Pingala Series

AI21BTECH11017

1. JEE 2019

Let

$$a_n = \frac{\alpha^n - \beta^n}{\alpha - \beta}, \quad n \geq 1 \tag{1.1}$$

$$b_n = a_{n-1} + a_{n+1}, \quad n \geq 2, \quad b_1 = 1 \tag{1.2}$$

Verify the following using a python code.

1.1
$$\sum_{k=1}^{n} a_k = a_{n+2} - 1, \quad n \geq 1 \tag{1.3}$$

1.2
$$\sum_{k=1}^{\infty} \frac{a_k}{10^k} = \frac{10}{89} \tag{1.4}$$

1.3
$$b_n = \alpha^n + \beta^n, \quad n \geq 1 \tag{1.5}$$

1.4
$$\sum_{k=1}^{\infty} \frac{b_k}{10^k} = \frac{8}{89} \tag{1.6}$$

Solution: Download the following Python code that verifies the above

```
wget https://github.com/Ankit−Saha−2003/EE3900
    /raw/main/Pingala/codes/1.py
```

Run the code by executing

```
python 1.py
```

It turns out that everything apart from the last equation is true

2. Pingala Series

2.1 The *one sided* Z-transform of $x(n)$ is defined as

$$X^{+}(z) = \sum_{n=0}^{\infty} x(n) z^{-n}, \quad z \in \mathbb{C} \tag{2.1}$$

2.2 The *Pingala* series is generated using the difference equation

$$x(n+2) = x(n+1) + x(n) \tag{2.2}$$

$$x(0) = x(1) = 1, n \geq 0 \tag{2.3}$$

Generate a stem plot for $x(n)$.
Solution: Download the following Python code that plots Fig. 2.2

```
wget https://github.com/Ankit−Saha−2003/
    EE3900/raw/main/Pingala/codes/2.2.py
```

Run the code by executing

```
python 2.2.py
```

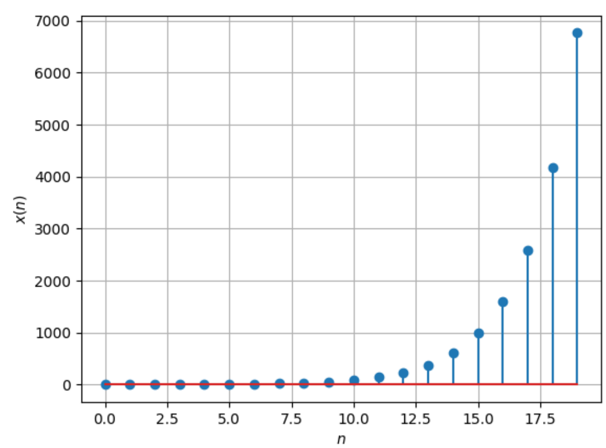

Fig. 2.2. Plot of $x(n)$

2.3 Find $X^+(z)$

Solution:

We have

$$x(n+2) = x(n+1) + x(n) \qquad n \geq 0 \tag{2.4}$$

On taking the one-sided Z-transform on both sides of the equation, we get

$$\mathcal{Z}^+\{x(n+2)\} = \mathcal{Z}^+\{x(n+1) + x(n)\} \tag{2.5}$$

Since, the one-sided Z-transform is a linear transformation

$$\implies \mathcal{Z}^+\{x(n+2)\} = \mathcal{Z}^+\{x(n+1)\} + \mathcal{Z}^+\{x(n)\} \tag{2.6}$$

$$\implies \sum_{n=0}^{\infty} x(n+2)z^{-n} = \sum_{n=0}^{\infty} x(n+1)z^{-n} + \sum_{n=0}^{\infty} x(n)z^{-n} \tag{2.7}$$

But

$$\sum_{n=0}^{\infty} x(n+2)z^{-n} = z^2 \sum_{k=2}^{\infty} x(k)z^{-k} \tag{2.8}$$

$$= z^2\left(X^+(z) - x(0) - x(1)z^{-1}\right) \tag{2.9}$$

$$= z^2X^+(z) - z^2 - z \tag{2.10}$$

$$\sum_{n=0}^{\infty} x(n+1)z^{-n} = z\sum_{k=1}^{\infty} x(k)z^{-k} \tag{2.11}$$

$$= z\left(X^+(z) - x(0)\right) \tag{2.12}$$

$$= zX^+(z) - z \tag{2.13}$$

Thus

$$\implies z^2X^+(z) - z^2 - z = zX^+(z) - z + X^+(z) \tag{2.14}$$

$$\implies \left(z^2 - z - 1\right)X^+(z) = z^2 \tag{2.15}$$

$$\implies X^+(z) = \frac{1}{1 - z^{-1} - z^{-2}} \tag{2.16}$$

$$\therefore X^+(z) = \frac{1}{(1-\alpha z^{-1})(1-\beta z^{-1})} \qquad |z| > \max(|\alpha| \tag{2.17}$$

2.4 Find $x(n)$

Solution: Using partial fraction decomposition

$$X^+(z) = \frac{1}{(\alpha - \beta)z^{-1}}\left(\frac{1}{1-\alpha z^{-1}} - \frac{1}{1-\beta z^{-1}}\right) \tag{2.18}$$

We know that

$$\frac{1}{1-\alpha z^{-1}} \overset{\mathcal{Z}}{\rightleftharpoons} \alpha^n u(n) \tag{2.19}$$

$$\frac{1}{1-\beta z^{-1}} \overset{\mathcal{Z}}{\rightleftharpoons} \beta^n u(n) \tag{2.20}$$

$$zF(z) \overset{\mathcal{Z}}{\rightleftharpoons} f(n+1) \tag{2.21}$$

Since we are dealing with the unit step function here, the Z-transform and the one-sded Z-transform are both the same

$$x(n) = \frac{1}{\alpha - \beta}\left(\alpha^{n+1}u(n+1) - \beta^{n+1}u(n+1)\right) \tag{2.22}$$

Since at $n = -1$, $x(n)$ is zero anyway, we can write

$$x(n) = \frac{\alpha^{n+1} - \beta^{n+1}}{\alpha - \beta}u(n) \tag{2.23}$$

2.5 Sketch

$$y(n) = x(n-1) + x(n+1) \qquad n \geq 0 \tag{2.24}$$

Solution: Download the following Python code that plots Fig. 2.5

```
wget https://github.com/Ankit−Saha−2003/
    EE3900/raw/main/Pingala/codes/2.5.py
```

Run the code by executing

```
python 2.5.py
```

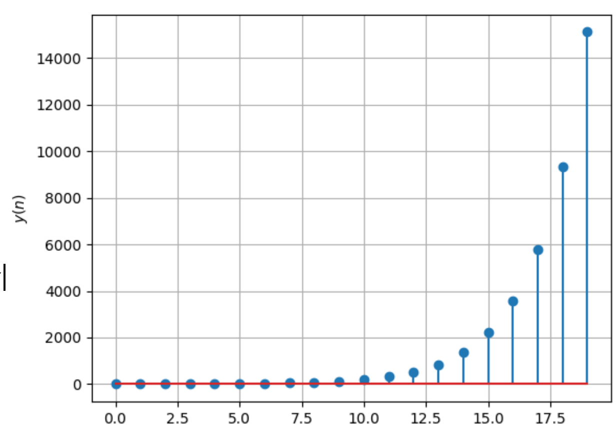

Fig. 2.5. Plot of $y(n)$

2.6 Find $Y^+(z)$

Solution: We have

$$y(n) = x(n-1) + x(n+1) \quad n \geq 0 \tag{2.25}$$

On taking the one-sided Z-transform on both sides of the equation, we get

$$\mathcal{Z}^+\{y(n)\} = \mathcal{Z}^+\{x(n-1) + x(n+1)\} \tag{2.26}$$
$$= \mathcal{Z}^+\{x(n+1)\} + \mathcal{Z}^+\{x(n-1)\} \tag{2.27}$$
$$= \sum_{n=0}^{\infty} x(n+1)z^{-n} + \sum_{n=0}^{\infty} x(n-1)z^{-n} \tag{2.28}$$

But

$$\sum_{n=0}^{\infty} x(n-1)z^{-n} = z^{-1}\sum_{k=-1}^{\infty} x(k)z^{-k} \tag{2.29}$$
$$= z^{-1}\left(X^+(z) + x(-1)z\right) \tag{2.30}$$
$$= z^{-1}X^+(z) \tag{2.31}$$

Thus

$$Y^+(z) = zX^+(z) - z + z^{-1}X^+(z) \tag{2.32}$$
$$= \frac{z + z^{-1}}{1 - z^{-1} - z^{-2}} - z \tag{2.33}$$
$$= \frac{z + z^{-1} - z + 1 + z^{-1}}{1 - z^{-1} - z^{-2}} \tag{2.34}$$
$$= \frac{1 + 2z^{-1}}{1 - z^{-1} - z^{-2}} \qquad |z| > \max\left(|\alpha|, |\beta|\right) \tag{2.35}$$

2.7 Find $y(n)$
Solution:

$$y(n) = x(n+1) + x(n-1) \quad n \geq 0 \tag{2.36}$$
$$= \frac{\alpha^{n+2} - \beta^{n+2}}{\alpha - \beta}u(n+1) + \frac{\alpha^n - \beta^n}{\alpha - \beta}u(n-1) \tag{2.37}$$

We have

$$y(n) = 0 \qquad \forall n < 0 \tag{2.38}$$
$$y(0) = x(1) + x(-1) = 1 + 0 = 1 \tag{2.39}$$

For $n > 0$

$$y(n) = \frac{\alpha^{n+2} - \beta^{n+2} + \alpha^n - \beta^n}{\alpha - \beta} \tag{2.40}$$

Since $\alpha\beta = -1$

$$y(n) = \frac{\alpha^{n+2} - (\alpha\beta)\alpha^n - \beta^{n+2} + (\alpha\beta)\beta^n}{\alpha - \beta} \tag{2.41}$$
$$= \frac{\alpha^{n+1}(\alpha - \beta) + \beta^{n+1}(\alpha - \beta)}{\alpha - \beta} \tag{2.42}$$
$$= \alpha^{n+1} + \beta^{n+1} \tag{2.43}$$

If we plug in $n = 0$ in this equation, we get

$$y(0) = \alpha + \beta = 1 \tag{2.44}$$

which is consistent with what we obtained earlier
Therefore

$$y(n) = \left(\alpha^{n+1} + \beta^{n+1}\right)u(n) \tag{2.45}$$

3. Power of the Z transform

3.1 Show that

$$\sum_{k=1}^{n} a_k = \sum_{k=0}^{n-1} x(k) = x(n) * u(n-1) \tag{3.1}$$

Solution: We have obtained that $a_n = x(n-1)$

$$\sum_{k=1}^{n} a_k = \sum_{k=1}^{n} x(k-1) \tag{3.2}$$
$$= \sum_{k=0}^{n-1} x(k) \tag{3.3}$$

Also

$$x(k) = \begin{cases} 1 & k \geq 0 \\ 0 & k < 0 \end{cases} \tag{3.4}$$

$$u(n-1-k) = \begin{cases} 1 & k \leq n-1 \\ 0 & k > n-1 \end{cases} \tag{3.5}$$

Thus

$$\sum_{k=0}^{n-1} x(k) = \sum_{k=-\infty}^{\infty} x(k)u(n-1-k) \tag{3.6}$$
$$= x(n) * u(n-1) \tag{3.7}$$

3.2 Show that

$$a_{n+2} - 1 \qquad n \geq 1 \tag{3.8}$$

can be expressed as

$$\left(x(n+1) - 1\right)u(n) \tag{3.9}$$

Solution:

$$a_{n+2} - 1 \qquad n \geq 1 \tag{3.10}$$

can be written as

$$(a_{n+2} - 1)u(n-1) \tag{3.11}$$

Since the system defined by a is time-invariant and a_n is just $x(n)$ shifted forward by one unit, we can shift this backward by one unit to get the expression in terms of x

$$a_{n+2} \rightleftharpoons x(n+1) \tag{3.12}$$
$$u(n-1) \rightleftharpoons u(n) \tag{3.13}$$

Therefore, the above expression can be expressed as

$$(x(n+1) - 1)u(n) \tag{3.14}$$

3.3 Show that

$$\sum_{k=1}^{\infty} \frac{a_k}{10^k} = \frac{1}{10}\sum_{k=0}^{\infty} \frac{x(k)}{10^k} = \frac{1}{10}X^+(10) \tag{3.15}$$

Solution:

$$\sum_{k=1}^{\infty} \frac{a_k}{10^k} = \sum_{k=0}^{\infty} \frac{a_{k+1}}{10^{k+1}} \qquad k \to k+1 \tag{3.16}$$
$$= \frac{1}{10}\sum_{k=0}^{\infty} \frac{a_{k+1}}{10^k} \tag{3.17}$$
$$= \frac{1}{10}\sum_{k=0}^{\infty} \frac{x(k)}{10^k} \qquad \because a_{k+1} = x(k) \tag{3.18}$$
$$= \frac{1}{10}\sum_{k=0}^{\infty} x(k)10^{-k} \tag{3.19}$$
$$= \frac{1}{10}X^+(10) \tag{3.20}$$

3.4 Show that

$$\alpha^n + \beta^n \qquad n \geq 1 \tag{3.21}$$

can be expressed as

$$w(n) = \left(\alpha^{n+1} + \beta^{n+1}\right)u(n) \tag{3.22}$$

and find $W(z)$

Solution:

$$\alpha^n + \beta^n \qquad n \geq 1 \tag{3.23}$$

can be written as

$$(\alpha^n + \beta^n)u(n-1) \tag{3.24}$$

On shifiting this backward by one unit, we get

$$(\alpha^{n+1} + \beta^{n+1})u(n) \tag{3.25}$$

by time-invariance

Now

$$w(n) = \alpha \cdot \alpha^n u(n) + \beta \cdot \beta^n u(n) \tag{3.26}$$

We know that

$$\alpha^n u(n) \overset{\mathcal{Z}}{\rightleftharpoons} \frac{1}{1-\alpha z^{-1}} \qquad |z| > |\alpha| \tag{3.27}$$
$$\beta^n u(n) \overset{\mathcal{Z}}{\rightleftharpoons} \frac{1}{1-\beta z^{-1}} \qquad |z| > |\beta| \tag{3.28}$$

Therefore

$$W(z) = \frac{\alpha}{1-\alpha z^{-1}} + \frac{\beta}{1-\beta z^{-1}} \qquad |z| > \max(|\alpha|, |\beta|) \tag{3.29}$$
$$= \frac{\alpha(1-\beta z^{-1}) + \beta(1-\alpha z^{-1})}{(1-\alpha z^{-1})(1-\beta z^{-1})} \tag{3.30}$$
$$= \frac{\alpha + \beta - 2\alpha\beta z^{-1}}{1 - (\alpha+\beta)z^{-1} + \alpha\beta z^{-2}} \tag{3.31}$$
$$= \frac{1 + 2z^{-1}}{1 - z^{-1} - z^{-2}} \tag{3.32}$$

3.5 Show that

$$\sum_{k=1}^{\infty} \frac{b_k}{10^k} = \frac{1}{10}\sum_{k=0}^{\infty} \frac{y(k)}{10^k} = \frac{1}{10}Y^+(10) \tag{3.33}$$

Solution: Note that

$$b_n = a_{n-1} + a_{n+1} \qquad n \geq 2,\ b_1 = 1 \tag{3.34}$$
$$= x(n-2) + x(n) \tag{3.35}$$
$$= y(n-1) \qquad n \geq 2 \tag{3.36}$$

At $n = 1$, $y(0) = 1$ anyway, thus

$$b_n = y(n-1) \qquad n \geq 1 \tag{3.37}$$

Now

$$\sum_{k=1}^{\infty} \frac{b_k}{10^k} = \sum_{k=0}^{\infty} \frac{b_{k+1}}{10^{k+1}} \qquad k \to k+1 \tag{3.38}$$
$$= \frac{1}{10}\sum_{k=0}^{\infty} \frac{b_{k+1}}{10^k} \tag{3.39}$$
$$= \frac{1}{10}\sum_{k=0}^{\infty} \frac{y(k)}{10^k} \qquad \because b_{k+1} = y(k) \tag{3.40}$$
$$= \frac{1}{10}\sum_{k=0}^{\infty} y(k)10^{-k} \tag{3.41}$$
$$= \frac{1}{10}Y^+(10) \tag{3.42}$$

3.6 Solve the JEE 2019 problem.
Solution: We have shown that

$$\sum_{k=1}^{n} a_k = x(n) * u(n-1) \tag{3.43}$$

On taking the Z-transform on both sides of the equation

$$\mathcal{Z}\left\{\sum_{k=1}^{n} a_k\right\} = X(z)\left(z^{-1}U(z)\right) \tag{3.44}$$

$$= \frac{z^{-1}}{(1-z^{-1}-z^{-2})(1-z^{-1})} \tag{3.45}$$

$$= \frac{1}{z^{-1}}\left(\frac{1}{1-z^{-1}-z^{-2}} - \frac{1}{1-z^{-1}}\right) \tag{3.46}$$

On taking the inverse Z-transform on both sides of this equation, we get

$$\sum_{k=1}^{n} a_k = x(n+1)u(n) - u(n+1) \tag{3.47}$$

For $n \geq 1$

$$\sum_{k=1}^{n} a_k = x(n+1) - 1 \tag{3.48}$$

$$= a_{n+2} - 1 \qquad n \geq 1 \tag{3.49}$$

We have proved that

$$\sum_{k=1}^{\infty} \frac{a_k}{10^k} = \frac{1}{10}X^+(10) \tag{3.50}$$

$$= \frac{1}{10}\frac{1}{(1-10^{-1}-10^{-2})} \tag{3.51}$$

$$= \frac{1}{10}\frac{100}{100-10-1} \tag{3.52}$$

$$= \frac{10}{89} \tag{3.53}$$

We have also shown that

$$b_n = y(n-1) \qquad n \geq 1 \tag{3.54}$$

But

$$y(n) = (\alpha^{n+1} + \beta^{n+1})u(n) \tag{3.55}$$

Thus

$$b_n = (\alpha^n + \beta^n)u(n-1) \qquad n \geq 1 \tag{3.56}$$

$$= \alpha^n + \beta^n \qquad n \geq 1 \tag{3.57}$$

We have also proved that

$$\sum_{k=1}^{\infty} \frac{b_k}{10^k} = \frac{1}{10}Y^+(10) \tag{3.58}$$

$$= \frac{1}{10}\frac{1+2(10)^{-1}}{(1-10^{-1}-10^{-2})} \tag{3.59}$$

$$= \frac{1}{10}\frac{100+20}{100-10-1} \tag{3.60}$$

$$= \frac{12}{89} \tag{3.61}$$

Therefore, all of the options except the last option are correct