Mathematical concepts for computer science

Sets

- Sets are used to **group objects** together.
  - All the students currently taking a course in discrete mathematics at any school
- A *set is an unordered collection of objects, called elements or members of the set.*
- A set is said to contain its elements.
- $a \in A$ to denote that a is an element of the set A.
  - $4 \in \{1, 2, 3, 4\}$
- $a \notin A$ denotes that a is not an element of the set A.
  - $7 \notin \{1, 2, 3, 4\}$

Roster method.

- **Sets** are usually represented by a **capital letter** (A, B, S, etc.)
- **Elements** are usually represented by an **italic lower-case letter** (*a*, *x*, *y*, etc.)
  - A={*a, b, c, d*}
  - The set V of all vowels in the English alphabet can be written as **V = {*a, e, i, o, u*}.**
  - The set O of odd positive integers less than 10 can be expressed by **O = {*1, 3, 5, 7, 9*}**.
  - The set of positive integers less than 100 can be denoted by **{*1, 2, 3, . . . , 99*}**.

set builder notation

- We characterize all those elements in the set by **stating the property or properties** they must have to be members.
  - O = {x | x is an odd positive integer less than 10}
  - The universe as the set of positive integers, as

$$O = \{x \in \mathbf{Z^+} \mid \mathbf{x \text{ is odd and } x < 10}\}.$$

Sets

$\mathbf{N} = \{0, 1, 2, 3, \ldots\}$, the set of **natural numbers**
$\mathbf{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$, the set of **integers**
$\mathbf{Z}^+ = \{1, 2, 3, \ldots\}$, the set of **positive integers**
$\mathbf{Q} = \{p/q \mid p \in \mathbf{Z}, q \in \mathbf{Z}, \text{ and } q \neq 0\}$, the set of **rational numbers**
$\mathbf{R}$, the set of **real numbers**
$\mathbf{R}^+$, the set of **positive real numbers**
$\mathbf{C}$, the set of **complex numbers**.

Equal Sets

- Two sets are equal if and only if they have the same elements.
- Therefore, if A and B are sets, then A and B are equal if and only if $\forall x(x \in A \leftrightarrow x \in B)$.
- We write $A = B$ if A and B are equal sets.
- Is the sets $\{1, 3, 5\}$ and $\{3, 5, 1\}$ are equal ?

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- We write $A = B$ if A and B are equal sets.
- Is the sets $\{1, 3, 3, 3, 5, 5, 5, 5\}$ and $\{1, 3, 5\}$ are equal ?
- **Yes**

THE EMPTY SET & singleton set

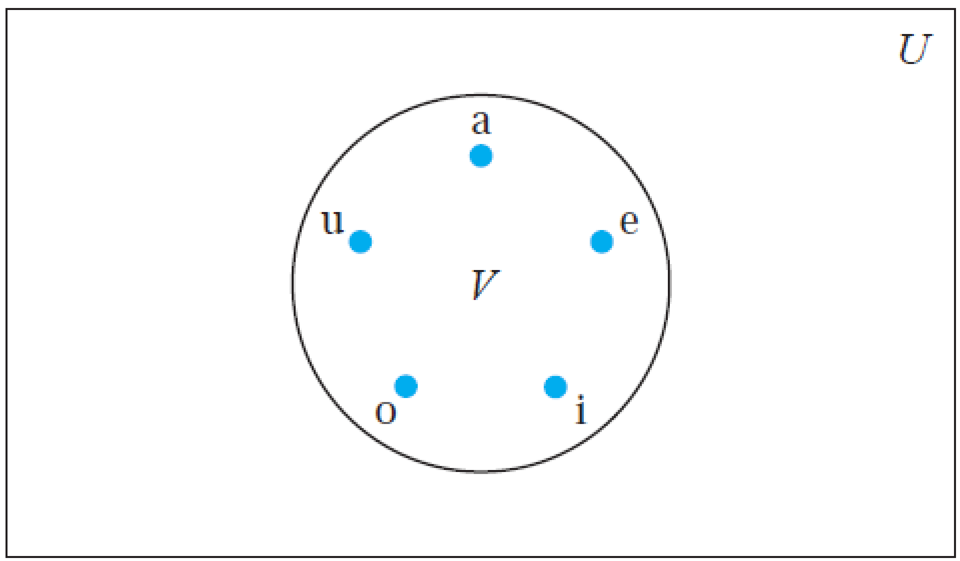
- There is a special set that has no elements.
- This set is called the empty set, or null set, and is denoted by **∅**. The empty set can also be denoted by **{ }**
- A set with one element is called a **singleton set.**
- A common error is to confuse the empty **{∅}** has one more element than ∅. set ∅ with the set {∅}, which is a singleton set.

Venn Diagrams

- Sets can be represented graphically using **Venn diagrams**, named after the English mathematician **John Venn**, who introduced their use in **1881**.
- In Venn diagrams the **universal set U, which** contains all the objects under consideration, is represented by a **rectangle**. (Note that the universal set varies depending on which objects are of interest.)
- Inside this rectangle, **circles or other geometrical** figures are used to represent **sets**.
- Sometimes **points** are used to represent the **particular elements** of the set.

Venn Diagrams

- Venn diagram that represents V, **the set of vowels** in the English alphabet.



Subsets

- The set **A is a subset of B** if and only if **every element of A is also an element of B**.
- We use the notation **A ⊆ B** to indicate that A is a subset of the set B.

$$\forall x (x \in A \rightarrow x \in B)$$

The set of all odd positive integers less than 10 is a subset of the set of all positive integers less than 10

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Showing that A is a Subset of B To show that A ⊆ B, show that if x belongs to A then x also belongs to B.
Showing that A is Not a Subset of B, find a single x ∈ A such that x ∉ B.

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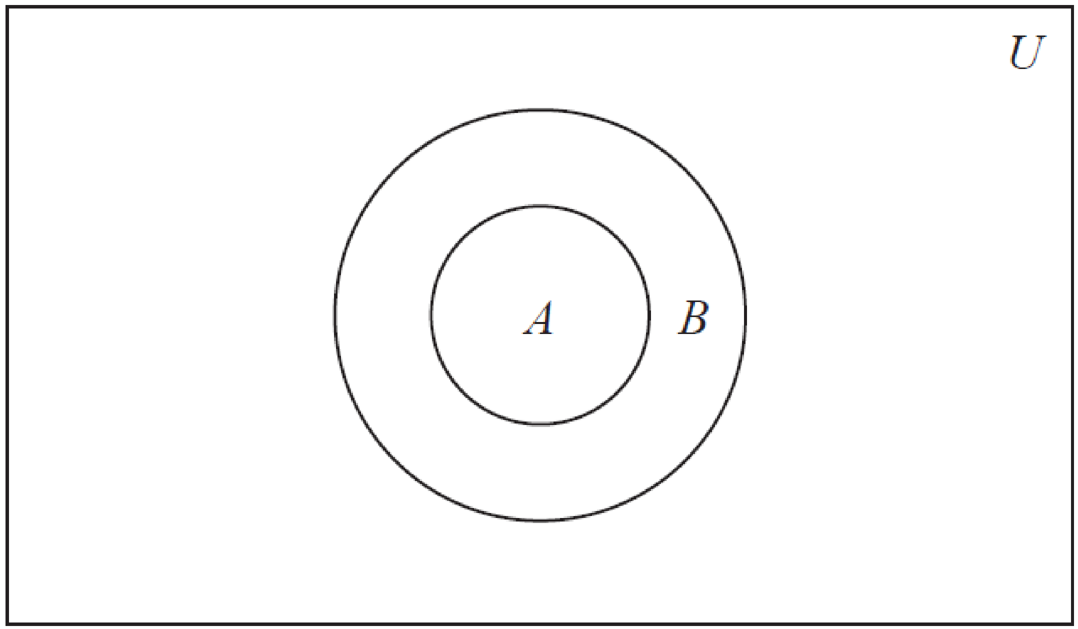

Venn Diagram Showing that *A Is a Subset of B.*

Subsets

Theorem:

For every set S, (i) $\emptyset \subseteq S$ and (ii) $S \subseteq S$.

– Every nonempty set S is guaranteed to have at least two subsets, the empty set and the set S itself, that is, $\emptyset \subseteq S$ and $S \subseteq S$.

- **Proof**

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- **Proof**
- Let **S** be a **set**. To show that $\emptyset \subseteq S$, we must show that $\forall x(x \in \emptyset \rightarrow x \in S)$ is true.

Subsets

Theorem:

For every set S, (i) $\emptyset \subseteq S$ and (ii) $S \subseteq S$.

- Let **S** be a **set**. To show that **$\emptyset \subseteq S$**, we must show that **$\forall x(x \in \emptyset \rightarrow x \in S)$** is true.
- Because the **empty set contains no elements**, it follows that **$x \in \emptyset$ is always false**.
- It follows that the conditional statement **$x \in \emptyset \rightarrow x \in S$ is always true**, because its **hypothesis is always false** and a conditional statement with a false hypothesis is true. Therefore, **$\forall x(x \in \emptyset \rightarrow x \in S)$ is true**.
- This completes the proof of (i). Note that this is an example of a **vacuous proof**.

Subsets

Theorem:

For every set S, (i) Ø ⊆ S and (ii) S ⊆ S.

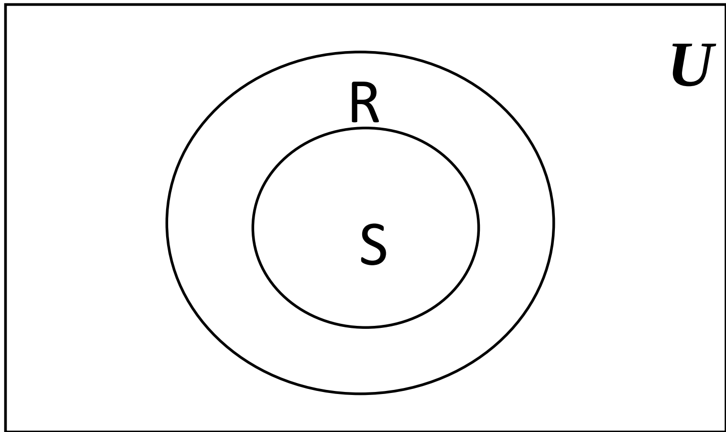
- Let **S** be a **set**. To show that **S ⊆ S**, we must show that **∀x(x ∈ S → x ∈ S)** is true.
- **T →T or F →F**
- **On both case it is true hence proved**

proper subset

- When we wish to emphasize that a set A is a subset of a set B but that A ≠ B,
- we write A ⊂ B and say that A is a **proper subset of B.**

$$\forall x(x \in A \rightarrow x \in B) \wedge \exists x(x \in B \wedge x \notin A)$$

$S \subset R$



Showing **Two Sets are Equal** To show that two sets A and B are equal, show that **A ⊆ B and B ⊆ A**.

The Size of a Set

- Let **S** be a set. If there are exactly **n** distinct elements in **S** where **n** is a nonnegative integer, we say that **S** is a finite set and that **n** is the **cardinality of S**.
- The cardinality of S is denoted by **|S|**.
- A set is said to be infinite if it is not finite.

- Let **A** be the set of odd positive integers less than 10. Then **|A|** = ?.

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- Let **A** be the set of odd positive integers less than 10. Then **|A|** = 5.

Power Sets

- Given a set **S**, the **power set of S** is the set of all subsets of the set S.
- The power set of S is denoted by P(S).

  S={0, 1, 2}

  – **P({0, 1, 2}) = {Ø, {0}, {1}, {2}, {0, 1}, {0, 2}, {1, 2}, {0, 1, 2}}.**

  P(Ø) = {Ø}.

  P({Ø}) = {Ø, {Ø}}.

Cartesian Products

- Let **A** and **B** be sets. The Cartesian product of A and B, denoted by **A × B**, is the set of all ordered pairs (a, b), where a ∈ A and b ∈ B. Hence, **A × B = {(a, b) | a ∈ A ∧ b ∈ B}.**
- Let **A** represent the **set of all students at a university**, and let **B** represent the **set of all courses offered at the university**. What is the Cartesian product **A × B** and how can it be used?

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- **All possible enrollments of students in courses at the university.**

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- **Cartesian product of A = {1, 2} and B = {a, b, c}?**

A × B = {(1, a), (1, b), (1, c), (2, a), (2, b), (2, c)}.

Cartesian Products

- Let **A** and **B** be sets. The Cartesian product of A and B, denoted by **$A \times B$**, is the set of all ordered pairs (a, b), where $a \in A$ and $b \in B$. Hence, **$A \times B = \{(a, b) \mid a \in A \wedge b \in B\}$.**

$$\mathbf{B \times A \neq A \times B}$$

Cartesian Products

- The **Cartesian product** of the sets $\mathbf{A_1, A_2, . . , A_n}$, denoted by $\mathbf{A_1 \times A_2 \times \cdot \cdot \times A_n}$, is the **set of ordered n-tuples (a_1, a_2, ... , a_n)**, where a_i belongs to A_i for $i = 1, 2, . . . , n$.

$$\mathbf{A_1 \times A_2 \times \cdot \cdot \times A_n = \{(a_1, a_2, . . . , a_n) \mid a_i \in A_i \text{ for } i = 1, 2, . . . , n\}.}$$

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Cartesian product $A \times B \times C$, where $A = \{0, 1\}$, $B = \{1, 2\}$, and $C = \{0, 1, 2\}$?

$A \times B \times C = \{(0, 1, 0), (0, 1, 1), (0, 1, 2), (0, 2, 0), (0, 2, 1), (0, 2, 2), (1, 1, 0), (1, 1, 1), (1, 1, 2), (1, 2, 0), (1, 2, 1), (1, 2, 2)\}$.

Cartesian Products

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A = {1, 2}.

$$A^2 = \{(1,1), (1,2), (2,1), (2,2)\}$$

$$\mathrm{A}^3 = (1,1,2), (1,2,1), (1,2,2), (2,1,1), (2,1,2), (2,2,1), (2,2,2)\}$$

ordered pairs

- What are the ordered pairs in the less than or equal to relation, which contains (a, b) if **$a \le b$**, on the set **{0, 1, 2, 3}**?
- **(0,0), (0,1), (0,2), (0,3), (1,1), (1,2), (1,3), (2,2), (2, 3), and (3, 3)**

Using Set Notation with Quantifiers

- **$\forall x \in S(P(x))$** is shorthand for **$\forall x(x \in S \rightarrow P(x))$.**
- **$\exists x \in S(P(x))$** is shorthand for **$\exists x(x \in S \wedge P(x))$.**

$$\forall x \in R\,(x^2 \geq 0)$$

$$\exists x \in Z\,(x^2 = 1)$$

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$$\forall x \in R\,(x^2 \geq 0)$$

"The square of every real number is nonnegative."

$$\exists x \in Z\,(x^2 = 1)$$

"There is an integer whose square is 1."

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Truth Sets and Quantifiers

- Given a predicate P, and a domain D, we define the **truth set of P to be the set of elements x in D for which P(x) is true**. The truth set of P(x) is denoted by **$\{x \in D \mid P(x)\}$.**

- What are the truth sets of the predicates P(x), Q(x), and R(x), where the domain is the set of integers.

$$P(x) \text{ is "} |x| = 1, \text{" } Q(x) \text{ is "} x^2 = 2, \text{" and } R(x) \text{ is "} |x| = x. \text{"}$$

Truth Sets and Quantifiers

- What are the truth sets of the predicates P(x), Q(x), and R(x), where the domain is the set of integers.

P(x) is "$|x| = 1$," Q(x) is "$x^2 = 2$," and R(x) is "$|x| = x$."

The truth set of $P: \{x \in Z / |x| = 1\} \Rightarrow \{-1, 1\}$.

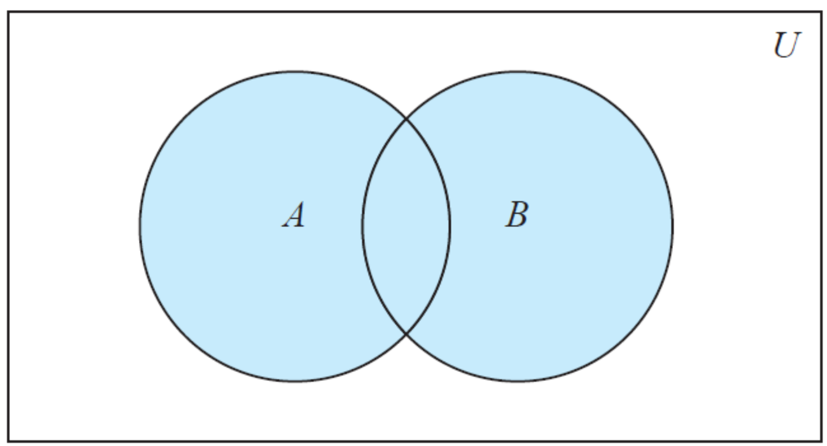
The truth set of $Q, \{x \in Z / x^2 = 2\} \Rightarrow \phi$

The truth set of R, {x¸ Z / |x| = x} => the set of nonnegative integers.

Set Operations

- Let **A** and **B** be sets. The union of the sets A and B, denoted by **A ∪ B**, is the set that contains those elements that are **either in A or in B, or in both**.

$$A \cup B = \{x \mid x \in A \vee x \in B\}.$$

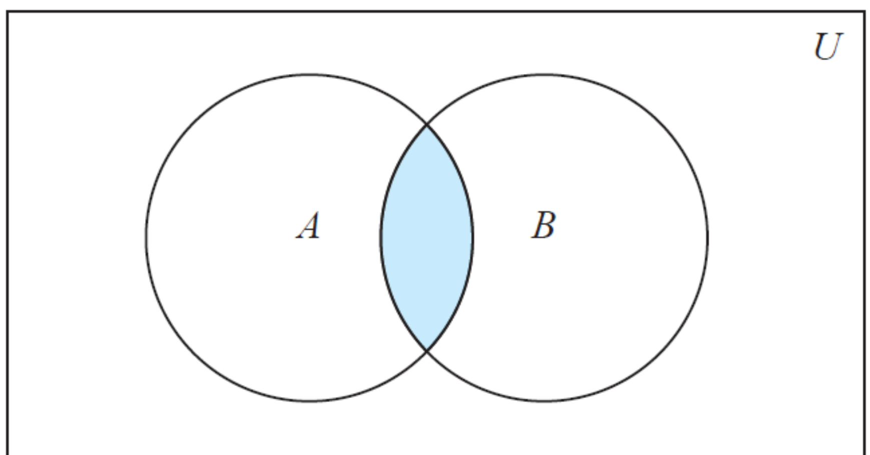


$A \cup B$ is shaded.

Set Operations

- Let **A** and **B** be sets. The intersection of the sets A and B, denoted by **A ∩ B**, is the set containing those elements in **both A and B**.

$$A \cap B = \{x \mid x \in A \wedge x \in B\}.$$



$A \cap B$ is shaded.

Set Operations

- **A={1, 3, 5} and B= {1, 2, 3}**

- **A ∪ B= {1, 2, 3, 5}**

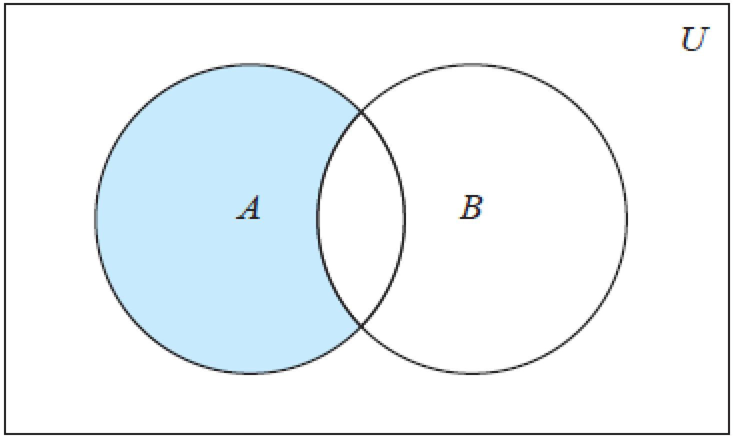
- **A ∩ B= {1, 3}**

- Two sets are called **disjoint** if their **intersection is the empty set**.

Set Operations

- Let **A** and **B** be sets. The difference of **A and B**, denoted by **A – B**, is the **set containing those elements that are in A but not in B**.
- The difference of A and B is also called the complement of B with respect to A.

$$A - B = \{x \mid x \in A \land x \notin B\}$$



$A - B$ is shaded.

Set Operations

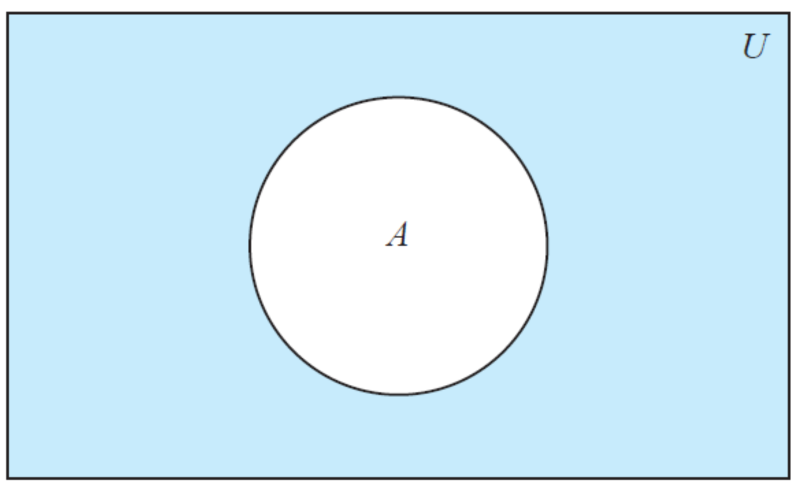
- **A={1, 3, 5} and B= {1, 2, 3}**

- **A - B= {5}**

Set Operations

- Let **U** be the universal set. The complement of the set **A**, denoted by $\bar{A}$, is the complement of A with respect to U. Therefore, the complement of the set A is U − A.

$$\overline{A} = \{x \in U \mid x \notin A\}$$



$\overline{A}$ is shaded.

Set Operations

- Let **A = {a, e, i, o, u}** (where the universal set is the set of letters of the English alphabet).
- Then **$\bar{A}$ = {b, c, d, f, g, h, j, k, l,m, n, p, q, r, s, t, v, w, x, y, z}**.

Set laws

TABLE 1 Set Identities.

Identity	*Name*
$A \cap U = A$ $A \cup \emptyset = A$	Identity laws
$A \cup U = U$ $A \cap \emptyset = \emptyset$	Domination laws
$A \cup A = A$ $A \cap A = A$	Idempotent laws
$\overline{(\overline{A})} = A$	Complementation law
$A \cup B = B \cup A$ $A \cap B = B \cap A$	Commutative laws

Set laws

$A \cup (B \cup C) = (A \cup B) \cup C$ $A \cap (B \cap C) = (A \cap B) \cap C$	Associative laws
$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$	Distributive laws
$\overline{A \cap B} = \overline{A} \cup \overline{B}$ $\overline{A \cup B} = \overline{A} \cap \overline{B}$	De Morgan's laws
$A \cup (A \cap B) = A$ $A \cap (A \cup B) = A$	Absorption laws
$A \cup \overline{A} = U$ $A \cap \overline{A} = \emptyset$	Complement laws

Set laws

Prove that $\overline{A \cap B} = \overline{A} \cup \overline{B}$.

Proof by contradiction

$\overline{A \cap B} = \overline{A} \cup \overline{B}$.

Assume premises is true and conclusion is false

Assume predicates $p : x \in \overline{A \cap B}$ and $r : x \notin \overline{A} \cup \overline{B}$

If $x \in \overline{A \cap B}$

$=> x \notin A \cap B$

$=> x \notin A$ or $x \notin B$, $x \notin A$ and $x \notin B$

case 1: $x \notin A$ but $x \in B$

$x \in \overline{A}$, so $x \in \overline{A} \cup \overline{B}$

case 2: $x \in A$ but $x \notin B$

$x \in \overline{B}$, so $x \in \overline{A} \cup \overline{B}$

case 3: $x \notin A$ and $x \notin B$

$x \in \overline{A}$, $x \in \overline{B}$, so $x \in \overline{A} \cup \overline{B}$

This contradicts the predicate r. Hence Proved $\overline{A \cap B} -> \overline{A} \cup \overline{B}$.

Set laws

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Assume predicates $r : x \notin \overline{A \cap B}$ and $p : x \in \overline{A} \cup \overline{B}$

If $x \in \overline{A} \cup \overline{B}$

$=> x \in \overline{A}$ or $x \in \overline{B}, x \in \overline{A}$ and $x \in \overline{B}$

$=> x \notin A$ or $x \notin B$, $x \notin A$ and $x \notin B$

case 1: $x \notin A$ but $x \in B$

$x \notin A \cap B$, so $x \in \overline{A \cap B}$

case 2: $x \in A$ but $x \notin B$

$x \notin A \cap B$, so $x \in \overline{A \cap B}$

case 3: $x \notin A$ and $x \notin B$

$x \notin A \cap B$, so $x \in \overline{A \cap B}$

This contradicts the predicate r. Hence Proved $\overline{A} \cup \overline{B} -> \overline{A \cap B}$

Set laws

Prove that $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$ for all sets A, B, and C..

Proof by contradiction

$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

Assume premises is true and conclusion is false

Assume predicates $r : x \notin (A \cap B) \cup (A \cap C)$ and $p : x \in A \cap (B \cup C)$

If $x \in A \cap (B \cup C)$

$\Rightarrow x \in A$ and, $x \in B$ or $x \in C$

case 1: $x \in A, x \in B, x \in C$

$x \in (A \cap B) \cup (A \cap C)$

case 2: $x \in A, x \in B, x \notin C$

$x \in (A \cap B) \cup (A \cap C)$

case 3: $x \in A, x \notin B, x \in C$

$x \in (A \cap B) \cup (A \cap C)$

This contradicts the predicate r. Hence Proved $A \cap (B \cup C) -> (A \cap B) \cup (A \cap C)$

Vice versa...

Set laws

Set identities can also be proved using **membership tables.**

Use a membership table to show that **$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$**.

TABLE 2 A Membership Table for the Distributive Property.

A	B	C	$B \cup C$	$A \cap (B \cup C)$	$A \cap B$	$A \cap C$	$(A \cap B) \cup (A \cap C)$
1	1	1	1	1	1	1	1
1	1	0	1	1	1	0	1
1	0	1	1	1	0	1	1
1	0	0	0	0	0	0	0
0	1	1	1	0	0	0	0
0	1	0	1	0	0	0	0
0	0	1	1	0	0	0	0
0	0	0	0	0	0	0	0

Set laws

Let A, B, and C be sets. Show that

$$\overline{\mathbf{A} \cup (\mathbf{B} \cap \mathbf{C})} = (\overline{\mathbf{C}} \cup \overline{\mathbf{B}}) \cap \overline{\mathbf{A}}.$$

Set laws

Let A, B, and C be sets. Show that

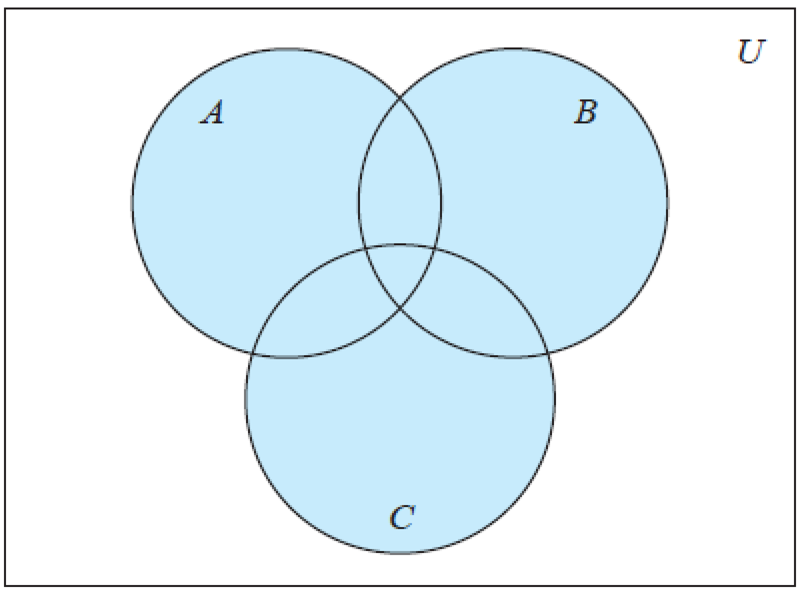
$\overline{A \cup (B \cap C)} = (\overline{C} \cup \overline{B}) \cap \overline{A}$.

$$\begin{aligned}
\overline{A \cup (B \cap C)} &= \overline{A} \cap (\overline{B \cap C}) && \text{by the first De Morgan law} \\
&= \overline{A} \cap (\overline{B} \cup \overline{C}) && \text{by the second De Morgan law} \\
&= (\overline{B} \cup \overline{C}) \cap \overline{A} && \text{by the commutative law for intersections} \\
&= (\overline{C} \cup \overline{B}) \cap \overline{A} && \text{by the commutative law for unions.}
\end{aligned}$$

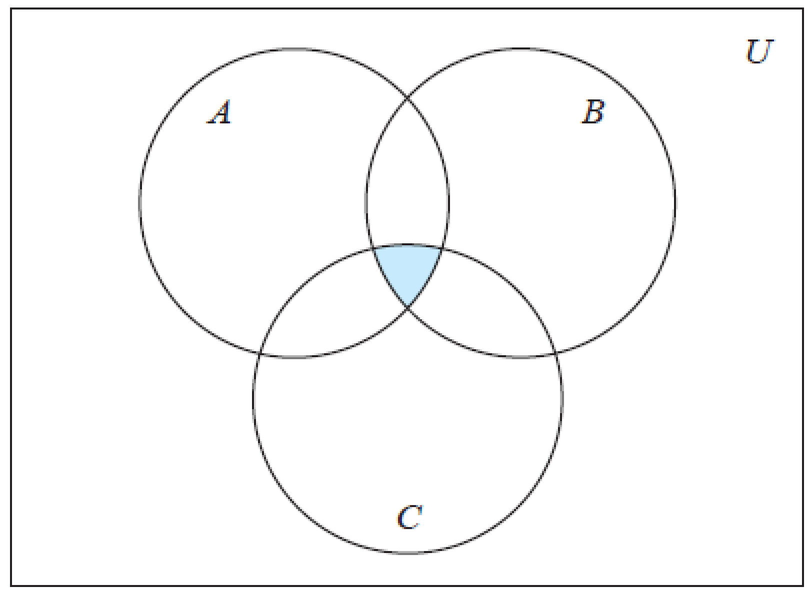
Generalized Unions and Intersections

- The union of a collection of sets is the set that contains those elements that are **members of at least one set** in the collection.

$$A_1 \cup A_2 \cup \cdots \cup A_n = \bigcup_{i=1}^{n} A_i$$



(a) $A \cup B \cup C$ is shaded.

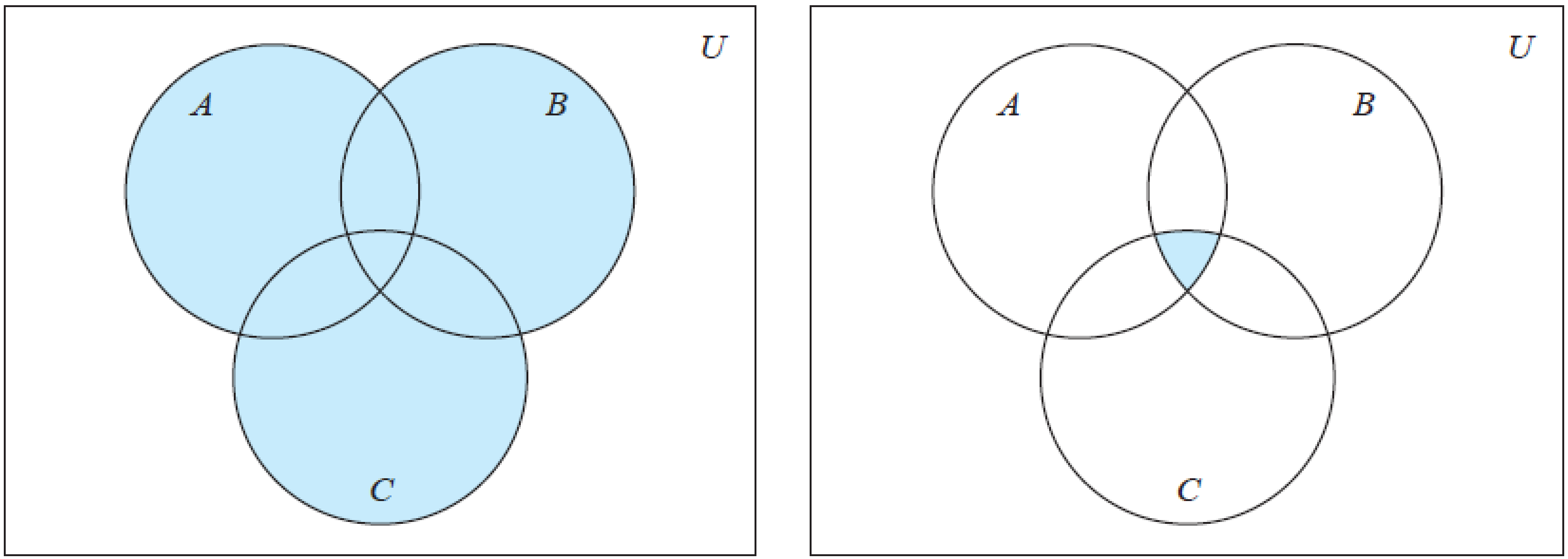


(b) $A \cap B \cap C$ is shaded.

Generalized Unions and Intersections

- The intersection of a collection of sets is the set that contains those elements that are **members of all the sets** in the collection.

$$A_1 \cap A_2 \cap \cdots \cap A_n = \bigcap_{i=1}^{n} A_i$$



(a) $A \cup B \cup C$ is shaded.

(b) $A \cap B \cap C$ is shaded.

Russell's paradox

- In the foundations of mathematics, **Russell's paradox** (also known as **Russell's antinomy**), discovered by Bertrand Russell in 1901
- As per the naïve set theory
  - Given any property there exists a set containing all objects that have that property

$$\text{Let } R = \{S : S \notin S\}$$

$$\text{Does } R \text{ contain itself ?}$$

Russell's paradox

$\text{Let } R = \{S : S \notin S\}$

$\text{Does R contain itself ?}$

$\text{Case 1: Let } R \in R$

$R \in \{S : S \notin S\}$

$R \notin R$

$\text{Case 2: Let } R \notin R$

$R \notin \{S : S \notin S\}$

$R \in R$

$\textit{Neither } R \in R \text{ nor } R \notin R$

Russell's paradox

- In the foundations of mathematics, **Russell's paradox** (also known as **Russell's antinomy**), discovered by Bertrand Russell in 1901
- The barber is the "**one who shaves all those, and those only, who do not shave themselves**".[**Barber paradox**]
- The question is, **does the barber shave himself**?
- Answering this question results in a contradiction. **The barber cannot shave himself** as he only shaves those who do not shave themselves. As such, **if he shaves himself he ceases to be the barber**. Conversely, **if the barber does not shave himself**, then he fits into the group of people who would be shaved by the barber, and thus, as the barber, **he must shave himself**.

Reference

- **Rosen, Kenneth H., and Kamala Krithivasan. Discrete mathematics and its applications: with combinatorics and graph theory. Tata McGraw-Hill Education, 2016.**