

Digital Signal Processing

EE3900

AI21BTECH11017

CONTENTS

- 1 Definitions
- 2 Laplace Transform
- 3 Initial Conditions

1. DEFINITIONS

1. The unit step function is

$$u(t) = \begin{cases} 1 & t > 0 \\ \frac{1}{2} & t = 0 \\ 0 & t < 0 \end{cases} \quad (1.1)$$

2. The Laplace transform of $g(t)$ is defined as

$$G(s) = \int_{-\infty}^{\infty} g(t)e^{-st} dt \quad (1.2)$$

2. LAPLACE TRANSFORM

1. In the circuit, the switch S is connected to position P for a long time so that the charge on the capacitor becomes $q_1 \mu C$. Then S is switched to position Q. After a long time, the charge on the capacitor is $q_2 \mu C$.
2. Draw the circuit using latex-tikz.

Solution: The following code yields Fig.2.1

```
wget https://github.com/Pradeep8802/EE3900
-Digital-Signal-Processing/blob/main/
cktsig/Tikz%20Circuits/2.2.tex
```

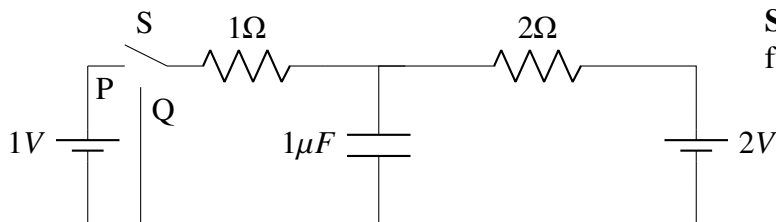


Fig. 2.1. Given Circuit

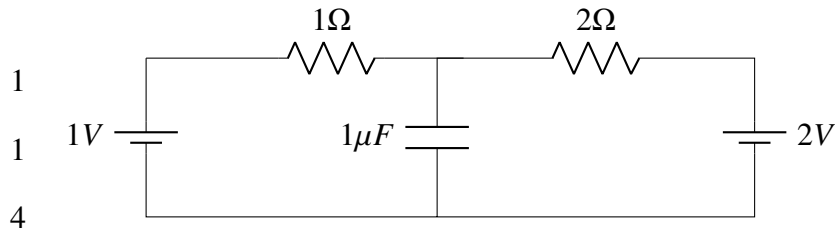


Fig. 2.2. Before switching S to Q

3. Find q_1 .

Solution: Before switching S to Q: At steady state, which achieved when switch S is at P for long time capacitor behaves as an open switch, hence current through capacitor is 0, Let i be the current flowing in the circuit at steady state. Applying KVL ,

$$1 - i - 2i - 2 = 0 \quad (2.1)$$

$$3i = -1 \Rightarrow i = \frac{-1}{3} A \quad (2.2)$$

Potential Difference across the capacitor at steady state is

$$1 - \left(\frac{-1}{3} \right) = \frac{4}{3} V \quad (2.3)$$

$$q_1 = \frac{4}{3} \cdot 1 \quad (2.4)$$

$$= \frac{4}{3} \mu C \quad (2.5)$$

4. Show that the Laplace transform of $u(t)$ is $\frac{1}{s}$ and find the ROC.

Solution: We know that Laplace Transform for function $f(t)$ is given as $F(s)$,

$$F(s) = \int_0^{\infty} f(t)e^{-st} dt \quad (2.6)$$

$$(2.7)$$

For $u(t)$, we have,

$$F(s) = \int_0^{\infty} u(t)e^{-st} dt \quad (2.8)$$

Using (1.1),

$$F(s) = \int_0^{\infty} u(t)e^{-st} dt \quad (2.9)$$

$$= \int_0^{\infty} e^{-st} dt \quad (2.10)$$

$$= -\left(0 - \frac{1}{s}\right) \quad (2.11)$$

$$= \frac{1}{s} \quad (2.12)$$

ROC is $Re(s) > 0$ since for $s > 0$, $e^{-st} < \infty$ for $t \rightarrow \infty$

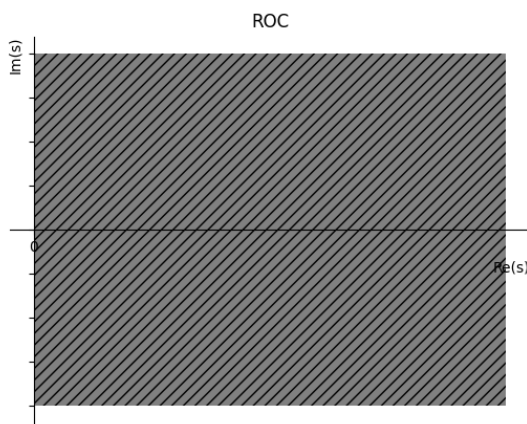


Fig. 2.3.

5. Show that

$$e^{-at}u(t) \xleftrightarrow{\mathcal{L}} \frac{1}{s+a}, \quad a > 0 \quad (2.13)$$

and find the ROC.

Solution: From 2.6,

$$F(s) = \int_0^{\infty} u(t)e^{-at}e^{-st} dt \quad (2.14)$$

$$= \int_0^{\infty} u(t)e^{-(s+a)t} dt \quad (2.15)$$

$$= \int_0^{\infty} e^{-(s+a)t} dt \quad (2.16)$$

$$= -\left(0 - \frac{1}{s+a}\right) \quad (2.17)$$

$$= \frac{1}{s+a} \quad (2.18)$$

ROC is

$$Re(s) + a > 0 \Rightarrow Re(s) > -a \quad (2.19)$$

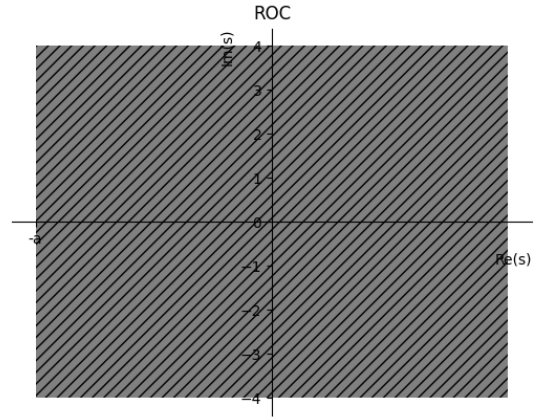


Fig. 2.4.

6. Now consider the following resistive circuit transformed from Fig. ?? where

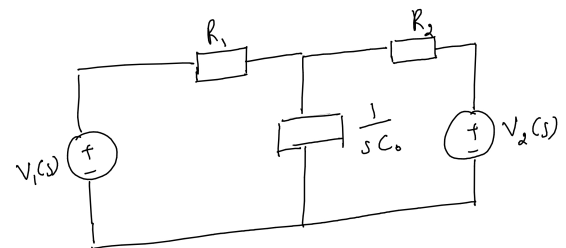


Fig. 2.5.

$$u(t) \xleftrightarrow{\mathcal{L}} V_1(s) \quad (2.20)$$

$$2u(t) \xleftrightarrow{\mathcal{L}} V_2(s) \quad (2.21)$$

Find the voltage across the capacitor $V_{C_0}(s)$.

Solution:

$$R_{eff} = \frac{1}{1 + \frac{1}{2}} = \frac{2}{3} \Omega \quad (2.22)$$

$$V_{eff} = \frac{1}{1 + \frac{1}{2}} = \frac{2}{3} V \quad (2.23)$$

$$V_{C_0}(s) = V_S(s) \frac{C_0}{C_0 + R_{eff}} \quad (2.24)$$

$$= \left(\frac{4}{3s}\right) \left(\frac{\frac{1}{s}}{\frac{1}{s} + \frac{2}{3}}\right) \quad (2.25)$$

$$= \frac{3 + 4s}{3s\left(s + \frac{3}{2}\right)} \quad (2.26)$$

7. Find $v_{C_0}(t)$. Plot using python.

Solution: Running the following code gives the plot.

```
wget https://github.com/Pradeep8802/EE3900
-Digital-Signal-Processing/tree/main/
cktsig/codes/2.7.py
```

Using (2.26),

$$\frac{3 + 4s}{3s(s + \frac{3}{2})} = \frac{2}{3s} + \frac{2}{3(\frac{3}{2} + s)} \quad (2.27)$$

Apply inverse Laplacian Transform,

$$V_{C_0}(s) \xleftrightarrow{\mathcal{L}^{-1}} V_{C_0}(t) \quad (2.28)$$

$$\mathcal{L}^{-1}[V_{C_0}(s)] = \mathcal{L}^{-1}\left[\frac{2}{3s} + \frac{2}{3(\frac{3}{2} + s)}\right] \quad (2.29)$$

$$= \mathcal{L}^{-1}\left[\frac{2}{3s}\right] - \frac{2}{3}\mathcal{L}^{-1}\left[\frac{1}{\frac{3}{2} + s}\right] \quad (2.30)$$

Since,

$$\mathcal{L}^{-1}\left[\frac{1}{s}\right] = u(t) \quad (2.31)$$

$$\mathcal{L}^{-1}\left[\frac{1}{s - a}\right] = e^{at}u(t) \quad (2.32)$$

Using the above equations,

$$V_{C_0}(t) = \frac{2}{3}\left(1 + e^{-\frac{3}{2}t}\right)u(t) \quad (2.33)$$

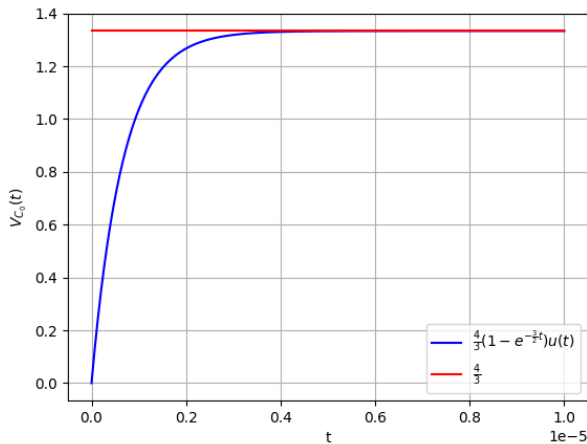


Fig. 2.6. Plot of $V_{C_0}(t)$

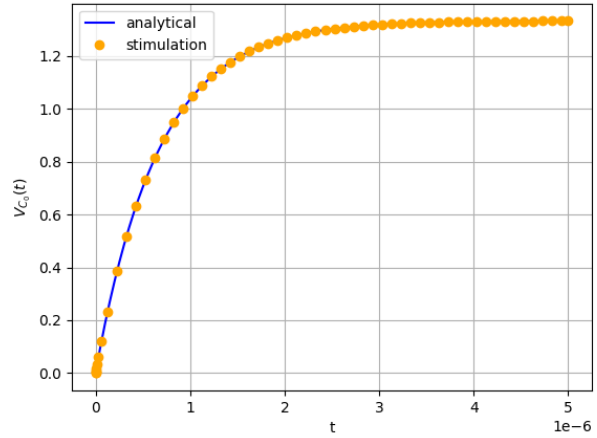


Fig. 2.7.

8. Verify your result using ngspice.

Solution:

9. Obtain Fig. 2.6 using the equivalent differential equation

Solution: Results obtained can be verified by running the following code.

```
wget https://github.com/Pradeep8802/EE3900
-Digital-Signal-Processing/tree/main/
cktsig/codes/2.8.cir
```

And is plotted using the below code.

```
wget https://github.com/Pradeep8802/EE3900
-Digital-Signal-Processing/tree/main/
cktsig/codes/2.8.py
```

Using Kirchoff's junction law

$$\frac{v_c(t) - v_1(t)}{R_1} + \frac{v_c(t) - v_2(t)}{R_2} + \frac{dq}{dt} = 0 \quad (2.34)$$

where $q(t)$ is the charge on the capacitor

On taking the Laplace transform on both sides of this equation

$$\frac{V_c(s) - V_1(s)}{R_1} + \frac{V_c(s) - V_2(s)}{R_2} + (sQ(s) - q(0^-)) = 0 \quad (2.35)$$

But $q(0^-) = 0$ and

$$q(t) = C_0 v_c(t) \quad (2.36)$$

$$\Rightarrow Q(s) = C_0 V_c(s) \quad (2.37)$$

Thus

$$\frac{V_c(s) - V_1(s)}{R_1} + \frac{V_c(s) - V_2(s)}{R_2} + sC_0 V_c(s) = 0 \quad (2.38)$$

$$\Rightarrow \frac{V_c(s) - V_1(s)}{R_1} + \frac{V_c(s) - V_2(s)}{R_2} + \frac{V_c(s) - 0}{\frac{1}{sC_0}} = 0 \quad (2.39)$$

which is the same equation as the one we obtained from Fig. 2.6

3. INITIAL CONDITIONS

1. Find q_2 in Fig. 2.1.

Solution: At steady state capacitor behaves as an open switch. Hence $V_{C_0} = V_{1\Omega}$.

Let i be the current in the circuit. Using KVL,

$$2 - 2i - i = 0 \Rightarrow i = \frac{2}{3} \quad (3.1)$$

$$V_{1\Omega} = i \times 1 = \frac{2}{3} V \quad (3.2)$$

$$V_{C_0} = \frac{q_2}{C_0} = V_{1\Omega} = \frac{2}{3} \quad (3.3)$$

$$\Rightarrow q_2 = \frac{2}{3} \mu C \quad (3.4)$$

2. Draw the equivalent s -domain resistive circuit when S is switched to position Q. Use variables R_1, R_2, C_0 for the passive elements. Use latex-tikz.

Solution:

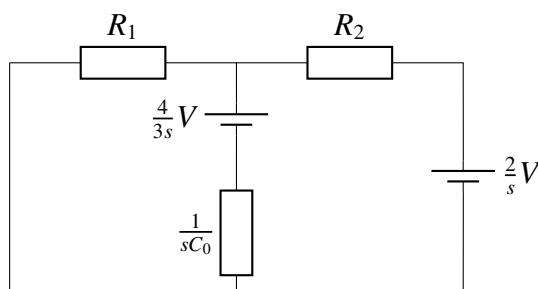


Fig. 3.1. After switching S to Q

3. $V_{C_0}(s) = ?$

Solution: Let voltage across capacitor be V . Using KCL at node in Fig. 3.1

$$\frac{V - 0}{R_1} + \frac{V - \frac{2}{s}}{R_2} + sC_0 \left(V - \frac{4}{3s} \right) = 0 \quad (3.5)$$

$$\Rightarrow V_{C_0}(s) = \frac{\frac{2}{sR_2} + \frac{4C_0}{3}}{\frac{1}{R_1} + \frac{2}{R_2} + sC_0} \quad (3.6)$$

4. $v_{C_0}(t) = ?$ Plot using python.

Solution: Running the following code gives the plot.

```
wget https://github.com/Pradeep8802/EE3900
-Digital-Signal-Processing/tree/main/
cktsig/codes/3.4.py
```

From (3.6),

$$V_{C_0}(s) = \frac{4}{3} \left(\frac{1}{\frac{1}{C_0} \left(\frac{1}{R_1} + \frac{1}{R_2} \right) + s} \right) + \frac{2}{R_2 \left(\frac{1}{R_1} + \frac{1}{R_2} \right)} \left(\frac{1}{s} - \frac{1}{\frac{1}{C_0} \left(\frac{1}{R_1} + \frac{1}{R_2} \right) + s} \right) \quad (3.7)$$

Taking an inverse Laplace Transform,

$$v_{C_0}(t) = \frac{4}{3} e^{-\left(\frac{1}{R_1} + \frac{1}{R_2}\right) \frac{t}{C_0}} u(t) + \frac{2}{R_2 \left(\frac{1}{R_1} + \frac{1}{R_2} \right)} \left(1 - e^{-\left(\frac{1}{R_1} + \frac{1}{R_2}\right) \frac{t}{C_0}} \right) u(t) \quad (3.8)$$

Substituting values gives

$$v_{C_0}(t) = \frac{2}{3} \left(1 + e^{-(1.5 \times 10^6)t} \right) u(t) \quad (3.9)$$

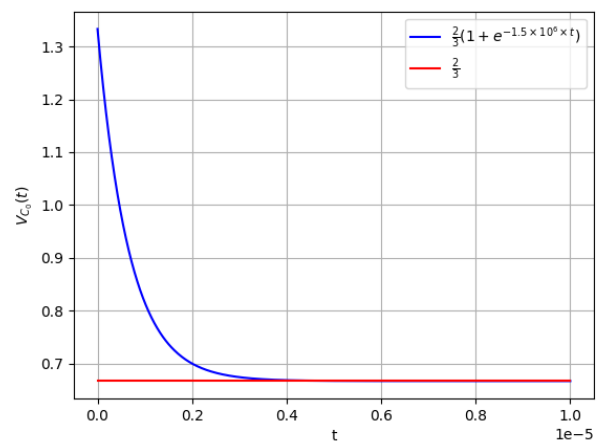


Fig. 3.2. Plot of $V_{C_0}(t)$

5. Verify your result using ngspice.

Solution: Results obtained can be verified by running the following code.

```
wget https://github.com/Pradeep8802/EE3900
-Digital-Signal-Processing/tree/main/
cktsig/codes/3.5.cir
```

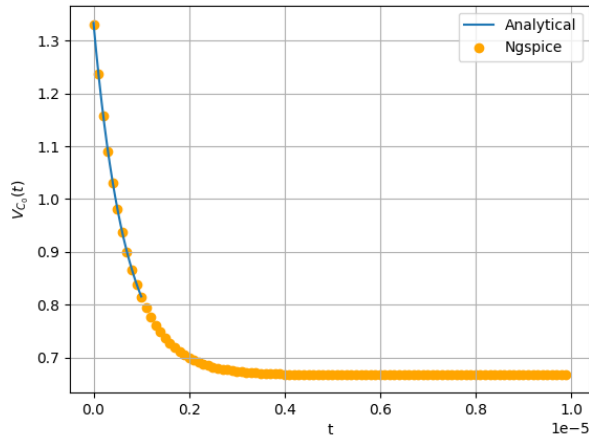


Fig. 3.3. ngspice plot of $V_{C_0}(t)$

Running the below code plots the figure 3.3, and verifies our result.

```
wget https://github.com/Pradeep8802/EE3900
-Digital-Signal-Processing/tree/main/
cktsig/codes/3.5.py
```

6. Find $v_{C_0}(0^-)$, $v_{C_0}(0^+)$ and $v_{C_0}(\infty)$.

Solution: From the initial conditions,

$$v_{C_0}(0^-) = \frac{q_1}{C} = \frac{4}{3}V \quad (3.10)$$

Using (3.9),

$$v_{C_0}(0^+) = \lim_{t \rightarrow 0^+} v_{C_0}(t) = \frac{4}{3}V \quad (3.11)$$

$$v_{C_0}(\infty) = \lim_{t \rightarrow \infty} v_{C_0}(t) = \frac{2}{3}V \quad (3.12)$$

7. Obtain Fig. 3.2 using the equivalent differential equation

Solution: Using Kirchoff's junction law

$$\frac{v_c(t) - 0}{R_1} + \frac{v_c(t) - v_2(t)}{R_2} + \frac{dq}{dt} = 0 \quad (3.13)$$

where $q(t)$ is the charge on the capacitor. On taking the Laplace transform on both sides of the equation (3.13), we get,

$$\frac{V_c(s) - 0}{R_1} + \frac{V_c(s) - V_2(s)}{R_2} + sQ(s) - q(0^-) = 0 \quad (3.14)$$

But $q(0^-) = \frac{4}{3}C_0$ and

$$q(t) = C_0 v_c(t) \quad (3.15)$$

$$\Rightarrow Q(s) = C_0 V_c(s) \quad (3.16)$$

Thus

$$\frac{V_c(s) - 0}{R_1} + \frac{V_c(s) - V_2(s)}{R_2} + \left(sC_0 V_c(s) - \frac{4}{3}C_0 \right) = 0 \quad (3.17)$$

$$\Rightarrow \frac{V_c(s) - 0}{R_1} + \frac{V_c(s) - V_2(s)}{R_2} + \frac{V_c(s) - \frac{4}{3s}}{\frac{1}{sC_0}} = 0 \quad (3.18)$$

which is the same equation as the one we obtained from Fig. 3.2