Mathematical concepts for computer science

Propositional Equivalences

- An important type of step used in a mathematical argument is the **replacement of a statement with another statement** with the same truth value.
- A compound proposition that is **always true**, no matter what the truth values of the propositional variables that occur in it, is called a **tautology**.
- A compound proposition that is **always false** is called a **contradiction**.
- A compound proposition that is **neither a tautology nor a contradiction** is called a **contingency**.

Propositional Equivalences

- Check whether the following arguments are tautology or contradiction
- **$p \vee \neg p$**
- **$p \wedge \neg p$**

Propositional Equivalences

- Check whether the following arguments are tautology or contradiction
- **$p \vee \neg p$** - tautology
- **$p \wedge \neg p$ -** contradiction

TABLE 1 Examples of a Tautology and a Contradiction.

p	$\neg p$	$p \vee \neg p$	$p \wedge \neg p$
T	F	T	F
F	T	T	F

Logical Equivalences

- Compound propositions that have the **same truth values** in all possible cases are called **logically equivalent.**
- The compound propositions **p** and **q** are called **logically equivalent** if **p ↔ q** is a **tautology**.
- The notation **p ≡ q (p**⇔q**)**denotes that **p and q are logically equivalent**.

TABLE 2 De Morgan's Laws.
$\neg(p \wedge q) \equiv \neg p \vee \neg q$
$\neg(p \vee q) \equiv \neg p \wedge \neg q$

Logical Equivalences

- Show that **¬(p ∨ q) and ¬p ∧¬q** are logically equivalent.

TABLE 2 De Morgan's Laws.
$\neg(p \wedge q) \equiv \neg p \vee \neg q$
$\neg(p \vee q) \equiv \neg p \wedge \neg q$

Logical Equivalences

- Show that $\neg(p \lor q)$ and $\neg p \land \neg q$ are logically equivalent.

TABLE 3 Truth Tables for $\neg(p \vee q)$ and $\neg p \wedge \neg q$.

p	q	$p \vee q$	$\neg(p \vee q)$	$\neg p$	$\neg q$	$\neg p \wedge \neg q$
T	T	T	F	F	F	F
T	F	T	F	F	T	F
F	T	T	F	T	F	F
F	F	F	T	T	T	T

TABLE 2 De Morgan's Laws.
$\neg(p \wedge q) \equiv \neg p \vee \neg q$
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Logical Equivalences

- Show that **p → q and ¬p V q** are logically equivalent.

Logical Equivalences

- Show that **p → q and ¬p ∨ q** are logically equivalent.

TABLE 4 Truth Tables for $\neg p \vee q$ and $p \rightarrow q$.

p	q	$\neg p$	$\neg p \vee q$	$p \rightarrow q$
T	T	F	T	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

Distributive law of disjunction over conjunction.

TABLE 5 A Demonstration That $p \vee (q \wedge r)$ and $(p \vee q) \wedge (p \vee r)$ Are Logically Equivalent.

p	q	r	$q \wedge r$	$p \vee (q \wedge r)$	$p \vee q$	$p \vee r$	$(p \vee q) \wedge (p \vee r)$
T	T	T	T	T	T	T	T
T	T	F	F	T	T	T	T
T	F	T	F	T	T	T	T
T	F	F	F	T	T	T	T
F	T	T	T	T	T	T	T
F	T	F	F	F	T	F	F
F	F	T	F	F	F	T	F
F	F	F	F	F	F	F	F

Logical Equivalences

TABLE 6 Logical Equivalences.

Equivalence	*Name*
$p \wedge \mathbf{T} \equiv p$ $p \vee \mathbf{F} \equiv p$	Identity laws
$p \vee \mathbf{T} \equiv \mathbf{T}$ $p \wedge \mathbf{F} \equiv \mathbf{F}$	Domination laws
$p \vee p \equiv p$ $p \wedge p \equiv p$	Idempotent laws
$\neg(\neg p) \equiv p$	Double negation law
$p \vee q \equiv q \vee p$ $p \wedge q \equiv q \wedge p$	Commutative laws

Logical Equivalences

$(p \vee q) \vee r \equiv p \vee (q \vee r)$ $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$	Associative laws
$p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$ $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$	Distributive laws
$\neg(p \wedge q) \equiv \neg p \vee \neg q$ $\neg(p \vee q) \equiv \neg p \wedge \neg q$	De Morgan's laws
$p \vee (p \wedge q) \equiv p$ $p \wedge (p \vee q) \equiv p$	Absorption laws
$p \vee \neg p \equiv \mathbf{T}$ $p \wedge \neg p \equiv \mathbf{F}$	Negation laws

Logical Equivalences

TABLE 7 Logical Equivalences Involving Conditional Statements.

$$p \to q \equiv \neg p \lor q$$

$$p \to q \equiv \neg q \to \neg p$$

$$p \lor q \equiv \neg p \to q$$

$$p \land q \equiv \neg(p \to \neg q)$$

$$\neg(p \to q) \equiv p \land \neg q$$

$$(p \to q) \land (p \to r) \equiv p \to (q \land r)$$

$$(p \to r) \land (q \to r) \equiv (p \lor q) \to r$$

$$(p \to q) \lor (p \to r) \equiv p \to (q \lor r)$$

$$(p \to r) \lor (q \to r) \equiv (p \land q) \to r$$

Logical Equivalences

TABLE 8 **Logical Equivalences Involving Biconditional Statements.**

$p \leftrightarrow q \equiv (p \rightarrow q) \wedge (q \rightarrow p)$
$p \leftrightarrow q \equiv \neg p \leftrightarrow \neg q$
$p \leftrightarrow q \equiv (p \wedge q) \vee (\neg p \wedge \neg q)$
$\neg(p \leftrightarrow q) \equiv p \leftrightarrow \neg q$

Constructing New Logical Equivalences

Show that **$\neg(p \rightarrow q)$** and **$p \land \neg q$** are logically **equivalent without using truth table**.

Constructing New Logical Equivalences

Show that **¬(p → q)** and **p ∧¬q** are logically **equivalent without using truth table**.

$$
\begin{aligned}
\neg(p \to q) &\equiv \neg(\neg p \vee q) \\
&\equiv \neg(\neg p) \wedge \neg q \\
&\equiv p \wedge \neg q
\end{aligned}
$$

Constructing New Logical Equivalences

- Show that $\neg(\mathbf{p} \vee (\neg\mathbf{p} \wedge \mathbf{q}))$ and $\neg\mathbf{p} \wedge \neg\mathbf{q}$ are logically equivalent by developing a series of logical equivalences.

Constructing New Logical Equivalences

- Show that **¬(p ∨ (¬p ∧ q))** and **¬p ∧¬q** are logically equivalent by developing a series of logical equivalences.

$$
\begin{aligned}
\neg(p \vee (\neg p \wedge q)) &\equiv \neg p \wedge \neg(\neg p \wedge q) && \text{by the second De Morgan law} \\
&\equiv \neg p \wedge [\neg(\neg p) \vee \neg q] && \text{by the first De Morgan law} \\
&\equiv \neg p \wedge (p \vee \neg q) && \text{by the double negation law} \\
&\equiv (\neg p \wedge p) \vee (\neg p \wedge \neg q) && \text{by the second distributive law} \\
&\equiv \mathbf{F} \vee (\neg p \wedge \neg q) && \text{because } \neg p \wedge p \equiv \mathbf{F} \\
&\equiv (\neg p \wedge \neg q) \vee \mathbf{F} && \text{by the commutative law for disjunction} \\
&\equiv \neg p \wedge \neg q && \text{by the identity law for } \mathbf{F}
\end{aligned}
$$

Constructing New Logical Equivalences

- Show that **$(p \land q) \rightarrow (p \lor q)$** is a **tautology**.

Constructing New Logical Equivalences

- Show that **(p ∧ q) → (p ∨ q)** is a **tautology**.

$$
\begin{aligned}
(p \wedge q) \rightarrow (p \vee q) &\equiv \neg(p \wedge q) \vee (p \vee q) \\
&\equiv (\neg p \vee \neg q) \vee (p \vee q) && \text{by the first De Morgan law} \\
&\equiv (\neg p \vee p) \vee (\neg q \vee q) && \text{by the associative and commutative} \\
& && \text{laws for disjunction} \\
&\equiv \mathbf{T} \vee \mathbf{T} \\
&\equiv \mathbf{T} && \text{by the domination law}
\end{aligned}
$$

Propositional Satisfiability

- A compound proposition is **satisfiable**, if there is an assignment of truth values to its variables that **makes it true**.
- When no such assignments exists, that is, when the compound proposition **is false for all assignments** of truth values to its variables, the compound proposition is **unsatisfiable.**
- To show that a compound proposition is unsatisfiable, we need **to show that every assignment of truth values** to its variables **makes it false**.

Propositional Satisfiability

- Determine whether each of the compound propositions $(p \lor \neg q) \land (q \lor \neg r) \land (r \lor \neg p)$ is satisfiable.

Propositional Satisfiability

- Determine whether each of the compound propositions **$(p \lor \neg q) \land (q \lor \neg r) \land (r \lor \neg p)$** is satisfiable.

- **True** when the three variable p, q, and r have the **same truth value**

Propositional Satisfiability

- Determine whether each of the compound propositions $(\mathbf{p} \lor \mathbf{q} \lor \mathbf{r}) \land (\neg\mathbf{p} \lor \neg\mathbf{q} \lor \neg\mathbf{r})$, is satisfiable.

Propositional Satisfiability

- Determine whether each of the compound propositions **$(p \lor q \lor r) \land (\neg p \lor \neg q \lor \neg r)$,** is satisfiable.

- **Satisfiable,** when **at least one of p, q, and r is true** and **at least one is false**

Propositional Satisfiability

- Determine whether each of the compound propositions $(\mathbf{p} \vee \neg \mathbf{q}) \wedge (\mathbf{q} \vee \neg \mathbf{r}) \wedge (\mathbf{r} \vee \neg \mathbf{p}) \wedge (\mathbf{p} \vee \mathbf{q} \vee \mathbf{r}) \wedge (\neg \mathbf{p} \vee \neg \mathbf{q} \vee \neg \mathbf{r})$ is satisfiable.

Propositional Satisfiability

- Determine whether each of the compound propositions **$(p \lor \neg q) \land (q \lor \neg r) \land (r \lor \neg p) \land (p \lor q \lor r) \land (\neg p \lor \neg q \lor \neg r)$** is satisfiable.

- **$(p \lor \neg q) \land (q \lor \neg r) \land (r \lor \neg p)$** and
- **$(p \lor q \lor r) \land (\neg p \lor \neg q \lor \neg r)$** must both be true.

Propositional Satisfiability

- **$(p \lor \neg q) \land (q \lor \neg r) \land (r \lor \neg p)$** and
- **$(p \lor q \lor r) \land (\neg p \lor \neg q \lor \neg r)$** must both be true.

- **For the first to be true**, the three variables must have the same truth values.
- **For the second to be true**, at least one of three variables must be true and at least one must be false.
- **These conditions are contradictory.**
- **It is unsatisfiable**

Applications of Satisfiability

	2	9				4		
			5			1		
	4							
				4	2			
6							7	
5								
7			3					5
	1			9				
							6	

A 9 x 9 Sudoku puzzle.

- **SUDOKU**
- The puzzle is solved by assigning a number to each blank cell so that every row, every column, and every one of the nine 3 × 3 blocks **contains each of the nine possible numbers**.
- Note that instead of using a 9 × 9 grid, Sudoku puzzles can be based on $n^2 \times n^2$ grids, for any positive integer n, with the $n^2 \times n^2$ grid made up of n^2 $n \times n$ sub grids.

Applications of Satisfiability

- To encode a Sudoku puzzle, let **p(i, j, n)** denote the proposition that is true when the number **n** is in the cell in **the ith row and j th column**.
- There are **9 × 9 × 9 = 729** such propositions, as i, j , and n all range from 1 to 9.

p(5, 1, 6) is true,
p(5, j, 6) is false for j = 2, 3, . . . , 9.

	2	9				4		
			5			1		
	4							
				4	2			
6							7	
5								
7			3					5
	1			9				
							6	

Applications of Satisfiability

- For each cell with a given value, we assert **p(i, j, n)** when the cell in **row i** and **column j** has the given **value n**.
- We assert that every row contains every number:

$$\bigwedge_{i=1}^{9} \bigwedge_{n=1}^{9} \bigvee_{j=1}^{9} p(i, j, n)$$

	2	9				4		
			5			1		
	4							
				4	2			
6							7	
5								
7			3					5
	1			9				
							6	

Applications of Satisfiability

- We assert that every column contains every number:

	2	9				4		
			5			1		
	4							
				4	2			
6							7	
5								
7			3					5
	1			9				
							6	

Applications of Satisfiability

- We assert that every column contains every number:

$$\bigwedge_{j=1}^{9} \bigwedge_{n=1}^{9} \bigvee_{i=1}^{9} p(i, j, n)$$

	2	9				4		
			5			1		
	4							
				4	2			
6							7	
5								
7			3					5
	1			9				
							6	

Applications of Satisfiability

- We assert that each of the nine 3 × 3 blocks contains every number:

$$\bigwedge_{r=0}^{2} \bigwedge_{s=0}^{2} \bigwedge_{n=1}^{9} \bigvee_{i=1}^{3} \bigvee_{j=1}^{3} p(3r+i, 3s+j, n)$$

	2	9				4		
			5			1		
	4							
				4	2			
6							7	
5								
7			3					5
	1			9				
							6	

Solving Satisfiability Problems

- A **truth table** can be used to determine whether a compound proposition is satisfiable.
- When the number of variables grows, this becomes **impractical**. $2^{20} = 1{,}048{,}576$ rows
- **No procedure** is known that a computer can follow to determine **in a reasonable amount of time** whether an arbitrary compound proposition in such a large number of variables **is satisfiable**.

Reference

- **Rosen, Kenneth H., and Kamala Krithivasan. Discrete mathematics and its applications: with combinatorics and graph theory. Tata McGraw-Hill Education, 2016.**