Probability and Statistics (IT302)

7th September 2020 Monday 09:45 AM - 10:15AM

Class 14

Means and Variances of Linear Combinations of Random Variables

Theorem 4.5: If a and b are constants, then

$$E(aX + b) = aE(X) + b.$$

***Proof*:** By the definition of expected value,

$$E(aX + b) = \int_{-\infty}^{\infty} (ax + b) f(x)\ dx = a \int_{-\infty}^{\infty} x f(x)\ dx + b \int_{-\infty}^{\infty} f(x)\ dx.$$

The first integral on the right is $E(X)$ and the second integral equals 1. Therefore, we have

$$E(aX + b) = aE(X) + b.$$

Corollary 4.1: Setting $a = 0$, we see that $E(b) = b$.

Corollary 4.2: Setting $b = 0$, we see that $E(aX) = aE(X)$.

Source : Probability & Statistics for Engineers & Scientists, by Ronald E. Walpole, Raymond H. Myers, Sharon L. Myers, Keying Ye, 9th Edition, Prentice Hall.

Example 4.17

Assume that the number of cars X that pass through a car wash between 4:00 P.M. and 5:00 P.M. on any sunny Friday has the following probability distribution:

x	4	5	6	7	8	9
$P(X = x)$	1/12	1/12	¼	¼	1/6	1/6

Apply Theorem 4.5 to the discrete random variable $f(X) = 2X - 1$.

Solution: According to Theorem 4.5, we can write

$$E(2X - 1) = 2E(X) - 1.$$

Now

$$\mu = E(X) = \sum_{x=4}^{9} x f(x)$$

$$= (4)\left(\frac{1}{12}\right) + (5)\left(\frac{1}{12}\right) + (6)\left(\frac{1}{4}\right) + (7)\left(\frac{1}{4}\right) + (8)\left(\frac{1}{6}\right) + (9)\left(\frac{1}{6}\right) = \frac{41}{6}.$$

Therefore,

$$\mu_{2X-1} = (2)\left(\frac{41}{6}\right) - 1 = \$12.67,$$

Source : Probability & Statistics for Engineers & Scientists, by Ronald E. Walpole, Raymond H. Myers, Sharon L. Myers, Keying Ye, 9th Edition, Prentice Hall.

Example 4.18

Let X be a random variable with density function

$$f(x) = \begin{cases} \frac{x^2}{3}, & -1 < x < 2, \\ 0, & \text{elsewhere.} \end{cases}$$

Apply Theorem 4.5 to the continuous random variable $g(X) = 4X + 3$,

***Solution* :** use Theorem 4.5 to write $E(4X + 3) = 4E(X) + 3$.

$$E(X) = \int_{-1}^{2} x \left(\frac{x^2}{3}\right) dx = \int_{-1}^{2} \frac{x^3}{3}\, dx = \frac{5}{4}.$$

Therefore,

$$E(4X + 3) = (4)\left(\frac{5}{4}\right) + 3 = 8,$$

as before.

Source : Probability & Statistics for Engineers & Scientists, by Ronald E. Walpole, Raymond H. Myers, Sharon L. Myers, Keying Ye, 9th Edition, Prentice Hall.

Theorem 4.6

The expected value of the sum or difference of two or more functions of a random variable X is the sum or difference of the expected values of the functions. That is,

$$E[g(X) \pm h(X)] = E[g(X)] \pm E[h(X)].$$

***Proof*:** By definition,

$$\begin{aligned} E[g(X) \pm h(X)] &= \int_{-\infty}^{\infty} [g(x) \pm h(x)] f(x)\, dx \\ &= \int_{-\infty}^{\infty} g(x) f(x)\, dx \pm \int_{-\infty}^{\infty} h(x) f(x)\, dx \\ &= E[g(X)] \pm E[h(X)]. \end{aligned}$$

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Example 4.19

Let X be a random variable with probability distribution as follows:

x	0	1	2	3
$f(x)$	1/3	½	0	1/6

Find the expected value of $Y = (X - 1)^2$.

Solution **:** Applying Theorem 4.6 to the function $Y = (X - 1)^2$, we can write

$$E[(X-1)^2] = E(X^2 - 2X + 1) = E(X^2) - 2E(X) + E(1).$$

From Corollary 4.1, $E(1) = 1$, and by direct computation,

$$E(X) = (0)\left(\frac{1}{3}\right) + (1)\left(\frac{1}{2}\right) + (2)(0) + (3)\left(\frac{1}{6}\right) = 1 \text{ and}$$

$$E(X^2) = (0)\left(\frac{1}{3}\right) + (1)\left(\frac{1}{2}\right) + (4)(0) + (9)\left(\frac{1}{6}\right) = 2.$$

Hence,

$$E[(X-1)^2] = 2 - (2)(1) + 1 = 1.$$

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Example 4.20

The weekly demand for a certain drink, in thousands of liters, at a chain of convenience stores is a continuous random variable $g(X) = X^2 + X - 2$, where X has the density function

$$f(x) = \begin{cases} 2(x-1), & 1 < x < 2, \\ 0, & \text{elsewhere.} \end{cases}$$

Find the expected value of the weekly demand for the drink.

Solution **:** Applying Theorem 4.6,we write

$$E(X^2 + X - 2) = E(X^2) + E(X) - E(2).$$

From Corollary 4.1, $E(2) = 2$, and by direct integration,

$$E(X) = \int_1^2 2x(x-1)\,dx = \frac{5}{3} \text{ and } E(X^2) = \int_1^2 2x^2(x-1)\,dx = \frac{17}{6}.$$

Now

$$E(X^2 + X - 2) = \frac{17}{6} + \frac{5}{3} - 2 = \frac{5}{2},$$

so the average weekly demand for the drink from this chain of efficiency stores is 2500 liters.

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Theorem 4.7

The expected value of the sum or difference of two or more functions of the random variables *X* and *Y* is the sum or difference of the expected values of the functions.

That is,

$$E[g(X,Y) \pm h(X,Y)] = E[g(X,Y)] \pm E[h(X,Y)].$$

Proof: By Definition 4.2,

$$E[g(X,Y) \pm h(X,Y)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} [g(x,y) \pm h(x,y)] f(x,y)\, dx\, dy$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y) f(x,y)\, dx\, dy \pm \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(x,y) f(x,y)\, dx\, dy$$

$$= E[g(X,Y)] \pm E[h(X,Y)].$$

Corollary 4.3 and 4.4

Corollary 4.3: Setting $g(X,Y) = g(X)$ and $h(X,Y) = h(Y)$, we see that

$$E[g(X) \pm h(Y)] = E[g(X)] \pm E[h(Y)].$$

Corollary 4.4: Setting $g(X,Y) = X$ and $h(X,Y) = Y$, we see that

$$E[X \pm Y] = E[X] \pm E[Y].$$

If X represents the daily production of some item from machine A and Y the daily production of the same kind of item from machine B, then $X + Y$ represents the total number of items produced daily by both machines. Corollary 4.4 states that the average daily production for both machines is equal to the sum of the average daily production of each machine.

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Theorem 4.8

Let X and Y be two independent random variables. Then $E(XY) = E(X)E(Y)$.

***Proof*:** By Definition 4.2,

$$E(XY) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xyf(x,y)\ dx\ dy.$$

Since X and Y are independent, we may write

$$f(x,y) = g(x)h(y),$$

where $g(x)$ and $h(y)$ are the marginal distributions of X and Y, respectively. Hence,

$$E(XY) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xyg(x)h(y)\ dx\ dy = \int_{-\infty}^{\infty} xg(x)\ dx \int_{-\infty}^{\infty} yh(y)\ dy$$
$$= E(X)E(Y).$$

Corollary 4.5: Let X and Y be two independent random variables. Then $\sigma_{XY} = 0$.

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Example 4.21

It is known that the ratio of gallium to arsenide does not affect the functioning of gallium-arsenide wafers, which are the main components of microchips. Let X denote the ratio of gallium to arsenide and Y denote the functional wafers retrieved during a 1-hour period. X and Y are independent random variables with the joint density function

$$f(x,y) = \begin{cases} \frac{x(1+3y^2)}{4}, & 0 < x < 2,\ 0 < y < 1, \\ 0, & \text{elsewhere.} \end{cases}$$

Show that $E(XY) = E(X)E(Y)$

Solution: By definition,

$$E(XY) = \int_0^1 \int_0^2 \frac{x^2 y(1+3y^2)}{4}\, dx dy = \frac{5}{6},\ E(X) = \frac{4}{3},\ \text{and } E(Y) = \frac{5}{8}.$$

Hence,

$$E(X)E(Y) = \left(\frac{4}{3}\right)\left(\frac{5}{8}\right) = \frac{5}{6} = E(XY).$$

Source : Probability & Statistics for Engineers & Scientists, by Ronald E. Walpole, Raymond H. Myers, Sharon L. Myers, Keying Ye, 9th Edition, Prentice Hall.

Theorem 4.9

Theorem 4.9: If X and Y are random variables with joint probability distribution $f(x, y)$ and a, b, and c are constants, then

$$\sigma^2_{aX+bY+c} = a^2\sigma^2_X + b^2\sigma^2_Y + 2ab\sigma_{XY}.$$

***Proof*:** By definition, $\sigma^2_{aX+bY+c} = E\{[(aX + bY + c) - \mu_{aX+bY+c}]^2\}$. Now

$$\mu_{aX+bY+c} = E(aX + bY + c) = aE(X) + bE(Y) + c = a\mu_X + b\mu_Y + c,$$

by using Corollary 4.4 followed by Corollary 4.2. Therefore,

$$\begin{aligned}\sigma^2_{aX+bY+c} &= E\{[a(X - \mu_X) + b(Y - \mu_Y)]^2\}\\ &= a^2E[(X - \mu_X)^2] + b^2E[(Y - \mu_Y)^2] + 2abE[(X - \mu_X)(Y - \mu_Y)]\\ &= a^2\sigma^2_X + b^2\sigma^2_Y + 2ab\sigma_{XY}.\end{aligned}$$

Source : Probability & Statistics for Engineers & Scientists, by Ronald E. Walpole, Raymond H. Myers, Sharon L. Myers, Keying Ye, 9th Edition, Prentice Hall.

Using Theorem 4.9, we have the following corollaries

Corollary 4.6: Setting $b = 0$, we see that

$$\sigma^2_{aX+c} = a^2\sigma^2_X = a^2\sigma^2.$$

Corollary 4.7: Setting $a = 1$ and $b = 0$, we see that

$$\sigma^2_{X+c} = \sigma^2_X = \sigma^2.$$

Corollary 4.8: Setting $b = 0$ and $c = 0$, we see that

$$\sigma^2_{aX} = a^2\sigma^2_X = a^2\sigma^2.$$

Source : Probability & Statistics for Engineers & Scientists, by Ronald E. Walpole, Raymond H. Myers, Sharon L. Myers, Keying Ye, 9th Edition, Prentice Hall.

Using Theorem 4.9, we have the following corollaries Contd.

Corollary 4.9: If X and Y are independent random variables, then

$$\sigma^2_{aX+bY} = a^2\sigma^2_X + b^2\sigma^2_Y.$$

The result stated in Corollary 4.9 is obtained from Theorem 4.9 by invoking Corollary 4.5.

Corollary 4.10: If X and Y are independent random variables, then

$$\sigma^2_{aX-bY} = a^2\sigma^2_X + b^2\sigma^2_Y.$$

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Using Theorem 4.9, we have the following corollary Contd.

Corollary 4.11: If $X_1, X_2, \ldots, X_n$ are independent random variables, then

$$\sigma^2_{a_1X_1+a_2X_2+\cdots+a_nX_n} = a_1^2\sigma^2_{X_1} + a_2^2\sigma^2_{X_2} + \cdots + a_n^2\sigma^2_{X_n}.$$

Source : Probability & Statistics for Engineers & Scientists, by Ronald E. Walpole, Raymond H. Myers, Sharon L. Myers, Keying Ye, 9th Edition, Prentice Hall.

Example 4.22

Example 4.22: If X and Y are random variables with variances $\sigma_X^2 = 2$ and $\sigma_Y^2 = 4$ and covariance $\sigma_{XY} = -2$, find the variance of the random variable $Z = 3X - 4Y + 8$.

Solution:

$$\sigma_Z^2 = \sigma_{3X-4Y+8}^2 = \sigma_{3X-4Y}^2 \qquad \text{(by Corollary 4.6)}$$

$$= 9\sigma_X^2 + 16\sigma_Y^2 - 24\sigma_{XY} \qquad \text{(by Theorem 4.9)}$$

$$= (9)(2) + (16)(4) - (24)(-2) = 130.$$

Source : Probability & Statistics for Engineers & Scientists, by Ronald E. Walpole, Raymond H. Myers, Sharon L. Myers, Keying Ye, 9th Edition, Prentice Hall.

Example 4.23

Example 4.23: Let X and Y denote the amounts of two different types of impurities in a batch of a certain chemical product. Suppose that X and Y are independent random variables with variances $\sigma_X^2 = 2$ and $\sigma_Y^2 = 3$. Find the variance of the random variable $Z = 3X - 2Y + 5$.

Solution:

$$\begin{aligned}
\sigma_Z^2 &= \sigma_{3X-2Y+5}^2 = \sigma_{3X-2Y}^2 && \text{(by Corollary 4.6)} \\
&= 9\sigma_x^2 + 4\sigma_y^2 && \text{(by Corollary 4.10)} \\
&= (9)(2) + (4)(3) = 30.
\end{aligned}$$

Source : Probability & Statistics for Engineers & Scientists, by Ronald E. Walpole, Raymond H. Myers, Sharon L. Myers, Keying Ye, 9th Edition, Prentice Hall.