Mathematical concepts for computer science

Propositional Equivalences

- An important type of step used in a mathematical argument is the replacement of a statement with another statement with the same truth value.
- A compound proposition that is always true, no matter what the truth values of the propositional variables that occur in it, is called a tautology.
- A compound proposition that is always false is called a contradiction.
- A compound proposition that is neither a tautology nor a contradiction is called a contingency.

Propositional Equivalences

- Check whether the following arguments are tautology or contradiction
- p ∨¬p
- p ∧¬p

Propositional Equivalences

- Check whether the following arguments are tautology or contradiction
- **pV¬p** tautology
- p ∧¬p contradiction

TABLE 1 Examples of a Tautology and a Contradiction.									
p	$p \qquad \neg p \qquad p \lor \neg p \qquad p \land \neg p$								
T	F T F								
F	T	T	F						

- Compound propositions that have the same truth values in all possible cases are called logically equivalent.
- The compound propositions p and q are called logically equivalent if p ←> q is a tautology.
- The notation p ≡ q (p⇔q)denotes that p and q are logically equivalent.

TABLE 2 De Morgan's Laws.

$$\neg (p \land q) \equiv \neg p \lor \neg q$$

Show that ¬(p ∨ q) and ¬p ∧¬q are logically equivalent.

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TABI	TABLE 3 Truth Tables for $\neg(p \lor q)$ and $\neg p \land \neg q$.										
p	\boldsymbol{q}	$p \vee q$	$\neg (p \lor q)$	$\neg p$	$\neg q$	$\neg p \land \neg q$					
T	T	T	F	F	F	F					
T	F	T	F	F	T	F					
F	T	T	F	T	F	F					
F	F	F	T	T	T	T					

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$$\neg(p \land q) \equiv \neg p \lor \neg q$$

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• Show that $\mathbf{p} \rightarrow \mathbf{q}$ and $\neg \mathbf{p} \lor \mathbf{q}$ are logically equivalent.

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TABLE 4 Truth Tables for $\neg p \lor q$ and $p \to q$.											
p q $\neg p$ $\neg p \lor q$ $p \to q$											
T	T	F	T	T							
T	F	F	F	F							
F	T	T	T	T							
F	F	T	T	T							

Distributive law of disjunction over conjunction.

TABLE 5 A Demonstration That $p \lor (q \land r)$ and $(p \lor q) \land (p \lor r)$ Are Logically Equivalent.												
p	\boldsymbol{q}	r	$q \wedge r$	$p \vee (q \wedge r)$	$p \vee q$	$p \vee r$	$(p \vee q) \wedge (p \vee r)$					
T	T	T	T	Т	T	Т	T					
T	T	F	F	T	T	T	T					
T	F	T	F	T	T	T	T					
T	F	F	F	T	T	T	T					
F	T	T	T	T	T	T	T					
F	T	F	F	F	T	F	F					
F	F	T	F	F	F	T	F					
F	F	F	F	F	F	F	F					

TABLE 6 Logical Equivalences.	
Equivalence	Name
$p \wedge \mathbf{T} \equiv p$	Identity laws
$p \vee \mathbf{F} \equiv p$	
$p \vee \mathbf{T} \equiv \mathbf{T}$	Domination laws
$p \wedge \mathbf{F} \equiv \mathbf{F}$	
$p \vee p \equiv p$	Idempotent laws
$p \wedge p \equiv p$	
$\neg(\neg p) \equiv p$	Double negation law
$p \vee q \equiv q \vee p$	Commutative laws
$p \wedge q \equiv q \wedge p$	

$(p \lor q) \lor r \equiv p \lor (q \lor r)$ $(p \land q) \land r \equiv p \land (q \land r)$	Associative laws
$p \lor (q \land r) \equiv (p \lor q) \land (p \lor r)$ $p \land (q \lor r) \equiv (p \land q) \lor (p \land r)$	Distributive laws
$\neg(p \land q) \equiv \neg p \lor \neg q$ $\neg(p \lor q) \equiv \neg p \land \neg q$	De Morgan's laws
$p \lor (p \land q) \equiv p$ $p \land (p \lor q) \equiv p$	Absorption laws
$p \lor \neg p \equiv \mathbf{T}$ $p \land \neg p \equiv \mathbf{F}$	Negation laws

TABLE 7 Logical Equivalences Involving Conditional Statements.

$$p \to q \equiv \neg p \lor q$$

$$p \to q \equiv \neg q \to \neg p$$

$$p \lor q \equiv \neg p \to q$$

$$p \land q \equiv \neg (p \to \neg q)$$

$$\neg (p \to q) \equiv p \land \neg q$$

$$(p \to q) \land (p \to r) \equiv p \to (q \land r)$$

$$(p \to r) \land (q \to r) \equiv (p \lor q) \to r$$

$$(p \to q) \lor (p \to r) \equiv p \to (q \lor r)$$

$$(p \to r) \lor (q \to r) \equiv (p \land q) \to r$$

TABLE 8 Logical Equivalences Involving Biconditional Statements.

$$p \leftrightarrow q \equiv (p \to q) \land (q \to p)$$

$$p \leftrightarrow q \equiv \neg p \leftrightarrow \neg q$$

$$p \leftrightarrow q \equiv (p \land q) \lor (\neg p \land \neg q)$$

$$\neg (p \leftrightarrow q) \equiv p \leftrightarrow \neg q$$

Show that $\neg(p \rightarrow q)$ and $p \land \neg q$ are logically equivalent without using truth table.

Show that $\neg(p \rightarrow q)$ and $p \land \neg q$ are logically equivalent without using truth table.

$$\neg(p \to q) \equiv \neg(\neg p \lor q)$$
$$\equiv \neg(\neg p) \land \neg q$$
$$\equiv p \land \neg q$$

• Show that $\neg(p \lor (\neg p \land q))$ and $\neg p \land \neg q$ are logically equivalent by developing a series of logical equivalences.

 Show that ¬(p V (¬p Λ q)) and ¬p Λ¬q are logically equivalent by developing a series of logical equivalences.

$$\neg (p \lor (\neg p \land q)) \equiv \neg p \land \neg (\neg p \land q) \qquad \text{by the second De Morgan law}$$

$$\equiv \neg p \land [\neg (\neg p) \lor \neg q] \qquad \text{by the first De Morgan law}$$

$$\equiv \neg p \land (p \lor \neg q) \qquad \text{by the double negation law}$$

$$\equiv (\neg p \land p) \lor (\neg p \land \neg q) \qquad \text{by the second distributive law}$$

$$\equiv \mathbf{F} \lor (\neg p \land \neg q) \qquad \text{because } \neg p \land p \equiv \mathbf{F}$$

$$\equiv (\neg p \land \neg q) \lor \mathbf{F} \qquad \text{by the commutative law for disjunction}$$

$$\equiv \neg p \land \neg q \qquad \text{by the identity law for } \mathbf{F}$$

• Show that $(p \land q) \rightarrow (p \lor q)$ is a **tautology**.

• Show that $(p \land q) \rightarrow (p \lor q)$ is a **tautology**.

 $\equiv T$

$$(p \land q) \rightarrow (p \lor q) \equiv \neg (p \land q) \lor (p \lor q)$$

$$\equiv (\neg p \lor \neg q) \lor (p \lor q) \text{ by the first De Morgan law}$$

$$\equiv (\neg p \lor p) \lor (\neg q \lor q) \text{ by the associative and commutative laws for disjunction}$$

$$\equiv \mathbf{T} \lor \mathbf{T}$$

by the domination law

- A compound proposition is satisfiable, if there is an assignment of truth values to its variables that makes it true.
- When no such assignments exists, that is, when the compound proposition is false for all assignments of truth values to its variables, the compound proposition is unsatisfiable.
- To show that a compound proposition is unsatisfiable, we need to show that every assignment of truth values to its variables makes it false.

 Determine whether each of the compound propositions (p V¬q) ∧ (q V¬r) ∧ (r V¬p) is satisfiable.

 Determine whether each of the compound propositions (p V¬q) Λ (q V¬r) Λ (r V¬p) is satisfiable.

 True when the three variable p, q, and r have the same truth value

• Determine whether each of the compound propositions (p V q V r) Λ (¬p V¬q V¬r), is satisfiable.

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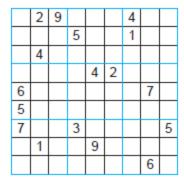
 Satisfiable, when at least one of p, q, and r is true and at least one is false

Determine whether each of the compound propositions (p V¬q) Λ (q V¬r) Λ (r V¬p) Λ (p V q V r) Λ (¬p V¬q V¬r) is satisfiable.

Determine whether each of the compound propositions (p V¬q) Λ (q V¬r) Λ (r V¬p) Λ (p V q V r) Λ (¬p V¬q V¬r) is satisfiable.

- (p V¬q) ∧ (q V¬r) ∧ (r V¬p) and
- (p V q V r) ∧ (¬p V¬q V¬r) must both be true.

- (p V¬q) ∧ (q V¬r) ∧ (r V¬p) and
- (p ∨ q ∨ r) ∧ (¬p ∨¬q ∨¬r) must both be true.
- For the first to be true, the three variables must have the same truth values.
- For the second to be true, at least one of three variables must be true and at least one must be false.
- These conditions are contradictory.
- It is unsatisfiable



A 9 x 9 Sudoku puzzle.

SUDOKU

- The puzzle is solved by assigning a number to each blank cell so that every row, every column, and every one of the nine 3 × 3 blocks contains each of the nine possible numbers.
- Note that instead of using a 9 \times 9 grid, Sudoku puzzles can be based on $n^2 \times n^2$ grids, for any positive integer n, with the $n^2 \times n^2$ grid made up of n^2 $n \times n$ sub grids.

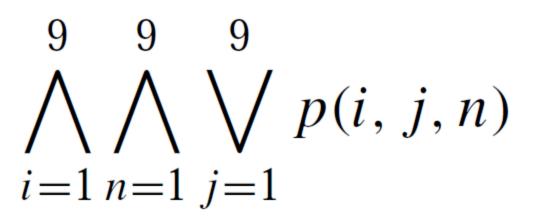
- To encode a Sudoku puzzle, let p(i, j, n) denote the proposition that is true when the number n is in the cell in the ith row and j th column.
- There are 9 × 9 × 9 = 729 such propositions, as i, j, and n all range from 1 to 9.

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p(5, 1, 6) is true,

p(5, j, 6) is false for j = 2, 3, ..., 9.
```

	2	9				4		
			5			1		
	4							
				4	2			
6							7	
6 5								
7			3					5
	1			9				
							6	

- For each cell with a given value, we assert p(i, j, n) when the cell in row i and column j has the given value n.
- We assert that every row contains every number:

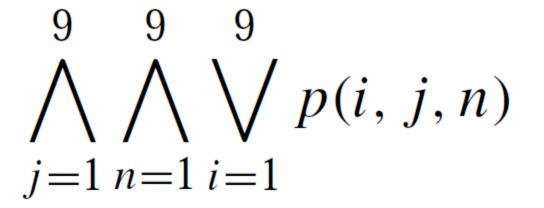


	2	9				4		
			5			1		
	4							
				4	2			
6 5							7	
5								
7			3					5
	1			9				
							6	

We assert that every column contains every number:

	2	9				4		
			5			1		
	4							
				4	2			
6							7	
6 5 7								
7			3					5
	1			9				
							6	

We assert that every column contains every number:



	2	9				4		
			5			1		
	4							
				4	2			
6							7	
6 5								
7			3					5
	1			9				
							6	

 We assert that each of the nine 3 × 3 blocks contains every number:

$$\bigwedge_{r=0}^{2} \bigwedge_{s=0}^{2} \bigwedge_{n=1}^{9} \bigvee_{i=1}^{3} \bigvee_{j=1}^{3} p(3r+i, 3s+j, n)$$

	2	9				4		
			5			1		
	4							
				4	2			
6							7	
6 5								
7			3					5
	1			9				
							6	

Solving Satisfiability Problems

- A truth table can be used to determine whether a compound proposition is satisfiable.
- When the number of variables grows, this becomes impractical. $2^{20} = 1,048,576 \text{ rows}$
- No procedure is known that a computer can follow to determine in a reasonable amount of time whether an arbitrary compound proposition in such a large number of variables is satisfiable.

Reference

 Rosen, Kenneth H., and Kamala Krithivasan. Discrete mathematics and its applications: with combinatorics and graph theory. Tata McGraw-Hill Education, 2016.