Algorithm strategies: Greedy algorithms

Optimization problems

- An **optimization problem** is one in which you want to find, not just a solution, but the best solution
- A "**greedy algorithm**" sometimes works well for optimization problems
- A greedy algorithm works in phases. At each phase:
  - You take the **best you can get right now**, without regard for future consequences
  - You hope that by **choosing a local optimum at each step**, you will end up at a global optimum

Greedy algorithms

- The **greedy approach** suggests **constructing a solution** through a sequence of steps, **each expanding a partially constructed solution** obtained so far, **until a complete solution** to the problem is reached.

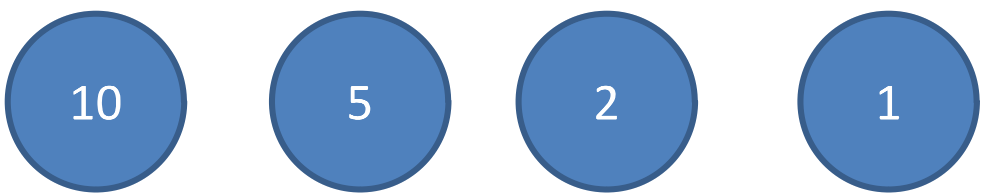
Greedy algorithms

- The central point of this technique—the choice made must be:
- Feasible, i.e., it has to satisfy the problem's constraints
- Locally optimal, i.e., it has to be the best local choice among all feasible choices available on that step
- Irrevocable, i.e., once made, it cannot be changed on subsequent steps of the algorithm

Greedy algorithms

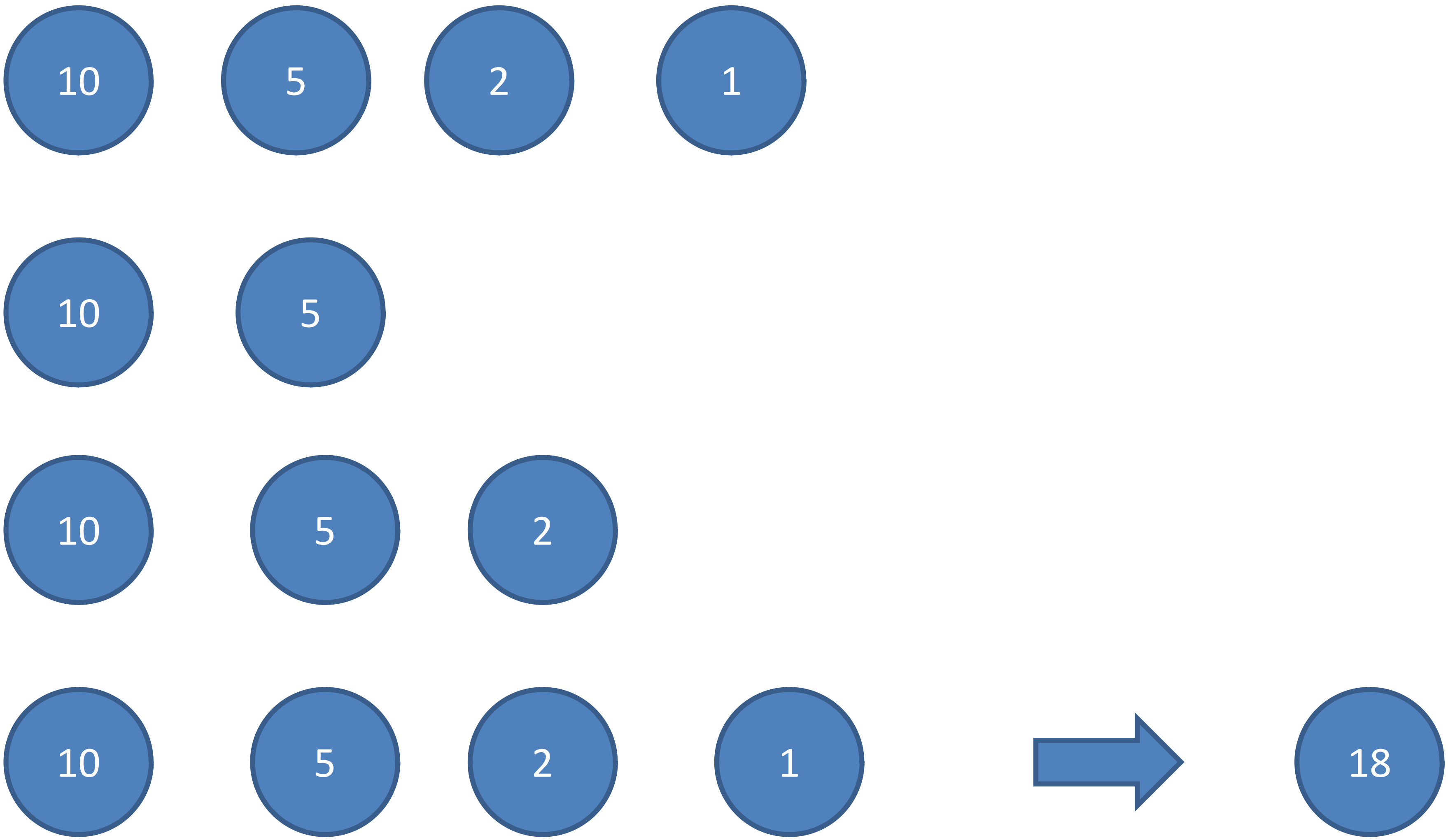
Change-making problem

- Give change for a specific amount **n** with the least number of coins of the denominations

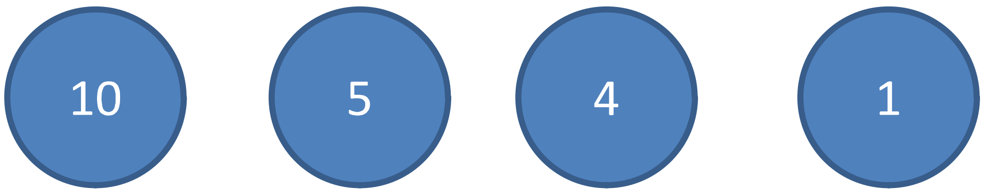


- Get 18 with least number of coins

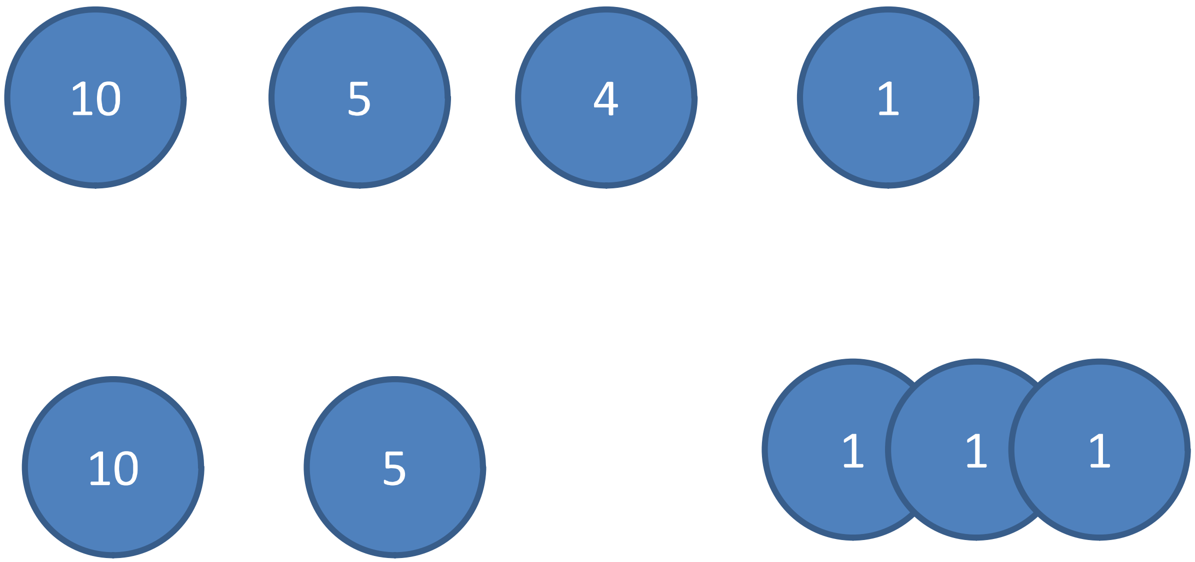
Greedy algorithms



Greedy algorithms

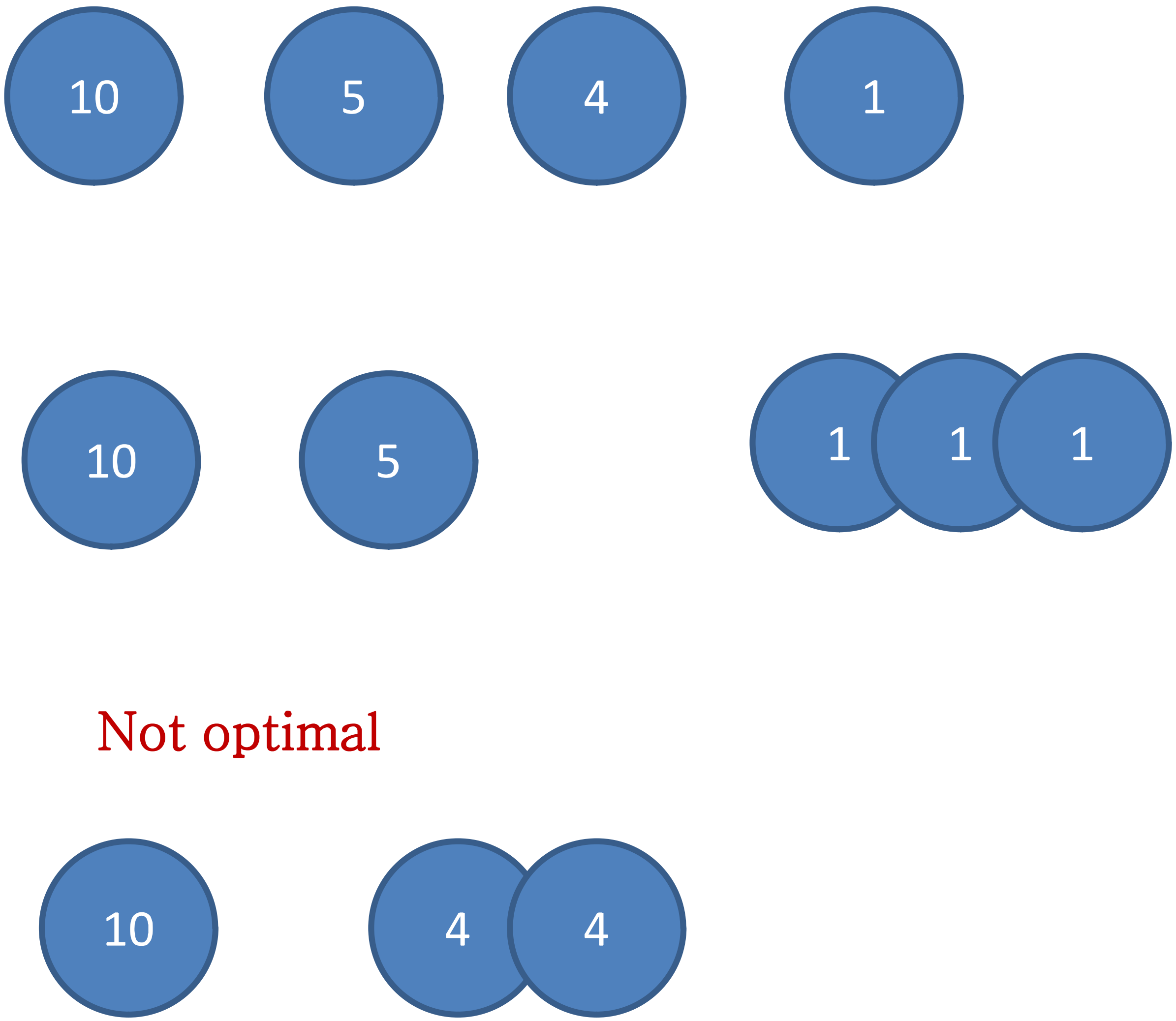


Greedy algorithms



Is this optimal ?

Greedy algorithms



Knapsack Problem

- Given n items of known **weights w1, . . . , wn** and **values v1, . . . , vn** and a knapsack of capacity **W**,
- find the most valuable subset of the items that fit into the knapsack.
- Given **a set of items**, each with a **weight and a value**, determine a **subset of items** to include in a collection so that the **total weight is less than or equal** to a given limit and the **total value is as large as possible**.
- The knapsack problem is in combinatorial optimization problem.

Problem Scenario

- A thief is robbing a store and can carry a maximal weight of **W** into his knapsack.
- There are **n** items available in the store and weight of **i**th item is $\mathbf{w_i}$ and its profit is $\mathbf{p_i}$. What items should the thief take?
- In this context, the items should be selected in such a way that the thief will carry those items for which he will **gain maximum profit**.
- Based on the nature of the items, Knapsack problems are categorized as
  - **Fractional Knapsack**
  - **0-1 Knapsack**

0-1 Knapsack

Let us consider that the capacity of the knapsack **W = 60** and the list of provided items are shown in the following table.

Item	A	B	C	D
Profit (pi)	280	100	120	50
Weight(wi)	40	10	20	10
Ratio (pi/wi)	7	10	6	5

0-1 Knapsack

After sorting, the items are as shown in the following table.

Item	B	A	C	D
Profit (pi)	100	280	120	50
Weight(wi)	10	40	20	10
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Ratio (pi/wi)	**10**	**7**	**6**	**5**

The total weight of the selected items is **10 + 40 + 10= 60**
And the total profit is **100 + 280 + 50= 380 + 50 = 430**

Fractional Knapsack

After sorting, the items are as shown in the following table.

Item	B	A	C	D
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The total weight of the selected items is **10 + 40 + 20 * (10/20) = 60**
And the total profit is **100 + 280 + 120 * (10/20) = 380 + 60 = 440**

Fractional Knapsack

Algorithm: Greedy-Fractional-Knapsack (w[1..n], p[1..n], W)

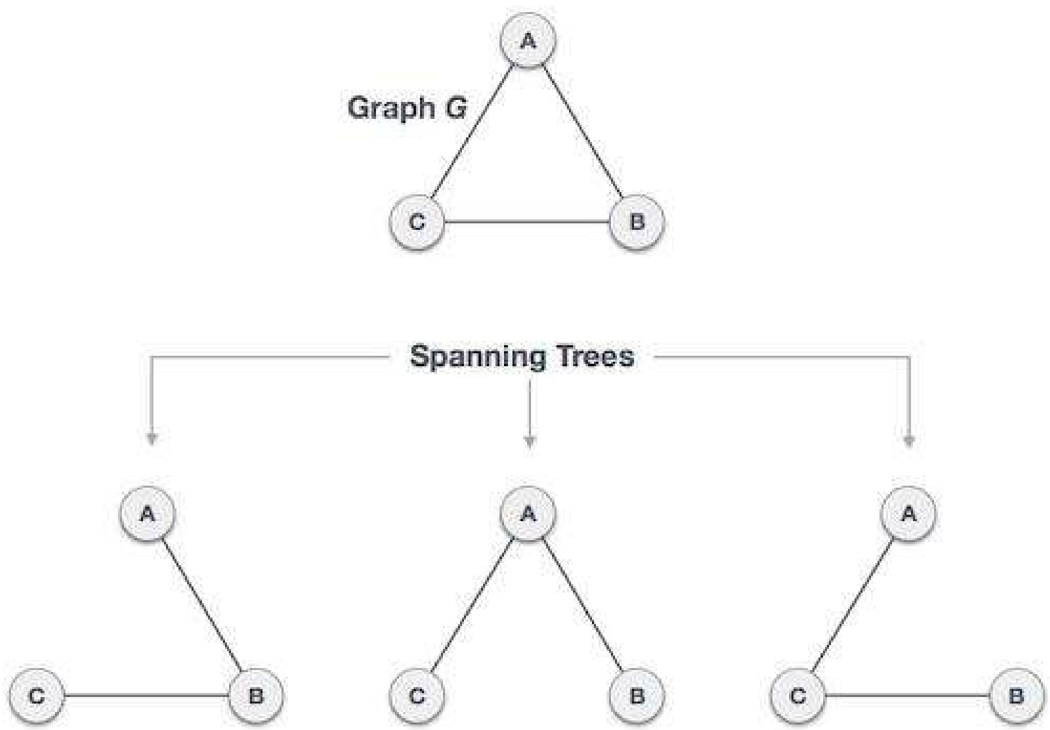
```
for i = 1 to n do
        x[i] = 0
weight = 0
for i = 1 to n
        if weight + w[i] ≤ W then
                x[i] = 1
                weight = weight + w[i]
        else
                x[i] = (W - weight) / w[i]
                weight = W
                break
return x
```

O(n)

O(n logn)

spanning tree

- A spanning tree of an undirected connected graph is its connected acyclic sub graph (i.e., a tree) that contains all the vertices of the graph.

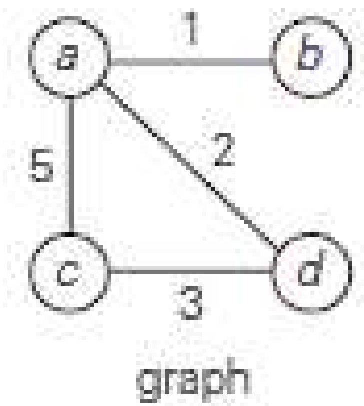


minimum spanning tree problem

- If a undirected connected graph has weights assigned to its edges, a minimum spanning tree is its spanning tree of the smallest weight, where the weight of a tree is defined as the sum of the weights on all its edges.

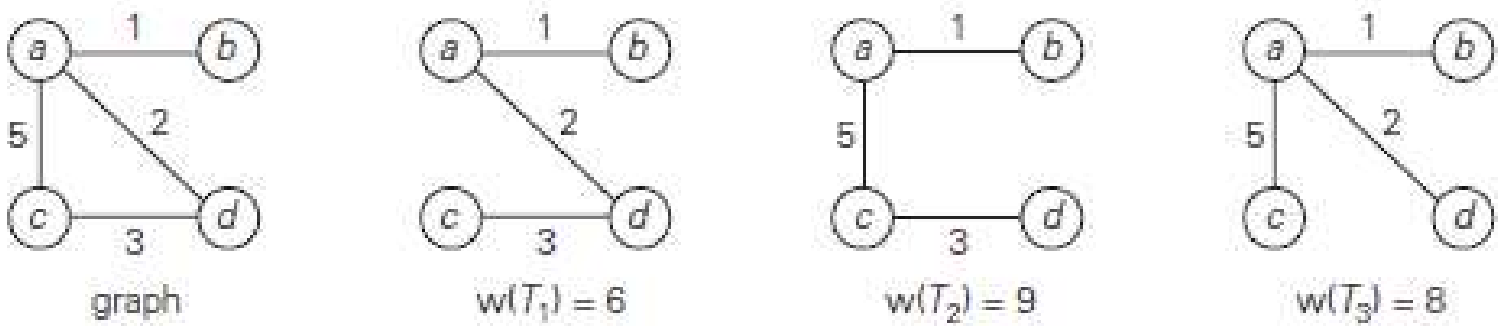
- The **minimum spanning tree problem** is the problem of finding a minimum spanning tree for a given weighted connected graph.

minimum spanning tree problem



Find minimum spanning tree using brute force approach.

minimum spanning tree problem



T_1 is the minimum spanning tree.

Prim’s Algorithm

- Prim’s algorithm constructs a minimum spanning tree through a sequence of expanding sub-trees.
- The initial sub-tree in such a sequence consists of a single vertex selected arbitrarily from the set V of the graph’s vertices.
- On each iteration, the algorithm expands the current tree in the greedy manner by simply attaching to it the nearest vertex not in that tree. (By the nearest vertex, we mean a vertex not in the tree connected to a vertex in the tree by an edge of the smallest weight.)
- The algorithm stops after all the graph’s vertices have been included in the tree being constructed.

Prim's Algorithm

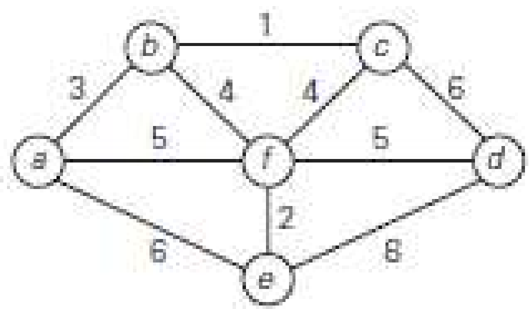
ALGORITHM *Prim(G)*

//Prim's algorithm for constructing a minimum spanning tree
//Input: A weighted connected graph $G = \langle V, E \rangle$
//Output: E_T, the set of edges composing a minimum spanning tree of G
$V_T \leftarrow \{v_0\}$ //the set of tree vertices can be initialized with any vertex
$E_T \leftarrow \varnothing$
for $i \leftarrow 1$ **to** $|V| - 1$ **do**
    find a minimum-weight edge $e^* = (v^*, u^*)$ among all the edges (v, u)
    such that v is in V_T and u is in $V - V_T$
    $V_T \leftarrow V_T \cup \{u^*\}$
    $E_T \leftarrow E_T \cup \{e^*\}$
return E_T

After we have identified a vertex $u*$ to be added to the tree, we need to perform two operations:

1. Move u∗ from the set $V - V_T$ to the set of tree vertices V_T.
2. For each remaining vertex u in $V - V_T$ that is connected to u∗ by a shorter edge than the u's current distance label, update its labels by u∗ and the weight of the edge between u∗ and u, respectively.

Prim’s Algorithm



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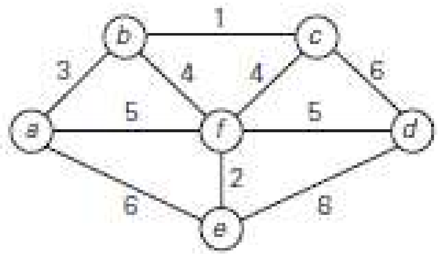
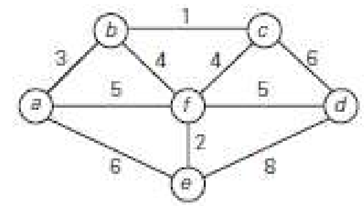
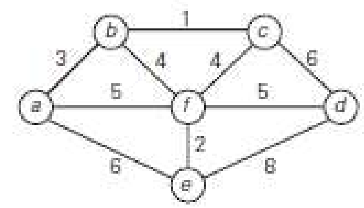
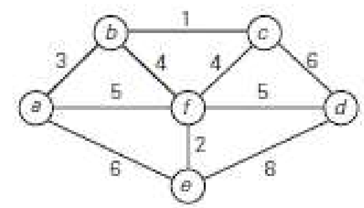
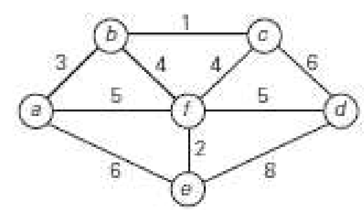
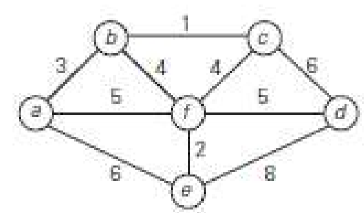
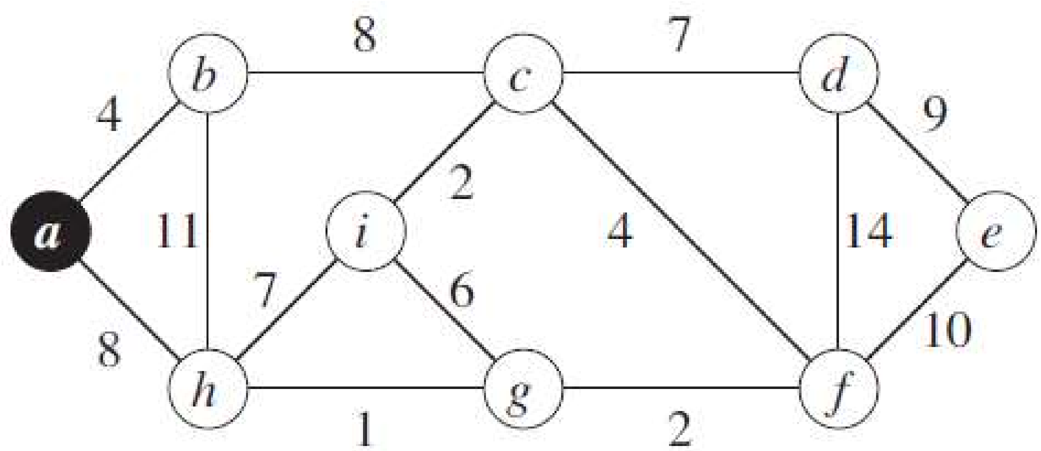


Tree vertices	Remaining vertices	Illustration
a(−, −)	**b(a, 3)** c(−, ∞) d(−, ∞) e(a, 6) f(a, 5)	
b(a, 3)	**c(b, 1)** d(−, ∞) e(a, 6) f(b, 4)	
c(b, 1)	d(c, 6) e(a, 6) **f(b, 4)**	
f(b, 4)	d(f, 5) **e(f, 2)**	
e(f, 2)	**d(f, 5)**	
d(f, 5)		

Prim’s Algorithm



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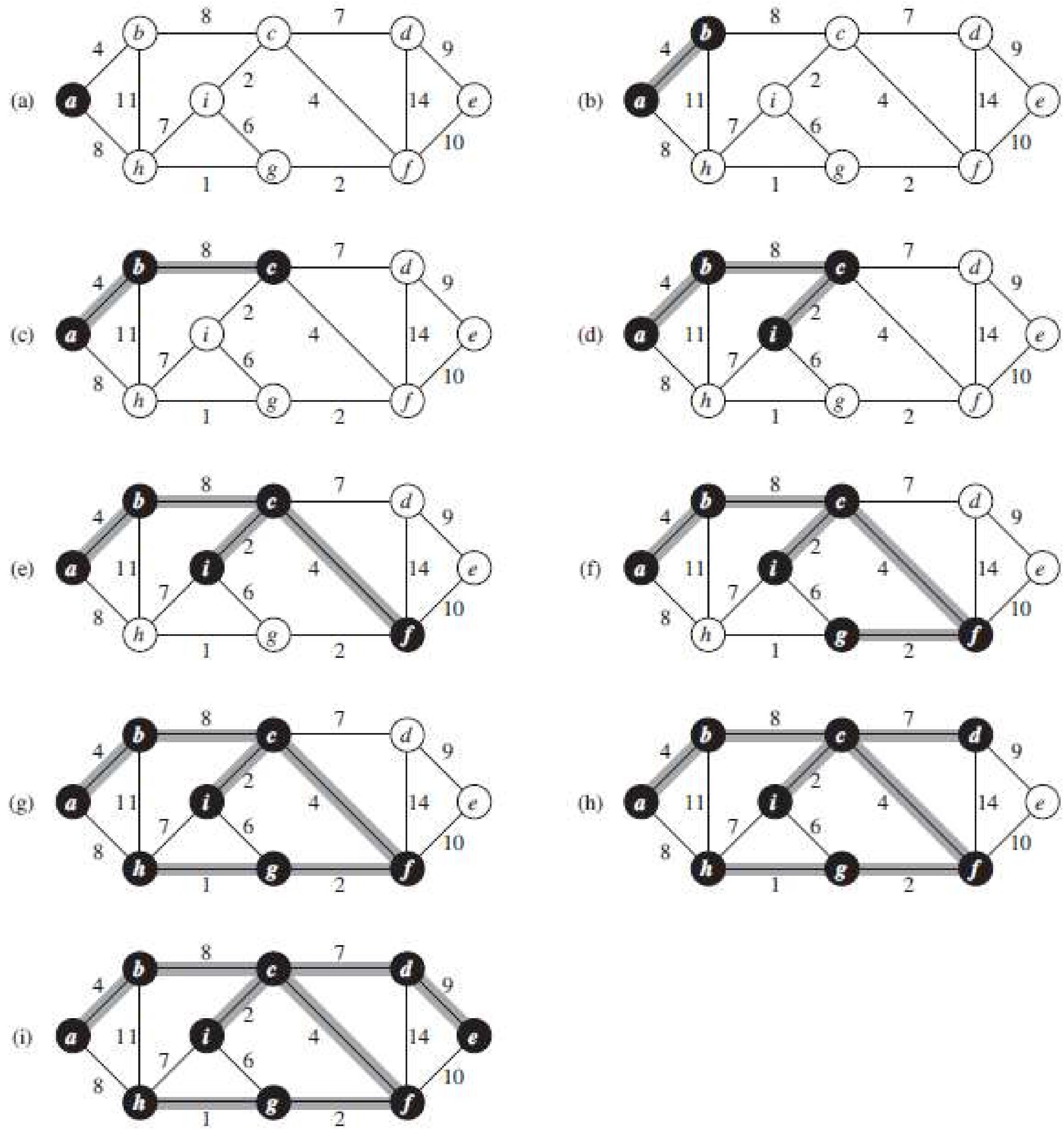


Proof of Prim's Algorithm

- Does Prim's algorithm always yield a minimum spanning tree?

Proof of Prim's Algorithm

- Does Prim's algorithm always yield a minimum spanning tree?

The answer to this question is yes.

Proof of Prim’s Algorithm

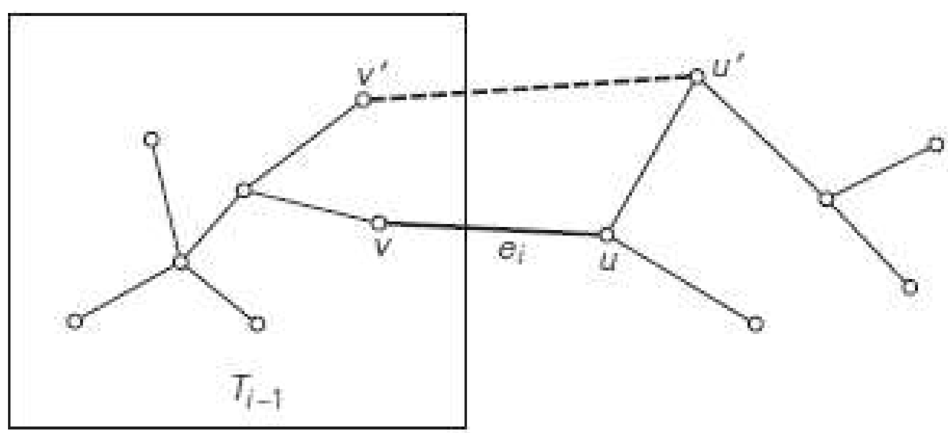
- Let us prove by induction that each of the sub-trees T_i,

$i = 0, \ldots, n - 1$, generated by Prim’s algorithm is a part (i.e., a sub-graph) of some minimum spanning tree.

- T_0 consists of a single vertex and hence must be a part of any minimum spanning tree.
- For the inductive step, let us assume that T_{i-1} is part of some minimum spanning tree T.
- We need to prove that T_i, generated from T_{i-1} by Prim’s algorithm, is also a part of a minimum spanning tree

Proof of Prim's Algorithm

- By contradiction by assuming that **no minimum spanning tree of the graph can contain T_i.**



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Compute the efficiency of the algorithm

MST-PRIM(G, w, r)

for each $u \in G.V$
    $u.key = \infty$
    $u.\pi =$ NIL
$r.key = 0$
$Q = G.V$
while $Q \neq \emptyset$
    $u =$ EXTRACT-MIN(Q)
    for each $v \in G.Adj[u]$
        if $v \in Q$ and $w(u, v) < v.key$
            $v.\pi = u$
            $v.key = w(u, v)$

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Compute the efficiency of the algorithm

The answer depends on the data structures chosen for the graph itself and for the priority queue of the set V − VT whose vertex priorities are the distances to the nearest tree vertices.

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If a graph is represented by its weight matrix and the priority queue is implemented as an unordered array

The algorithm’s running time will be in $O(|V|^2)$

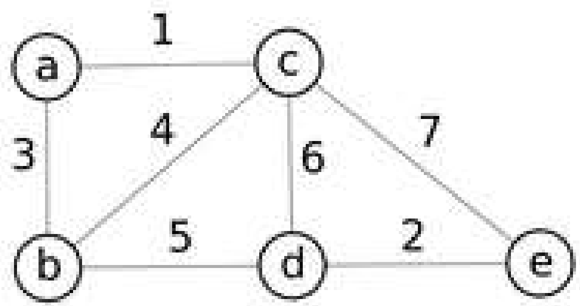
On each of the $|V| - 1$ iterations, the array implementing the priority queue is traversed to find and delete the minimum and then to update, if necessary, the priorities of the remaining vertices.

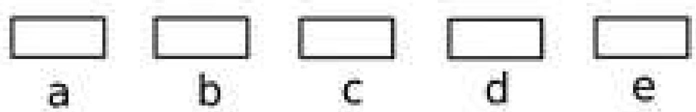
Kruskal's algorithm

```
MST-KRUSKAL(G, w)
A = ∅
for each vertex v ∈ G.V
    MAKE-SET(v)
sort the edges of G.E into nondecreasing order by weight w
for each edge (u, v) ∈ G.E, taken in nondecreasing order by weight
    if FIND-SET(u) ≠ FIND-SET(v)
        A = A ∪ {(u, v)}
        UNION(u, v)
return A
```

$O(E \lg E)$.

Kruskal’s algorithm





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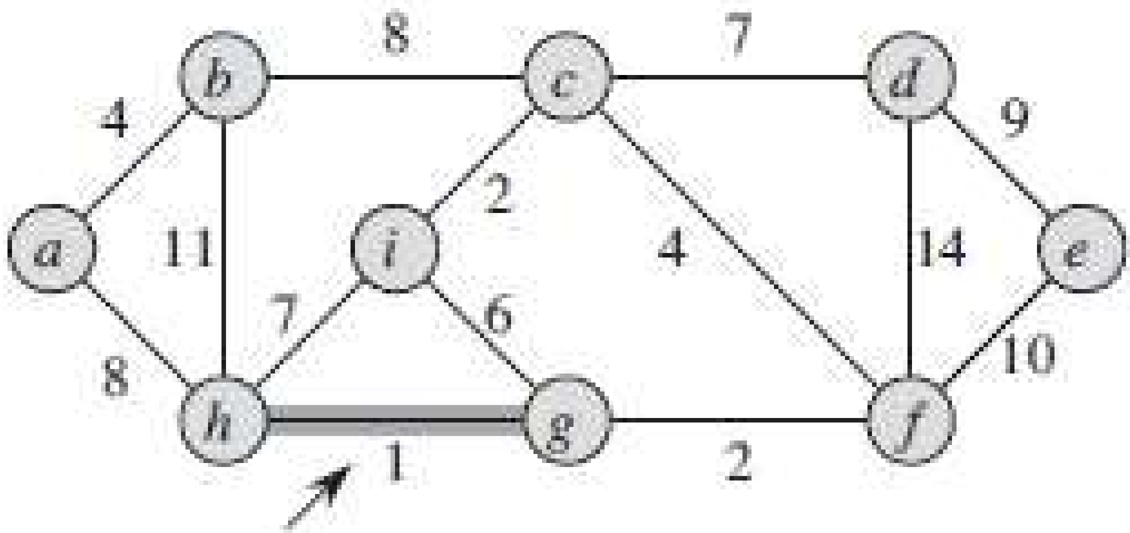

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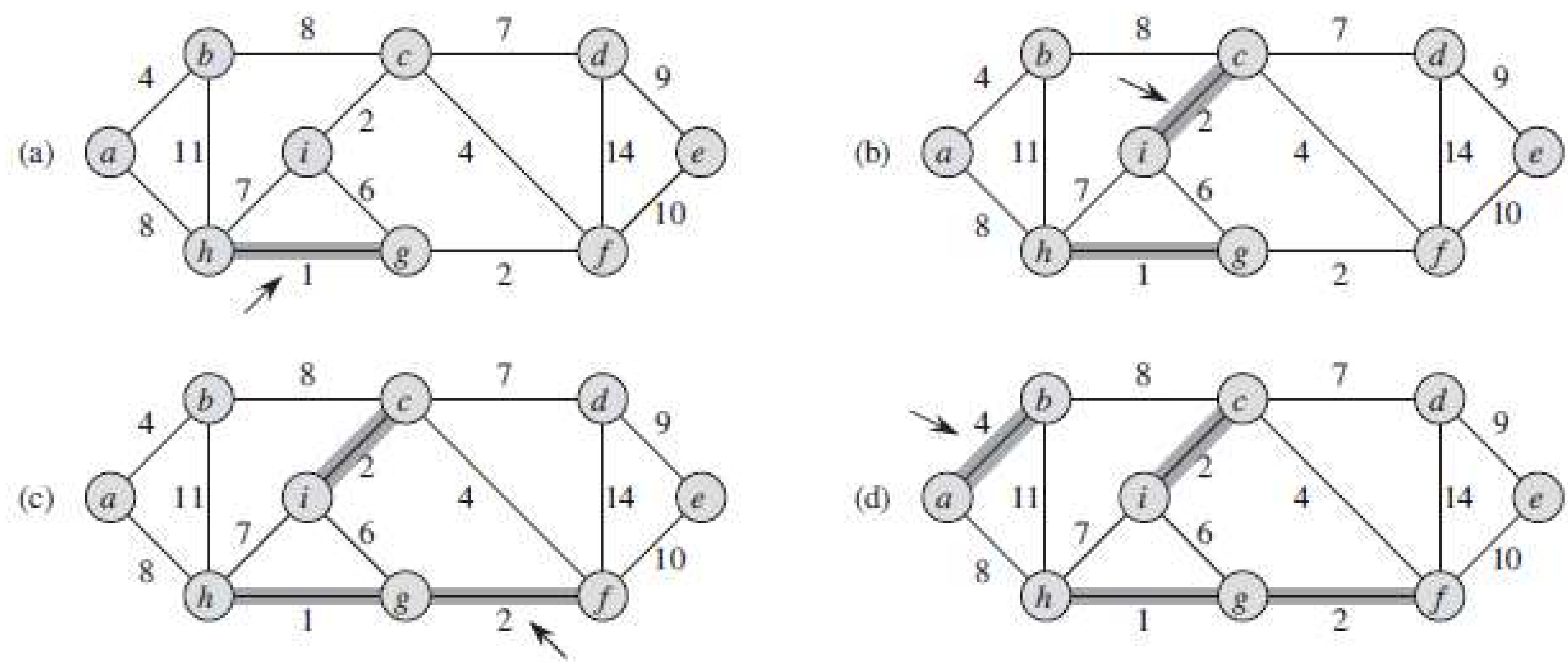

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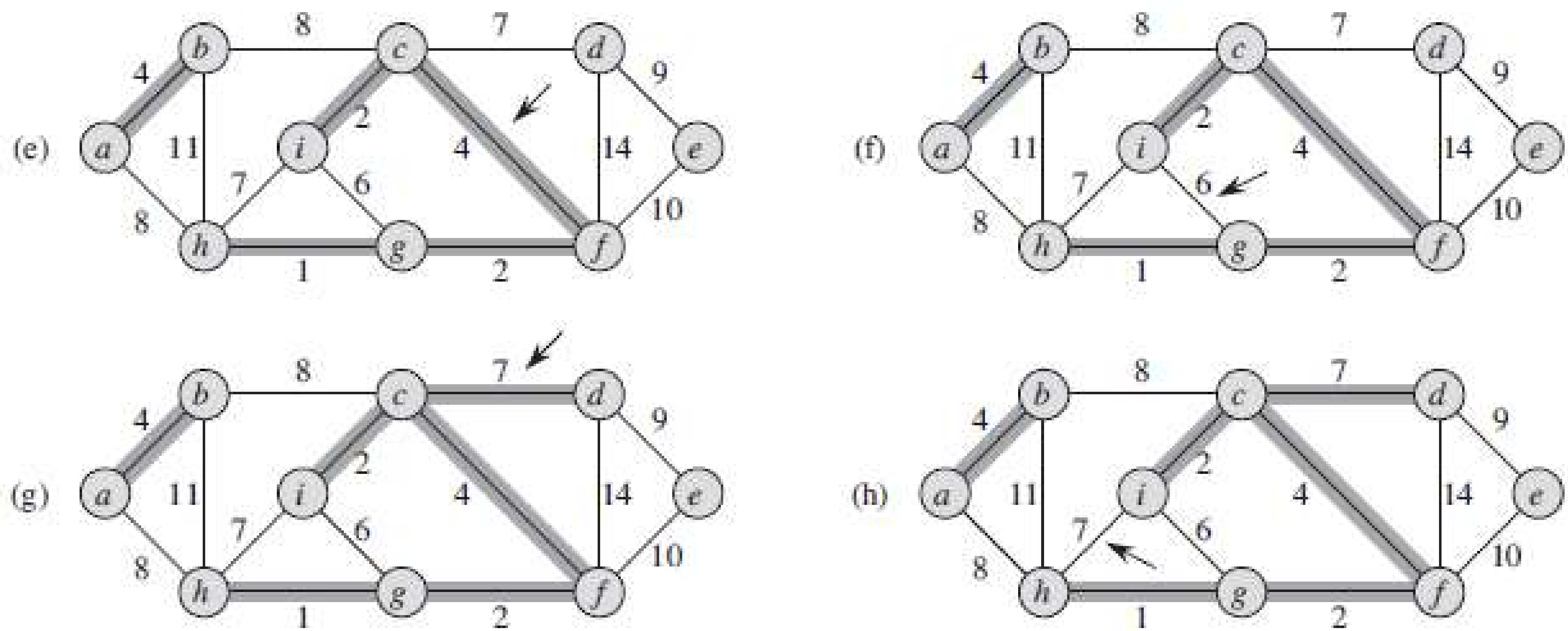

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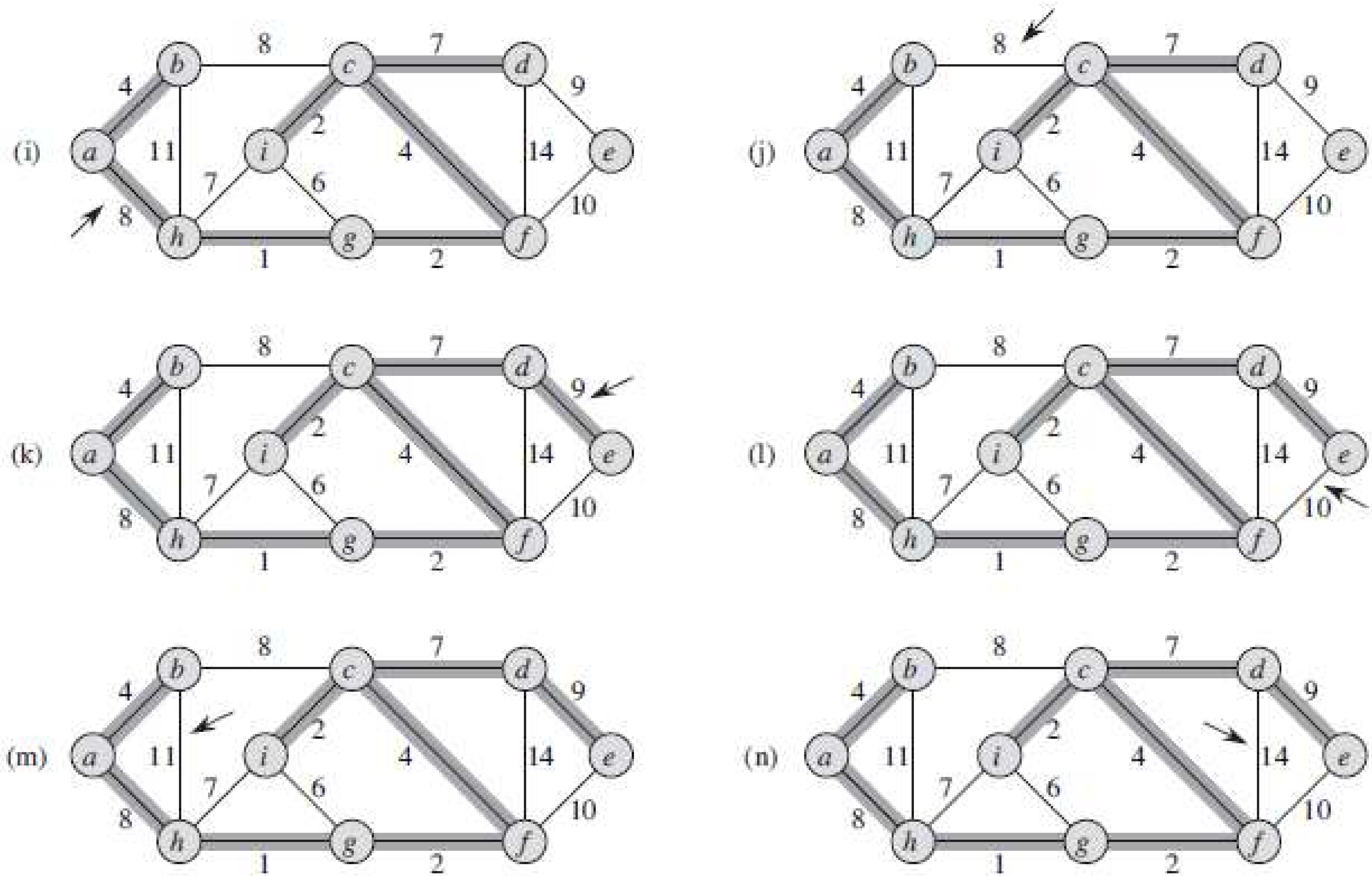

Thank you