Mathematical concepts for computer science

So far…

1. Propositional Logic
2. Applications of Propositional Logic
3. Propositional Equivalences

Predicates and Quantifiers

Predicates

“$x > 3$”

“$x = y + 3$”

“$x + y = z$”

“computer x is under attack by an intruder”

“computer x is functioning properly”

These statements are **neither true nor false** when the values of the variables are not specified.

Predicates

Predicates

“x > 3”

- **Variable x**, is the **subject** of the statement.
- The **predicate**, “**is greater than 3**”—refers to a property that the subject of the statement can have.

Predicates

Predicates

P(x) : “x > 3”

- Once **a value has been assigned to the variable x**, the statement **P(x) becomes a proposition** and **has a truth value**.
- **P(4)=true** and **P(2)=False.**

Predicates

Predicates

P(x, y) : *"x = y + 3"*

- Once **a value has been assigned to the variable x and y**, the statement **P(x,y) becomes a proposition** and **has a truth value**.
- **P(4,1)=true** and **P(2,3)=False.**

Predicates

Predicates

A(x):“Computer x is under attack by an intruder.”

- Suppose that of the computers on campus, only **CS2** and **MATH1** are currently under attack by intruders.
- What are truth values of **A(CS1)**, **A(CS2)**, and **A(MATH1)**?

Predicates

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- Suppose that of the computers on campus, only **CS2** and **MATH1** are currently under attack by intruders.
- What are truth values of **A(CS1)**, **A(CS2)**, and **A(MATH1)**?
- **A(CS1)=false, A(CS2)=True, A(MATH1)=True.**

Predicates

Predicates

A(c, n):"Computer *c is connected to network n,"*

- where **c** is a variable representing a **computer** and **n** is a variable representing a **network**.
- Suppose that the computer **MATH1** is connected to network **CAMPUS2**, but not to network CAMPUS1.
- What are the values of **A(MATH1, CAMPUS1)** and **A(MATH1, CAMPUS2)**?

Predicates

Predicates

A(c, n):"Computer *c is connected to network n,"*

- where **c** is a variable representing a **computer** and **n** is a variable representing a **network**.
- Suppose that the computer **MATH1** is connected to network **CAMPUS2**, but not to network CAMPUS1.
- What are the values of **A(MATH1, CAMPUS1)** and **A(MATH1, CAMPUS2)**?
- **A(MATH1, CAMPUS1) is false.**
- **A(MATH1, CAMPUS2) is true**

n-place predicate or a n-ary predicate

- In general, a statement involving the **n** variables **x1, x2, . . . , xn** can be denoted by **P(x1, x2, . . . , xn).**
- A statement of the form **P(x1, x2, . . . , xn)** is the value of the **propositional function P at the n-tuple (x1, x2, . . . , xn)**, and **P is also called an n-place predicate or a n-ary predicate**.

R(x, y, z): “x + y = z.”

R(1, 2, 3) is true

R(0, 0, 1) is false

Predicates

- Consider the statement

if ***x > 0 then x := x + 1.***

P(x): x > 0

- If **P(x) is true** for this value of x, the assignment statement x := x + 1 is executed, so the value of **x is increased by 1**.
- If **P(x) is false** for this value of x, the assignment statement is not executed, so the value of **x is not changed**.

PRECONDITIONS AND POSTCONDITIONS

- Predicates are also used to establish the **correctness of computer programs**, that is, to show that **computer programs always produce the desired output when given valid input**.
- The statements that describe valid input are known as **preconditions.**
- The conditions that the output should satisfy when the program has run are known as **post-conditions.**

PRECONDITIONS AND POSTCONDITIONS

- Consider the following program, designed to **interchange** the values of two variables x and y.

```
temp := x
x := y
y := temp
```

- Find predicates that we can use as the precondition and the post-condition to verify the correctness of this program.

PRECONDITIONS AND POSTCONDITIONS

- Consider the following program, designed to **interchange** the values of two variables x and y.

temp := x

x := y

y := temp

P(x, y): “x = a and y = b,”

Q(x, y):“x = b and y = a.”

PRECONDITIONS AND POSTCONDITIONS

P(x, y): "x = a and y = b,"

temp := x *// x = a, temp = a, and y = b.*

x := y *//x = b, temp = a, and y = b.*

y := temp *//x = b, temp = a, and y = a.*

Q(x, y):"x = b and y = a."

Quantifiers

- **Quantification** expresses the extent to which a predicate is true over a range of elements.
- **all, some, many, none, and few** are used in quantifications.
- The area of logic that deals with predicates and quantifiers is called the **predicate calculus.**

THE UNIVERSAL QUANTIFIER

- Many mathematical statements assert that a **property is true for all values of a variable in a particular domain**, called the domain of discourse (or the universe of discourse), often just referred to as the domain. Such a statement is expressed using universal quantification.
- The **universal quantification of P(x)** for a **particular domain** is the proposition that asserts that **P(x) is true for all values of x in this domain**.
- **Domain specifies the possible values** of the **variable *x***.

THE UNIVERSAL QUANTIFIER

- The **universal quantification of P(x)** is the statement "**P(x) for all values of x in the domain**."
- The notation **∀xP(x)** denotes the universal quantification of P(x). Here **∀** is called the universal quantifier.
- We read ∀xP(x) as "for all xP(x)" or "for every xP(x)."
- An element for which P(x) is false is called a counter example of ∀xP(x).

THE UNIVERSAL QUANTIFIER

- A statement $\forall x P(x)$ is false, where $P(x)$ is a propositional function, if and only if $P(x)$ is not always true when x is in the domain.

Suppose **Q(x) :“$x < 2$.”**

What is the **truth value of the quantification $\forall x Q(x)$,** where the domain consists of all **real numbers**?

THE UNIVERSAL QUANTIFIER

- A statement $\forall x P(x)$ is false, where $P(x)$ is a propositional function, if and only if $P(x)$ is not always true when x is in the domain.

Suppose **$Q(x)$:"$x < 2$."**

What is the **truth value of the quantification $\forall x Q(x)$,** where the domain consists of all **real numbers**?

- **$Q(x)$ is not true** for every real number x, because, for instance, **$Q(3)$ is false**. That is, $x = 3$ is a counterexample for the statement $\forall x Q(x)$.

THE UNIVERSAL QUANTIFIER

- **∀xP(x) is the same as the conjunction P(x1) ∧ P(x2) ∧ · · · ∧ P(xn),** because this conjunction is true if and only if **P(x1), P(x2), . . . , P (xn) are all true**.

What is the **truth value of ∀xP(x),** where **P(x)** is the statement " $x^2 < 10$ " and the domain consists of the **positive integers not exceeding 4**?

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What is the **truth value of ∀xP(x),** where **P(x)** is the statement " $x^2 < 10$ " and the domain consists of the **positive integers not exceeding 4**?

- **P(1) ∧ P(2) ∧ P(3) ∧ P(4)**, because the domain consists of the integers 1, 2, 3, and 4.
- Because P(4), which is the statement "16 < 10," is false, it follows that **∀xP(x) is false**.

THE EXISTENTIAL QUANTIFIER

- Many mathematical statements assert that **there is an element with a certain property**. Such statements are expressed using existential quantification.
- With existential quantification, we form a proposition that **is true if and only if *P(x) is true for* at least one value of *x in the domain*.**

THE EXISTENTIAL QUANTIFIER

- The existential quantification of P(x) is the proposition “**There exists an element x in the domain such that P(x)**.”
- We use the notation **$\exists x P(x)$** for the existential quantification of P(x). Here **$\exists$** is called the **existential quantifier**.

“There is an x such that P(x),”

“There is at least one x such that P(x),”

or

“For some xP(x).”

THE EXISTENTIAL QUANTIFIER

- The existential quantification of P(x) is the proposition "**There exists an element x in the domain such that P(x)**."
- We use the notation **∃xP(x)** for the existential quantification of P(x). Here **∃** is called the **existential quantifier**.

TABLE 1 Quantifiers.

Statement	*When True?*	*When False?*
$\forall x P(x)$	$P(x)$ is true for every x.	There is an x for which $P(x)$ is false.
$\exists x P(x)$	There is an x for which $P(x)$ is true.	$P(x)$ is false for every x.

THE EXISTENTIAL QUANTIFIER

- Let P(x) denote the statement “**x > 3**.” What is the truth value of the quantification **∃xP(x)**, where the domain consists of all real numbers?

THE EXISTENTIAL QUANTIFIER

- Let P(x) denote the statement “**x > 3**.” What is the truth value of the quantification **∃xP(x)**, where the domain consists of all real numbers?
- Because “**x > 3**” is **sometimes true**—for instance, when **x = 4 -** the existential quantification of P(x), which is ∃xP(x), is true.

THE EXISTENTIAL QUANTIFIER

- Let $Q(x)$ denote the statement "$x = x + 1$." What is the truth value of the quantification $\exists x Q(x)$, where the domain consists of all real numbers?

THE EXISTENTIAL QUANTIFIER

- Let $Q(x)$ denote the statement "$x = x + 1$."What is the truth value of the quantification $\exists xQ(x)$,where the domain consists of all real numbers?
- Because **$Q(x)$ is false** for every real number x, the existential quantification of $Q(x)$, which is $\exists xQ(x)$, is false.

THE EXISTENTIAL QUANTIFIER

- When all elements in the domain can be listed—say, **x1, x2, . . . , xn**—the existential quantification **∃xP(x)** is the same as the **disjunction P(x1) ∨ P(x2) ∨ · · · ∨ P(xn)**, because this disjunction is true if and only if at least one of P(x1), P(x2), . . . , P (xn) is true.
- What is the truth value of ∃xP(x), where P(x) is the statement “ $x^2 > 10$ ” and the universe of discourse consists of the positive integers not exceeding 4?

THE EXISTENTIAL QUANTIFIER

- What is the truth value of ∃xP(x), where P(x) is the statement " $x^2 > 10$ " and the universe of discourse consists of the positive integers not exceeding 4?
- Because the domain is {1, 2, 3, 4}, the proposition ∃xP(x) is the same as the disjunction P(1) ∨ P(2) ∨ P(3) ∨ P(4).
- Because **P(4)**, which is the statement "**4×4 > 10**," is true, it follows that **∃xP(x) is true**.

THE UNIQUENESS QUANTIFIER

- **Uniqueness quantifier, denoted by $\exists!$ or $\exists 1$.**
- The notation **$\exists!xP(x)$** [or **$\exists_1 xP(x)$**] states "There exists a unique x such that P(x) is true." (Other phrases for uniqueness quantification include "**there is exactly one**" and "**there is one and only one**.")
- For instance, **$\exists!x(x - 1 = 0)$**, where the domain is the set of real numbers, states that **there is a unique real number x such that $x - 1 = 0$**.

THE UNIQUENESS QUANTIFIER

- **Uniqueness quantifier, denoted by ∃! or ∃1.**
- The notation **∃!xP(x)** [or **$\exists_1$xP(x)**] states "There exists a unique x such that P(x) is true." (Other phrases for uniqueness quantification include "**there is exactly one**" and "**there is one and only one**.")
- For instance, **∃!x(x − 1 = 0)**, where the domain is the set of real numbers, states that **there is a unique real number x such that x − 1 = 0**.
- This is a **true** statement, as **x = 1 is the unique real number such that x − 1 = 0**.

Quantifiers with Restricted Domains

- An abbreviated notation is often used to restrict the domain of a quantifier.

$$\forall x < 0\,(x^2 > 0),\ \forall y \neq 0\,(y^3 \neq 0),\ \text{and}\ \exists z > 0\,(z^2 = 2)$$

- The domain in each case consists of the real numbers

$$\forall x < 0\,(x^2 > 0)$$

- The square of a negative real number is positive

$$\forall x\,(x < 0 \rightarrow x^2 > 0)$$

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- The domain in each case consists of the real numbers

$$\forall y \neq 0\,(y^3 \neq 0)$$

- The cube of every nonzero real number is nonzero.

$$\forall y(y \neq 0 \rightarrow y^3 \neq 0).$$

Quantifiers with Restricted Domains

- An abbreviated notation is often used to restrict the domain of a quantifier.

$$\forall x < 0\,(x^2 > 0),\ \forall y \neq 0\,(y^3 \neq 0),\text{ and } \exists z > 0\,(z^2 = 2)$$

- The domain in each case consists of the real numbers

$$\exists z > 0\,(z^2 = 2)$$

- There is a positive square root of 2.

$$\exists z(z > 0 \wedge z^2 = 2)$$

Precedence of Quantifiers

- The quantifiers $\forall$ and $\exists$ have **higher precedence than all logical operators** from propositional calculus.
- **$\forall x P(x)$** $\vee$ **$Q(x)$** means ?

Precedence of Quantifiers

- The quantifiers $\forall$ and $\exists$ have **higher precedence than all logical operators** from propositional calculus.
- **$\forall x P(x)$** $\vee$ **$Q(x)$** means **$(\forall x P(x))$** $\vee$ **$Q(x)$**

Binding Variables

- When a **quantifier is used on the variable x**, we say that this occurrence of the variable is **bound.**
- An occurrence of a variable that is not bound by a quantifier is said to be **free.**
- The part of a logical expression to which a quantifier is applied is called the **scope of this** quantifier.
- $\exists x(x + y = 1)$ identify the bound and free variable?

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- $\exists x(x + y = 1)$ identify the bound and free variable?

x is bound, but y is free.

Logical Equivalences Involving Quantifiers

- Definition: Statements involving predicates and quantifiers are logically equivalent **if and only if they have the same truth value** no matter which predicates are substituted into these statements and which domain of discourse is used for the variables in these propositional functions.
- We use the notation **S ≡ T** to indicate that two statements S and T involving predicates and quantifiers are logically equivalent.

Logical Equivalences Involving Quantifiers

- Show that $\forall x(P(x) \wedge Q(x))$ and $\forall x P(x) \wedge \forall x Q(x)$ are logically equivalent.

Logical Equivalences Involving Quantifiers

- Suppose we have **particular predicates P and Q**, with a **common domain**.
- We can show that $\forall x(P(x) \land Q(x))$ and $\forall xP(x) \land \forall xQ(x)$ are logically equivalent by **doing two things**.
- First, we show that **if $\forall x(P(x) \land Q(x))$ is true**, then **$\forall xP(x) \land \forall xQ(x)$ is true**.
- Second, we show that **if $\forall xP(x) \land \forall xQ(x)$** is true, then **$\forall x(P(x) \land Q(x))$ is true**.

Logical Equivalences Involving Quantifiers

- First, we show that **if $\forall x(P(x) \land Q(x))$ is true**, then **$\forall xP(x) \land \forall xQ(x)$ is true**.
- Suppose that **$\forall x(P(x) \land Q(x))$ is true**.
- This means that **if a is in the domain, then $P(a) \land Q(a)$ is true**. Hence, $P(a)$ is true and $Q(a)$ is true. Because $P(a)$ is true and $Q(a)$ is true **for every element in the domain**, we can conclude that **$\forall xP(x)$ and $\forall xQ(x)$ are both true**.
- This means that **$\forall xP(x) \land \forall xQ(x)$ is true**.

Logical Equivalences Involving Quantifiers

- Second, we show that **if ∀xP(x) ∧ ∀xQ(x)** is true, then **∀x(P(x) ∧ Q(x)) is true**.
- Suppose that **∀xP(x) ∧ ∀xQ(x) is true**.
- It follows that **∀xP(x) is true** and **∀xQ(x) is true**. Hence, **if a is in the domain**, then **P(a) is true** and **Q(a) is true.**
- It follows that **for all a, P(a) ∧ Q(a) is true**.

$$\forall x(P(x) \wedge Q(x)) \equiv \forall x P(x) \wedge \forall x Q(x)$$

Negating Quantified Expressions

- Consider the negation of the statement "**Every student in your class has taken a course in calculus**."
- This statement is a universal quantification, namely,

$$\forall x P(x),$$

- where **P(x)** is the statement "**x has taken a course in calculus**" and **the domain** consists of **the students in your class**.
- What is the negation of this statement ?

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- It is not the case that every student in your class has taken a course in calculus

"There is a student in your class who has not taken a course in calculus."

$$\exists x \neg P(x)$$

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"There is a student in your class who has not taken a course in calculus."

$$\neg \forall x P(x) \equiv \exists x \neg P(x)$$

Negating Quantified Expressions

Prove that $\neg\forall x P(x) \equiv \exists x \neg P(x)$.

Negating Quantified Expressions

Prove that **$\neg\forall x P(x) \equiv \exists x \neg P(x)$.**

- **$\neg\forall x P(x)$ is true if and only if $\forall x P(x)$ is false**.
- **$\forall x P(x)$ is false if and only if there is an element x in the domain for which P(x) is false**.
- **This holds if and only if there is an element x in the domain for which $\neg P(x)$ is true**.
- Finally, note that there is an element x in the domain for which **$\neg P(x)$ is true if and only if $\exists x \neg P(x)$ is true**.
- Putting these steps together, we can conclude that **$\neg\forall x P(x)$ is true if and only if $\exists x \neg P(x)$ is true**.
- It follows that **$\neg\forall x P(x)$ and $\exists x \neg P(x)$ are logically equivalent**.

Negating Quantified Expressions

- Consider the negation of the statement "**There is a student in this class who has taken a course in calculus**."
- This statement is a existential quantification, namely,

$$\exists x Q(x)$$

where **Q(x)** is the statement "**x has taken a course in calculus**" and **the domain** consists of **the students in your class**.

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- The negation of this statement is the proposition "**It is not the case that there is a student in this class who has taken a course in calculus**. "

"Every student in this class has not taken calculus"

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"Every student in this class has not taken calculus"

$$\forall x \neg Q(x)$$

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Prove that $\neg \exists x Q(x) \equiv \forall x \neg Q(x)$.

Negating Quantified Expressions

Prove that **$\neg\exists x Q(x) \equiv \forall x \neg Q(x)$.**

$\neg\exists x Q(x)$ is true if and only if **$\exists x Q(x)$ is false**.

No x exists in the domain for which Q(x) is true if and only if Q(x) is false for every x in the domain.

Q(x) is false for every x in the domain if and only if **$\neg Q(x)$ is true for all x in the domain**, which holds if and only if **$\forall x \neg Q(x)$ is true**.

Negating Quantified Expressions

TABLE 2 **De Morgan's Laws for Quantifiers.**

Negation	*Equivalent Statement*	*When Is Negation True?*	*When False?*
$\neg\exists x P(x)$	$\forall x \neg P(x)$	For every x, $P(x)$ is false.	There is an x for which $P(x)$ is true.
$\neg\forall x P(x)$	$\exists x \neg P(x)$	There is an x for which $P(x)$ is false.	$P(x)$ is true for every x.

Negating Quantified Expressions

- What is the negation of the statement "**There is an honest politician**"?

Negating Quantified Expressions

- What is the negation of the statement "**There is an honest politician**"?
- **H(x)**:"x is honest."
- **∃xH(x)**:There is an honest politician, where the domain consists of all politicians.
- $\neg \exists x H(x)$ equal to $\forall x \neg H(x)$
- "**Every politician is dishonest**."

Negating Quantified Expressions

- What is the negation of the statement "**All Americans eat cheeseburgers**"?

Negating Quantified Expressions

- What is the negation of the statement "**All Americans eat cheeseburgers**"?
- **C(x)**:"x eats cheeseburgers."
- $\boldsymbol{\forall x C(x)}$ **:** "All Americans eat cheeseburgers"
- $\boldsymbol{\neg\forall x C(x)}$ equivalent to $\boldsymbol{\exists x \neg C(x)}$
- "**Some American does not eat cheeseburgers**."

Negating Quantified Expressions

- What are the negations of the statements

$$\forall x(x^2 > x)$$

$$\exists x(x^2 = 2)$$

Negating Quantified Expressions

- What are the negations of the statements

$$\forall x(x^2 > x)$$

$$\neg\forall x(x^2 > x)$$

$$\exists x\neg(x^2 > x)$$

$$\exists x(x^2 \le x)$$

Negating Quantified Expressions

- What are the negations of the statements

$$\exists x(x^2 = 2)$$

$$\neg\exists x(x^2 = 2)$$

$$\forall x\neg(x^2 = 2)$$

$$\forall x(x^2 \neq 2)$$

Negating Quantified Expressions

- What are the negations of the statements

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$$\forall x(x^2 \neq 2)$$

Translating from English into Logical Expressions

- Express the statement "**Every student in this class has studied calculus**" using predicates and quantifiers.
- **"For every student x in this class, x has studied calculus."**
- C(x): "x has studied calculus."
- The **domain** for **x** consists of **the students in the class**

$$\forall x C(x)$$

Translating from English into Logical Expressions

- Express the statement "**Every student in this class has studied calculus**" using predicates and quantifiers.
- **"For every person x, if person x is a student in this class then x has studied calculus."**
- The **domain** for **x** consists of **the students in the class**
- **S(x)** : "Person **x** is in this class"
- **C(x)**: "**x** has studied calculus."

Translating from English into Logical Expressions

- Express the statement "**Every student in this class has studied calculus**" using predicates and quantifiers.
- **"For every person x, if person x is a student in this class then x has studied calculus."**
- The **domain** for **x** consists of **the students in the class**
- **S(x)** : "Person **x** is in this class"
- **C(x)**: "**x** has studied calculus."

$$\forall x(S(x) \rightarrow C(x))$$

Nested Quantifiers

- **One quantifier is within the scope of another.**

$$\forall x \exists y(x + y = 0)$$

says that for every real number **x** there is a real number **y** such that **x + y = 0**. (additive inverse)

$$\forall x \forall y(x + y = y + x)$$

says that **x + y = y + x** for all real numbers **x** and **y**.

(commutative law for addition)

The Order of Quantifiers

- **P(x, y):"x + y = y + x."** What are the truth values of the quantifications **$\forall x \forall y P(x, y)$** and **$\forall y \forall x P(x, y)$** where the domain for all variables consists of all real numbers?
- **Both are true.**
- The order of nested universal quantifiers in a statement without other quantifiers **can be changed without changing the meaning** of the quantified statement.

The Order of Quantifiers

- Let **Q(x, y)** denote **“x + y = 0.”** What are the truth values of the quantifications **∃y∀xQ(x, y)** and **∀x∃yQ(x, y)**, where the domain for all variables consists of all **real numbers**?
- **∃y∀xQ(x, y)** denotes the proposition “**There is a real number y such that for every real number x**, **Q(x, y)**.”

The Order of Quantifiers

- Let **Q(x, y)** denote **"x + y = 0."** What are the truth values of the quantifications **∃y∀xQ(x, y)** and **∀x∃yQ(x, y)**, where the domain for all variables consists of all **real numbers**?
- **∃y∀xQ(x, y)** denotes the proposition "**There is a real number y such that for every real number x**, **Q(x, y)**."
- No matter what value of y is chosen, **there is only one value of x for which x + y = 0**. Because **there is no real number y such that x + y = 0 for all real numbers x**, the statement **∃y∀xQ(x, y)** is **false**.

The Order of Quantifiers

- Let **$Q(x, y)$** denote **"$x + y = 0$."** What are the truth values of the quantifications **$\exists y \forall x Q(x, y)$** and **$\forall x \exists y Q(x, y)$**, where the domain for all variables consists of all **real numbers**?
- **$\forall x \exists y Q(x, y)$** denotes the proposition "**For every real number x there is a real number y such that $Q(x, y)$**."

The Order of Quantifiers

- Let **Q(x, y)** denote **“x + y = 0.”** What are the truth values of the quantifications **∃y∀xQ(x, y)** and **∀x∃yQ(x, y)**, where the domain for all variables consists of all **real numbers**?
- **∀x∃yQ(x, y)** denotes the proposition “**For every real number x there is a real number y such that Q(x, y)**.”
- Given a real number **x**, there is a real number y such that x + y = 0; namely, **y = −x**. Hence, the statement **∀x∃yQ(x, y)** is **true**.

The Order of Quantifiers

- Let **$Q(x, y, z)$** be the statement "**$x + y = z$**." What are the truth values of the statements **$\forall x \forall y \exists z Q(x, y, z)$** and **$\exists z \forall x \forall y Q(x, y, z)$**, where the domain of all variables consists of **all real numbers**?

The Order of Quantifiers

- Let **Q(x, y, z)** be the statement "**x + y = z**." What are the truth values of the statements **∀x∀y∃zQ(x, y, z)** and **∃z∀x∀yQ(x, y, z)**, where the domain of all variables consists of **all real numbers**?
- **∀x∀y∃zQ(x, y, z):**"For all real numbers x and for all real numbers y there is a real number z such that x + y = z,"- **True**

The Order of Quantifiers

- Let **$Q(x, y, z)$** be the statement "**$x + y = z$**." What are the truth values of the statements **$\forall x \forall y \exists z Q(x, y, z)$** and **$\exists z \forall x \forall y Q(x, y, z)$**, where the domain of all variables consists of **all real numbers**?
- **$\exists z \forall x \forall y Q(x, y, z)$** :"There is a real number z such that for all real numbers x and for all real numbers y it is true that $x + y = z$,"- **False**

The Order of Quantifiers

TABLE 1 **Quantifications of Two Variables.**

Statement	*When True?*	*When False?*
$\forall x \forall y P(x, y)$ $\forall y \forall x P(x, y)$	$P(x, y)$ is true for every pair x, y.	There is a pair x, y for which $P(x, y)$ is false.
$\forall x \exists y P(x, y)$	For every x there is a y for which $P(x, y)$ is true.	There is an x such that $P(x, y)$ is false for every y.
$\exists x \forall y P(x, y)$	There is an x for which $P(x, y)$ is true for every y.	For every x there is a y for which $P(x, y)$ is false.
$\exists x \exists y P(x, y)$ $\exists y \exists x P(x, y)$	There is a pair x, y for which $P(x, y)$ is true.	$P(x, y)$ is false for every pair x, y.

Translating Mathematical Statements into Statements Involving Nested Quantifiers

- Translate the statement "**The sum of two positive integers is always positive**" into a logical expression.

Translating Mathematical Statements into Statements Involving Nested Quantifiers

- Translate the statement "**The sum of two positive integers is always positive**" into a logical expression.
- "For every two integers, if these integers are both positive, then the sum of these integers is positive."
- "For all positive integers x and y, x + y is positive."
- $\forall x \forall y((x > 0) \wedge (y > 0) \rightarrow (x + y > 0))$, where the **domain** for both variables consists of all **integers**.
- $\forall x \forall y(x + y > 0)$, where the **domain** for both variables consists of **all positive integers**.

Translating Mathematical Statements into Statements Involving Nested Quantifiers

- Translate the statement “**Every real number except zero has a multiplicative inverse**.” (A multiplicative inverse of a real number x is a real number y such that xy = 1.)

Translating Mathematical Statements into Statements Involving Nested Quantifiers

- Translate the statement "**Every real number except zero has a multiplicative inverse**." (A multiplicative inverse of a real number x is a real number y such that xy = 1.)
- "For every real number x except zero, x has a multiplicative inverse."
- "For every real number x, if x ≠ 0, then there exists a real number y such that xy = 1."
- **$\forall x((x \neq 0) \rightarrow \exists y(xy = 1))$**

Translating from Nested Quantifiers into English

- Translate the statement $\forall x(C(x) \lor \exists y(C(y) \land F(x, y)))$ into English,
- where **C(x)** is **“x has a computer”**
- **F(x, y)** is **“x and y are friends”**
- **The domain** for both x and y consists of **all students in your school**.

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- **F(x, y)** is **“x and y are friends”**
- **The domain** for both x and y consists of **all students in your school**.
- For every student x in your school, x has a computer or there is a student y such that y has a computer and x and y are friends.
- **Every student in your school has a computer or has a friend who has a computer.**

Translating from Nested Quantifiers into English

- Translate the statement into English

$$\exists x \forall y \forall z((F(x, y) \land F(x, z) \land (y = z)) \rightarrow \neg F(y,z))$$

- **F(a,b)** means **a and b are friends** and the **domain** for x, y, and z consists of **all students in your school**.

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$\exists x \forall y \forall z((F(x, y) \land F(x, z) \land (y \neq z)) \rightarrow \neg F(y,z))$

- **F(a,b)** means **a and b are friends** and the **domain** for x, y, and z consists of **all students in your school**.
- If students x and y are friends, and students x and z are friends, and furthermore, if y and z are not the same student, then y and z are not friends.
- There is a student *x such that for all* students *y and all students z other than y, if x and y are friends and x and z are friends, then y* and *z are not friends.*
- There is a case such that, if two students have a common friend then the students are not mutual friends.

Translating English Sentences into Logical Expressions

- Express the statement **“If a person is female and is a parent, then this person is someone’s mother”** as a logical expression involving predicates, quantifiers with a domain consisting of all people, and logical connectives.
- **F(x)** to represent **“x is female,” P(x)** to represent **“x is a parent,”** and **M(x, y)** to represent **“x is the mother of y.”**

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- **F(x)** to represent **"x is female,"** **P(x)** to represent **"x is a parent,"** and **M(x, y)** to represent **"x is the mother of y."**
- $\forall x((F(x) \land P(x)) \rightarrow \exists y M(x, y))$

Negating Nested Quantifiers

- Statements involving nested quantifiers can be negated by successively applying the rules for negating statements involving a single quantifier.
- Express the negation of the statement **$\forall x \exists y(xy = 1)$** so that no negation precedes a quantifier.

TABLE 2 De Morgan's Laws for Quantifiers.

Negation	*Equivalent Statement*	*When Is Negation True?*	*When False?*
$\neg\exists x P(x)$	$\forall x \neg P(x)$	For every x, $P(x)$ is false.	There is an x for which $P(x)$ is true.
$\neg\forall x P(x)$	$\exists x \neg P(x)$	There is an x for which $P(x)$ is false.	$P(x)$ is true for every x.

Negating Nested Quantifiers

- Statements involving nested quantifiers can be negated by successively applying the rules for negating statements involving a single quantifier.
- Express the negation of the statement **$\forall x \exists y(xy = 1)$** so that no negation precedes a quantifier.
- **$\neg \forall x \exists y(xy = 1) = \exists x \neg \exists y(xy = 1) = \exists x \forall y \neg (xy = 1)$**
- **$\exists x \forall y(xy \neq 1)$**

Negating Nested Quantifiers

- Use quantifiers to express the statement that **“There does not exist a woman who has taken a flight on every airline in the world.”**
- **P(w, f)** is **“w has taken f ”** and **Q(f, a)** is **“f is a flight on a**.”

Negating Nested Quantifiers

- Use quantifiers to express the statement that **"There does not exist a woman who has taken a flight on every airline in the world."**
- $\neg\exists w \forall a \exists f\ (P(w, f) \wedge Q(f, a))$, where **P(w, f)** is **"w has taken f "** and **Q(f, a)** is **"f is a flight on a**."

Negating Nested Quantifiers

- $\neg\exists w\forall a\exists f\,(P(w,f)\land Q(f,a))$, where **P(w, f)** is **“w has taken f ”** and **Q(f, a)** is **“f is a flight on a**.”

$\neg\exists w\forall a\exists f\,(P(w,f)\land Q(f,a))$

$\equiv \forall w\neg\forall a\exists f\,(P(w,f)\land Q(f,a))$

$\equiv \forall w\exists a\neg\exists f\,(P(w,f)\land Q(f,a))$

$\equiv \forall w\exists a\forall f\neg(P(w,f)\land Q(f,a))$

$\equiv \forall w\exists a\forall f\,(\neg P(w,f)\lor\neg Q(f,a))$.

- “For every woman there is an airline such that for all flights, this woman has not taken that flight or that flight is not on this airline.”

Reference

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