

**PHILOSOPHIÆ
NATURALIS
PRINCIPIA
MATHEMATICA**

Al Caveman

November 1, 2015

Contents

1	Integration	2
1.1	Areas of Lots of Extremely Tiny Rectangular Columns	3
1.2	Approximating the Area	5
1.3	The Precise Area Under The Bitch	7
1.4	Generalizing That	7
2	Differentiation	9
2.1	Theorems	9
3	Multivariate Stuff	11
3.1	Functions that are sum of other linear functions	11

Chapter 1

Integration

The goal here is to integrate the area under $f(x)$ from when $x = 0$ until we reach $x = x_{end}$. What follows here is bunch of steps that shows my thinking process because *shadowdaemon* asked for it (and also because I am happy now).

1.1 Areas of Lots of Extremely Tiny Rectangular Columns

So to integrate $f(x) = x^2$ we have to keep summing extremely skinny columns¹ of, each of width d .

$$\begin{aligned}
 & (x + d - x)(x)^2 + \\
 & (x + 2d - (x + d))(x + d)^2 + \\
 & (x + 3d - (x + 2d))(x + 2d)^2 + \\
 & (x + 4d - (x + 3d))(x + 3d)^2 + \\
 & (x + 5d - (x + 4d))(x + 4d)^2 + \\
 & (x + 6d - (x + 5d))(x + 5d)^2 + \\
 & (x + 7d - (x + 6d))(x + 6d)^2 + \\
 & (x + 8d - (x + 7d))(x + 7d)^2 + \\
 & (x + 9d - (x + 8d))(x + 8d)^2 + \\
 & (x + 10d - (x + 9d))(x + 9d)^2 + \\
 & (x + 11d - (x + 10d))(x + 10d)^2 + \\
 & \dots
 \end{aligned}$$

You see, if d is extremely tiny (near zero), then we will have to sum an infinite number of those tiny skinny areas. But for simplicity I put \dots instead.

¹**Note:** *fefelix* of *freenode/#gentoo-chat-exile* tried to look smart by attacking my rigor by saying that the term *skinny columns* is wrong and that it must be replaced by the term *infinitesimal* (facepalm moment here). He also tried to look even smarter by using the phrase *In the realm of \mathbb{R}* . Obviously, any half-assed mathematician knows that \mathbb{R} has only a single infinitesimal which is zero. Yep, zero bitch. So the term *infinitesimal* is absolutely wrong in this context, thus even worse than *skinny columns*. The only exception is if *fefelix* wishes to live in the 1600s.

We can simplify things:

$$\begin{aligned}
 & d(x)^2 + \\
 & d(x+d)^2 + \\
 & d(x+2d)^2 + \\
 & d(x+3d)^2 + \\
 & d(x+4d)^2 + \\
 & d(x+5d)^2 + \\
 & d(x+6d)^2 + \\
 & d(x+7d)^2 + \\
 & d(x+8d)^2 + \\
 & d(x+9d)^2 + \\
 & d(x+10d)^2 + \\
 & \dots
 \end{aligned}$$

Let's expand those squares:

$$\begin{aligned}
 & d(x)(x) + \\
 & d(x+d)(x+d) + \\
 & d(x+2d)(x+2d) + \\
 & d(x+3d)(x+3d) + \\
 & d(x+4d)(x+4d) + \\
 & d(x+5d)(x+5d) + \\
 & d(x+6d)(x+6d) + \\
 & d(x+7d)(x+7d) + \\
 & d(x+8d)(x+8d) + \\
 & d(x+9d)(x+9d) + \\
 & d(x+10d)(x+10d) + \\
 & \dots
 \end{aligned}$$

Let's multiply them square bitches:

$$\begin{aligned}
& dx^2 + \\
& d(x^2 + 2dx + d^2) + \\
& d(x^2 + 4dx + 4d^2) + \\
& d(x^2 + 6dx + 9d^2) + \\
& d(x^2 + 8dx + 16d^2) + \\
& d(x^2 + 10dx + 25d^2) + \\
& d(x^2 + 12dx + 36d^2) + \\
& d(x^2 + 14dx + 49d^2) + \\
& d(x^2 + 16dx + 64d^2) + \\
& d(x^2 + 18dx + 81d^2) + \\
& d(x^2 + 20dx + 100d^2) + \\
& \dots
\end{aligned}$$

Let's now multiply those bitches with d so that the shit gets spread even more:

$$\begin{aligned}
& dx^2 + \\
& dx^2 + 2d^2x + d^3 + \\
& dx^2 + 4d^2x + 4d^3 + \\
& dx^2 + 6d^2x + 9d^3 + \\
& dx^2 + 8d^2x + 16d^3 + \\
& dx^2 + 10d^2x + 25d^3 + \\
& dx^2 + 12d^2x + 36d^3 + \\
& dx^2 + 14d^2x + 49d^3 + \\
& dx^2 + 16d^2x + 64d^3 + \\
& dx^2 + 18d^2x + 81d^3 + \\
& dx^2 + 20d^2x + 100d^3 + \\
& \dots
\end{aligned}$$

1.2 Approximating the Area

Now this is a critical point. Below is basically saying that each row is an approximation for the area under $f(x)$ from $x = 0$ till $x = x_{end}$. So the 1st one is a shit approximation where we approximate the area under that curve by only one big fat column; so d is so huge here, in fact $d = x_{end}$.

Then, the 2nd line show a slightly less shit approximation where we approximate the area under the curve by two fat ass rectangular columns. So here $d = \frac{x_{end}}{2}$.

So the approximation of the area under the curve gets more and more accurate in each line.

$$\begin{aligned}
& dx^2 \\
& 2dx^2 + 2d^2x + d^3 \\
& 3dx^2 + 6d^2x + 5d^3 \\
& 4dx^2 + 12d^2x + 14d^3 \\
& 5dx^2 + 20d^2x + 30d^3 \\
& 6dx^2 + 30d^2x + 55d^3 \\
& 7dx^2 + 42d^2x + 91d^3 \\
& 8dx^2 + 56d^2x + 140d^3 \\
& 9dx^2 + 72d^2x + 204d^3 \\
& 10dx^2 + 90d^2x + 285d^3 \\
& 11dx^2 + 110d^2x + 385d^3
\end{aligned}$$

So basically, you see there is a pattern. The coefficient of the 1^{st} term is easy peasy (just incrementing from 1 to ∞). The coefficient from the 2^{nd} term is kinda interesting, it follows the equation $(i^2 + i)$ where i is the line number. Note that we start counting lines from 0. So the 1^{st} has $i = 0$ and the 2^{nd} line has $i = 1$, etc. Finally, the last term is kinda cool, it follows the pattern $\frac{(i^2+i)(2i+1)}{6}$.

Now you may ask, how did I find these patterns? Well these are well known number series. You can look them up in the On-Line Encyclopedia of Integer Sequences².

So, the area under the curve of $f(x)$ from $x = 0$ up to x_{end} , by any d (and its corresponding i) is:

$$dx^2 + (i^2 + i)d^2x + \frac{(i^2 + i)(2i + 1)}{6}d^3$$

Now we are almost done. We know that $x = 0$, so we can cancel a few terms:

$$\begin{aligned}
d0^2 + (i^2 + i)d^20 + \frac{(i^2 + i)(2i + 1)}{6}d^3 \\
\frac{(i^2 + i)(2i + 1)}{6}d^3
\end{aligned} \tag{1.1}$$

Of course, we could've canceled those terms that multiply against zero earlier, but I didn't for random reasons. I just didn't. That's the randomness of life. But it's all mathematically correct as my caveman balls tell.

You can code a simple script that you give it x_{end} and d , by which it automatically finds $i = x_{end}/d$. You will notice that as d gets smaller, you end up approaching some limit after which reduction in d does not cause any change in the estimated area under the curve.

²<http://oeis.org/>

1.3 The Precise Area Under The Bitch

Now let's find the ultimate precision in the limit as $i \rightarrow \infty$ which also means that $d \rightarrow 0$. But how about not? Cause it's too hard to solve the limit when two variables are approaching different limits.

To simplify the limits in an easier way, let's represent i in terms of d and x_{end} as follows $i = x_{end}/d$. Then the same equation would become as follows:

$$\begin{aligned} & \frac{((x_{end}/d)^2 + (x_{end}/d))(2(x_{end}/d) + 1)}{6} d^3 \\ & \frac{(\frac{x_{end}^2}{d^2} + \frac{x_{end}}{d})(\frac{2x_{end}}{d} + 1)}{6} d^3 \\ & \frac{(\frac{x_{end}^2}{d^2} + \frac{x_{end}}{d})(\frac{2x_{end}}{d} + 1)}{6} d^3 \\ & \frac{\frac{1}{d}(\frac{x_{end}^2}{d} + x_{end})(\frac{2x_{end}}{d} + 1)}{6} d^3 \\ & \frac{(\frac{x_{end}^2}{d} + x_{end})(\frac{2x_{end}}{d} + 1)}{6} d^2 \\ & \frac{\frac{2x_{end}^3}{d^2} + \frac{x_{end}^2}{d} + \frac{2x_{end}^2}{d} + x_{end}}{6} d^2 \\ & \frac{\frac{2x_{end}^3 d^2}{d^2} + \frac{x_{end}^2 d^2}{d} + \frac{2x_{end}^2 d^2}{d} + x_{end} d^2}{6} \\ & \frac{2x_{end}^3 + x_{end}^2 d + 2x_{end}^2 d + x_{end} d^2}{6} \end{aligned}$$

Now, it's super easy. We have to find the limit of that equation as a single variable approaches 0 (we got rid of i). The equation becomes:

$$\begin{aligned} & \frac{2x_{end}^3}{6} \\ & \frac{x_{end}^3}{3} \end{aligned}$$

That's it. Integration re-invented bitch :) — $\frac{x_{end}^2}{3}$.
Q.E. freaking DEE.

1.4 Generalizing That

Let's integrate x^c where $x, c \in \mathbb{R}$. How easy is that? Let's try.

So, back to the business of summing skinny columns:

$$\begin{aligned}
 & d \times d^c + \\
 & d \times (2d)^c + \\
 & d \times (3d)^c + \\
 & d \times (4d)^c + \\
 & d \times (5d)^c + \\
 & \dots
 \end{aligned}$$

Simplified to:

$$\begin{aligned}
 & d \times d^c + \\
 & d \times 2^c \times d^c + \\
 & d \times 3^c \times d^c + \\
 & d \times 4^c \times d^c + \\
 & d \times 5^c \times d^c + \\
 & \dots
 \end{aligned}$$

As we try to estimate that sum better we get:

$$nd \times \left(\frac{n(n+1)}{2}\right)^c \times d^c$$

Chapter 2

Differentiation

Here we want to find the slope of $f(x)$ at point x . This is easy so I won't say much here.

Differentiate x^2 .

$$\frac{f(x+d)-f(x)}{d}$$

$$\frac{(x+d)^2-x^2}{d}$$

$$\frac{(x+d)^2-x^2}{d}$$

$$\frac{(x+d)(x+d)-x^2}{d}$$

$$\frac{x^2+xd+xd+d^2-x^2}{d}$$

$$\frac{x^2}{d} + \frac{2xd}{d} + \frac{d^2}{d} - \frac{x^2}{d}$$

$$\frac{x^2}{d} + 2x + d - \frac{x^2}{d}$$

$$2x + d$$

Now, as $d \rightarrow 0$, it becomes $2x$. Done.

2.1 Theorems

My symbol for differentiation is diff . That's it. Now, if t, x, c are any numbers, except c is just a non-zero natural number, then:

Theorem 1. $\text{diff } tx^c = ctx^{c-1}$.

Okay let's do this... let's assume that $f(x) = tx^c$ for a wise reason that I need not explain. Just trust me that I am doing the right thing.

Proof.

$$\begin{aligned}
\text{diff } f(x) &= \frac{f(x+d) - f(x)}{(x+d) - x} \\
&= \frac{f(x+d) - f(x)}{d} \\
&= \frac{t(x+d)^c - tx^c}{d} \\
&= \frac{t(\sum_{n=0}^c \binom{c}{c-n} x^{c-n} d^n) - tx^c}{d} \\
&= \frac{(\sum_{n=0}^c t \binom{c}{c-n} x^{c-n} d^n) - tx^c}{d} \\
&= \frac{t \binom{c}{c-0} x^{c-0} d^0 + (\sum_{n=1}^c t \binom{c}{c-n} x^{c-n} d^n) - tx^c}{d} \\
&= \frac{tx^c + (\sum_{n=1}^c t \binom{c}{c-n} x^{c-n} d^n) - tx^c}{d} \\
&= \frac{\sum_{n=1}^c t \binom{c}{c-n} x^{c-n} d^n}{d} \\
&= \sum_{n=1}^c \frac{t \binom{c}{c-n} x^{c-n} d^n}{d} \\
&= \sum_{n=1}^c t \binom{c}{c-n} x^{c-n} d^{n-1} \\
&= t \binom{c}{c-1} x^{c-1} d^{1-1} + \sum_{n=2}^c t \binom{c}{c-n} x^{c-n} d^{n-1} \\
&= tcx^{c-1} + \sum_{n=2}^c t \binom{c}{c-n} x^{c-n} d^{n-1}
\end{aligned}$$

Then:

$$\lim_{d \rightarrow 0} tcx^{c-1} + \sum_{n=2}^c t \binom{c}{c-n} x^{c-n} d^{n-1} = tcx^{c-1}$$

OMG it's the Q.E.D. baby!

□

Chapter 3

Multivariate Stuff

3.1 Functions that are sum of other linear functions

Let's say that we got linear functions $b(n)$ and $c(m)$. By definition, this means that:

$$b(n) = c_1 + w_1 n \quad (3.1)$$

$$c(m) = c_2 + w_2 m \quad (3.2)$$

Theorem 2. If $a(n, m) = b(n) + c(m)$, then $\lim_{(n,m) \rightarrow (x,y)} a(n, m) = \lim_{n \rightarrow x} b(n) + \lim_{m \rightarrow y} c(m)$

Proof. Limits of b and c :

$$\begin{aligned} \lim_{n \rightarrow x} b(n) &= \lim_{n \rightarrow x} c_1 + w_1 n \\ &= c_1 + w_1 x \end{aligned} \quad (3.3)$$

$$\begin{aligned} \lim_{m \rightarrow y} c(m) &= \lim_{m \rightarrow y} c_2 + w_2 m \\ &= c_2 + w_2 y \end{aligned} \quad (3.4)$$

Limit of a by theorem:

$$\lim_{(n,m) \rightarrow (x,y)} a(n, m) = c_1 + w_1 x + c_2 + w_2 y \quad (3.5)$$

By substitution we get expanded a :

$$a(n, m) = c_1 + w_1 n + c_2 + w_2 m \quad (3.6)$$

Limit of expanded a :

$$\begin{aligned} \lim_{(n,m) \rightarrow (x,y)} a(n, m) &= \lim_{(n,m) \rightarrow (x,y)} c_1 + w_1 n + c_2 + w_2 m \\ &= c_1 + w_1 x + c_2 + w_2 y \end{aligned} \quad (3.7)$$

Guess wat, limit of expanded a is the same as the original one. Who could've thought? \square