Mathematical concepts for computer science

Sequences

- A sequence is a **discrete structure** used to represent an **ordered list**.
- A sequence is a **function** **from a subset of the set of integers** (usually either the set {0, 1, 2, . . .} or the set {1, 2, 3, . . .}) to **a set S**.
- The notation a_n to denote the image of the integer n.
- a_n is a term of the sequence.

$$a_n = \frac{1}{n} \qquad a_1, a_2, a_3, a_4, \ldots, \quad 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots.$$

Geometric progression

A *geometric progression* is a sequence of the form

$$a, ar, ar^2, \ldots, ar^n, \ldots$$

where the *initial term a* and the *common ratio r* are real numbers.

$b_n = (-1)^n \qquad 1, -1, 1, -1, 1, \ldots; \qquad n = 0$

initial term and common ratio equal to 1 and -1

$c_n = 2 \cdot 5^n \qquad 2, 10, 50, 250, 1250, \ldots; \qquad$ 2 and 5

$d_n = 6 \cdot (1/3)^n \quad 6, 2, \frac{2}{3}, \frac{2}{9}, \frac{2}{27}, \ldots. \qquad$ 6 and 1/3

Arithmetic progression

An *arithmetic progression* is a sequence of the form

$$a, a + d, a + 2d, \ldots, a + nd, \ldots$$

where the *initial term* a and the *common difference* d are real numbers.

$$s_n = -1 + 4n \qquad -1, 3, 7, 11, \ldots, \qquad n = 0$$

$$t_n = 7 - 3n \qquad 7, 4, 1, -2, \ldots.$$

Summations

$$a_m, a_{m+1}, \ldots, a_n$$

$$\sum_{j=m}^{n} a_j, \qquad \textstyle\sum_{j=m}^{n} a_j, \qquad \text{or} \qquad \textstyle\sum_{m \le j \le n} a_j$$

$$a_m + a_{m+1} + \cdots + a_n.$$

Use summation notation to express the sum of the first 100 terms of the sequence $\{a_j\}$, where $a_j = 1/j$ for $j = 1, 2, 3, \ldots$.

$$\sum_{j=1}^{100} \frac{1}{j}.$$

Summations

$$\sum_{j=1}^{5} j^2 = 1^2 + 2^2 + 3^2 + 4^2 + 5^2$$
$$= 1 + 4 + 9 + 16 + 25$$
$$= 55.$$

$$\sum_{k=4}^{8} (-1)^k$$

Summations

$$\begin{aligned}\sum_{k=4}^{8}(-1)^k &= (-1)^4 + (-1)^5 + (-1)^6 + (-1)^7 + (-1)^8 \\ &= 1 + (-1) + 1 + (-1) + 1 \\ &= 1.\end{aligned}$$

Sums of terms of geometric progressions

If a and r are real numbers and $r \neq 0$, then

$$\sum_{j=0}^{n} ar^j = \begin{cases} \dfrac{ar^{n+1} - a}{r - 1} & \text{if } r \neq 1 \\ (n+1)a & \text{if } r = 1. \end{cases}$$

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Proof: Let

$$S_n = \sum_{j=0}^{n} ar^j.$$

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substituting summation formula for S

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$$= \sum_{j=0}^{n} ar^{j+1} \quad \text{by the distributive property}$$

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$$= \left(\sum_{k=0}^{n} ar^k \right) + (ar^{n+1} - a) \quad \text{removing } k = n + 1 \text{ term and adding } k = 0 \text{ term}$$

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$$= S_n + (ar^{n+1} - a)$$ substituting S for summation formula

$$rS_n = S_n + (ar^{n+1} - a)$$

Solving for S_n shows that if $r \neq 1$, then

$$S_n = \frac{ar^{n+1} - a}{r - 1}.$$

If $r = 1$, then the $S_n = \sum_{j=0}^{n} ar^j = \sum_{j=0}^{n} a = (n+1)a$.

Sum	*Closed Form*
$\sum_{k=0}^{n} ar^k \ (r \neq 0)$	$\frac{ar^{n+1} - a}{r - 1}, r \neq 1$
$\sum_{k=1}^{n} k$	$\frac{n(n+1)}{2}$
$\sum_{k=1}^{n} k^2$	$\frac{n(n+1)(2n+1)}{6}$
$\sum_{k=1}^{n} k^3$	$\frac{n^2(n+1)^2}{4}$
$\sum_{k=0}^{\infty} x^k, \lvert x\rvert < 1$	$\frac{1}{1-x}$
$\sum_{k=1}^{\infty} kx^{k-1}, \lvert x\rvert < 1$	$\frac{1}{(1-x)^2}$

Sums

What is the value of $\sum_{s \in \{0,2,4\}} s$?

$$\sum_{s \in \{0,2,4\}} s = 0 + 2 + 4 = 6.$$

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$$\sum_{k=1}^{n} k^2 = n(n+1)(2n+1)/6$$

$$\sum_{k=50}^{100} k^2 = \frac{100 \cdot 101 \cdot 201}{6} - \frac{49 \cdot 50 \cdot 99}{6} = 338{,}350 - 40{,}425 = 297{,}925$$

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Arithmetic sum

- **To sum up** the terms of this arithmetic sequence:

a + (a+d) + (a+2d) + (a+3d) + ...

$$\sum_{k=0}^{n-1}(a+kd)=\frac{n}{2}(2a+(n-1)d)$$

Add up the first 10 terms of the arithmetic sequence:

{ 1, 4, 7, 10, 13, ... }

a = 1 (the first term)

d = 3 (the "common difference" between terms)

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Arithmetic sum-proof

$S = a + (a + d) + \ldots + (a + (n-2)d) + (a + (n-1)d)$

S in reverse order

$S = (a + (n-1)d) + (a + (n-2)d) + \ldots + (a + d) + a$

Add both and get 2S

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$2S = (2a + (n-1)d) + (2a + (n-1)d) + \ldots + (2a + (n-1)d)$

Each term is the same! And there are "n" of them so ...

$2S = n \times (2a + (n-1)d)$

$S = (n/2) \times (2a + (n-1)d)$

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Harmonic Sum

$$\sum_{k=1}^{\infty}\frac{1}{k}=1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{n}$$

$$\sum_{k=1}^{\infty}\frac{1}{k}=\frac{1}{a}+\frac{1}{a+d}+\frac{1}{a+2d}+\ldots+\frac{1}{a+(n-1)d}$$

Sum = 1/d (ln(2a + (2n – 1)d) / (2a – d))
(Approximation)

Proof is a reading assignment

https://brilliant.org/wiki/harmonic-progression/

Reference

- **Rosen, Kenneth H., and Kamala Krithivasan. Discrete mathematics and its applications: with combinatorics and graph theory. Tata McGraw-Hill Education, 2016.**