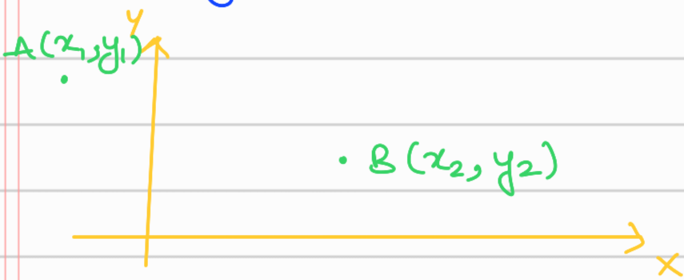
In Calculus of Variation our intention is to find a Stationary Functions of a Functional $I[f]$, [Function of functions)

Deriving the Euler Lagrange Equations:-

Find $y = f(x)$ such that the functional

$$\boxed{I = \int_{x_1}^{x_2} F(x, y, y')\, dx}$$ is stationary —(a)

Boundary conditions:- $y(x_1) = y_1$ & $y(x_2) = y_2$



Suppose $y(x)$ makes I stationary and satisfies the above boundary conditions

$y(x)$ = Extremal func^n

- Introduce a Func^n $\eta(x)$, $\eta(x_1) = \eta(x_2) = 0$

Implicit:- All func^n's have continuous 2^{nd} derivatives.

- We Define $\bar{y}(x) = y(x) + \varepsilon\eta(x)$

Here '$\bar{y}$' is an Extension of $y(x)$, the only restrictions being put on '$\bar{y}$' are the one that are already being put on 'y' and 'η' earlier usally Boundary Conditions.

$$\bar{y}(x_1) = y(x_1) + \varepsilon\eta(x_1) = y(x_1) + \varepsilon(0) = y(x_1) = y_1$$

$$\Rightarrow \boxed{\bar{y}(x_1) = y_1}$$

and

$$\bar{y}(x_2) = y(x_2) + \varepsilon\eta(x_2) = y(x_2) + \varepsilon \times (0) = y(x_2) = y_2$$

$$\Rightarrow \boxed{\bar{y}(x_2) = y_2}$$

$\Rightarrow \boxed{\bar{y}(x_2) = y_2}$

From the Above Highlighted box we can clearly see that boundary conditions of y & $\bar{y}$ are same.

→ '$\bar{y}$' can be interpreted as a mathematical Entity of a family of curves

* Here our goal is to find a particular member of this family of curves ($\bar{y}$) which make '$I = \int_{x_1}^{x_2} F(x, \bar{y}, \bar{y}')\,dx$' stationary $\left(\bar{y}' = \frac{d\bar{y}}{dx}\right)$

'$I(\varepsilon)$' Here I is a func^n of epsilon (ε) only because After the Integral I finally gets Integrated only 'ε' remains as lower & upper Limits are put.

Our primary object is to make 'I' stationary this can be achieved By letting $\left.\frac{dI}{d\varepsilon}\right|_{\varepsilon=0} = 0$

Because as at $\varepsilon = 0$ '$\bar{y}$' becomes 'y' which is in our start is the func^n which makes I stationary corresponds to '$\varepsilon = 0$'

$$\Rightarrow I = \int_{x_1}^{x_2} F(x, \bar{y}, \bar{y}')\,dx$$

$$\Rightarrow \frac{dI}{d\varepsilon} = \frac{d}{d\varepsilon}\int_{x_1}^{x_2} F(x, \bar{y}, \bar{y}')\,dx$$

$$\Rightarrow \frac{dI}{d\varepsilon} = \int_{x_1}^{x_2}\left\{\frac{\partial}{\partial\varepsilon}\left(F(x, \bar{y}, \bar{y}')\right)\right\}dx$$

Here '$\bar{y}$' and '$\bar{y}'$' are only variables in F which depends upon ε but not x $\Rightarrow \frac{\partial x}{\partial \varepsilon} = 0$

x_2

$$\Rightarrow \frac{dI}{d\varepsilon} = \int_{x_1} \left\{ \frac{\partial F}{\partial \bar{y}} \cdot \frac{\partial \bar{y}}{\partial \varepsilon} + \frac{\partial F}{\partial \bar{y}'} \cdot \frac{\partial \bar{y}}{\partial \varepsilon} \right\} dx$$

Now $\bar{y} = y + \varepsilon \eta(x) \Rightarrow \bar{y}' = y' + \varepsilon \eta'(x)$

$$\bar{y}' = \frac{\partial y'}{\partial x} \Rightarrow \frac{\partial \bar{y}}{\partial \varepsilon} = \eta(x) \text{ and } \frac{\partial \bar{y}'}{\partial \varepsilon} = \eta'(x)$$

$$\Rightarrow \left.\frac{dI}{d\varepsilon}\right|_{\varepsilon=0} = \int_{x_1}^{x_2} \left\{ \frac{\partial F}{\partial \bar{y}} \cdot \eta + \underbrace{\frac{\partial F}{\partial \bar{y}'} \eta'}_{} \right\} dx \Big|_{\varepsilon=0} = 0$$

Integrating this term By 'By parts'

$$\Rightarrow \int_{x_1}^{x_2} \frac{\partial F}{\partial \bar{y}'} \eta' \, dx = \left[\frac{\partial F}{\partial \bar{y}'} \cancel{\eta} \right]_{x_1}^{x_2} - \int \frac{\partial}{\partial x}\left(\frac{\partial F}{\partial \bar{y}'}\right) \eta \, dx$$

$\eta(x_1) = \eta(x_2) = 0$

$$\boxed{\Rightarrow \int_{x_1}^{x_2} \frac{\partial F}{\partial \bar{y}'} \eta' dx = -\int_{x_1}^{x_2} \eta \frac{d}{dx}\left(\frac{\partial F}{\partial \bar{y}'}\right) dx}$$

$$\Rightarrow \left.\frac{dI}{d\varepsilon}\right|_{\varepsilon=0} = \int_{x_1}^{x_2} \left\{ \frac{\partial F}{\partial \bar{y}} \cdot \eta - \eta \frac{d}{dx}\left(\frac{\partial F}{\partial \bar{y}'}\right) \right\} dx \Big|_{\varepsilon=0} = 0$$

$$\left.\frac{dI}{d\varepsilon}\right|_{\varepsilon=0} = \int_{x_1}^{x_2} \eta \left[\frac{\partial F}{\partial \bar{y}} - \frac{d}{dx}\left(\frac{\partial F}{\partial \bar{y}'}\right) \right] dx \Big|_{\varepsilon=0} = 0$$

Here The Above Integral eq^n can only be zero if The term in the brackets Becomes Zero as $\eta(x)$ is Some arbitrary func^n

& At $\varepsilon = 0$ $\bar{y} = y$ & $\bar{y}' = y'$

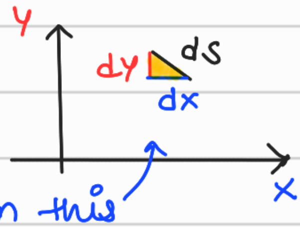
$$\boxed{\Rightarrow \frac{\partial F}{\partial y} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0}$$

Euler Lagrange's Eq^n

Not a Sufficient Condition

→ Now if we Need to find geodesic for two points on a plane surface

Here small arc length ds can we Given as $\boxed{ds^2 = dx^2 + dy^2}$

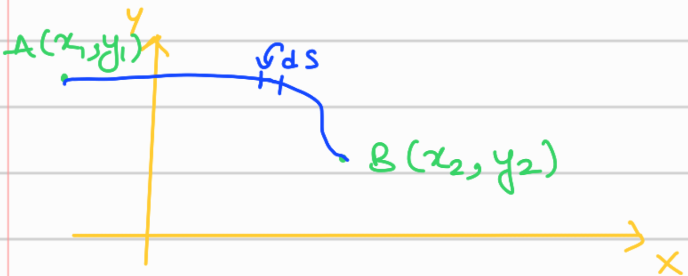


(using Pythagorus theorem in this Elemental triangle

$$\Rightarrow \quad ds = \sqrt{dx^2 + dy^2}$$ (As Length is always positive)

$$\Rightarrow \quad ds = \left(\sqrt{1+\left(\frac{dy}{dx}\right)^2}\right) dx$$

Now Inorder to find arc length between two points Along some particular path we have



$$I = \int_A^B ds$$

$$I = \int_A^B \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx$$

Now on comparing this Eqⁿ with eqⁿ (4) we get

$$F(x, y, y') = \sqrt{1+(y')^2} \qquad \left(\text{Here } y' = \frac{dy}{dx}\right)$$

Now putting value of 'F' in Euler lagrange equation we get.

$$\frac{\partial F}{\partial x} - \frac{d}{dx}\left(\frac{\partial F}{\partial y'}\right) = 0$$

$$\Rightarrow \quad \frac{\partial F}{\partial x} = 0$$ ($\because$ 'F' in this case is not a funcⁿ of x hence it's partial derivative is zero)

$$\& \quad \frac{\partial F}{\partial y'} = \frac{1}{2\sqrt{1+(y')^2}} \cdot (2y') = \frac{y'}{\sqrt{1+(y')^2}}$$

$$\Rightarrow \quad 0 - \frac{d}{dx}\left(\frac{y'}{\sqrt{1+(y')^2}}\right) = 0$$

$$\Rightarrow \quad \frac{y'}{\sqrt{1+(y')^2}} = k \quad \text{(some constant)}$$

(Because derivative of a constant is zero)

$$\Rightarrow \quad (y')^2 = k^2(1+(y')^2)$$

$$\Rightarrow \quad (y')^2 = \frac{k^2}{1-k^2}$$

$$\Rightarrow \quad y' = \frac{dy}{dx} = \sqrt{\frac{k^2}{1-k^2}}$$

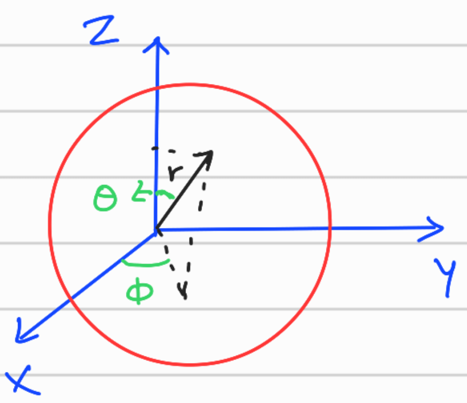
Let $\sqrt{\frac{k^2}{1-k^2}} = c$ (some another arbitrary constant $\forall\ k \neq \pm 1$)

$$\Rightarrow \quad \frac{dy}{dx} = c \qquad \Rightarrow \int dy = \int c\,dx$$

$$\Rightarrow \quad \boxed{y = cx + D}$$

Eq^n of a straight line (Here D is some arbitrary constant)

→ Now if we need to find geodesic for two points on a spherical surface



From the figure we can clearly see that

$$x = r\sin\theta\cos\phi = x(r,\theta,\phi) - (A_0)$$

$$y = r\sin\theta\sin\phi = y(r,\theta,\phi) - (B_0)$$

$$z = r\cos\theta = z(r,\theta) - (C_0)$$

Now, Arc length in Cartesian co-ordinate is given as $\boxed{ds^2 = dx^2 + dy^2 + dz^2}$ — (δ)

Now from eq^n (A_0) we get

$$dx = \frac{\partial x}{\partial r} dr + \frac{\partial x}{\partial \theta} d\theta + \frac{\partial x}{\partial \phi} d\phi$$

$$\Rightarrow \boxed{dx = \sin\theta\cos\phi\, dr + r\cos\theta\sin\phi\, d\theta - r\sin\theta\sin\phi\, d\phi} \quad -(\alpha)$$

From eq^n (B_0)

$$dy = \frac{\partial y}{\partial r} dr + \frac{\partial y}{\partial \theta} d\theta + \frac{\partial y}{\partial \phi} d\phi$$

$$\Rightarrow \boxed{dy = \sin\theta\sin\phi\, dr + r\cos\theta\sin\phi\, d\theta + r\sin\theta\cos\phi\, d\phi} \quad -(\beta)$$

From eq^n (C_0)

$$dz = \frac{\partial z}{\partial r} dr + \frac{\partial z}{\partial \theta} d\theta$$

$$\Rightarrow \boxed{dz = \cos\theta\, dr - r\sin\theta\, d\theta} \quad -(\gamma)$$

Now putting values of 'dx', 'dy' and 'dz' from (α), (β) and (γ) in eq^n (δ)

$$\Rightarrow ds^2 = r^2 d\theta^2 + r^2\sin^2\theta\, d\phi^2$$

(Here as along the surface of sphere radius remains constant $\Rightarrow$ $r = R$ (constant) $\Rightarrow$ $dr = 0$)

$$\Rightarrow I = \int_A^B ds = \int_{\theta_1}^{\theta_2} \sqrt{r^2 + r^2\sin^2\theta\left(\frac{d\phi}{d\theta}\right)^2}\, d\theta$$

$$I = \int_{\theta_1}^{\theta_2} r\sqrt{1 + \sin^2\theta(\phi')^2}\, d\theta \qquad \left[\text{Here } \phi' = \frac{d\phi}{d\theta}\right]$$

On comparing this eq^n with eq^n (ω) we get

$$F(\theta, \phi) = r\sqrt{1 + \sin^2\theta \cdot (\phi')^2}$$

(Here variables (x, y) are

Now putting the value of F in Euler Lagrange's Eq^n we get

Replaced by (θ, ϕ)

$$\Rightarrow \frac{\partial F}{\partial \phi} - \frac{d}{d\theta}\left(\frac{\partial F}{\partial \phi'}\right) = 0$$

Now, $\frac{\partial F}{\partial \phi} = 0$

AND

$$\frac{\partial F}{\partial \phi'} = \frac{1}{2\sqrt{1+\sin^2\theta \cdot (\phi')^2}} \cdot (2\sin^2\theta(\phi')) = \frac{\sin^2\theta(\phi')}{\sqrt{1+\sin^2\theta(\phi')^2}}$$

$$\Rightarrow 0 - \frac{d}{d\theta}\left[\frac{\sin^2\theta(\phi')}{\sqrt{1+\sin^2\theta(\phi')^2}}\right] = 0$$

$$\Rightarrow \frac{\sin^2\theta(\phi')}{\sqrt{1+\sin^2\theta(\phi')^2}} = k$$ (Arbitrary constant)

$$\Rightarrow \left(\sin^2\theta(\phi')\right)^2 = \left\{k\left(\sqrt{1+\sin^2\theta(\phi')^2}\right)\right\}^2$$

$$\Rightarrow \sin^4\theta(\phi')^2 = k^2\left(1+\sin^2\theta(\phi')^2\right)$$

$$\Rightarrow (\sin^2\theta - k^2)\sin^2\theta(\phi')^2 = k^2$$

$$\Rightarrow \phi' = \sqrt{\frac{k^2}{(\sin^2\theta - k^2)\sin^2\theta}}$$

$$\Rightarrow \phi' = \frac{d\phi}{d\theta} = \frac{k}{\sin\theta\sqrt{\sin^2\theta - k^2}}$$

$$\Rightarrow d\phi = \frac{k\, d\theta}{\sin\theta\sqrt{\sin^2\theta - k^2}}$$

Now Integrating both sides

Now let $\boxed{\omega = \cot\theta}$ — (B₁) $\Rightarrow$ $\boxed{d\omega = -\operatorname{cosec}^2\theta\, d\theta}$ — (A₁)

$$\Rightarrow \int d\phi = \int \frac{k}{\sin^2\theta} \frac{d\theta}{\sqrt{1-(k^2/\sin^2\theta)}}$$

Now as $\frac{1}{\sin\theta} = \operatorname{cosec}\theta$

$$\Rightarrow \int d\phi = \int \frac{k \operatorname{cosec}^2\theta\, d\theta}{\sqrt{1-k^2\operatorname{cosec}^2\theta}}$$

$$\Rightarrow \int d\phi = \int \frac{-k(-\operatorname{cosec}^2\theta\, d\theta)}{\sqrt{1-k^2(1+\cot^2\theta)}}$$

Now from eqⁿ (A₁) and eqⁿ (B₁) we get

$$\Rightarrow \int d\phi = \int \frac{-k\, d\omega}{\sqrt{1-k^2(1+\omega^2)}}$$

$$\int d\phi = \int \frac{-k\, d\omega}{\sqrt{(1-k^2) - k^2\omega^2}}$$

Now dividing L.H.S's numerator & Denominator by k

$$\Rightarrow \int d\phi = \int \frac{-d\omega}{\sqrt{\left(\frac{1-k^2}{k^2}\right) - \omega^2}}$$

let $\frac{(1-k^2)}{k^2} = \psi^2$

$$\Rightarrow \int d\phi = -\int \frac{d\omega}{\sqrt{\psi^2 - \omega^2}}$$

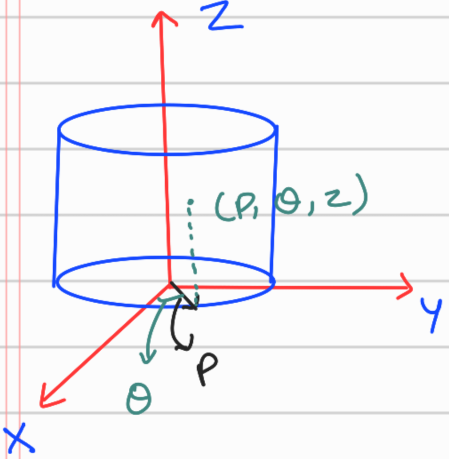
$$\phi = \cos\left(\frac{\omega}{\psi}\right) + \phi_0$$

ϕ_0 → Integration constant

$$\Rightarrow \boxed{\omega = \psi \cos(\phi - \phi_0)}$$

This is the eq^n of Great circle.

→ Now if we Need to find geodesic for two points on a cylinderical surface



we can clearly see from the diagram that

$$x = \rho\cos\theta = x(\rho,\theta) \quad - (A_2)$$
$$y = \rho\sin\theta = y(\rho,\theta) \quad - (B_2)$$
$$z = z = z(z) \quad - (C_2)$$

Now From eq^n (A_2) we get

$$dx = \frac{\partial x}{\partial \rho}d\rho + \frac{\partial x}{\partial \theta}d\theta$$

$$\boxed{dx = \cos\theta\, d\rho - \rho\sin\theta\, d\theta} \quad - (A_3)$$

Now From eq^n (B_2) we get

$$dy = \frac{\partial y}{\partial \rho}d\rho + \frac{\partial y}{\partial \theta}d\theta$$

$$\Rightarrow \boxed{dy = \sin\theta\, d\rho + \rho\cos\theta\, d\theta} \quad - (B_3)$$

Now From eq^n (C_2) we get

$$dz = dz$$

Now From eq^n (δ) we get

Now from eq (?) we get

$ds^2 = dx^2 + dy^2 + dz^2$

Now putting values from eqⁿ A_3, B_3 and C_3 of dx, dy and dz in the Above eqⁿ we get

$\Rightarrow ds^2 = (dp)^2 + (p\,d\theta)^2 + (dz)^2$

Now As we Have to move along on the surface of this cylinder Hence we have 'p = constant' $\Rightarrow$ $\boxed{dp = 0}$

$$\Rightarrow ds^2 = p^2 d\theta^2 + (dz)^2$$

$$\Rightarrow I = \int_{\theta_1}^{\theta_2} ds = \int_{\theta_1}^{\theta_2} \sqrt{p^2 d\theta + dz^2}$$

$$\Rightarrow I = \int_{\theta_1}^{\theta_2} \sqrt{p^2 + \left(\frac{dz}{d\theta}\right)^2}\, d\theta$$

$$\Rightarrow I = \int_{\theta_1}^{\theta_2} \sqrt{p^2 + (z')^2}\, d\theta$$

$z' = \dfrac{dz}{d\theta}$

On comparing the above eqⁿ with eqⁿ (4) we get

$$\Rightarrow F = \sqrt{p^2 + (z')^2}$$

Now putting the value of F in Euler Lagrange's eqⁿ we get

$$\frac{\partial F}{\partial z} - \frac{d}{d\theta}\left(\frac{\partial F}{\partial z'}\right) = 0$$

$$\frac{\partial F}{\partial z} = 0 \quad \text{and} \quad \frac{\partial F}{\partial z'} = \frac{1 \quad (2z')}{\sqrt{p^2 + (z')^2}} = \frac{z'}{\sqrt{p^2 + (z')^2}}$$

$$\Rightarrow 0 - \frac{d}{d\theta}\left(\frac{z'}{\sqrt{p^2+(z')^2}}\right) = 0$$

$$\Rightarrow \frac{z'}{\sqrt{p^2+(z')^2}} = k$$ (Arbitrary constant)

$$\Rightarrow (z')^2 = k^2\left(p^2+(z')^2\right)$$

$$\Rightarrow z' = \sqrt{\frac{k^2p^2}{1-k^2}}$$

$$\Rightarrow \frac{dz}{d\theta} = \frac{kp}{\sqrt{1-k^2}}$$

$$\Rightarrow dz = \frac{kp \, d\theta}{\sqrt{1-k^2}}$$

$$\Rightarrow \boxed{z = \left(\frac{kp}{\sqrt{1-k^2}}\right)\theta + z_0}$$

(Integration constant)

eq^n of a Helical Path.