability & Statistics for Engineers & Scientists, by Ronald E. Walpole, Raymond H. Myers, Sharon L. Myers, Keying Ye, 9th Edition, Prentice Hall.

Mean of a Random Variable Contd.

The method described previously for calculating the expected number of heads per toss of 2 coins suggests that the **mean, or expected value, of any discrete random variable** may be obtained by multiplying each of the values $x1, x2, \ldots, xn$ of the random variable X by its corresponding probability $f(x1), f(x2), \ldots, f(xn)$ and summing the products. This is true, however, only if the random variable is discrete.

In the case of continuous random variables, the definition of an expected value is essentially the same with summations replaced by integrations.

Definition of The mean, of expected value, of Random Variable

Let X be a random variable with probability distribution $f(x)$. The **mean**, or **expected value**, of X is

$$\mu = E(X) = \sum_x x f(x)$$

if X is discrete, and

$$\mu = E(X) = \int_{-\infty}^{\infty} x f(x)\, dx$$

if X is continuous.

Source : Probability & Statistics for Engineers & Scientists, by Ronald E. Walpole, Raymond H. Myers, Sharon L. Myers, Keying Ye, 9^{th} Edition, Prentice Hall.

Example 4.1

A salesperson for a medical device company has two appointments on a given day. At the first appointment, he believes that he has a 70% chance to make the deal, from which he can earn \$1000 commission if successful. On the other hand, he thinks he only has a 40% chance to make the deal at the second appointment, from which, if successful, he can make \$1500. What is his expected commission based on his own probability belief? Assume that the appointment results are independent of each other.

Solution: First, we know that the salesperson, for the two appointments, can have 4 possible commission totals: \$0, \$1000, \$1500, and \$2500. We then need to calculate their associated probabilities. By independence, we obtain

$$f(\$0) = (1 - 0.7)(1 - 0.4) = 0.18, \quad f(\$2500) = (0.7)(0.4) = 0.28,$$
$$f(\$1000) = (0.7)(1 - 0.4) = 0.42, \text{ and } f(\$1500) = (1 - 0.7)(0.4) = 0.12.$$

Therefore, the expected commission for the salesperson is

$$\begin{aligned} E(X) &= (\$0)(0.18) + (\$1000)(0.42) + (\$1500)(0.12) + (\$2500)(0.28) \\ &= \$1300. \end{aligned}$$

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Example 4.2

A lot containing 7 components is sampled by a quality inspector; the lot contains 4 good components and 3 defective components. A sample of 3 is taken by the inspector. Find the expected value of the number of good components in this sample.

Solution: Let X represent the number of good components in the sample. The probability distribution of X is

$$f(x) = \frac{\binom{4}{x}\binom{3}{3-x}}{\binom{7}{3}}, \qquad x = 0, 1, 2, 3.$$

Simple calculations yield $f(0) = 1/35$, $f(1) = 12/35$, $f(2) = 18/35$, and $f(3) = 4/35$. Therefore,

$$\mu = E(X) = (0)\left(\frac{1}{35}\right) + (1)\left(\frac{12}{35}\right) + (2)\left(\frac{18}{35}\right) + (3)\left(\frac{4}{35}\right) = \frac{12}{7} = 1.7.$$

Thus, if a sample of size 3 is selected at random over and over again from a lot of 4 good components and 3 defective components, it will contain, on average, 1.7 good components.

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Example 4.3

Let X be the random variable that denotes the life in hours of a certain electronic device. The probability density function is

Find the expected life of this type of device.

Solution: Using Definition 4.1, we have

$$\mu = E(X) = \int_{100}^{\infty} x \frac{20,000}{x^3}\, dx = \int_{100}^{\infty} \frac{20,000}{x^2}\, dx = 200.$$

Therefore, we can expect this type of device to last, *on average*, 200 hours. Now let us consider a new random variable $g(X)$, which depends on X; that is, each value of $g(X)$ is determined by the value of X. For instance, $g(X)$ might be X^2 or $3X - 1$, and whenever X assumes the value 2, $g(X)$ assumes the value $g(2)$. In particular, if X is a discrete random variable with probability distribution $f(x)$, for $x = -1, 0, 1, 2$, and $g(X) = X^2$, then

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Example 4.3 Contd.

$$P[g(X) = 0] = P(X = 0) = f(0),$$
$$P[g(X) = 1] = P(X = -1) + P(X = 1) = f(-1) + f(1),$$
$$P[g(X) = 4] = P(X = 2) = f(2),$$

and so the probability distribution of $g(X)$ may be written

$g(x)$	0	1	4
$P[g(X) = g(x)]$	$f(0)$	$f(-1) + f(1)$	$f(2)$

By the definition of the expected value of a random variable, we obtain

$$\begin{aligned}\mu_{g(X)} &= E[g(x)] = 0f(0) + 1[f(-1) + f(1)] + 4f(2)\\ &= (-1)^2 f(-1) + (0)^2 f(0) + (1)^2 f(1) + (2)^2 f(2) = \sum_x g(x)f(x).\end{aligned}$$

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Example 4.3 Contd.

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Theorem 4.1

Let X be a random variable with probability distribution $f(x)$. The expected value of the random variable $g(X)$ is

$$\mu_{g(X)} = E[g(X)] = \sum_{x} g(x)f(x)$$

if X is discrete, and

$$\mu_{g(X)} = E[g(X)] = \int_{-\infty}^{\infty} g(x)f(x)\,dx$$

if X is continuous.

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Example 4.4

Example 4.4: Suppose that the number of cars X that pass through a car wash between 4:00 P.M. and 5:00 P.M. on any sunny Friday has the following probability distribution:

x	4	5	6	7	8	9
$P(X=x)$	$\frac{1}{12}$	$\frac{1}{12}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{6}$	$\frac{1}{6}$

Let $g(X) = 2X - 1$ represent the amount of money, in dollars, paid to the attendant by the manager. Find the attendant's expected earnings for this particular time period.

***Solution*:** By Theorem 4.1, the attendant can expect to receive

$$E[g(X)] = E(2X-1) = \sum_{x=4}^{9}(2x-1)f(x)$$

$$= (7)\left(\frac{1}{12}\right) + (9)\left(\frac{1}{12}\right) + (11)\left(\frac{1}{4}\right) + (13)\left(\frac{1}{4}\right)$$

$$+ (15)\left(\frac{1}{6}\right) + (17)\left(\frac{1}{6}\right) = \$12.67.$$

Source : Probability & Statistics for Engineers & Scientists, by Ronald E. Walpole, Raymond H. Myers, Sharon L. Myers, Keying Ye, 9th Edition, Prentice Hall.

Example 4.5

Example 4.5: Let X be a random variable with density function

$$f(x) = \begin{cases} \frac{x^2}{3}, & -1 < x < 2, \\ 0, & \text{elsewhere.} \end{cases}$$

Find the expected value of $g(X) = 4X + 3$.

***Solution*:** By Theorem 4.1, we have

$$E(4X+3) = \int_{-1}^{2} \frac{(4x+3)x^2}{3}\, dx = \frac{1}{3}\int_{-1}^{2} (4x^3 + 3x^2)\, dx = 8.$$

We shall now extend our concept of mathematical expectation to the case of two random variables X and Y with joint probability distribution $f(x, y)$.

Source : Probability & Statistics for Engineers & Scientists, by Ronald E. Walpole, Raymond H. Myers, Sharon L. Myers, Keying Ye, 9th Edition, Prentice Hall.

Definition 4.2

Let X and Y be random variables with joint probability distribution $f(x, y)$. The mean, or expected value, of the random variable $g(X, Y)$ is

$$\mu_{g(X,Y)} = E[g(X,Y)] = \sum_x \sum_y g(x,y)f(x,y)$$

if X and Y are discrete, and

$$\mu_{g(X,Y)} = E[g(X,Y)] = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x,y)f(x,y)\, dx\, dy$$

if X and Y are continuous.

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Example 4.6:

Let X and Y be the random variables with joint probability distribution indicated in below Table. Find the expected value of $g(X, Y) = XY$. The table is reprinted here for convenience.

	$f(x,y)$	$x=0$	$x=1$	$x=2$	Row Totals
y	0	$\frac{3}{28}$	$\frac{9}{28}$	$\frac{3}{28}$	$\frac{15}{28}$
	1	$\frac{3}{14}$	$\frac{3}{14}$	0	$\frac{3}{7}$
	2	$\frac{1}{28}$	0	0	$\frac{1}{28}$
Column Totals		$\frac{5}{14}$	$\frac{15}{28}$	$\frac{3}{28}$	1

Solution: By Definition 4.2, we write

$$\begin{aligned}
E(XY) &= \sum_{x=0}^{2}\sum_{y=0}^{2} xyf(x,y) \\
&= (0)(0)f(0,0) + (0)(1)f(0,1) \\
&\quad + (1)(0)f(1,0) + (1)(1)f(1,1) + (2)(0)f(2,0) \\
&= f(1,1) = \frac{3}{14}.
\end{aligned}$$

Source : Probability & Statistics for Engineers & Scientists, by Ronald E. Walpole, Raymond H. Myers, Sharon L. Myers, Keying Ye, 9th Edition, Prentice Hall.

Example 4.7

Example 4.7: Find $E(Y/X)$ for the density function

$$f(x,y) = \begin{cases} \frac{x(1+3y^2)}{4}, & 0 < x < 2,\ 0 < y < 1, \\ 0, & \text{elsewhere.} \end{cases}$$

Solution: We have

$$E\left(\frac{Y}{X}\right) = \int_0^1 \int_0^2 \frac{y(1+3y^2)}{4}\, dx dy = \int_0^1 \frac{y+3y^3}{2}\, dy = \frac{5}{8}.$$

Note that if $g(X,Y) = X$ in Definition 4.2, we have

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Example 4.7

$$E(X) = \begin{cases} \sum_x \sum_y x f(x,y) = \sum_x x g(x) & \text{(discrete case)}, \\ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x f(x,y)\, dy\, dx = \int_{-\infty}^{\infty} x g(x)\, dx & \text{(continuous case)}, \end{cases}$$

where $g(x)$ is the marginal distribution of X. Therefore, in calculating $E(X)$ over a two-dimensional space, one may use either the joint probability distribution of X and Y or the marginal distribution of X. Similarly, we define

$$E(Y) = \begin{cases} \sum_y \sum_x y f(x,y) = \sum_y y h(y) & \text{(discrete case)}, \\ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} y f(x,y)\, dx dy = \int_{-\infty}^{\infty} y h(y)\, dy & \text{(continuous case)}, \end{cases}$$

where $h(y)$ is the marginal distribution of the random variable Y.