Probability and Statistics (IT302)

4th August 2020 (10:30AM-11:00AM) Class

Counting Sample Points

In many cases, we shall be able to solve a probability problem by counting the number of points in the sample space without actually listing each element.

The fundamental principle of counting, often referred to as the **Multiplication Rule.**

Multiplication Rule: If an operation can be performed in n_1 ways, and if for each of these ways a second operation can be performed in n_2 ways, then the two operations can be performed together in $n_1 \times n_2$ ways.

Source : Probability & Statistics for Engineers & Scientists, by Ronald E. Walpole, Raymond H. Myers, Sharon L. Myers, Keying Ye, 9th Edition, Prentice Hall

Multiplication Rule

Example 2.13: How many sample points are there in the sample space when a pair of dice is thrown once?

Solution: The first die can land face-up in any one of $n_1 = 6$ ways. For each of these 6 ways, the second die can also land face-up in $n_2 = 6$ ways. Therefore, the pair of dice can land in $n_1 n_2 = (6)(6) = 36$ possible ways.

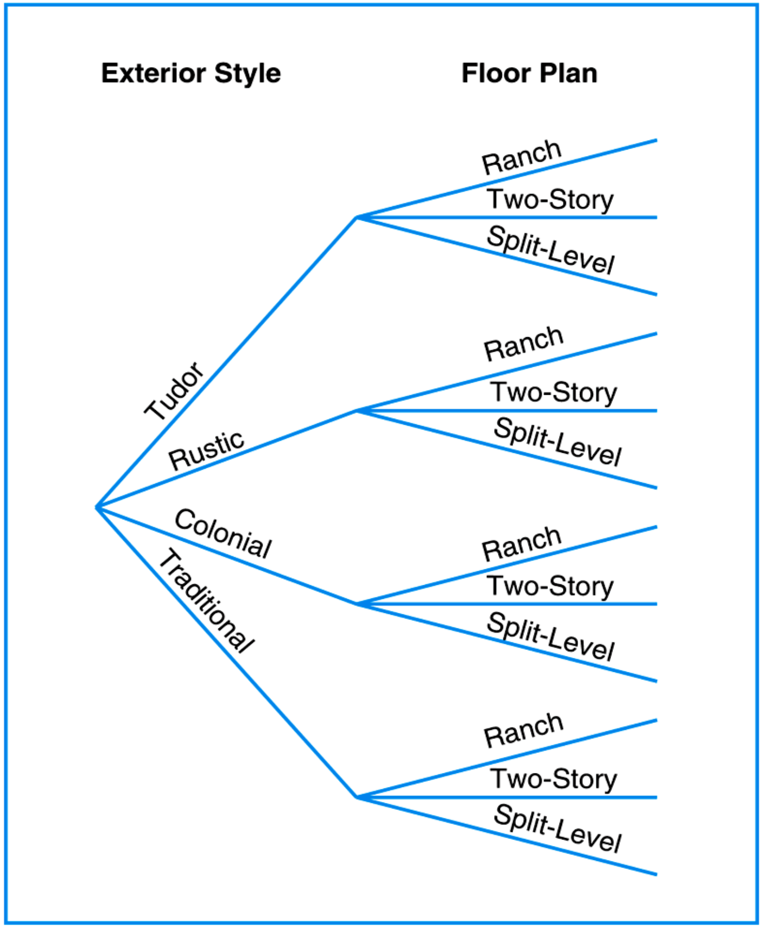
Source : Probability & Statistics for Engineers & Scientists, by Ronald E. Walpole, Raymond H. Myers, Sharon L. Myers, Keying Ye, 9th Edition, Prentice Hall

Multiplication Rule Contd.

Example 2.14: A developer of a new subdivision offers prospective home buyers a choice of **Tudor**, **Rustic**, **Colonial**, and **Traditional** exterior styling in **Ranch**, **Two-story**, and **Split-level** floor plans. In how many different ways can a buyer order one of these homes?

Solution: Since n_1=4 and n_2=3, a buyer must choose from n_1n_2=(4)(3)=12 possible homes.

The answers to the two preceding examples can be verified by constructing Tree diagrams and counting the various paths along the branches.



Source : Probability & Statistics for Engineers & Scientists, by Ronald E. Walpole, Raymond H. Myers, Sharon L. Myers, Keying Ye, 9th Edition, Prentice Hall

Multiplication Rule Contd.

Example 2.15: If a 22-member club needs to elect a chair and a treasurer, how many different ways can these two to be elected?

Solution: For the chair position, there are 22 total possibilities. For each of those 22 possibilities, there are 21 possibilities to elect the treasurer. Using the Multiplication Rule, we obtain $n_1 \times n_2 = 22 \times 21 = 462$ different ways.

Source : Probability & Statistics for Engineers & Scientists, by Ronald E. Walpole, Raymond H. Myers, Sharon L. Myers, Keying Ye, 9th Edition, Prentice Hall

Multiplication Rule Contd.

The Multiplication Rule may be extended to cover any number of operations. Suppose, for instance, that a customer wishes to buy a new cell phone and can choose from n_1=5 brands, n_2=5 sets of capability, and n_3=4 colors. These three classifications result in $n_1 n_2 n_3 = (5)(5)(4) = 100$ different ways for a customer to order one of these phones.

The **generalized Multiplication Rule** covering k operations is stated in the following.

If an operation can be performed in n_1 ways, and if for each of these a second operation can be performed in n_2 ways, and for each of the first two a third operation can be performed in n_3 ways, and so forth, then the sequence of k operations can be performed in $n_1 n_2 \cdot \cdot \cdot n_k$ ways.

Source : Probability & Statistics for Engineers & Scientists, by Ronald E. Walpole, Raymond H. Myers, Sharon L. Myers, Keying Ye, 9th Edition, Prentice Hall

Multiplication Rule Contd.

Example 2.16: Sam is going to assemble a computer by himself. He has the choice of chips from two brands, a hard drive from four, memory from three, and an accessory bundle from five local stores. How many different ways can Sam order the parts?

Solution: Since $n_1=2$, $n_2=4$, $n_3=3$, and $n_4=5$, there are
$n_l \times n_2 \times n_3 \times n_4 = 2 \times 4 \times 3 \times 5 = 120$ different ways to order the parts.

Source : Probability & Statistics for Engineers & Scientists, by Ronald E. Walpole, Raymond H. Myers, Sharon L. Myers, Keying Ye, 9th Edition, Prentice Hall

Multiplication Rule Contd.

Example 2.17: How many even four-digit numbers can be formed from the digits 0, 1, 2, 5, 6, and 9 if each digit can be used only once?

Solution: Since the number must be even, we have only $n_1=3$ choices for the units position. However, for a four-digit number the thousands position cannot be 0. Hence, we consider the units position in two parts, 0 or not 0. If the units position is 0 (i.e., $n_1=1$), we have $n_2=5$ choices for the thousands position, $n_3=4$ for the hundreds position, and $n_4=3$ for the tens position. Therefore, in this case we have a total of 60 even four-digit numbers.

$$n_1 n_2 n_3 n_4 = (1)(5)(4)(3) = 60$$

Source : Probability & Statistics for Engineers & Scientists, by Ronald E. Walpole, Raymond H. Myers, Sharon L. Myers, Keying Ye, 9th Edition, Prentice Hall

Multiplication Rule Contd.

Example 2.17 Contd.: On the other hand, if the units position is not 0 (i.e., n_1=2), we have n_2=4 choices for the thousands position, n_3=4 for the hundreds position, and n_4=3 for the tens position. In this situation, there are a total of 96

$$n_1 n_2 n_3 n_4 = (2)(4)(4)(3) = 96$$

Since the two cases are mutually exclusive, the total number of even four-digit numbers can be calculated as 60 + 96 = 156.

Source : Probability & Statistics for Engineers & Scientists, by Ronald E. Walpole, Raymond H. Myers, Sharon L. Myers, Keying Ye, 9^{th} Edition, Prentice Hall

Permutation

A **permutation** is an arrangement of all or part of a set of objects.

Consider the three letters *a*, *b*, and *c*. The possible permutations are *abc*, *acb*, *bac*, *bca*, *cab*, and *cba*. Thus, there are 6 distinct arrangements.

Using Multiplication Rule, we could arrive at the answer 6 without actually listing the different orders by the following arguments: There are $n_1=3$ choices for the first position. No matter which letter is chosen, there are always $n_2=2$ choices for the second position. No matter which two letters are chosen for the first two positions, there is only $n_3=1$ choice for the last position, giving a total of $n_1 n_2 n_3=(3)(2)(1)=6$ permutations.

In general, n distinct objects can be arranged in $n(n-1)(n-2) \cdot \cdot \cdot (3)(2)(1)$ ways.

Source : Probability & Statistics for Engineers & Scientists, by Ronald E. Walpole, Raymond H. Myers, Sharon L. Myers, Keying Ye, 9th Edition, Prentice Hall

Permutation Contd.

Theorem 2.2: The number of permutations of *n* distinct objects taken *r* at a time is

The number of permutations of n distinct objects taken r at a time is

$$
{}_nP_r = \frac{n!}{(n-r)!}.
$$

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Permutation Contd.

Example 2.18: In one year, three awards (research, teaching, and service) will be given to a class of 25 graduate students in a statistics department. If each student can receive at most one award, how many possible selections are there?

Solution: Since the awards are distinguishable, it is a permutation problem. The total number of sample points is

$${}_{25}P_3 = \frac{25!}{(25-3)!} = \frac{25!}{22!} = (25)(24)(23) = 13,800$$

Source : Probability & Statistics for Engineers & Scientists, by Ronald E. Walpole, Raymond H. Myers, Sharon L. Myers, Keying Ye, 9th Edition, Prentice Hall

Theorem 2.4

Theorem 2.4: The number of distinct permutations of n things of which n_1 are of one kind, n_2 of a second kind, . . . , n_k of a k^{th} kind is

The number of distinct permutations of n things of which n_1 are of one kind, n_2 of a second kind, $\ldots$, n_k of a kth kind is

$$\frac{n!}{n_1!n_2!\cdots n_k!}.$$

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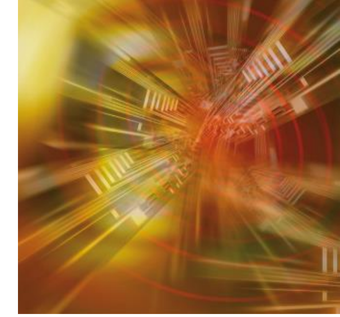
Example 2.20

Example 2.20: In a college football training session, the defensive coordinator needs to have 10 players standing in a row. Among these 10 players, there are 1 freshman, 2 sophomores, 4 juniors, and 3 seniors. How many different ways can they be arranged in a row if only their class level will be distinguished?

Solution: Directly using Theorem 2.4, we find that the total number of arrangements is

$$\frac{10!}{1!\,2!\,4!\,3!} = 12,600.$$

Source : Probability & Statistics for Engineers & Scientists, by Ronald E. Walpole, Raymond H. Myers, Sharon L. Myers, Keying Ye, 9th Edition, Prentice Hall



Often we are concerned with the number of ways of partitioning a set of n objects into r subsets called **cells**.

A partition has been achieved if the intersection of every possible pair of the r subsets is the empty set φ and if the union of all subsets gives the original set. The order of the elements within a cell is of no importance.

Consider the set *{a, e, i, o, u}*. **The possible partitions into two cells in which the first cell contains 4 elements** and **the second cell 1 element are** {(a, e, i, o), (u)}, {(a, i, o, u), (e)}, {(e, i, o, u), (a)}, {(a, e, o, u), (i)}, {(a, e, i, u), (o)}.

There are 5 ways to partition a set of 4 elements into two subsets, or cells, containing 4 elements in the first cell and 1 element in the second.

Source : Probability & Statistics for Engineers & Scientists, by Ronald E. Walpole, Raymond H. Myers, Sharon L. Myers, Keying Ye, 9th Edition, Prentice Hall

Theorem 2.5

The number of ways of partitioning a set of n objects into r cells with n_1 elements in the first cell, n_2 elements in the second, and so forth, is where $n_1 + n_2 + \cdot \cdot \cdot + n_r = n$.

The number of ways of partitioning a set of n objects into r cells with n_1 elements in the first cell, n_2 elements in the second, and so forth, is

$$\binom{n}{n_1, n_2, \ldots, n_r} = \frac{n!}{n_1! n_2! \cdots n_r!},$$

where $n_1 + n_2 + \cdots + n_r = n$.

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Example 2.21

Example 2.21: In how many ways can 7 graduate students be assigned to 1 triple and 2 double hotel rooms during a conference?

Solution: The total number of possible partitions would be

$$\binom{7}{3,2,2} = \frac{7!}{3!\,2!\,2!} = 210.$$

Source : Probability & Statistics for Engineers & Scientists, by Ronald E. Walpole, Raymond H. Myers, Sharon L. Myers, Keying Ye, 9th Edition, Prentice Hall

Theorem 2.6

The number of combinations of n distinct objects taken r at a time is

$$\binom{n}{r} = \frac{n!}{r!(n-r)!}.$$

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Exercises

Question 2.21: Registrants at a large convention are offered 6 sightseeing tours on each of 3 days. In how many ways can a person arrange to go on a sightseeing tour planned by this convention?

Answer: With $n_1 = 6$ sightseeing tours each available on $n_2 = 3$ different days, the multiplication rule gives $n_1 n_2 = (6)(3) = 18$ ways for a person to arrange a tour.

Source : Probability & Statistics for Engineers & Scientists, by Ronald E. Walpole, Raymond H. Myers, Sharon L. Myers, Keying Ye, 9th Edition, Prentice Hall

Exercises

Question 2.22: In a medical study, patients are classified in 8 ways according to whether they have blood type AB^+, AB^-, A^+, A^-, B^+, B^-, O^+, or O^-, and also according to whether their blood pressure is low, normal, or high. Find the number of ways in which a patient can be classified.

Answer: With $n_1 = 8$ blood types and $n_2 = 3$ classifications of blood pressure, the multiplication rule gives $n_1 n_2 = (8)(3) = 24$ classifications.

Source : Probability & Statistics for Engineers & Scientists, by Ronald E. Walpole, Raymond H. Myers, Sharon L. Myers, Keying Ye, 9th Edition, Prentice Hall

Exercises

Question 2.23 If an experiment consists of throwing a die and then drawing a letter at random from the English alphabet, how many points are there in the sample space?

Answer: Since the die can land in $n_1 = 6$ ways and a letter can be selected in $n_2 = 26$ ways, the multiplication rule gives $n_1 n_2 = (6)(26) = 156$ points in S.

Source : Probability & Statistics for Engineers & Scientists, by Ronald E. Walpole, Raymond H. Myers, Sharon L. Myers, Keying Ye, 9th Edition, Prentice Hall

Exercises

Question 2.24 Students at a private liberal arts college are classified as being freshmen, sophomores, juniors, or seniors, and also according to whether they are male or female. Find the total number of possible classifications for the students of that college.

Answer: Since a student may be classified according to $n_1=4$ class standing and $n_2=2$ gender classifications, the multiplication rule gives $n_1 n_2=(4)(2)=8$ possible classifications for the students.

Source : Probability & Statistics for Engineers & Scientists, by Ronald E. Walpole, Raymond H. Myers, Sharon L. Myers, Keying Ye, 9th Edition, Prentice Hall

Exercises

Question 2.25 A certain brand of shoes comes in 5 different styles, with each style available in 4 distinct colors. If the store wishes to display pairs of these shoes showing all of its various styles and colors, how many different pairs will the store have on display?

Answer: With n_1=5 different shoe styles in n_2=4 different colors, the multiplication rule gives n_1n_2=(5)(4)=20 different pairs of shoes.

Source : Probability & Statistics for Engineers & Scientists, by Ronald E. Walpole, Raymond H. Myers, Sharon L. Myers, Keying Ye, 9th Edition, Prentice Hall

Exercises

Question 2.27 A developer of a new subdivision offers a prospective home buyer a choice of 4 designs, 3 different heating systems, a garage or carport, and a patio or screened porch. How many different plans are available to this buyer?

Answer: Using the generalized multiplication rule, there are $n_1 \times n_2 \times n_3 \times n_4 = (4)(3)(2)(2) = 48$ different house plans available

Source : Probability & Statistics for Engineers & Scientists, by Ronald E. Walpole, Raymond H. Myers, Sharon L. Myers, Keying Ye, 9th Edition, Prentice Hall

Exercises

Question 2.29 In a fuel economy study, each of 3 race cars is tested using 5 different brands of gasoline at 7 test sites located in different regions of the country. If 2 drivers are used in the study, and test runs are made once under each distinct set of conditions, how many test runs are needed?

Answer: With n_1=3 race cars, n_2=5 brands of gasoline, n_3=7 test sites, and n_4=2 drivers, the generalized multiplication rule yields (3)(5)(7)(2)=210 test runs.

Source : Probability & Statistics for Engineers & Scientists, by Ronald E. Walpole, Raymond H. Myers, Sharon L. Myers, Keying Ye, 9th Edition, Prentice Hall

Exercises

Question 2.30 In how many different ways can a true-false test consisting of 9 questions be answered

Answer: With n_1=2 choices for the first question, n_2=2 choices for the second question, and so forth, the generalized multiplication rule yields $n_1 n_2 \cdots n_9 = 2^9 = 512$ ways to answer the test.

Source : Probability & Statistics for Engineers & Scientists, by Ronald E. Walpole, Raymond H. Myers, Sharon L. Myers, Keying Ye, 9th Edition, Prentice Hall

Exercises

Question 2.33 If a multiple-choice test consists of 5 questions, each with 4 possible answers of which only 1 is correct,

(a) in how many different ways can a student check off one answer to each question?

Answer: With n_1=4 possible answers for the first question, n_2=4 possible answers for the second question, and so forth, the generalized multiplication rule yields $4^5 = 1024$ ways to answer the test.

(b) in how many ways can a student check off one answer to each question and get all the answers wrong?

Answer: With n_1=3 wrong answers for the first question, n_2=3 wrong answers for the second question, and so forth, the generalized multiplication rule yields $n_1 n_2 n_3 n_4 n_5 = (3)(3)(3)(3)(3) = 3^5 = 243$ ways to answer the test and get all questions wrong.

Source : Probability & Statistics for Engineers & Scientists, by Ronald E. Walpole, Raymond H. Myers, Sharon L. Myers, Keying Ye, 9th Edition, Prentice Hall

Exercises

Question 2.35: A contractor wishes to build 9 houses, each different in design. In how many ways can he place these houses on a street if 6 lots are on one side of the street and 3 lots are on the opposite side?

Answer: The first house can be placed on any of the n_1=9 lots, the second house on any of the remaining n_2=8 lots, and so forth. Therefore, there are 9! = 362,880 ways to place the 9 homes on the 9 lots

Source : Probability & Statistics for Engineers & Scientists, by Ronald E. Walpole, Raymond H. Myers, Sharon L. Myers, Keying Ye, 9th Edition, Prentice Hall

Exercises

Question 2.3: In how many ways can 4 boys and 5 girls sit in a row if the boys and girls must alternate?

Answer: The first seat must be filled by any of 5 girls and the second seat by any of 4 boys. Continuing

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Exercises

Question 2.38 Four married couples have bought 8 seats in the same row for a concert. In how many different ways can they be seated

(a) with no restrictions?

Answer: $8! = 40320$.

(b) if each couple is to sit together?

Answer: There are $4!$ ways to seat 4 couples and then each member of a couple can be interchanged resulting in $2^4(4!) = 384$ ways.

Source : Probability & Statistics for Engineers & Scientists, by Ronald E. Walpole, Raymond H. Myers, Sharon L. Myers, Keying Ye, 9th Edition, Prentice Hall

Exercises

Question 2.39: In a regional spelling bee, the 8 finalists consist of 3 boys and 5 girls. Find the number of sample points in the sample space *S* for the number of possible orders at the conclusion of the contest for all 8 finalists.

Answer: Any of the $n_1 = 8$ finalists may come in first, and of the $n_2=7$ remaining finalists can then come in second, and so forth. there $8!=40320$ possible orders in which 8 finalists may finish the spelling bee.

Source : Probability & Statistics for Engineers & Scientists, by Ronald E. Walpole, Raymond H. Myers, Sharon L. Myers, Keying Ye, 9th Edition, Prentice Hall