1 Experiment No. 11

2 Experiment Title

Open-ended lab: Modeling and solving an RLC circuit using numerical ODE methods.

2.1 Objectives

1. To model a series RLC circuit and derive its governing differential equation
2. To convert the second-order ODE into a system of first-order ODEs
3. To implement numerical solutions using four different methods
4. To compare the accuracy and computational efficiency of each method
5. To analyze the transient response of the RLC circuit

2.2 Theory

RLC Circuit Modeling

Consider a series RLC circuit with:

1. Resistor $R = 10\Omega$
2. Inductor $L = 0.5$ H
3. Capacitor $C = 0.01$ F
4. Voltage source $V(t) = 10\sin(2t)$

The governing differential equation derived from Kirchhoff's Voltage Law is:

$$L\frac{d^2i}{dt^2} + R\frac{di}{dt} + \frac{1}{C}i = \frac{dV}{dt}$$

First-Order System Conversion

Let:

$$x_1 = i \quad \text{(current)}$$
$$x_2 = \frac{di}{dt} \quad \text{(rate of change of current)}$$

Given the voltage source:

$$V(t) = 10\sin(2t)$$

We compute its derivative to use in the ODE:

$$\frac{dV(t)}{dt} = \frac{d}{dt}[10\sin(2t)] = 10 \cdot 2\cos(2t) = 20\cos(2t)$$

This explains the appearance of the $20\cos(2t)$ term in the state-space equation:

$$\frac{dx_2}{dt} = \frac{1}{L}\left(20\cos(2t) - Rx_2 - \frac{1}{C}x_1\right)$$

This converts the second-order ODE into a system of first-order ODEs:

$$\frac{dx_1}{dt} = x_2$$
$$\frac{dx_2}{dt} = \frac{1}{L}\left(\frac{dV}{dt} - Rx_2 - \frac{1}{C}x_1\right)$$

Substituting the given parameters and $V(t)$:

$$\frac{d\mathbf{x}}{dt} = \begin{bmatrix} x_2 \\ \frac{20\cos(2t) - 10x_2 - 100x_1}{0.5} \end{bmatrix}$$

Analytical Solution of the RLC Circuit

We solve the second-order non-homogeneous differential equation:

$$\frac{d^2i}{dt^2} + 20\frac{di}{dt} + 200i = 40\cos(2t)$$

Step 1: Homogeneous solution
The characteristic equation is:

$$r^2 + 20r + 200 = 0 \Rightarrow r = -10 \pm 10i$$

So the homogeneous solution is:

$$i_h(t) = e^{-10t}(A\cos(10t) + B\sin(10t))$$

Step 2: Particular solution
We try a particular solution of the form:

$$i_p(t) = C\cos(2t) + D\sin(2t)$$

Substituting and solving, we get:

$$C \approx 0.196, \quad D \approx 0.04$$

General solution:

$$i(t) = e^{-10t}(A\cos(10t) + B\sin(10t)) + 0.196\cos(2t) + 0.04\sin(2t)$$

Using the initial conditions:

$$i(0) = 0, \quad \frac{di}{dt}(0) = 0$$

We find:

$$A = -0.196, \quad B \approx -0.276$$

Final analytical solution:

$$i(t) = e^{-10t}(-0.196\cos(10t) - 0.276\sin(10t)) + 0.196\cos(2t) + 0.04\sin(2t)$$

2.3 Algorithm

Initialization

1. Define circuit parameters: R, L, C
2. Define voltage source function $V(t)$
3. Set initial conditions: $i(0) = 0$, $\frac{di}{dt}(0) = 0$
4. Set time parameters: $t_{\text{start}} = 0$, $t_{\text{end}} = 10$ s, step size $h = 0.01$ s
5. Initialize solution arrays for each method

Numerical Methods

For each time step t_i from 0 to 10 s with increment h:

Euler's Method

$$\mathbf{x}_{i+1} = \mathbf{x}_i + h \cdot \mathbf{f}(t_i, \mathbf{x}_i)$$

Heun's Method

$$\begin{aligned}
\mathbf{k}_1 &= \mathbf{f}(t_i, \mathbf{x}_i) \\
\mathbf{k}_2 &= \mathbf{f}(t_i + h, \mathbf{x}_i + h\mathbf{k}_1) \\
\mathbf{x}_{i+1} &= \mathbf{x}_i + \frac{h}{2}(\mathbf{k}_1 + \mathbf{k}_2)
\end{aligned}$$

RK2 Method

$$\begin{aligned}
\mathbf{k}_1 &= \mathbf{f}(t_i, \mathbf{x}_i) \\
\mathbf{k}_2 &= \mathbf{f}\left(t_i + \frac{h}{2}, \mathbf{x}_i + \frac{h}{2}\mathbf{k}_1\right) \\
\mathbf{x}_{i+1} &= \mathbf{x}_i + h\mathbf{k}_2
\end{aligned}$$

RK4 Method

$$\begin{aligned}
\mathbf{k}_1 &= \mathbf{f}(t_i, \mathbf{x}_i) \\
\mathbf{k}_2 &= \mathbf{f}\left(t_i + \frac{h}{2}, \mathbf{x}_i + \frac{h}{2}\mathbf{k}_1\right) \\
\mathbf{k}_3 &= \mathbf{f}\left(t_i + \frac{h}{2}, \mathbf{x}_i + \frac{h}{2}\mathbf{k}_2\right) \\
\mathbf{k}_4 &= \mathbf{f}(t_i + h, \mathbf{x}_i + h\mathbf{k}_3) \\
\mathbf{x}_{i+1} &= \mathbf{x}_i + \frac{h}{6}(\mathbf{k}_1 + 2\mathbf{k}_2 + 2\mathbf{k}_3 + \mathbf{k}_4)
\end{aligned}$$

2.4 MATLAB Program

```
% RLC Circuit ODE Solver
clc; clear; close all;

% Circuit parameters
R = 10;      % Resistance (Ohms)
L = 0.5;     % Inductance (H)
C = 0.01;    % Capacitance (F)
V = @(t) 10*sin(2*t);  % Voltage source

% ODE system: dx/dt = f(t,x)
f = @(t,x) [x(2);
(2*10*cos(2*t) - R*x(2) - x(1)/C)/L];

% Initial conditions and time span
x0 = [0; 0];     % [i(0); di/dt(0)]
h = 0.01;          % Step size
t = 0:h:10;

% Initialize solution arrays
n = length(t);
x_euler = zeros(2,n);
x_euler(:,1) = x0;
x_heun = zeros(2,n);
x_heun(:,1) = x0;
x_rk2 = zeros(2,n);
x_rk2(:,1) = x0;
x_rk4 = zeros(2,n);
x_rk4(:,1) = x0;
% Analytical solution
A = -0.196;
B = -0.276;
C = 0.196;
D = 0.04;
i_analytical = exp(-10*t).*(A*cos(10*t) + B*sin(10*t)) + C*cos(2*t) + D*
sin(2*t);

% Numerical solutions
for i = 1:n-1
% Euler's method
x_euler(:,i+1) = x_euler(:,i) + h*f(t(i),x_euler(:,i));

% Heun's method
k1 = f(t(i), x_heun(:,i));
k2 = f(t(i)+h, x_heun(:,i) + h*k1);
x_heun(:,i+1) = x_heun(:,i) + h/2*(k1 + k2);

% RK2 method
k1 = f(t(i), x_rk2(:,i));
k2 = f(t(i)+h/2, x_rk2(:,i) + h/2*k1);
x_rk2(:,i+1) = x_rk2(:,i) + h*k2;

% RK4 method
k1 = f(t(i), x_rk4(:,i));
k2 = f(t(i)+h/2, x_rk4(:,i) + h/2*k1);
k3 = f(t(i)+h/2, x_rk4(:,i) + h/2*k2);
k4 = f(t(i)+h, x_rk4(:,i) + h*k3);
x_rk4(:,i+1) = x_rk4(:,i) + h/6*(k1 + 2*k2 + 2*k3 + k4);
```

```matlab
    end

    % Plot results - Current (x1) vs Time
    figure;
    hold on;
    plot(t, x_euler(1,:), 'r-', 'LineWidth', 1.5);
    plot(t, x_heun(1,:), 'g--', 'LineWidth', 1.5);
    plot(t, x_rk2(1,:), 'b-.', 'LineWidth', 1.5);
    plot(t, x_rk4(1,:), 'm:', 'LineWidth', 2);
    plot(t, i_analytical, 'k-', 'LineWidth', 2);  % Analytical solution in black

    xlabel('Time (s)');
    ylabel('Current (A)');
    title('RLC Circuit Current Response Comparison');
    legend('Euler', 'Heun', 'RK2', 'RK4', 'Analytical', 'Location', 'northeast');
    % Zoomed-in box (inset axes)
    inset = axes('Position', [0.6, 0.2, 0.3, 0.3]);  % [x, y, width, height]
    box on; hold on;
    plot(t, x_euler(1,:), '-r.', 'MarkerSize', 6, 'LineWidth', 1.5);
    plot(t, x_heun(1,:), '-g.', 'MarkerSize', 6, 'LineWidth', 1.5);
    plot(t, x_rk2(1,:), '-b.', 'MarkerSize', 6, 'LineWidth', 1.5);
    plot(t, x_rk4(1,:), '-m.', 'MarkerSize', 6, 'LineWidth', 1.5);
    plot(t, i_analytical, '-k.', 'MarkerSize', 8, 'LineWidth', 2);

    % Set zoom-in region (adjust based on where you want to zoom)
    xlim([3.14 3.36]);    % Zoom in around t = 2s
    ylim([0.195 0.205]);      % Adjust y-axis range accordingly
    grid on;

```

Listing 1: MATLAB Code for RLC Circuit Simulation

2.5 Results

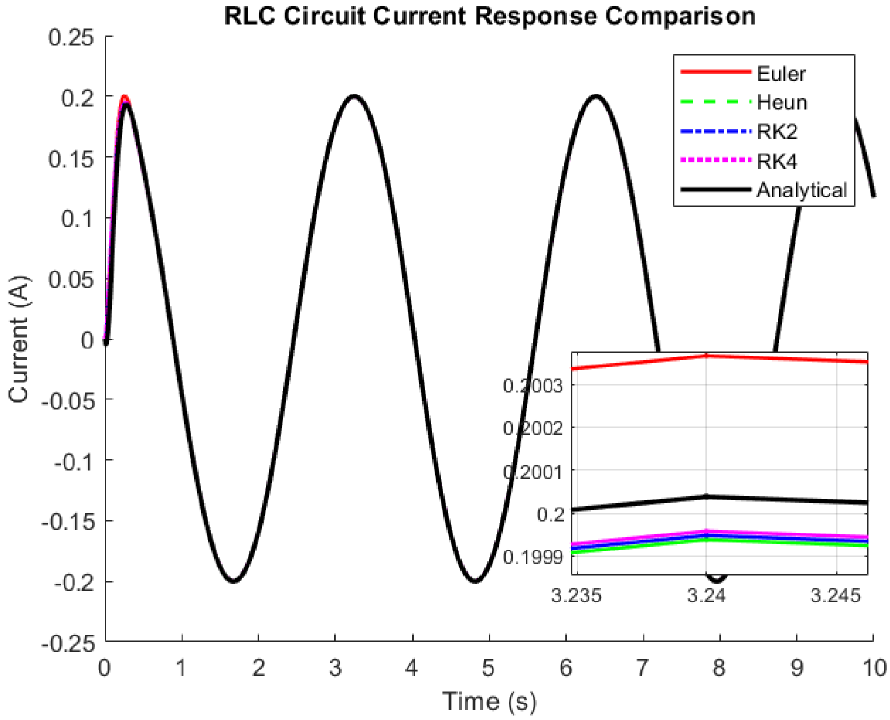

Figure 1: Current response in RLC circuit simulated using four numerical methods. The RK4 method (magenta) shows the smoothest response, while Euler's method (red) exhibits significant numerical oscillation.

2.6 Discussion

The experiment successfully modeled a series RLC circuit and solved its governing differential equation using four numerical methods. Key observations:

- **Method Comparison**:
  1. *Euler's method* showed significant error accumulation and oscillation, especially during rapid changes in current
  2. *Heun's method* provided moderate improvement but still exhibited phase errors
  3. *RK2 method* demonstrated good balance between accuracy and computation
  4. *RK4 method* produced the most accurate and stable solution

While Euler's method is simplest to implement, its accuracy limitations make it unsuitable for sensitive electrical system simulations. For most engineering applications, RK4 provides the best compromise between accuracy and computational requirements. This experiment demonstrates how numerical methods enable analysis of complex electrical systems that lack closed-form solutions.