Mathematical concepts for computer science

Set Theory and Probability

- Probability Spaces
  - A countable sample space S is a nonempty countable set. An element ω ϵ S is called an outcome. A subset of S is called an event.
  - A probability function on a sample space S is a total function Pr: $S \rightarrow \mathbb{R}$ such that
    - $\Pr[\omega] \geq 0$ for all $\omega \in \mathcal{S}$, and
    - $\sum_{\omega \in \mathcal{S}} \Pr[\omega] = 1.$

Probability Spaces

A sample space together with a probability function is called a *probability space*. For any event $E \subseteq \mathcal{S}$, the *probability of* E is defined to be the sum of the probabilities of the outcomes in E:

$$\Pr[E] ::= \sum_{\omega \in E} \Pr[\omega]$$

Probability Rules from Set Theory

An immediate consequence of the definition of event probability is that for *disjoint* events E and F,

$$\Pr[E \cup F] = \Pr[E] + \Pr[F].$$

(Sum Rule). *If $E_0, E_1, \ldots, E_n, \ldots$ are pairwise disjoint events, then*

$$\Pr\left[\bigcup_{n \in \mathbb{N}} E_n\right] = \sum_{n \in \mathbb{N}} \Pr[E_n]$$

For example, if the probability that a randomly chosen MIT student is native to the United States is 60%, to Canada is 5%, and to Mexico is 5%, then the probability that a random MIT student is native to one of these three countries is 70%.

Probability Rules from Set Theory

$\Pr[\overline{A}] = 1 - \Pr[A].$ (Complement Rule)

$\Pr[B - A] = \Pr[B] - \Pr[A \cap B],$ (Difference Rule)

$\Pr[A \cup B] = \Pr[A] + \Pr[B] - \Pr[A \cap B],$ (Inclusion-Exclusion)

$\Pr[A \cup B] \leq \Pr[A] + \Pr[B],$ (Boole's Inequality)

If $A \subseteq B$, then $\Pr[A] \leq \Pr[B].$ (Monotonicity Rule)

$\Pr[E_1 \cup \cdots \cup E_n \cup \cdots] \leq \Pr[E_1] + \cdots + \Pr[E_n] + \cdots .$ (Union Bound)

Uniform Probability Spaces

A finite probability space, $\mathcal{S}$, is said to be *uniform* if $\Pr[\omega]$ is the same for every outcome $\omega \in \mathcal{S}$

As we saw in the strange dice problem, uniform sample spaces are particularly easy to work with. That's because for any event $E \subseteq \mathcal{S}$,

$$\Pr[E] = \frac{|E|}{|\mathcal{S}|}$$

This means that once we know the cardinality of E and S, we can immediately obtain Pr[E]

Hands with a Full House

A *Full House* is a hand with three cards of one rank and two cards of another rank. Here are some examples:

$$\{2\spadesuit,\ 2\clubsuit,\ 2\diamondsuit,\ J\clubsuit,\ J\diamondsuit\}$$

$$\{5\diamondsuit,\ 5\clubsuit,\ 5\heartsuit,\ 7\heartsuit,\ 7\clubsuit\}$$

For example, suppose that you select five cards at random from a standard deck of 52 cards. What is **the probability of having a full house**?

Hands with a Full House

The probability of having a full house?

$$|\mathcal{S}| = \binom{52}{5} \qquad\qquad |E| = 13 \cdot \binom{4}{3} \cdot 12 \cdot \binom{4}{2}$$

$$\begin{aligned}
\Pr[E] &= \frac{13 \cdot 12 \cdot \binom{4}{3} \cdot \binom{4}{2}}{\binom{52}{5}} \\
&= \frac{13 \cdot 12 \cdot 4 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2}{52 \cdot 51 \cdot 50 \cdot 49 \cdot 48} = \frac{18}{12495} \\
&\approx \frac{1}{694}.
\end{aligned}$$

Infinite Probability Spaces

- Two players take turns flipping a fair coin. Whoever flips heads first is declared the winner. What is the probability that the first player wins?

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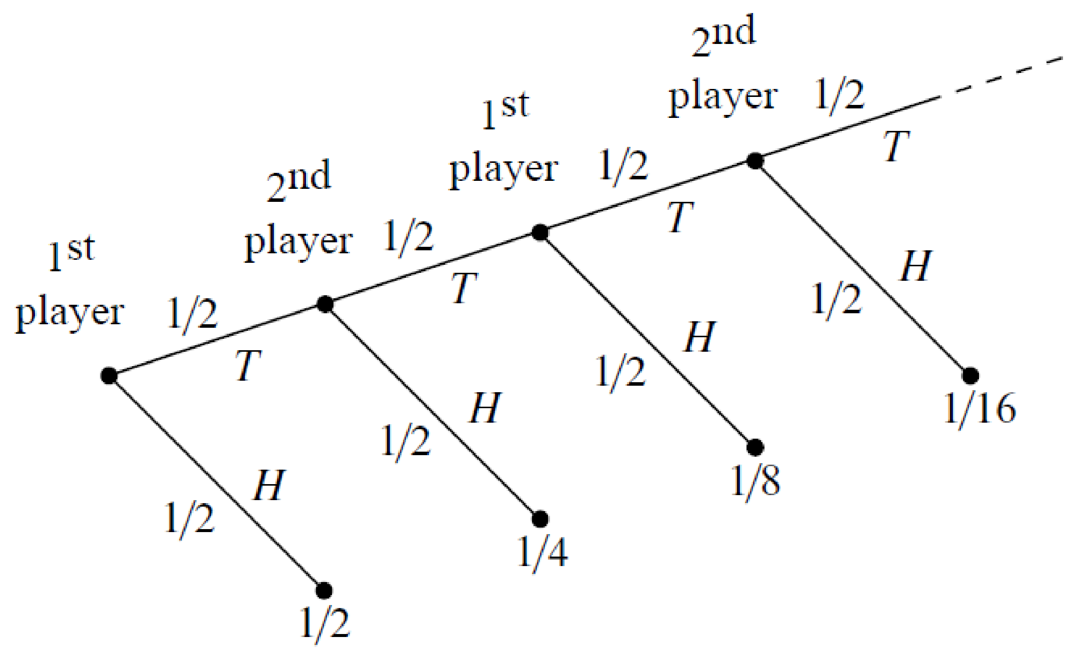
\begin{aligned}
\Pr[\text{first player wins}] &= \frac{1}{2} + \frac{1}{8} + \frac{1}{32} + \frac{1}{128} + \cdots \\
&= \frac{1}{2} \sum_{n=0}^{\infty} \left(\frac{1}{4}\right)^n \\
&= \frac{1}{2}\left(\frac{1}{1-1/4}\right) = \frac{2}{3}.
\end{aligned}
$$



Reference

- Eric Lehman, F Thomson Leighton, Albert R Meyer, Mathematics for Computer Science, 1e, MIT, 2010.