ASHESI

Discrete Structures and Theory (Spring 2024)
Revision for Final Exam

Exercise 1:

Write each of these statements in the form "if p, then q" in English.

a) I will remember to send you the address only if you send me an e-mail message.
b) To be a citizen of this country, it is sufficient that you were born in the United States.
c) If you keep your textbook, it will be a useful reference in your future courses.
d) The Red Wings will win the Stanley Cup if their goalie plays well.
e) That you get the job implies that you had the best credentials.
f) The beach erodes whenever there is a storm.
g) It is necessary to have a valid password to log on to the server.
h) You will reach the summit unless you begin your climb too late.

Exercise 2:

Show that each of these conditional statements is a tautology using logical equivalences.

a) $(p \wedge q) \rightarrow p$
b) $\neg p \rightarrow (p \rightarrow q)$
c) $(p \wedge q) \rightarrow (p \rightarrow q)$

Exercise 3:

Express the negation of these propositions using quantifiers, and then express the negation in English.

a) Some drivers do not obey the speed limit.
b) All Swedish movies are serious.
c) No one can keep a secret.
d) There is someone in this class who does not have a good attitude.

Exercise 4:

Let $T(x, y)$ mean that student x likes cuisine y, where the domain for x consists of all students at your school and the domain for y consists of all cuisines. Express each of these statements by a simple English sentence.

a) $\neg T\ (Abdallah\ Hussein, Japanese)$
b) $\exists x T\ (x, Korean) \wedge \forall x T\ (x, Mexican)$
c) $\exists y (T\ (Monique\ Arsenault, y) \vee T\ (Jay\ Johnson, y))$
d) $\forall x \forall z \exists y ((x \neq z) \rightarrow \neg (T\ (x, y) \wedge T\ (z, y)))$
e) $\exists x \exists z \forall y (T\ (x, y) \leftrightarrow T\ (z, y))$

Exercise 5:

Use rules of inference to show that the hypotheses "If it does not rain or if it is not foggy, then the sailing race will be held and the lifesaving demonstration will go on," "If the sailing race is held, then the trophy will be awarded," and "The trophy was not awarded" imply the conclusion "It rained."

Exercise 6:
Use rules of inference to show that the hypotheses "All movies produced by John Sayles are wonderful", and "John Sayles produced a movie about coal miners" imply the conclusion "There is a wonderful movie about coal miners."

Exercise 7:
Show that if x is rational, then $3x - 1$ is rational.

Exercise 8:
Let n be an integer. Show that n is even if and only if $n^3 + 5$ is odd.

Exercise 9:
Determine whether each of these statements is true or false.

a) $0 \in \emptyset$
b) $\emptyset \in \{0\}$
c) $\{0\} \subset \emptyset$
d) $\emptyset \subset \{0\}$
e) $\{0\} \in \{0\}$
f) $\{0\} \subset \{0\}$
g) $\{\emptyset\} \subseteq \{\emptyset\}$

Exercise 10:
Let A and B be sets. Show that:

a) $(A \cap B) \subseteq A$.
b) $A \subseteq (A \cup B)$.
c) $A - B \subseteq A$.

Exercise 11:
Let A and B be sets. Using set builder notation, show that:

a) $A \cap \emptyset = \emptyset$.
b) $A \cup A = U$.
c) $A \cap (B - A) = \emptyset$.

Exercise 12:

a) Use a truth table to express the values of the Boolean function $F(x, y, z) = x + yz$.
b) Using the result from (a), write $F(x, y, z)$ in conjunctive normal form.
c) Using the result from (a), write $F(x, y, z)$ in disjunctive normal form.
d) Using Boolean identities, write $F(x, y, z)$ in disjunctive normal form.

Exercise 13:
Construct circuits from inverters, AND gates, and OR gates to produce these outputs.

a) $\bar{x} + y$
b) $xyz + \bar{x}\bar{y}\bar{z}$

Exercise 14:

Use mathematical induction to prove that 43 divides $6^{n+1} + 7^{2n-1}$ for every positive integer n.

Exercise 15:

Prove by induction that $\sum_{j=0}^{n} \left(-\frac{1}{2}\right)^j = \frac{2^{n+1}+(-1)^n}{3\times 2^n}$, whenever n is a nonnegative integer.

Exercise 16:

Suppose you have 30 books (15 novels, 10 history books, and 5 math books). Assume that all 30 books are different.

a) In how many ways can you put the 30 books in a row on a shelf?

b) In how many ways can you get a bunch of four books to give to a friend?

c) In how many ways can you get a bunch of three history books and seven novels to give to a friend?

d) In how many ways can you put the 30 books in a row on a shelf if the novels are on the left, the math books are in the middle, and the history books are on the right?

Exercise 17:

What is the minimum number of students, each of whom comes from one of the 50 states of the USA, who must be enrolled in a university to guarantee that there are at least 100 who come from the same state?

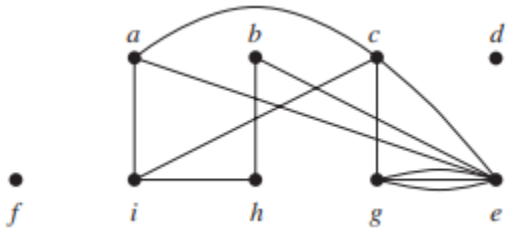
Exercise 18:

a) Find the expansion of $(x + y)^6$

b) Find the coefficient of x^6y^9 in the expansion of $(3x - 2y)^{15}$?

Exercise 19:

Find the number of vertices, the number of edges, and the degree of each vertex in the following graph.



Exercise 20:

Can a simple graph exist with 15 vertices each of degree five?

Exercise 21:

In a round-robin tournament the Tigers beat the Blue Jays, the Tigers beat the Cardinals, the Tigers beat the Orioles, the Blue Jays beat the Cardinals, the Blue Jays beat the Orioles, and the Cardinals beat the Orioles.

a) Model this outcome with a directed graph.
b) What do the in-degree and the out-degree of a vertex in this graph represent?

A complete bipartite graph $K_{m,n}$ is a graph that has its vertex set partitioned into two subsets of m and n vertices, respectively with an edge between two vertices if and only if one vertex is in the first subset and the other vertex is in the second subset.

Exercise 22:

Draw these graphs:

a) K_7
b) $K_{1,8}$
c) $K_{4,4}$
d) C_7
e) W_7

Exercise 23:

The complementary graph $\bar{G}$ of a simple graph G has the same vertices as G. Two vertices are adjacent in $\bar{G}$ if and only if they are not adjacent in G.

Describe each of these graphs:

a) $\overline{K_n}$
b) $\overline{K_{m,n}}$
c) $\overline{C_n}$

Exercise 24:

For which values of n are these graphs bipartite?

a) K_n
b) C_n
c) W_n