PHILOSOPHIÆ NATURALIS **PRINCIPIA** MATHEMATICA

Al Caveman

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Contents

Chapter 1

Integration

The goal here is to integrate the area under $f(x)$ from when $x = 0$ until we reach $x = x_{end}$. What follows here is bunch of steps that shows my thinking process because *shadowdaemon* asked for it (and also because I am happy now).

1.1 Areas of Lots of Extremely Tiny Rectangular Columns

So to integrate $f(x) = x^2$ we have to keep summing extremely skinny columns[1] of, each of width d.

$$
\begin{aligned}
&(x + d - x)(x)^2 + \\
&(x + 2d - (x + d))(x + d)^2 + \\
&(x + 3d - (x + 2d))(x + 2d)^2 + \\
&(x + 4d - (x + 3d))(x + 3d)^2 + \\
&(x + 5d - (x + 4d))(x + 4d)^2 + \\
&(x + 6d - (x + 5d))(x + 5d)^2 + \\
&(x + 7d - (x + 6d))(x + 6d)^2 + \\
&(x + 8d - (x + 7d))(x + 7d)^2 + \\
&(x + 9d - (x + 8d))(x + 8d)^2 + \\
&(x + 10d - (x + 9d))(x + 9d)^2 + \\
&(x + 11d - (x + 10d))(x + 10d)^2 + \\
&\dots
\end{aligned}
$$

You see, if d is extremely tiny (near zero), then we will have to sum an infinite number of those tiny skinny areas. But for simplicity I put ... instead.

[1] **Note:** *fefelix* of *freenode/#gentoo-chat-exile* tried to look smart by attacking my rigor by saying that the term *skinny columns* is wrong and that it must be replaced by the term *infinitesimal* (facepalm moment here). He also tried to look even smarter by using the phrase *In the realm of* $\mathbb{R}$. Obviously, any half-assed mathematician knows that $\mathbb{R}$ has only a single infinitesimal which is zero. Yep, zero bitch. So the term *infinitesimal* is absolutely wrong in this context, thus even worse than *skinny columns*. The only exception is if *fefelix* wishes to live in the 1600s.

We can simplify things:

$$\begin{aligned}
d(x)^2 &+ \\
d(x+d)^2 &+ \\
d(x+2d)^2 &+ \\
d(x+3d)^2 &+ \\
d(x+4d)^2 &+ \\
d(x+5d)^2 &+ \\
d(x+6d)^2 &+ \\
d(x+7d)^2 &+ \\
d(x+8d)^2 &+ \\
d(x+9d)^2 &+ \\
d(x+10d)^2 &+ \\
&\ldots
\end{aligned}$$

Let's expand those squares:

$$\begin{aligned}
d(x)(x) &+ \\
d(x+d)(x+d) &+ \\
d(x+2d)(x+2d) &+ \\
d(x+3d)(x+3d) &+ \\
d(x+4d)(x+4d) &+ \\
d(x+5d)(x+5d) &+ \\
d(x+6d)(x+6d) &+ \\
d(x+7d)(x+7d) &+ \\
d(x+8d)(x+8d) &+ \\
d(x+9d)(x+9d) &+ \\
d(x+10d)(x+10d) &+ \\
&\ldots
\end{aligned}$$

Let's multiply them square bitches:

$$
\begin{aligned}
dx^2&+\\
d(x^2+2dx+d^2)&+\\
d(x^2+4dx+4d^2)&+\\
d(x^2+6dx+9d^2)&+\\
d(x^2+8dx+16d^2)&+\\
d(x^2+10dx+25x^2)&+\\
d(x^2+12dx+36d^2)&+\\
d(x^2+14dx+49d^2)&+\\
d(x^2+16dx+64d^2)&+\\
d(x^2+18dx+81d^2)&+\\
d(x^2+20dx+100d^2)&+\\
\ldots&
\end{aligned}
$$

Let's now multiply those bitches with d so that the shit gets spread even more:

$$
\begin{aligned}
dx^2&+\\
dx^2+2d^2x+d^3&+\\
dx^2+4d^2x+4d^3&+\\
dx^2+6d^2x+9d^3&+\\
dx^2+8d^2x+16d^3&+\\
dx^2+10d^2x+25x^3&+\\
dx^2+12d^2x+36d^3&+\\
dx^2+14d^2x+49d^3&+\\
dx^2+16d^2x+64d^3&+\\
dx^2+18d^2x+81d^3&+\\
dx^2+20d^2x+100d^3&+\\
\ldots&
\end{aligned}
$$

1.2 Approximating the Area

Now this is a critical point. Below is basically saying that each row is an approximation for the area under $f(x)$ from $x = 0$ till $x = x_{end}$. So the 1^{st} one is a shit approximation where were approximate the area under that curve by only one big fat column; so d is so huge here, in fact $d = x_{end}$.

Then, the 2nd line show a slightly less shit approximation where we approximate the area under the curve by two fat ass rectangular columns. So here $d = \frac{x_{end}}{2}$.

So the approximation of the area under the curve gets more and more accurate in each line.

$$\begin{aligned}
dx^2\\
2dx^2 + 2d^2x + d^3\\
3dx^2 + 6d^2x + 5d^3\\
4dx^2 + 12d^2x + 14d^3\\
5dx^2 + 20d^2x + 30d^3\\
6dx^2 + 30d^2x + 55d^3\\
7dx^2 + 42d^2x + 91d^3\\
8dx^2 + 56d^2x + 140d^3\\
9dx^2 + 72d^2x + 204d^3\\
10dx^2 + 90d^2x + 285d^3\\
11dx^2 + 110d^2x + 385d^3
\end{aligned}$$

So basically, you see there is a pattern. The coefficient of the 1^{st} term is easy peasy (just incrementing from 1 to ∞). The coefficient from the 2^{nd} term is kinda interesting, it follows the equation $(i^2 + i)$ where i is the line number. Note that we start counting lines from 0. So the 1^{st} has $i = 0$ and the 2^{nd} line has $i = 1$, etc. Finally, the last term is kinda cool, it follows the pattern $\frac{(i^2+i)(2i+1)}{6}$.

Now you may ask, how did I find these patterns? Well these are well known number series. You can look them up in the On-Line Encyclopedia of Integer Sequences[2].

So, the area under the curve of $f(x)$ from $x = 0$ up to x_{end}, by any d (and its corresponding i) is:

$$dx^2 + (i^2 + i)d^2x + \frac{(i^2 + i)(2i + 1)}{6}d^3$$

Now we are almost done. We know that $x = 0$, so we can cancel a few terms:

$$\begin{aligned}
d0^2 + (i^2 + i)d^2 0 + \frac{(i^2 + i)(2i + 1)}{6}d^3\\
\frac{(i^2 + i)(2i + 1)}{6}d^3
\end{aligned} \tag{1.1}$$

Of course, we could've canceled those terms that multiply against zero earlier, but I didn't for random reasons. I just didn't. That's the randomness of life. But it's all mathematically correct as my caveman balls tell.

You can code a simple script that you give it x_{end} and d, by which it automatically finds $i = x_{end}/d$. You will notice that as d gets smaller, you end up approaching some limit after which reduction in d does not cause any change in the estimated area under the curve.

[2] http://oeis.org/

1.3 The Precise Area Under The Bitch

Now let's find the ultimate precision in the limit as $i \to \infty$ which also means that $d \to 0$. But how about not? Cause it's too hard to solve the limit when two variables are approaching different limits.

To simplify the limits in an easier way, let's represent i in terms of d and x_{end} as follows $i = x_{end}/d$. Then the same equation would become as follows:

$$\frac{((x_{end}/d)^2 + (x_{end}/d))(2(x_{end}/d) + 1)}{6} d^3$$
$$\frac{(\frac{x_{end}^2}{d^2} + \frac{x_{end}}{d})(\frac{2x_{end}}{d} + 1)}{6} d^3$$
$$\frac{(\frac{x_{end}^2}{d^2} + \frac{x_{end}}{d})(\frac{2x_{end}}{d} + 1)}{6} d^3$$
$$\frac{\frac{1}{d}(\frac{x_{end}^2}{d} + x_{end})(\frac{2x_{end}}{d} + 1)}{6} d^3$$
$$\frac{(\frac{x_{end}^2}{d} + x_{end})(\frac{2x_{end}}{d} + 1)}{6} d^2$$
$$\frac{\frac{2x_{end}^3}{d^2} + \frac{x_{end}^2}{d} + \frac{2x_{end}^2}{d} + x_{end}}{6} d^2$$
$$\frac{\frac{2x_{end}^3 d^2}{d^2} + \frac{x_{end}^2 d^2}{d} + \frac{2x_{end}^2 d^2}{d} + x_{end} d^2}{6}$$
$$\frac{2x_{end}^3 + x_{end}^2 d + 2x_{end}^2 d + x_{end} d^2}{6}$$

Now, it's super easy. We have to find the limit of that equation as a single variable approaches 0 (we got rid of i). The equation becomes:

$$\frac{2x_{end}^3}{6}$$
$$\frac{x_{end}^3}{3}$$

That's it. Integration re-invented bitch :) — $\frac{x_{end}^2}{3}$.

Q.E. freaking DEE.

1.4 Generalizing That

Let's integrate x^c where $x, c \in \mathbb{R}$. How easy is that? Let's try.

So, back to the business of summing skinny columns:

$$\begin{aligned} d \times d^c + \\ d \times (2d)^c + \\ d \times (3d)^c + \\ d \times (4d)^c + \\ d \times (5d)^c + \\ \ldots \end{aligned}$$

Simplified to:

$$\begin{aligned} d \times d^c + \\ d \times 2^c \times d^c + \\ d \times 3^c \times d^c + \\ d \times 4^c \times d^c + \\ d \times 5^c \times d^c + \\ \ldots \end{aligned}$$

As we try to estimate that sum better we get:

$$nd \times (\frac{n(n+1)}{2})^c \times d^c$$

Chapter 2

Differentiation

Here we want to find the slope of $f(x)$ at point x. This is easy so I won't say much here.

Differentiate x^2.
$\frac{f(x+d)-f(x)}{x+d-x}$
$\frac{(x+d)^2-x^2}{x+d-x}$
$\frac{(x+d)^2-x^2}{d}$
$\frac{(x+d)(x+d)-x^2}{d}$
$\frac{x^2+xd+xd+d^2-x^2}{d}$
$\frac{x^2}{d}+\frac{2xd}{d}+\frac{d^2}{d}-\frac{x^2}{d}$
$\frac{x^2}{d}+2x+d-\frac{x^2}{d}$
$2x+d$
Now, as $d \to 0$, it becomes $2x$. Done.

2.1 Theorems

See, Gottfried Leibniz was nuts, he picked bad notations ($\frac{dy}{dx}$ dafuqq!). Others were even more nuts and picked worse notations. But having nutter nuts exist is no execuse for letting lesser nuts exist. So I hereby fix all your problems by introducing the symbol diff. That's it. Easy. Just diff.

If t, x, c are any numbers, except c is just a natty number.

Theorem 1. $\operatorname{diff} tx^c = ctx^{c-1}$.

Okay let's do this... let's assume that $f(x) = tx^c$ for a wise reason that I need not explain. Just trust me that I am doing the right thing.

Proof.

$$\begin{aligned}
\operatorname{diff} f(x) &= \frac{f(x+d)-f(x)}{(x+d)-x} \\
&= \frac{f(x+d)-f(x)}{d} \\
&= \frac{t(x+d)^c - tx^c}{d} \\
&= \frac{t(\sum_{n=0}^{c}\binom{c}{c-n}x^{c-n}d^n) - tx^c}{d} \\
&= \frac{(\sum_{n=0}^{c} t\binom{c}{c-n}x^{c-n}d^n) - tx^c}{d} \\
&= \frac{t\binom{c}{c-0}x^{c-0}d^0 + (\sum_{n=1}^{c} t\binom{c}{c-n}x^{c-n}d^n) - tx^c}{d} \\
&= \frac{tx^c + (\sum_{n=1}^{c} t\binom{c}{c-n}x^{c-n}d^n) - tx^c}{d} \\
&= \frac{\sum_{n=1}^{c} t\binom{c}{c-n}x^{c-n}d^n}{d} \\
&= \sum_{n=1}^{c} \frac{t\binom{c}{c-n}x^{c-n}d^n}{d} \\
&= \sum_{n=1}^{c} t\binom{c}{c-n}x^{c-n}d^{n-1} \\
&= t\binom{c}{c-1}x^{c-1}d^{1-1} + \sum_{n=2}^{c} t\binom{c}{c-n}x^{c-n}d^{n-1} \\
&= tcx^{c-1} + \sum_{n=2}^{c} t\binom{c}{c-n}x^{c-n}d^{n-1}
\end{aligned}$$

Then:

$$\lim_{d\to 0} tcx^{c-1} + \sum_{n=2}^{c} t\binom{c}{c-n}x^{c-n}d^{n-1} = tcx^{c-1}$$

OMG it's the Q.E.D. baby! □