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Modeling and managing portfolios including listed private equity

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A B S T R A C T

Listed private equity (LPE) provides investors with a liquid means of considering private equity in their portfolios. This paper presents a first-order autoregressive Markov-switching model (ARMS) which is able to capture the characteristics of the asset classes bonds, stocks, and LPE, such as heavy tails and autocorrelation. Optimizing a portfolio between bonds, stocks, and LPE shows that an investor benefits from including LPE due to the high diversification effects, which also holds for a very risk-averse investor. Allocating a portfolio with the presented Markov-switching optimization can help to significantly outperform a portfolio which is optimized assuming an underlying geometric Brownian motion (GBM) – even during the financial crisis: The terminal value of a portfolio of a model investor with medium risk aversion was on average 8.7% higher over the three years 2007–2009 than the GBM portfolio.

1. Introduction

Due to its interesting risk-return profile, private equity has increasingly attracted attention of institutional investors. However, the access to investing in private equity is limited, and private equity does not belong to the liquid asset classes, because there exists no such secondary market as it is the case for stocks and bonds. Therefore, listed private equity (LPE) vehicles provide for a more liquid means of investing in this promising asset class to a broad range of investors. But what are the effects of including LPE in a portfolio, and which fraction of his portfolio should an investor allot to LPE? So far, there has not been much academic research on this topic. The most recent studies are those of Zimmermann et al. [1] and the dissertation of Bilo [2]. In both studies, a basket (or index) of publicly traded private equity vehicles is constructed and both had to deal with liquidity constraints. Furthermore, the aforementioned studies did not account for the significant non-normal higher moments in return distributions of LPE. Although Bilo identified a negative skewness and an excess kurtosis, these facts were not heeded for performance measurement.

This study is based on the LPX 50 which represents the listed private equity market. The LPX 50 is a global index consisting of the 50 largest liquid LPE companies published by LPX GmbH. There are two main advantages by examining this index: First, there is no need to create a basket of LPE companies, as the index already represents the LPE market. Second – what is even more important – problems of illiquidity are negligible because there is a growing number of liquid vehicles such as open end index tracker certificates and even exchange traded funds.

As the set of the examined return time series is significantly non-normally distributed and/or autocorrelated, it is not sufficient to describe the assets with a GBM model (geometric Brownian motion) using normally distributed log-returns. Thus, we employ a Markov-switching model which can map the stylized facts of LPE. With the help of a GBM, we analyze the following questions: How can we determine the optimal fraction of LPE in a portfolio? What are the impacts of including LPE in a portfolio, what are the diversification effects? How did a portfolio perform which was allocated according to the rules derived from the model during the financial crisis, when stocks and, in particular, LPE were suffering high losses?

2. Markov-switching model

The most straight-forward and common approach to model the returns of an asset class is via a geometric Brownian motion (GBM) as it is the case, for example, in the Black–Scholes model (see [3]). This model assumes the log-returns to be normally distributed. However, history has shown – in particular recent history with the financial crisis – that the normal distribution underestimates the occurrence of extreme events in the tails. Furthermore, a geometric Brownian motion does not capture the characteristics that the returns of an asset class like listed private equity display such a high kurtosis (corresponding to the fatter tails) and autocorrelation (see Section 4).

For this purpose, a Markov-switching model is used. As motivated above, the model is able to take both autocorrelation

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and non-normality into account at the same time. There are numerous studies emphasizing the benefits of Markov-switching models in describing economic time series.[1]

2.1. The market model

The Markov-switching process which is used to model the asset class returns is defined as follows:

$$r_{t,a} = \mu_{s_t,a} + \phi \cdot (r_{t-1,a} - \mu_{s_{t-1},a}) + \varepsilon_{t,a}, \tag{1}$$

where s_t denotes the state of markets at time t, $\mu_{s_t,a}$ is the mean return of asset $a \in \{1,2,3\}$ (where $1 \mathrel{\hat=}$ bonds, $2 \mathrel{\hat=}$ stocks, $3 \mathrel{\hat=}$ LPE in the application depicted in this study) in state s_t, $\varepsilon_t = (\varepsilon_{t,1},\varepsilon_{t,2},\varepsilon_{t,3})^T$ the innovation at time t, $\varepsilon_t \sim N(0,\Sigma_{s_t})$, and ϕ the autocorrelation parameter satisfying $|\phi| < 1$. This describes a Markov-switching model allowing for state-independent (first lag) autoregressive dynamics.

The first-order autoregressive Markov-switching (ARMS) model allows for two possible states in this study: For $s_t = 1$, the markets are in state 1, and if they are in state 2 then $s_t = 2$. The changes of the state s_t over time are modeled by a Markov chain. The transition matrix containing the probabilities of "switching" from state i at time $t-1$ into state j at time t is the following:

$$P = \begin{pmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{pmatrix} \quad \text{where} \quad P(s_t = j | s_{t-1} = i) = p_{ij}.$$

The entire process is then described by the parameter vector:

$$\Theta = (\mu_1, \mu_2, p_{12}, p_{21}, \phi, \Sigma_1, \Sigma_2) \tag{2}$$

with $\mu_1, \mu_2 \in \mathbb{R}^3$, $p_{12}, p_{21} \in [0,1]$, $\phi \in [-1,1]^3$, and $\Sigma_1, \Sigma_2 \in \mathbb{R}^{3\times 3}$ the covariance matrices for the two states 1 and 2.

This model slightly differs from the models which were used in other data studies so far. For example, the Markov-switching model in Höcht et al. [6] first applies univariate processes for each asset and then includes the correlation structure between the asset returns by generating multivariate normally distributed residuals. In other words, whereas the model in this paper uses overall market states, Höcht et al. [6] resorts to single states for each asset class. This allows for all possible combinations of different individual market states, where each asset class can be in one of two states.[2]

Depending on the data that is to be described, this can bear one drawback: The correlation can only be captured in the residuals, namely the Σ_{s_t}. Therefore, the model applied by Höcht et al. [6] does not allow for correlations as high as measured between stocks and private equity, which is due to the fact that the final correlation only partially depends on the "innovation" and the states are uncorrelated.[3]

Also in order to allow for a high correlation between asset classes, instead of modeling independent Markov chains, in this study, only two meta-states are modeled: $s_t = 1$ and 2, i.e. all assets are in the same state at each point in time t. Of course, the more market states are taken into account (it is also possible to model three or even more market states), the better the empirical moments and autocorrelations may be matched. However, the (quadratic) increase of the parameter vector itself causes the fitting method to be less efficient. Furthermore, over-fitting problems are more likely to occur. Therefore, as few states as possible should be used which still map the characteristics of the assets classes well. As it can be seen in Table 3, two market states already enable the model to match the empirical parameters sufficiently well.

[1] For more details, consult Timmermann [4]; applications of Markov-switching models can be found in, for example, Brunner and Hafner [5] or Höcht et al. [6].

[2] For n assets, where each asset can be in two different states independent of each other, there are 2^n "meta" states.

[3] Applying this approach to the problem set depicted in this study would yield to theoretical correlations (the actual theoretical correlations for the conducted study are presented in Table 4) which are only about 50% of the empirical correlations.

2.2. Estimation of the model parameters

In order to fit the parameters Θ of the model to the time series, the method of moments is applied: The optimal set of parameters is chosen such that the theoretical statistics of the Markov-switching process match the empirical statistics best possible (in terms of squared deviation).

Table 2 in Section 4 displays the fitted parameters of the ARMS model (1) for three assets. The parameters of the three-dimensional ARMS model were gained the following way: Timmermann [4] provides estimators for the first four central moments, the autocorrelation, as well as the bivariate correlations. These moment estimators are functions of the parameters which are to be fitted. The method of moments now minimizes the error function which is defined as the sum of squared residuals between the empirical moments and the theoretical moments subject of the fitted parameters. No specific constraints are imposed for the fitting algorithm other than "numerical" boundaries, such as probabilities between 0 and 1, state-dependent means between -1 and 1, volatilities between 0 and 1, and correlations between -1 and 1. However, no solution is a boundary solution, i.e. the boundaries remained untouched for the optimal set of parameters. Positive semidefinite is also not claimed by constraints, yet the resulting matrices are checked to fulfill this condition. Due to the high complexity of this optimization problem, it is solved 100,000 times, each time with a random set of initial parameters. Finally, the optimal solution is that set of parameters among all with the smallest error.

3. Portfolio optimization

The aim of this study is to determine the optimal allocation of a portfolio. So far, we can describe the processes of the asset classes as an ARMS processes (1) via the parameters Θ. In order to derive the optimal allocation, multiple return series are simulated by Monte Carlo methods. The procedure is as follows: A model describing the return time series as well as possible is built. In this study, we model returns as depicted in (1). As future return time series are not predictable, this model hinges on random processes, namely the $\varepsilon_{t,a}$ and switching of states. In a next step, numerous (in this case 10,000) return paths are simulated applying this model. These return paths exhibit the same stylized facts as the historical time series which are taken into account by the model. However, instead of having only a limited amount of historical returns, a sufficiently fine return distribution can be gained by the means of Monte Carlo simulation. Finally, the portfolio can be optimized with respect to the simulated time series.

3.1. Mean–variance framework

The most popular method of constructing optimal portfolios by Markowitz [7] assumes that mean and standard deviation embody sufficient information about the return distribution of a portfolio (which is the case when dealing with normality). The notion of an optimal portfolio in this framework is quite simple: A portfolio is mean–variance efficient if there exists no other portfolio which has the same (or less) risk and a higher expected return, or the same (or a higher) expected return accompanied by lower risk.

To ensure that the expected return does not fall short of a chosen target return μ_{target} that the entire capital is invested (full-investment constraint), and that short-selling is not allowed[4] (each portfolio weight has to be non-negative), a set Z of all possible portfolios is defined:

$$Z := \left\{ w \in \mathbb{R}^3 | w^T \mu \geq \mu_{target},\ \sum_i w_i = 1,\ w_i \geq 0 \right\} \tag{3}$$

with $\mu = (\mu_1, \mu_2, \mu_3)^T$, where μ_i denotes the expected return of asset i, μ_{target} is the target return of the portfolio, and the vector $w = (w_1, w_2, w_3)^T$ denoting the portfolio weights.

The definition of mean–variance efficiency implies that there exists a unique μ_{target} for each portfolio with a certain standard deviation such that this portfolio is efficient in terms of the mean–variance framework. For a set of target returns, a large number of efficient portfolios can be obtained forming a concave efficient frontier in the risk-return space which is referred to as the "efficient frontier" (see, for example, Fig. 5). Among the set of mean-variance-efficient portfolios, it depends on the risk aversion of an investor to choose one of these portfolios. As only risk-averse investors are considered, this choice is a trade-off between risk (measured in terms of standard deviation) and return. The actual optimization problem can be formulated as:

$$\max_{w \in Z} w^T \mu - \frac{\lambda}{2} \cdot w^T \Sigma w \tag{4}$$

with the covariance-matrix Σ and the risk-aversion parameter λ. The question of how to choose this risk-aversion parameter is addressed in Section 4.

3.2. Mean-CVaR framework

Facing positive excess kurtosises in the return distributions (see Table 1), a performance measure is introduced which is sensitive to fat tails. This performance measure is based on the conditional value-at-risk, known as the CVaR (see e.g. Artzner et al. [8,9] and Pflug [10]):

$$CVaR(w^T R) = \mathbb{E}[w^T R | w^T R < VaR(w^T R)], \tag{5}$$

where $VaR(R)$ is the value-at-risk of the portfolio return $R(w) := w^T R$ with $R = (R_1, R_2, R_3)^T$ and R_i is the density of the random return of asset i at the confidence level α (here, $\alpha = 1\%$). The value-at-risk is given by the α-quantile of the return distribution. Then, the mean-CVaR is defined the following way:

$$MCVaR(w^T R) = w^T \mu_R - \lambda_{MCVaR} \cdot CVaR(w^T R) \tag{6}$$

with $\mu = (\mu_1, \mu_2, \mu_3)^T$ where μ_i denotes the expected return of asset i, and λ_{MCVaR} is the risk-aversion parameter of the investor.

This yields a performance measure with the structure "$reward - risk\ aversion \cdot risk$". As the risk is measured in terms of the CVaR, it focuses on the worst case scenarios and thus is particularly sensitive to fat-tailed distributions. For the optimization with respect to the mean-CVaR, the same restrictions apply as for the mean–variance framework, i.e. $w \in Z$.

As already mentioned, the mean–variance optimization only accounts for the first two moments. Despite its intuitive formula, it has one major shortcoming: As it only uses the first two moments of the return distribution, it ignores characteristics of non-normally distributed returns as opposed to the mean-CVaR framework. Therefore, it might not be sufficient for return distributions which show non-normal characteristics, such as it is the case especially for listed private equity (see Section 4).

[4] The short-selling restriction is imposed for practical reasons only. Although the problem can also be solved accordingly without the no-short-selling constraint, we seek to reflect the position of an asset manager or private investor who is not allowed or not willing to take short positions.

Table 1
Empirical statistics of monthly log-returns 1997–2006.

Empirical statistics	Bonds	Stocks	LPE
Mean	0.0043	0.0061	0.0081
Standard deviation	0.0189	0.0421	0.0802
Skewness	0.2724	−0.7805	0.0094
Excess kurtosis	0.0117	0.8665	0.9654
Autocorrelation: lag 1	0.1470	0.0482	0.3869

4. Listed private equity in a portfolio

In Sections 4 and 5, two applications of the Markov-switching model will be presented and the following questions will be tackled:

1. What are the influences for a bond/stock-investor to include LPE in his portfolio? How much listed private equity should be in an investor's portfolio?
2. Was a Markov-switching optimization able to limit the losses of a portfolio including listed private equity during the financial crisis?

4.1. Data

For this study, we consider a portfolio consisting of the asset classes bonds, stocks and listed private equity. The analysis is based on monthly log-returns of the following indices representing the three asset classes:

- *Bonds:* JPMorgan Global Government Bond Index,
- *Stocks:* MSCI World Index,
- *Listed Private Equity:* LPX 50 Index.

The performance of these three indices since 1997 is visualized in Fig. 1. The graph clearly displays the growing dotcom-bubble in 1999 and its bursting from 2000 on. Within these years, LPE and stocks show a highly correlated behavior, whereas the variance seems to be much higher for LPE. The grey-shaded area indicates the years of the financial crisis which will be examined in more detail in Section 5. While LPE could gain high returns in the booming years 2003–2006, it was also experiencing higher losses during the financial crisis.

4.1.1. Listed private equity and the LPX 50

Whereas the access to private equity investments is rather restrictive and illiquid, listed private equity provides institutional and even private investors with exposure to this asset class while maintaining a decent level of liquidity. As private equity companies primarily hold the corporate structure of a limited liability partnership, they are not publicly quoted. But there are some listed companies whose core business is private equity (with, for example, Blackstone being among the most prominent ones). This way, investors can gain access to private equity exposure via the shares of a listed company. However, it has to be mentioned that the stock price of e.g. Blackstone is not only driven by the company's private equity investments but is also exposed to market movements on the stock markets. Thus, listed private equity exhibits a higher correlation to stocks than it is the case for direct private equity investments. Since the latter does not yield a

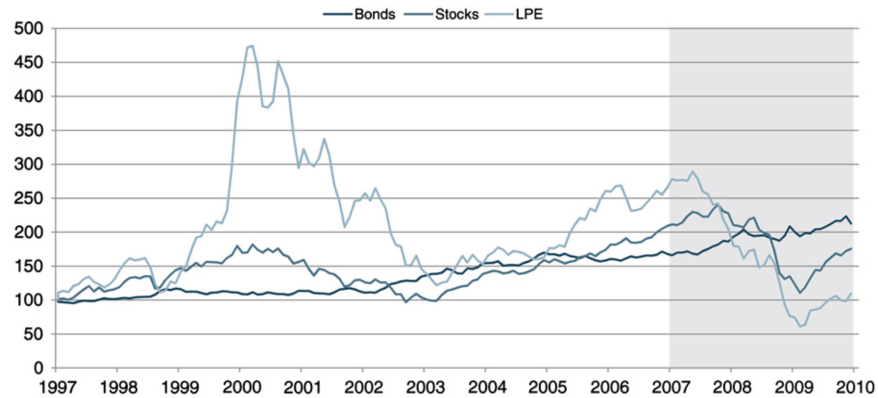


Fig. 1. Performances of indices.

time series which can be used for portfolio optimization, we resort to the LPX 50 Index. For a detailed discussion of the asset class private equity, we refer the interested reader to Aigner et al. [11].

The LPX 50 Index is composed of such listed private equity companies. The index is issued by LPX GmbH, a company based in Switzerland that specializes in constructing indices based on LPE. The LPX 50 is a global index that consists of the 50 largest liquid LPE companies covered by LPX. As the distinction between LPE and "regular" public equity is very blurred (companies might invest in both), LPX regards only those listed companies whose total assets comprise at least 50% unregistered securities. A detailed description of the choice of index constituents and the calculation method can be found in [12].

Investing in listed private equity has become much easier during the last years. Nowadays, there are numerous index tracker certificates and even exchange traded funds available which are tradable on every single trading day. The LPX indices are officially published since 2004, however they are calculated according to the index description since 1994.

4.1.2. Statistics of the indices

The motivation for a Markov-switching model, as already outlined, is that the data exhibits non-normal characteristics in its distribution. Table 1 summarizes the empirical statistics of monthly log-returns for a time span of ten years between 1997 and 2006 (which corresponds to the non-shaded area in Fig. 1). First, the table shows that the standard deviation of monthly LPE returns is about twice the standard deviation of stock returns. Furthermore, due to the values of skewness (in particular for stocks) and excess kurtosis deviating from zero (which would be the case for a normal distribution), a normal distribution already seems unlikely. A Jarque Bera test on the higher moments of the distribution gives statistical evidence that the returns are not normally distributed. Furthermore, a Durbin Watson test on autocorrelation detects a highly significant autocorrelation (with a p-value of 4.59E$-$6) for the LPE time series. Thus, we can conclude that a geometric Brownian motion (based on the assumption of normality only accounting for mean and standard deviation and neglecting autocorrelation) cannot sufficiently capture the characteristics of the return distributions, in particular for LPE.

4.2. Calibration of the model

For the calibration of the model, it is important to decide on the data the model shall be calibrated to. On the one hand, the longer the time series (i.e. the more data points), the better the parameters can be estimated. On the other hand, the more data points there are in the time series, the less influence new observations have (especially of relevance, when the evolution of the model is examined, as done in Section 5). In the following, we use five years of data for the calibration.[5] In this section, we show the estimation and results of the calibration and portfolio optimization for one point in time (which is January 2007) in the ARMS model in detail. In Section 5, this exercise is performed each month over three years. We will examine if there are any benefits from the model if a portfolio is managed according to the obtained optimal allocations (compared to a common approach assuming GBM for the assets).

The input for fitting the model parameters is the monthly returns of the three assets classes bonds, stocks, and LPE. Then, the parameters are chosen such that they best match the mean, standard deviation, skewness, kurtosis, and autocorrelation as well as the correlation structure, as it is described in Section 2.2. However, if we, just for example, consider the five years from 2000 to 2004, LPE obviously exhibits a strongly negative mean return which would result in a zero allocation for LPE in the portfolio. In general, the question arises whether the mean of the past returns can be used as indicator for the future. For that purpose, we conduct a linear shift of the return distribution to best reflect the current market view on the development of the asset classes. In other words, we adjust the means, yet the other moments remain as observed from history.

Since our focus lies on demonstrating the benefits of the ARMS model in portfolio optimization, we will not discuss various approaches to mending this problem, which also occurs with any other forward-looking portfolio optimization procedure. For the sake of objectivity, we do not consider an individual investor's view on future performance, which is the case e.g. in the Black-Litterman approach (see [13]), yet solely rely on given market information: For the expected return of bonds, we take the five-year U.S. treasury rate of the current interest rate curve which coincides with the duration of the bond index of approximately five years.[6] Furthermore, we assume an equity premium of 3.5% which is often considered as long-term mean for the equity premium (according to e.g. Donaldson et al. [14] or Dimson et al. [15]). In general, a premium reflects the higher risk an investor is taking and can be regarded as the additional reward. In

[5] Taking into account that the LPX 50 Index is officially published since 2004, yet three years are too short, we found five years to be long enough for the calibration, yet reasonable in terms of true data availability at the same time. However, of course any other time span deemed suitable can be analyzed with the model.

[6] Note that, for ease of exposition only, U.S. treasury rates are used for representing a global government bond index. Given the composition of the index, this is a rough proxy, yet not an implausible assumption.

Table 2
Model parameters for the assets (2002–2006).

Parameter	μ_1	μ_2	σ_1	σ_2	ϕ
Bonds	−0.00257	0.01166	0.01683	0.02013	0.06734
Stocks	0.01984	−0.00875	0.01575	0.04621	0.09448
LPE	0.03921	−0.02826	0.03565	0.06310	0.21839

Correlation	Bonds	Stocks	LPE
Correlation in state 1			
Bonds	1	0.0380	−0.9746
Stocks	0.0380	1	−0.0354
LPE	−0.9746	−0.0354	1
Correlation in state 2			
Bonds	1	0.1234	−0.2860
Stocks	0.1234	1	0.6333
LPE	−0.2860	0.6333	1

$$P=\begin{pmatrix}0.6068 & 0.3932\\ 0.4705 & 0.5295\end{pmatrix}$$

addition to that, we add another 2% as premium for LPE over stocks which is the approximate average premium between 1997 and 2006. For other possible outcomes when considering a different value as LPE premium, Fig. 4 shows the results of a sensitivity analysis toward the LPE premium.

4.2.1. Model parameters

Table 2 gives the parameter vector Θ for the Markov-switching model (1) calibrated to the five years from 2002 to 2006. Thus, all the information used is available on January 1, 2007, which is the first point in time we want to examine in detail. One can make some interesting observations from the parameters and find characteristics of the model and the markets comprised in them. First of all, state 1 describes a market scenario in which the stock and LPE markets perform strongly and bonds have a slightly negative expected return. On the contrary, state 2 features a well performing bond market with stocks and LPE falling at the same time.

Having a closer look at the estimated parameters, the expected return in state 1, μ_1, is about twice as high for LPE compared to stocks (note that the parameters are in monthly terms because the model is fitted to monthly return data). In state 2, the expected return for LPE is even more than three times (however, both of negative sign) the return of stocks. Thus, in bearish markets, one can experience severe losses with LPE. Comparing the standard deviation parameters of the two states, σ_1 and σ_2, we only observe a slight difference for bonds ($\sigma_1=1.683\%$ opposed to $\sigma_2=2.013\%$). On the other hand, for stocks and LPE, the standard deviation approximately doubles in the falling markets of state 2 ($\sigma_1=1.575\%$ and $\sigma_2=4.621\%$ for stocks, and $\sigma_1=3.565\%$ and $\sigma_2=6.310\%$ for LPE). This observation matches reality where we are facing higher volatilities in falling markets. Another interesting peculiarity in real markets is that the correlation between assets increases in times of a crisis. This feature can also be observed in the parameters. For the purpose of better illustration, instead of the covariance matrices Σ_1 and Σ_2, Table 2 gives the corresponding correlation matrices. For example, the correlation of the error term between stocks and LPE is -0.0354 in state 1 and 0.6333 in state 2. Thus, in times of a crisis, losses in both asset classes are more likely to coincide. Furthermore, as one would expect, the autocorrelation parameter ϕ is the highest for LPE, as LPE also shows the highest autocorrelation.

Ergodic Markov chains display the useful characteristic of having a unique stationary distribution. The unconditional probability of such a process being in state 1 is given by $\pi_1=p_{21}/(p_{12}+p_{21})$ and the probability of it being in state 2 is respectively given by $\pi_2=p_{12}/(p_{12}+p_{21})$. From the transition matrix P we also know the overall probabilities for the two states. In this case, $\pi_1=54.47\%$ and $\pi_2=45.53\%$. In the simulation of paths later on, a Bernoulli distributed random variable is drawn to decide on the initial state.

4.2.2. Comparison of empirical and theoretical statistics

Tables 3 and 4 report the empirical statistics and correlations of the input time series 2002–2006 as well as the theoretical statistics resulting from the parameters in Table 2. Note, that the mean has been shifted according to the scheme outlined above. Therefore, the theoretical mean is not given (indicated by (*) in Table 3) as it does not match the empirical mean but the adjusted mean. However, it should be noted that the deviations of the theoretical means from their target values are smaller than 1E−6 in relative terms. For all statistics, the relative deviation, i.e. the difference between the empirical and theoretical statistics divided by the empirical statistics, is always smaller than 4E−5. Therefore, if we now simulate multiple time series, the monthly returns have the same distribution features as in the empirical time series.

Furthermore, it is worthwhile to have another look at the correlation structure in Table 4. As it has already been mentioned before, this correlation structure could not be mapped by a Markov-switching model with two states for each asset individually. In such a model, we were not able to receive a correlation higher than 0.35 between stocks and LPE. Thus, only correlating the error terms is not sufficient. Therefore, in our model, we also “correlate” the states in the sense that we have only two market states, so stocks cannot be in their “good” state while LPE is in its “bad” state.

4.3. Optimal portfolio allocation

In the next step, we used the estimated parameters to simulate 10,000 time series (according to the ARMS model only) for the three asset classes. For optimization purposes, we considered the distribution after five years (i.e. 60 simulated months) as we assume an investment horizon of five years. We also tested the sensitivity of the resulting portfolio toward the investment horizon and found that the final allocations do not deviate more than 1% in the weight of the asset classes.

Table 3
Empirical and theoretical statistics of monthly log-returns 2002–2006.

		Mean	Std.	Skewness	Ex. kurt.	Autocorr.
Bonds	Empirical	0.006649	0.01976	0.1761	0.0620	0.0762
	Theoretical	(*)	0.01976	0.1761	0.0621	0.0762
Stocks	Empirical	0.007918	0.03633	−0.8491	1.5962	0.1009
	Theoretical	(*)	0.03633	−0.8491	1.5962	0.1009
LPE	Empirical	0.001195	0.06132	−0.6236	0.3320	0.1937
	Theoretical	(*)	0.06132	−0.6236	0.3321	0.1937

Table 4
Empirical and theoretical correlation structure of monthly log-returns 2002–2006.

Correlation	Bonds	Stocks	LPE
Empirical			
Bonds	1	−0.059595	−0.601762
Stocks	−0.059595	1	0.595155
LPE	−0.601762	0.595155	1
Theoretical			
Bonds	1	−0.059596	−0.601772
Stocks	−0.059596	1	0.595158
LPE	−0.601772	0.595158	1

4.3.1. Optimization results

Figs. 2 and 3 visualize the optimal allocation between bonds, stocks, and LPE in the mean–variance and mean-CVaR framework, respectively. The allocation is plotted with respect to the per-annum standard deviation of the portfolio. On the right scale, the according risk-aversion parameter for the portfolio of a certain standard deviation is plotted. In both frameworks, we can observe a similar scheme: The higher the risk-aversion parameter the lower the share of risky assets (stocks and LPE) and the higher the share of bonds. The share of bonds does not exceed a certain level in both frameworks which corresponds to the minimum-variance portfolio in the mean–variance framework. The minimum-variance portfolio consists of 19% LPE and 81% stocks. The reason that the minimum-variance portfolio consists of bonds and LPE instead of bonds and stocks lies in the higher diversification achieved by LPE (considering the much lower correlation between bonds and LPE compared to bonds and stocks), despite the higher risk of LPE.

In order to derive an optimal allocation from the graphs, an investor would have to quantify his risk-aversion parameter, which seems far from reality. The question investors can actually answer is how they would allocate their portfolio between bonds and stocks if these two were the only asset classes they could invest in. Thus, we assign an investor that risk-aversion parameter which would lead to his preferred bonds/stocks allocation if only bonds and stocks are considered in the optimization. For example, a model investor in the examined ARMS model with a 50/50 allocation would have a $\lambda = 4.979$ and $\lambda_{MCVaR} = 0.378$.

The portfolios resulting from these risk-aversion parameters in the two frameworks are summarized in Table 5. In general, both portfolios show a very similar allocation. However, one can already see in this table that the mean–variance framework results in a riskier portfolio, as the share in stocks and LPE is higher than in the mean-CVaR framework. The portion invested in risky assets (stocks and LPE) amounts to 38.33% as opposed to 32.76%. Accordingly, the investor following the mean-CVaR approach allocates a higher share in bonds. Below the portfolio weights, Table 5 also gives the statistics of these portfolios for the considered investment horizon of five years. The portfolios would lead to expected (continuous) returns of 6.5% and 6.3% per-annum (the values in Table 5 are on a five-year basis). Due to the higher share of risky assets, the investor can expect a higher return in the mean–variance framework. Considering values-at-risk (VaR) and conditional values-at-risk (CVaR) which measure the risk in the tails, one can see the other side of the medal: The mean–variance portfolio bears higher tail risks than the mean-CVaR portfolio. And the higher the significance of the measures the bigger the difference becomes. For example, the CVaR 1%, which is accounted for in the mean-CVaR framework, is -8.43% in the

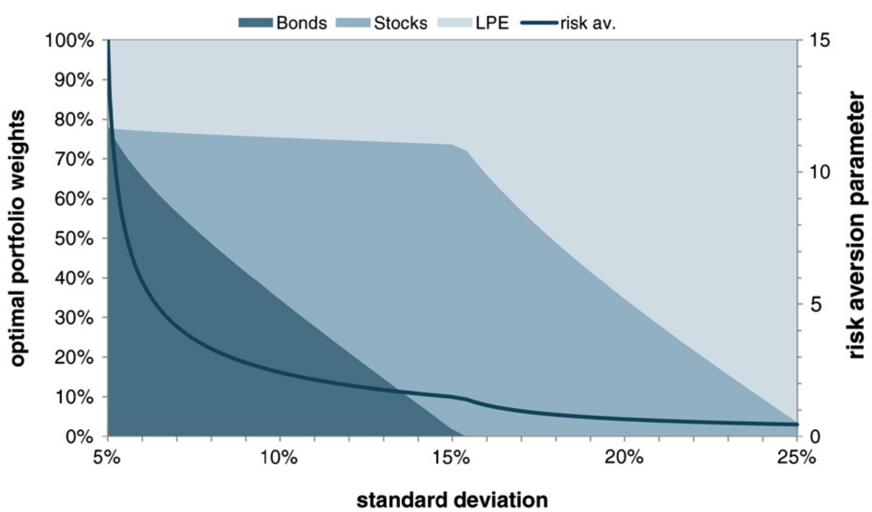


Fig. 2. Optimal portfolio weights in the mean–variance framework.

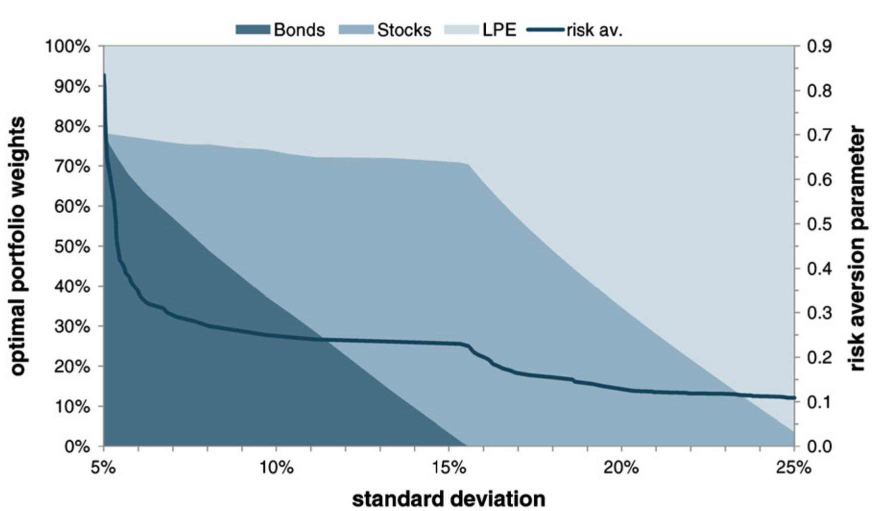


Fig. 3. Optimal portfolio weights in the mean-CVaR framework.

mean–variance and −5.09% in the mean-CVaR portfolio. This means that, considering only the 1% worst cases, an investor can expect to lose on average 8.43% or 5.09%, respectively, of his wealth during the investment horizon of five years.

4.3.2. Sensitivity toward LPE premium

As mentioned above, it might be of interest how far the optimal portfolio is changing subject to the LPE premium which we assumed to be 2% based on the history of data. Therefore, Fig. 4 gives an answer to that question, as it illustrates the optimal portfolio weights in the mean–variance framework subject to the LPE premium. For this analysis, we once more consider our 50/50 model investor in the mean–variance framework, i.e. an investor who would invest 50% of his portfolio in bonds and 50% in stocks if he could only allocate between these two asset classes.

It comes as no surprise that the share in LPE increases with the LPE premium. However, even for a premium of 0%, i.e. stocks and LPE have the same expected returns, the portfolio still includes LPE despite its higher risks compared to stocks. This is again due to the higher diversification effects of LPE. If the assumed LPE premium is higher than 3.85%, the portfolio completely neglects stocks, i.e. the model investor's allocation only consists of bonds and LPE. For the "barrier premium" of 3.85% the portfolio consists of 66.73% bonds and 33.28% LPE.

The same analysis has been conducted for the mean-CVaR framework which leads to similar results as in the mean–variance framework.

Table 5
Allocations for a 50/50 model investor on January 1, 2007, and their statistics.

Optimization	Mean–variance (%)	Mean-CVaR (%)
Bonds	61.67	67.24
Stocks	15.15	11.19
LPE	23.18	21.57
Exp. return	32.58	31.45
Std. dev.	14.46	13.02
VaR 10%	13.76	14.54
VaR 5%	8.61	9.85
VaR 1%	−2.54	0.10
CVaR 10%	6.66	8.14
CVaR 5%	1.85	3.85
CVaR 1%	−8.43	−5.09

4.3.3. Diversification effects of LPE

To conclude this section dealing with the optimization at one point in time, Fig. 5 visualizes the diversification effect of including LPE in a portfolio. We see the expected return over the investment

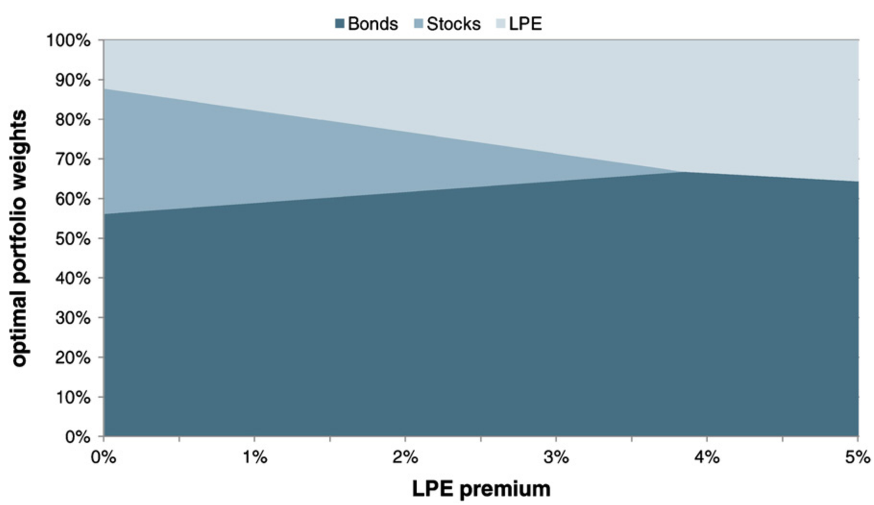


Fig. 4. Optimal portfolio weights in the mean–variance framework for the 50/50 model investor subject to the LPE premium.

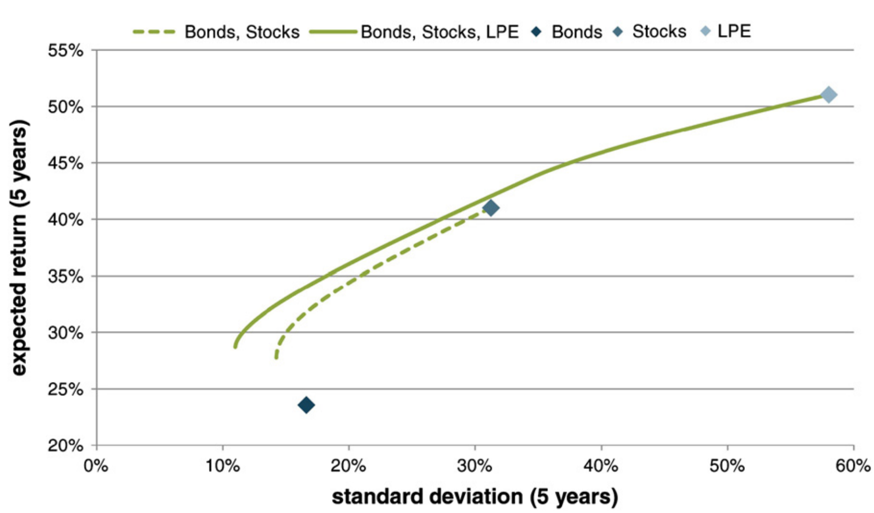


Fig. 5. Efficient frontiers (mean–variance) with and without LPE.

horizon of five years over the standard deviation for the same time horizon. The three diamonds mark the three asset classes in the mean–variance diagram, in other words they stand for a portfolio consisting of 100% bonds, stocks, or LPE, respectively. The dotted line displays the efficient frontier of portfolios of bonds and stocks only. The upper right end of the dotted line represents a portfolio consisting of 100% stocks and the lower left end the minimum-variance portfolio. The solid line represents the efficient portfolios containing bonds, stocks, and LPE. The minimum-variance portfolio including LPE has a lower standard deviation, yet promises a higher return at the same time. Thus, even risk-averse investors should include LPE in their portfolios for diversification effects.

The mean–variance diagram in Fig. 5 is the most popular and common risk-reward diagram, with risk measured in terms of standard deviation in this case. Fig. 6 displays a mean-CVaR diagram corresponding to the mean-CVaR optimization framework. The notion of the efficient frontier is exactly the same, however with risk measured in terms of the conditional value-at-risk. Here, the CVaR on the x-axis is given in terms of loss as opposed to return. (In other words, we plot $-$CVaR instead of CVaR, where the CVaR is defined as in (5).) Note that the CVaR of -106.0% (i.e. loss of 106.0%) is given in continuous terms corresponding to a discrete CVaR of 65.4%. The conclusions that can be drawn from the mean-CVaR diagram are the same as from the mean–variance diagram discussed above.

5. Listed private equity during the financial crisis: advantages of the Markov-switching optimization

In the previous section, we demonstrated the positive effects of including LPE in a portfolio. This was done by means of an ARMS model. As already outlined, we believe that the ARMS approach is better suited for return distributions such as those of LPE which display autocorrelation and non-normality in its returns. In this section, we will actually demonstrate that the ARMS model outperforms the currently mostly used GBM model. This is done by an out-of-sample test for both the ARMS and the GBM approach. LPE was experiencing severe losses during the financial crisis. Thus, we find that time span of particular interest for such an analysis.

5.1. Setting up an out-of-sample test

In the following, we always consider two portfolios consisting of bonds, stocks, and LPE between 2007 and 2009 (over three

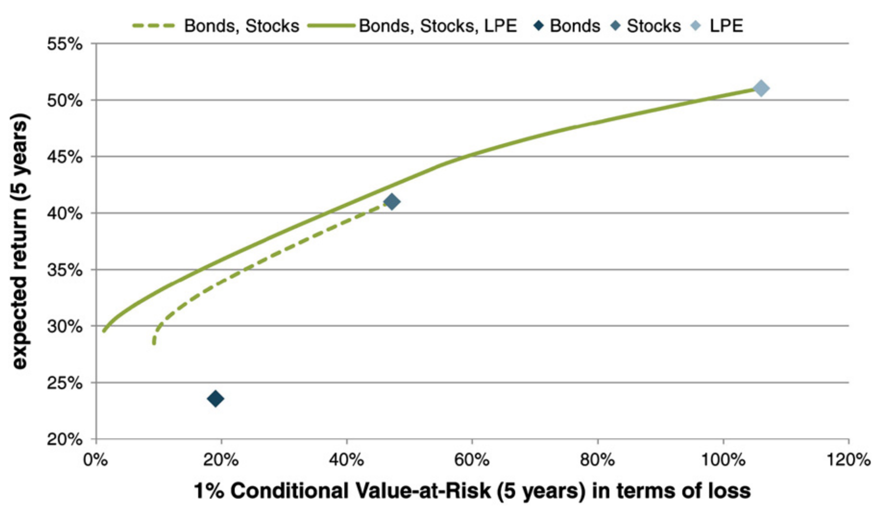


Fig. 6. Efficient frontiers (mean-CVaR) with and without LPE.

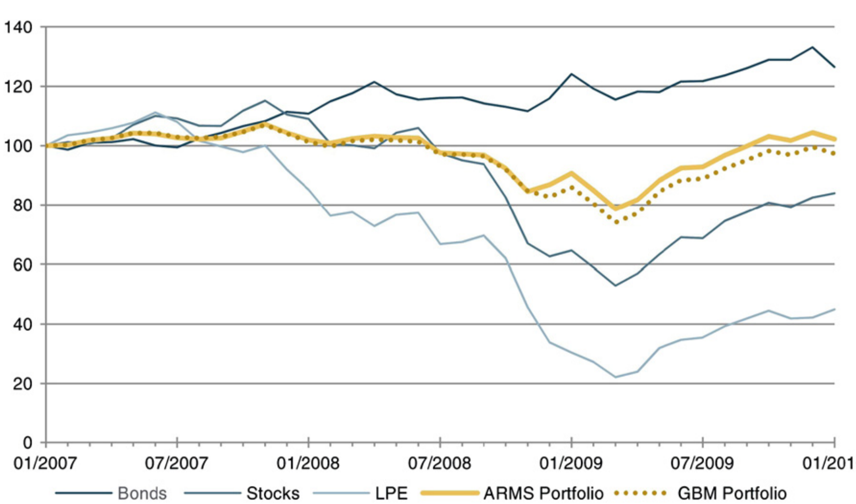


Fig. 7. Performance with reset of risk-aversion parameter each year (mean–variance framework).

entire years): One portfolio is optimized based on simulations performed with the ARMS model, as outlined in the previous section. The second portfolio is optimized based on simulations according to the common model assuming a geometric Brownian motion, so that we are able to study the effects of the ARMS model compared to a GBM. Both portfolios are reallocated monthly. The methodology is as follows:

- Fit the model parameters Θ of the ARMS model to the return data of the previous five years. All information which is used has been observable at the time we optimize the portfolio for. Thus, a real out-of-sample test is conducted.
- Calculate the standard deviation of the return time series (based on the same length of data as for the ARMS model) for the GBM portfolio.
- Simulate 10,000 monthly return time series over five years (1) for a common GBM model using mean and standard deviation, and (2) for the ARMS model. In both cases, the expected returns are adjusted as described in Section 4.
- Determine the optimal portfolio allocation for both sets of simulated return time series with respect to the according optimization model (mean–variance or mean-CVaR) regarding the risk aversion of the investor and an investment horizon of five years. In order to be consistent within the frameworks, a different risk-aversion parameter is derived for each of the two models (ARMS and GBM).

For this analysis, we again resort to our model investor who would allocate half of his wealth in bonds and the other half in stocks if these two would be the only investment alternatives. We examine two different approaches of reallocation: In the first, the model investor is assumed to adjust his risk-aversion parameter according to his 50/50 benchmark portfolio every year (similar to an annual adjustment of investment objectives in the asset management business), in the second he does not. Thus, in the first approach, we have an updated risk-aversion parameter each year. In the second approach, we use the same risk-aversion parameter for the entire three years.

5.2. Portfolio with annual reset

Fig. 7 shows the performance of the portfolios with annual reset of the risk-aversion parameter and monthly reallocation. These portfolios have been optimized according to the mean–variance framework. For comparison, also the performances of the three asset classes are plotted. The dotted line represents the performance of the portfolio where a geometric Brownian motion was assumed for the underlying asset class processes, the solid line the Markov-switching approach. The terminal value of the GBM portfolio is 97.33%, the terminal value of the ARMS portfolio is 102.21%, i.e. the return over the three years is about 5% higher when using the Markov-switching optimization. As one could expect, the benefits are most significant in the year 2008, where the GBM portfolio loses 22.31%, the ARMS portfolio 17.66%: As the Markov-switching model better maps the higher risk (due to the effects of autocorrelation), its advantages take effect in particular in crashing markets. The standard deviation of the 36 monthly returns amounts to 3.56% in the GBM portfolio and 3.49% in the ARMS portfolio—despite the higher fluctuations in the weights of the ARMS portfolio.

Examining the weight charts in Fig. 8 shows that the ARMS model reacts faster to changes in the return characteristics of the three asset classes. For example, in November 2008, after both stocks and LPE were experiencing high losses, the Markov-switching approach limits the LPE exposure to 4.0% (with the remainder being invested in bonds), whereas the GBM approach still

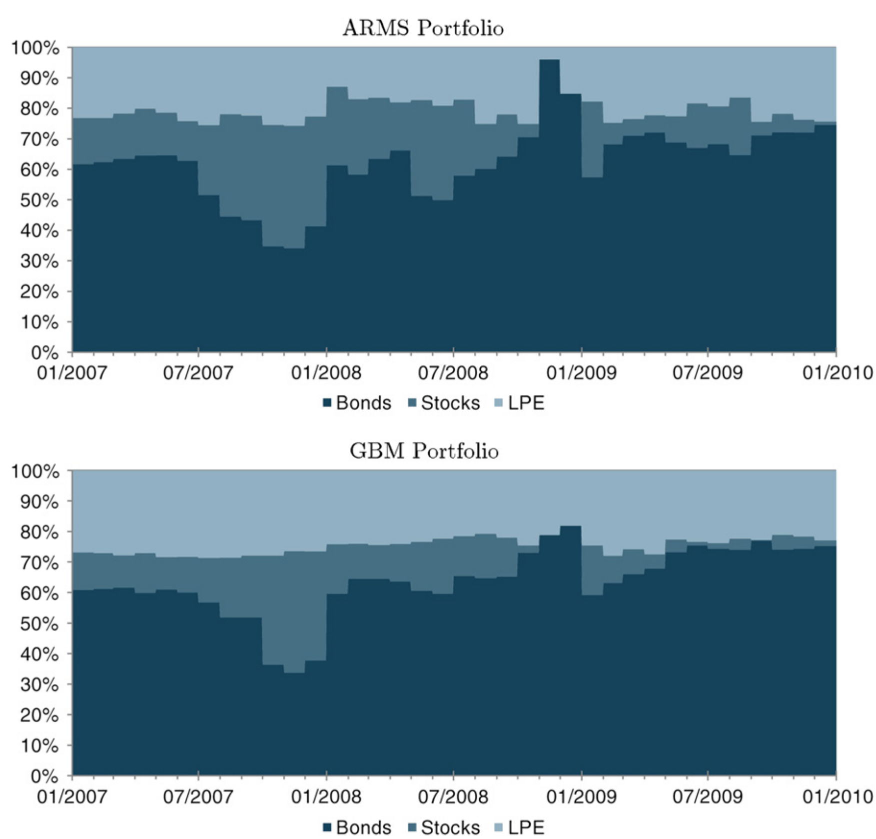


Fig. 8. Portfolio weights with reset of risk-aversion parameter each year (mean–variance framework).

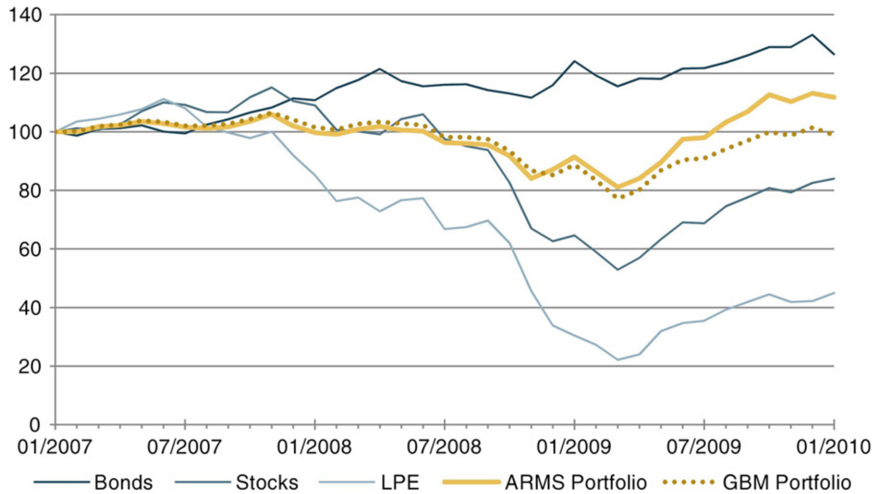


Fig. 9. Performance with reset of risk-aversion parameter each year (mean-CVaR framework).

allocates 18.2% in LPE. The average allocation in bonds over the three years was 63.5% for the GBM and 62.2% for the ARMS portfolio, 11.8% or 17.1% in stocks, and 24.7% or 20.7% in LPE. In other words, the average share in risky assets is about the same in both portfolios. However, the ARMS approach decreases the exposure to LPE compared to the GBM approach, as the Markov-switching model is better able to map the stylized facts of LPE such as fat tails (representing "severe losses"). Nevertheless, due to diversification effects, even in the ARMS approach, a minimum of 4.0% in LPE observable.

Figs. 9 and 10 show the according results for the mean-CVaR framework. In general, we obtain similar results as in the mean–variance framework. However, they are more distinct. For example, the terminal values are 98.82% (GBM) and 111.75% (ARMS)—a difference of approximately 13% over three years. Taking a closer look at the weight charts, one can see that the optimization in the mean-CVaR framework is even more sensitive. This is due to the fact that the risk component in the optimization criterion focuses on the worst 1% of the return distribution, and therefore is less robust than the standard deviation which considers the entire distribution. Whereas the effects of the ARMS model in the mean–variance framework are predominantly due to the incorporation of autocorrelation (effecting the risk in terms of the standard deviation), the mean-CVaR framework is additionally very well able, as described in Section 3.2, to account for the fat tails. Thus, it comes as no surprise that the differences between the ARMS and GBM model are higher in the mean-CVaR framework. The minimum and maximum allocations in risky assets are also extended in the ARMS portfolio, ranging from 0.8% to 100%. Yet, neither of the two frameworks can be claimed better in general. It rather depends on the investor's personal understanding of risk.

5.3. Portfolio without annual reset

As described, the risk-aversion parameter was adjusted at the beginning of each year to the investor's benchmark portfolio which is set up of 50% bonds and 50% stocks. This methodology seems to be practical, as it is a common approach in the asset management industry to adjust expectations and market views at least once every year. As it also might be of interest what the effect of this adjustment is, we conducted the exercise without changing the risk-aversion parameter over the three years. The results for the mean–variance framework are presented in Figs. 11 and 12 (the same conclusions can be drawn from the mean-CVaR framework). In both cases, GBM and ARMS, the portfolio performs worse without annual reset compared to the corresponding portfolio with annual reset: The terminal values are 81.37% (GBM) and 89.72% (ARMS), opposed to 97.33% and 102.21% when the risk aversion is reset on an annual basis.

Obviously, the performance in the year 2007 has to be exactly the same as for the portfolio with the reset. However, as the risk-aversion parameter is adjusted for 2008 and then again for 2009, these years show different performances. Comparing the weight charts of the two settings reveals the reason for this: The portfolio was allocated very defensively in 2009 and thus, could not participate in the bullish markets after the crisis has bottomed out. For the optimization in 2009, the events of the crisis were included when fitting the model, but the risk-aversion parameters were not adjusted to reflect this. Thus, the non-updated risk-aversion parameter consequently had to lead to a very low exposure in stocks and LPE.

6. Conclusion

Listed private equity provides an attractive investment alternative, due to its liquidity (compared to non-quoted private equity) and yet enhanced risk-return profile. Although the performance of LPE is not solely driven by the performance of the underlying direct private equity investments but also by the overall performance of markets in general (as the shares of LPE are "listed" and thus publicly traded), it still provides a true alternative to pure stock investments. And in contrast to direct private equity investments, LPE can be invested via numerous liquid market vehicles whose number has been continuously growing for the last years.

Considering listed private equity in asset allocation along with bonds and stocks requires a method which can account for high correlations (which is the case between stocks and listed private equity) as well as non-normal distributions and autocorrelation. This study presents a Markov-switching approach which is able to model the asset classes' returns while capturing their most important characteristics. It has been shown that the simulations thereby obtained very well match the original (empirical) characteristics of the observed return series of the asset classes. Applying this model, one can simulate return distributions which can be used for asset allocation purposes.

In order to answer the questions raised in the introduction of how to determine the optimal fraction of LPE in a portfolio and what the impacts of including LPE in a portfolio are, several

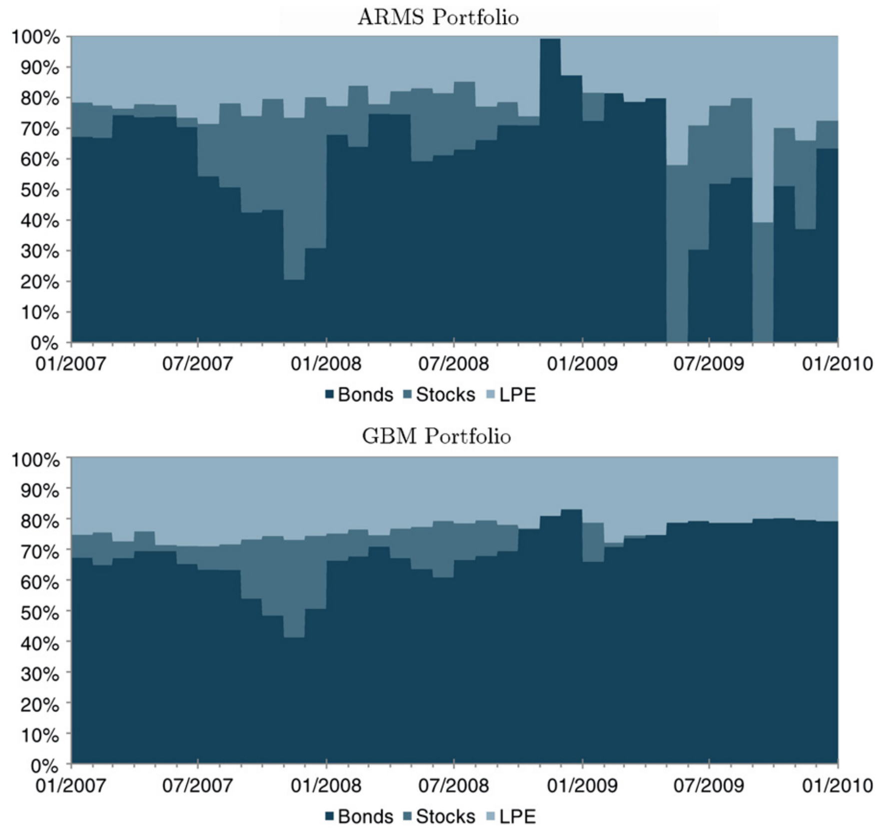


Fig. 10. Portfolio weights with reset of risk-aversion parameter each year (mean-CVaR framework).

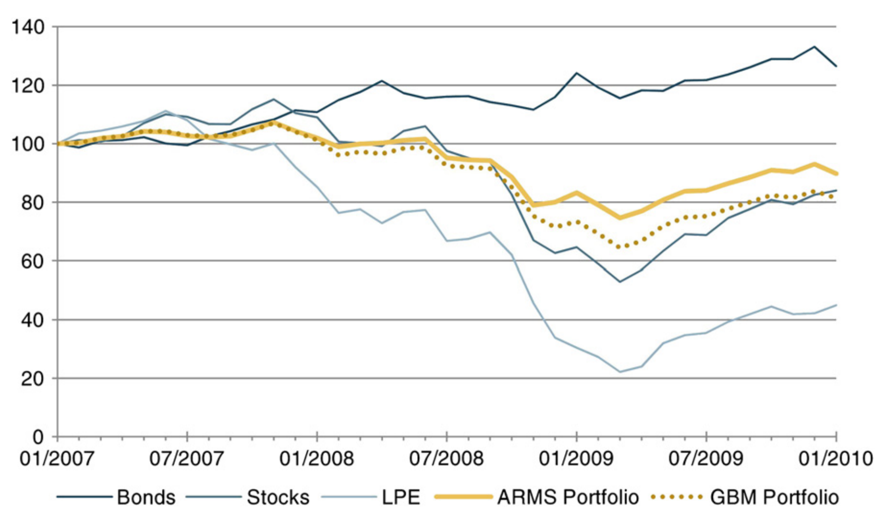
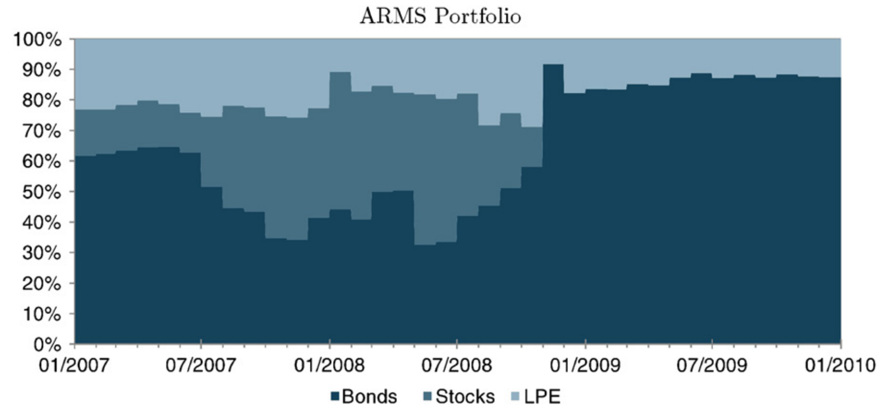


Fig. 11. Performance without reset of risk-aversion parameter each year (mean–variance framework).

optimization frameworks were applied. However, there cannot be a unique answer to that question, and it would be misleading to give one single percentage as a suggestion. Furthermore, in particular for an institutional investor, restrictions on the maximum share in alternative asset classes are likely to apply. Yet, we found that even a very risk-averse investor can benefit from the diversification effects of including LPE in his portfolio. Listed private equity can significantly ‘‘extend the efficient frontier’’.

In the last section, we answered the question how a portfolio including LPE was performing, which was allocated with respect to the rules of Markov-switching optimization. In an out-of-sample test, we showed the benefits of applying a first-order autoregressive Markov-switching (ARMS) model in asset allocation, especially when asset classes are considered which exhibit significant tail risks. Furthermore, we obtained the indication that adjusting one's risk-aversion parameter would have been useful to adjust to the changes in the markets. We were examining the performance of a portfolio allocated between bonds, stocks, and LPE according to the presented Markov-switching optimization during the years 2007, 2008, and 2009 in light of the financial crisis. We compared the performance of this portfolio to a portfolio allocated according to a common GBM model. In three different settings we presented that the ARMS



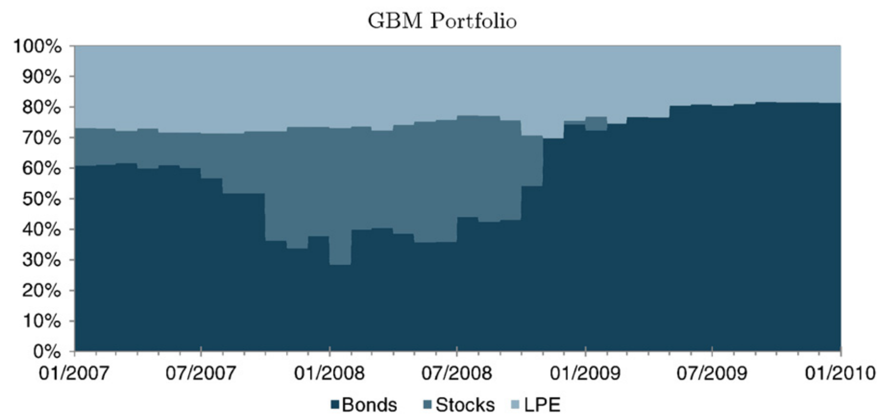


Fig. 12. Portfolio weights without reset of risk-aversion parameter each year (mean–variance framework).

portfolio was always performing better. The ARMS portfolios gained between 1.6% and 7.3% p.a. over the GBM portfolios.

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