Probability and Statistics (IT302) Class No. 23

7^{th} October 2020 Wednesday 11:15AM - 11:45AM

Poisson Distribution and the Poisson Process

Experiments yielding numerical values of a random variable X, the number of outcomes occurring during a given time interval or in a specified region, are called **Poisson experiments**. The given time interval may be of any length, such as a minute, a day, a week, a month, or even a year.

For example, a Poisson experiment can generate observations for the random variable X representing the number of telephone calls received per hour by an office, the number of days school is closed due to snow during the winter, or the number of games postponed due to rain during a baseball season.

The specified region could be a line segment, an area, a volume, or perhaps a piece of material. In such instances, X might represent the number of field mice per acre, the number of bacteria in a given culture, or the number of typing errors per page.

Source : Probability & Statistics for Engineers & Scientists, by Ronald E. Walpole, Raymond H. Myers, Sharon L. Myers, Keying Ye, 9th Edition, Prentice Hall.

Properties of the Poisson Process

A Poisson experiment is derived from the **Poisson process** and possesses the following properties.

1. The number of outcomes occurring in one time interval or specified region of space is independent of the number that occur in any other disjoint time interval or region. In this sense we say that the Poisson process has no memory.

2. The probability that a single outcome will occur during a very short time interval or in a small region is proportional to the length of the time interval or the size of the region and does not depend on the number of outcomes occurring outside this time interval or region.

3. The probability that more than one outcome will occur in such a short time interval or fall in such a small region is negligible.

Source : Probability & Statistics for Engineers & Scientists, by Ronald E. Walpole, Raymond H. Myers, Sharon L. Myers, Keying Ye, 9th Edition, Prentice Hall.

Poisson Random Variable

The number X of outcomes occurring during a Poisson experiment is called a **Poisson Random Variable**, and its probability distribution is called the **Poisson Distribution**.

The mean number of outcomes is computed from $\mu = \lambda t$, where t is the specific "time," "distance," "area," or "volume" of interest. Since the probabilities depend on λ, the rate of occurrence of outcomes, we shall denote them by $p(x; \lambda t)$.

Poisson Distribution

The probability distribution of the Poisson random variable X, representing the number of outcomes occurring in a given time interval or specified region denoted by t, is

$$p(x; \lambda t) = \frac{e^{-\lambda t}(\lambda t)^x}{x!}, \quad x = 0, 1, 2, \ldots,$$

where λ is the average number of outcomes per unit time, distance, area, or volume and $e = 2.71828 \ldots .$

Source : Probability & Statistics for Engineers & Scientists, by Ronald E. Walpole, Raymond H. Myers, Sharon L. Myers, Keying Ye, 9th Edition, Prentice Hall.

Example 5.17

During a laboratory experiment, the average number of radioactive particles passing through a counter in 1 millisecond is 4. What is the probability that 6 particles enter the counter in a given millisecond?

Solution : Using the Poisson distribution with x = 6 and $\lambda t = 4$ and referring to Table A.2 of below mentioned text book,

$$p(6;4) = \frac{e^{-4}4^6}{6!} = \sum_{x=0}^{6} p(x;4) - \sum_{x=0}^{5} p(x;4) = 0.8893 - 0.7851 = 0.1042.$$

Source : Probability & Statistics for Engineers & Scientists, by Ronald E. Walpole, Raymond H. Myers, Sharon L. Myers, Keying Ye, 9th Edition, Prentice Hall.

Example 5.18

Ten is the average number of oil tankers arriving each day at a certain port. The facilities at the port can handle at most 15 tankers per day. What is the probability that on a given day tankers have to be turned away?

Solution : Let X be the number of tankers arriving each day. Then, using Table A.2 of below mentioned text book,

$$P(X > 15) = 1 - P(X \leq 15) = 1 - \sum_{x=0}^{15} p(x; 10) = 1 - 0.9513 = 0.0487.$$

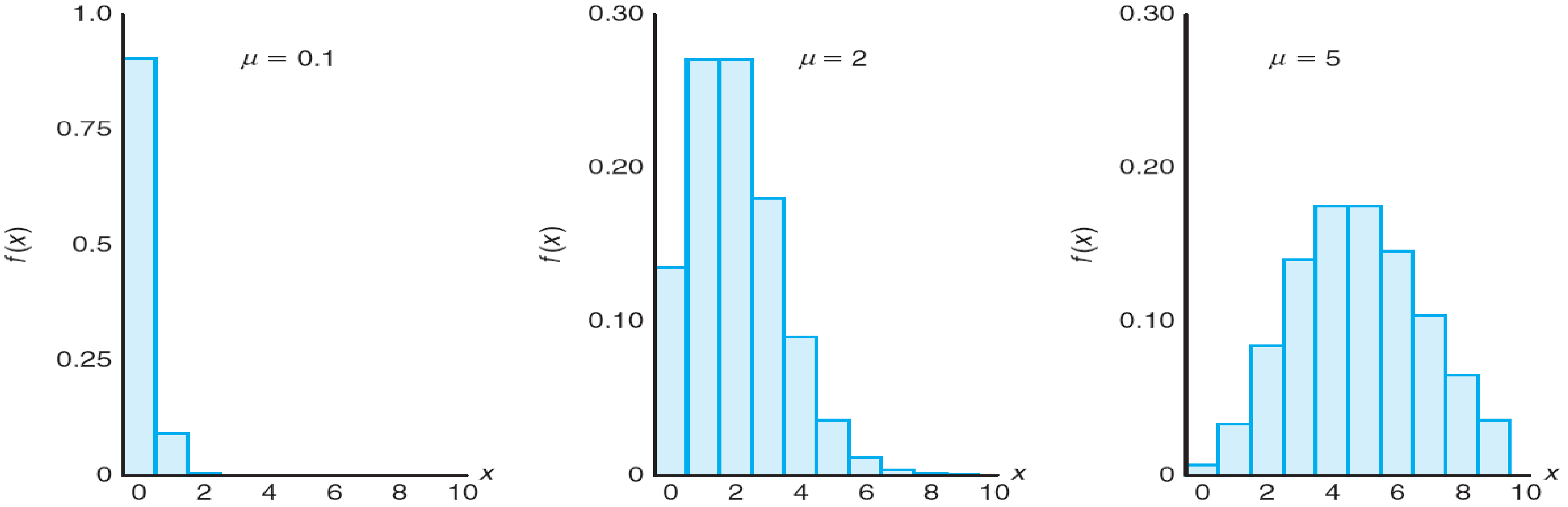
Theorem 5.4

Both the mean and the variance of the Poisson distribution p(x; λt) are **λt**.

Source : Probability & Statistics for Engineers & Scientists, by Ronald E. Walpole, Raymond H. Myers, Sharon L. Myers, Keying Ye, 9th Edition, Prentice Hall.

Poisson Density Functions for Different Means

Like so many discrete and continuous distributions, the form of the Poisson distribution becomes more and more symmetric, even bell-shaped, as the mean grows large. Below Figure illustrates this, showing plots of the probability function for $\mu = 0.1$, $\mu = 2$, and $\mu = 5$. Note the nearness to symmetry when μ becomes as large as 5.



Source : Probability & Statistics for Engineers & Scientists, by Ronald E. Walpole, Raymond H. Myers, Sharon L. Myers, Keying Ye, 9th Edition, Prentice Hall.

Approximation of Binomial Distribution by a Poisson Distribution

- It should be evident from the three principles of the Poisson process that the Poisson distribution is related to the binomial distribution. Although the Poisson usually finds applications in space and time problems, as illustrated by Examples 5.17 and 5.18, it can be viewed as a limiting form of the binomial distribution.

- In the case of the binomial, if n is quite large and p is small, the conditions begin to simulate the continuous space or time implications of the Poisson process. The independence among Bernoulli trials in the binomial case is consistent with principle 2 of the Poisson process.

- Allowing the parameter p to be close to 0 relates to principle 3 of the Poisson process. Indeed, if n is large and p is close to 0, the Poisson distribution can be used, with $\mu = np$, to approximate binomial probabilities. If p is close to 1, we can still use the Poisson distribution to approximate binomial probabilities by interchanging what we have defined to be a success and a failure, thereby changing p to a value close to 0.

Source : Probability & Statistics for Engineers & Scientists, by Ronald E. Walpole, Raymond H. Myers, Sharon L. Myers, Keying Ye, 9th Edition, Prentice Hall.

Theorem 5.5

Let X be a binomial random variable with probability distribution $b(x; n, p)$. When $n \to \infty$, $p \to 0$, and $np \overset{n\to\infty}{\longrightarrow} \mu$ remains constant,

$$b(x; n, p) \overset{n\to\infty}{\longrightarrow} p(x; \mu).$$

Source : Probability & Statistics for Engineers & Scientists, by Ronald E. Walpole, Raymond H. Myers, Sharon L. Myers, Keying Ye, 9th Edition, Prentice Hall.

Example 5.19

In a certain industrial facility, accidents occur infrequently. It is known that the probability of an accident on any given day is 0.005 and accidents are independent of each other.

a) What is the probability that in any given period of 400 days there will be an accident on one day?
b) What is the probability that there are at most three days with an accident?

Solution : Let X be a binomial random variable with n = 400 and p = 0.005. Thus, np = 2. Using the Poisson approximation,

$$\text{(a)}\quad P(X=1) = e^{-2}2^1 = 0.271 \text{ and}$$

$$\text{(b)}\quad P(X \le 3) = \sum_{x=0}^{3} e^{-2}2^x/x! = 0.857$$

Source : Probability & Statistics for Engineers & Scientists, by Ronald E. Walpole, Raymond H. Myers, Sharon L. Myers, Keying Ye, 9th Edition, Prentice Hall.

Example 5.20

In a manufacturing process where glass products are made, defects or bubbles occur, occasionally rendering the piece undesirable for marketing. It is known that, on average, 1 in every 1000 of these items produced has one or more bubbles. What is the probability that a random sample of 8000 will yield fewer than 7 items possessing bubbles?

Solution : This is essentially a binomial experiment with n = 8000 and p = 0.001. Since p is very close to 0 and n is quite large, we shall approximate with the Poisson distribution using

$$\mu = (8000)(0.001) = 8.$$

Hence, if X represents the number of bubbles, we have

$$P(X < 7) = \sum_{x=0}^{6} b(x; 8000, 0.001) \approx p(x; 8) = 0.3134$$

Source : Probability & Statistics for Engineers & Scientists, by Ronald E. Walpole, Raymond H. Myers, Sharon L. Myers, Keying Ye, 9th Edition, Prentice Hall.

Additional Material

Poisson Distribution

- Poisson distribution is for counts—if events happen at a constant rate over time, the Poisson distribution gives the probability of X number of events occurring in time T.

Example

The Poisson distribution models counts, such as the number of new cases of SARS that occur in women in New England next month.

The distribution tells you the probability of all possible numbers of new cases, from 0 to infinity.

If X= # of new cases next month and X ~ Poisson (λ), then the probability that X=k (a particular count) is:

$$\boxed{p(X = k) = \frac{\lambda^k e^{-\lambda}}{k!}}$$

Example : Poisson Distribution

Suppose that a rare disease has an incidence of 1 in 1000 person-years. Assuming that members of the population are affected independently, find the probability of k cases in a population of 10,000 (followed over 1 year) for k=0,1,2.

The expected value (mean) =λ = .001*10,000 = 10
10 new cases expected in this population per year→

$$P(X=0)=\frac{(10)^0 e^{-(10)}}{0!}=.0000454$$

$$P(X=1)=\frac{(10)^1 e^{-(10)}}{1!}=.000454$$

$$P(X=2)=\frac{(10)^2 e^{-(10)}}{2!}=.00227$$

Source: https://web.stanford.edu/~kcobb/hrp259/lecture5.ppt

Poisson Process

"Poisson Process" (rates)

Note that the Poisson parameter λ can be given as the mean number of events that occur in a defined time period OR, equivalently, λ can be given as a rate, such as λ=2/month (2 events per 1 month) that must be multiplied by *t*=time (called a "Poisson Process") →

X ~ Poisson (λ)

$$P(X = k) = \frac{(\lambda t)^k e^{-\lambda t}}{k!}$$

E(X) = λt

Var(X) = λt

For **example**, if new cases of West Nile in New England are occurring at a rate of about 2 per month, then what's the probability that exactly 4 cases will occur in the next 3 months? X ~ Poisson (λ=2/month)

$$\mathrm{P}(\mathrm{X} = 4 \text{ in } 3 \text{ months}) = \frac{(2*3)^4 e^{-(2*3)}}{4!} = \frac{6^4 e^{-(6)}}{4!} = 13.4\%$$

Exactly 6 cases?

$$\mathrm{P}(\mathrm{X} = 6 \text{ in } 3 \text{ months}) = \frac{(2*3)^6 e^{-(2*3)}}{6!} = \frac{6^6 e^{-(6)}}{6!} = 16\%$$

Source: https://web.stanford.edu/~kcobb/hrp259/lecture5.ppt

Poisson Distribution

If events happen independently of each other, with average number of events in some fixed interval λ, then the distribution of the number of events k in that interval is Poisson.

A random variable X has the Poisson distribution with parameter $\lambda(> 0)$ if

$$P(X = k) = \frac{e^{-\lambda}\lambda^k}{k!} \quad (k = 0,1,2, \dots)$$

Examples of possible Poisson distributions

1) Number of messages arriving at a telecommunications system in a day
2) Number of flaws in a metre of fibre optic cable
3) Number of radio-active particles detected in a given time
4) Number of photons arriving at a CCD pixel in some exposure time (e.g. astronomy observations)

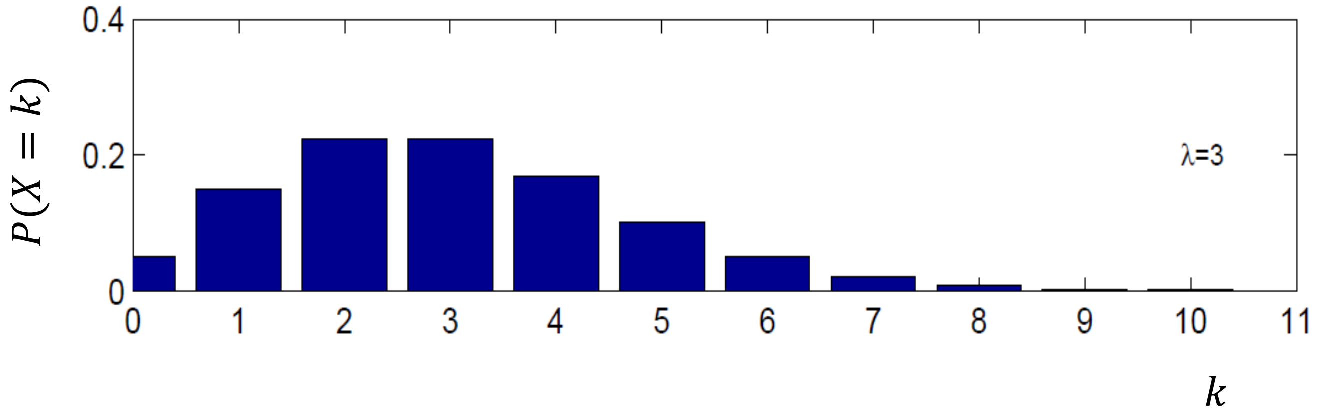
Sum of Poisson Variables

If X is Poisson with average number λ_X and Y is Poisson with average number λ_Y
Then $X + Y$ is Poisson with average number $\lambda_X + \lambda_Y$
The probability of events per unit time does not have to be constant for the total number of events to be Poisson – can split up the total into a sum of the number of events in smaller intervals.

Source: https://cosmologist.info/teaching/STAT/Statistics4.pptx

Example: On average lightning kills three people each year in the UK, $\lambda = 3$. What is the probability that only one person is killed this year?

Answer: Assuming these are independent random events, the number of people killed in a given year therefore has a Poisson distribution:



Let the random variable X be the number of people killed in a year.

Poisson distribution $P(X = k) = \frac{e^{-\lambda}\lambda^k}{k!}$ with $\lambda = 3$

$$\Rightarrow P(X = 1) = \frac{e^{-3}3^1}{1!} \approx 0.15$$

Source: https://cosmologist.info/teaching/STAT/Statistics4.pptx

Poisson Distribution

Question: Suppose that trucks arrive at a receiving dock with an average arrival rate of 3 per hour. What is the probability exactly 5 trucks will arrive in a two-hour period?

Answer : $\frac{e^{-6}6^5}{5!}$

$Reminder: P(X = k) = \frac{e^{-\lambda}\lambda^k}{k!}$

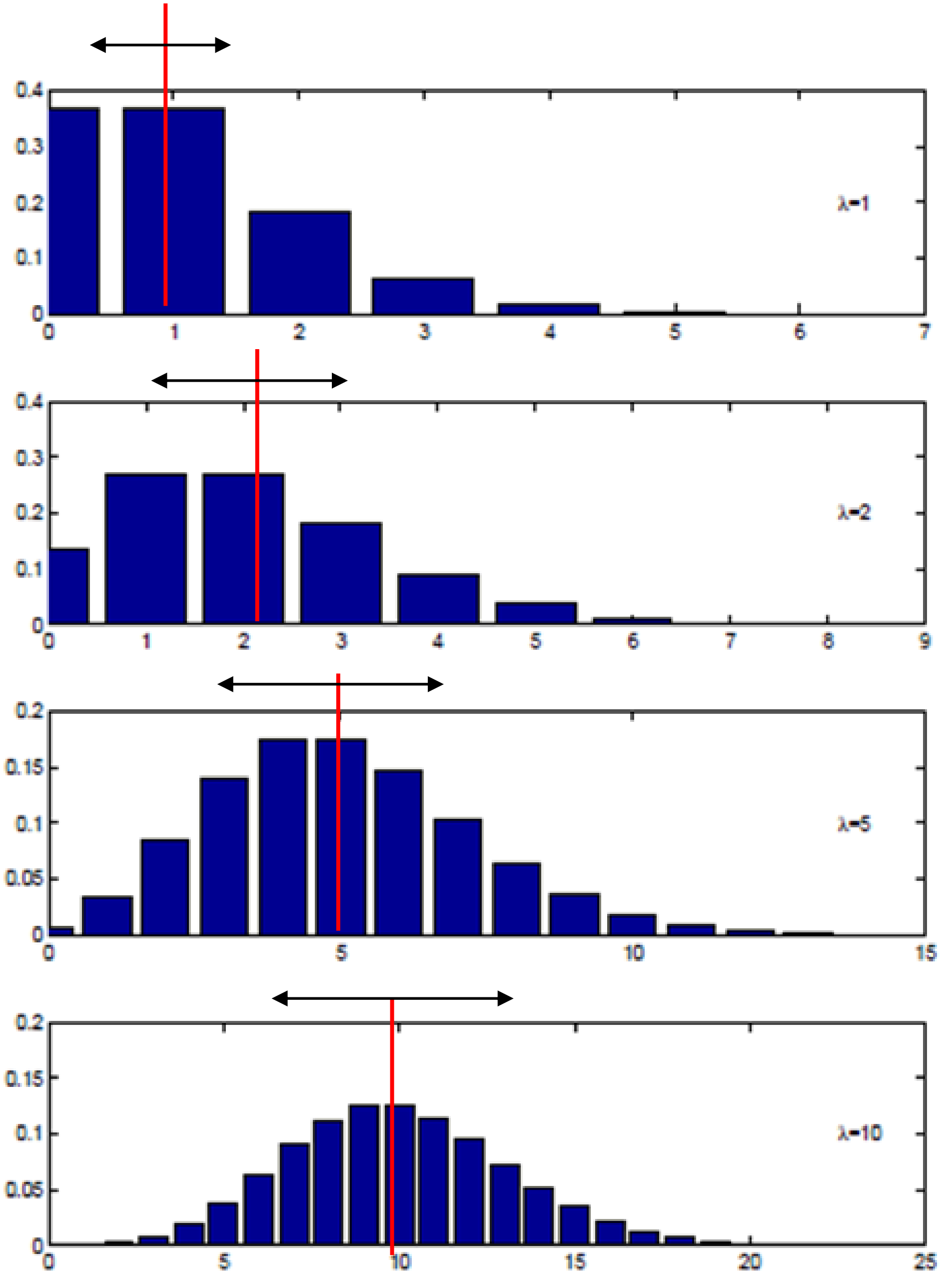
Question : Suppose that trucks arrive at a receiving dock with an average arrival rate of 3 per hour. What is the probability exactly 5 trucks will arrive in a two-hour period?

Answer:

In two hours mean number is $\lambda = 2 \times 3 = 6.$

$$P(X = k = 5) = \frac{e^{-\lambda}\lambda^k}{k!} = \frac{e^{-6}6^5}{5!}$$

Source: https://cosmologist.info/teaching/STAT/Statistics4.pptx



Mean and Variance

If $X \sim$ Poisson with mean λ, then

$$\mu = E(X) = \lambda$$

$$\sigma^2 = \mathrm{var}(X) = \lambda$$

Source: https://cosmologist.info/teaching/STAT/Statistics4.pptx

Example: ***Telecommunications***

Messages arrive at a switching centre at random and at an average rate of 1.2 per second.

a) Find the probability of 5 messages arriving in a 2-sec interval.

Answer:

Times of arrivals form a Poisson process, rate $\nu = 1.2/\text{sec}$.

(a) Let Y = number of messages arriving in a 2-sec interval. Then Y ~ Poisson, mean number $\lambda = \nu t = 1.2 \times 2 = 2.4$

$$P(Y = k = 5) = \frac{e^{-\lambda}\lambda^k}{k!} = \frac{e^{-2.4}2.4^5}{5!} = 0.060$$

Source: https://cosmologist.info/teaching/STAT/Statistics4.pptx