Probability and Statistics (IT302)

15th September 2020 Tuesday 10:30 AM - 11:00AM Class 16

16th September 2020 Wednesday 11:15 AM - 11:45AM Class 17

Exercise 4.59

If a random variable X is defined such that $E[(X-1)^2] = 10$ and $E[(X-2)^2] = 6$, find μ and σ^2.

Solution

The equations $E[(X-1)^2] = 10$ and $E[(X-2)^2] = 6$ may be written in the form:

$$E[(X-1)^2] = 10 \qquad E[(X-2)^2] = 6$$

$$E[(X^2 + 1 - 2X)] = 10 \qquad E[X^2 + 4 - 2*2*X)] = 6$$

$$E[(X^2)] - 2E[(X)] = 9 \qquad E[X^2 - 2*2*X)] = 2$$

$$E(X^2) - 2E(X) = 9, \qquad E(X^2) - 4E(X) = 2.$$

Solving these two equations simultaneously give $E(X) = 7/2$, and $E(X^2) = 16$.

Hence $\mu = 7/2$ and $\sigma^2 = 16 - (7/2)^2 = 15/4$.

Source : Probability & Statistics for Engineers & Scientists, by Ronald E. Walpole, Raymond H. Myers, Sharon L. Myers, Keying Ye, 9th Edition, Prentice Hall.

Exercise 4.60

Suppose that X and Y are independent random variables having the joint probability distribution

$f(x, y)$		x	
		2	4
y	1	0.10	0.15
	3	0.20	0.30
	5	0.10	0.15

Find (a) $E(2X - 3Y)$;
(b) $E(XY)$.

Solution:
$E(X) = (2)(0.40) + (4)(0.60) = 3.20$, and
$E(Y) = (1)(0.25) + (3)(0.50) + (5)(0.25) = 3$. So,

(a) $E(2X - 3Y) = 2E(X) - 3E(Y) = (2)(3.20) - (3)(3.00) = -2.60$.

(b) $E(XY) = E(X)E(Y) = (3.20)(3.00) = 9.60$.

Source : Probability & Statistics for Engineers & Scientists, by Ronald E. Walpole, Raymond H. Myers, Sharon L. Myers, Keying Ye, 9th Edition, Prentice Hall.

Exercise 4.61

If X and Y are independent Random Variables with variances $\sigma^2{}_X = 5$ and $\sigma^2{}_Y = 3$, find the variance of the random variable $Z = -2X + 4Y - 3$.

Solution: $\sigma^2{}_Z = \sigma^2{}_{-2X+4Y-3} = 4\sigma^2{}_X + 16\sigma^2{}_Y = (4)(5) + (16)(3) = 68.$

Corollary 4.9: If X and Y are independent random variables, then

$$\sigma^2_{aX+bY} = a^2\sigma^2_X + b^2\sigma^2_Y.$$

Corollary 4.6: Setting $b = 0$, we see that

$$\sigma^2_{aX+c} = a^2\sigma^2_X = a^2\sigma^2.$$

Source : Probability & Statistics for Engineers & Scientists, by Ronald E. Walpole, Raymond H. Myers, Sharon L. Myers, Keying Ye, 9th Edition, Prentice Hall.

Exercise 4.75

An electrical firm manufactures a 100-watt light bulb, which, according to specifications written on the package, has a mean life of 900 hours with a standard deviation of 50 hours. At most, what percentage of the bulbs fail to last even 700 hours? Assume that the distribution is symmetric about the mean.

Solution:

$\mu = 900$ hours and $\sigma = 50$ hours. Solving $\mu - k\sigma = 700$ we obtain $k = 4$.

So, using Chebyshev's theorem with $P(\mu - 4\sigma < X < \mu + 4\sigma) \geq 1 - 1/4^2 = 0.9375$, obtain $P(700 < X < 1100) \geq 0.9375$. Therefore, $P(X \leq 700) \leq 0.03125$.

Source : Probability & Statistics for Engineers & Scientists, by Ronald E. Walpole, Raymond H. Myers, Sharon L. Myers, Keying Ye, 9th Edition, Prentice Hall.

Exercise 4.76

Seventy new jobs are opening up at an automobile manufacturing plant, and 1000 applicants show up for the 70 positions. To select the best 70 from among the applicants, the company gives a test that covers mechanical skill, manual dexterity, and mathematical ability. The mean grade on this test turns out to be 60, and the scores have a standard deviation of 6. Can a person who scores 84 count on getting one of the jobs? [*Hint*: Use Chebyshev's theorem.] Assume that the distribution is symmetric about the mean.

Solution:

Using $\mu = 60$ and $\sigma = 6$ and Chebyshev's theorem $P(\mu - k\sigma < X < \mu + k\sigma) \geq 1 - 1/k^2$,
since from $\mu + k\sigma = 84$ we obtain $k = 4$.
So, $P(X < 84) \geq P(36 < X < 84) \geq 1 - 1/4^2 = 0.9375$. Therefore,
$P(X \geq 84) \leq 1 - 0.9375 = 0.0625$.
Since $1000(0.0625) = 62.5$, we claim that at most 63 applicants would have a score as 84 or higher. Since there will be 70 positions, the applicant will have the job.

Source : Probability & Statistics for Engineers & Scientists, by Ronald E. Walpole, Raymond H. Myers, Sharon L. Myers, Keying Ye, 9th Edition, Prentice Hall.

Exercise 4.77

A random variable X has a mean $\mu = 10$ and a variance $\sigma^2 = 4$. Using Chebyshev's theorem, find (a) $P(|X - 10| \geq 3)$; (b) $P(|X - 10| < 3)$;

Solution:

(a) $P(|X - 10| \geq 3) = 1 - P(|X - 10| < 3)$

$= 1 - P[10 - (3/2)(2) < X < 10 + (3/2)(2)] \leq 1 - [1 - 1/(3/2)^2] = 4/9$.

(b) $P(|X - 10| < 3) = 1 - P(|X - 10| \geq 3) \geq 1 - 4/9 = 5$

Exercise 4.55

Suppose that a grocery store purchases 5 cartons of skim milk at the wholesale price of \$1.20 per carton and retails the milk at \$1.65 per carton. After the expiration date, the unsold milk is removed from the shelf and the grocer receives a credit from the distributor equal to three-fourths of the wholesale price. If the probability distribution of the random variable X, the number of cartons that are sold from this lot, is

x	0	1	2	3	4	5
$f(x)$	1/15	2/15	2/15	3/15	4/15	3/15

find the expected profit.

Solution:

Let X = number of cartons sold and Y = profit.

We can write $Y = 1.65X + (0.90)(5 - X) - 6 = 0.75X - 1.50$. Now

$E(X) = (0)(1/15) + (1)(2/15) + (2)(2/15) + (3)(3/15) + (4)(4/15) + (5)(3/15) = 46/15$,

and $E(Y) = (0.75)E(X) - 1.50 = (0.75)(46/15) - 1.50 = \$0.80$.

Source : Probability & Statistics for Engineers & Scientists, by Ronald E. Walpole, Raymond H. Myers, Sharon L. Myers, Keying Ye, 9th Edition, Prentice Hall.

Example-1 related to Chebyshev’s Theorem

Example-1

A class of second graders has a mean height of five feet with a standard deviation of one inch. At least what percent of the class must be between 4’10” and 5’2”?

Solution

The heights that are given in the range above are within two standard deviations from the mean height of five feet. Chebyshev’s inequality says that at least $1 – 1/2^2 = 3/4 = 75\%$ of the class is in the given height range.

Source : https://www.thoughtco.com/worksheet-for-chebyshevs-inequality-solutions-3126519 Accessed on 14th September 2020

Example-2 related to Chebyshev's Theorem

Example-2

Computers from a particular company are found to last on average for three years without any hardware malfunction, with a standard deviation of two months. At least what percent of the computers last between 31 months and 41 months?

Solution

The mean lifetime of three years corresponds to 36 months. The times of 31 months to 41 months are each 5/2 = 2.5 standard deviations from the mean. By Chebyshev's inequality, at least $1 - 1/(2.5)^2 = 84\%$ of the computers last from 31 months to 41 months.

Source : https://www.thoughtco.com/worksheet-for-chebyshevs-inequality-solutions-3126519 Accessed on 14th September 2020

Example-3 related to Chebyshev's Theorem

Example-3

Bacteria in a culture live for an average time of three hours with a standard deviation of 10 minutes. At least what fraction of the bacteria live between two and four hours?

Solution

Two and four hours are each one hour away from the mean. One hour corresponds to six standard deviations. So at least $1 – 1/6^2 = 35/36 = 97\%$ of the bacteria live between two and four hours.

Source : https://www.thoughtco.com/worksheet-for-chebyshevs-inequality-solutions-3126519 Accessed on 14th September 2020

Example-4 related to Chebyshev’s Theorem

Example-4

What is the smallest number of standard deviations from the mean that we must go if we want to ensure that we have at least 50% of the data of a distribution?

Solution

Here Chebyshev’s inequality is used.

Requirement is 50% = 0.50 = 1/2 = 1 – $1/K^2$.

$$1/2 = 1/K^2.$$

$$2 = K^2.$$

Take the square root of both sides, and since K is a number of standard deviations, ignore the negative solution to the equation. This shows that ***K* is equal to the square root of two**. So at least 50% of the data is within approximately 1.4 standard deviations from the mean.

Source : https://www.thoughtco.com/worksheet-for-chebyshevs-inequality-solutions-3126519 Accessed on 14th September 2020

Example-5 related to Chebyshev's Theorem

Example #5

Bus route #25 takes a mean time of 50 minutes with a standard deviation of 2 minutes. A promotional poster for this bus system states that "95% of the time bus route #25 lasts from _____ to ______ minutes." What numbers would you fill in the blanks with?

Solution

This question is similar to the last one in that we need to solve for K, the number of standard deviations from the mean. Start by setting 95% = 0.95 = $1 - 1/K^2$. This shows that 1 - 0.95 = $1/K^2$. Simplify to see that 1/0.05 = 20 = K^2. So K = 4.47.

Now express this in the terms above. At least 95% of all rides are 4.47 standard deviations from the mean time of 50 minutes. Multiply 4.47 by the standard deviation of 2 to end up with nine minutes. So 95% of the time, bus route #25 takes between 41 and 59 minutes.

Source : https://www.thoughtco.com/worksheet-for-chebyshevs-inequality-solutions-3126519 Accessed on 14th September 2020