Computer Science Department:
Discrete Structure and Theory
FINAL EXAM PRACTICE SET
December 2nd, 2024

1. Show that $\neg(p \to q) \to \neg q$ is a tautology
   a) using a truth table.
   b) using logical equivalences.

2. Using logical equivalences, prove the second absorption law $p \wedge (p \vee q) \equiv p$.
   Do not use the first absorption law in your proof.

3. Translate these statements into English, where $C(x)$ is "x is a comedian" and $F(x)$ is "x is funny" and the domain consists of all people.
   a) $\forall x(C(x) \to F(x))$
   b) $\forall x(C(x) \wedge F(x))$
   c) $\exists x(C(x) \to F(x))$
   d) $\exists x(C(x) \wedge F(x))$

4. Suppose the domain of the propositional function $P(x, y)$ consists of pairs x and y, where x is 2, or 3 and y is $1, 2$, or 3. Write out these propositions using disjunctions and conjunctions.
   a) $\forall x \forall y P(x, y)$ b) $\exists x \exists y P(x, y)$
   c) $\exists x \forall y P(x, y)$ d) $\forall y \neg P(3, y)$

5. Assume that the universe for x is all people and the universe for y is the set of all movies. Write the English statement using the following predicates and any needed quantifiers:
   $S(x, y)$: x saw y
   $L(x, y)$: x liked y
   $A(y)$: y won an award
   $C(y)$: y is a comedy.

   a) No comedy won an award.
   b) Lois saw Casablanca but did not like it.
   c) Some people have seen every comedy.
   d) No one liked every movie he has seen.
   e) Ben has never seen a movie that won an award.

6. Show that the following argument is valid:
   "All movies produced by John Sayles are wonderful. John Sayles produced a movie about coal miners. Therefore, there is a wonderful movie about coal miners."

7. Show that the following argument is valid:
   "Nana, a student in this class, has visited Egypt. Everyone who visits Egypt takes a camel ride. Therefore, someone in this class has taken a camel ride."

8. Prove each of the following statements:
   a) The sum of two rational numbers is rational.
   b) Let n be an integer. Show that n is even if and only if $3n - 11$ is odd.

9. Show that if $A \subseteq C$ and $B \subseteq D$, then $A \times B \subseteq C \times D$.

10. Show that if A, B, and C are sets, then
$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$

11. The successor of the set A is the set $A \cup \{A\}$. Find the successors of the following sets.

    a) $\{1,2,3\}$ b) $\emptyset$

    c) $\{\emptyset\}$ d) $\{\emptyset, \{\emptyset\}\}$

12. Using set identities, prove the first absorption law $A \cup (A \cap B) = A$.
    Do not use the second absorption law in your proof.

13. Use set builder notation and logical equivalences to prove that $A \cap (B - A) = \emptyset$.

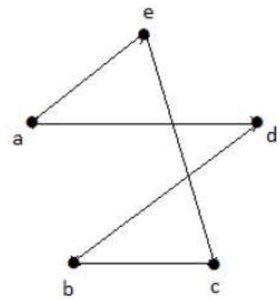
14. Consider the Boolean function $F(x, y, z) = x + y + z$.
a) Use a truth table to express the values of $F(x, y, z)$.
b) Using the result from (a), write $F(x, y, z)$ in disjunctive normal form.
c) Using the result from (a), write $F(x, y, z)$ in conjunctive normal form.

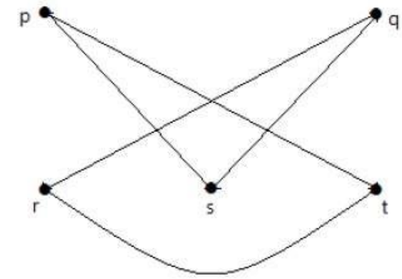
15. What values of the Boolean variables x and y satisfy $xy = x + y$?

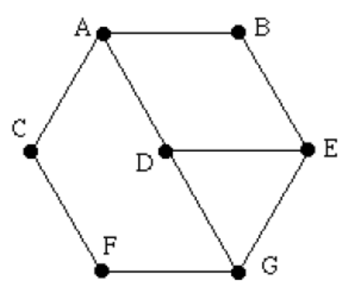
16. Construct a circuit from inverters, AND gates, and OR gates to produce the output:
$$(\bar{x} + y + \bar{z})\overline{yz}$$

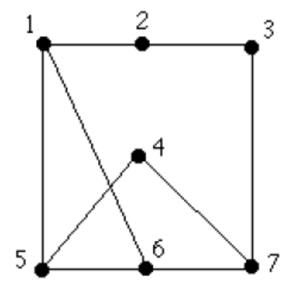
17. The complementary graph $\bar{G}$ of a simple graph G has the same vertices as G. Two vertices are adjacent in $\bar{G}$ if and only if they are not adjacent in G.
    If a simple graph G has n vertices and m edges, how many edges does $\bar{G}$ have?

18. Determine whether the given pair of graphs is isomorphic. Exhibit an isomorphism or provide a rigorous argument that none exists.



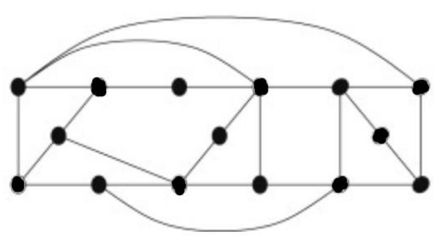






19. Draw the undirected graph with adjacency matrix $\begin{pmatrix} 0 & 1 & 3 & 0 & 4 \\ 1 & 2 & 1 & 3 & 0 \\ 3 & 1 & 1 & 0 & 1 \\ 0 & 3 & 0 & 0 & 2 \\ 4 & 0 & 1 & 2 & 3 \end{pmatrix}$

20. Is the following graph bipartite?



21. Use mathematical induction to prove that whenever n is a positive integer,

$$1^3 + 2^3 + \cdots + n^3 = (n(n+1)/2)^2.$$

22. Use mathematical induction to prove that whenever n is a nonnegative integer,

$$2 - 2 \times 7 + 2 \times 7^2 - \cdots + 2 \times (-7)^n = (1 - (-7)^{n+1})/4.$$

23. Prove by induction that $\sum_{j=1}^{n} \frac{1}{2^j} = \frac{(2^n - 1)}{2^n}$.

24. How many strings of eight uppercase English letters are there
a) if letters can be repeated?
b) if no letter can be repeated?
c) that start with X, if letters can be repeated?
d) that start with X, if no letter can be repeated?
e) that start and end with X, if letters can be repeated?
f) that start with the letters BO (in that order), if letters can be repeated?
g) that start and end with the letters BO (in that order), if letters can be repeated?
h) that start or end with the letters BO (in that order), if letters can be repeated?

25. Suppose that a password for a computer system must have at least 8, but no more than 12, characters, where each character in the password is a lowercase English letter, an uppercase English letter, a digit, or one of the six special characters $*, >, <, !, +$ and $=$.
   a) How many different passwords are available for this computer system?
   b) How many of these passwords contain at least one occurrence of at least one of the six special characters?
   c) Using your answer to part (a), determine how long it takes a hacker to try every possible password, assuming that it takes one nanosecond for a hacker to check each possible password.

26. What is the minimum number of students, each of whom comes from one of the 50 states, who must be enrolled in a university to guarantee that there are at least 100 who come from the same state?