Probability and Statistics (IT302)

17th August 2020 Monday 09:45AM-10:15AM

Class 6

Probability Function, Probability Mass Function or Probability Distribution

The set of ordered pairs $(x, f(x))$ is a **probability function**, **probability mass function**, or **probability distribution** of the discrete random variable X if, for each possible outcome x,

1. $f(x) \geq 0$,
2. $\sum_x f(x) = 1$,
3. $P(X = x) = f(x)$.

Source : Probability & Statistics for Engineers & Scientists, by Ronald E. Walpole, Raymond H. Myers, Sharon L. Myers, Keying Ye, 9th Edition, Prentice Hall, PP. 84

Probability Function, Probability Mass Function or Probability Distribution Contd.

Example 3.2: A stockroom clerk returns three safety helmets at random to three steel mill employees who had previously checked them. If Smith, Jones, and Brown, in that order, receive one of the three hats, list the sample points for the possible orders of returning the helmets, and find the value *m* of the random variable *M* that represents the number of correct matches.

Solution **:** If *S*, *J*, and *B* stand for Smith's, Jones's, and Brown's helmets, respectively, then the possible arrangements in which the helmets may be returned and the number of correct matches are

Sample Space	m
SJB	3
SBJ	1
BJS	1
JSB	1
JBS	0
BSJ	0

If one assumes equal weights for the simple events, the probability that no employee gets back the right helmet, that is, the probability that *M* assumes the value 0, is 1/3. The possible values *m* of *M* and their probabilities are

m	0	1	3
$P(M = m)$	$\frac{1}{3}$	$\frac{1}{2}$	$\frac{1}{6}$

Note that the values of *m* exhaust all possible cases and hence the probabilities add to 1. Frequently, it is convenient to represent all the probabilities of a random variable *X* by a formula. Therefore, we write $f(x) = P(X = x)$; that is, $f(3) = P(X = 3)$. The set of ordered pairs $(x, f(x))$ is called the **probability function**, **probability mass function**, or **probability distribution** of the discrete random variable *X*

Source : Probability & Statistics for Engineers & Scientists, by Ronald E. Walpole, Raymond H. Myers, Sharon L. Myers, Keying Ye, 9th Edition, Prentice Hall

Probability Function, Probability Mass Function or Probability Distribution Example

Example 3.8: A shipment of 20 similar laptop computers to a retail outlet contains 3 that are defective. If a school makes a random purchase of 2 of these computers, find the probability distribution for the number of defectives.

Solution **:** Let *X* be a random variable whose values *x* are the possible numbers of defective computers purchased by the school. Then *x* can only take the numbers 0, 1, and 2

2. Now

$$f(0) = P(X = 0) = \frac{\binom{3}{0}\binom{17}{2}}{\binom{20}{2}} = \frac{68}{95}, \quad f(1) = P(X = 1) = \frac{\binom{3}{1}\binom{17}{1}}{\binom{20}{2}} = \frac{51}{190}$$

$$f(2) = P(X = 2) = \frac{\binom{3}{2}\binom{17}{0}}{\binom{20}{2}} = \frac{3}{190}.$$

Thus, the probability distribution of X is

x	0	1	2
$f(x)$	$\frac{68}{95}$	$\frac{51}{190}$	$\frac{3}{190}$

Source : Probability & Statistics for Engineers & Scientists, by Ronald E. Walpole, Raymond H. Myers, Sharon L. Myers, Keying Ye, 9th Edition, Prentice Hall, PP. 85

Example

Example 3.9: If a car agency sells 50% of its inventory of a certain foreign car equipped with side airbags, find a formula for the probability distribution of the number of cars with side airbags among the next 4 cars sold by the agency.

***Solution*:** Since the probability of selling an automobile with side airbags is 0.5, the $2^4 = 16$ points in the sample space are equally likely to occur. Therefore, the denominator for all probabilities, and also for our function, is 16. To obtain the number of ways of selling 3 cars with side airbags, we need to consider the number of ways of partitioning 4 outcomes into two cells, with 3 cars with side airbags assigned to one cell and the model without side airbags assigned to the other. This can be done in $\binom{4}{3} = 4$ ways. In general, the event of selling x models with side airbags and $4 - x$ models without side airbags can occur in $\binom{4}{x}$ ways, where x can be 0, 1, 2, 3, or 4. Thus, the probability distribution $f(x) = P(X = x)$ is

$$f(x) = \frac{1}{16}\binom{4}{x}, \quad \text{for } x = 0, 1, 2, 3, 4.$$

Source : Probability & Statistics for Engineers & Scientists, by Ronald E. Walpole, Raymond H. Myers, Sharon L. Myers, Keying Ye, 9th Edition, Prentice Hall, PP. 85

Example

The number of patients seen in the ER in any given hour is a random variable represented by x. The probability distribution for x is:

x	10	11	12	13	14
$P(x)$	.4	.2	.2	.1	.1

Find the probability that in a given hour:

a. exactly 14 patients arrive $p(x=14)= .1$

b. At least 12 patients arrive $p(x\geq 12)= (.2 + .1 +.1) = .4$

c. At most 11 patients arrive $p(x\leq 11)= (.4 +.2) = .6$

Source : http://home.ubalt.edu/ntsbarsh/lecture2.ppt Accessed on 8th August 2020

Cumulative Distribution Function

There are many problems where we may wish to compute the probability that the observed value of a random variable X will be less than or equal to some real number x. Writing $F(x) = P(X \leq x)$ for every real number x, we define $F(x)$ to be the **Cumulative Distribution Function** of the random variable X.

The **cumulative distribution function** $F(x)$ of a discrete random variable X with probability distribution $f(x)$ is

$$F(x) = P(X \leq x) = \sum_{t \leq x} f(t), \quad \text{for } -\infty < x < \infty.$$

One should pay particular notice to the fact that the cumulative distribution function is a monotone nondecreasing function defined not only for the values assumed by the given random variable but for all real numbers.

Source : Probability & Statistics for Engineers & Scientists, by Ronald E. Walpole, Raymond H. Myers, Sharon L. Myers, Keying Ye, 9th Edition, Prentice Hall, PP. 85

Cumulative Distribution Function Contd.

For the random variable M, the number of correct matches in Example 3.2, we have

$$F(2) = P(M \leq 2) = f(0) + f(1) = \frac{1}{3} + \frac{1}{2} = \frac{5}{6}.$$

The cumulative distribution function of M is

$$F(m) = \begin{cases} 0, & \text{for } m < 0, \\ \frac{1}{3}, & \text{for } 0 \leq m < 1, \\ \frac{5}{6}, & \text{for } 1 \leq m < 3, \\ 1, & \text{for } m \geq 3. \end{cases}$$

Sample Space	m
SJB	3
SBJ	1
BJS	1
JSB	1
JBS	0
BSJ	0

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Cumulative Distribution Function Example 3.10:

Example 3.10: Find the cumulative distribution function of the random variable X in Example 3.9. Using $F(x)$, verify that $f(2) = 3/8$.

Solution **:** Direct calculations of the probability distribution of Example 3.9 give $f(0)=1/16$, $f(1) = 1/4$, $f(2)= 3/8$, $f(3)= 1/4$, and $f(4)= 1/16$. Therefore,

$F(0) = f(0) =1/16$

$F(1) = f(0) + f(1) =5/16$

$F(2) = f(0) + f(1) + f(2) =11/16$

$F(3) = f(0) + f(1) + f(2) + f(3) =15/16$

$F(4) = f(0) + f(1) + f(2) + f(3) + f(4) = 1.$

Hence,

$$F(x) = \begin{cases} 0, & \text{for } x < 0, \\ \frac{1}{16}, & \text{for } 0 \le x < 1, \\ \frac{5}{16}, & \text{for } 1 \le x < 2, \\ \frac{11}{16}, & \text{for } 2 \le x < 3, \\ \frac{15}{16}, & \text{for } 3 \le x < 4, \\ 1 & \text{for } x \ge 4. \end{cases}$$

Now $f(2) = F(2) - F(1) = 11/16 - 5/16 = 3/8.$

Source : Probability & Statistics for Engineers & Scientists, by Ronald E. Walpole, Raymond H. Myers, Sharon L. Myers, Keying Ye, 9th Edition, Prentice Hall

Probability Mass Function Plot

It is often helpful to look at a probability distribution in graphic form. One might plot the points (x, $f(x)$) of Example 3.9 ($f(0)=1/16$, $f(1) = 4/16$, $f(2)= 6/16$, $f(3)= 4/16$, and $f(4)= 1/16$) to obtain below Figure. By joining the points to the x axis either with a dashed or with a solid line, we obtain a **Probability Mass Function** plot. Figure makes it easy to see what values of X are most likely to occur, and it also indicates a perfectly symmetric situation in this case.

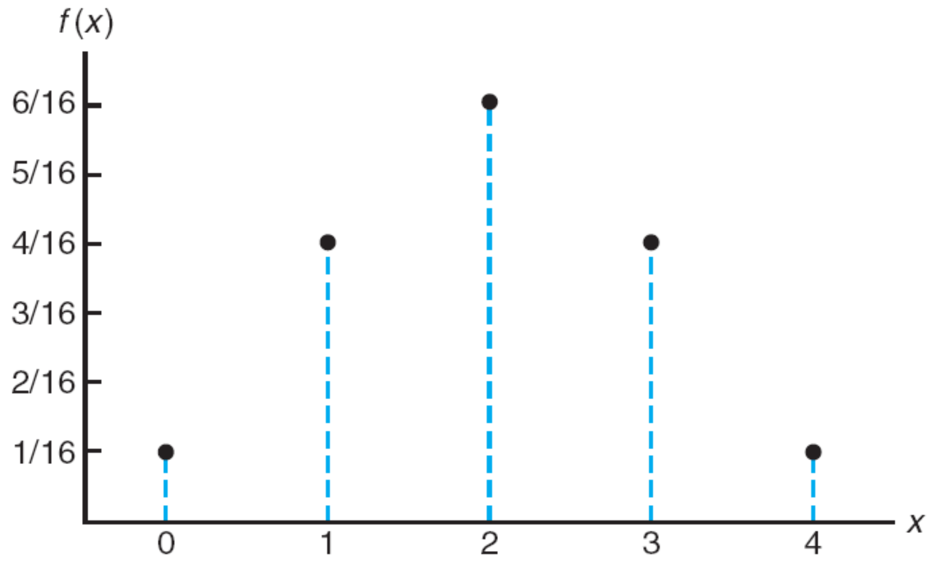
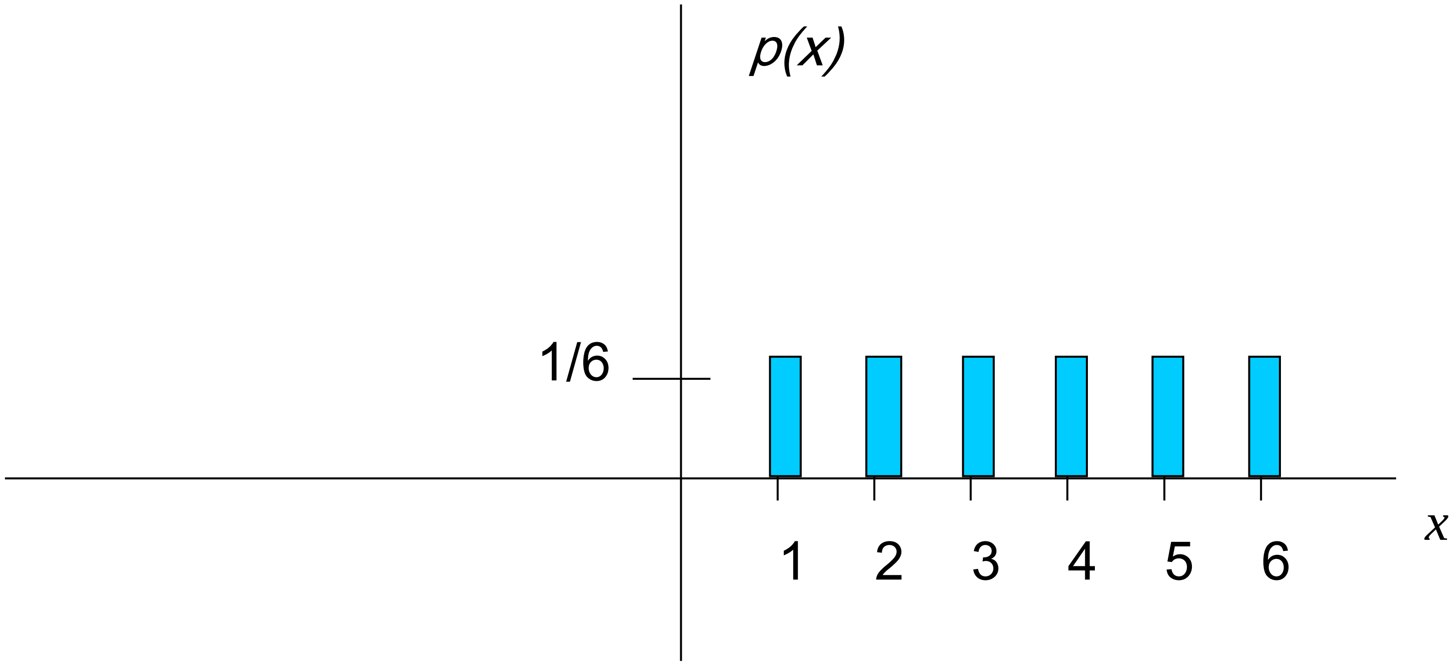


Figure Probability Mass Function plot

Source : Probability & Statistics for Engineers & Scientists, by Ronald E. Walpole, Raymond H. Myers, Sharon L. Myers, Keying Ye, 9th Edition, Prentice Hall

Probability Function, Probability Mass Function or Probability Distribution Example

Discrete example: roll of a die



x	$p(x)$
1	$p(x=1)=1/6$
2	$p(x=2)=1/6$
3	$p(x=3)=1/6$
4	$p(x=4)=1/6$
5	$p(x=5)=1/6$
6	$p(x=6)=1/6$
	1.0

$$\sum_{\text{all } x} P(x) = 1$$

Source : http://home.ubalt.edu/ntsbarsh/lecture2.ppt Accessed on 8th August 2020

Probability Histogram.

Instead of plotting the points $(x, f(x))$, more frequently construct rectangles, as in below Figure. Here the rectangles are constructed so that their bases of equal width are centered at each value x and their heights are equal to the corresponding probabilities given by $f(x)$. The bases are constructed so as to leave no space between the rectangles. Figure is called a **Probability Histogram**.

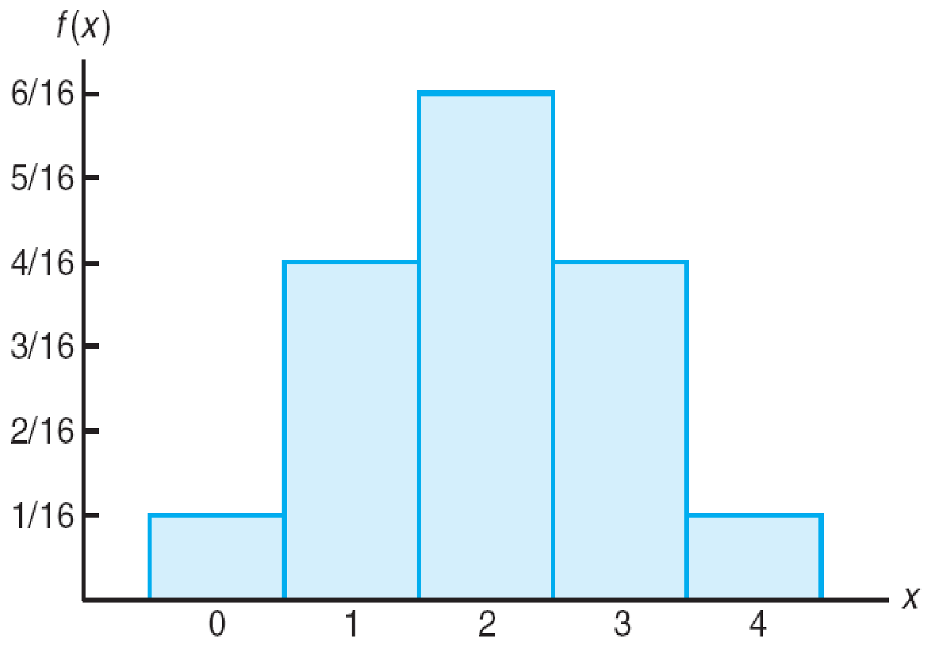
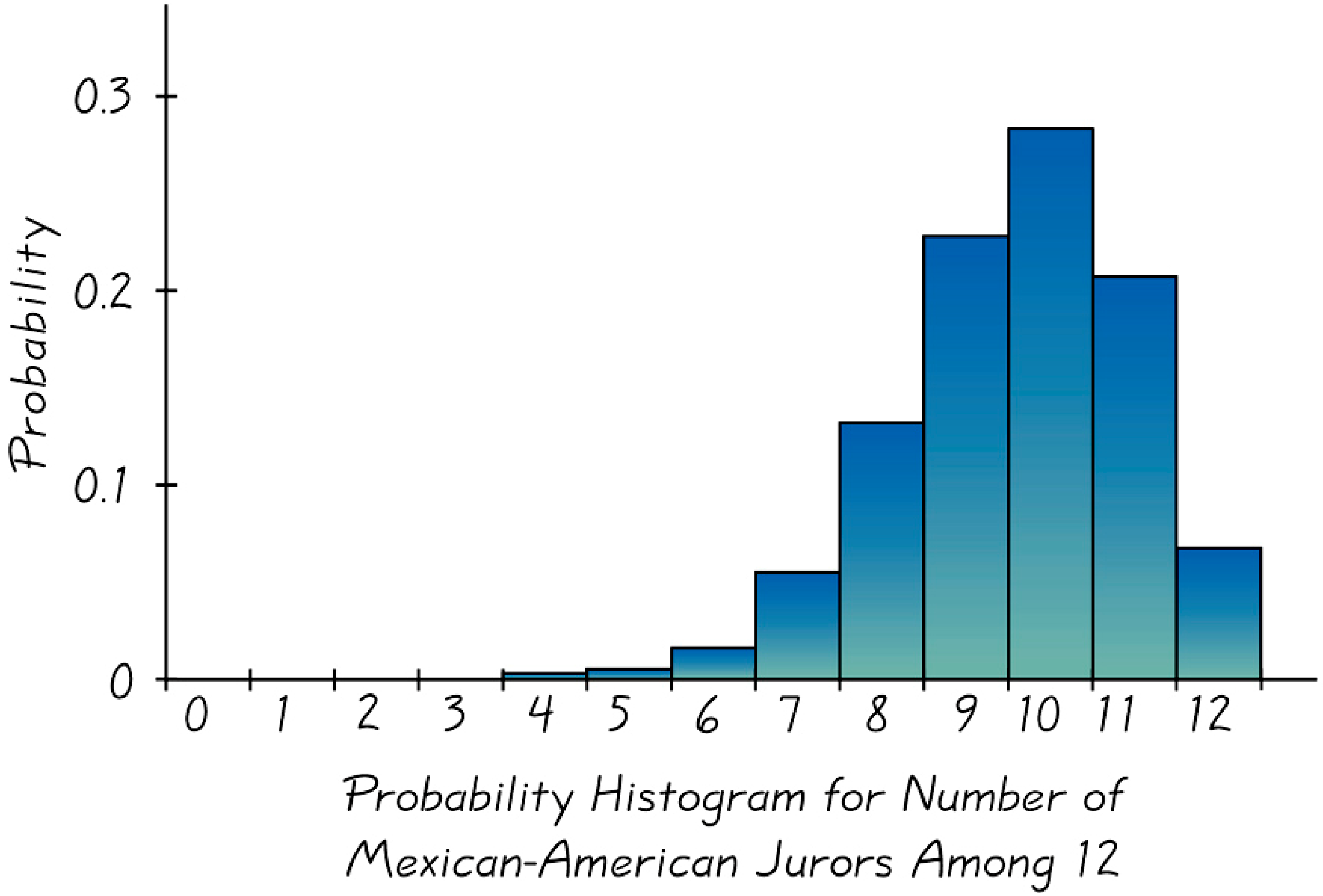


Figure Probability Histogram

Source : Probability & Statistics for Engineers & Scientists, by Ronald E. Walpole, Raymond H. Myers, Sharon L. Myers, Keying Ye, 9th Edition, Prentice Hall

Probability Histogram.

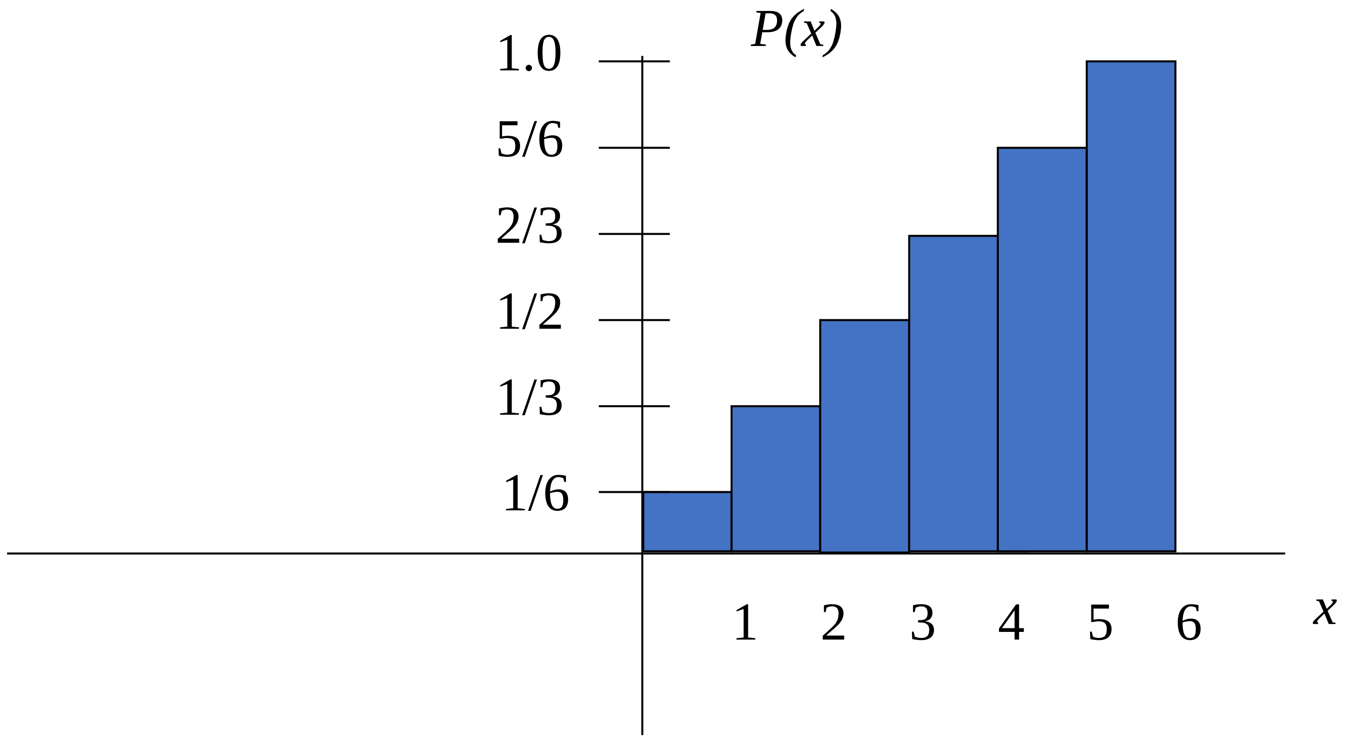
The Probability Histogram is very similar to a relative frequency histogram, but the vertical scale shows Probabilities.



Source: http://www.shsu.edu/~jga001/alford_chapter5.ppt Accessed on 8th August 2020

Cumulative Distribution Function Example

Discrete example: Roll of a Die



x	$P(x \le A)$
1	$P(x \le 1) = 1/6$
2	$P(x \le 2) = 2/6$
3	$P(x \le 3) = 3/6$
4	$P(x \le 4) = 4/6$
5	$P(x \le 5) = 5/6$
6	$P(x \le 6) = 6/6$

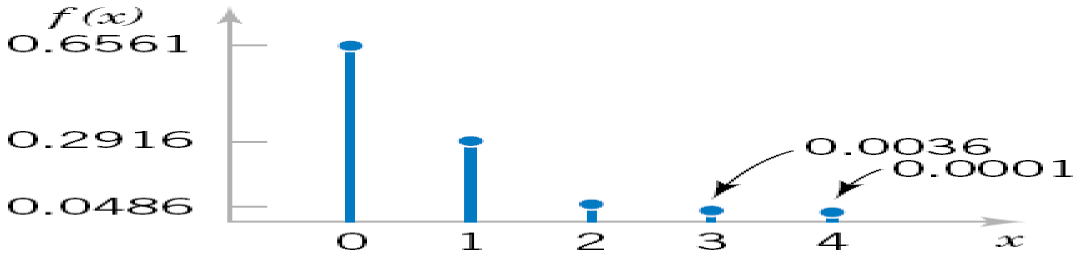
Source : http://home.ubalt.edu/ntsbarsh/lecture2.ppt Accessed on 8th August 2020

Probability Distribution in Graph Form

There is a chance that a bit transmitted through a digital transmission channel is received in error. Let X equal the number of bits in error in the next four bits transmitted. The possible values for X are {0, 1, 2, 3, 4}. Based on a model for the errors that is presented in the following section, probabilities for these values will be determined. Suppose that the probabilities are

$$P(X=0)=0.6561,\ P(X=1)=0.2916,\ P(X=2)=0.0486,\ P(X=3)=0.0036,\ P(X=4)=0.0001$$

The probability distribution of X is specified by the possible values along with the probability of each. A graphical description of the probability distribution of X is shown in Fig. 3.



Source : Applied Statistics and Probability for Engineers by Douglas C. Montgomery and George C. Runger, Third Edition, PP. 61

Exercise Problem No. 3.10

Question 3.10 Find a formula for the probability distribution of the random variable *X* representing the outcome when a single die is rolled once.

Solution : The die can land in 6 different ways each with probability 1/6. Therefore, $f(x)=1/6$, for $x = 1, 2, . . . , 6$.

Exercise Problem No. 3.11

Question 3.11 A shipment of 7 television sets contains 2 defective sets. A hotel makes a random purchase of 3 of the sets. If *x* is the number of defective sets purchased by the hotel, find the Probability Distribution of *X*. Express the results graphically as a probability histogram.

Solution :

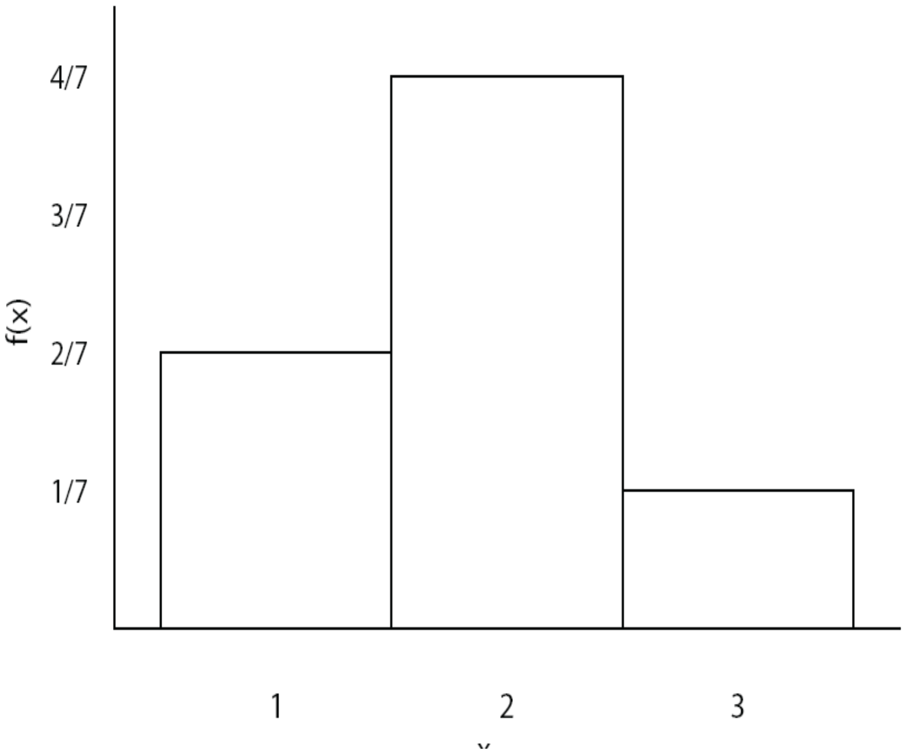
We can select x defective sets from 2, and $3-x$ good sets from 5 in $\binom{2}{x}\binom{5}{3-x}$ ways. A random selection of 3 from 7 sets can be made in $\binom{7}{3}$ ways. Therefore,

$$f(x) = \frac{\binom{2}{x}\binom{5}{3-x}}{\binom{7}{3}}, \qquad x = 0, 1, 2.$$

In tabular form

x	0	1	2
$f(x)$	2/7	4/7	1/7

The following is a probability histogram:



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Exercise Problem No. 3.11

Question 3.11 A shipment of 7 television sets contains 2 defective sets. A hotel makes a random purchase of 3 of the sets. If x is the number of defective sets purchased by the hotel, Find the cumulative distribution function of the random variable X representing the number of defective. Then using $F(x)$, find (a) $P(X = 1)$; (b) $P(0 < X \leq 2)$.

Solution :

The c.d.f. of X is

$$F(x) = \begin{cases} 0, & \text{for } x < 0, \\ 2/7, & \text{for } 0 \leq x < 1, \\ 6/7, & \text{for } 1 \leq x < 2, \\ 1, & \text{for } x \geq 2. \end{cases}$$

(a) $P(X = 1) = P(X \leq 1) - P(X \leq 0) = 6/7 - 2/7 = 4/7;$

(b) $P(0 < X \leq 2) = P(X \leq 2) - P(X \leq 0) = 1 - 2/7 = 5/7.$

Source : Probability & Statistics for Engineers & Scientists, by Ronald E. Walpole, Raymond H. Myers, Sharon L. Myers, Keying Ye, 9th Edition, Prentice Hall

Exercise Problem No. 3.13

Question 3.13 The probability distribution of *X*, the number of imperfections per 10 meters of a synthetic fabric in continuous rolls of uniform width, is given by

x	0	1	2	3	4
$f(x)$	0.41	0.37	0.16	0.05	0.01

Construct the cumulative distribution function of *X*.

Solution: CDF of X is

$$F(x) = \begin{cases} 0, & \text{for } x < 0, \\ 0.41, & \text{for } 0 \le x < 1, \\ 0.78, & \text{for } 1 \le x < 2, \\ 0.94, & \text{for } 2 \le x < 3, \\ 0.99, & \text{for } 3 \le x < 4, \\ 1, & \text{for } x \ge 4. \end{cases}$$

Source : Probability & Statistics for Engineers & Scientists, by Ronald E. Walpole, Raymond H. Myers, Sharon L. Myers, Keying Ye, 9th Edition, Prentice Hall

Additional Slides for Probability Mass Function (PMF)

3.1.2 Probability mass functions (PMFs)

For any discrete random variable X, we define the *probability mass function* (PMF) to be the function which gives the probability of each $x \in S_X$. Clearly we have

$$P(X = x) = \sum_{\{s \in S | X(s) = x\}} P(\{s\}).$$

That is, the probability of getting a particular number is the sum of the probabilities of all those outcomes which have that number associated with them. Also $P(X = x) \geq 0$ for each $x \in S_X$, and $P(X = x) = 0$ otherwise. The set of all pairs $\{(x, P(X = x)) | x \in S_X\}$ is known as the *probability distribution* of X.

Source: https://www.staff.ncl.ac.uk/d.j.wilkinson/teaching/mas131/notes.pdf, Accessed on 8th August 2020

Additional Slides for Probability Mass Function (PMF)

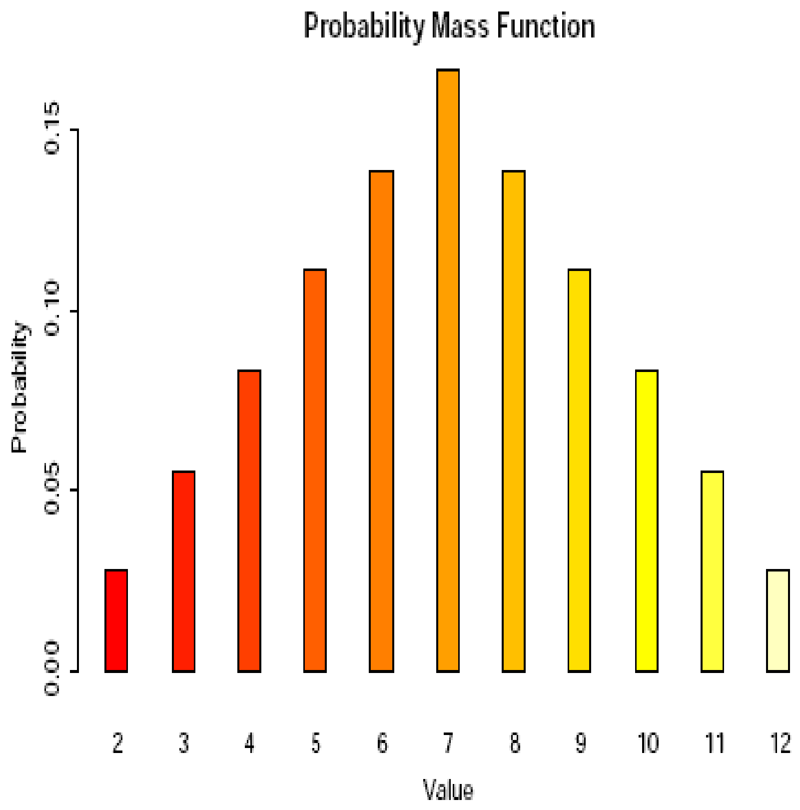
Example

For the example above concerning the sum of two dice, the probability distribution is

$$\{(2,1/36),(3,2/36),(4,3/36),(5,4/36),(6,5/36),(7,6/36),$$
$$(8,5/36),(9,4/36),(10,3/36),(11,2/36),(12,1/36)\}$$

and the probability mass function can be tabulated as

x	2	3	4	5	6	7	8	9	10	11	12
$P(X=x)$	1/36	2/36	3/36	4/36	5/36	6/36	5/36	4/36	3/36	2/36	1/36



Source: https://www.staff.ncl.ac.uk/d.j.wilkinson/teaching/mas131/notes.pdf, Accessed on 8th August 2020

Additional Slides for Cumulative Distribution Functions (CDFs)

For any discrete random quantity, X, we clearly have

$$\sum_{x \in S_X} \mathrm{P}(X = x) = 1$$

as every outcome has some number associated with it. It can often be useful to know the probability that your random number is no greater than some particular value. With that in mind, we define the *cumulative distribution function*,

$$F_X(x) = \mathrm{P}(X \leq x) = \sum_{\{y \in S_X | y \leq x\}} \mathrm{P}(X = y).$$

Source: https://www.staff.ncl.ac.uk/d.j.wilkinson/teaching/mas131/notes.pdf, Accessed on 8th August 2020

Additional Slides for Cumulative distribution functions (CDFs)

Example

For the sum of two dice, the CDF can be tabulated for the outcomes as

x	2	3	4	5	6	7	8	9	10	11	12
$F_X(x)$	1/36	3/36	6/36	10/36	15/36	21/36	26/36	30/36	33/36	35/36	36/36

but it is important to note that the CDF is defined *for all real numbers* — not just the possible values. In our example we have

$$\begin{aligned} F_X(-3) &= \mathrm{P}(X \leq -3) = 0, \\ F_X(4.5) &= \mathrm{P}(X \leq 4.5) = \mathrm{P}(X \leq 4) = 6/36, \\ F_X(25) &= \mathrm{P}(X \leq 25) = 1. \end{aligned}$$

Source: https://www.staff.ncl.ac.uk/d.j.wilkinson/teaching/mas131/notes.pdf, Accessed on 8th August 2020