Probability and Statistics (IT302)

24th August 2020 Monday 09:45 AM-10:15 AM Class 8

25th August 2020 Tuesday 10:30 AM-11:00 AM Class 9

Introduction to Joint Probability Distributions or Probability Mass Function

If X and Y are **two discrete random variables**, the probability distribution for their simultaneous occurrence can be represented by a function with values $f(x, y)$ for any pair of values (x, y) within the range of the random variables X and Y . It is customary to refer to this function as the **joint probability distribution** of X and Y. Hence, in the discrete case,

$$f(x, y) = P(X = x, Y = y);$$

that is, the values $f(x, y)$ give the probability that outcomes x and y occur at the same time. For example, if an 18-wheeler is to have its tires serviced and X represents the number of miles these tires have been driven and Y represents the number of tires that need to be replaced, then $f(30000, 5)$ is the probability that the tires are used over 30,000 miles and the truck needs 5 new tires.

Source : Probability & Statistics for Engineers & Scientists, by Ronald E. Walpole, Raymond H. Myers, Sharon L. Myers, Keying Ye, 9th Edition, Prentice Hall.

Introduction to Joint Probability Distributions or Probability Mass Function Contd.

The function $f(x, y)$ is a **joint probability distribution** or **probability mass function** of the discrete random variables X and Y if

1. $f(x, y) \geq 0$ for all (x, y),
2. $\sum_x \sum_y f(x, y) = 1$,
3. $P(X = x, Y = y) = f(x, y)$.

For any region A in the xy plane, $P[(X, Y) \in A] = \sum\sum_A f(x, y)$.

Source : Probability & Statistics for Engineers & Scientists, by Ronald E. Walpole, Raymond H. Myers, Sharon L. Myers, Keying Ye, 9th Edition, Prentice Hall

Introduction to Joint Probability Distributions Contd.

Suppose X and Y are two **Discrete Random Variables** and that X takes values $\{x_1, x_2, \ldots, x_n\}$ and Y takes values $\{y_1, y_2, \ldots, y_m\}$. The ordered pair (X, Y) take values in the product $\{(x_1, y_1),(x_1, y_2), \ldots (x_n, y_m)\}$. The **Joint Pobability Mass Function (joint pmf)** of X and Y is the function $p(x_i, y_j)$ giving the probability of the joint outcome $X = x_i$, $Y = y_j$.

Organize this in a Joint Probability Table as shown:

$X\backslash Y$	y_1	y_2	$\cdots$	y_j	$\cdots$	y_m
x_1	$p(x_1, y_1)$	$p(x_1, y_2)$	$\cdots$	$p(x_1, y_j)$	$\cdots$	$p(x_1, y_m)$
x_2	$p(x_2, y_1)$	$p(x_2, y_2)$	$\cdots$	$p(x_2, y_j)$	$\cdots$	$p(x_2, y_m)$
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$
x_i	$p(x_i, y_1)$	$p(x_i, y_2)$	$\cdots$	$p(x_i, y_j)$	$\cdots$	$p(x_i, y_m)$
$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	$\cdots$	
x_n	$p(x_n, y_1)$	$p(x_n, y_2)$	$\cdots$	$p(x_n, y_j)$	$\cdots$	$p(x_n, y_m)$

Source : https://ocw.mit.edu/courses/mathematics/18-05-introduction-to-probability-and-statistics-spring-2014/readings/MIT18_05S14_Reading7a.pdf

Introduction to Joint Probability Distributions Example-1

Example-1:- Roll two dice. Let X be the value on the First die and let Y be the value on the second die. Then both X and Y take values 1 to 6 and the joint pmf is p(i, j) = 1/36 for all i and j between 1 and 6. Here is the Joint Probability Table:

$X\backslash Y$	1	2	3	4	5	6
1	1/36	1/36	1/36	1/36	1/36	1/36
2	1/36	1/36	1/36	1/36	1/36	1/36
3	1/36	1/36	1/36	1/36	1/36	1/36
4	1/36	1/36	1/36	1/36	1/36	1/36
5	1/36	1/36	1/36	1/36	1/36	1/36
6	1/36	1/36	1/36	1/36	1/36	1/36

Source : https://ocw.mit.edu/courses/mathematics/18-05-introduction-to-probability-and-statistics-spring-2014/readings/MIT18_05S14_Reading7a.pdf

Introduction to Joint Probability Distributions Example-2

Example-2. Roll two dice. Let X be the value on the first die and let T be the total on both dice. Here is the Joint Probability Table:

$X\backslash T$	2	3	4	5	6	7	8	9	10	11	12
1	1/36	1/36	1/36	1/36	1/36	1/36	0	0	0	0	0
2	0	1/36	1/36	1/36	1/36	1/36	1/36	0	0	0	0
3	0	0	1/36	1/36	1/36	1/36	1/36	1/36	0	0	0
4	0	0	0	1/36	1/36	1/36	1/36	1/36	1/36	0	0
5	0	0	0	0	1/36	1/36	1/36	1/36	1/36	1/36	0
6	0	0	0	0	0	1/36	1/36	1/36	1/36	1/36	1/36

A joint probability mass function must satisfy two properties:

1. $0 \leq p(x_i, y_j) \leq 1$
2. The total probability is 1. We can express this as a double sum:

$$\sum_{i=1}^{n} \sum_{j=1}^{m} p(x_i, y_j) = 1$$

Source : https://ocw.mit.edu/courses/mathematics/18-05-introduction-to-probability-and-statistics-spring-2014/readings/MIT18_05S14_Reading7a.pdf

Introduction to Joint Probability Distributions Example 3.14

Example 3.14 : Two ballpoint pens are selected at random from a box that contains 3 blue pens, 2 red pens, and 3 green pens. If X is the number of blue pens selected and Y is the number of red pens selected, find

1. the joint probability function $f(x, y)$,
2. $P[(X, Y) \in A]$, where A is the region $\{(x, y) | x + y \leq 1\}$.

***Solution*:** The possible pairs of values (x, y) are $(0, 0)$, $(0, 1)$, $(1, 0)$, $(1, 1)$, $(0, 2)$, and $(2, 0)$.

(a) Now, $f(0, 1)$, for example, represents the probability that a red and a green pens are selected. The total number of equally likely ways of selecting any 2 pens from the 8 is $\binom{8}{2} = 28$. The number of ways of selecting 1 red from 2 red pens and 1 green from 3 green pens is $\binom{2}{1}\binom{3}{1} = 6$. Hence, $f(0, 1) = 6/28 = 3/14$. Similar calculations yield the probabilities for the other cases, which are presented in Table 3.1. Note that the probabilities sum to 1. In Chapter

Source : Probability & Statistics for Engineers & Scientists, by Ronald E. Walpole, Raymond H. Myers, Sharon L. Myers, Keying Ye, 9th Edition, Prentice Hall

Introduction to Joint Probability Distributions Example 3.14 Contd.

5, it will become clear that the joint probability distribution of Table 3.1 can be represented by the formula

$$f(x, y) = \frac{\binom{3}{x}\binom{2}{y}\binom{3}{2-x-y}}{\binom{8}{2}},$$

for $x = 0, 1, 2$; $y = 0, 1, 2$; and $0 \leq x + y \leq 2$.

(b) The probability that (X, Y) fall in the region A is

$$P[(X, Y) \in A] = P(X + Y \leq 1) = f(0, 0) + f(0, 1) + f(1, 0)$$
$$= \frac{3}{28} + \frac{3}{14} + \frac{9}{28} = \frac{9}{14}.$$

Table 3.1: Joint Probability Distribution for Example 3.14

	$f(x, y)$	x = 0	1	2	Row Totals
y	0	$\frac{3}{28}$	$\frac{9}{28}$	$\frac{3}{28}$	$\frac{15}{28}$
	1	$\frac{3}{14}$	$\frac{3}{14}$	0	$\frac{3}{7}$
	2	$\frac{1}{28}$	0	0	$\frac{1}{28}$
Column Totals		$\frac{5}{14}$	$\frac{15}{28}$	$\frac{3}{28}$	1

Source : Probability & Statistics for Engineers & Scientists, by Ronald E. Walpole, Raymond H. Myers, Sharon L. Myers, Keying Ye, 9th Edition, Prentice Hall

Joint Probability Mass Function Example 4.3a

EXAMPLE 4.3a Suppose that 3 batteries are randomly chosen from a group of 3 new, 4 used but still working, and 5 defective batteries. If we let X and Y denote, respectively, the number of new and used but still working batteries that are chosen, then the **Joint Probability Mass Function** of X and Y, p(i, j) = P{X = i, Y = j}, is given by

$$p(0,0) = \binom{5}{3} \Big/ \binom{12}{3} = 10/220$$

$$p(0,1) = \binom{4}{1}\binom{5}{2} \Big/ \binom{12}{3} = 40/220$$

$$p(0,2) = \binom{4}{2}\binom{5}{1} \Big/ \binom{12}{3} = 30/220$$

$$p(0,3) = \binom{4}{3} \Big/ \binom{12}{3} = 4/220$$

$$p(1,0) = \binom{3}{1}\binom{5}{2} \Big/ \binom{12}{3} = 30/220$$

$$p(1,1) = \binom{3}{1}\binom{4}{1}\binom{5}{1} \Big/ \binom{12}{3} = 60/220$$

$$p(1,2) = \binom{3}{1}\binom{4}{2} \Big/ \binom{12}{3} = 18/220$$

$$p(2,0) = \binom{3}{2}\binom{5}{1} \Big/ \binom{12}{3} = 15/220$$

$$p(2,1) = \binom{3}{2}\binom{4}{1} \Big/ \binom{12}{3} = 12/220$$

$$p(3,0) = \binom{3}{3} \Big/ \binom{12}{3} = 1/220$$

Joint Probability Mass Function EXAMPLE 4.3a Contd.

$$p(0,0) = \binom{5}{3} \Big/ \binom{12}{3} = 10/220$$

$$p(0,1) = \binom{4}{1}\binom{5}{2} \Big/ \binom{12}{3} = 40/220$$

$$p(0,2) = \binom{4}{2}\binom{5}{1} \Big/ \binom{12}{3} = 30/220$$

$$p(0,3) = \binom{4}{3} \Big/ \binom{12}{3} = 4/220$$

$$p(1,0) = \binom{3}{1}\binom{5}{2} \Big/ \binom{12}{3} = 30/220$$

$$p(1,1) = \binom{3}{1}\binom{4}{1}\binom{5}{1} \Big/ \binom{12}{3} = 60/220$$

$$p(1,2) = \binom{3}{1}\binom{4}{2} \Big/ \binom{12}{3} = 18/220$$

$$p(2,0) = \binom{3}{2}\binom{5}{1} \Big/ \binom{12}{3} = 15/220$$

$$p(2,1) = \binom{3}{2}\binom{4}{1} \Big/ \binom{12}{3} = 12/220$$

$$p(3,0) = \binom{3}{3} \Big/ \binom{12}{3} = 1/220$$

TABLE 4.1 $P\{X = i, Y = j\}$

i \ j	0	1	2	3	Row Sum $= P\{X = i\}$
0	$\frac{10}{220}$	$\frac{40}{220}$	$\frac{30}{220}$	$\frac{4}{220}$	$\frac{84}{220}$
1	$\frac{30}{220}$	$\frac{60}{220}$	$\frac{18}{220}$	0	$\frac{108}{220}$
2	$\frac{15}{220}$	$\frac{12}{220}$	0	0	$\frac{27}{220}$
3	$\frac{1}{220}$	0	0	0	$\frac{1}{220}$
Column Sums = $P\{Y = j\}$	$\frac{56}{220}$	$\frac{112}{220}$	$\frac{48}{220}$	$\frac{4}{220}$	

Joint Probability Mass Function EXAMPLE 4.3b

EXAMPLE 4.3b Suppose that 15 percent of the families in a certain community have no children, 20 percent have 1, 35 percent have 2, and 30 percent have 3 children; suppose further that each child is equally likely (and independently) to be a boy or a girl. If a family is chosen at random from this community, then B, the number of boys, and G, the number of girls, in this family will have the joint probability mass function shown in Table 4.2.

TABLE 4.2 $P\{B = i, G = j\}$

$i \setminus j$	0	1	2	3	Row Sum $= P\{B = i\}$
0	.15	.10	.0875	.0375	.3750
1	.10	.175	.1125	0	.3875
2	.0875	.1125	0	0	.2000
3	.0375	0	0	0	.0375
Column Sum $= P\{G = j\}$	.3750	.3875	.2000	.0375	

Joint Probability Mass Function EXAMPLE 4.3b Contd.

TABLE 4.2 $P\{B = i, G = j\}$

$i \backslash j$	0	1	2	3	Row Sum $= P\{B = i\}$
0	.15	.10	.0875	.0375	.3750
1	.10	.175	.1125	0	.3875
2	.0875	.1125	0	0	.2000
3	.0375	0	0	0	.0375
Column Sum $= P\{G = j\}$	.3750	.3875	.2000	.0375	

These probabilities are obtained as follows:

$$
\begin{aligned}
P\{B = 0, G = 0\} &= P\{\text{no children}\} \\
&= .15 \\
P\{B = 0, G = 1\} &= P\{\text{1 girl and total of 1 child}\} \\
&= P\{\text{1 child}\}P\{\text{1 girl}|\text{1 child}\} \\
&= (.20)\left(\tfrac{1}{2}\right) = .1 \\
P\{B = 0, G = 2\} &= P\{\text{2 girls and total of 2 children}\} \\
&= P\{\text{2 children}\}P\{\text{2 girls}|\text{2 children}\} \\
&= (.35)\left(\tfrac{1}{2}\right)^2 = .0875 \\
P\{B = 0, G = 3\} &= P\{\text{3 girls and total of 3 children}\} \\
&= P\{\text{3 children}\}P\{\text{3 girls}|\text{3 children}\} \\
&= (.30)\left(\tfrac{1}{2}\right)^3 = .0375
\end{aligned}
$$

Joint Probability Mass Function EXAMPLE 4.1.1

Roll a pair of fair dice. For each of the 36 sample points with probability 1/36, let X denote the smaller and Y the larger outcome on the dice. For example, if the outcome is (3,2), then the observed values are $X = 2$, $Y = 3$. The event $\{X = 2, Y = 3\}$ could occur in one of two ways—(3,2) or (2,3)—so its probability is

$$\frac{1}{36} + \frac{1}{36} = \frac{2}{36}.$$

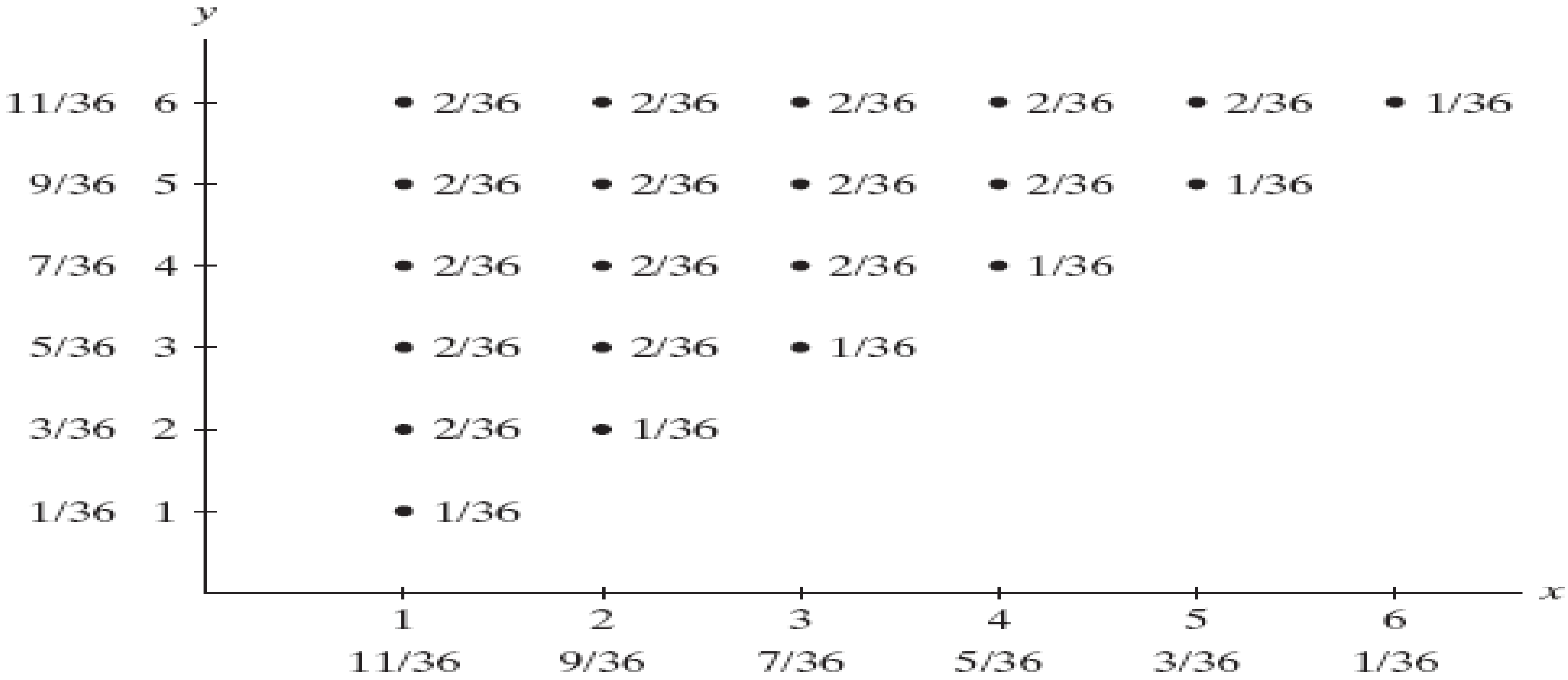
If the outcome is (2,2), then the observed values are $X = 2$, $Y = 2$. Since the event $\{X = 2, Y = 2\}$ can occur in only one way, $P(X = 2, Y = 2) = 1/36$. The joint pmf of X and Y is given by the probabilities

$$f(x, y) = \begin{cases} \dfrac{1}{36}, & 1 \le x = y \le 6, \\[2ex] \dfrac{2}{36}, & 1 \le x < y \le 6, \end{cases}$$

when x and y are integers. Figure 4.1-1 depicts the probabilities of the various points of the space S.

Source : PROBABILITY AND STATISTICAL INFERENCE by Robert V. Hogg, Elliot A. Tanis, Dale L. Zimmerman, 9th Edition

Joint Probability Mass Function EXAMPLE 4.1.1 Contd.



Source : PROBABILITY AND STATISTICAL INFERENCE by Robert V. Hogg, Elliot A. Tanis, Dale L. Zimmerman, 9th Edition

Joint Density Function of the Continuous Random Variables

The function $f(x, y)$ is a **joint density function** of the continuous random variables X and Y if

1. $f(x, y) \geq 0$, for all (x, y),
2. $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y)\, dx\, dy = 1$,
3. $P[(X, Y) \in A] = \int \int_A f(x, y)\, dx\, dy$, for any region A in the xy plane.

Source : Probability & Statistics for Engineers & Scientists, by Ronald E. Walpole, Raymond H. Myers, Sharon L. Myers, Keying Ye, 9th Edition, Prentice Hall

Joint Density Function of the Continuous Random Variables Examples

Example 3.15: A privately owned business operates both a drive-in facility and a walk-in facility. On a randomly selected day, let X and Y, respectively, be the proportions of the time that the drive-in and the walk-in facilities are in use, and suppose that the joint density function of these random variables is

$$f(x,y) = \begin{cases} \frac{2}{5}(2x+3y), & 0 \le x \le 1, 0 \le y \le 1, \\ 0, & \text{elsewhere.} \end{cases}$$

(a) Verify condition 2 of Definition 3.9.

(b) Find $P[(X,Y) \in A]$, where $A = \{(x,y) \mid 0 < x < \frac{1}{2}, \frac{1}{4} < y < \frac{1}{2}\}$.

Joint Density Function of the Continuous Random Variables Examples Contd.

Solution: (a) The integration of $f(x, y)$ over the whole region is

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy = \int_0^1\int_0^1 \frac{2}{5}(2x+3y)\,dx\,dy$$
$$= \int_0^1 \left(\frac{2x^2}{5} + \frac{6xy}{5}\right)\Big|_{x=0}^{x=1} dy$$
$$= \int_0^1 \left(\frac{2}{5} + \frac{6y}{5}\right) dy = \left(\frac{2y}{5} + \frac{3y^2}{5}\right)\Big|_0^1 = \frac{2}{5} + \frac{3}{5} = 1.$$

(b) To calculate the probability, we use

$$P[(X,Y) \in A] = P\left(0 < X < \frac{1}{2}, \frac{1}{4} < Y < \frac{1}{2}\right)$$
$$= \int_{1/4}^{1/2}\int_0^{1/2} \frac{2}{5}(2x+3y)\,dx\,dy$$
$$= \int_{1/4}^{1/2} \left(\frac{2x^2}{5} + \frac{6xy}{5}\right)\Big|_{x=0}^{x=1/2} dy = \int_{1/4}^{1/2} \left(\frac{1}{10} + \frac{3y}{5}\right) dy$$
$$= \left(\frac{y}{10} + \frac{3y^2}{10}\right)\Big|_{1/4}^{1/2}$$
$$= \frac{1}{10}\left[\left(\frac{1}{2} + \frac{3}{4}\right) - \left(\frac{1}{4} + \frac{3}{16}\right)\right] = \frac{13}{160}.$$

Marginal Distributions

The marginal distributions of X alone and of Y alone are

$$g(x) = \sum_{y} f(x, y) \quad \text{and} \quad h(y) = \sum_{x} f(x, y)$$

for the discrete case, and

$$g(x) = \int_{-\infty}^{\infty} f(x, y)\, dy \quad \text{and} \quad h(y) = \int_{-\infty}^{\infty} f(x, y)\, dx$$

for the continuous case.

Source : Probability & Statistics for Engineers & Scientists, by Ronald E. Walpole, Raymond H. Myers, Sharon L. Myers, Keying Ye, 9th Edition, Prentice Hall

Marginal Distributions Example

Example 3.16: Show that the column and row totals of Table 3.1 give the marginal distribution of X alone and of Y alone.

Solution: For the random variable X, we see that

$$g(0) = f(0,0) + f(0,1) + f(0,2) = \frac{3}{28} + \frac{3}{14} + \frac{1}{28} = \frac{5}{14},$$

$$g(1) = f(1,0) + f(1,1) + f(1,2) = \frac{9}{28} + \frac{3}{14} + 0 = \frac{15}{28},$$

and

$$g(2) = f(2,0) + f(2,1) + f(2,2) = \frac{3}{28} + 0 + 0 = \frac{3}{28},$$

which are just the column totals of Table 3.1. In a similar manner we could show that the values of $h(y)$ are given by the row totals. In tabular form, these marginal distributions may be written as follows:

x	0	1	2
$g(x)$	$\frac{5}{14}$	$\frac{15}{28}$	$\frac{3}{28}$

y	0	1	2
$h(y)$	$\frac{15}{28}$	$\frac{3}{7}$	$\frac{1}{28}$

Table 3.1: Joint Probability Distribution for Example 3.14

	$f(x,y)$	x = 0	1	2	Row Totals
y	0	$\frac{3}{28}$	$\frac{9}{28}$	$\frac{3}{28}$	$\frac{15}{28}$
	1	$\frac{3}{14}$	$\frac{3}{14}$	0	$\frac{3}{7}$
	2	$\frac{1}{28}$	0	0	$\frac{1}{28}$
Column Totals		$\frac{5}{14}$	$\frac{15}{28}$	$\frac{3}{28}$	1

Source : Probability & Statistics for Engineers & Scientists, by Ronald E. Walpole, Raymond H. Myers, Sharon L. Myers, Keying Ye, 9th Edition, Prentice Hall

Marginal Probability Mass Function

Definition 4.1-2

Let X and Y have the joint probability mass function $f(x, y)$ with space S. The probability mass function of X alone, which is called the **marginal probability mass function of** X, is defined by

$$f_X(x) = \sum_y f(x, y) = P(X = x), \qquad x \in S_X,$$

where the summation is taken over all possible y values for each given x in the x space S_X. That is, the summation is over all (x, y) in S with a given x value. Similarly, the **marginal probability mass function of** Y is defined by

$$f_Y(y) = \sum_x f(x, y) = P(Y = y), \qquad y \in S_Y,$$

where the summation is taken over all possible x values for each given y in the y space S_Y. The random variables X and Y are **independent** if and only if, for every $x \in S_X$ and every $y \in S_Y$,

$$P(X = x, Y = y) = P(X = x)P(Y = y)$$

or, equivalently,

$$f(x, y) = f_X(x)f_Y(y);$$

otherwise, X and Y are said to be **dependent**.

Source : PROBABILITY AND STATISTICAL INFERENCE by Robert V. Hogg, Elliot A. Tanis, Dale L. Zimmerman, 9th Edition

Marginal Probability Mass Function Example

Example 4.1-2

Let the joint pmf of X and Y be defined by

$$f(x,y) = \frac{x+y}{21}, \qquad x = 1,2,3, \qquad y = 1,2.$$

Then

$$f_X(x) = \sum_y f(x,y) = \sum_{y=1}^{2} \frac{x+y}{21}$$

$$= \frac{x+1}{21} + \frac{x+2}{21} = \frac{2x+3}{21}, \qquad x = 1,2,3,$$

and

$$f_Y(y) = \sum_x f(x,y) = \sum_{x=1}^{3} \frac{x+y}{21} = \frac{6+3y}{21} = \frac{2+y}{7}, \qquad y = 1,2.$$

Note that both $f_X(x)$ and $f_Y(y)$ satisfy the properties of a probability mass function. Since $f(x,y) \neq f_X(x)f_Y(y)$, X and Y are dependent.

Source : PROBABILITY AND STATISTICAL INFERENCE by Robert V. Hogg, Elliot A. Tanis, Dale L. Zimmerman, 9th Edition

Marginal Probability Mass Function Example Contd.

Example 4.1-3

Let the joint pmf of X and Y be

$$f(x, y) = \frac{xy^2}{30}, \qquad x = 1, 2, 3, \qquad y = 1, 2.$$

The marginal probability mass functions are

$$f_X(x) = \sum_{y=1}^{2} \frac{xy^2}{30} = \frac{x}{6}, \qquad x = 1, 2, 3,$$

and

$$f_Y(y) = \sum_{x=1}^{3} \frac{xy^2}{30} = \frac{y^2}{5}, \qquad y = 1, 2.$$

Then $f(x, y) = f_X(x)f_Y(y)$ for $x = 1, 2, 3$ and $y = 1, 2$; thus, X and Y are independent.

Source : PROBABILITY AND STATISTICAL INFERENCE by Robert V. Hogg, Elliot A. Tanis, Dale L. Zimmerman, 9th Edition

Additional Material

Source : http://homepage.stat.uiowa.edu/~rdecook/stat2020/notes/ch5_pt1.pdf

Accessed on 22nd August 2020

Recall a discrete probability distribution (or *pmf*) for a single *r.v.* X with the example below...

x	0	1	2
$f(x)$	0.50	0.20	0.30

Sometimes we're simultaneously interested in two or more variables in a random experiment. We're looking for a relationship between the two variables.

Examples for discrete *r.v.*'s

- Year in college vs. Number of credits taken
- Number of cigarettes smoked per day vs. Day of the week

Examples for continuous *r.v.*'s

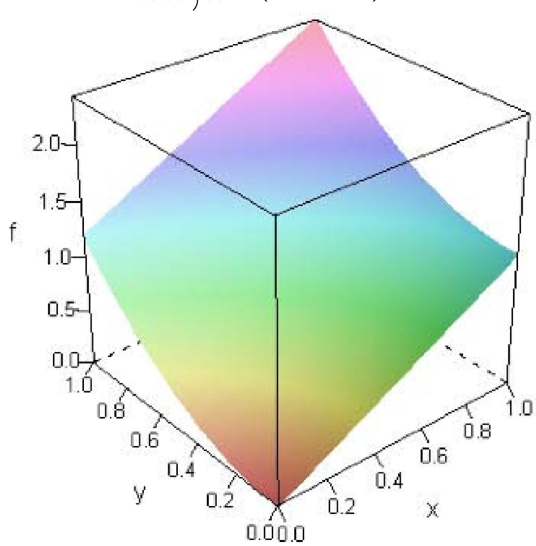
- Time when bus driver picks you up vs. Quantity of caffeine in bus driver's system
- Dosage of a drug (ml) vs. Blood compound measure (percentage)

In general, if X and Y are two random variables, the probability distribution that defines their simultaneous behavior is called a joint probability distribution.

Shown here as a table for two discrete random variables, which gives $P(X = x, Y = y)$.

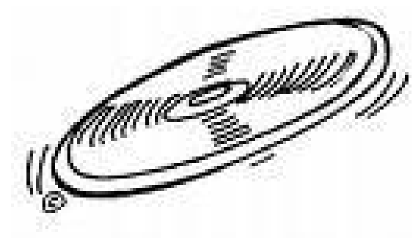
		x		
		1	2	3
	1	0	1/6	1/6
y	2	1/6	0	1/6
	3	1/6	1/6	0

Shown here as a graphic for two continuous random variables as $f_{X,Y}(x, y)$.



If X and Y are discrete, this distribution can be described with a joint probability mass function.

If X and Y are continuous, this distribution can be described with a joint probability density function.



- **Example**: Plastic covers for CDs (Discrete joint pmf)

  Measurements for the length and width of a rectangular plastic covers for CDs are rounded to the nearest mm (so they are discrete).

  Let X denote the length and
  Y denote the width.

  The possible values of X are 129, 130, and 131 mm. The possible values of Y are 15 and 16 mm (Thus, both X and Y are discrete).

There are 6 possible pairs (X, Y).

We show the probability for each pair in the following table:

x=length

		129	130	131
y=width	15	0.12	0.42	0.06
	16	0.08	0.28	0.04

The sum of all the probabilities is 1.0.

The combination with the highest probability is $(130, 15)$.

The combination with the lowest probability is $(131, 16)$.

The <u>joint probability mass function</u> is the function $f_{XY}(x, y) = P(X = x, Y = y)$. For example, we have $f_{XY}(129, 15) = 0.12$.

If we are given a joint probability distribution for X and Y, we can obtain the individual probability distribution for X or for Y (and these are called the **Marginal Probability Distributions**)...

- **Example**: Continuing plastic covers for CDs

  Find the probability that a CD cover has length of $129mm$ (i.e. $X = 129$).

		x= length		
		129	130	131
y=width	15	**0.12**	0.42	0.06
	16	**0.08**	0.28	0.04

$$\begin{aligned} P(X = 129) = & \; P(X = 129 \text{ and } Y = 15) \\ & + P(X = 129 \text{ and } Y = 16) \\ = & \; 0.12 \; + \; 0.08 = 0.20 \end{aligned}$$

What is the probability distribution of X?

		x= length		
		129	130	131
y=width	15	0.12	0.42	0.06
	16	0.08	0.28	0.04
column totals		**0.20**	**0.70**	**0.10**

The probability distribution for X appears in the column totals...

x	129	130	131
$f_X(x)$	0.20	0.70	0.10

* NOTE: We've used a subscript X in the probability mass function of X, or $f_X(x)$, for clarification since we're considering more than one variable at a time now.

We can do the same for the Y random variable:

		x= length			**row totals**
		129	130	131	
y=width	15	0.12	0.42	0.06	**0.60**
	16	0.08	0.28	0.04	**0.40**
column totals		**0.20**	**0.70**	**0.10**	**1**

y	15	16
$f_Y(y)$	0.60	0.40

Because the the probability mass functions for X and Y appear in the margins of the table (i.e. column and row totals), they are often referred to as the **Marginal Distributions** for X and Y.

When there are two random variables of interest, we also use the term **bivariate probability distribution** or **bivariate distribution** to refer to the joint distribution.

- **Joint Probability Mass Function**
  The joint probability mass function of the discrete random variables X and Y, denoted as $f_{XY}(x, y)$, satisfies

$$(1)\ \ f_{XY}(x, y) \geq 0$$
$$(2)\ \ \sum_x \sum_y f_{XY}(x, y) = 1$$
$$(3)\ \ f_{XY}(x, y) = P(X = x, Y = y)$$

For when the r.v.'s are discrete.

(Often shown with a 2-way table.)

		x= length		
		129	**130**	**131**
y=width	**15**	0.12	0.42	0.06
	16	0.08	0.28	0.04

- **Marginal Probability Mass Function**
  If X and Y are discrete random variables with joint probability mass function $f_{XY}(x,y)$, then the marginal probability mass functions of X and Y are

$$f_X(x) = \sum_y f_{XY}(x,y)$$

and

$$f_Y(y) = \sum_x f_{XY}(x,y)$$

where the sum for $f_X(x)$ is over all points in the range of (X,Y) for which $X = x$ and the sum for $f_Y(y)$ is over all points in the range of (X,Y) for which $Y = y$.

We found the marginal distribution for X in the CD example as...

x	129	130	131
$f_X(x)$	0.20	0.70	0.10

HINT: When asked for $E(X)$ or $V(X)$ (i.e. values related to only 1 of the 2 variables) but you are given a joint probability distribution, first calculate the marginal distribution $f_X(x)$ and work it as we did before for the univariate case (i.e. for a single random variable).

- **Example**: Batteries
  Suppose that 2 batteries are randomly chosen without replacement from the following group of 12 batteries:

  3 new
  4 used (working)
  5 defective

  Let X denote the number of new batteries chosen.

  Let Y denote the number of used batteries chosen.

a) Find $f_{XY}(x, y)$
{i.e. the joint probability distribution}.

~~b) Find $E(X)$.~~

ANS:

a) Though X can take on values 0, 1, and 2, and Y can take on values 0, 1, and 2, when we consider them jointly, $X + Y \leq 2$. So, not all combinations of (X, Y) are possible.

There are 6 possible cases...

CASE: no new, no used (so all defective)

$$f_{XY}(0,0) = \frac{\binom{5}{2}}{\binom{12}{2}} = 10/66$$

CASE: no new, 1 used

$$f_{XY}(0,1) = \frac{\binom{4}{1}\binom{5}{1}}{\binom{12}{2}} = 20/66$$

CASE: no new, 2 used

$$f_{XY}(0,2) = \frac{\binom{4}{2}}{\binom{12}{2}} = 6/66$$

CASE: 1 new, no used

$$f_{XY}(1,0) = \frac{\binom{3}{1}\binom{5}{1}}{\binom{12}{2}} = 15/66$$

CASE: 2 new, no used

$$f_{XY}(2,0) = \frac{\binom{3}{2}}{\binom{12}{2}} = 3/66$$

CASE: 1 new, 1 used

$$f_{XY}(1,1) = \frac{\binom{3}{1}\binom{4}{1}}{\binom{12}{2}} = 12/66$$

The joint distribution for X and Y is...

		x= number of *new* chosen		
		0	1	2
y=number of	0	10/66	15/66	3/66
used	1	20/66	12/66	
chosen	2	6/66		

There are 6 possible (X, Y) pairs.
And, $\sum_x \sum_y f_{XY}(x,y) = 1$.

- **Joint Probability Density Function**

  A <u>joint probability density function</u> for the continuous random variable X and Y, denoted as $f_{XY}(x,y)$, satisfies the following properties:

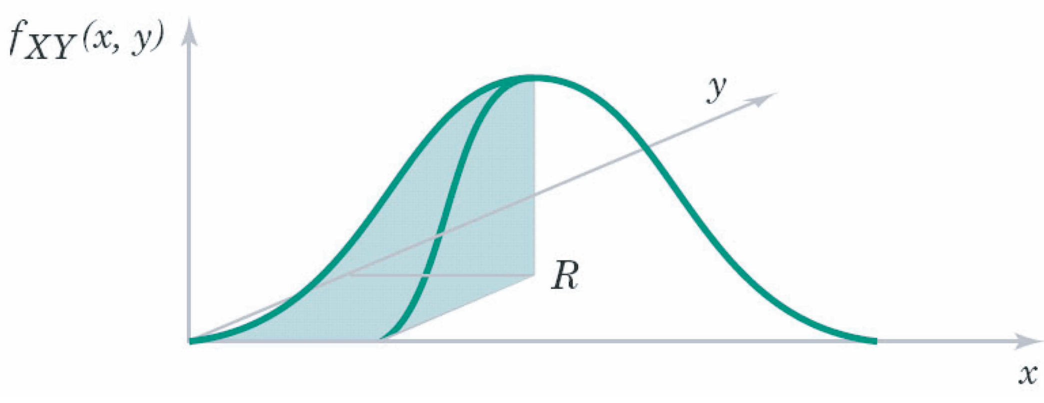
  1. $f_{XY}(x,y) \geq 0$ for all x, y

  2. $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{XY}(x,y)\,dx\,dy = 1$

  3. For any region R of 2-D space

  $$P((X,Y) \in R) = \int\int_R f_{XY}(x,y)\,dx\,dy$$

For when the r.v.'s are continuous.



- **Example**: Movement of a particle

  An article describes a model for the movement of a particle. Assume that a particle moves within the region A bounded by the x axis, the line $x = 1$, and the line $y = x$. Let (X, Y) denote the position of the particle at a given time. The joint density of X and Y is given by

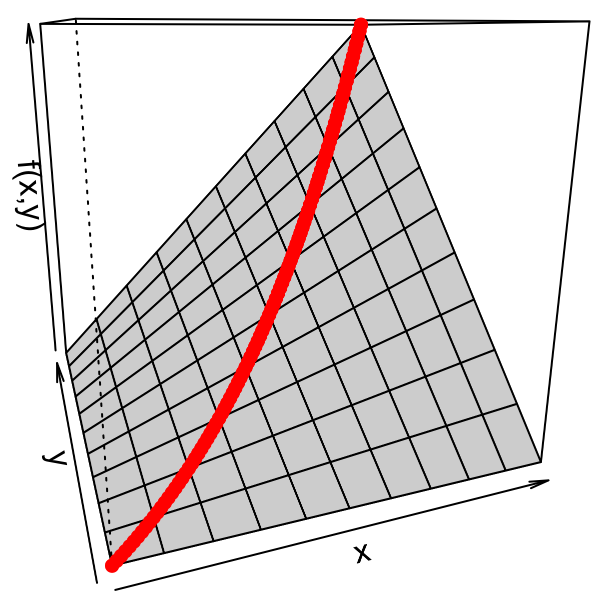
$$f_{XY}(x, y) = 8xy \quad \text{for} \quad (x, y) \in A$$

  a) Graphically show the region in the XY plane where $f_{XY}(x, y)$ is nonzero.

The probability density function $f_{XY}(x, y)$ is shown graphically below.

Without the information that $f_{XY}(x, y) = 0$ for (x, y) outside of A, we could plot the full surface, but the particle is only found in the given triangle A, so the joint probability density function is shown on the right.

This gives a volume under the surface that is above the region A equal to 1.



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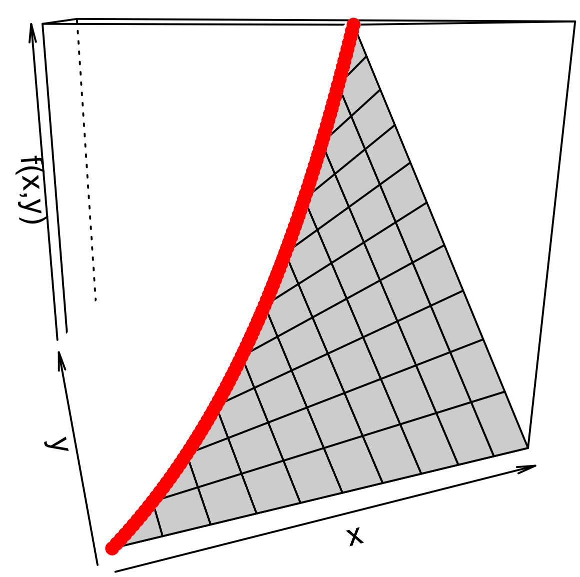

A *pdf*

- **Marginal Probability Density Function**

  If X and Y are continuous random variables with joint probability density function $f_{XY}(x, y)$, then the marginal density functions for X and Y are

  $$f_X(x) = \int_y f_{XY}(x, y)\, dy$$

  and

  $$f_Y(y) = \int_x f_{XY}(x, y)\, dx$$

  where the first integral is over all points in the range of (X, Y) for which $X = x$, and the second integral is over all points in the range of (X, Y) for which $Y = y$.