Mathematical concepts for computer science

Pigeon hole principle

- Suppose that a flock of 20 pigeons flies into a set of 19 pigeonholes to roost.
- Because there are 20 pigeons but only 19 pigeonholes, a least one of these 19 pigeonholes must have at least two pigeons in it.
- If there are more pigeons than pigeonholes, then there must be at least one pigeonhole with at least two pigeons in it.

Pigeon hole principle

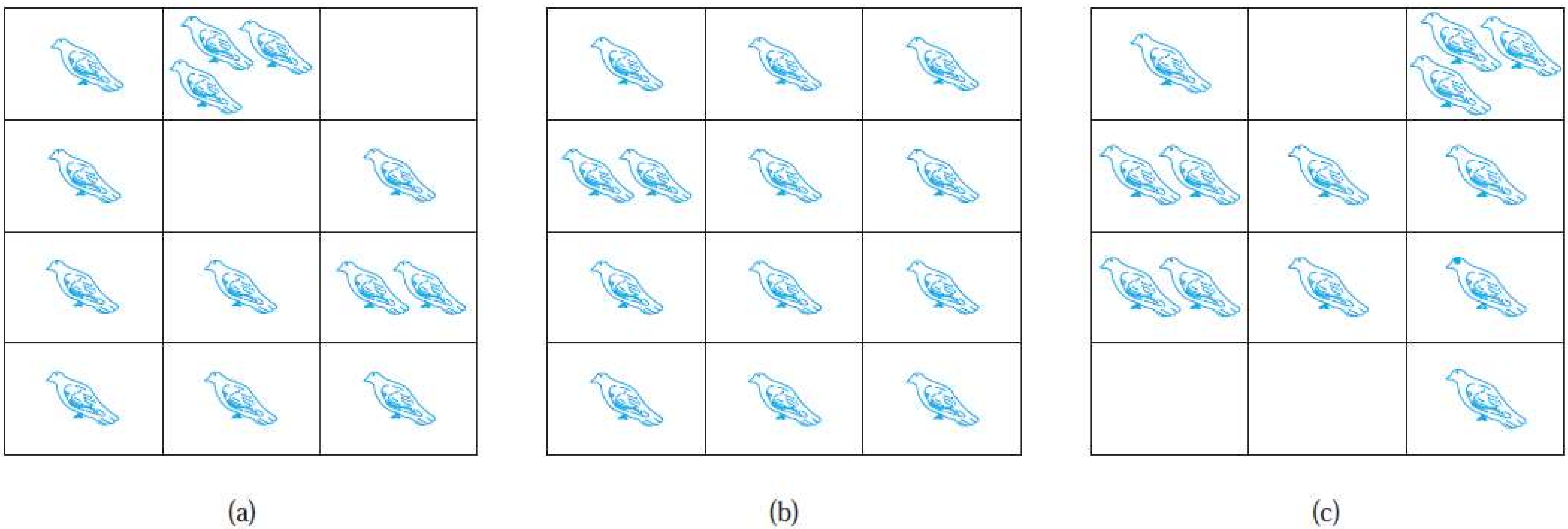

FIGURE 1 There Are More Pigeons Than Pigeonholes.

Proof: We prove the pigeonhole principle using a proof by contraposition.
Suppose that none of the k boxes contains more than one object. Then the total number of objects would be at most k.
This is a contradiction, because there are at least k + 1 objects.

Pigeon hole principle

- A function **f** from a set with **k + 1** or more elements to a set with **k** elements is **not one-to-one**.
- Proof: Suppose that for each element y in the co-domain of f **we have a box that contains all elements x of the domain of f such that f (x) = y**. Because the **domain contains k + 1 or more elements** and the **co-domain contains only k elements**, the pigeonhole principle tells us that one of these boxes contains two or more elements x of the domain. This means that f cannot be one-to-one.

Pigeon hole principle Applications

- Among any group of 367 people, there must be at least two with the same birthday, because there are only 366 possible birthdays.
- In any group of 27 English words, there must be at least two that begin with the same letter, because there are 26 letters in the English alphabet.
- How many students must be in a class to guarantee that at least two students receive the same score on the final exam, if the exam is graded on a scale from 0 to 100 points?

Pigeon hole principle Applications

- How many students must be in a class to guarantee that at least two students receive the same score on the final exam, if the exam is graded on a scale from 0 to 100 points?

Solution: There are 101 possible scores on the final. The pigeonhole principle shows that among any 102 students there must be at least 2 students with the same score.

The Generalized Pigeonhole Principle

- If **N objects** are placed into **k boxes**, then there is at least one box containing at least $\lceil N/k \rceil$ **objects**.
- For instance, among any set of **21 decimal digits** there must be 3 that are the same. This follows because when **21 objects are distributed into 10 boxes**, one box must have more than 2 objects.

The Generalized Pigeonhole Principle

THE GENERALIZED PIGEONHOLE PRINCIPLE If N objects are placed into k boxes, then there is at least one box containing at least $\lceil N/k \rceil$ objects.

Proof: We will use a proof by contraposition. Suppose that none of the boxes contains more than $\lceil N/k \rceil - 1$ objects. Then, the total number of objects is at most

$$k\left(\left\lceil \frac{N}{k} \right\rceil - 1\right) < k\left(\left(\frac{N}{k} + 1\right) - 1\right) = N,$$

where the inequality $\lceil N/k \rceil < (N/k) + 1$ has been used. This is a contradiction because there are a total of N objects. ◁

The Generalized Pigeonhole Principle

Among 100 people there are at least $\lceil 100/12 \rceil = 9$ who were born in the same month.

What is the minimum number of students required in a discrete mathematics class to be sure that at least six will receive the same grade, if there are five possible grades, A, B, C, D, and F?

The Generalized Pigeonhole Principle

What is the minimum number of students required in a discrete mathematics class to be sure that at least six will receive the same grade, if there are five possible grades, A, B, C, D, and F?

Solution: The minimum number of students needed to ensure that at least six students receive the same grade is the smallest integer N such that $\lceil N/5 \rceil = 6$. The smallest such integer is $N = 5 \cdot 5 + 1 = 26$. If you have only 25 students, it is possible for there to be five who have received each grade so that no six students have received the same grade. Thus, 26 is the minimum number of students needed to ensure that at least six students will receive the same grade. ◀

Reference

- **Rosen, Kenneth H., and Kamala Krithivasan. Discrete mathematics and its applications: with combinatorics and graph theory. Tata McGraw-Hill Education, 2016.**