Digital Signal Processing

AI21BTECH11017
Akshitha Kola

Contents

***Abstract*—This manual provides a simple introduction to digital signal processing.**

1 Software Installation

Run the following commands

```
sudo apt−get update
sudo apt−get install libffi−dev libsndfile1 python3
    −scipy python3−numpy python3−matplotlib
sudo pip install cffi pysoundfile
```

2 Digital Filter

2.1 Download the sound file from

https://github.com/AkshithaKola/EE3900/blob/main/**filter**/code/Sound%20Noise.wav

2.2 You will find a spectrogram at https://academo.org/demos/spectrum-analyzer. Upload the sound file that you downloaded in Problem 2.1 in the spectrogram and play. Observe the spectrogram. What do you find?
Solution: There are a lot of yellow lines between 440 Hz to 5.1 KHz. These represent the synthesizer key tones. Also, the key strokes are audible along with background noise.

2.3 Write the python code for removal of out of band noise and execute the code.
Solution: The following is the python code to remove band noise

https://github.com/AkshithaKola/EE3900/blob/main/**filter**/code/2.3.py

2.4 The output of the python script in Problem 2.3 is the audio file Sound_With_ReducedNoise.wav. Play the file in the spectrogram in Problem 2.2. What do you observe?
Solution: The key strokes as well as background noise is subdued in the audio. Also, the signal is blank for frequencies above 5.1 kHz.
The reduced noise audio file is

https://github.com/AkshithaKola/EE3900/blob/main/**filter**/code/Sound%20With%20ReducedNoise.wav

3 Difference Equation

3.1 Let

$$x(n) = \left\{\underset{\uparrow}{1}, 2, 3, 4, 2, 1\right\} \tag{3.1}$$

Sketch $x(n)$.
Solution:

https://github.com/AkshithaKola/EE3900/blob/main/**filter**/code/3.2.py

3.2 Let

$$y(n) + \frac{1}{2}y(n-1) = x(n) + x(n-2),$$
$$y(n) = 0, n < 0 \tag{3.2}$$

Sketch $y(n)$.

Solution: The following code yields Fig. 3.2

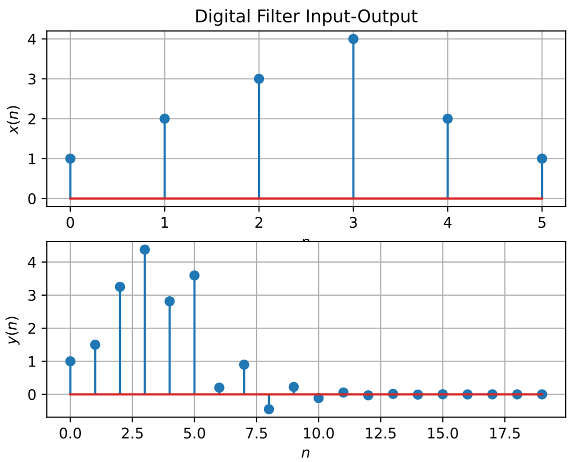

Fig. 3.2

4 Z-transform

4.1 The Z-transform of $x(n)$ is defined as

$$X(z) = \mathcal{Z}\{x(n)\} = \sum_{n=-\infty}^{\infty} x(n)z^{-n} \tag{4.1}$$

Show that

$$\mathcal{Z}\{x(n-1)\} = z^{-1}X(z) \tag{4.2}$$

and find

$$\mathcal{Z}\{x(n-k)\} \tag{4.3}$$

Solution: From (4.1),

$$\mathcal{Z}\{x(n-k)\} = \sum_{n=-\infty}^{\infty} x(n-1)z^{-n} \tag{4.4}$$

$$= \sum_{n=-\infty}^{\infty} x(n)z^{-n-1} = z^{-1}\sum_{n=-\infty}^{\infty} x(n)z^{-n} \tag{4.5}$$

resulting in (4.2). Similarly, it can be shown that

$$\mathcal{Z}\{x(n-k)\} = z^{-k}X(z) \tag{4.6}$$

4.2 Obtain $X(z)$ for $x(n)$ defined in problem 3.1.

Solution:

$$X(z) = \mathcal{Z}\{x(n)\} = \sum_{n=-\infty}^{\infty} x(n)z^{-n} \tag{4.7}$$

$$= 1 + 2z^{-1} + 3z^{-2} + 4z^{-3} + 2z^{-4} + z^{-5} \tag{4.8}$$

4.3 Find

$$H(z) = \frac{Y(z)}{X(z)} \tag{4.9}$$

from (3.2) assuming that the Z-transform is a linear operation.

Solution: Applying (4.6) in (3.2),

$$Y(z) + \frac{1}{2}z^{-1}Y(z) = X(z) + z^{-2}X(z) \tag{4.10}$$

$$\implies \frac{Y(z)}{X(z)} = \frac{1 + z^{-2}}{1 + \frac{1}{2}z^{-1}} \tag{4.11}$$

4.4 Find the Z transform of

$$\delta(n) = \begin{cases} 1 & n = 0 \\ 0 & \text{otherwise} \end{cases} \tag{4.12}$$

and show that the Z-transform of

$$u(n) = \begin{cases} 1 & n \geq 0 \\ 0 & \text{otherwise} \end{cases} \tag{4.13}$$

is

$$U(z) = \frac{1}{1 - z^{-1}}, \quad |z| > 1 \tag{4.14}$$

Solution: It is easy to show that

$$\delta(n) \overset{\mathcal{Z}}{\rightleftharpoons} 1 \tag{4.15}$$

and from (4.13),

$$U(z) = \sum_{n=0}^{\infty} z^{-n} \tag{4.16}$$

$$= \frac{1}{1 - z^{-1}}, \quad |z| > 1 \tag{4.17}$$

using the formula for the sum of an infinite geometric progression.

4.5 Show that

$$a^n u(n) \overset{\mathcal{Z}}{\rightleftharpoons} \frac{1}{1 - az^{-1}} \quad |z| > |a| \tag{4.18}$$

Solution: Let $a^n u(n) = v(n)$

$$V(z) = \sum_{n=0}^{\infty} a^n u(n) z^{-n} \tag{4.19}$$
$$= \sum_{n=0}^{\infty} a^n z^{-n} \tag{4.20}$$
$$= \sum_{n=0}^{\infty} (az^{-1})^n \tag{4.21}$$
$$= \frac{1}{1 - az^{-1}} \tag{4.22}$$

4.6 Let

$$H\left(e^{\jmath\omega}\right) = H\left(z = e^{\jmath\omega}\right). \tag{4.23}$$

Plot $\left|H\left(e^{\jmath\omega}\right)\right|$. Comment. $H(e^{\jmath\omega})$ is known as the *Discret Time Fourier Transform* (DTFT) of $x(n)$.
Solution: The following code plots Fig. 4.6.

https://github.com/AkshithaKola/EE3900/blob/main/**filter**/code/4.5.py

Using (4.11), we observe that $\left|H\left(e^{\jmath\omega}\right)\right|$ is given by

$$\left|H\left(e^{\jmath\omega}\right)\right| = \left|\frac{1 + e^{-2\jmath\omega}}{1 + \frac{1}{2}e^{-\jmath\omega}}\right| \tag{4.24}$$
$$= \sqrt{\frac{(1+\cos 2\omega)^2 + (\sin 2\omega)^2}{\left(1 + \frac{1}{2}\cos\omega\right)^2 + \left(\frac{1}{2}\sin\omega\right)^2}} \tag{4.25}$$
$$= \sqrt{\frac{2\,(1+\cos 2\omega)}{\frac{5}{4} + \cos\omega}} \tag{4.26}$$
$$= \sqrt{\frac{2\,(2\cos^2\omega)}{\frac{5}{4} + \cos\omega}} \tag{4.27}$$
$$= \frac{4|\cos\omega|}{\sqrt{5 + 4\cos\omega}} \tag{4.28}$$

The period of numerator is π and the period of denominator is 2π
$\therefore$The period of $\left|H\left(e^{\jmath\omega}\right)\right|$ is LCM(π,2π)

$$\left|H\left(e^{\jmath(\omega+2\pi)}\right)\right| = \frac{4\left|\cos(\omega + 2\pi)\right|}{\sqrt{5 + 4\cos(\omega + 2\pi)}} \tag{4.29}$$
$$= \frac{4\left|\cos\omega\right|}{\sqrt{5 + 4\cos\omega}} \tag{4.30}$$
$$\left|H\left(e^{\jmath(\omega+2\pi)}\right)\right| = \left|H\left(e^{\jmath\omega}\right)\right| \tag{4.31}$$

and so its fundamental period is 2π.

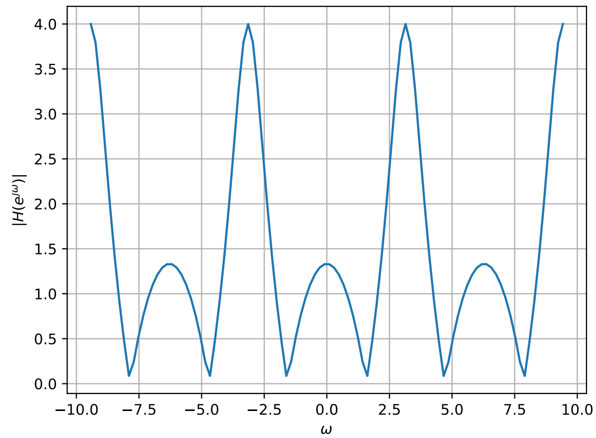

Fig. 4.6: $\left|H\left(e^{\jmath\omega}\right)\right|$

4.7 Express $h(n)$ in terms of $H(e^{\jmath\omega})$.
Solution: We have,

$$H(e^{\jmath\omega}) = \sum_{k=-\infty}^{\infty} h(k) e^{-\jmath\omega k} \tag{4.32}$$

on multiplying $e^{\jmath\omega n} d\omega$ on both sides

$$\implies H(e^{\jmath\omega}) e^{\jmath\omega n} d\omega = \sum_{k=-\infty}^{\infty} h(k) e^{-\jmath\omega k} e^{\jmath\omega n} d\omega \tag{4.33}$$

Now integrating on both sides from $-\pi$ to π
However,

$$\int_{-\pi}^{\pi} e^{\jmath\omega(n-k)} d\omega = \begin{cases} 2\pi & n = k \\ 0 & \text{otherwise} \end{cases} \tag{4.34}$$

and so,

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} H(e^{\jmath\omega}) e^{j\omega n} d\omega \tag{4.35}$$
$$= \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} \int_{-\pi}^{\pi} h(k) e^{\jmath\omega(n-k)} d\omega \tag{4.36}$$
$$= \frac{1}{2\pi} 2\pi h(n) = h(n) \tag{4.37}$$

Thus,

$$h(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} H(e^{\jmath\omega}) e^{\jmath\omega n} d\omega \tag{4.38}$$
$$\tag{4.39}$$

5 Impulse Response

5.1 Using long division, find $h(n)$, $n < 5$ for H(z) in (4.11).
Solution: From (4.11), we have

$$H(z) = \frac{1+z^{-2}}{1+\frac{1}{2}z^{-1}} \tag{5.1}$$

Substitute $z^{-1} = x$ to perform long division

$$\begin{array}{r} 2x-4 \\ \frac{1}{2}x+1 \overline{\big) \; x^2 \qquad +1} \\ \underline{-x^2-2x} \\ -2x+1 \\ \underline{2x+4} \\ 5 \end{array}$$

From above division we can write,

$$1+z^{-2} = (1+\frac{1}{2}z^{-1})(2z^{-1}-4)+5 \tag{5.2}$$

$$\frac{1+z^{-2}}{1+\frac{1}{2}z^{-1}} = 2z^{-1}-4+\frac{5}{1+\frac{1}{2}z^{-1}} \tag{5.3}$$

From (4.11), we can write

$$H(z) = -4+2z^{-1}+\frac{5}{1+\frac{1}{2}z^{-1}} \tag{5.4}$$

$$= -4+2z^{-1}+5\sum_{n=0}^{\infty}\left(-\frac{1}{2}\right)^n z^{-n} \tag{5.5}$$

$$= 1-\frac{1}{2}z^{-1}+5\sum_{n=2}^{\infty}\left(-\frac{1}{2}\right)^n z^{-n} \tag{5.6}$$

$$= \sum_{n=0}^{\infty}\left(-\frac{1}{2}\right)^n z^{-n}+4\sum_{n=2}^{\infty}\left(-\frac{1}{2}\right)^n z^{-n} \tag{5.7}$$

$$= \sum_{n=-\infty}^{\infty}u(n)\left(-\frac{1}{2}\right)^n z^{-n}+$$

$$\sum_{n=-\infty}^{\infty}u(n-2)\left(-\frac{1}{2}\right)^{n-2} z^{-n} \tag{5.8}$$

Therefore, from (4.1),

$$h(n) = \left(-\frac{1}{2}\right)^n u(n)+\left(-\frac{1}{2}\right)^{n-2}u(n-2) \tag{5.9}$$

5.2 Find an expression for $h(n)$ using $H(z)$, given that

$$h(n) \overset{\mathcal{Z}}{\rightleftharpoons} H(z) \tag{5.10}$$

and there is a one to one relationship between $h(n)$ and $H(z)$.
$h(n)$ is known as the *impulse response* of the system defined by (3.2).
Solution: From (4.11),

$$H(z) = \frac{1}{1+\frac{1}{2}z^{-1}}+\frac{z^{-2}}{1+\frac{1}{2}z^{-1}} \tag{5.11}$$

From (4.18),

$$\frac{1}{1+\frac{1}{2}z^{-1}} \overset{\mathcal{Z}}{\rightleftharpoons} \left(-\frac{1}{2}\right)^n u(n) \quad |z| > \frac{1}{2} \tag{5.12}$$

$$\frac{z^{-2}}{1+\frac{1}{2}z^{-1}} \overset{\mathcal{Z}}{\rightleftharpoons} \left(-\frac{1}{2}\right)^{n-2} u(n-2) \quad |z| > \frac{1}{2} \tag{5.13}$$

$$\Rightarrow H(z) \overset{\mathcal{Z}}{\rightleftharpoons} \left(-\frac{1}{2}\right)^n u(n)+\left(-\frac{1}{2}\right)^{n-2} u(n-2) \quad |z| > \frac{1}{2} \tag{5.14}$$

(5.15)

(Since Z-transform is a linear operator)

$$\therefore h(n) = \left(-\frac{1}{2}\right)^n u(n)+\left(-\frac{1}{2}\right)^{n-2} u(n-2) \tag{5.16}$$

From (5.11),Consider the first part:

$$\frac{1}{1+\frac{1}{2}z^{-1}} = \sum_{n=0}^{\infty}\left(-\frac{1}{2}\right)^n z^n \quad |z| > \frac{1}{2} \tag{5.17}$$

and

$$\frac{1}{1+\frac{1}{2}z^{-1}} = \frac{2z}{1+2z} \tag{5.18}$$

$$= \sum_{n=-\infty}^{-1}(2)^{-n}(-1)^{-n+1}z^{-n} \quad |z| < \frac{1}{2} \tag{5.19}$$

Therefore, ROC of $H(z)$ will be

$$|z| \neq \frac{1}{2} \tag{5.20}$$

5.3 Sketch $h(n)$. Is it bounded? Convergent?
Solution: The following code plots Fig. 5.3.

```
https://github.com/AkshithaKola/
    EE3900/blob/main/filter/code
    /5.3.py
```

and run the code using the following command

```
python3 5.3.py
```

From the plot, we can conclude that it is convergent to 0
$\therefore$ It is bounded as well.

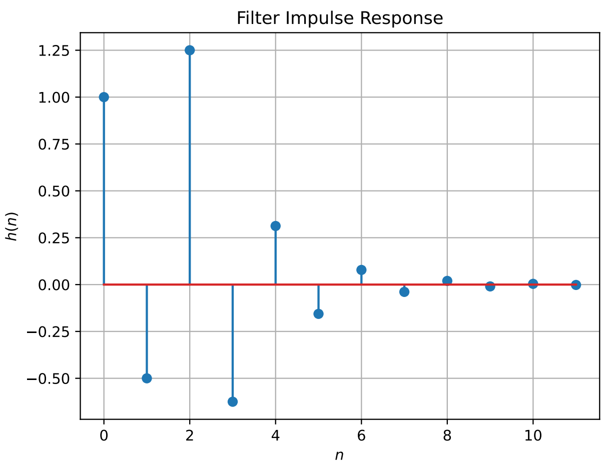

Fig. 5.3: $h(n)$ as the inverse of $H(z)$

5.4 Is it convergent? Justify using the ratio test.
Solution: Using the ratio test for convergence

$$\lim_{n\to\infty}\left|\frac{h(n+1)}{h(n)}\right| = \lim_{n\to\infty}\left|\frac{\left(-\frac{1}{2}\right)^{n-1}\left(1+\frac{1}{4}\right)}{\left(-\frac{1}{2}\right)^{n-2}\left(1+\frac{1}{4}\right)}\right| \tag{5.21}$$

$$= \lim_{n\to\infty}\left|-\frac{1}{2}\right| \tag{5.22}$$

$$= \frac{1}{2} < 1 \tag{5.23}$$

$\therefore$ h(n) is Convergent.

5.5 The system with $h(n)$ is defined to be stable if

$$\sum_{n=-\infty}^{\infty} h(n) < \infty \tag{5.24}$$

Is the system defined by (3.2) stable for the impulse response in (5.10)?

Solution: From 5.2,

$$\sum_{n=-\infty}^{\infty} h(n) = \sum_{n=-\infty}^{\infty}\left(-\frac{1}{2}\right)^n u(n) + \sum_{n=-\infty}^{\infty}\left(-\frac{1}{2}\right)^{n-2} u(n-2) \tag{5.25}$$

$$= \sum_{n=0}^{\infty}\left(-\frac{1}{2}\right)^n + \sum_{n=0}^{\infty}\left(-\frac{1}{2}\right)^{n-2} u(n-2) \tag{5.26}$$

$$= \frac{1}{1-\left(\frac{-1}{2}\right)} + \frac{1}{1-\left(\frac{-1}{2}\right)} \tag{5.27}$$

$$= \frac{4}{3} < \infty \tag{5.28}$$

using the fomula for the sum of an infinite geometric progression

$\therefore$ The system is stable.

5.6 Verify the above result using a python code.
Solution: The following code verifies whether the given system is stable or not

```
https://github.com/AkshithaKola/
    EE3900/blob/main/filter/code
    /5.6.py
```

run the code using the following command

```
python3 5.6.py
```

5.7 Compute and sketch $h(n)$ using

$$h(n) + \frac{1}{2}h(n-1) = \delta(n) + \delta(n-2), \tag{5.29}$$

This is the definition of $h(n)$.
Solution: The following code plots Fig. 5.7. Note that this is the same as Fig. 5.3.

```
https://github.com/AkshithaKola/
    EE3900/blob/main/filter/code
    /5.7.py
```

run the code using the following command

```
python3 5.7.py
```

5.8 Compute

$$y(n) = x(n) * h(n) = \sum_{k=-\infty}^{\infty} x(k)h(n-k) \tag{5.30}$$

Comment. The operation in (5.30) is known as *convolution*.

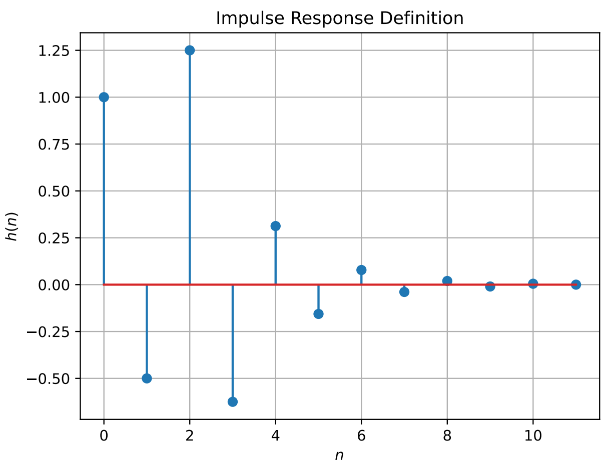

Fig. 5.7: $h(n)$ from the definition

Solution:

$$x(n) * h(n) = \sum_{k=-\infty}^{\infty} x(k)h(n-k) \tag{5.31}$$

$$= \sum_{k=0}^{5} x(k)h(n-k) \tag{5.32}$$

The following code plots Fig. 5.9.

```
https://github.com/AkshithaKola/
    EE3900/blob/main/filter/code
    /5.8.py
```

run the code using the following command

```
python3 5.8.py
```

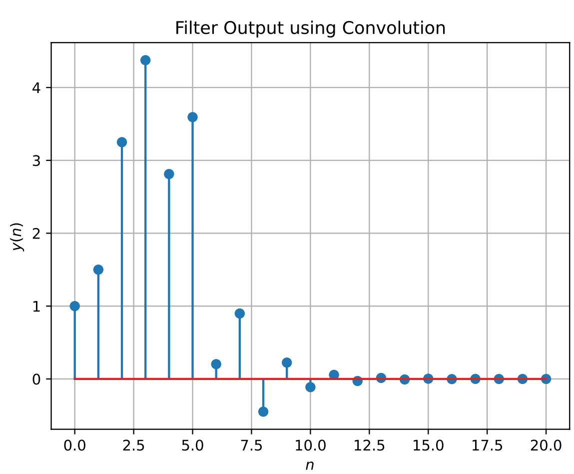

Fig. 5.8: $y(n)$ from the definition of convolution

This plot is same as y(n) in Fig. **??**

Therefore,

$$y(n) = x(n) * h(n) \tag{5.33}$$

5.9 Express the above convolution using a Teoplitz matrix.
Solution: From (3.1),we can write

$$\mathbf{x} = \begin{pmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 2 \\ 1 \end{pmatrix} \qquad \mathbf{h} = \begin{pmatrix} 1 \\ -0.5 \\ 1.25 \\ -0.62 \\ 0.31 \\ -0.16 \end{pmatrix} \tag{5.34}$$

Their convolution is given by the product of the following Toeplitz matrix $\mathbf{T}$

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ -0.5 & 1 & 0 & 0 & 0 & 0 \\ 1.25 & -0.5 & 1 & 0 & 0 & 0 \\ -0.62 & 1.25 & -0.5 & 1 & 0 & 0 \\ 0.31 & -0.62 & 1.25 & -0.5 & 1 & 0 \\ -0.16 & 0.31 & -0.62 & 1.25 & -0.5 & 1 \\ 0 & -0.16 & 0.31 & -0.62 & 1.25 & -0.5 \\ 0 & 0 & -0.16 & 0.31 & -0.62 & 1.25 \\ 0 & 0 & 0 & -0.16 & 0.31 & -0.62 \\ 0 & 0 & 0 & 0 & -0.16 & 0.31 \\ 0 & 0 & 0 & 0 & 0 & -0.16 \end{pmatrix} \tag{5.35}$$

$$\mathbf{y} = \mathbf{x} \circledast \mathbf{h} = \mathbf{T}\mathbf{x} = \begin{pmatrix} 1 \\ 1.5 \\ 3.25 \\ 4.38 \\ 2.81 \\ 3.59 \\ 0.12 \\ 0.78 \\ -0.62 \\ 0 \\ -0.16 \end{pmatrix} \tag{5.36}$$

The following python code computes the convolution using Teoplitz matrix.

```
https://github.com/AkshithaKola/
    EE3900/blob/main/filter/code
    /5.9.py
```

run the code using the following command

```
python3 5.9.py
```

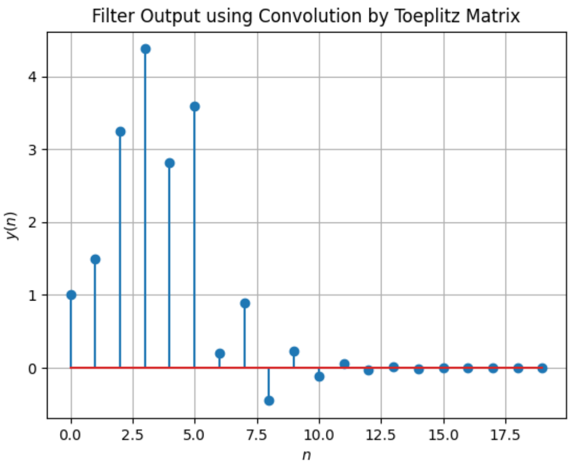

Fig. 5.9: $y(n)$ from the definition of convolution using Teoplitz matrix

5.10 Show that

$$y(n) = \sum_{k=-\infty}^{\infty} x(n-k)h(k) \tag{5.37}$$

Solution: From (5.30) we know that,

$$y(n) = x(n) * h(n) = \sum_{k=-\infty}^{\infty} x(k)h(n-k) \tag{5.38}$$

Substitute k = n-i

$$\sum_{k=-\infty}^{\infty} x(k)h(n-k) = \sum_{n-i=-\infty}^{\infty} x(n-i)h(n-(n-i)) \tag{5.39}$$

$$= \sum_{i=\infty}^{-\infty} x(n-i)h(i) \tag{5.40}$$

$$= \sum_{i=-\infty}^{\infty} x(n-i)h(i) \tag{5.41}$$

Since, the order of limit doesn't matter in case of summation.Therefore, now we have

$$\sum_{k=-\infty}^{\infty} x(k)h(n-k) = \sum_{k=-\infty}^{\infty} x(n-k)h(k) \tag{5.42}$$

from (5.30)

$$\therefore y(n) = \sum_{k=-\infty}^{\infty} x(n-k)h(k) \tag{5.43}$$

6 DFT and FFT

6.1 Compute

$$X(k) \triangleq \sum_{n=0}^{N-1} x(n)e^{-j2\pi kn/N}, \quad k = 0, 1, \ldots, N-1 \tag{6.1}$$

and $H(k)$ using $h(n)$.

Solution: The following code plots Fig. 6.1.

```
https://github.com/AkshithaKola/
    EE3900/blob/main/filter/code
    /6.1.py
```

run the above code using the command.

```
python3 6.1.py
```

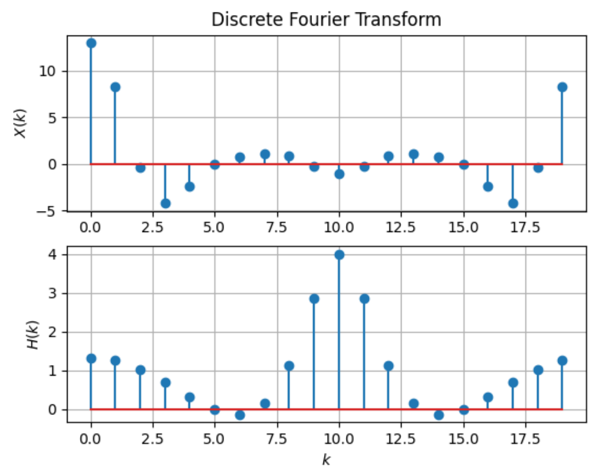

Fig. 6.1: Plots of the real parts of the DFT of $x(n)$ and $h(n)$

6.2 Compute

$$Y(k) = X(k)H(k) \tag{6.2}$$

Solution: Download and run the following code.

```
https://github.com/AkshithaKola/
    EE3900/blob/main/filter/code
    /6.2.py
```

run the above code using the command.

```
python3 6.2.py
```

6.3 Compute

$$y(n) = \frac{1}{N}\sum_{k=0}^{N-1} Y(k) \cdot e^{j2\pi kn/N}, \quad n = 0, 1, \ldots, N-1 \tag{6.3}$$

Solution: The following code plots Fig. 5.9. Note that this is the same as $y(n)$ in Fig. 3.2.

```
https://github.com/AkshithaKola/
    EE3900/blob/main/filter/code
    /6.3.py
```

run the above code using the command.

```
python3 6.3.py
```

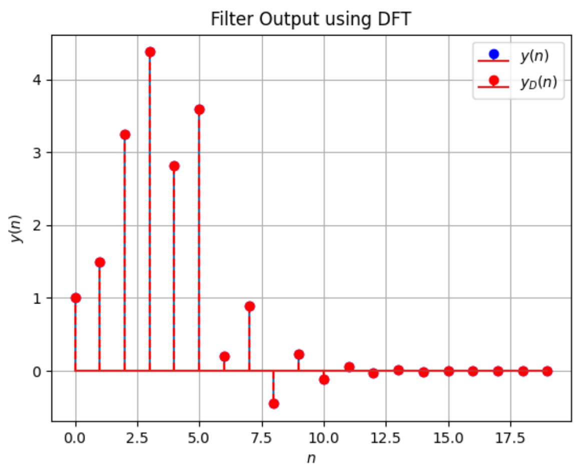

Fig. 6.3: $y(n)$ from the DFT

6.4 Repeat the previous exercise by computing $X(k), H(k)$ and $y(n)$ through FFT and IFFT.
Solution: Download the code from

```
https://github.com/AkshithaKola/
    EE3900/blob/main/filter/code
    /6.4.py
```

and execute it using

```
$ python3 6.4.py
```

Observe that Fig. (6.4) is the same as $y(n)$ in Fig. (3.2).

6.5 Wherever possible, express all the above equations as matrix equations.
Solution: We use the DFT Matrix, where $\omega = e^{-\frac{j2k\pi}{N}}$, which is given by

$$\mathbf{W} = \begin{pmatrix} \omega^0 & \omega^0 & \dots & \omega^0 \\ \omega^0 & \omega^1 & \dots & \omega^{N-1} \\ \vdots & \vdots & \ddots & \vdots \\ \omega^0 & \omega^{N-1} & \dots & \omega^{(N-1)(N-1)} \end{pmatrix} \tag{6.4}$$

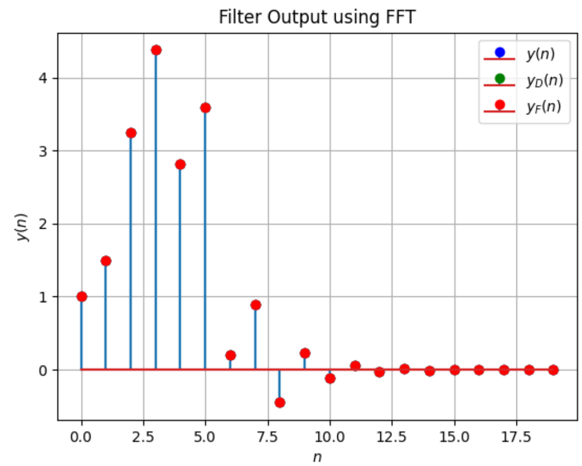

Fig. 6.4: $y(n)$ using FFT and IFFT

i.e. $W_{jk} = \omega^{jk}$, $0 \le j,k < N$. Hence, we can write any DFT equation as

$$\mathbf{X} = \mathbf{W}\mathbf{x} = \mathbf{x}\mathbf{W} \tag{6.5}$$

where

$$\mathbf{x} = \begin{pmatrix} x(0) \\ x(1) \\ \vdots \\ x(n-1) \end{pmatrix} \tag{6.6}$$

Using (6.3), the inverse Fourier Transform is given by

$$\mathbf{x} = \mathcal{F}^{-1}(\mathbf{X}) = \mathbf{W}^{-1}\mathbf{X} = \frac{1}{N}\mathbf{W}^{\mathbf{H}}\mathbf{X} = \frac{1}{N}\mathbf{X}\mathbf{W}^{\mathbf{H}} \tag{6.7}$$

$$\implies \mathbf{W}^{-1} = \frac{1}{N}\mathbf{W}^{\mathbf{H}} \tag{6.8}$$

where H denotes hermitian operator. We can rewrite (6.2) using the element-wise multiplication operator as

$$\mathbf{Y} = \mathbf{H} \cdot \mathbf{X} = (\mathbf{W}\mathbf{h}) \cdot (\mathbf{W}\mathbf{x}) \tag{6.9}$$

6.6 Verify the above equation by generating the DFT matrix in python.
Solution: Download the code from

```
https://github.com/AkshithaKola/
    EE3900/blob/main/filter/code
    /6.6.py
```

and execute it using

```
$ python3 6.6.py
```

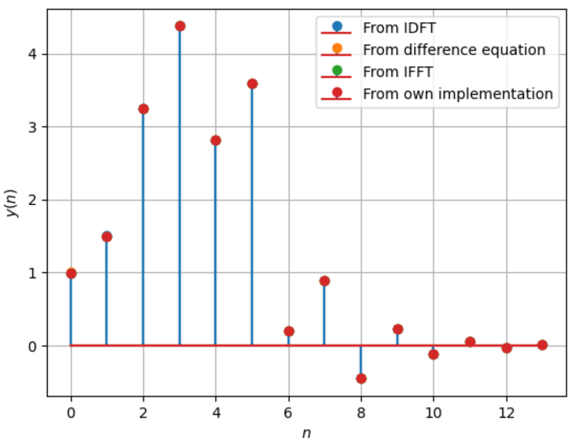

Fig. 6.6: $y(n)$ using DFT matrix

The above code plots (6.6)

6.7 Compute the 8-point FFT in C.

Solution: Download the following C code

```
https://github.com/AkshithaKola/
    EE3900/blob/main/filter/code
    /6.7.c
```

run the above C code using the command

```
$ gcc 6.7.c −lm −o 6.7.out
$ ./6.7.out
```

The above code generates the text files that are loaded in the following code

Download the following code

```
https://github.com/AkshithaKola/
    EE3900/blob/main/filter/code
    /6.7.1.py
https://github.com/AkshithaKola/
    EE3900/blob/main/filter/code
    /6.7.2.py
```

run the above code using the command

```
$ python3 6.7.1_plot.py
$ python3 6.7.2_plot.py
```

The above code plots the graphs 6.7 and 6.7

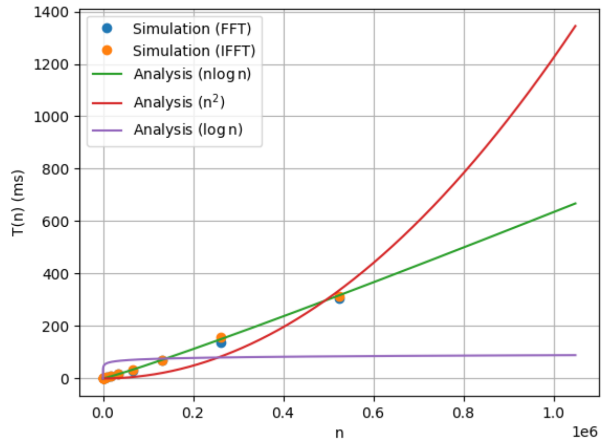

Fig. 6.7: Complexity of FFT/IFFT is O(n log n)

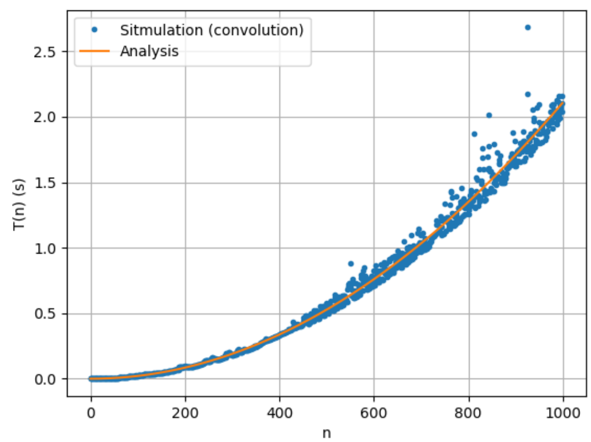

Fig. 6.7: Complexity of convolution is O(n^2)

7 FFT

7.1 Definitions

1. The DFT of $x(n)$ is given by

$$X(k) \triangleq \sum_{n=0}^{N-1} x(n) e^{-j2\pi kn/N}, \quad k = 0, 1, \dots, N-1 \tag{7.1}$$

2. Let

$$W_N = e^{-j2\pi/N} \tag{7.2}$$

Then the N-point *DFT matrix* is defined as

$$\mathbf{F}_N = [W_N^{mn}] \tag{7.3}$$

where W_N^{mn} are the elements of $\mathbf{F}_N$.

3. Let
$$\mathbf{I}_4 = \begin{pmatrix}\mathbf{e}_4^1 & \mathbf{e}_4^2 & \mathbf{e}_4^3 & \mathbf{e}_4^4\end{pmatrix} \tag{7.4}$$
be the 4×4 identity matrix. Then the 4 point *DFT permutation matrix* is defined as
$$\mathbf{P}_4 = \begin{pmatrix}\mathbf{e}_4^1 & \mathbf{e}_4^3 & \mathbf{e}_4^2 & \mathbf{e}_4^4\end{pmatrix} \tag{7.5}$$
4. The 4 point *DFT diagonal matrix* is defined as
$$\mathbf{D}_4 = diag\begin{pmatrix}W_N^0 & W_N^1 & W_N^2 & W_N^3\end{pmatrix} \tag{7.6}$$

7.2 Problems

1. Show that
$$W_N^2 = W_{N/2} \tag{7.7}$$
Solution: From defination
$$W_N = e^{-j2\pi/N} \tag{7.8}$$
$$W_N^2 = \left(e^{-j2\pi/N}\right)^2 \tag{7.9}$$
$$= e^{-j2\pi/N/2} \tag{7.10}$$
$$= W_{N/2} \tag{7.11}$$
2. Find $\mathbf{P}_6$.
Solution: $\mathbf{P}_6 = \begin{pmatrix}\mathbf{e}_4^1 & \mathbf{e}_4^3 & \mathbf{e}_4^5 & \mathbf{e}_4^2 & \mathbf{e}_4^4 & \mathbf{e}_4^6\end{pmatrix}$
$$\mathbf{P}_6 = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 1 & 0 & 0\\ 0 & 1 & 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 1 & 0\\ 0 & 0 & 1 & 0 & 0 & 0\\ 0 & 0 & 0 & 0 & 0 & 1\end{pmatrix} \tag{7.12}$$
3. Find $\mathbf{D}_3$.
Solution:
$$\mathbf{D}_3 = diag\begin{pmatrix}W_3^0 & W_3^1 & W_3^2\end{pmatrix} \tag{7.13}$$
$$= \begin{pmatrix} 1 & 0 & 0\\ 0 & e^{-j2\pi/3} & 0\\ 0 & 0 & (e^{-j2\pi/3})^2\end{pmatrix} \tag{7.14}$$
4. Show that
$$\mathbf{F}_4 = \begin{bmatrix}\mathbf{I}_2 & \mathbf{D}_2\\ \mathbf{I}_2 & -\mathbf{D}_2\end{bmatrix}\begin{bmatrix}\mathbf{F}_2 & 0\\ 0 & \mathbf{F}_2\end{bmatrix}\mathbf{P}_4 \tag{7.15}$$
Solution:
$$\mathbf{F}_4\mathbf{P}_4 = \begin{bmatrix}\mathbf{I}_2 & \mathbf{D}_2\\ \mathbf{I}_2 & -\mathbf{D}_2\end{bmatrix}\begin{bmatrix}\mathbf{F}_2 & 0\\ 0 & \mathbf{F}_2\end{bmatrix} \tag{7.16}$$
$$= \begin{bmatrix}\mathbf{I}_2\mathbf{F}_2 & \mathbf{D}_2\mathbf{F}_2\\ \mathbf{I}_2\mathbf{F}_2 & -\mathbf{D}_2\mathbf{F}_2\end{bmatrix} \tag{7.17}$$
$$= \begin{bmatrix}\mathbf{F}_2 & \mathbf{D}_2\mathbf{F}_2\\ \mathbf{F}_2 & -\mathbf{D}_2\mathbf{F}_2\end{bmatrix} \tag{7.18}$$
$$= \begin{bmatrix}W_2^0 & W_2^0 & W_2^0 & W_2^0\\ W_2^0 & W_2^1 & W_2^1 & W_2^2\\ W_2^0 & W_2^0 & -W_2^0 & -W_2^0\\ W_2^0 & W_2^1 & -W_2^1 & -W_2^2\end{bmatrix} \tag{7.19}$$
$$= \begin{bmatrix}1 & 1 & 1 & 1\\ 1 & W_2^1 & W_2^1 & W_2^2\\ 1 & W_2^0 & -W_2^0 & -W_2^0\\ 1 & W_2^1 & -W_2^1 & -W_2^2\end{bmatrix} \tag{7.20}$$
from eqn(7.7) we get $W_2 = W_4^2$
$$\mathbf{F}_4\mathbf{P}_4 = \begin{bmatrix}W_4^0 & W_4^0 & W_4^0 & W_4^0\\ W_4^0 & W_4^2 & W_4^1 & W_4^3\\ W_4^0 & W_4^4 & W_4^2 & W_4^6\\ W_4^0 & W_4^6 & W_4^3 & W_4^9\end{bmatrix} \tag{7.21}$$
5. Show that
$$\mathbf{F}_N = \begin{bmatrix}\mathbf{I}_{N/2} & \mathbf{D}_{N/2}\\ \mathbf{I}_{N/2} & -\mathbf{D}_{N/2}\end{bmatrix}\begin{bmatrix}\mathbf{F}_{N/2} & 0\\ 0 & \mathbf{F}_{N/2}\end{bmatrix}\mathbf{P}_N \tag{7.22}$$
Solution: Observe that for even N and letting $\mathbf{f}_N^i$ denote the i^{th} column of $\mathbf{F}_N$,
$$\begin{pmatrix}\mathbf{D}_{N/2}\mathbf{F}_{N/2}\\ -\mathbf{D}_{N/2}\mathbf{F}_{N/2}\end{pmatrix} = \begin{pmatrix}\mathbf{f}_N^2 & \mathbf{f}_N^4 & \dots & \mathbf{f}_N^N\end{pmatrix} \tag{7.23}$$
and
$$\begin{pmatrix}\mathbf{I}_{N/2}\mathbf{F}_{N/2}\\ \mathbf{I}_{N/2}\mathbf{F}_{N/2}\end{pmatrix} = \begin{pmatrix}\mathbf{f}_N^1 & \mathbf{f}_N^3 & \dots & \mathbf{f}_N^{N-1}\end{pmatrix} \tag{7.24}$$
Thus,
$$\begin{pmatrix}\mathbf{I}_2\mathbf{F}_2 & \mathbf{D}_2\mathbf{F}_2\\ \mathbf{I}_2\mathbf{F}_2 & -\mathbf{D}_2\mathbf{F}_2\end{pmatrix} = \begin{pmatrix}\mathbf{I}_{N/2} & \mathbf{D}_{N/2}\\ \mathbf{I}_{N/2} & -\mathbf{D}_{N/2}\end{pmatrix}\begin{pmatrix}\mathbf{F}_{N/2} & 0\\ 0 & \mathbf{F}_{N/2}\end{pmatrix}$$
$$= \begin{pmatrix}\mathbf{f}_N^1 & \dots & \mathbf{f}_N^{N-1} & \mathbf{f}_N^2 & \dots & \mathbf{f}_N^N\end{pmatrix} \tag{7.25}$$
and so,
$$\begin{pmatrix}\mathbf{I}_{N/2} & \mathbf{D}_{N/2}\\ \mathbf{I}_{N/2} & -\mathbf{D}_{N/2}\end{pmatrix}\begin{pmatrix}\mathbf{F}_{N/2} & 0\\ 0 & \mathbf{F}_{N/2}\end{pmatrix}\mathbf{P}_N$$
$$= \begin{pmatrix}\mathbf{f}_N^1 & \mathbf{f}_N^2 & \dots & \mathbf{f}_N^N\end{pmatrix} = \mathbf{F}_N \tag{7.26}$$
6. Find
$$\mathbf{P}_4\mathbf{x} \tag{7.27}$$

Solution: We have,

$$\mathbf{P}_4\mathbf{x} = \begin{pmatrix}\mathbf{e}_4^1 & \mathbf{e}_4^3 & \mathbf{e}_4^2 & \mathbf{e}_4^4\end{pmatrix}\begin{pmatrix}x(0)\\x(1)\\x(2)\\x(3)\end{pmatrix} = \begin{pmatrix}x(0)\\x(2)\\x(1)\\x(3)\end{pmatrix} \tag{7.28}$$

7. Show that

$$\mathbf{X} = \mathbf{F}_N\mathbf{x} \tag{7.29}$$

where $\mathbf{x}, \mathbf{X}$ are the vector representations of $x(n), X(k)$ respectively.
Solution: Writing the terms of X,

$$X(0) = x(0) + x(1) + \ldots + x(N-1) \tag{7.30}$$

$$X(1) = x(0) + x(1)e^{-\frac{\jmath 2\pi}{N}} + \ldots + + x(N-1)e^{-\frac{\jmath 2(N-1)\pi}{N}} \tag{7.31}$$

$$\vdots$$

$$X(N-1) = x(0) + x(1)e^{-\frac{\jmath 2(N-1)\pi}{N}} + \ldots + + x(N-1)e^{-\frac{\jmath 2(N-1)(N-1)\pi}{N}} \tag{7.32}$$

Clearly, the term in the m^{th} row and n^{th} column is given by ($0 \le m \le N-1$ and $0 \le n \le N-1$)

$$T_{mn} = x(n)e^{-\frac{\jmath 2mn\pi}{N}} \tag{7.33}$$

and so, we can represent each of these terms as a matrix product

$$\mathbf{X} = \mathbf{F}_N\mathbf{x} \tag{7.34}$$

where $\mathbf{F}_N = \left[e^{-\frac{-\jmath 2mn\pi}{N}}\right]_{mn}$ for $0 \le m \le N-1$ and $0 \le n \le N-1$.

8. Derive the following Step-by-step visualisation of 8-point FFTs into 4-point FFTs and so on

$$\begin{bmatrix}X(0)\\X(1)\\X(2)\\X(3)\end{bmatrix} = \begin{bmatrix}X_1(0)\\X_1(1)\\X_1(2)\\X_1(3)\end{bmatrix} + \begin{bmatrix}W_8^0 & 0 & 0 & 0\\0 & W_8^1 & 0 & 0\\0 & 0 & W_8^2 & 0\\0 & 0 & 0 & W_8^3\end{bmatrix}\begin{bmatrix}X_2(0)\\X_2(1)\\X_2(2)\\X_2(3)\end{bmatrix} \tag{7.35}$$

$$\begin{bmatrix}X(4)\\X(5)\\X(6)\\X(7)\end{bmatrix} = \begin{bmatrix}X_1(0)\\X_1(1)\\X_1(2)\\X_1(3)\end{bmatrix} - \begin{bmatrix}W_8^0 & 0 & 0 & 0\\0 & W_8^1 & 0 & 0\\0 & 0 & W_8^2 & 0\\0 & 0 & 0 & W_8^3\end{bmatrix}\begin{bmatrix}X_2(0)\\X_2(1)\\X_2(2)\\X_2(3)\end{bmatrix} \tag{7.36}$$

4-point FFTs into 2-point FFTs

$$\begin{bmatrix}X_1(0)\\X_1(1)\end{bmatrix} = \begin{bmatrix}X_3(0)\\X_3(1)\end{bmatrix} + \begin{bmatrix}W_4^0 & 0\\0 & W_4^1\end{bmatrix}\begin{bmatrix}X_4(0)\\X_4(1)\end{bmatrix} \tag{7.37}$$

$$\begin{bmatrix}X_1(2)\\X_1(3)\end{bmatrix} = \begin{bmatrix}X_3(0)\\X_3(1)\end{bmatrix} - \begin{bmatrix}W_4^0 & 0\\0 & W_4^1\end{bmatrix}\begin{bmatrix}X_4(0)\\X_4(1)\end{bmatrix} \tag{7.38}$$

$$\begin{bmatrix}X_2(0)\\X_2(1)\end{bmatrix} = \begin{bmatrix}X_5(0)\\X_5(1)\end{bmatrix} + \begin{bmatrix}W_4^0 & 0\\0 & W_4^1\end{bmatrix}\begin{bmatrix}X_6(0)\\X_6(1)\end{bmatrix} \tag{7.39}$$

$$\begin{bmatrix}X_2(2)\\X_2(3)\end{bmatrix} = \begin{bmatrix}X_5(0)\\X_5(1)\end{bmatrix} - \begin{bmatrix}W_4^0 & 0\\0 & W_4^1\end{bmatrix}\begin{bmatrix}X_6(0)\\X_6(1)\end{bmatrix} \tag{7.40}$$

$$P_8\begin{bmatrix}x(0)\\x(1)\\x(2)\\x(3)\\x(4)\\x(5)\\x(6)\\x(7)\end{bmatrix} = \begin{bmatrix}x(0)\\x(2)\\x(4)\\x(6)\\x(1)\\x(3)\\x(5)\\x(7)\end{bmatrix} \tag{7.41}$$

$$P_4\begin{bmatrix}x(0)\\x(2)\\x(4)\\x(6)\end{bmatrix} = \begin{bmatrix}x(0)\\x(4)\\x(2)\\x(6)\end{bmatrix} \tag{7.42}$$

$$P_4\begin{bmatrix}x(1)\\x(3)\\x(5)\\x(7)\end{bmatrix} = \begin{bmatrix}x(1)\\x(5)\\x(3)\\x(7)\end{bmatrix} \tag{7.43}$$

Therefore,

$$\begin{bmatrix}X_3(0)\\X_3(1)\end{bmatrix} = F_2\begin{bmatrix}x(0)\\x(4)\end{bmatrix} \tag{7.44}$$

$$\begin{bmatrix}X_4(0)\\X_4(1)\end{bmatrix} = F_2\begin{bmatrix}x(2)\\x(6)\end{bmatrix} \tag{7.45}$$

$$\begin{bmatrix}X_5(0)\\X_5(1)\end{bmatrix} = F_2\begin{bmatrix}x(1)\\x(5)\end{bmatrix} \tag{7.46}$$

$$\begin{bmatrix}X_6(0)\\X_6(1)\end{bmatrix} = F_2\begin{bmatrix}x(3)\\x(7)\end{bmatrix} \tag{7.47}$$

9. For

$$\mathbf{x} = \begin{pmatrix}1\\2\\3\\4\\2\\1\end{pmatrix} \tag{7.48}$$

compte the DFT using (7.29)
Solution: Download the code from

https://github.com/AkshithaKola/ EE3900/blob/main/**filter**/code

```
/7.9.py
```

and execute it using

```
$ python3 7.9.py
```

10. Repeat the above exercise using the FFT after zero padding **x**.
Solution: Download the code from

```
https://github.com/AkshithaKola/
    EE3900/blob/main/filter/code
    /7.10.py
```

and execute it using

```
$ python3 7.10.py
```

11. Write a C program to compute the 8-point FFT.
Solution: Download the code from

```
https://github.com/AkshithaKola/
    EE3900/blob/main/filter/code
    /7.11.c
```

compile it using

```
$ gcc 7.11.c −lm
```

and execute it using

```
$ ./a.out
```