Control of Robotic Systems

On

Inverted Pendulum on cart

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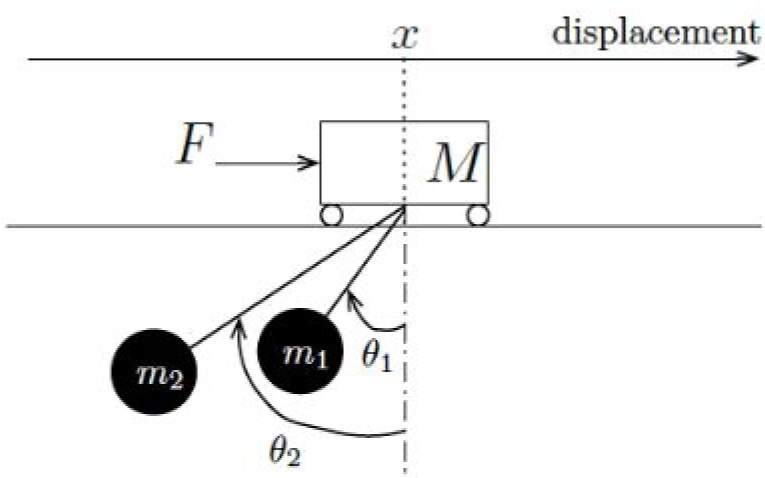
Section 0201

First Component :

Consider a crane that moves along a one-dimensional track. It behaves as a frictionless cart with mass M actuated by an external force F that constitutes the input of the system. There are two loads suspended from cables attached to the crane. The loads have mass m_1 and m_2, and the lengths of the cables are l_1 and l_2, respectively. The following figure depicts the crane and associated variables used throughout this project.



A)Obtain the equations of motion for the system and the corresponding nonlinear state-space representation.

For mass m_1

$$X = x - l_1 s_1$$

$$Y = -l_1 c_1$$

$$\dot{X} = \dot{x} - l_1 c_1 \dot{\theta}_1$$

$$\dot{Y} = l_1 c_1 \dot{\theta}_1$$

$$v_1^2 = \left(\dot{x} - l_1 c_1 \dot{\theta}_1\right)^2 + \left(l_1 c_1 \dot{\theta}_1\right)^2 = x^2 + l_1^2 \dot{\theta}_1^2 - 2\dot{x} l_1 \dot{\theta}_1 c_1$$

where $s_1 = \sin(\Theta_1)$ and $c_1 = \text{Cos}(\Theta_1)$ and l_1 & l_2 are lengths of cables connected to respective masses.

Same procedure will be followed by 2nd mass and we will derive the equation of motion. For mass 2 equations are as below.

For mass m_2,

$$X = x - l_2 s_2$$

$$Y = -l_2 c_2$$

$$\dot{X} = \dot{x} - l_2 c_2 \dot{\theta}_2$$

$$\dot{Y} = l_2 c_2 \dot{\theta}_2$$

$$v_2^2 = x^2 + l_2^2 \dot{\theta}_2^2 - 2\dot{x} l_2 \dot{\theta}_2 c_2$$

where $s_2 = \text{Sin}(\Theta_2)$ and $c_2 = \text{Cos}(\Theta_2)$.

Kinetic Energy of the system

$$K.E = \tfrac{1}{2} m_1 v_1^2 + \tfrac{1}{2} m_2 v_2^2 + \tfrac{1}{2} M \dot{x}^2$$

$$= \tfrac{1}{2} m_1 \left(\dot{x}^2 + l_1^2 \dot{\theta}_1^2 - 2\dot{x} l_1 \dot{\theta}_1 c_1 \right) + \tfrac{1}{2} m_2 \left(\dot{x}^2 + l_2^2 \dot{\theta}_2^2 - 2\dot{x} l_2 \dot{\theta}_2 c_2 \right) + \tfrac{1}{2} M \dot{x}^2$$

$$= \tfrac{1}{2} \dot{x}^2 (m_1 + m_2 + M) + \tfrac{1}{2} m_1 l_1 \dot{\theta}_1^2 + \tfrac{1}{2} m_2 l_2 \dot{\theta}_2^2 - m_1 l_1 c_1 \dot{\theta}_1 \dot{x} - m_2 l_2 c_2 \dot{\theta}_2 \dot{x}_2$$

Potential Energy of the system

$$P.E = -m_1 g l_1 c_1 - m_2 g l_2 c_2$$

Lagrangian

L = Kinetic Energy – Potential Energy

$$L = \tfrac{1}{2}\dot{x}^2\,(m_1 + m_2 + M) + \tfrac{1}{2}m_1 l_1 \dot{\theta}_1^2 + \tfrac{1}{2}m_2 l_2 \dot{\theta}_2^2 - m_1 l_1 c_1 \dot{\theta}_1 \dot{x} - m_2 l_2 c_2 \dot{\theta}_2 \dot{x}_2 - m_1 g l_1 c_1 + m_2 g l_2 c_2$$

$$\tfrac{\partial L}{\partial \dot{x}} = \dot{x}\,(m_1 + m_2 + M) + m_1 l_1 \dot{\theta}_1^2 - m_2 l_2 \dot{\theta}_2^2$$

$$\tfrac{d}{dt}\left(\tfrac{\partial L}{\partial \dot{x}}\right) = \ddot{x}\,(m_1 + m_2 + M) - m_1 l_1 \left(\ddot{\theta} c_1 - s_1 \dot{\theta}_1^2\right) - m_2 l_2 \left(\ddot{\theta} c_2 - s_2 \dot{\theta}_2^2\right) = F$$

$$\tfrac{\partial L}{\partial \dot{\theta}_1} = m_1 l_1^2 \dot{\theta}_1 - m_1 l_1 c_1 \dot{x}$$

$$\tfrac{d}{dt}\left(\tfrac{\partial L}{\partial \dot{\theta}_1}\right) = m_1 l_1^2 \ddot{\theta}_1 - m_1 l_1 (c_1 \ddot{x} - s_1 \dot{x} \dot{\theta}_1)$$

$$\tfrac{\partial L}{\partial \theta_1} = s_1 m_1 l_1 \dot{x} \dot{\theta}_1 - m_1 g l_1 s_1$$

$$\tfrac{\partial L}{\partial \theta_2} = s_2 m_2 l_2 \dot{x} \dot{\theta}_2 - m_2 g l_2 s_2$$

$$\tfrac{\partial L}{\partial \dot{\theta}_2} = m_2 l_2^2 \dot{\theta}_2 - m_2 l_2 c_2 \dot{x}$$

$$\tfrac{d}{dt}\left(\tfrac{\partial L}{\partial \dot{\theta}_2}\right) = m_2 l_2^2 \ddot{\theta}_2 - m_2 l_2 (c_2 \dot{x} - s_2 \dot{x} \dot{\theta}_2)$$

Derived Equation of Motion for the system:

$$\ddot{x} = \frac{F - m_1 l_1(\,\dot{\theta}_1^2 sin(\theta_1) - \ddot{\theta}_1 cos(\theta_1)\,) - m_2 l_2(\,\dot{\theta}_2^2 sin(\theta_2) - \ddot{\theta}_2 cos(\theta_2)\,)}{(M + m_1 + m_2)}$$

$$\ddot{\theta}_1 l_1 - \ddot{x}\cos(\theta_1) + g\sin(\theta_1) = 0$$

$$\ddot{\theta}_2 l_2 - \ddot{x}\cos(\theta_2) + g\sin(\theta_2) = 0$$

Deriving NonLinear State Space for the System

$$\ddot{x} = \frac{F - m_1\left(g s_1 c_1 + l_1 s_1 \dot{\theta}_1^2\right) - m_2\left(g s_2 c_2 + l_2 s_2 \dot{\theta}_2^2\right)}{(M + m_1 s_1^2 + m_2 s_2^2)}$$

$$\ddot{\theta}_1 = \frac{c_1}{l_1}\left[\frac{F - m_1\left(g s_1 c_1 + l_1 s_1 \dot{\theta}_1^2\right) - m_2\left(g s_2 c_2 + l_2 s_2 \dot{\theta}_2^2\right)}{(M + m_1 s_1^2 + m_2 s_2^2)}\right] - \frac{g}{l_1} s_1$$

$$\ddot{\theta}_2 = \frac{c_2}{l_2}\left[\frac{F - m_1\left(g s_1 c_1 + l_1 s_1 \dot{\theta}_1^2\right) - m_2\left(g s_2 c_2 + l_2 s_2 \dot{\theta}_2^2\right)}{(M + m_1 s_1^2 + m_2 s_2^2)}\right] - \frac{g}{l_2} s_2$$

Choosing state as

$$x = [\, x_1,\ \dot{x}_1,\ \theta_1,\ \dot{\theta}_1,\ \theta_2,\ \dot{\theta}_2 \,] = [x_1,\ x_2,\ x_3,\ x_4,\ x_5,\ x_6]$$

NonLinear State equations is of the form:-

$$\dot{x}_1 = x_2$$

$$\dot{x}_2 = \frac{F - (m_1 l_1 x_4^2 \sin(x_3) + m_1 g \sin(x_3)\cos(x_3) + m_2 l_2 x_6^2 \sin(x_5) + m_2 g \sin(x_5)\cos(x_5))}{(M + m_1 \sin^2(x_3) + m_2 \sin^2(x_5))}$$

$$\dot{x}_3 = x_4$$

$$\dot{x}_4 = \frac{F - (\ (m_1 + M) g \sin(x_3) + m_1 l_1 x_4^2 \sin(x_3)\cos(x_3) + m_2 l_2 x_6^2 \sin(x_5)\cos(x_3) + m_2 g \sin(x_5)\cos(x_3 - x_5)\ )}{(M + m_1 \sin^2(x_3) + m_2 \sin^2(x_5)) l_1}$$

$$\dot{x}_5 = x_6$$

$$\dot{x}_6 = \frac{F - (\ (m_2 + M) g \sin(x_5) + m_1 l_1 x_4^2 \sin(x_3)\cos(x_3) + m_2 l_2 x_6^2 \sin(x_5)\cos(x_3) + m_1 g \sin(x_3)\cos(x_3 - x_5)\ )}{(M + m_1 \sin^2(x_3) + m_2 \sin^2(x_5)) l_2}$$

B) Obtain the linearized system around the equilibrium point specified by x= 0 and θ1=θ2= 0.Write the state-space representation of the linearized system.

After linearization we get A_F.

$$A_F = \nabla_{\vec{x}} \left| F(\vec{x}(t), \vec{u}(t)) \right.$$

$$A_F = \left. \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \frac{\partial f_1}{\partial x_3} & \frac{\partial f_1}{\partial x_4} & \frac{\partial f_1}{\partial x_5} & \frac{\partial f_1}{\partial x_6} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdot & \cdot & \cdot & \cdot \\ \frac{\partial f_3}{\partial x_1} & \frac{\partial f_3}{\partial x_2} & \cdot & \cdot & \cdot & \cdot \\ \frac{\partial f_4}{\partial x_1} & \frac{\partial f_4}{\partial x_2} & \cdot & \cdot & \cdot & \cdot \\ \frac{\partial f_5}{\partial x_1} & \frac{\partial f_5}{\partial x_2} & \cdot & \cdot & \cdot & \cdot \\ \frac{\partial f_6}{\partial x_1} & \frac{\partial f_6}{\partial x_2} & \cdot & \cdot & \cdot & \frac{\partial f_6}{\partial x_6} \end{bmatrix} \right|_{x=0}$$

We have $f_1 = x_2$

So, $\frac{\partial f_1}{\partial x_1} = \frac{\partial f_1}{\partial x_3} = \frac{\partial f_1}{\partial x_4} = \frac{\partial f_1}{\partial x_5} = \frac{\partial f_1}{\partial x_6} = 0$

and $\frac{\partial f_1}{\partial x_2} = 1$

$$f_2 = \dot{x}_2 = \frac{-\{m_1 l_1 \dot{x}_4^2 \sin x_3 + m_1 g \sin x_3 \cos x_3 + m_2 l_2 x_6^2 \sin x_5 + m_2 g \sin x_5 \cos x_5 + F\}}{M + m_1 \sin^2 x_3 + m_2 \sin^2 x_5}$$

$$\frac{\partial f_2}{\partial x_1} = \frac{\partial f_2}{\partial x_2} = 0$$

$$\left. \frac{\partial f_2}{\partial x_3} \right|_{x=0} = \frac{-(m_1 g (\cos(0))^2 M - 0)}{M^2} = \frac{-m_1 g}{M}$$

$$\left. \frac{\partial f_2}{\partial x_4} \right|_{x=0} = 0$$

$$\left. \frac{\partial f_2}{\partial x_5} \right|_{x=0} = \frac{-(m_2 g (\cos(0)^2)^2 . M)}{M^2} = \frac{-m_2 g}{M}$$

$$\left. \frac{\partial f_2}{\partial x_6} \right|_{x=0} = 0$$

$$f_3 = x_4$$

$$\therefore \frac{\partial f_3}{\partial x_1} = \frac{\partial f_3}{\partial x_2} = \frac{\partial f_3}{\partial x_3} = \frac{\partial f_3}{\partial x_5} = \frac{\partial f_3}{\partial f_6} = 0$$

and $\frac{\partial f_3}{\partial x_4} = 1$.

$$f_4 = \frac{-\left\{(M+m_1)g\sin x_3 + m_1 l_1 x_4^2 \sin x_3 \cos x_3 + m_2 l_2 x_6^2 \sin x_5 \cos x_3 + m_2 g \sin x_5 \cos(x_3 - x_5) + F\right\}}{(M + m_1 \sin^2 x_3 + m_2 \sin^2 x_5)\cdot l_1}$$

$$\frac{\partial f_4}{\partial x_1} = \frac{\partial f_4}{\partial x_2} = 0.$$

$$\left.\frac{\partial f_4}{\partial x_3}\right|_{x=0} = -\frac{(M+m_1)g}{M l_1}$$

$$\left.\frac{\partial f_4}{\partial x_4}\right|_{x=0} = 0.$$

$$\left.\frac{\partial f_4}{\partial x_5}\right|_{x=0} = \frac{-(m_2 g)}{M l_1}$$

$$\left.\frac{\partial f_4}{\partial x_6}\right|_{x=0} = 0$$

$$f_5 = x_6.$$

$$\therefore \frac{\partial f_5}{\partial x_1} = \frac{\partial f_5}{\partial x_2} = \frac{\partial f_5}{\partial x_3} = \frac{\partial f_5}{\partial x_4} = \frac{\partial f_5}{\partial x_5} = 0$$

and $\frac{\partial f_5}{\partial x_6} = 1$.

$$f_6 = \frac{-\left\{m_1 l_1 x_4^2 \sin x_3 \cos x_5 + m_1 g \sin x_3 \cos(x_3 - x_5) + (M+m_2) g \sin x_5 + m_2 l_2 x_6^2 \sin x_5 \cos x_5 + F\right\}}{(M + m_1 \sin^2 x_3 + m_2 \sin^2 x_5)\cdot l_2}$$

$$\frac{\partial f_6}{\partial x_1} = \frac{\partial f_6}{\partial x_2} = 0.$$

$$\left.\frac{\partial f_6}{\partial x_3}\right|_{x=0} = \frac{\{-m_1 g\cos(0)\cos(0) - 0\}\cancel{M l_2}}{M^{\cancel{2}} l_2^{\cancel{2}}}$$

$$= -\left(\frac{m_1 g}{M l_2}\right)$$

$$\left.\frac{\partial f_6}{\partial x_4}\right|_{x=0} = 0$$

$$\frac{\partial f_6}{\partial x_5} = \frac{-g(M+m_2)}{M l_2}$$

$$\frac{\partial f_6}{\partial x_6} = 0$$

$$\therefore\ A_F = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\frac{(m_1 g)}{M} & 0 & -\frac{(m_2 g)}{M} & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -\frac{g(M+m_1)}{M l_1} & 0 & -\frac{(m_2 g)}{m l_2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & -\frac{(m_1 g)}{M l_2} & 0 & \frac{-g(M+m_2)}{M l_2} & 0 \end{bmatrix}$$

$$B_F = \nabla_{\vec{u}} \left| f(\vec{x}(t), \vec{u}(t)) \right)$$

$$B_F = \left.\begin{bmatrix} \frac{\partial f_1}{\partial u} & \frac{\partial f_2}{\partial u} & \frac{\partial f_3}{\partial u} & \frac{\partial f_4}{\partial u} & \frac{\partial f_5}{\partial u} & \frac{\partial f_6}{\partial u} \end{bmatrix}^T\right|_{x=0} ; \quad u = F.$$

$$\frac{\partial f_1}{\partial u} = 0 \qquad \left.\frac{\partial f_2}{\partial u}\right|_{x=0} = \frac{1}{M}$$

$$\frac{\partial f_3}{\partial u} = 0 \qquad \frac{\partial f_4}{\partial u} = \frac{1}{M l_1} \qquad \frac{\partial f_5}{\partial u} = 0 \qquad \frac{\partial f_6}{\partial u} = \frac{1}{M l_2}$$

$$\therefore\ B_F = \begin{bmatrix} 0 \\ \frac{1}{M} \\ 0 \\ \frac{1}{M l_1} \\ 0 \\ \frac{1}{M l_2} \end{bmatrix}$$

With A_F =

```
[0 1            0               0       0              0;
 0 0        -m1*g/M             0    -m2*g/M           0;
 0 0            0               1       0              0;
 0 0    -g*(M+m1)/(M*l1)        0 -(m2*g)/(M*l1)       0;
 0 0            0               0       0              1;
 0 0    -(m1*g)/(M*l2)          0 -(g*(M+m2))/(M*l2) 0];
```

and B_F =

```
[   0   ;
   1/M  ;
    0   ;
 1/(M*l1);
    0   ;
 1/(M*l2)];
```

Linearized State Space Equations is as:

$$\dot{X} = A_F X + B_F u$$

where $X = [\, x_1,\ \dot{x}_1,\ \theta_1,\ \dot{\theta}_1,\ \theta_2,\ \dot{\theta}_2 \,] = [x_1, x_2, x_3, x_4, x_5, x_6]$ and u = F

$$\begin{bmatrix} \dot{x} \\ \ddot{x} \\ \dot{\theta}_1 \\ \ddot{\theta}_1 \\ \dot{\theta}_2 \\ \ddot{\theta}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{-(m_1 g)}{M} & 0 & \frac{-m_2 g}{M} & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{-g(M+m)}{M l_1} & 0 & \frac{-m_2 g}{M l_1} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & \frac{-m_1 g}{M l_2} & 0 & \frac{-g(M+m_2)}{M l_2} & 0 \end{bmatrix} \begin{bmatrix} x \\ \dot{x} \\ \theta_1 \\ \dot{\theta}_1 \\ \theta_2 \\ \dot{\theta}_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \frac{1}{M} \\ 0 \\ \frac{1}{M l_1} \\ 0 \\ \frac{1}{M l_2} \end{bmatrix} F$$

C) Obtain conditions on M, m_1, m_2, l_1, l_2 for which the linearized system is controllable.

For A =

```
[0 1            0             0        0              0;
 0 0        -m1*g/M           0    -m2*g/M            0;
 0 0            0             1        0              0;
 0 0    -g*(M+m1)/(M*l1)      0 -(m2*g)/(M*l1)        0;
 0 0            0             0        0              1;
 0 0    -(m1*g)/(M*l2)        0 -(g*(M+m2))/(M*l2) 0];
```

B =

```
[   0   ;
  1/M  ;
   0   ;
 1/(M*l1);
   0   ;
 1/(M*l2)];
```

Controllability Matrix, $C = [B \;\; AB \;\; A^2B \;\; A^3B \;\; A^4B \;\; A^5B]$

For Linearized system to be controllable $det(C) \neq 0$ *(i.e. rank of C should be Full Rank(6))*

Using Matlab, $det(C) = (2g^6 l_1 l_2 - g^6 l_1{}^2 - g^6 l_2{}^2) / (M^6 l_1{}^6 l_2{}^6) \neq 0$

On Solving we get, $l_1 \neq l_2$

D) Choose M=1000Kg, m_1=m_2=100Kg, l_1=20m and l_2=10m. Check that the system is controllable and obtain an LQR controller. Simulate the resulting response to initial conditions when the controller is applied to the linearized system and also to the original nonlinear system. Adjust the parameters of the LQR cost until you obtain a suitable response. Use Lyapunov's indirect method to certify stability (locally or globally) of the closed-loop system.

On substituting values in A:

A =

0	1.0000	0	0	0	0
0	0	-0.9810	0	-0.9810	0
0	0	0	1.0000	0	0
0	0	-0.5395	0	-0.0491	0
0	0	0	0	0	1.0000
0	0	-0.0981	0	-1.0791	0

B =

1.0e-03 *

0
1.0000
0
0.0500
0
0.1000

Controllability Matrix, C =

1.0e-03 *

0	1.0000	0	-0.1472	0	0.1419
1.0000	0	-0.1472	0	0.1419	0
0	0.0500	0	-0.0319	0	0.0227
0.0500	0	-0.0319	0	0.0227	0
0	0.1000	0	-0.1128	0	0.1249
0.1000	0	-0.1128	0	0.1249	0

$det(C) = -1.3926e-24 \neq 0$. Hence System is controllable!

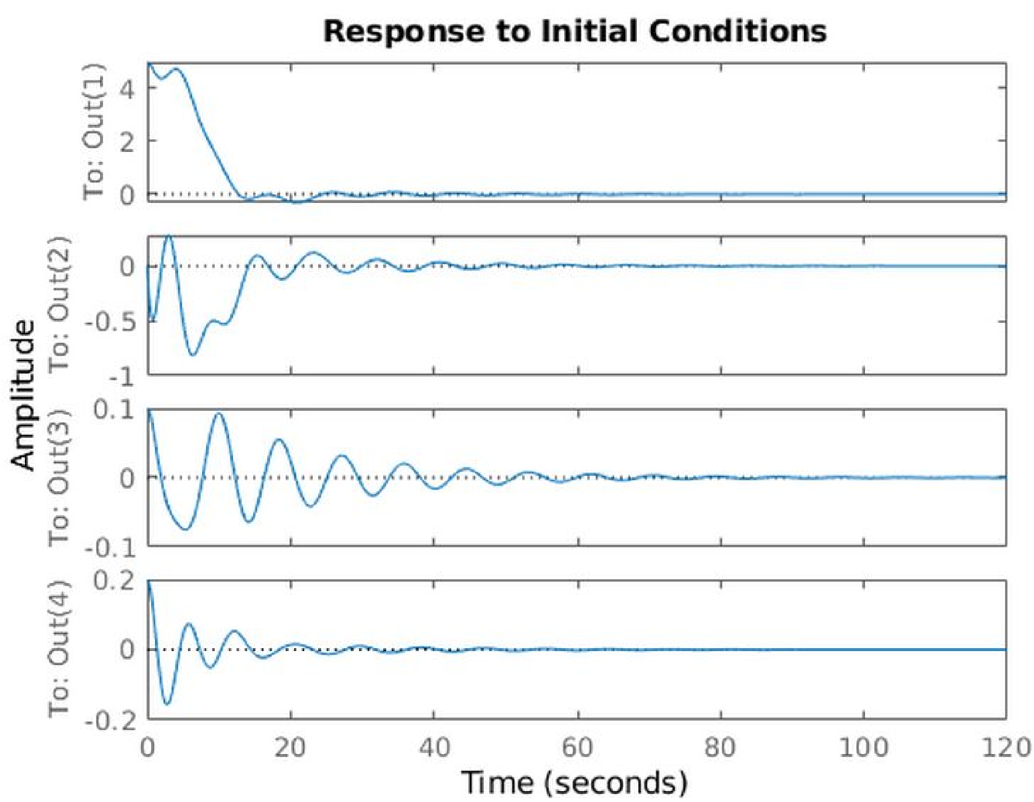
From trial and error, I choose Q and R as follows:

Q =

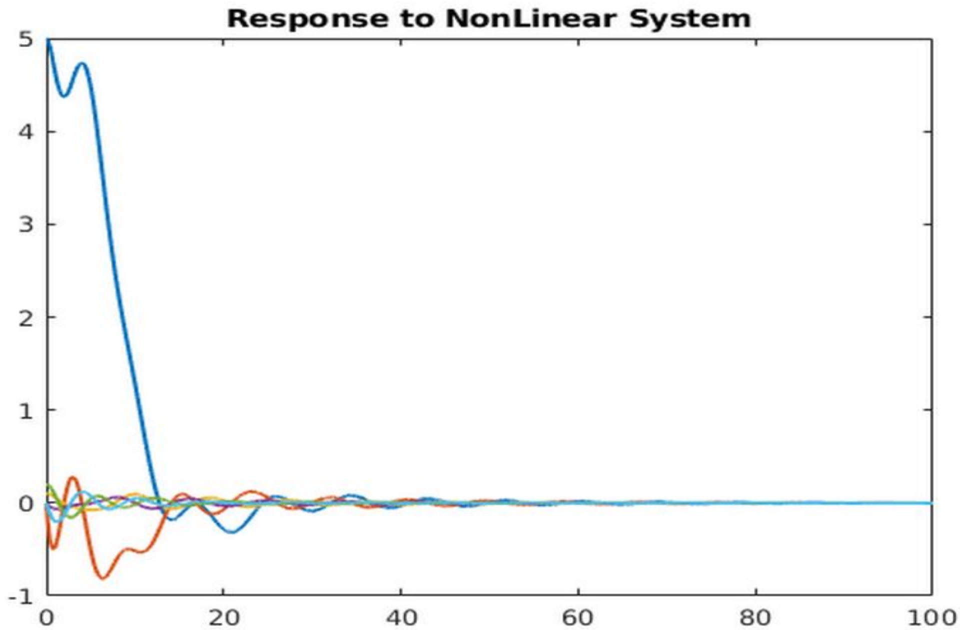
1	0	0	0	0	0
0	1	0	0	0	0
0	0	10	0	0	0
0	0	0	1000	0	0
0	0	0	0	200	0
0	0	0	0	0	2000

$$R = 1.0000e-04$$

Response to Initial Condition for Linearized System:



Response to Initial Condition for NonLinear System:



<u>Lyapunov's indirect method for Stability Check</u>

I received the following eigenvalues for A-BK:

```
-0.2428 + 1.0205i
-0.2428 - 1.0205i
-0.2051 + 0.2028i
-0.2051 - 0.2028i
-0.0548 + 0.7229i
-0.0548 - 0.7229i
```

All of them lie in the left half plane, hence the linearized system is stable. Thus NonLinear system is locally stable near the equilibrium point.

Second Component

Consider the parameters selected in C above.

E) Suppose that you can select the following output vectors: x(t),(θ_1(t),θ_2(t)), (x(t),θ_2(t)) or (x(t),θ_1(t),θ_2(t)). Determine for which output vectors the linearized system is observable.

Case I: Output Vector : x(t)

C = [1 0 0 0 0 0];

disp(rank(obsv(A,C))); % Observable as Rank = 6(Full Rank)

% Case II: Output Vector : (θ 1 (t), θ 2 (t))

C = [0 0 0 1 0 0; 0 0 0 0 1 0];

disp(rank(obsv(A,C)));% Not Observable as Rank = 4 < 6

% Case III: Output Vector : (x(t), θ 2 (t))

C = [1 0 0 0 0 0; 0 0 0 0 1 0];

disp(rank(obsv(A,C)));% Observable as Rank = 6 (Full Rank)

% Case IV: Output Vector : (x(t), θ 1 (t), θ 2 (t))

C = [1 0 0 0 0 0; 0 0 1 0 0 0; 0 0 0 0 0 1];

disp(rank(obsv(A,C)));% Observable as Rank = 6 (Full Rank)

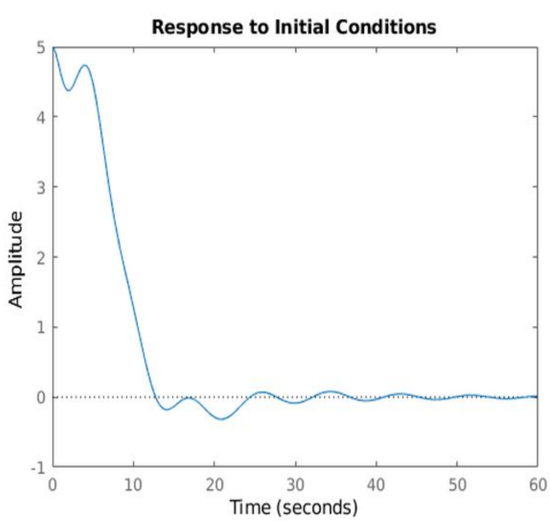
F) Obtain your "best" Luenberger observer for each one of the output vectors for which the system is observable and simulate its response to initial conditions and unit step input. The simulation should be done for the observer applied to both the linearized system and the original nonlinear system.

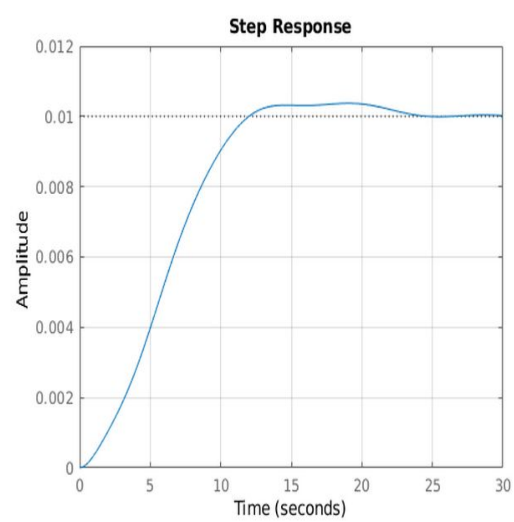
For resultant closed loop system with observer, poles are placed at [-2 -3 -4 -5 -6 -7]. K is same as LQR design. Initial Condition is [5; 0; 0.1; 0; 0.2; 0]

Luenberger Observer for Output x(t)

Observer Gain Matrix, L:

```
1.0e+04 *

 0.0027
 0.0293
-1.4999
-0.5404
 1.3346
 0.0686
```

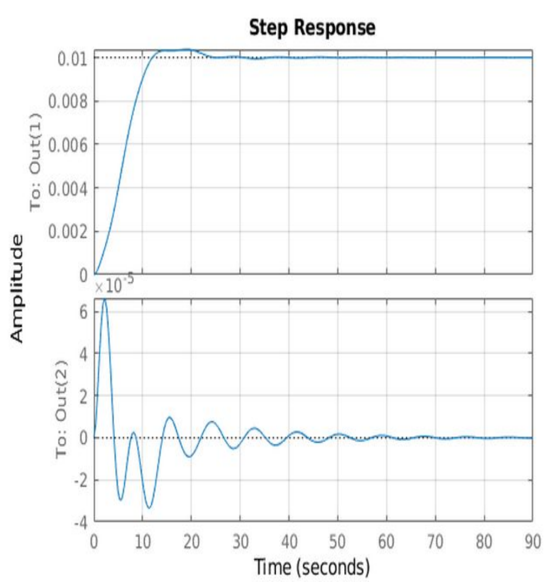


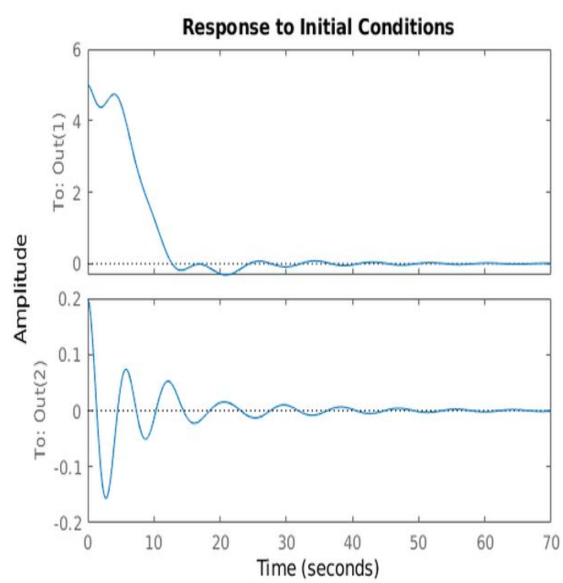


Luenberger Observer for Output [x(t), theta2(t)]

Observer Gain Matrix, L:

```
  Column 1

   17.2418
  103.7782
 -256.4987
 -174.6426
    0.6195
    7.4369

  Column 2

   -1.9388
  -26.1469
  101.6392
  110.3637
    9.7582
   20.1529
```

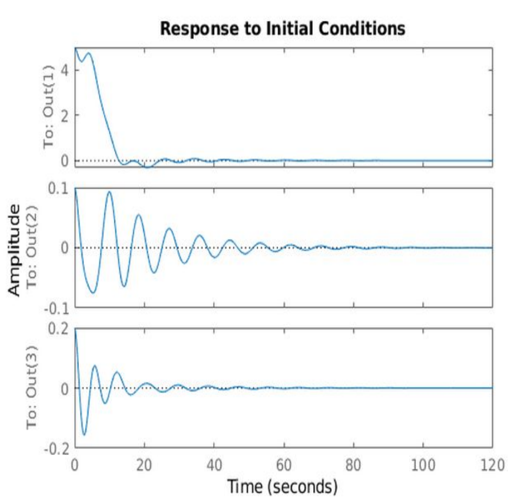


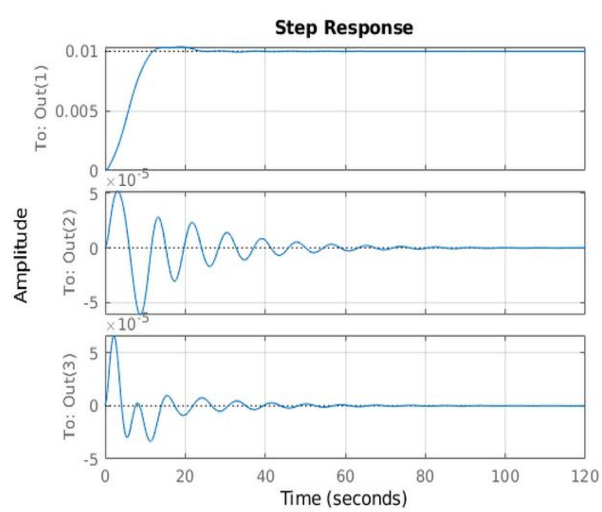


Luenberger Observer for Output [x(t) theta1(t) theta2(t)]

Observer Gain Matrix, L:

```
Column 1

 10.4331
 26.3638
 -0.8468
 -4.6605
  0.0000
  0.0000

Column 2

 -0.8011
 -5.3594
 11.5669
 32.0963
 -0.0000
 -0.0981

Column 3

  0.0000
 -0.9810
 -0.0000
 -0.0492
  5.0000
  4.9209
```



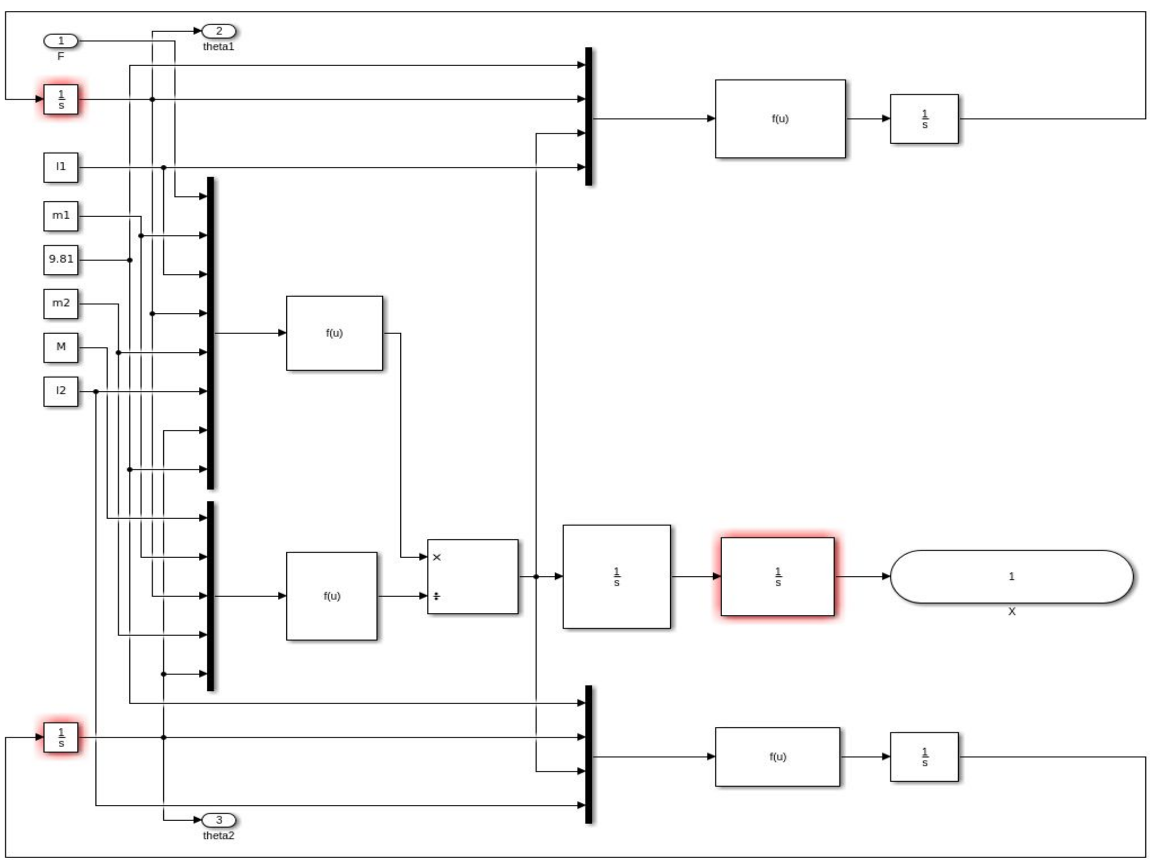


G) Design an output feedback controller for your choice of the "smallest" output vector. Use the LQG method and apply the resulting output feedback controller to the original nonlinear system. Obtain your best design and illustrate its performance in simulation. How would you reconfigure your controller to asymptotically track a constant reference on x? Will your design reject constant force disturbances applied on the cart ?

For LQG, I took the following values for the parameters:

1. *disturbance = 0.1 * eye(1); % Covariance of Disturbance*
2. *noise = 1; % Covariance of Noise*

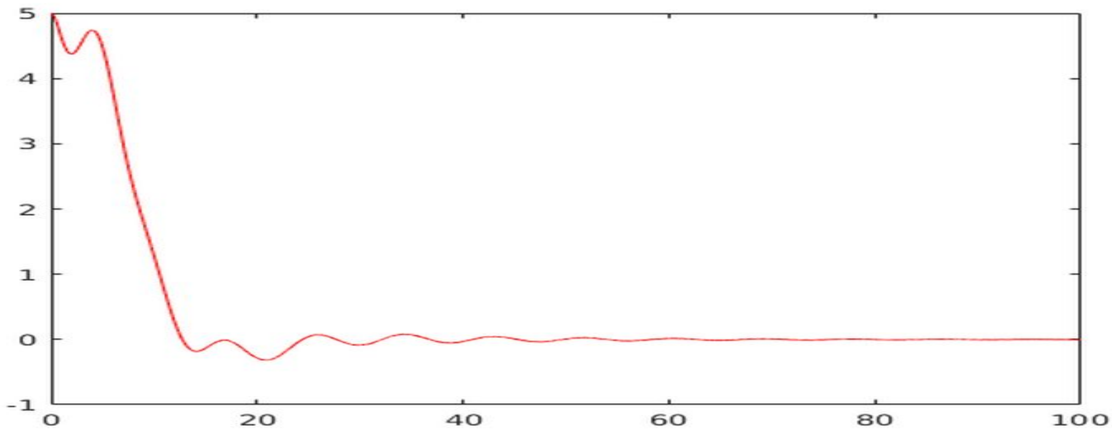
I modeled the system in Simulink as follows:



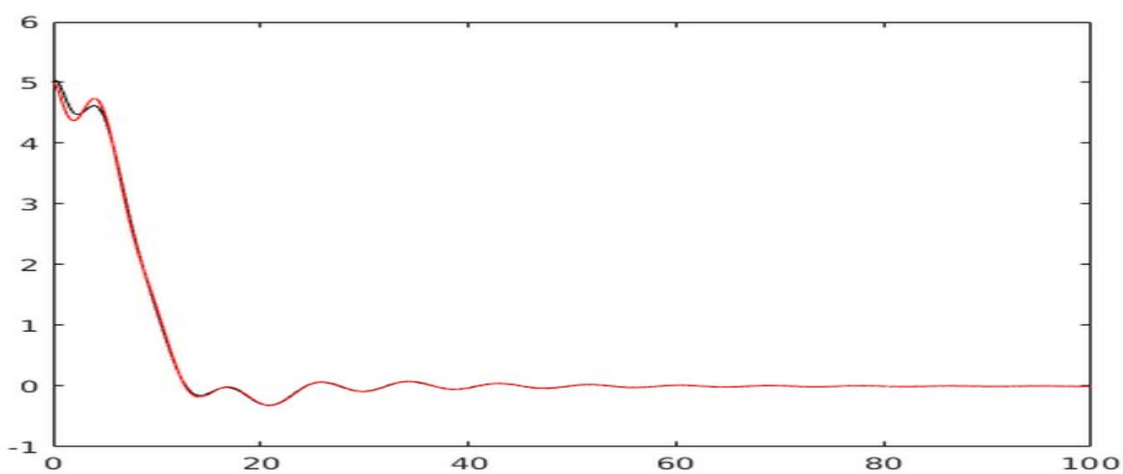
Obtained Gain Matrix, L =

$$\begin{matrix} 0.0230 \\ 0.0003 \\ 0.0000 \\ 0.0000 \\ 0.0000 \\ 0.0000 \end{matrix}$$

NonLinear Response with LQG for Initial Condition as previous



Error Tracking



LQG takes care of Steady State Error in Reference tracking as it keeps account of disturbance and noise as well. Amplifying the reference solves this issue.

Appendix

```
%% ENPM 667 Controls of Robotic Systems
%% Info Section

% Author: Akshitha Pothamshetty
% Section: 0201
%% Part D

% Choose M = 1000Kg, m 1 = m 2 = 100Kg, l 1 = 20m and l 2 = 10m. Check that
% the system is controllable and obtain an LQR controller. Simulate the
resulting response to initial
% conditions when the controller is applied to the linearized system and
also to the original nonlinear
% system. Adjust the parameters of the LQR cost until you obtain a suitable
response. Use Lyapunov's
% indirect method to certify stability (locally or globally) of the
closed-loop system.


% Setting Given Variables

m1 = 100;
m2 = 100;
M = 1000;
l1 = 20;
l2 = 10;
g = 9.81;

% Check if System is controllable?

A = [0 1            0           0       0           0;
     0 0        -m1*g/M         0   -m2*g/M         0;
     0 0            0           1       0           0;
     0 0    -g*(M+m1)/(M*l1)    0 -(m2*g)/(M*l1)    0;
     0 0            0           0       0           1;
     0 0    -(m1*g)/(M*l2)      0 -(g*(M+m2))/(M*l2) 0];
```

```
% disp(A);

B= [  0  ;
     1/M ;
      0  ;
    1/(M*l1);
      0  ;
    1/(M*l2)];

disp(rank(ctrb(A,B))); % Rank = 6(Full Rank) Hence Controllable.

% Obtaining an LQR Controller

C=[1 0 0 0 0 0;
   0 1 0 0 0 0;
   0 0 1 0 0 0;
   0 0 0 0 1 0];

D = [0];

% Selecting appropriate Q and R Values
Q = [1 0 0 0 0 0;
     0 1 0 0 0 0;
     0 0 10 0 0 0;
     0 0 0 1000 0 0;
     0 0 0 0 200 0;
     0 0 0 0 0 2000];

R = 0.0001;

% Gain Matrix
K = lqr(A, B, Q, R);
% disp(K);

disp(eig(A - B*K)); % Lyapunov Indirect Method: All eigenValues are on the left hand side, so system is stable.
sys = ss(A-B*K, B, C, D); % Create a state-space system.

intialState = [5;
               0;
               0.1;
               0;
```

```
                0.2;
                0];

initial(sys, intialState)


% Response to Non-Linear System

t = 0:0.01:100; % TimeSteps
[t,x]=ode45(@State,t,intialState);
plot(t,x,'linewidth',1.5);
title('Response to NonLinear System');
%% Part E

A = [0 1            0           0       0            0;
     0 0        -m1*g/M         0   -m2*g/M          0;
     0 0            0           1       0            0;
     0 0    -g*(M+m1)/(M*l1)    0 -(m2*g)/(M*l1)     0;
     0 0            0           0       0            1;
     0 0    -(m1*g)/(M*l2)      0 -(g*(M+m2))/(M*l2) 0];

% Case I: Output Vector : x(t)
C = [1 0 0 0 0 0];
disp(rank(obsv(A,C))); % Observable as Rank = 6(Full Rank)

% Case II: Output Vector : (θ 1 (t), θ 2 (t))
C = [0 0 0 1 0 0;
     0 0 0 0 1 0];
disp(rank(obsv(A,C)));% Not Observable as Rank = 4 < 6

% Case III: Output Vector : (x(t), θ 2 (t))
C = [1 0 0 0 0 0;
     0 0 0 0 1 0];
disp(rank(obsv(A,C)));% Observable as Rank = 6(Full Rank)

% Case IV: Output Vector : (x(t), θ 1 (t), θ 2 (t))
C = [1 0 0 0 0 0;
     0 0 1 0 0 0;
     0 0 0 0 0 1];
disp(rank(obsv(A,C)));% Observable as Rank = 6(Full Rank)
%% Part F
```

```
% Obtain your "best" Luenberger observer for each one of the output vectors
for which the system is
% observable and simulate its response to initial conditions and unit step
input. The simulation should
% be done for the observer applied to both the linearized system and the
original nonlinear system.
%% Luenberger Observer for Output x(t)

% Setting Given Variables
m1 = 100;
m2 = 100;
M = 1000;
l1 = 20;
l2 = 10;
g = 9.81;

% Check if System is controllable?

A = [0 1            0       0       0       0;
     0 0        -m1*g/M     0   -m2*g/M     0;
     0 0            0       1       0       0;
     0 0 -g*(M+m1)/(M*l1)   0 -(m2*g)/(M*l1) 0;
     0 0            0       0       0        1;
     0 0 -(m1*g)/(M*l2) 0 -(g*(M+m2))/(M*l2) 0];

% disp(A);

B= [ 0; 1/M ;0; 1/(M*l1) ;0 ;1/(M*l2)];

% disp(rank(ctrb(A,B))); % Rank = 6(Full Rank) Hence Controllable.

% Obtaining an LQR Controller

C=[1 0 0 0 0 0];

D = [0];

% Selecting appropriate Q and R Values
Q = [1 0 0 0 0 0;
     0 1 0 0 0 0;
     0 0 10 0 0 0;
     0 0 0 1000 0 0;
```

```
     0 0 0 0 200 0;
     0 0 0 0 0 2000];

R = 0.0001;

% Gain Matrix
K = lqr(A, B, Q, R);
% disp(K);

p = [-2 -3 -4 -5 -6 -7];
L = place(A.', C.', p);
L = L.';
disp(L);

eig(A-L*C);

Ac = [(A-B*K) (B*K); zeros(size(A)) (A-L*C)];
Bc=[ B ;zeros(size(B))];
Cc= [C zeros(size(C))];
Dc=[0];

sys = ss(Ac,Bc,Cc,Dc);

initialState = [ 5 ; 0 ; 0.1 ; 0 ;0.2 ; 0 ; 0 ; 0 ; 0 ; 0 ; 0 ; 0];

initial(sys, initialState);
step(sys);
grid;
%% Luenberger Observer for Output [x(t), theta2(t)]

C=[1 0 0 0 0 0 ; 0 0 0 0 1 0];

p = [-2 -3 -4 -5 -6 -7];
L = place(A.', C.', p);
L = L.';
disp(L);

eig(A-L*C);

Ac = [(A-B*K) (B*K); zeros(size(A)) (A-L*C)];
Bc=[ B ;zeros(size(B))];
Cc= [C zeros(size(C))];
```

```
Dc=[0];

sys = ss(Ac,Bc,Cc,Dc);

initialState = [ 5 ; 0 ; 0.1 ; 0 ;0.2 ; 0 ; 0 ; 0 ; 0 ; 0 ; 0 ; 0];

initial(sys, initialState);
step(sys);
grid;

%% Luenberger Observer for Output [x(t) theta1(t) theta2(t)]

C=[1 0 0 0 0 0 ; 0 0 1 0 0 0; 0 0 0 0 1 0];

p = [-2 -3 -4 -5 -6 -7];
L = place(A.', C.', p);
L = L.';
disp(L);

eig(A-L*C);

Ac = [(A-B*K) (B*K); zeros(size(A)) (A-L*C)];
Bc=[ B ;zeros(size(B))];
Cc= [C zeros(size(C))];
Dc=[0];

sys = ss(Ac,Bc,Cc,Dc);

initialState = [ 5 ; 0 ; 0.1 ; 0 ;0.2 ; 0 ; 0 ; 0 ; 0 ; 0 ; 0 ; 0];

initial(sys, initialState);
step(sys);
grid;

%% Part G

% Design an output feedback controller for your choice of the "smallest"
output vector. Use the LQG
% method and apply the resulting output feedback controller to the original
nonlinear system. Obtain
% your best design and illustrate its performance in simulation. How would
you reconfigure your con-
```

```
% troller to asymptotically track a constant reference on x ? Will your
design reject constant force
% disturbances applied on the cart ?


% Setting Given Variables
m1 = 100;
m2 = 100;
M = 1000;
l1 = 20;
l2 = 10;
g = 9.81;

% Setting Parameters for LQE & LQR
disturbance = 0.1 * eye(1);
noise = 1;
sensors = [1];
known = [1];
states =
{'x','x_dot','theta1','theta1_dot','theta2','theta2_dot','e_1','e_2','e_3',
'e_4','e_5','e_6'};
inputs = {'F'};
outputs = {'x'};
initialState = [ 5 ; 0 ; 0.1 ; 0 ;0.2 ; 0 ; 0 ; 0 ; 0 ; 0 ; 0 ; 0];

% Check if System is controllable?

A = [0 1            0       0       0       0;
     0 0        -m1*g/M     0    -m2*g/M     0;
     0 0            0       1       0       0;
     0 0 -g*(M+m1)/(M*l1)   0 -(m2*g)/(M*l1) 0;
     0 0            0       0       0        1;
     0 0 -(m1*g)/(M*l2) 0 -(g*(M+m2))/(M*l2) 0];

% disp(A);

B= [ 0; 1/M ;0; 1/(M*l1) ;0 ;1/(M*l2)];

% disp(rank(ctrb(A,B))); % Rank = 6(Full Rank) Hence Controllable.

% Obtaining an LQR Controller
```

```
C=[1 0 0 0 0 0];

D = [0];

% Selecting appropriate Q and R Values
Q = [1 0 0 0 0 0;
     0 1 0 0 0 0;
     0 0 10 0 0 0;
     0 0 0 1000 0 0;
     0 0 0 0 200 0;
     0 0 0 0 0 2000];

R = 0.0001;

% Gain Matrix
K = lqr(A, B, Q, R);
% disp(K);

% LQE Design
system = ss(A,[B B],C,[zeros(1,1) zeros(1,1)]);
[~,L,~] = kalman(system,disturbance,noise,[],sensors,known); % filter gain
L
disp(L);

% LQR Design
Ac = [A-B*K B*K;zeros(size(A)) A-L*C];
Bc = zeros(12,1);
Cc = [C zeros(size(C))];
LQG = ss(Ac,Bc,Cc,D,
'statename',states,'inputname',inputs,'outputname',outputs);

time = 0:0.01:100; % Timesteps
Force = zeros(size(time)); % External Force

% Simulate Time Response
[Y,~,X] = lsim(LQG,Force,time,initialState);
figure
plot(time,Y(:,1),'r');

u = zeros(size(time)); % Input

for i = 1:size(X,1)
```

```
    u(i) = K * (X(i,1:6))';
end

Xhat = X(:,1) - X(:,6); % Error Tracking
figure;
plot(time,Xhat, 'k'), hold on, plot(time,X(:,1),'r');


%%
function xDot = State(t,x)


% Setting Given Variables

m1 = 100;
m2 = 100;
M = 1000;
l1 = 20;
l2 = 10;
g = 9.81;


A = [0 1            0       0       0       0;
     0 0        -m1*g/M     0   -m2*g/M     0;
     0 0            0       1       0       0;
     0 0 -g*(M+m1)/(M*l1)   0 -(m2*g)/(M*l1) 0;
     0 0            0       0       0        1;
     0 0 -(m1*g)/(M*l2) 0 -(g*(M+m2))/(M*l2) 0];

% disp(A);

B= [ 0; 1/M ;0; 1/(M*l1) ;0 ;1/(M*l2)];

% Selecting appropriate Q and R Values
Q = [1 0 0 0 0 0;
     0 1 0 0 0 0;
     0 0 10 0 0 0;
     0 0 0 1000 0 0;
     0 0 0 0 200 0;
     0 0 0 0 0 2000];

R = 0.0001;
```

```
K = lqr(A, B, Q, R);
u = -K*x;

xDot = zeros(6,1);

xDot(1)= x(2);
xDot(2)= -((-u) + m1*l1*x(4)^2*sin(x(3)) + m1*g*sin(x(3))*cos(x(3))+
m2*l2*x(6)^2*sin(x(5))+m2*g*sin(x(5))*cos(x(5)))/(M+m1*sin(x(3)^2)+
m2*sin(x(5)^2));
xDot(3)= x(4);
xDot(4)= -((-u) +(M+m1)*g*sin(x(3)) + m1*l1*x(4)^2*sin(x(3))*cos(x(3)) +
m2*l2*x(6)^2*sin(x(5))*cos(x(3)) + m2*g*sin(x(5))*cos(x(3)- x(5)))/(( M+
m1*sin(x(3)^2)+ m2*sin(x(5)^2))*l1);
xDot(5)= x(6);
xDot(6)= -((-u) + m1*l1*x(4)^2*sin(x(3))*cos(x(5))
+m1*g*sin(x(3))*cos(x(3)-x(5)) +
(M+m1)*g*sin(x(5))+m2*l2*x(6)^2*sin(x(5))*cos(x(5)))/(( M+m1*sin(x(3)^2)+
m2*sin(x(5)^2))*l2);
end
```