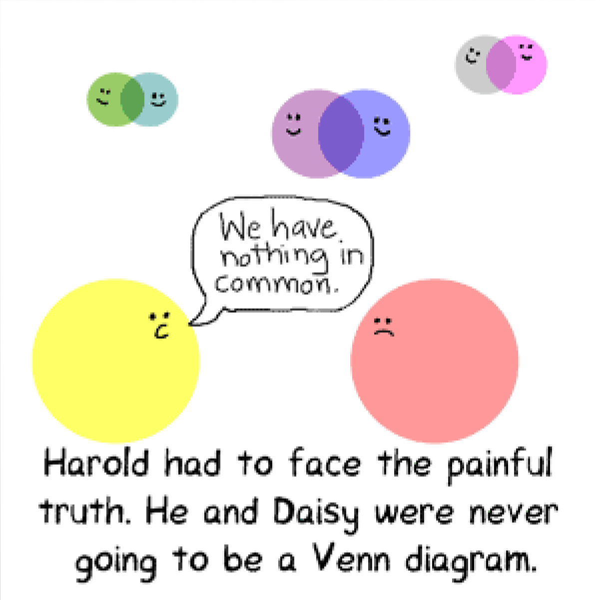


Sets and set operations - 2.1 & 2.2

SECTION SUMMARY

- Definition of sets
- Describing Sets
  - Roster Method
  - Set-Builder Notation
- Some Important Sets in Mathematics
- Empty Set and Universal Set
- Subsets and Set Equality
- Cardinality of Sets
- Tuples
- Cartesian Product

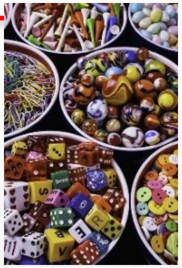
WHY SETS

- Sets are one of the basic building blocks for the types of objects considered in discrete mathematics.
  - Important for counting.
  - Programming languages have set operations.
- Set theory is an important branch of mathematics.
  - Many different systems of axioms have been used to develop set theory.
  - Here we are not concerned with a formal set of axioms for set theory. Instead, we will use what is called naïve set theory.

SETS-DEFINITION

- A *set* is an unordered collection of objects.
  - the students in this class
  - the chairs in this room
- The objects in a set are called the *elements*, or *members* of the set. A set is said to *contain* its elements.
- The notation $a \in A$ denotes that a is an element of the set A.
- If a is not a member of A, write $a \notin A$

DESCRIBING A SET: ROSTER METHOD



- $S = \{a,b,c,d\}$
- Order is not important $S = \{a,b,c,d\} = \{b,c,a,d\}$
- Each distinct object is either a member or not; listing more than once does not change the set. $S = \{a,b,c,d\} = \{a,b,c,b,c,d\}$
- Elipses (…) may be used to describe a set without listing all of the members when the pattern is clear. $S = \{a,b,c,d, \ldots,z\}$

$\mathbf{N}$ = *natural numbers* = $\{0,1,2,3,\ldots\}$

$\mathbf{Z}$ = *integers* = $\{\ldots,-3,-2,-1,0,1,2,3,\ldots\}$

$\mathbf{Z}^+$ = *positive integers* = $\{1,2,3,\ldots\}$

$\mathbf{R}$ = set of *real numbers*

$\mathbf{R}^+$ = set of *positive real numbers*

$\mathbf{C}$ = set of *complex numbers.*

$\mathbf{Q}$ = set of rational numbers

SET NOTATION

- **<u>Set builder notation</u>**
- Specify the property or properties that all members must satisfy:

$S = \{x \mid x \text{ is a positive integer less than } 100\}$

$O = \{x \in \mathbf{Z}^{+} \mid x \text{ is odd and } x < 10\}$

- A predicate may be used:

$S = \{x \mid P(x)\}$ Example: $S = \{x \mid \mathrm{Prime}(x)\}$

- **<u>Interval notation</u>**

$[a,b] = \{x \mid a \le x \le b\}$

$[a,b) = \{x \mid a \le x < b\}$

$(a,b] = \{x \mid a < x \le b\}$

$(a,b) = \{x \mid a < x < b\}$

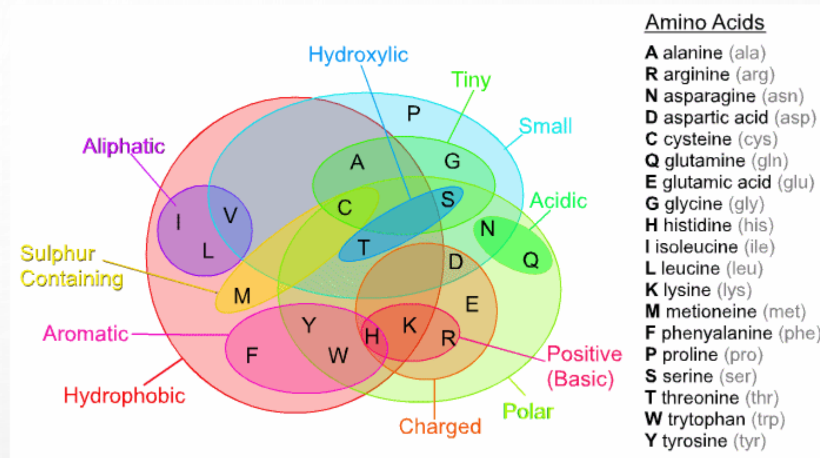
closed interval [a,b]

open interval (a,b)

UNIVERSAL SET AND EMPTY SET

John Venn (1834-1923)
Cambridge, UK

- The *universal set U* is the set containing everything currently under consideration.
  - Sometimes implicit
  - Sometimes explicitly stated.
  - Contents depend on the context.



- The empty set is the set with no elements. Symbolized ∅, but {} also used.

Sets can be elements of sets.

$\{\{1,2,3\},a, \{b,c\}\}$

$\{\mathbf{N},\mathbf{Z},\mathbf{Q},\mathbf{R}\}$

The empty set is different from a set containing the empty set.

$\emptyset \neq \{ \emptyset \}$

SET EQUALITY & SUBSETS

Definition: Two sets are *equal* if and only if they have the same elements.

- Therefore if A and B are sets, then A and B are equal if and only if
  - $\forall x(x \in A \leftrightarrow x \in B)$
- We write $A = B$ if A and B are equal sets.

$\{1,3,5\} = \{3, 5, 1\}$ $\{1,5,5,5,3,3,1\} = \{1,3,5\}$

Definition: The set A ***is a subset*** of B, if and only if every element of A is also an element of B.

The notation $A \subseteq B$ is used to indicate that A is a subset of the set B.
$A \subseteq B$ holds if and only if $\forall x(x \in A \rightarrow x \in B)$ is true.

1. Because $a \in \emptyset$ is always false, $\emptyset \subseteq S$, for every set S.
2. Because $a \in S \rightarrow a \in S$, $S \subseteq S$, for every set S.

SETS AND SUBSETS

- **<u>Showing that A is a Subset of B</u>**: To show that $A \subseteq B$, show that if x belongs to A, then x also belongs to B.
- **<u>Showing that A is not a Subset of B</u>**: To show that A is not a subset of B, $A \nsubseteq B$, find an element $x \in A$ with $x \notin B$. (Such an x is a counterexample to the claim that $x \in A$ implies $x \in B$.)

Recall that two sets A and B are *equal*, denoted by $A = B$, iff $\forall x(x \in A \leftrightarrow x \in B)$

Using logical equivalences we have that $A = B$ iff

$$\forall x[(x \in A \rightarrow x \in B) \wedge (x \in B \rightarrow x \in A)]$$

This is equivalent to

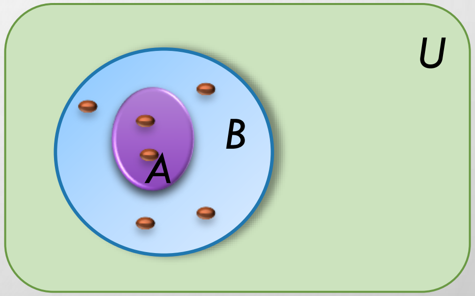
$$A \subseteq B \quad \text{and} \quad B \subseteq A$$

PROPER SUBSETS AND CARDINALITY

Definition: If $A \subseteq B$, but $A \neq B$, then we say A is a *proper subset* of B, denoted by $A \subset B$. If $A \subset B$, then $\forall x(x \in A \rightarrow x \in B) \wedge \exists x(x \in B \wedge x \notin A)$ is true.

Definition: If there are exactly n distinct elements in S where n is a nonnegative integer, we say that S is *finite*. Otherwise it is *infinite*.

Definition: The *cardinality* of a finite set A, denoted by $|A|$, is the number of (distinct) elements of A.



Examples:

1. **$|\emptyset| = 0$**
2. Let S be the letters of the English alphabet. Then $|S| = 26$
3. **$|\{\emptyset\}| = 1$**
4. The set of integers is infinite.

POWER SETS & TUPLE

Definition: The set of all subsets of a set A, denoted $\mathcal{P}(A)$, is called the *power set* of A.

Example: If $A = \{a,b\}$ then $\mathcal{P}(A) = \{ø, \{a\},\{b\},\{a,b\}\}$

- If a set has n elements, then the cardinality of the power set is 2^n.

- The *ordered n-tuple* $(a_1,a_2,\ldots,a_n)$ is the ordered collection that has a_1 as its first element and a_2 as its second element and so on until a_n as its last element.
- Two n-tuples are equal **if and only if** their corresponding elements are equal. $(1,2)\neq(2,1)$
- 2-tuples are called ***ordered pairs***.
- The ordered pairs (a,b) and (c,d) are equal if and only if $a = c$ and $b = d$.

CARTESIAN PRODUCT

René Descartes
(1596-1650)

Definition: The ***Cartesian Product*** of two sets A and B, denoted by $A \times B$ is the set of ordered pairs (a,b) where $a \in A$ and $b \in B$.

Example:

$$A \times B = \{(a, b) | a \in A \wedge b \in B\}$$

$A = \{a, b\}$ $B = \{1,2,3\}$

$A \times B = \{(a, 1), (a, 2), (a, 3), (b, 1), (b, 2), (b, 3)\}$

Definition: A subset R of the Cartesian product $A \times B$ is called a ***relation*** from the set A to the set B.

Definition: The cartesian products of the sets $A_1, A_2, \dots, A_n$, denoted by $A_1 \times A_2 \times \dots \times An$, is the set of ordered n-tuples $(a_1, a_2, \dots, a_n)$ where a_i belongs to A_i for $i = 1, \dots n$.

$$A_1 \times A_2 \times \cdots \times A_n = \{(a_1, a_2, \dots, a_n) | a_i \in A_i \text{ for } i = 1, 2, \dots n\}$$

Example: What is $A \times B \times$ C where $A = \{0,1\}$, $B = \{1,2\}$ and C $= \{0,1,2\}$

Solution: $A \times B \times$ C = {(0,1,0), (0,1,1), (0,1,2),(0,2,0), (0,2,1), (0,2,2),(1,1,0), (1,1,1), (1,1,2), (1,2,0), (1,2,1), (1,2,2)}

TRUTH SETS OF PREDICATES

- Given a predicate *P* and a domain *D*, we define the ***truth set of P*** to be the set of elements in *D* for which *P*(x) is true. The truth set of *P*(x) is denoted by

$$\{x \in D | P(x)\}$$

Example: The truth set of P where $P(x)$ is "$|x| = 1$" and the domain is the integers is the set $\{x \in Z|\ |x| = 1\} = \{-1,1\}$

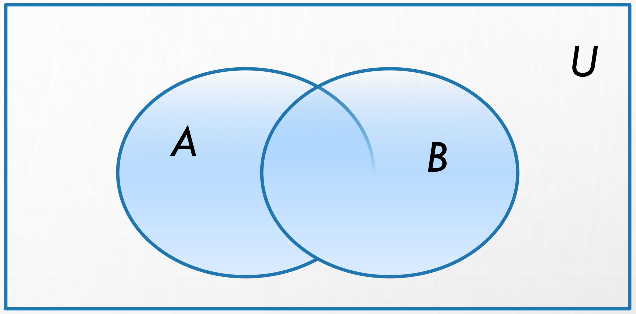
Example: What is the truth set of P (x) , $\{x \in Z|\ x^2 = 1.3\}$

SET OPERATIONS

- Set Operations
  - Union
  - Intersection
  - Complementation
  - Difference
- Set Identities
- Proving Identities
- Membership Tables

UNION VS. INTERSECTION

Definition: Let *A* and *B* be sets. The *union* of the sets *A* and *B*, denoted by *A* ∪ *B*, is the set: $\{x|x \in A \vee x \in B\}$



Example: What is $\{1,2,3\} \cup \{3, 4, 5\}$?

Solution: $\{1,2,3,4,5\}$

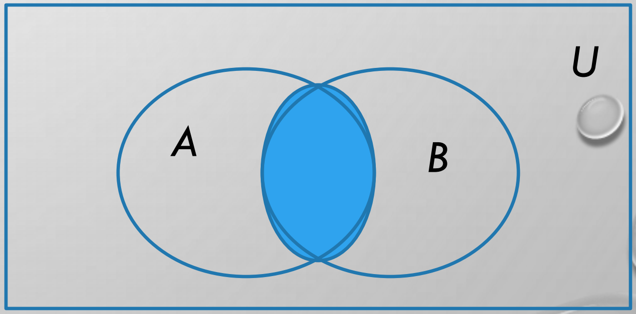
- **Definition:** The *intersection* of sets *A* and *B*, denoted by *A* ∩ *B*, is

$$\{x|x \in A \wedge x \in B\}$$

<u>**Note**</u> if the intersection is empty, then *A* and *B* are said to be *disjoint*.

Example: What is $\{1,2,3\} \cap \{3,4,5\}$?

Solution: $\{3\}$



Example: What is $\{1,2,3\} \cap \{4,5,6\}$?

Solution: ∅

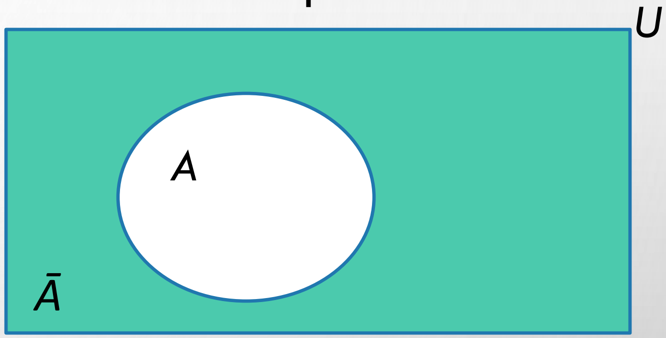
COMPLEMENT & DIFFERENCE

Definition: If A is a set, then the complement of the A (with respect to U), denoted by $\bar{A}$ is the set $U - A$ $\bar{A} = \{x \in U \mid x \notin A\}$

(The complement of A is sometimes denoted by A^c .)

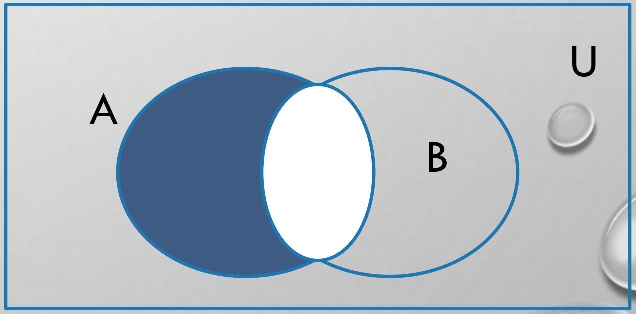
Example: If U is the positive integers less than 100, what is the complement of $\{x \mid x > 70\}$

Solution: $\{x \in U \mid x \leq 70\} = \{1,2,3,\ldots,70\}$

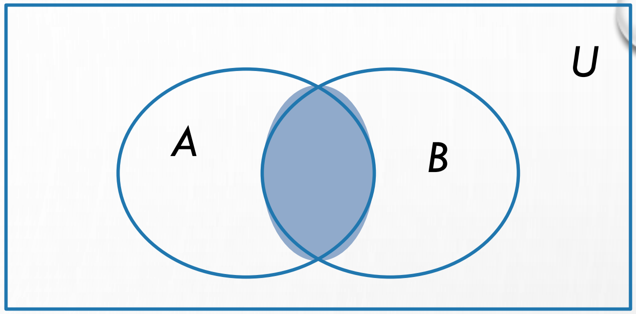


Definition: Let A and B be sets. The ***difference*** of A and B, denoted by $A - B$, is the set containing the elements of A that are not in B. The difference of A and B is also called the complement of B with respect to A.

$$A - B = \{x \mid x \in A \wedge x \notin B\} = A \cap \overline{B}$$



THE CARDINALITY OF THE UNION OF TWO SETS



Inclusion-Exclusion

$|A \cup B| = |A| + |B| - |A \cap B|$

Example: Let A be the math majors in your class and B be the CS majors.

To count the number of students who are either math majors or CS majors, add the number of math majors and the number of CS majors, and subtract the number of joint CS/math majors.

Example: $U = \{0,1,2,3,4,5,6,7,8,9,10\}$ $A = \{1,2,3,4,5\}$, $B = \{4,5,6,7,8\}$

1. $A \cup B$

   Solution: $\{1,2,3,4,5,6,7,8\}$
2. $A \cap B$

   Solution: $\{4,5\}$
3. $\bar{A}$

   Solution: $\{0,6,7,8,9,10\}$
4. $\bar{B}$

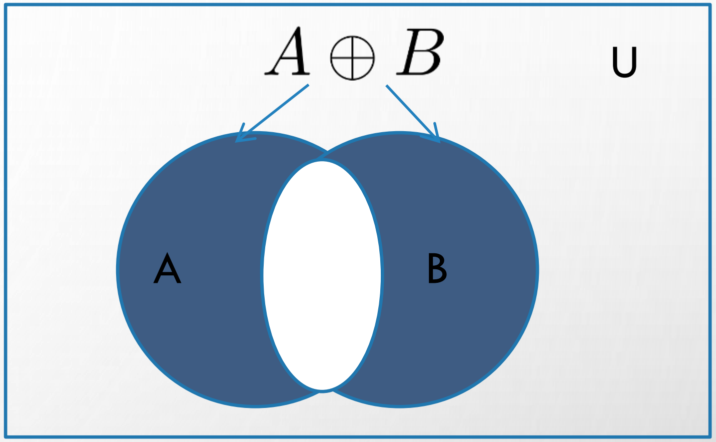
   Solution: $\{0,1,2,3,9,10\}$
5. $A - B$

   Solution: $\{1,2,3\}$
6. $B - A$

   Solution: $\{6,7,8\}$

SYMMETRIC DIFFERENCE

Definition: The *symmetric difference* of **A** and **B**, denoted by $A \oplus B$ is the set $(A - B) \cup (B - A)$



Example:

$U = \{0,1,2,3,4,5,6,7,8,9,10\}$

$A = \{1,2,3,4,5\}$ $B = \{4,5,6,7,8\}$

What is $A \oplus B$:

Solution: $\{1,2,3,6,7,8\}$

SET IDENTITIES

- **Identity laws**

$$A \cup \emptyset = A \quad A \cap U = A$$

- **Domination laws**

$$A \cup U = U \quad A \cap \emptyset = \emptyset$$

- **Idempotent laws**

$$A \cup A = A \quad A \cap A = A$$

- **Complementation law**

$$\overline{(\overline{A})} = A$$

- **De Morgan's laws**

$$\overline{A \cup B} = \overline{A} \cap \overline{B} \quad \overline{A \cap B} = \overline{A} \cup \overline{B}$$

- **Absorption laws**

$$A \cup (A \cap B) = A \quad A \cap (A \cup B) = A$$

- **Commutative laws**

$$A \cup B = B \cup A$$

$$A \cap B = B \cap A$$

- **Associative laws**

$$A \cup (B \cup C) = (A \cup B) \cup C$$

$$A \cap (B \cap C) = (A \cap B) \cap C$$

- **Distributive laws**

$$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$$

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

- **Complement laws**

$$A \cup \overline{A} = U$$

$$A \cap \overline{A} = \emptyset$$

PROVING SET IDENTITIES

Different ways to prove set identities:

1. Prove that each set (side of the identity) is a subset of the other.
2. Use set builder notation and propositional logic.
3. **Membership Tables:** Verify that elements in the same combination of sets always either belong or do not belong to the same side of the identity. Use 1 to indicate it is in the set and a 0 to indicate that it is not.

Example: Prove that $\overline{A \cap B} = \overline{A} \cup \overline{B}$

Solution: We prove this identity by showing that:

$\overline{A \cap B} \subseteq \overline{A} \cup \overline{B}$ and $\overline{A} \cup \overline{B} \subseteq \overline{A \cap B}$

PROOF OF SECOND DE MORGAN LAW

$\overline{A \cap B} \subseteq \overline{A} \cup \overline{B}$

$x \in \overline{A \cap B}$	by assumption
$x \notin A \cap B$	defn. of complement
$\neg((x \in A) \wedge (x \in B))$	defn. of intersection
$\neg(x \in A) \vee \neg(x \in B)$	1st De Morgan Law for Prop Logic
$x \notin A \vee x \notin B$	defn. of negation
$x \in \overline{A} \vee x \in \overline{B}$	defn. of complement
$x \in \overline{A} \cup \overline{B}$	defn. of union

$\overline{A} \cup \overline{B} \subseteq \overline{A \cap B}$

$x \in \overline{A} \cup \overline{B}$	by assumption
$(x \in \overline{A}) \vee (x \in \overline{B})$	defn. of union
$(x \notin A) \vee (x \notin B)$	defn. of complement
$\neg(x \in A) \vee \neg(x \in B)$	defn. of negation
$\neg((x \in A) \wedge (x \in B))$	by 1st De Morgan Law for Prop Logic
$\neg(x \in A \cap B)$	defn. of intersection
$x \in \overline{A \cap B}$	defn. of complement

SET-BUILDER NOTATION: SECOND DE MORGAN LAW

$$
\begin{array}{rcll}
\overline{A \cap B} & = & \{x | x \notin A \cap B\} & \text{by defn. of complement} \\
& = & \{x | \neg(x \in (A \cap B))\} & \text{by defn. of does not belong symbol} \\
& = & \{x | \neg(x \in A \wedge x \in B\} & \text{by defn. of intersection} \\
& = & \{x | \neg(x \in A) \vee \neg(x \in B)\} & \text{by 1st De Morgan law} \\
& & & \text{for Prop Logic} \\
& = & \{x | x \notin A \vee x \notin B\} & \text{by defn. of not belong symbol} \\
& = & \{x | x \in \overline{A} \vee x \in \overline{B}\} & \text{by defn. of complement} \\
& = & \{x | x \in \overline{A} \cup \overline{B}\} & \text{by defn. of union} \\
& = & \overline{A} \cup \overline{B} & \text{by meaning of notation}
\end{array}
$$

MEMBERSHIP TABLE

Example: Construct a membership table to show that the distributive law holds.

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

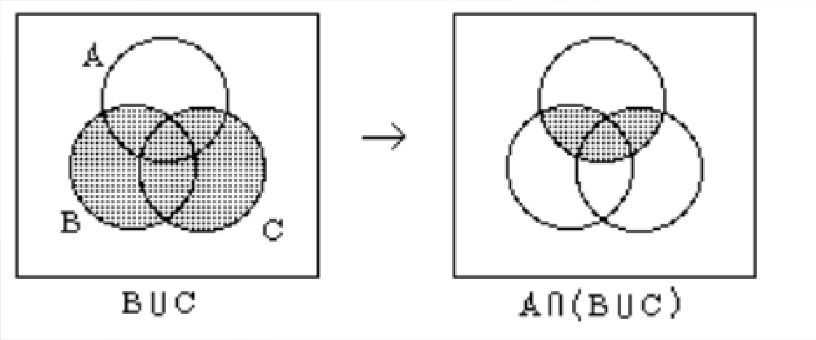
Solution:

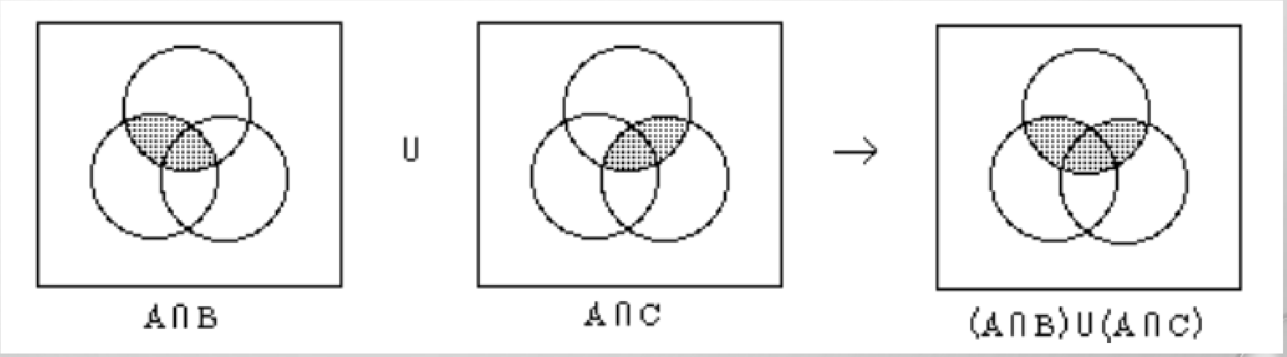
A	B	C	$B \cap C$	$A \cup (B \cap C)$	$A \cup B$	$A \cup C$	$(A \cup B) \cap (A \cup C)$
1	1	1	1	1	1	1	1
1	1	0	0	1	1	1	1
1	0	1	0	1	1	1	1
1	0	0	0	1	1	1	1
0	1	1	1	1	1	1	1
0	1	0	0	0	1	0	0
0	0	1	0	0	0	1	0
0	0	0	0	0	0	0	0

PROOF USING VENN DIAGRAM

Example: Prove that $A \cap (BUC) = (A \cap B)U\ (A \cap C)$

Solution:





GENERALIZED UNIONS AND INTERSECTIONS

- Let A_1, A_2 ,…, A_n be an indexed collection of sets.

  We define:
  $$\bigcup_{i=1}^{n} A_i = A_1 \cup A_2 \cup \ldots \cup A_n$$
  $$\bigcap_{i=1}^{n} A_i = A_1 \cap A_2 \cap \ldots \cap A_n$$

  These are well defined, since union and intersection are associative.

- For i = 1,2,…, let A_i = {i, i + 1, i + 2, ….}. Then,

$$\bigcup_{i=1}^{n} A_i = \bigcup_{i=1}^{n} \{i, i+1, i+2, ...\} = \{1, 2, 3, ...\}$$

$$\bigcap_{i=1}^{n} A_i = \bigcap_{i=1}^{n} \{i, i+1, i+2, ...\} = \{n, n+1, n+2, ...\} = A_n$$