# Random Numbers

## AI21BTECH11017

### CONTENTS

*Abstract*—**This manual provides a simple introduction to the generation of random numbers**

### 1 UNIFORM RANDOM NUMBERS

Let $U$ be a uniform random variable between 0 and 1.

1.1 Generate $10^6$ samples of $U$ using a C program and save into a file called uni.dat .
**Solution:** Download the following file and execute the C program.

> https://github.com/AkshithaKola/
>     Random_numbers/blob/main/Codes/1.1.c

1.2 Load the uni.dat file into python and plot the empirical CDF of $U$ using the samples in uni.dat. The CDF is defined as

$$F_U(x) = \Pr\left(U \le x\right) \quad (1.1)$$

**Solution:** The following code plots Fig. 1.2

> https://github.com/AkshithaKola/
>     Random_numbers/blob/main/Codes/1.2.
>     py

1.3 Find a theoretical expression for $F_U(x)$.
**Solution:** The CDF of $U$ is

$$F_U(x) = \Pr\left(U \le x\right) = \int_{-\infty}^{x} p_U(x) \, \mathrm{d}x \quad (1.2)$$

The PDF of $U$ is

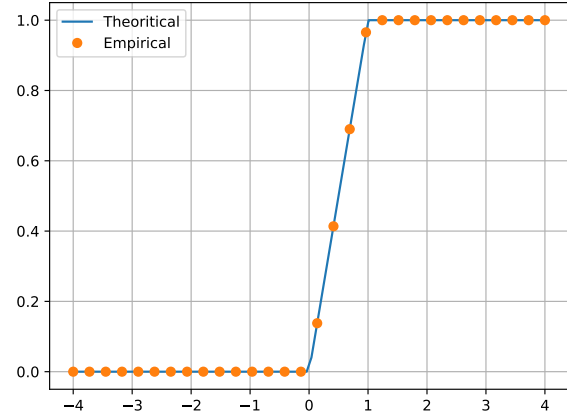$$p_U(x) = \begin{cases} 1 & x \in [0, 1] \\ 0 & \text{otherwise} \end{cases} \quad (1.3)$$



Fig. 1.2: The CDF of $U$

If $x < 0$,

$$\int_{-\infty}^{x} p_U(x) \, \mathrm{d}x = \int_{-\infty}^{x} 0 \, \mathrm{d}x = 0 \quad (1.4)$$

If $0 \le x < 1$,

$$\int_{-\infty}^{x} p_U(x) \, \mathrm{d}x = \int_{-\infty}^{0} 0 \, \mathrm{d}x + \int_{0}^{x} 1 \, \mathrm{d}x \quad (1.5)$$

$$= 0 + x = x \quad (1.6)$$

$$(1.7)$$

If $x \ge 1$,

$$\int_{-\infty}^{x} p_U(x) \, \mathrm{d}x$$

$$= \int_{-\infty}^{0} 0 \, \mathrm{d}x + \int_{0}^{1} 1 \, \mathrm{d}x + \int_{1}^{x} 0 \, \mathrm{d}x \quad (1.8)$$

$$\int_{-\infty}^{x} p_U(x) \, \mathrm{d}x = 0 + 1 + 0 \quad (1.9)$$

$$= 1 \quad (1.10)$$

So, the CDF of $U$ is

$$F_U(x) = \begin{cases} 0 & x < 0 \\ x & 0 \le x \le 1 \\ 1 & x > 1 \end{cases} \qquad (1.11)$$

1.4 The mean of $U$ is defined as

$$E[U] = \frac{1}{N} \sum_{i=1}^{N} U_i \qquad (1.12)$$

and its variance as

$$\text{var}[U] = E[U - E[U]]^2 \qquad (1.13)$$

Write a C program to find the mean and variance of $U$.
**Solution:** The following code gives the mean and variance.

https://github.com/AkshithaKola/
    Random_numbers/blob/main/Codes/1.4.c

1.5 Verify your result theoretically given that

$$E\left[U^k\right] = \int_{-\infty}^{\infty} x^k dF_U(x) \qquad (1.14)$$

**Solution:** Given $E\left[U^k\right] = \int_{-\infty}^{\infty} x^k dF_U(x)$

$$mean = E[U] \qquad (1.15)$$

$$E[U] = \int_{-\infty}^{\infty} x dF_U(x) \qquad (1.16)$$

$$p_U(x) = \frac{dF_U(x)}{dx} \qquad (1.17)$$

$$dF_U(x) = p_U(x)dx \qquad (1.18)$$

$$dF_U(x) = \begin{cases} 0 & x < 0 \\ dx & 0 \le x \le 1 \\ 0 & x > 1 \end{cases} \qquad (1.19)$$

$$\therefore E[U] = \int_0^1 x \, dx = \frac{1}{2} = 0.5 \qquad (1.20)$$

$$\text{variance} = E(U - E[U])^2$$

$$= E(U^2 - 2U(E[U]) + (E[U])^2) \qquad (1.21)$$

$$= E[U^2] + E[-2UE[U]] + [(E[(U)^2]] \qquad (1.22)$$

$$= E[U^2] - 2E[U]E[U] + (E[U])^2 \qquad (1.23)$$

$$= E[U^2] - (E[U])^2 \qquad (1.24)$$

$$\qquad (1.25)$$

$$E[U^2] = \int_0^1 x^2 \, dx = \frac{1}{3} \qquad (1.26)$$

$\therefore$ variance is

$$= \frac{1}{3} - \left(\frac{1}{2}\right)^2 \qquad (1.27)$$

$$= \frac{1}{12} \approx 0.083333 \qquad (1.28)$$

## 2 CENTRAL LIMIT THEOREM

2.1 Generate $10^6$ samples of the random variable

$$X = \sum_{i=1}^{12} U_i - 6 \qquad (2.1)$$

using a C program, where $U_i, i = 1, 2, \dots, 12$ are a set of independent uniform random variables between 0 and 1 and save in a file called gau.dat
**Solution:** The following is the code to generate X a

https://github.com/AkshithaKola/
    Random_numbers/blob/main/Codes/2.1.c

2.2 Load gau.dat in python and plot the empirical CDF of $X$ using the samples in gau.dat. What properties does a CDF have?
**Solution:** The following code plots Fig. 2.2

https://github.com/AkshithaKola/
    Random_numbers/blob/main/Codes/2.2.
    py

The CDF of $X$ is plotted in Fig. 2.2
 We can say that every CDF is a non decreasing function and is always in the range $[0, 1]$

2.3 Load gau.dat in python and plot the empirical PDF of $X$ using the samples in gau.dat. The PDF of $X$ is defined as

$$p_X(x) = \frac{d}{dx} F_X(x) \qquad (2.2)$$

What properties does the PDF have?
**Solution:** The PDF of $X$ is plotted in Fig. 2.3 using the code below

https://github.com/AkshithaKola/
    Random_numbers/blob/main/Codes/2.3.
    py

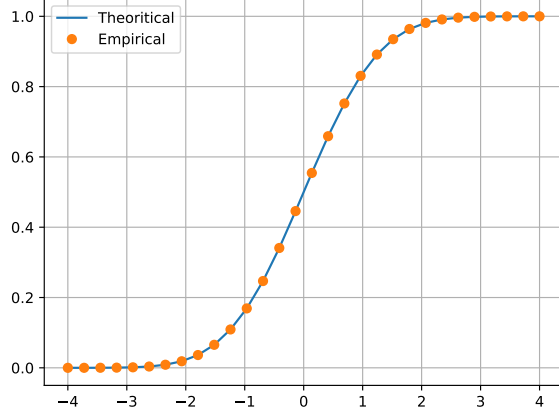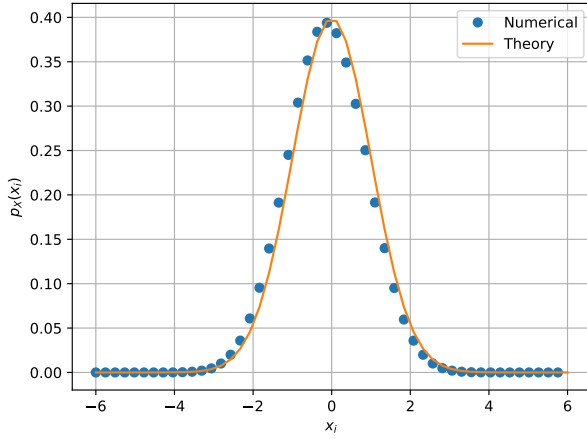2.4 Find the mean and variance of $X$ by writing a C program.

Fig. 2.2: The CDF of $X$



Fig. 2.3: The PDF of $X$

**Solution:** The following is the code for mean and variance of $X$

https://github.com/AkshithaKola/
Random_numbers/blob/main/Codes/2.4.c

2.5  Given that

$$p_X(x) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right), -\infty < x < \infty, \quad (2.3)$$

repeat the above exercise theoretically.
**Solution:** The mean of $X$ is

$$E[X] = \int_{-\infty}^{\infty} x p_X(x) \mathrm{d}x \quad (2.4)$$

$$= \int_{-\infty}^{\infty} \frac{x}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) \mathrm{d}x \quad (2.5)$$

Let

$$f(x) = \frac{x}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) \quad (2.6)$$

$$f(-x) = -f(x) \quad (2.7)$$

$\therefore$ $f(x)$ is an odd function.
So,

$$E[X] = \int_{-\infty}^{\infty} f(x)\mathrm{d}x = 0 \quad (2.8)$$

Now,

$$E[X^2] = \int_{-\infty}^{\infty} x^2 p_X(x)\mathrm{d}x \quad (2.9)$$

$$= \int_{-\infty}^{\infty} \frac{x^2}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right)\mathrm{d}x \quad (2.10)$$

$$(2.11)$$

let,

$$g(x) = \int_{-\infty}^{\infty} \frac{x^2}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right) \quad (2.12)$$

$$g(x) = g(-x) \quad (2.13)$$

$\therefore$ $g(x)$ is an even function.
So we can write

$$E[X^2] = 2 \int_{0}^{\infty} \frac{x^2}{\sqrt{2\pi}} \exp\left(-\frac{x^2}{2}\right)\mathrm{d}x \quad (2.14)$$

Using integration by parts,

$$E[X^2] = \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} x \cdot x \exp\left(-\frac{x^2}{2}\right)\mathrm{d}x \quad (2.15)$$

$$= \sqrt{\frac{2}{\pi}} \left( x \int x \exp\left(-\frac{x^2}{2}\right) \mathrm{d}x \right) \Bigg|_{0}^{\infty}$$

$$- \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} 1 \cdot \int x \exp\left(-\frac{x^2}{2}\right)\mathrm{d}x \quad (2.16)$$

Substitute $t = \frac{x^2}{2} \implies \mathrm{d}t = x\mathrm{d}x$

$$\int x \exp\left(-\frac{x^2}{2}\right)\mathrm{d}x = \int \exp(-t)\mathrm{d}t \quad (2.17)$$

$$= -\exp(-t) \quad (2.18)$$

$$= -\exp\left(-\frac{x^2}{2}\right) \quad (2.19)$$

As,

$$\lim_{x \to \infty} x \exp\left(-\frac{x^2}{2}\right) = \lim_{x \to \infty} \frac{x}{\exp\left(\frac{x^2}{2}\right)} = 0 \quad (2.20)$$

$$\implies \left. -x \exp\left(-\frac{x^2}{2}\right)\right|_0^\infty = 0 - 0 = 0 \quad (2.21)$$

$$(2.22)$$

And,

$$\int_0^\infty -\exp\left(-\frac{x^2}{2}\right) dx \quad (2.23)$$

$$\xleftrightarrow{x = t\sqrt{2}} \int_0^\infty -\exp(-t^2) dt \sqrt{2} \quad (2.24)$$

$$= -\sqrt{2} \int_0^\infty \exp(-t^2) dt \quad (2.25)$$

$$= -\sqrt{2} \frac{\sqrt{\pi}}{2} \quad (2.26)$$

$$= -\sqrt{\frac{\pi}{2}} \quad (2.27)$$

Therefore,

$$E[X^2] = 0 - \sqrt{\frac{2}{\pi}} \left(-\sqrt{\frac{\pi}{2}}\right) \quad (2.28)$$

$$= 1 \quad (2.29)$$

$$\therefore var[X] = E[X^2] - (E[X])^2 \quad (2.30)$$

$$= 1 - 0 \quad (2.31)$$

$$= 1 \quad (2.32)$$

## 3 FROM UNIFORM TO OTHER

### 3.1 Generate samples of

$$V = -2\ln(1 - U) \quad (3.1)$$

and plot its CDF.
**Solution:** The following are the codes of samples of V and the CDF plot of V.

https://github.com/AkshithaKola/
 Random_numbers/blob/main/Codes/3.1.c

https://github.com/AkshithaKola/
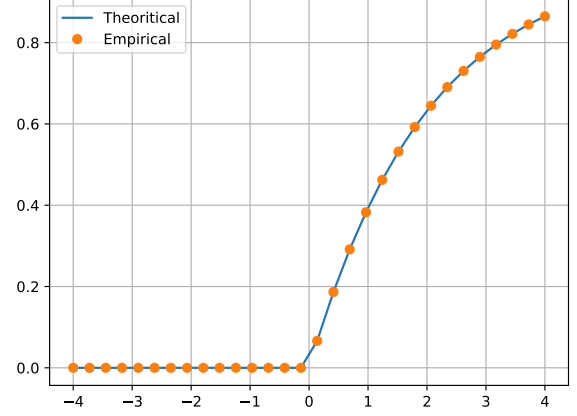 Random_numbers/blob/main/Codes/3.1.
 py



Fig. 3.1: The CDF of $V$

### 3.2 Find a theoretical expression for $F_V(x)$.
**Solution:**

$$F_V(x) = \Pr(V \le x) \quad (3.2)$$

$$= \Pr(-2\ln(1 - U) \le x) \quad (3.3)$$

$$= \Pr\left(\ln(1 - U) \ge -\frac{x}{2}\right) \quad (3.4)$$

$$= \Pr\left(1 - U \ge \exp\left(-\frac{x}{2}\right)\right) \quad (3.5)$$

$$= \Pr\left(U \le 1 - \exp\left(-\frac{x}{2}\right)\right) \quad (3.6)$$

$$= F_U\left(1 - \exp\left(-\frac{x}{2}\right)\right) \quad (3.7)$$

As,

$$F_U(x) = \begin{cases} 0 & x < 0 \\ x & 0 \le x \le 1 \\ 1 & x > 1 \end{cases} \quad (3.8)$$

Now,

$$0 \le 1 - \exp\left(-\frac{x}{2}\right) < 1 \qquad \text{if } x \ge 0 \quad (3.9)$$

$$1 - \exp\left(-\frac{x}{2}\right) < 0 \qquad \text{if } x < 0 \quad (3.10)$$

So

$$F_V(x) = \begin{cases} 1 - \exp\left(-\frac{x}{2}\right) & x \ge 0 \\ 0 & x < 0 \end{cases} \quad (3.11)$$