Random Numbers

AI21BTECH11017

Contents

***Abstract*—This manual provides a simple introduction to the generation of random numbers**

1 Uniform Random Numbers

Let U be a uniform random variable between 0 and 1.

1.1 Generate 10^6 samples of U using a C program and save into a file called uni.dat .
Solution: Download the following file and execute the C program.

```
https://github.com/AkshithaKola/
    Random_numbers/blob/main/Codes/1.1.c
```

1.2 Load the uni.dat file into python and plot the empirical CDF of U using the samples in uni.dat. The CDF is defined as

$$F_U(x) = \Pr(U \leq x) \tag{1.1}$$

Solution: The following code plots Fig. 1.2

```
https://github.com/AkshithaKola/
    Random_numbers/blob/main/Codes/1.2.
    py
```

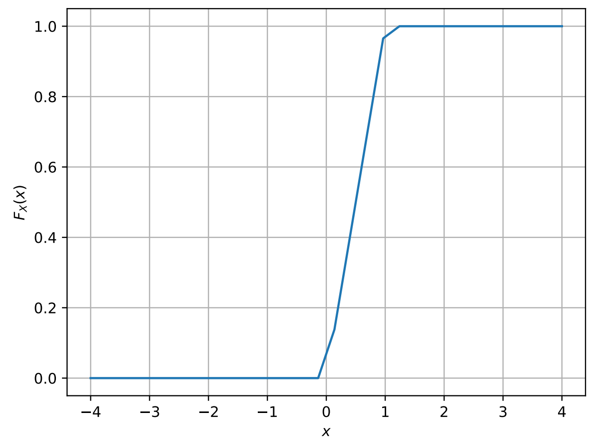

Fig. 1.2: The CDF of U

1.3 Find a theoretical expression for $F_U(x)$.
Solution: The CDF of U is

$$F_U(x) = \Pr(U \leq x) = \int_{-\infty}^{x} p_U(x)\, dx \tag{1.2}$$

The PDF of U is

$$p_U(x) = \begin{cases} 1 & x \in [0,1] \\ 0 & \text{otherwise} \end{cases} \tag{1.3}$$

If $x < 0$,

$$\int_{-\infty}^{x} p_U(x)\, dx = \int_{-\infty}^{x} 0\, dx = 0 \tag{1.4}$$

If $0 \leq x < 1$,

$$\int_{-\infty}^{x} p_U(x)\, dx = \int_{-\infty}^{0} 0\, dx + \int_{0}^{x} 1\, dx \tag{1.5}$$

$$= 0 + x = x \tag{1.6}$$

$$\tag{1.7}$$

If $x \geq 1$,

$$\int_{-\infty}^{x} p_U(x)\, dx$$

$$= \int_{-\infty}^{0} 0\, dx + \int_{0}^{1} 1\, dx + \int_{1}^{x} 0\, dx \tag{1.8}$$

$$\int_{-\infty}^{x} p_U(x)\, dx = 0 + 1 + 0 \tag{1.9}$$

$$= 1 \tag{1.10}$$

So, the CDF of U is

$$F_U(x) = \begin{cases} 0 & x < 0 \\ x & 0 \le x \le 1 \\ 1 & x > 1 \end{cases} \tag{1.11}$$

1.4 The mean of U is defined as

$$E[U] = \frac{1}{N}\sum_{i=1}^{N} U_i \tag{1.12}$$

and its variance as

$$\text{var}[U] = E[U - E[U]]^2 \tag{1.13}$$

Write a C program to find the mean and variance of U.
Solution: The following code gives the mean and variance.

https://github.com/AkshithaKola/Random_numbers/blob/main/Codes/1.4.c

1.5 Verify your result theoretically given that

$$E\left[U^k\right] = \int_{-\infty}^{\infty} x^k dF_U(x) \tag{1.14}$$

Solution: Given $E\left[U^k\right] = \int_{-\infty}^{\infty} x^k dF_U(x)$

$$mean = E[U] \tag{1.15}$$

$$E[U] = \int_{-\infty}^{\infty} x dF_U(x) \tag{1.16}$$

$$p_U(x) = \frac{\mathrm{d}F_U(x)}{dx} \tag{1.17}$$

$$\mathrm{d}F_U(x) = p_U(x)\mathrm{d}x \tag{1.18}$$

$$\mathrm{d}F_U(x) = \begin{cases} 0 & x < 0 \\ \mathrm{d}x & 0 \le x \le 1 \\ 0 & x > 1 \end{cases} \tag{1.19}$$

$$\therefore E[U] = \int_0^1 x \ \mathrm{d}x = \frac{1}{2} = 0.5 \tag{1.20}$$

variance $= E(U - E[U])^2$

$$= E(U^2 - 2U(E[U]) + (E[U])^2 \tag{1.21}$$

$$= E[U^2] + E[-2UE[U]] + [(E[(U)^2] \tag{1.22}$$

$$= E[U^2] - 2E[U]E[U] + (E[U])^2 \tag{1.23}$$

$$= E[U^2] - (E[U])^2 \tag{1.24}$$

$$\tag{1.25}$$

$$E[U^2] = \int_0^1 x^2 \ \mathrm{d}x = \frac{1}{3} \tag{1.26}$$

$\therefore$ variance is

$$= \frac{1}{3} - \left(\frac{1}{2}\right)^2 \tag{1.27}$$

$$= \frac{1}{12} \approx 0.083333 \tag{1.28}$$

2 Central Limit Theorem

2.1 Generate 10^6 samples of the random variable

$$X = \sum_{i=1}^{12} U_i - 6 \tag{2.1}$$

using a C program, where $U_i, i = 1, 2, \ldots, 12$ are a set of independent uniform random variables between 0 and 1 and save in a file called gau.dat
Solution: The following is the code to generate X a

https://github.com/AkshithaKola/Random_numbers/blob/main/Codes/2.1.c

2.2 Load gau.dat in python and plot the empirical CDF of X using the samples in gau.dat. What properties does a CDF have?
Solution: The following code plots Fig. 2.2

https://github.com/AkshithaKola/Random_numbers/blob/main/Codes/2.2.py

The CDF of X is plotted in Fig. 2.2
We can say that every CDF is a non decreasing function and is always in the range $[0, 1]$

2.3 Load gau.dat in python and plot the empirical PDF of X using the samples in gau.dat. The PDF of X is defined as

$$p_X(x) = \frac{d}{dx}F_X(x) \tag{2.2}$$

What properties does the PDF have?
Solution: The PDF of X is plotted in Fig. 2.3 using the code below

https://github.com/AkshithaKola/Random_numbers/blob/main/Codes/2.3.py

2.4 Find the mean and variance of X by writing a C program.

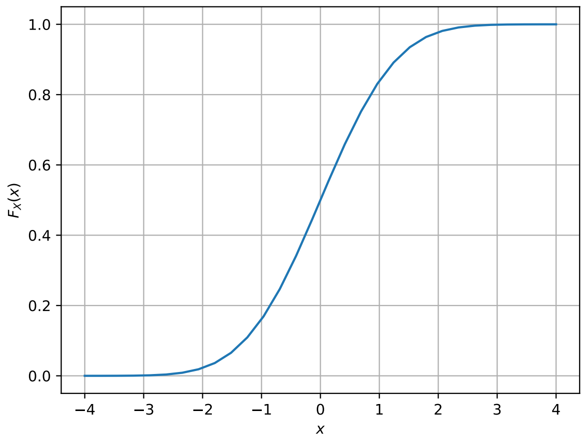

Fig. 2.2: The CDF of X

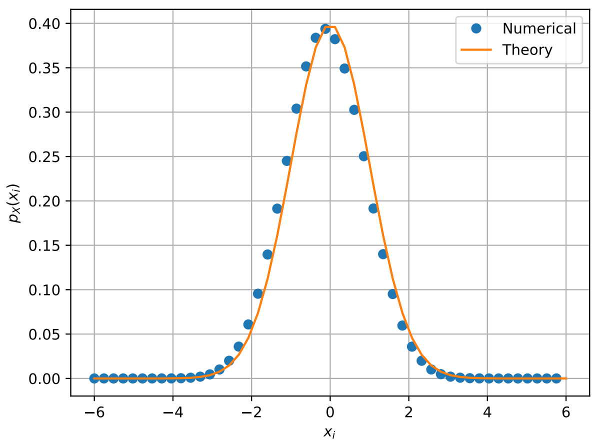

Fig. 2.3: The PDF of X

Solution: The following is the code for mean and variance of X

https://github.com/AkshithaKola/Random_numbers/blob/main/Codes/2.4.c

2.5 Given that

$$p_X(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right), -\infty < x < \infty, \tag{2.3}$$

repeat the above exercise theoretically.
Solution: The mean of X is

$$E[X] = \int_{-\infty}^{\infty} x p_X(x) dx \tag{2.4}$$

$$= \int_{-\infty}^{\infty} \frac{x}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) dx \tag{2.5}$$

Let

$$f(x) = \frac{x}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) \tag{2.6}$$

$$f(-x) = -f(x) \tag{2.7}$$

$\therefore f(x)$ is an odd function.
So,

$$E[X] = \int_{-\infty}^{\infty} f(x) dx = 0 \tag{2.8}$$

Now,

$$E[X^2] = \int_{-\infty}^{\infty} x^2 p_X(x) dx \tag{2.9}$$

$$= \int_{-\infty}^{\infty} \frac{x^2}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) dx \tag{2.10}$$

$$\tag{2.11}$$

let,

$$g(x) = \int_{-\infty}^{\infty} \frac{x^2}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) \tag{2.12}$$

$$g(x) = g(-x) \tag{2.13}$$

$\therefore g(x)$ is an even function.
So we can write

$$E[X^2] = 2 \int_0^{\infty} \frac{x^2}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) dx \tag{2.14}$$

Using integration by parts,

$$E[X^2] = \sqrt{\frac{2}{\pi}} \int_0^{\infty} x \cdot x \exp\left(-\frac{x^2}{2}\right) dx \tag{2.15}$$

$$= \sqrt{\frac{2}{\pi}} \left(x \int x \exp\left(-\frac{x^2}{2}\right) dx \right)\Bigg|_0^{\infty} - \sqrt{\frac{2}{\pi}} \int_0^{\infty} 1 \cdot \int x \exp\left(-\frac{x^2}{2}\right) dx \tag{2.16}$$

Substitute $t = \frac{x^2}{2} \implies dt = x dx$

$$\int x \exp\left(-\frac{x^2}{2}\right) dx = \int \exp(-t) dt \tag{2.17}$$

$$= -\exp(-t) \tag{2.18}$$

$$= -\exp\left(-\frac{x^2}{2}\right) \tag{2.19}$$

As,

$$\lim_{x\to\infty} x\exp\left(-\frac{x^2}{2}\right) = \lim_{x\to\infty} \frac{x}{\exp\left(\frac{x^2}{2}\right)} = 0 \quad (2.20)$$

$$\implies -x\exp\left(-\frac{x^2}{2}\right)\Big|_0^\infty = 0 - 0 = 0 \quad (2.21)$$

$$(2.22)$$

And,

$$\int_0^\infty -\exp\left(-\frac{x^2}{2}\right)\mathrm{d}x \quad (2.23)$$

$$\overset{x=t\sqrt{2}}{\longleftrightarrow} \int_0^\infty -\exp(-t^2)\mathrm{d}t\,\sqrt{2} \quad (2.24)$$

$$= -\sqrt{2}\int_0^\infty \exp(-t^2)\mathrm{d}t \quad (2.25)$$

$$= -\sqrt{2}\frac{\sqrt{\pi}}{2} \quad (2.26)$$

$$= -\sqrt{\frac{\pi}{2}} \quad (2.27)$$

Therefore,

$$E[X^2] = 0 - \sqrt{\frac{2}{\pi}}\left(-\sqrt{\frac{\pi}{2}}\right) \quad (2.28)$$

$$= 1 \quad (2.29)$$

$$\therefore var[X] = E[X^2] - (E[X])^2 \quad (2.30)$$

$$= 1 - 0 \quad (2.31)$$

$$= 1 \quad (2.32)$$

3 From Uniform to Other

3.1 Generate samples of

$$V = -2\ln(1-U) \quad (3.1)$$

and plot its CDF.
Solution: The following are the codes of samples of V and the CDF plot of V.

https://github.com/AkshithaKola/Random_numbers/blob/main/Codes/3.1.c

https://github.com/AkshithaKola/Random_numbers/blob/main/Codes/3.1.py

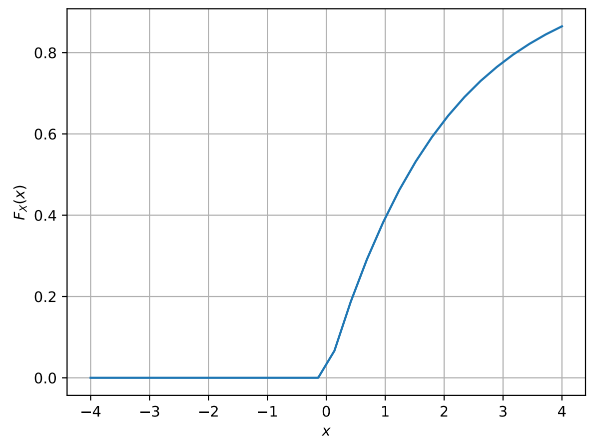

Fig. 3.1: The CDF of V

3.2 Find a theoretical expression for $F_V(x)$.
Solution:

$$F_V(x) = \Pr(V \le x) \quad (3.2)$$

$$= \Pr(-2\ln(1-U) \le x) \quad (3.3)$$

$$= \Pr\left(\ln(1-U) \ge -\frac{x}{2}\right) \quad (3.4)$$

$$= \Pr\left(1-U \ge \exp\left(-\frac{x}{2}\right)\right) \quad (3.5)$$

$$= \Pr\left(U \le 1-\exp\left(-\frac{x}{2}\right)\right) \quad (3.6)$$

$$= F_U\left(1-\exp\left(-\frac{x}{2}\right)\right) \quad (3.7)$$

As,

$$F_U(x) = \begin{cases} 0 & x < 0 \\ x & 0 \le x \le 1 \\ 1 & x > 1 \end{cases} \quad (3.8)$$

Now,

$$0 \le 1-\exp\left(-\frac{x}{2}\right) < 1 \qquad \text{if } x \ge 0 \quad (3.9)$$

$$1-\exp\left(-\frac{x}{2}\right) < 0 \qquad \text{if } x < 0 \quad (3.10)$$

So

$$F_V(x) = \begin{cases} 1-\exp\left(-\frac{x}{2}\right) & x \ge 0 \\ 0 & x < 0 \end{cases} \quad (3.11)$$