Mathematical concepts for computer science

Number Theory

- **The part of mathematics devoted to the study of the set of integers and their properties is known as number theory**

Divisibility and Modular Arithmetic

- Division of an integer by a positive integer produces a **quotient and a remainder**.
- Working with these remainders leads to **modular arithmetic**, which plays an important role in mathematics and which is used throughout computer science.
- Generating pseudorandom numbers, assigning computer memory locations to files, constructing check digits, and encrypting messages.

Division

If a and b are integers with $a \neq 0$, we say that *a divides b* if there is an integer c such that $b = ac$, or equivalently, if $\frac{b}{a}$ is an integer. When a divides b we say that a is a *factor* or *divisor* of b, and that b is a *multiple* of a. The notation $a \mid b$ denotes that a divides b. We write $a \nmid b$ when a does not divide b.

Determine whether $3 \mid 7$ and whether $3 \mid 12$.

Solution: We see that $3 \nmid 7$, because $7/3$ is not an integer. On the other hand, $3 \mid 12$ because $12/3 = 4$. ◀

Division

Let a, b, and c be integers, where $a \neq 0$. Then

(*i*) if $a \mid b$ and $a \mid c$, then $a \mid (b + c)$;

(*ii*) if $a \mid b$, then $a \mid bc$ for all integers c;

(*iii*) if $a \mid b$ and $b \mid c$, then $a \mid c$.

If a, b, and c are integers, where $a \neq 0$, such that $a \mid b$ and $a \mid c$, then $a \mid mb + nc$ whenever m and n are integers.

THE DIVISION ALGORITHM Let a be an integer and d a positive integer. Then there are unique integers q and r, with $0 \leq r < d$, such that $a = dq + r$.

In the equality given in the division algorithm, d is called the *divisor*, a is called the *dividend*, q is called the *quotient*, and r is called the *remainder*. This notation is used to express the quotient and remainder:

$$q = a \textbf{ div } d, \quad r = a \textbf{ mod } d.$$

Division

What are the quotient and remainder when 101 is divided by 11?

Solution: We have

$$101 = 11 \cdot 9 + 2.$$

Hence, the quotient when 101 is divided by 11 is $9 = 101$ **div** 11, and the remainder is $2 = 101$ **mod** 11.

What are the quotient and remainder when -11 is divided by 3?

Solution: We have

$$-11 = 3(-4) + 1.$$

Hence, the quotient when -11 is divided by 3 is $-4 = -11$ **div** 3, and the remainder is $1 = -11$ **mod** 3.

Modular Arithmetic

If a and b are integers and m is a positive integer, then a is *congruent to b modulo m* if m divides $a - b$. We use the notation $a \equiv b \pmod{m}$ to indicate that a is congruent to b modulo m. We say that $a \equiv b \pmod{m}$ is a **congruence** and that m is its **modulus** (plural **moduli**). If a and b are not congruent modulo m, we write $a \not\equiv b \pmod{m}$.

Let a and b be integers, and let m be a positive integer. Then $a \equiv b \pmod{m}$ if and only if $a \textbf{ mod } m = b \textbf{ mod } m$.

Determine whether 17 is congruent to 5 modulo 6 and whether 24 and 14 are congruent modulo 6.

Solution: Because 6 divides $17 - 5 = 12$, we see that $17 \equiv 5 \pmod{6}$. However, because $24 - 14 = 10$ is not divisible by 6, we see that $24 \not\equiv 14 \pmod{6}$. ◀

Modular Arithmetic

Let m be a positive integer. The integers a and b are congruent modulo m if and only if there is an integer k such that $a = b + km$.

Proof: If $a \equiv b \pmod{m}$, by the definition of congruence (Definition 3), we know that $m \mid (a - b)$. This means that there is an integer k such that $a - b = km$, so that $a = b + km$. Conversely, if there is an integer k such that $a = b + km$, then $km = a - b$. Hence, m divides $a - b$, so that $a \equiv b \pmod{m}$. ◁

Modular Arithmetic

Let m be a positive integer. If $a \equiv b \pmod{m}$ and $c \equiv d \pmod{m}$, then

$$a + c \equiv b + d \pmod{m} \qquad \text{and} \qquad ac \equiv bd \pmod{m}.$$

Because $7 \equiv 2 \pmod{5}$ and $11 \equiv 1 \pmod{5}$, it follows from Theorem 5 that

$$18 = 7 + 11 \equiv 2 + 1 = 3 \pmod{5}$$

and that

$$77 = 7 \cdot 11 \equiv 2 \cdot 1 = 2 \pmod{5}.$$

Modular Arithmetic

Let m be a positive integer. If $a \equiv b \pmod{m}$ and $c \equiv d \pmod{m}$, then

$$a + c \equiv b + d \pmod{m} \qquad \text{and} \qquad ac \equiv bd \pmod{m}.$$

Proof: We use a direct proof. Because $a \equiv b \pmod{m}$ and $c \equiv d \pmod{m}$, by Theorem 4 there are integers s and t with $b = a + sm$ and $d = c + tm$. Hence,

$$b + d = (a + sm) + (c + tm) = (a + c) + m(s + t)$$

and

$$bd = (a + sm)(c + tm) = ac + m(at + cs + stm).$$

Hence,

$$a + c \equiv b + d \pmod{m} \qquad \text{and} \qquad ac \equiv bd \pmod{m}.$$

$\triangleleft$

Arithmetic Modulo *m*

We can define arithmetic operations on $\mathbf{Z}_m$, the set of nonnegative integers less than m, that is, the set $\{0, 1, \ldots, m-1\}$. In particular, we define addition of these integers, denoted by $+_m$ by

$$a +_m b = (a + b) \textbf{ mod } m,$$

where the addition on the right-hand side of this equation is the ordinary addition of integers, and we define multiplication of these integers, denoted by $\cdot_m$ by

$$a \cdot_m b = (a \cdot b) \textbf{ mod } m,$$

where the multiplication on the right-hand side of this equation is the ordinary multiplication of integers. The operations $+_m$ and $\cdot_m$ are called addition and multiplication modulo m and when we use these operations, we are said to be doing **arithmetic modulo** m.

A familiar use of modular arithmetic is in the 12-hour clock, in which the day is divided into two 12-hour periods. If the time is 7:00 now, then 8 hours later it will be 3:00. Simple addition would result in 7 + 8 = 15, 15 is *congruent* to 3 modulo 12

Arithmetic Modulo *m*

Use the definition of addition and multiplication in $\mathbf{Z}_m$ to find $7 +_{11} 9$ and $7 \cdot_{11} 9$.

Solution: Using the definition of addition modulo 11, we find that

$$7 +_{11} 9 = (7 + 9) \ \mathbf{mod}\ 11 = 16 \ \mathbf{mod}\ 11 = 5,$$

and

$$7 \cdot_{11} 9 = (7 \cdot 9) \ \mathbf{mod}\ 11 = 63 \ \mathbf{mod}\ 11 = 8.$$

Hence $7 +_{11} 9 = 5$ and $7 \cdot_{11} 9 = 8$.

Arithmetic Modulo *m*

Closure If a and b belong to $\mathbf{Z}_m$, then $a +_m b$ and $a \cdot_m b$ belong to $\mathbf{Z}_m$.

Associativity If a, b, and c belong to $\mathbf{Z}_m$, then $(a +_m b) +_m c = a +_m (b +_m c)$ and $(a \cdot_m b) \cdot_m c = a \cdot_m (b \cdot_m c)$.

Commutativity If a and b belong to $\mathbf{Z}_m$, then $a +_m b = b +_m a$ and $a \cdot_m b = b \cdot_m a$.

Identity elements The elements 0 and 1 are identity elements for addition and multiplication modulo m, respectively. That is, if a belongs to $\mathbf{Z}_m$, then $a +_m 0 = 0 +_m a = a$ and $a \cdot_m 1 = 1 \cdot_m a = a$.

Additive inverses If $a \neq 0$ belongs to $\mathbf{Z}_m$, then $m - a$ is an additive inverse of a modulo m and 0 is its own additive inverse. That is $a +_m (m - a) = 0$ and $0 +_m 0 = 0$.

Distributivity If a, b, and c belong to $\mathbf{Z}_m$, then $a \cdot_m (b +_m c) = (a \cdot_m b) +_m (a \cdot_m c)$ and $(a +_m b) \cdot_m c = (a \cdot_m c) +_m (b \cdot_m c)$.

Primes and Greatest Common Divisors

An integer p greater than 1 is called *prime* if the only positive factors of p are 1 and p. A positive integer that is greater than 1 and is not prime is called *composite*.

The integer 7 is prime because its only positive factors are 1 and 7, whereas the integer 9 is composite because it is divisible by 3.

THE FUNDAMENTAL THEOREM OF ARITHMETIC Every integer greater than 1 can be written uniquely as a prime or as the product of two or more primes where the prime factors are written in order of nondecreasing size.

The prime factorizations of 100, 641, 999, and 1024 are given by

$$100 = 2 \cdot 2 \cdot 5 \cdot 5 = 2^2 5^2,$$

$$641 = 641,$$

$$999 = 3 \cdot 3 \cdot 3 \cdot 37 = 3^3 \cdot 37,$$

$$1024 = 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 \cdot 2 = 2^{10}.$$

Primes and Greatest Common Divisors

If n is a composite integer, then n has a prime divisor less than or equal to $\sqrt{n}$.

THE FUNDAMENTAL THEOREM OF ARITHMETIC Every integer greater than 1 can be written uniquely as a prime or as the product of two or more primes where the prime factors are written in order of nondecreasing size.

Proof: If n is composite, by the definition of a composite integer, we know that it has a factor a with $1 < a < n$. Hence, by the definition of a factor of a positive integer, we have $n = ab$, where b is a positive integer greater than 1. We will show that $a \leq \sqrt{n}$ or $b \leq \sqrt{n}$. If $a > \sqrt{n}$ and $b > \sqrt{n}$, then $ab > \sqrt{n} \cdot \sqrt{n} = n$, which is a contradiction. Consequently, $a \leq \sqrt{n}$ or $b \leq \sqrt{n}$. Because both a and b are divisors of n, we see that n has a positive divisor not exceeding $\sqrt{n}$. This divisor is either prime or, by the fundamental theorem of arithmetic, has a prime divisor less than itself. In either case, n has a prime divisor less than or equal to $\sqrt{n}$. ◁

Primes and Greatest Common Divisors

If n is a composite integer, then n has a prime divisor less than or equal to $\sqrt{n}$.

Show that 101 is prime.

Solution: The only primes not exceeding $\sqrt{101}$ are 2, 3, 5, and 7. Because 101 is not divisible by 2, 3, 5, or 7 (the quotient of 101 and each of these integers is not an integer), it follows that 101 is prime. ◀

Primes and Greatest Common Divisors

THE FUNDAMENTAL THEOREM OF ARITHMETIC Every integer greater than 1 can be written uniquely as a prime or as the product of two or more primes where the prime factors are written in order of nondecreasing size.

Find the prime factorization of 7007.

Solution: To find the prime factorization of 7007, first perform divisions of 7007 by successive primes, beginning with 2. None of the primes 2, 3, and 5 divides 7007. However, 7 divides 7007, with $7007/7 = 1001$. Next, divide 1001 by successive primes, beginning with 7. It is immediately seen that 7 also divides 1001, because $1001/7 = 143$. Continue by dividing 143 by successive primes, beginning with 7. Although 7 does not divide 143, 11 does divide 143, and $143/11 = 13$. Because 13 is prime, the procedure is completed. It follows that $7007 = 7 \cdot 1001 = 7 \cdot 7 \cdot 143 = 7 \cdot 7 \cdot 11 \cdot 13$. Consequently, the prime factorization of 7007 is $7 \cdot 7 \cdot 11 \cdot 13 = 7^2 \cdot 11 \cdot 13$. ◀

Greatest Common Divisors and Least Common Multiples

Let a and b be integers, not both zero. The largest integer d such that $d \mid a$ and $d \mid b$ is called the *greatest common divisor* of a and b. The greatest common divisor of a and b is denoted by $\gcd(a, b)$.

What is the greatest common divisor of 24 and 36?

Solution: The positive common divisors of 24 and 36 are 1, 2, 3, 4, 6, and 12. Hence, $\gcd(24, 36) = 12$. ◀

What is the greatest common divisor of 17 and 22?

Solution: The integers 17 and 22 have no positive common divisors other than 1, so that $\gcd(17, 22) = 1$. ◀

The integers a and b are *relatively prime* if their greatest common divisor is 1.

Greatest Common Divisors and Least Common Multiples

The integers $a_1, a_2, \ldots, a_n$ are *pairwise relatively prime* if $\gcd(a_i, a_j) = 1$ whenever $1 \leq i < j \leq n$.

Determine whether the integers 10, 17, and 21 are pairwise relatively prime and whether the integers 10, 19, and 24 are pairwise relatively prime.

Greatest Common Divisors and Least Common Multiples

The integers $a_1, a_2, \ldots, a_n$ are *pairwise relatively prime* if $\gcd(a_i, a_j) = 1$ whenever $1 \leq i < j \leq n$.

Determine whether the integers 10, 17, and 21 are pairwise relatively prime and whether the integers 10, 19, and 24 are pairwise relatively prime.

Solution: Because $\gcd(10, 17) = 1$, $\gcd(10, 21) = 1$, and $\gcd(17, 21) = 1$, we conclude that 10, 17, and 21 are pairwise relatively prime.

Because $\gcd(10, 24) = 2 > 1$, we see that 10, 19, and 24 are not pairwise relatively prime. ◀

Greatest Common Divisors and Least Common Multiples

Because the prime factorizations of 120 and 500 are $120 = 2^3 \cdot 3 \cdot 5$ and $500 = 2^2 \cdot 5^3$, the greatest common divisor is

$$\gcd(120, 500) = 2^{\min(3,\,2)} 3^{\min(1,\,0)} 5^{\min(1,\,3)} = 2^2 3^0 5^1 = 20.$$

$$\gcd(a, b) = p_1^{\min(a_1,\, b_1)} p_2^{\min(a_2,\, b_2)} \cdots p_n^{\min(a_n,\, b_n)}$$

Greatest Common Divisors and Least Common Multiples

The *least common multiple* of the positive integers a and b is the smallest positive integer that is divisible by both a and b. The least common multiple of a and b is denoted by $\text{lcm}(a, b)$.

$$\text{lcm}(a, b) = p_1^{\max(a_1, b_1)} p_2^{\max(a_2, b_2)} \cdots p_n^{\max(a_n, b_n)}$$

What is the least common multiple of $2^3 3^5 7^2$ and $2^4 3^3$?

Solution: We have

$$\text{lcm}(2^3 3^5 7^2, 2^4 3^3) = 2^{\max(3, 4)} 3^{\max(5, 3)} 7^{\max(2, 0)} = 2^4 3^5 7^2.$$

Greatest Common Divisors and Least Common Multiples

Let a and b be positive integers. Then

$$ab = \gcd(a, b) \cdot \operatorname{lcm}(a, b).$$

The Euclidean Algorithm

- Computing the greatest common divisor of two integers directly from the prime factorizations of these integers is inefficient.
- The reason is that it is time-consuming to find prime factorizations.
- We will give a more efficient method of finding the greatest common divisor, called the **Euclidean algorithm.**
- **This algorithm has been known since ancient times.**
- It is named after the ancient **Greek mathematician Euclid**

The Euclidean Algorithm

- Find gcd*(91, 287)*
- 287 = 91 ▪ 3 + 14
- 91 = 14 ▪ 6 + 7
- gcd(287, 91) =gcd(91, 14) = gcd(14, 7) = 7

Let $a = bq + r$, where a, b, q, and r are integers. Then $\gcd(a, b) = \gcd(b, r)$.

The Euclidean Algorithm

Find the greatest common divisor of 414 and 662 using the Euclidean algorithm.

Let $a = bq + r$, where a, b, q, and r are integers. Then $\gcd(a, b) = \gcd(b, r)$.

The Euclidean Algorithm

Find the greatest common divisor of 414 and 662 using the Euclidean algorithm.

Solution: Successive uses of the division algorithm give:

$$
\begin{aligned}
662 &= 414 \cdot 1 + 248 \\
414 &= 248 \cdot 1 + 166 \\
248 &= 166 \cdot 1 + 82 \\
166 &= 82 \cdot 2 + 2 \\
82 &= 2 \cdot 41.
\end{aligned}
$$

Hence, $\gcd(414, 662) = 2$, because 2 is the last nonzero remainder.

Let $a = bq + r$, where a, b, q, and r are integers. Then $\gcd(a, b) = \gcd(b, r)$.

The Euclidean Algorithm

ALGORITHM 1 The Euclidean Algorithm.

procedure $gcd(a, b$: positive integers)
$x := a$
$y := b$
while $y \neq 0$
    $r := x$ **mod** y
    $x := y$
    $y := r$
return $x\{\text{gcd}(a, b) \text{ is } x\}$

Reference

- **Rosen, Kenneth H., and Kamala Krithivasan. Discrete mathematics and its applications: with combinatorics and graph theory. Tata McGraw-Hill Education, 2016.**